4. ALGEBRAIC GROUPS ACTING ON SCHEMES

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The aim of this section is to give a short outline of the theory of algebraic groups (and group schemes) acting on schemes. Our aim is to develop methods to prove the existence and projectivity of certain quotients. At the end we have to restrict ourselves to reductive groups over a field of characteristic zero, but we set up the general theory needed for the moduli of surfaces over Spec $\mathbb{Z}$. For a more general discussion of some of the topics see [Mumford-Fogarty82; Kollár95; Keel-Mori95]. We start with the basic definitions.

??.1 Basic Definitions

??.1.1 Definition. An *algebraic group* (over an algebraically closed field) is a variety G which is also a group such that the multiplication map $\mu : G \times G \to G$ and the inverse $b : G \to G$ are morphisms of varieties.

More generally, one can define the notion of a *group scheme* to be a scheme G/S together with morphisms

multiplication: $\mu : G \times_S G \to G$,

inverse: $b : G \to G$, and

unit: $e : S \to G$,

which satisfy the obvious group laws in the scheme theoretic setting (see [Mumford-Fogarty82, 0.1]).

For the applications in these notes we only need the examples $GL(n,k)$ and $PGL(n,k)$ over a field k. In order to indicate the level of generality, we talk about group schemes when the results work in that setting, but one can always think about the above examples.

In order to avoid some bad examples, we always assume that G is flat and of finite type over S. This is crucial, for instance, in (??.1.3.2).

??.1.2 Definition. (??.1.2.1) Let G be an algebraic group and X a variety (over an algebraically closed field). An *action of* G *on* X is a morphism $m_X : (g,x) \mapsto gx$ such that

$ex = x$ for every $x \in X$, and

$(g_1 g_2)x = g_1(g_2 x)$ for every $g_1, g_2 \in G$ and $x \in X$.

We usually say that X is a G-*variety* if the action is understood.

In the scheme theoretic version, let G/S be a group scheme and X/S a scheme. An *action of* G *on* X is a morphism $m_X : G \times_S X \to X$ such that

$m_X(e, id_X) : X \to X$ is the identity, and

$m_X(\mu(p_1,p_2),p_3) = m_X(p_1, m_X(p_2,p_3))$, where the p_i are the coordinate projections of $G \times_S G \times_S X$.

We usually say that X is a *G-scheme* if the action is understood.

If $\psi : U \to G$ and $f : U \to X$ are morphisms then $m_X(\psi, f) : U \to X$ is usually denoted by $\psi \cdot f$.

The *trivial* G action on X is given by $m_X = p_2$ where $p_2 : G \times_S X \to X$ is the second projection.

(??.1.2.2) Let G be a group, X a G-variety and F a sheaf on X. An *action of G on F* is a collection of isomorphisms $\sigma(g) : g^*F \cong F$ which satisfy the so called cocycle condition:

$$\sigma(gg') = \sigma(g) \circ g^*\sigma(g').$$

We say that F is a *G-sheaf* if the action is understood.

If G is an algebraic group then we also require that the $\sigma(g)$ "depend algebraically" on g. The easiest way to make this precise is by requiring that the $\sigma(g)$ be the sections of a single isomorphism $\sigma : m_X^*F \cong p_2^*F$ where $p_2 : G \times X \to X$ is the second projection.

If the F_i are G-sheaves, then the usual tensor operations (like $F^{\otimes m}$ or $Hom(F_1, F_2)$) are also G-sheaves.

It is easy to see that the G action on F induces an action on the Čech cohomology groups $H^i(X, F)$ (this is called the dual action in [Mumford-Fogarty82, p.32]).

In the scheme theoretic version, let G/S be a group scheme, X/S a G-scheme and F a sheaf on X. An *action of G on F* is an isomorphism $\sigma : m_X^*F \cong p_2^*F$ which satisfies the scheme theoretic analog of the cocycle condition; see [Mumford-Fogarty82, p.30] for details.

(??.1.2.3) If X is a scheme with trivial G-action and F a (quasi coherent) G-sheaf on X then

$$X \supset U \mapsto H^0(U, F|U)^G$$

is a (quasi coherent) sheaf on X. It is denoted by F^G. (For group schemes this can be made precise using the notion of a dual action [Mumford-Fogarty82, p.25].)

??.1.3 Definition. (??.1.3.1) Let $m_i : G \times X_i \to X_i$ be G-schemes. A morphism $f : X_1 \to X_2$ is called a *G-morphism* if the following diagram is commutative:

$$\begin{array}{ccc} G \times X_1 & \xrightarrow{m_1} & X_1 \\ {\scriptstyle id_G \times f}\downarrow & & \downarrow{\scriptstyle f} \\ G \times X_2 & \xrightarrow{m_2} & X_2. \end{array}$$

(??.1.3.2) Assume that G/S is flat and let F be a G-sheaf on X_1. m_2 and p_2 are flat, thus we have isomorphisms

$$m_2^*(R^i f_* F) \cong R^i(id_G \times f)_*(m_1^* F) \cong R^i(id_G \times f)_*(p_1^* F) \cong p_2^*(R^i f_* F).$$

This shows that the higher direct images $R^i f_* F$ are G-sheaves on X_2.

For any G-scheme X, we would like to construct another scheme which can be reasonably called the quotient of X by G. As we saw in (!!), this gives moduli spaces for certain classes of polarized varieties. To start with, we introduce the various quotient concepts:

??.1.4 Definition. Let G/S be a group scheme and X/S a G-scheme. Let Z be a scheme with trivial G-action and $q : X \to Z$ a G-morphism. Assume for simplicity that X is of finite type.

(??.1.4.1) $q : X \to Z$ is called a *categorical quotient* of X by G if, for every scheme U with trivial G-action and G-morphism $p : X \to U$, there is a morphism $u : Z \to U$ such that $p = u \circ q$.

(??.1.4.2) $q : X \to Z$ is called a *geometric quotient* of X by G if the following conditions are satisfied:

(??.1.4.2.1) For every algebraically closed field K the natural map $X(K)/G(K) \to Z(K)$ is an isomorphism (of sets).

(??.1.4.2.2) q is universally submersive (i.e. $U \subset Z$ is open iff $q^{-1}(U) \subset X$ is, and this also holds after any base change $Z' \to Z$).

(??.1.4.2.3) $\mathcal{O}_Z = (q_* \mathcal{O}_X)^G$.

??.1.4.3 Remark. It is not hard to see that a geometric quotient is also a categorical quotient, see [Mumford-Fogarty82, 0.1]. In particular, it is unique up to isomorphism. (As it was observed in [Kollár95] this fails for algebraic spaces.)

In general not all actions have quotients, and we need some conditions that guarantee that the group action is reasonable. There are many different types of conditions. We concentrate on those that come up naturally in the study of moduli of polarized schemes:

??.1.5 Definition–Proposition. Let X be a G-scheme. We say that the action of G on X is *proper* if the following equivalent conditions are satisfied:

(??.1.5.1) The morphism

$$\Psi_X : G \times_S X \xrightarrow{(m_X, p_2)} X \times_S X \quad \text{is proper,}$$

where $p_2 : G \times_S X \to X$ is the second projection.

