2. BIRATIONAL GEOMETRY OF LOG SURFACES

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The detailed study of log surfaces (especially for $b_i = 1$) was initiated by the Iitaka school, in particular [Miyanishi81], [Tsunoda83], [Miyanishi-Tsunoda83], [Fujita84] and [Sakai84,85]. (For more references see [Miyanishi81].) Recent interest arouse when 3-dimensional problems were reduced to questions regarding log surfaces with fractional coefficients (cf. [Kollár et al.92], [Shokurov93]).

This chapter is devoted to the elementary properties of the birational geometry of log surfaces and as an important part of that, the Minimal Model Program for log surfaces.

Throughout the chapter we work over an algebraically closed field of any characteristic.

2.0.1 Definition. A *log surface* is a pair (S, B), where S is a normal surface and $B = \sum_{i=1}^{s} b_i B_i$, B_i are irreducible and reduced curves and $0 \leq b_i \leq 1$.

2.0.2 Notation. Let X be a scheme. A $\mathbb{Q}$-Cartier divisor D on X is called *nef* if $D.C \geq 0$ for every proper curve $C \subset X$. D is called *big* if X is proper and $|\,mD\,|$ gives a birational map for some $m > 0$. In particular ample implies nef and big.

$Z_1(X)$ denotes the free abelian group generated by irreducible reduced curves on X and $Z_1(X)_{\mathbb{R}} = Z_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. All cycles numerically equivalent to the zero cycle form a subgroup of $Z_1(X)_{\mathbb{R}}$ and the quotient is denoted by $N_1(X)_{\mathbb{R}}$.

The effective 1-cycles generate a subsemigroup $NE(X) \subset N_1(X)_{\mathbb{R}}$. It is called the *cone of curves of* X. The *closed cone of curves* of X, denoted by $\overline{NE}(X)$, is the closure of $NE(X)$ in the Euclidean topology of $Z_1(X)_{\mathbb{R}}$.

If D is an arbitrary $\mathbb{Q}$-divisor, then $NE(X)_{D\geq 0}$ denotes the set of vectors $\xi \in NE(X)$ such that $\xi \cdot D \geq 0$.

A *log Del Pezzo* surface is a log surface (S, B) such that $-(K_S + B)$ is ample.

For the definition of *Minimal Model Program (MMP)*, *log-Minimal Model Program (log-MMP)* and *log resolution* see [Kollár et al.92, §2].

2.0.2.1 Remark. For surfaces numerical equivalence can be defined in two ways. Either requiring equal intersection numbers for all curves or only for those, that are $\mathbb{Q}$-Cartier divisors. These two different definitions result in different cones. Adopting the first one gives a possibly larger dimensional cone. The results of this chapter hold for both versions.

§2.1 Cone Theorem for Log Surfaces

The aim of this section is to prove the Cone Theorem for log surfaces. (In higher dimensions further assumptions are needed cf. (2.1.5).)

2.1.1 Cone Theorem for Log Surfaces. *Let (X, B) be a projective log surface, $f : Y \to X$ a minimal resolution of singularities of X, H an ample divisor on X and $\varepsilon > 0$. Then*

$$\overline{NE}(Y) = \overline{NE}(Y)_{f^*(K_Y+B+\varepsilon H)\geq 0} + \sum_{\substack{C \text{ smooth rational curve.} \\ \text{If } \rho(Y)\geq 3, \text{ then } C^2<0.}} \mathbb{R}^+ [\, C\,] \tag{2.1.1.1}$$

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B+\varepsilon H\geq 0} + \sum_{\substack{C \text{ smooth rational curve.} \\ \text{If } \rho(Y)\geq 3, \text{ then } C^2<0.}} \mathbb{R}^+ [\, f(C)\,] \tag{2.1.1.2}$$

Proof. If $f : Y \to X$ is a proper dominant morphism, then $f_* : \overline{NE}(Y) \to \overline{NE}(X)$ is surjective, so (2.1.1.2) follows from (2.1.1.1).