(??.1.5.2) Let T be the spectrum of a DVR with generic point t_g and $u_1, u_2 : T \to X$ two morphisms. Assume that there is a morphism $\psi_g : t_g \to G$ such that $u_1(t_g) = \psi_g(t_g) \cdot u_2(t_g)$. Then there is a morphism $\psi : T \to G$ extending $\psi_g : t_g \to G$ such that $u_1 = \psi \cdot u_2$.

Informally speaking, any morphism $u_g : t_g \to X$ has at most one extension to $u : T \to X$, up to translation by G.

Proof. A morphism $u = (u_1, u_2) : T \to X \times X$ plus a lifting over the generic point $t_g \mapsto (\psi_g(t_g), u_2(t_g))$ corresponds to a pair of morphisms $u_1, u_2 : T \to X$ such that $u_1(t_g) = \psi_g(t_g) \cdot u_2(t_g)$. We have a diagram

$$\begin{tikzcd} t_g \arrow[r, "(\psi_g, u_2)"] \arrow[d] & G \times X \arrow[d, "\Psi_X"] \\ T \arrow[r, "(u_1, u_2)"] & X \times X, \end{tikzcd}$$

and Ψ_X is proper iff the diagram can be completed with the arrow $(\psi, u_2) : T \to G \times X$ (while keeping it commutative). (??.1.5.2) precisely says that such a ψ exists. $\square$

??.1.5.3 Lemma. *Assume that G/S is affine and G acts on X properly. Then the stabilizer subgroup of any closed point is finite.*

Proof. By definition, the stabilizer of a point $x \in X$ is $\Psi_X^{-1}(x,x)$, as a subgroup of G. It is both proper and affine so finite. □

??.1.6 Definition. We say that X mod G is *complete* (or *proper*) if X is of finite type, the G-action on X is proper and the following version of the valuative criterion of properness is satisfied:

Let T be the spectrum of a DVR with generic point t_g. Let $u_g : t_g \to X$ be a morhism. Then there are

(??.1.6.1) T', the spectrum of a DVR with generic point t'_g,

(??.1.6.2) a finite and surjective morphism $h : T' \to T$,

(??.1.6.3) a morphism $\phi'_g : t'_g \to G$,

such that $u_g \circ (h|t'_g) = \phi'_g \cdot (u \circ h|t'_g)$.

Informally speaking, any morphism $u_g : t_g \to X$ extends to $u : T \to X$, at least after enlarging T and after a suitable translation by the group.

??.1.7 Remarks. The conditions (??.1.5–6) are cooked up to imply the separatedness (resp. properness) of quotients. This is shown by (??.1.8).

It is somewhat unfortunate that the name "proper" is already used in (??.1.5) to describe a condition which guarantees the separatedness of the quotient. We use the adjective "complete", to avoid any confusion.

??.1.8 Proposition. *Assume that X is of finite type over S and that a geometric quotient $q : X \to X/G$ exists.*

(??.1.8.1) If the G-action on X is proper then X/G is separated.

(??.1.8.2) If X mod G is complete then X/G is proper.

Proof. Let T be the spectrum of a DVR and $u : T \to X/G$ a morphism. Since q is universally submersive, there is a finite and surjective morphism $h : T' \to T$ (where T' is the spectrum of a DVR) such that $u \circ h : T' \to X/G$ can be lifted to a morphism $\bar{u} : T' \to X$.

Let $u_1, u_2 : T \to X/G$ be two morphisms which agree on the generic point. We find $T' \to T$ such that both of them lift to $\bar{u}_i : T' \to X$. $\bar{u}_1|t'_g$ and $\bar{u}_2|t'_g$ lift the same morphism $t'_g \to X/G$, thus they differ by translation by G. Since the G action on X is proper, the two morphisms $\bar{u}_i$ differ only by a translation by G. Thus $q \circ \bar{u}_1 = q \circ \bar{u}_2$, hence X/G is separated.

The second part is even easier. □

??.1.8.3 Remarks. [Mumford-Fogarty82, p.11] gives an example where X/G is separated but the G action on X is not proper. The above proof works also for algebraic spaces.

Both of these conditions behave well with respect to proper morphism:

??.1.9 Proposition. *Let $f : X \to Y$ be a proper G-morphism of G-schemes.*

(??.1.9.1) If G acts properly on Y then G acts properly on X and the converse also holds if f is surjective.

(??.1.9.2) If Y mod G is complete then X mod G is complete and the converse also holds if f is surjective.

Proof. In order to see the first assertion, look at the commutative diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\Psi_X} & X \times X \\ {\scriptstyle id_G \times f}\downarrow & & \downarrow{\scriptstyle f \times f} \\ G \times Y & \xrightarrow{\Psi_Y} & Y \times Y. \end{array}$$

If Ψ_Y is proper then so is $\Psi_Y \circ (id_G \times f)$. This implies that Ψ_X is proper. The converse also holds if f is surjective.

In order to see the second part let $u_g : t_g \to X$ be a morphism. By assumption there is a finite and surjective morphism $h : T' \to T$ and $\phi'_g : t'_g \to G$ such that $\phi'_g \cdot (f \circ u_g \circ h) \colon t'_g \to Y$ extends to a morphism $u'_Y : T' \to Y$. Since f is proper, there is a unique lifting of u'_Y to a morphism $u' : T' \to X$ which agrees with $\phi'_g \cdot (u_g \circ h)$ on t'_g.

Finally assume that f is surjective and let $u_g : t_g \to Y$ be a morphism. Since f is surjective, there is a finite and surjective morphism $h : T' \to T$ and $u'_g : t'_g \to X$ such that $f \circ u'_g = u_g \circ h$. Since X mod G is complete, there is a finite and surjective morphism $h' : T'' \to T'$ and $\phi''_g : t''_g \to G$ such that $\phi''_g \cdot (u'_g \circ h') \colon t''_g \to X$ extends to a morphism $u''_X : T'' \to X$. $u''_Y := f \circ u''_X : T'' \to Y$ shows that Y mod G is complete. $\square$

The usual constructions of normalization and fiber product can be performed with G-schemes in most cases.

??.1.10 Definition. Let $f_i : X_i \to Y$ be G-morphisms of G-schemes. Let $\Delta : G \to G \times G$ be the diagonal imbedding. Then the fiber product $X_1 \times_Y X_2$ is a G-scheme via the morphism

$$m : G \times (X_1 \times_Y X_2) \xrightarrow{(\Delta, id)} G \times G \times (X_1 \times_Y X_2) \cong (G \times X_1) \times_Y (G \times X_2) \xrightarrow{(m_1, m_2)} X_1 \times_Y X_2.$$

The coordinate projections $X_1 \times_Y X_2 \to X_i$ are G-morphisms.

The following properties are easy to see:

??.1.11 Lemma. *Notation as above. Assume that G acts properly on X_i (resp. X_i mod G is complete) for $i = 1, 2$. Then G acts properly on $X_1 \times_Y X_2$ (resp. $(X_1 \times_Y X_2)$ mod G is complete).* $\square$

??.1.12 Lemma. [Seshadri72, Lemma 6.1] *Let $p : X_1 \to X_2$ be a quasifinite G-morphism of G-schemes of finite type such that X_1 is normal. Assume that G is smooth over S.*

Then there is a unique G-scheme $X'_1 \supset X_1$ such that X'_1 is normal, X_1 is dense in X'_1 and p extends to a finite G-morphism $p' : X'_1 \to X_2$.

Proof. Let $X'_1 \supset X_1$ be the normalization of the closure of $p(X_1)$ in the field of rational functions of X_1. X_1 is dense in X'_1 and p extends to a finite morphism $p' : X'_1 \to X_2$. We need to prove that X'_1 is a G-scheme and that p' is a G-morphism.

Let $\Gamma_i \subset (G \times X_i) \times X_i$ be the graph of the multiplication morphism m_{X_i} and $\Gamma_1^* \subset (G \times X_1') \times X_1'$ the preimage of Γ_2. Then $\Gamma_1^* \to \Gamma_2$ is finite, hence the first projection $\Gamma_1^* \to G \times X_1'$ is finite. Let $\Gamma_1' \subset \Gamma_1^*$ be the closure of Γ_1. Then $\Gamma_1' \to G \times X_1'$ is finite and birational on every irreducible component, thus an isomorphism since X_1' is normal. ($G \times X_1'$ is also normal since G is smooth over S.) This shows that m_{X_1} extends to a morphism $m_{X_1'} : G \times X_1' \to X_1'$.

We need to check that $m_{X_1'}$ defines an action of G on X_1' and that p' is a G-morphism. This requires the verification that certain diagrams are commutative. By assumption they are commutative when restricted to X_1. Since all occurring schemes are separated, and they have no embedded points in $X_1' - X_1$, this implies commutativity. $\square$

Applying (??.1.12) to the case when $X_1 \subset X_2$ is the normal locus of $\operatorname{red} X_2$ we obtain:

??.1.13 Corollary. *Let X be a G-scheme and $n : \bar{X} \to X$ the normalization. Assume that X is of finite type and G is smooth over S. Then $\bar{X}$ has a unique structure of a G-scheme such that n is a G-morphism.* $\square$

The simplest examples of G-schemes are principal G-bundles. They play a crucial role in subsequent constructions, so we consider them in some detail.

??.2 Principal G-bundles

??.2.1 Definition. A flat and surjective morphism of G-schemes $f : X \to Y$ is called a *principal fiber bundle over Y with group G* (or a principal G-bundle for short) if the G-action on Y is trivial and

$$\Psi_X : G \times_S X \to X \times_Y X \quad \text{is an isomorphism.}$$

The simplest example of a principal G-bundle is the trivial one:

??.2.2 Definition. A G-scheme X is called a *trivial G-bundle* if there is a subscheme $Z \subset X$ such that the composite

$$G \times Z \hookrightarrow G \times X \xrightarrow{m_X} X \quad \text{is an isomorphism.}$$

Let $p : X \to Z$ be the coordinate projection. It is easy to see that Z is the geometric quotient of X by G, in particular it is unique (up to a canonical isomorphism). We denote it by X/G.

Equivalently, we also say that $f : X \to Y$ is a *trivial G-bundle* if $X \cong G \times Y$, as G-schemes over Y.

A somewhat more complicated class of principal G-bundles is the following:

??.2.3 Definition. Let $f : X \to Y$ be a principal G-bundle. We say that X is a *Zariski locally trivial principal G-bundle* if there is an open cover $\cup Y_i = Y$ such that $f^{-1}(Y_i) \to Y_i$ is a trivial G-bundle for every i.

In more concrete terms, this means that there is an open cover $Y = \cup Y_i$, G-isomorphisms $g_i : X_i := f^{-1}Y_i \cong Y_i \times G$ and morphisms $\phi_{ij} : Y_i \cap Y_j \to G$ such that the following diagrams are commutative

$$\begin{array}{ccc} f^{-1}(Y_i \cap Y_j) & = & f^{-1}(Y_i \cap Y_j) \\ {\scriptstyle g_i}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle g_j}\big\downarrow{\scriptstyle \cong} \\ G \times (Y_i \cap Y_j) & \xrightarrow{l(\phi_{ji})} & G \times (Y_i \cap Y_j), \end{array}$$

where $l(\phi_{ji})$ is left multiplication by ϕ_{ji}.

If $\rho : G \to H$ is a homomorphism then to $f : X \to Y$ one can associate a Zariski locally trivial principal H-bundle by using the patching functions $\rho \circ \phi_{ji} : Y_i \cap Y_j \to H$.

It is easy to describe all G-sheaves on a principal G-bundle:

??.2.4 Lemma. *Let $p : X \to Y$ be a principal G-bundle. Then $F \mapsto p^*F$ provides an equivalence between the category of coherent sheaves on Y and the category of coherent G-sheaves on X.*

Proof. This follows from faithfully flat descent (see, for instance, [BLR90, 6.1]). In the special case of Zariski locally trivial principal G-bundles the construction is easier. Let E be a G-sheaf on X. We have local sections $p_i : Y_i \to X_i$, giving sheaves p_i^*E on Y_i. Multiplication by ϕ_{ij} gives the patching data on the sheaves p_i^*E. Thus we obtain a sheaf E_Y such that $E \cong p^*E_Y$. □

Trivial G-bundles are in some sense the "largest" schemes with G-actions:

??.2.5 Lemma. *Let $f : V \to X$ be a G-morphism of G-schemes.*

(??.2.5.1) If X is a trivial G-bundle then so is V.

(??.2.5.2) If X is a Zariski locally trivial principal G-bundle then so is V.

Proof. Let $Z_X \subset X$ show that X is a trivial G-bundle. Set $Z_V := V \times_X Z_X \subset V$. Then

$$G \times Z_V = G \times Z_X \times_X V \cong X \times_X V = V.$$

This shows (??.2.5.1) which implies (??.2.5.2). □

The following result gives a more local way of recognizing Zariski locally trivial principal G-bundles:

??.2.6 Proposition. *Let X be a separated G-scheme. Assume that there is a G-invariant open cover $\cup X_i = X$ such that every X_i is a trivial G-bundle. Assume that G acts properly on X.*

Then the geometric quotient $Z = X/G$ exists and X is a Zariski locally trivial principal G-bundle over Z.

Proof. The first step is the construction of a candidate for X/G:

??.2.6.1 Construction. Let X be a G-scheme. Assume that there is a G-invariant open cover $\cup X_i = X$ such that every X_i is a trivial G-bundle. Let $Z_i := X_i/G$ and $Z_{ij} := (X_i \cap X_j)/G$. The inclusions $Z_i \supset Z_{ij} \subset Z_j$ define patching data for the collection of schemes Z_i. This yields a scheme $Z = \cup Z_i$ and a morphism $p : X \to Z$.

Next we show that Z is separated if G acts properly on X. To see this, let T be the spectrum of a DVR with generic point t_g. We need to show that if $u_g : t_g \to Z_{ij}$ is a morphism which can be extended to $u^i : T \to Z_i$ and also to $u^j : T \to Z_j$ then $u_i = u_j$ (and they both map to Z_{ij}).

The pair of extensions gives a morphism $(u^i, u^j) : T \to X \times X$. The trivializations of $Z_i \times G \cong X_i$ and $Z_j \times G \cong X_j$ give morphisms $e^i : Z_i \to X_i$ and $e^j : Z_j \to X_j$. Over $X_i \cap X_j$ these differ by translation by ψ_{ij}. In particular

$$(u^i, u^j)(t_g) = \Psi_X(\psi_{ij} \cdot u_g(t_g), u_g(t_g)).$$

By assumption Ψ_X is proper, thus there is an extension $u : T \to X$ such that $(u^i, u^j) = \Psi_X(\psi_{ij} \cdot u, u)$. Thus $u_i = p \circ u = u_j$ which shows that Z is separated.