2.1.2 Lemma. *Let Y be a projective variety, where the usual Cone Theorem (see e.g.* [CKM88, (9.6)]*) holds, H an ample $\mathbb{Q}$-Cartier divisor and $\varepsilon > 0$. Then $\overline{NE}(Y)$ is the smallest closed convex cone that contains the following two sets:*

$$\begin{aligned} \mathcal{P} &= \big\{[\, C\,] \mid C \subset Y \textit{ irreducible and } C \cdot (K_Y + \varepsilon H) \geq 0\big\} \\ \mathcal{N} &= \big\{[\, C\,] \mid \mathbb{R}^+ [\, C\,] \textit{ is an extremal ray of } \overline{NE}(Y) \textit{ such that } C \cdot (K_Y + \varepsilon H) < 0\big\} \end{aligned}$$

Proof. Let $\eta \in \overline{NE}(Y)$ be an extremal vector and let ε' be such that $\varepsilon > \varepsilon' > 0$. If $\eta \cdot (K_Y + \varepsilon' H) < 0$, then by the Cone Theorem some multiple of η is represented by an irreducible curve, so it is in $\mathcal{P}$ or $\mathcal{N}$. Then we may assume that $\eta \cdot (K_Y + \varepsilon' H) \geq 0$, so we have $\eta \cdot (K_Y + \varepsilon H) > 0$.

η is extremal, so there exist sequences of irreducible curves C_k and positive rational numbers α_k such that $\eta = \lim \alpha_k [\, C_k\,]$. Now

$$\lim \alpha_k [\, C_k\,] \cdot (K_Y + \varepsilon H) > 0,$$

so $C_k \cdot (K_Y + \varepsilon H) > 0$ for $k \gg 1$ and then η is in the smallest closed convex cone that contains $\mathcal{P}$. $\square$

2.1.2.1 Remark. With $\varepsilon = 0$ this Lemma is unknown.

2.1.3 Corollary. *With the same notation and assumptions as in* (2.1.2) *further assume that* $K_Y + \varepsilon H_Y = M - D$*, where* D *is an effective* $\mathbb{Q}$*-divisor (*$\mathbb{Q}$*-Cartier, if* $\dim Y > 2$*). Then*

$$\overline{NE}(Y) = \overline{NE}(Y)_{\substack{M \geq 0 \\ K_Y + \varepsilon H_Y \geq 0}} + \sum_{\substack{C \text{ is extremal} \\ C \cdot (K_Y + \varepsilon H_Y) < 0}} \mathbb{R}^+ [\,C\,] + \operatorname{im}\big[\,\overline{NE}(\operatorname{Supp} D) \to \overline{NE}(Y)\,\big]$$

Proof. The right hand side is a closed cone, so it is enough to show, that it contains the sets $\mathcal{P}$ and $\mathcal{N}$. It contains $\mathcal{N}$ trivially, so assume that C is an irreducible curve such that $C \cdot (M - D) = C \cdot (K_Y + \varepsilon H_Y) \geq 0$ and $C \cdot M < 0$. Then $C \cdot D < 0$, which implies, that $[\,C\,] \in \operatorname{im}\big[\,\overline{NE}(\operatorname{Supp} D) \to \overline{NE}(Y)\,\big]$. $\square$

Now we are ready to prove (2.1.3). (X, B) is a log surface, $f : Y \to X$ a minimal resolution of singularities of X, H an ample divisor on X and $\varepsilon > 0$.