Consider the following diagram:

$$\begin{array}{ccccc}
G \times X & \xrightarrow{\Psi} & X \times X & \longleftarrow & X \times_Z X \\
\downarrow & & \downarrow & & \downarrow \\
Z & = & Z & = & Z,
\end{array}$$

where the vertical arrows are projection to the second factor composed with p. Restricting to Z_i we see that $\Psi_X|G \times X_i$ is a closed immersion whose image coincides with $X_i \times_Z X_i$. Since Z is separated, this implies that Ψ_X is a closed immersion whose image coincides with $X \times_Z X$. $\square$

Up to a finite ambiguity, every quotient can be obtained this way:

??.2.7 Theorem. [Seshadri72, Theorem 6.1] *Let Y be an scheme of finite type and G an affine algebraic group, smooth over S. Let $m_Y : G \times Y \to Y$ be a proper G-action on Y.*

Then there is a Zariski locally trivial principal G-bundle $p : X \to Z$ and a finite and surjective G-morphism $f : X \to Y$.

Proof. In order to simplify the proof, we assume that G is irreducible. This is the only case that we use later.

The construction is done in several steps. Let $n : \bar{Y} \to Y$ be the normalization. By (??.1.13) $\bar{Y}$ is a G-scheme and n is finite and surjective. Thus we may assume to start with that Y is normal. G acts on each irreducible component of Y and it is sufficient to prove (??.2.7) for each irreducible component.

Therefore it is sufficient to prove (??.2.7) in the case when Y is normal and irreducible.

??.2.7.1 Claim. *Notation and assumptions as above. For every $y \in Y$ there are:*
a normal and irreducible G-scheme $X(y)$,
a G-invariant open set $U(y) \subset X(y)$ such that $U(y)$ is a trivial G-bundle, and
a finite and surjective G-morphism $p(y) : X(y) \to Y$ such that $y \in p(U(y))$.

Proof. For every closed point $y \in Y$ choose a locally closed subvariety $y \in Z'_y \subset Y$ such that $\dim Z'_y = \dim Y - \dim G$ and $Z'_y \cap Gy$ is zero dimensional. This implies that (e, y) is an isolated point of $m_Y^{-1}(y) \cap G \times Z'_y$. $\dim G \times Z'_y = \dim Y$, thus

$m_Y : G \times Z'_y \to Y$ is dominant and quasifinite at (e, y). By shrinking Z'_y we may assume that it is quasifinite. Let $Z_y \to Z'_y$ be the normalization and set $U(y) = G \times Z_y$. $m_y : U(y) \to Y$ is a dominant and quasifinite G-morphism of G-schemes. Apply (??.1.12) to obtain (??.2.7.1). □

$p(y)(U(y)) \subset Y$ is open and finitely many of these cover Y. Thus we obtain the following:

??.2.7.2 Claim. *There are finitely many normal and irreducible G-schemes $X_1, \dots, X_m$, G-invariant open subsets $U_i \subset X_i$, and finite, surjective G-morphisms $p_i : X_i \to Y$ such that*

(??.2.7.2.1) every U_i is a trivial G-bundle, and

(??.2.7.2.2) $Y = \cup_i p_i(U_i)$. □

Let $F \supset k(Y)$ be the normal closure of the field extensions $k(X_i) \supset k(Y)$ and X the normalization of Y in F. ($F \supset k(Y)$ need not be separable.) We prove that X satisfies the requirements of (??.2.7).

Let $h_{ij} : k(X_i) \to k(X), j = 1, \dots, d_i$ be the different embeddings over $k(Y)$. X can be obtained as an irreducible component of the normalization of the fiber product

$$\underbrace{X_1 \times_Y \cdots \times_Y X_1}_{d_1\text{-times}} \times_Y \cdots \times_Y \underbrace{X_m \times_Y \cdots \times_Y X_m}_{d_m\text{-times}}. \tag{??.2.7.2.3}$$

This shows that X is a G-scheme and $f : X \to Y$ a finite and surjective G-morphism. The G action on X is proper by (?.1.9.1). The Galois group $Gal(F/k(Y))$ acts on X as a subquotient of the permutation group $S_{d_1} \times \cdots \times S_{d_m}$ which acts on the product (??.2.7.2.3) by permuting the factors. Thus the $Gal(F/k(Y))$ action commutes with the G-action.

We would like to apply (??.2.6) to show that X is a Zariski localy trivial principal G-bundle. Thus we have to show that for every closed point $x \in X$ there is an open neighborhood $x \in U(x) \subset X$ which is a trivial G-bundle.

Set $y = f(x)$ and pick an index i such that $y \in p_i(U_i)$. The morphism $f : X \to Y$ factors (in different ways) as

$$f : X \xrightarrow{f_i} X_i \to Y$$

By (??.2.5.1) we see that $U'(x) := f_i^{-1}(U_i) \subset X$ is a trivial G-bundle. We do not know if $x \in U'(x)$, but we know that

$$U'(x) \cap f^{-1}(f(x)) \neq \emptyset.$$

$Gal(F/k(Y))$ acts transitively on every set theoretic fiber of f. Moving $U'(x)$ around by the Galois group we obtain $x \in U(x) \subset X$ as required.

By (??.2.6) the geometric quotient $Z := X/G$ exists and X is a Zariski localy trivial principal G-bundle over Z. □

?? 3. Cohomology of G-sheaves

Let X be a G-scheme and F a G-sheaf on X. The functor of G-invariant global sections $H^0(X,F)^G$ is left exact. We would like to study the higher cohomology groups that give the expected long exact sequences etc. The corresponding higher cohomology groups are called the G-equivariant cohomology groups. Unfortunately, the definition of an equivariant cohomology theory is not yet settled; see [Bernstein-Lutz94] for some cases.

Even in simple cases (like finite groups acting on schemes) the equivariant cohomology behaves in a rather complicated way, especially in positive characteristic. Since I am unable to understand the general case, I consider the most naive definition of equivariant cohomology (??.3.5). This definition does not even give long exact cohomology sequences, unless we make two strong assumptions. These assumptions are satisfied in several applications.

??.3.1 Assumptions. For the rest of this section assume that $S = \operatorname{Spec} k$ is the spectrum of a field, and that G is a linearly reductive algebraic group over $\operatorname{Spec} k$.

??.3.2 Definition. Let G be an affine algebraic group over $\operatorname{Spec} k$. We say that G is *linearly reductive* over $\operatorname{Spec} k$ if the following holds:

If V is a finite dimensional k-vectorspace and $\rho : G \to GL(V)$ a representation then V is the direct sum of irreducible G-subrepresentations.

??.3.3 Examples of linearly reductive groups. (??.3.3.1) Any finite group whose order is relatively prime to the characteristic.

(??.3.3.2) $GL(n,k)$ and $PGL(n,k)$ if $\operatorname{char} k = 0$. For a proof see, for instance, [Weyl39].

(??.3.3.3) More generally, over a field of characteristic zero, G is linearly reductive iff it is reductive (see, for instance [Fogarty69, p.178]).

(??.3.3.4) In postive characteristic, a connected algebraic group is linearly reductive iff it is a torus [Nagata61].

Linear reductivity is frequently used through the following properties:

??.3.4 Lemma. *Let G be a linearly reductive algebraic group over* $\operatorname{Spec} k$.

(??.3.4.1) Let $\cdots \to F_i \to F_{i+1} \to F_{i+2} \to \cdots$ be an exact sequence of (possibly infinite dimensional) k-vector spaces with G action. Then the sequence of invariant subspaces

$$\cdots \to F_i^G \to F_{i+1}^G \to F_{i+2}^G \to \cdots \qquad \textit{is also exact.}$$

(??.3.4.2) Let X be a k-variety with trivial G-action and F a G-sheaf. Then F^G is a direct summand of F.

(??.3.4.3) Let X be a k-variety with trivial G-action and

$$\cdots \to F_i \to F_{i+1} \to F_{i+2} \to \cdots$$

an exact sequence of quasi coherent G-sheaves on X. Then the sequence of invariant subsheaves

$$\cdots \to F_i^G \to F_{i+1}^G \to F_{i+2}^G \to \cdots \qquad \textit{is also exact.}$$

Proof. By a result of Cartier, a quasi coherent G-sheaf is the union of its coherent G-subsheaves. (See [Mumford-Fogarty82, p.25] for vector spaces and [Seshadri77, !!] in general.) This allows us to reduce each statement to the coherent case. The latter follows directly from the definition. $\square$

Using this result, we can define an equivariant cohomology theory, at least for groups satisfying our assumptions:

??.3.5 Definition. Let X be a G-scheme and $\mathfrak{Qco}(X,G)$ the category of quasi coherent G-sheaves on X. (??.3.4.1) implies that

$$\mathfrak{Qco}(X,G) \ni F \mapsto \{H^i(X,F)^G : i = 0,1,\dots\}$$

is an exact δ-functor. $H^i(X,F)^G$ is called the i^{th}-*invariant cohomology* of F on X. (I do not know if this is a universal δ-functor or not.)

Invariant cohomology is especially easy to understand on principal G-bundles:

??.3.6 Theorem. *Let G be a linearly reductive algebraic group over a field k and $p : X \to Y$ a principal G-bundle. Let F be a quasi coherent sheaf on Y. Then*

$$H^i(X, p^*F)^G = H^i(Y,F).$$

Proof. p is affine, thus for any sheaf E on X we have $H^i(X,E) = H^i(Y,p_*E)$. If E is a G-sheaf then p_*E is also a G-sheaf and $H^i(X,E)^G = H^i(Y,p_*E)^G$. By (??.3.7) we see that $H^i(Y,p_*E)^G = H^i(Y,(p_*E)^G)$. If $E = p^*F$ then $p_*E = F \otimes p_*\mathcal{O}_X$ and $(p_*E)^G = F \otimes (p_*\mathcal{O}_X)^G = F$. Thus

$$H^i(X,p^*F)^G = H^i(Y,p_*(p^*F))^G = H^i(Y,(p_*(p^*F))^G) = H^i(Y,F). \quad \square$$

??.3.7 Proposition. *Let G be a linearly reductive group and X a scheme with trivial G-action. Let F be a quasi coherent G-sheaf on X. Then*

$$H^i(X,F)^G = H^i(X,F^G).$$

Proof. By (??.3.4.2) it is sufficient to prove that if F is a sheaf such that $F^G = 0$ then $H^i(X,F)^G = 0$ for every i. Let $U_i \subset X$ be an open cover. We compute the cohomology of F as the homology of the Čech complex:

$$0 \to \sum_i H^0(U_i, F|U_i) \to \sum_{i<j} H^0(U_{ij}, F|U_{ij}) \to \cdots.$$

This is a sequence of G-modules, and none of them have nonzero invariants since $F^G = 0$. Thus the same holds for the homology. $\square$

We would like to use (??.3.6) and (??.2.7) to get information about the invariant cohomology on any scheme Y with a proper G-action. In order to do this, we need to show that suitable cohomological properties descend through finite and surjective morphisms.

Results of this type are well known. For instance, ampleness of a line bundle can be checked after a finite and surjective base change (see e.g. [Hartshorne77, III.5.7.Ex]). I could not find a reference where the statement (or its proof) is general enough for our needs. For this reason, I give details of the argument in a fairly general setting. The main steps are the following two technical results:

??.3.8 Proposition. *Let A be an Abelian category where every object is Noetherian. (For instance A can be the category of all coherent sheaves on a scheme.) Let $0 \in S_0 \subset S_1 \subset \cdots \subset ObA$ be subclasses with the following properties:*

(??.3.8.1) $S_n = ObA$ for $n \gg 1$.

(??.3.8.2) Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence. Then $F_1 \in S_{i+1}$ and $F_2 \in S_i$ implies that $F_3 \in S_i$.

(??.3.8.3) With the above notation, $F_1, F_3 \in S_i$ implies that $F_2 \in S_i$.

(??.3.8.4) For any $0 \neq F \in ObA$ there is a $G \in S_0$ and a nonzero morphism $q : G \to F$.

Then $S_i = ObA$ for every $i \geq 0$.

Proof. The proof is by descending induction on i. By (??.3.8.1) the assertion is true for $i \gg 0$. Assume that $S_{i+1} = ObA$.

Given any $F \in ObA$ there is a maximal subobject $F_1 \subset F$ such that $F_1 \in S_i$. If $F_1 \neq F$ then by (??.3.8.4) there is a $G \in S_0$ and a nonzero map $q : G \to F/F_1$; set $F_3 = \operatorname{im} q$ and $K = \ker q$. We have an exact sequence $0 \to K \to G \to F_3 \to 0$. By assumption $K \in S_{i+1}$ and $G \in S_0 \subset S_i$. Thus $F_3 \in S_i$ by (??.3.8.2). Setting $F_2 = q^{-1}F_3$ we obtain a sequence

$$0 \to F_1 \to F_2 \to F_3 \to 0.$$

By (??.3.8.3) this implies that $F_2 \in S_i$. Also, $F_1 \subsetneq F_2 \subset F$. This contradicts the choice of F_1. $\square$

??.3.9 Proposition. *Let $f : X \to Y$ be a finite and surjective morphism of schemes. For every coherent sheaf F_Y on Y there is a coherent sheaf F_X on X and a nonzero morphism $q : f_*F_X \to F_Y$.*

If f is a G-morphism of G-schemes and F_Y is a G-sheaf then we can choose F_X to be a G-sheaf and q a G-map.

Proof. Let $F_X = Hom_Y(f_*\mathcal{O}_X, F_Y)$ as an $\mathcal{O}_X$-module. We have a natural morphism

$$q\colon f_*F_X = Hom_Y(f_*\mathcal{O}_X, F_Y) \to Hom_Y(\mathcal{O}_Y, F_Y) \cong F_Y.$$

It remains to check that this is not zero. We may localize at a generic point of $\operatorname{Supp} F_Y$. Thus we are reduced to the case when Y is local with maximal ideal m_Y, residue field L and F_Y is Artinian. We have an injection $j\colon L \hookrightarrow F_Y$ thus it is sufficient to show that q is nonzero for $F_Y = L$.