There are E_1, E_2 and E_3 effective $\mathbb{Q}$-divisors on Y such that

$$\begin{aligned} K_Y &\equiv f^*K_X - E_1 \\ f_*^{-1}B &\equiv f^*B - E_2 \\ H_Y &\equiv f^*H - E_3, \end{aligned}$$

where H_Y is ample on Y. (The existence of H_Y and E_3 follows from (2.2.4).) Now

$$K_Y + \varepsilon H_Y \equiv \underbrace{f^*(K_X + B + \varepsilon H)}_{M} - \underbrace{(E_1 + E_2 + \varepsilon E_3) - f_*^{-1}B}_{D}.$$

Let F_j, $j = 1, 2, \dots, r$ be the f exceptional irreducible curves, $f_*^{-1}B = \sum_{i=1}^s b_i B_i$ and $E_k = \sum_{j=1}^r e_{k,j} F_j$ for $k = 1, 2, 3$ be the irreducible decompositions of $f_*^{-1}B$ and E_k. Then by (2.1.3) we have

$$\begin{aligned} \overline{NE}(Y) = \overline{NE}(Y)_{\substack{K_Y + \varepsilon H \geq 0 \\ f^*(K_X + B + \varepsilon H) \geq 0}} &+ \sum_{\substack{C \cdot (K_Y + \varepsilon H) < 0 \\ C \text{ is extremal}}} \mathbb{R}^+ [\,C\,] + \\ &+ \sum_{i=1}^s \mathbb{R}^+ [\,B_i\,] + \sum_{j=1}^r \mathbb{R}^+ [\,F_j\,]. \end{aligned} \tag{2.1.3.1}$$

Since F_j are exceptional,

$$\sum_{j=1}^r \mathbb{R}^+ [\,F_j\,] + \overline{NE}(Y)_{\substack{K_Y + \varepsilon H \geq 0 \\ f^*(K_X + B + \varepsilon H) \geq 0}} \subset \overline{NE}(Y)_{f^*(K_X + B + \varepsilon H) \geq 0}.$$

If $B_i \cdot f^*(K_X + B + \varepsilon H) \geq 0$, then $[\,B_i\,] \in \overline{NE}(Y)_{f^*(K_X + B + \varepsilon H) \geq 0}$. Next consider a B_i such that $B_i \cdot f^*(K_X + B + \varepsilon H) < 0$. If $B_i \cdot f^*B < 0$, then $B_i^2 < 0$ and

$$0 > B_i \cdot f^*(K_X + B + \varepsilon H) \geq B_i \cdot (f^*K_X - E_1 + B_i) + B_i \cdot (f^*B - B_i + \varepsilon f^*H) \geq$$
$$\geq B_i \cdot (K_Y + B_i) = 2p_a(B_i) - 2,$$

so $B_i \simeq \mathbb{P}^1$ and it is an extremal ray by [CKM88, (4.5)].

If $B_i \cdot f^*B \geq 0$, then

$$B_i \cdot (K_Y + \varepsilon H_Y) \leq B_i \cdot f^*(K_X + \varepsilon H) \leq B_i \cdot f^*(K_X + B + \varepsilon H) < 0.$$

If $[\,B_i\,]$ is an extremal vector of $\overline{NE}(Y)$, then $\mathbb{R}^+\,[\,B_i\,]$ is an ordinary extremal ray. If $[\,B_i\,]$ is not an extremal vector of the cone, then it can be dropped from the right hand side of (2.1.3.1).

Finally let C be a $(K_Y + \varepsilon H_Y)$-negative extremal ray. If $C^2 > 0$, then $\rho(X) = 1$ by [CKM88, (4.4)] and the statement is trivial, otherwise $C^2 \leq 0$, so $2p_a(C) - 2 = C \cdot (K_Y + C) < 0$, thus C is a smooth rational curve. If $C^2 = 0$, then $\rho(X) = 2$ by [CKM88, (3.7)] (cf. (2.3.4)).