L is a direct summand of $f_*\mathcal{O}_X/(m_y \cdot f_*\mathcal{O}_X)$ as an $\mathcal{O}_Y$-module and we have an isomorphism $Hom_Y(f_*\mathcal{O}_X, L) \cong Hom_Y(f_*\mathcal{O}_X/(m_y \cdot f_*\mathcal{O}_X), L)$. Thus the identity map $L \to L$ lifts to a map $f_*\mathcal{O}_X \to L$. $\square$

As a first application of descent, let us prove a coherence theorem:

??.3.10 Proposition. *Let Y be a G-scheme over k such that $Y \bmod G$ is complete. Let F be a coherent G-sheaf on Y. Then $H^i(Y, F)^G$ is finite dimensional for every i.*

Proof. By (??.2.7) there is a principal G-bundle $p : X \to Z$ which admits a finite and surjective morphism $f : X \to Y$. By (??.1.9.2) $X \bmod G$ is complete, hence Z

is proper by (??.1.8.2). Let E be a coherent G-sheaf on X. Then $E = p^*E_Z$ for a suitable coherent sheaf on Z by (??.2.4) and $H^i(X,E)^G \cong H^i(Z,E_Z)$ by (??.3.6). Thus $H^i(X,E)^G$ is finite dimensional.

Now we descend the coherence from X to Y. Apply (??.3.8) for

$$A = \{\text{coherent } G\text{-sheaves on } Y\} \quad \text{and} \quad S_i = \{F \mid \dim H^j(Y,F)^G < \infty \text{ for } j \geq i\}.$$

If E is a G-sheaf on X then $f_*E \in S_0$. (??.3.9) shows that (??.3.8.4) is satisfied. (??.3.8.1) holds if $n > \dim Y$ and (??.3.8.2–3) follow from the exact cohomology sequence. Thus (??.3.8) implies (??.3.10). □

Two similar applications are left as an exercise:

??.3.11 Exercise. Let $f : X \to Y$ be a finite and surjective morphism of schemes.

(??.3.11.1) (Chevalley) Show that if X is affine then so is Y. (Set $A = \mathfrak{Coh}(X)$ and $S_i = \{F|H^j(Y,F) = 0 \text{ for } j \geq i+1\}$.)

(??.3.11.2) Assume that Y is proper and let L be a line bundle on Y. Show that if f^*L is ample then so is L. (Set $S_i = \{F|H^j(Y,F\otimes L^m) = 0 \text{ for } j \geq i+1,\ m \gg 1\}$.)

The following is the equivariant version of the descent of ampleness:

??.3.12. Theorem. *Let G be a linearly reductive group over a field k. Let X,Y be G-schemes and $f : X \to Y$ a G-equivariant, finite and surjective morphism. Let L be a G-line bundle on Y. The following are equivalent:*

(??.3.12.1) $H^i(Y,F\otimes L^m)^G = 0$ for $i > 0, m \gg 1$ and for every coherent G-sheaf F on Y.

(??.3.12.2) $H^i(X,E\otimes f^*L^m)^G = 0$ for $i > 0, m \gg 1$ and for every coherent G-sheaf E on X.

Proof. Let E be a coherent G-sheaf on X. Then f_*E is a coherent G-sheaf on Y and $H^i(X,E\otimes f^*L^m)^G = H^i(Y,f_*E\otimes L^m)^G$. Thus (??.3.12.1) implies (??.3.12.2). This implication is valid even if f is not surjective.

For the proof of the converse set

$$S_i = \{F|H^j(Y,F\otimes L^m)^G = 0 \text{ for } j \geq i+1,\ m \gg 1\}.$$

If E is coherent G-sheaf on X then $f_*E \in S_0$. (??.3.8.1) is satisfied as soon as $n > \dim Y$. (??.3.8.2–3) are obvious from the long exact cohomology sequence. Finally, (??.3.9) implies (??.3.8.4). Thus by (??.3.8) every coherent G sheaf on Y is in S_0. □

The following example shows that (??.3.6) fails for nonreductive groups:

??.3.13 Principal G_a-bundles. Fix a field k, and let G_a denote the additive group of k. Principal G_a-bundles over Y are classified by $H^1(Y,\mathcal{O}_Y)$ and can be constructed as follows. Let

$$0 \to \mathcal{O}_Y \to E \to \mathcal{O}_Y \to 0 \tag{??.3.13.1}$$

be the extension corresponding to an element $\eta \in H^1(Y,\mathcal{O}_Y)$. To E one can associate a $\mathbb{P}^1$-bundle $p : \mathbb{P}(E) \to Y$ with natural section $D \subset \mathbb{P}(E)$. $X = X(\eta) := \mathbb{P}(E) - D$ is a principal G_a-bundle.

Let $E_{n+1} := p_*\mathcal{O}_{\mathbb{P}(E)}(nD)$. Then $E_1 = \mathcal{O}_X$, $E_2 = E$, $E_{n+1} = S^nE_2$ and there are natural exact sequences

$$0 \to E_n \to E_{n+m} \to E_m \to 0, \quad \text{for } n,m \geq 1. \tag{??.3.13.2}$$

??.3.13.3 Claim. *Assume that Y is proper, integral of dimension d and $\eta \neq 0$. Then, for every n we have $H^0(Y, E_n) = k = H^d(Y, E_n \otimes \omega_Y)$.*

Proof. The sequnce (??.3.13.1) gives an edge homomorphism $\delta : H^0(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$ whose image is exactly $k\eta$. Thus if $\eta \neq 0$ then $q_2 : H^0(Y, E_2) \to H^0(Y, \mathcal{O}_Y)$ is the zero map.

Let us prove by induction that $H^0(Y, E_n) = k$. Using (??.3.13.2) for $m = 1$ it is sufficient to show that

$$q_{n+1} : H^0(Y, E_{n+1}) \to H^0(Y, E_1 = \mathcal{O}_Y) \quad \text{is the zero map.}$$

Using (??.3.13.2) for (n-1,2) this can be factored as

$$q_{n+1} : H^0(Y, E_{n+1}) \to H^0(Y, E_2) \xrightarrow{q_2} H^0(Y, E_1 = \mathcal{O}_Y).$$

This shows that $q_{n+1} = 0$.

Similarly, using the sequence (??.3.13.2) for $n = 1$ we obtain

$$H^d(Y, \omega_Y) \xrightarrow{p_{n+1}} H^d(Y, E_{n+1} \otimes \omega_Y) \to H^d(Y, E_n \otimes \omega_Y) \to 0.$$

We intend to show that $p_{n+1} = 0$. For $n = 1$ this follows from the vanishing of q_2 using duality. As before, p_{n+1} factors through p_2, hence it is zero. $\square$

In positive characteristic there is no perfect duality between $S^m V$ and $S^m(V^*)$, thus the two assertions of (??.3.13.3) do not imply each other via duality.

??.3.13.4 Corollary. *Notation as above. Assume that Y is a smooth projective variety of dimension d and $\eta \neq 0$. Then*

$$H^d(X, p^*\omega_Y) = 0 \quad \textit{but} \quad H^d(Y, \omega_Y) = k.$$

Proof. Look at the exact sequence

$$0 \to E_n \otimes \omega_Y \xrightarrow{\tau_n} E_{n+1} \otimes \omega_Y \to \otimes\omega_Y \to 0.$$

By taking cohomologies we obtain

$$H^d(Y, E_n \otimes \omega_Y) \xrightarrow{H^d(\tau_n)} H^d(Y, E_{n+1} \otimes \omega_Y) \to H^d(Y, \otimes\omega_Y) \to 0.$$

By (??) all three groups occurring in the above sequence are isomorphic to k. This implies that $H^d(\tau_n)$ is the zero map. Thus

$$H^d(X, p^*\omega_Y) = H^d(Y, \omega_Y \otimes p^*\mathcal{O}_Y) = \varinjlim H^d(Y, E_n \otimes \omega_Y) = 0. \quad \square$$

??.4 Existence of Geometric Quotients

The aim of this section is to prove the following theorem which enables us to construct geometric quotients in some cases. While the scope of applications is limited, it is strong enough to obtain the existence of the coarse moduli space of surfaces of general type.

??.4.1 Theorem. *Let G be a linearly reductive group over a field k and X a G-scheme. Assume that X mod G is complete. Let L be a G-line bundle on X. The following are equivalent:*

(??.4.1.1) A geometric quotient $q : X \to Z$ exists and there is an ample line bundle M on Z such that $L^r \cong q^*M$ for some $r > 0$ (as G-line bundles).

(??.4.1.2) $H^i(X, F \otimes L^m)^G = 0$ for $i > 0, m \gg 1$ and for any coherent G-sheaf F on X.

(??.4.1.3) L is ample on X and every point of X is stable (with respect to L).

??.4.1.4 Remark. (??.4.1.1) and (??.4.1.3) are equivalent by [Mumford-Fogarty82, Ch.1 §4]. (??.4.1.3) is included solely to compare (??.4.1.1–2) with the standard constructions of geometric invariant theory.

For us the main point is (??.4.1.2) $\Rightarrow$ (??.4.1.1).

Proof. First we prove that (??.4.1.1) $\Rightarrow$ (??.4.1.2). The shortest way is to use (??.4.1.3). [Mumford-Fogarty82, p.38] implies that q has the following two properties:

(??.4.2.1) q is affine, and

(??.4.2.2) If F is a coherent G-sheaf on X then $(f_*F)^G$ is a coherent sheaf on Z.

(It is also possible to prove these directly using (??.2.7) without relying on (??.4.1.3). We do not include a proof since this direction of the implication is not used elsewhere.)

Let F be a coherent G-sheaf on X. Then

$$H^i(X, F \otimes (L^r)^m)^G = H^i(Z, M^m \otimes q_*F)^G = H^i(Z, M^m \otimes (q_*F)^G),$$

where first we used (??.4.2.1) and then (??.3.7). By (??.4.2.2) $(q_*F)^G$ is coherent, thus

$$H^i(X, F \otimes (L^r)^m)^G = 0 \quad \text{for } i > 0, m \gg 1.$$

In order to get (??.4.1.2) we need to apply the above argument to each of the sheaves $F \otimes L^j : j = 0, \dots, r-1$.

The main part of the proof is (??.4.1.2) $\Rightarrow$ (??.4.1.1). It is clearer to separate the arguments into two steps:

??.4.3 Theorem. *Notation and assumptions as in (??.4.1.2). Then:*

(??.4.3.1) For every algebraically closed field $K \supset k$, the $G(K)$-orbits on $X(K)$ are separated by the sections $H^0(X, L^m)^G \otimes_k K$ for m sufficiently large and divisible.

(??.4.3.2) $\sum_{m=0}^{\infty} H^0(X, L^m)^G$ is a finitely generated k-algebra.

Proof. As a first step we show that L^m is generated by G-invariant global sections for some $m > 0$.

Pick any closed point $x \in X$ and let $s(x)$ be the order of the stabilizer of x. The stabilizer of x acts trivially on $L^{s(x)} \otimes k(x)$, hence $L^{s(x)}|Gx$ is the trivial line bundle with a G-invariant constant section f_x. Let $I_x \subset \mathcal{O}_X$ be the ideal sheaf of Gx; I_x is a G-sheaf. For $r(x) \gg 1$ we obtain an exact sequence

$$H^0(X, L^{r(x)s(x)})^G \to H^0(Gx, L^{r(x)s(x)}|Gx)^G \xrightarrow{\delta} H^1(X, L^{r(x)s(x)} \otimes I_x)^G = 0.$$

Thus $f_x^{r(x)}$ lifts to a global G-invariant section F_x of $L^{r(x)s(x)}$. Finitely many of these sections (or rather, their suitable powers)

$$F_{x_1}^{d_1}, \dots, F_{x_n}^{d_n} \in H^0(X, L^m)^G, \quad \text{where } d_i = \frac{m}{r(x_i)s(x_i)}$$

generate L^m for some $m > 0$.

A similar argument shows that G-orbits are separated. Pick $x_1, x_2 \in X(K)$ which lie in different G-orbits. Pick a natural number s divisible by $s(x_1)$ and $s(x_2)$. Then $L^{rs}|Gx_1 \cup Gx_2$ is a trivial line bundle. Furthermore

$$H^0(Gx_1 \cup Gx_2, L^{rs}|Gx_1 \cup Gx_2)^G = \langle f_1, f_2 \rangle,$$

where f_i vanishes on Gx_{3-i} and is nowhere zero on Gx_i. As before, for $r \gg 1$ these sections lift to G-invariant sections of L^{rs} which separate the two orbits.

Next we show that $\sum_{r=0}^{\infty} H^0(X, L^r)^G$ is a finitely generated k-algebra. Each $H^0(X, L^r)^G$ is finite dimensional by (??.3.10). Fix $m > 0$ such that L^m is generated by G-invariant global sections and let $\{f_i\}$ be a basis of $H^0(X, L^m)^G$. Finite generation follows once we show that the images of the multiplication map

$$H^0(X, L^r)^G \xrightarrow{f_i} H^0(X, L^{m+r})^G$$

span the target for $r \gg 0$. This is implied by the following:

??.4.3.3 Lemma. *(cf.* [Mumford66, 14]*) Notation as above. Assume that*

$$H^j(X, L^{r-jm})^G = 0 \quad \text{for every } j \geq 1.$$

Then

$$\sum_i \operatorname{im}\left[H^0(X, L^r)^G \xrightarrow{f_i} H^0(X, L^{m+r})^G\right] = H^0(X, L^{m+r})^G.$$

Proof. Look at the Koszul complex which, in this case, is an exact sequence of G-sheaves:

$$\cdots \to \bigwedge^2 \Big(\sum_i L^{-m}\Big) \to \sum_i L^{-m} \to \mathcal{O}_X \to 0.$$

Tensor it with L^{m+r} and split it up into short exact sequences

$$0 \to K_j \to L^{m+r} \otimes \bigwedge^j \Big(\sum_i L^{-m}\Big) \to K_{j-1} \to 0.$$

We need to prove that $H^1(X, K_1)^G = 0$. Descending induction shows that $H^j(X, K_j)^G = 0$ for $j > 0$. □

This completes the proof of (??.4.3). □

This is enough to conclude that a geometric quotient exists. It is better to formulate this part of the argument in an abstract setting.

??.4.4 Theorem. *Let G be a linearly reductive group over a field k and X a G-scheme such that X mod G is complete. Let L be a G-line bundle on X. Assume that:*

(??.4.4.1) For every algebraically closed field $K \supset k$, the $G(K)$-orbits on $X(K)$ are separated by the sections $H^0(X, L^m)^G \otimes_k K$ for m sufficiently large and divisible.

(??.4.4.2) $\sum_{m=0}^{\infty} H^0(X, L^m)^G$ is a finitely generated k-algebra.

Then the natural morphism

$$q : X \to \operatorname{Proj} \sum_{m=0}^{\infty} H^0(X, L^m)^G \quad \textit{is the geometric quotient of } X \textit{ by } G.$$

Proof. Set $Z := \operatorname{Proj} \sum_{m=0}^{\infty} H^0(X, L^m)^G$. This is a projective scheme over k. Choose $r > 0$ such that $H^0(X, L^r)^G$ generates $\sum_{m=0}^{\infty} H^0(X, L^{rm})^G$. The choice of r gives a projective embedding $Z \subset \mathbb{P}^N$ ($N = \dim_k H^0(X, L^r)^G - 1$) such that $L^r = q^* \mathcal{O}_Z(1)$.

We need to check that $q : X \to Z$ satisfies the conditions (??.1.4.2.1–3).

If $m \gg 1$ then $H^0(Z, \mathcal{O}_Z(m)) = H^0(X, L^{rm})^G$ injects to $H^0(X, L^{rm})$, thus $q : X \to Z$ has dense image.

First we show (??.1.4.2.1). $H^0(X, L^m)^G$ commutes with any field extension $K \supset k$, thus we may assume that everything is defined over an algebraically closed field K. By assumption q separates $G(K)$-orbits, that is, $X(K)/G(K) \to Z(K)$ is an injection.

In order to show that it is surjective, pick any $z \in Z(K)$. Let T be the spectrum of a DVR with generic point t_g and closed point t_0 such that there is a morphism $v : T \to Z$ where $v(t_g)$ is a generic point of Z and $v(t_0) = z$. $v(t_g) \in \operatorname{im} q$, thus, possibly after enlarging T, we may assume that there is a morphism $u_g : t_g \to X$ such that $q \circ u_g = v|t_g$. By (??.1.6) there is a morphism $u' : T' \to X$ such that $u'(t'_0) \in X$ is a point in $q^{-1}(z)$. Thus $q^{-1}(z)$ is nonempty, hence it also has K-points. Thus $X(K)/G(K) \to Z(K)$ is surjective.

(??.1.4.2.2) is a formal consequence of the other properties:

??.4.4.3 Claim. *Let X be a G-scheme such that X mod G is complete and $q : X \to Z$ a surjective G-morphism (with trivial G-action on Z). Then q is universally submersive.*

Proof. Let T be the spectrum of a DVR and $u : T \to Z$ a morphism. Since q is surjective, there is a surjective morphism $h : T' \to T$ such that $u|t_g$ can be lifted to $u'_g : t'_g \to X$. Hence by (??.1.6) there is a commutative diagram

(??.4.4.4)
$$\begin{array}{ccc} T' & \xrightarrow{u'} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle q} \\ T & \xrightarrow{u} & Z, \end{array}$$

where T' is the spectrum of a DVR and $h : T' \to T$ a surjective morphism.

Instead of (??.4.4.3) we prove the stronger assertion that if $q : X \to Z$ is surjective, of finite type and satisfies (??.4.4.4) then it is universally submersive (no

G-action is necessary). The advantage of this formulation is that (??.4.4.4) is stable under any base change $Z' \to Z$. Therefore it is sufficient to prove that q is submersive.

Let $V \subset Z$ be a subset such that $U = q^{-1}(V) \subset X$ is open. $V = q(U)$ is constructible, hence V is open iff it is closed under generalization.

Pick $z \in V$ and let z' be a generalization of z. Let $u : T \to Z$ be a morphism such that z' is the image of the generic point and z the image of the closed point. Lift u to $u' : T' \to X$ as in (??.4.4.4). Then $u'(t'_0) \in q^{-1}(z) \subset U$ and $u'(t'_g)$ is a generalization of $u'(t'_0)$. Thus $u'(t'_g) \in U$ and therefore $z' = q(u'(t'_g)) \in V$. Thus q is submersive. □

Finally, the open sets

$$D(f) := \{z \in Z | f(z) \neq 0\} \quad \text{for} \quad f \in H^0(Z, \mathcal{O}_Z(m))$$

give a basis for the topology on Z. Thus it is enough to check (??.1.4.2.3) for these sets. We need to prove that for every $g \in H^0(q^{-1}D(f), \mathcal{O}_{q^{-1}D(f)})$ there is a section $h \in H^0(X, L^{rm})^G$ such that $(q^*f)^r g = h | q^{-1}D(f)$.

g can be viewed as a rational section of $\mathcal{O}_X \cong L^0$ which has poles of certain order along $q^*f = 0$. Thus $(q^*f)^r g$ gives a regular G-invariant section of L^{rm} for $r \gg 1$. □

The theorems (??.4.1) and (??.3.12) can be put together to obtain the following result about descending projective geometric quotients through finite and surjective morphisms:

??.4.5 Theorem. *Let G be a linearly reductive group over a field k. Let X, Y be G-schemes and $f : X \to Y$ a G-equivariant, finite and surjective morphism. Let L be a line bundle on Y. Assume that Y mod G is complete.*

The following are equivalent:

(??.4.5.1) A geometric quotient $q_X : X \to Z_X$ exists and there is an ample line bundle M_X on Z_X such that $f^*L^r \cong q^*M_X$ for some $r > 0$ (as G-line bundles).

(??.4.5.2) A geometric quotient $q_Y : Y \to Z_Y$ exists and there is an ample line bundle M_Y on Z_Y such that $L^r \cong q^*M_Y$ for some $r > 0$ (as G-line bundles). □

In general ampleness does not descend through finite and surjective morphisms of nonproper schemes, as shown by the following:

??.4.5.3 Example. Let $\bar{X}$ be a smooth projective surface containing two disjoint smooth rational curves $C_i \subset \bar{X}$. Pick points $p_i \in C_i$ and an isomorphism $h : C_1 \cong C_2$ such that $h(p_1) = p_2$. Let $\bar{f} : \bar{X} \to \bar{Y}$ be the surface obtained from $\bar{X}$ by gluing together C_1 and C_2 using h. Let p denote the common image of the points p_i. Set $Y = \bar{Y} - \{p\}$, $X = \bar{X} - \{p_1, p_2\}$ and $f : X \to Y$ the restriction of $\bar{f}$.

Let $\bar{L}$ be an ample line bundle on $\bar{X}$ and $L = \bar{L}|X$. Then there is a line bundle M on Y such that $f^*M \cong L$. Show that the following are equivalent:

(??.4.5.3.1) There is a line bundle $\bar{M}$ on $\bar{Y}$ such that $\bar{f}^*\bar{M} \cong \bar{L}$.

(??.4.5.3.2) M is ample on Y.

(??.4.5.3.3) $(\bar{L} \cdot C_1) = (\bar{L} \cdot C_2)$.