The condition on the Picard number and the self-intersection follows easily from the log-MMP (cf. (2.3.4.4)). □

2.1.4 Corollary. *Let (X, B) be a log Del Pezzo surface and $f : Y \to X$ a minimal resolution of singularities of X. Then*

$$\overline{NE}(Y) = \sum_{E \text{ is } f \text{ exceptional}} \mathbb{R}^+\,[\,E\,] + \sum_{\substack{C \text{ smooth rational curve.} \\ \text{If } \rho(Y) \geq 3, \text{ then } C^2 < 0.}} \mathbb{R}^+\,[\,C\,]$$

Proof. $-(K_X + B + \varepsilon H)$ is ample, so

$$\overline{NE}(Y)_{f^*(K_Y+B+\varepsilon H)\geq 0} = \overline{NE}(Y)_{f^*(K_Y+B+\varepsilon H)=0} = \sum_{E \text{ is } f \text{ exceptional}} \mathbb{R}^+\,[\,E\,]. \qquad \square$$

A higher dimensional variant of this Cone Theorem is:

2.1.5 Cone Theorem for Singular Varieties. *Let (X, B) be a $\mathbb{Q}$-factorial log variety and $Z \subset X$ the closed subscheme of X where (X, B) is not klt. Assume, that the log-MMP works in dimension* $\dim X$*. Then*

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B+\varepsilon H \geq 0} + \sum_{C\cdot(K_Y+\varepsilon H)<0} \mathbb{R}^+\,[\,C\,] + \operatorname{im}\left[\,\overline{NE}(Z) \to \overline{NE}(X)\,\right]$$

Proof. Exercise. □

Once the Cone Theorem is established, the Minimal Model Program proceeds to contract the extremal rays. In order to achieve that, one needs vanishing results, and that is the topic of the next section.

§2.2 A Vanishing Theorem for H^1

In dimension two, the assumptions of the generally used vanishing results (cf. [Grauert-Riemenschneider70], [Kawamata82], [Viehweg82]) can be weakened.

2.2.1 Theorem. *Let X be a 2-dimensional regular scheme, $C = \cup_{i=1}^s C_i$ a proper connected curve such that the matrix $(C_i \cdot C_j)$ is negative definite. Let L be a line bundle on X and assume, that there exist a $\mathbb{Q}$-divisor N and a set of rational numbers b_i, $i = 1, 2, \dots, s$ with the following properties for all i:*

$$N \cdot C_i \geq 0, \tag{2.2.1.1}$$

$$\text{one of the following} \begin{cases} 0 \leq b_i < 1, \\ 0 < b_i \leq 1 \quad \text{and} \quad \exists\, j,\ b_j \neq 1, \\ 0 < b_i \leq 1 \quad \text{and} \quad \exists\, j,\ N \cdot C_j > 0, \end{cases} \tag{2.2.1.2}$$

$$L \cdot C_i = (K_X + \sum_{j=1}^s b_j C_j + N) \cdot C_i. \tag{2.2.1.3}$$

Finally let $Z = \sum_{i=1}^s r_i C_i$, with $r_i \in \mathbb{N}$. Then

$$H^1(Z, L \otimes \mathcal{O}_Z) = 0.$$

Proof. Let C_i be contained in Z (i.e. $r_i > 0$), $Z_i = Z - C_i$ and consider the short exact sequence:

$$0 \to L \otimes \mathcal{O}_{C_i}(C_i - Z) \to L \otimes \mathcal{O}_Z \to L \otimes \mathcal{O}_{Z_i} \to 0.$$

By induction on $\sum r_i$, $H^1(Z_i, L \otimes \mathcal{O}_{Z_i}) = 0$.

2.2.2 Lemma. *In the situation above one can find a C_i, such that $r_i > 0$ and $H^1(C_i, L \otimes \mathcal{O}_{C_i}(C_i - Z)) = 0$.*

Proof. It is enough to see, that there exists an i such that

$$L \cdot C_i + (C_i - Z) \cdot C_i > 2p_a(C) - 2 = C_i^2 + K_X \cdot C_i.$$

By (2.2.1.3) this is equivalent to

$$N \cdot C_i + (\sum_{j=1}^s b_j C_j - Z) \cdot C_i > 0.$$

The following lemma provides the key step to find the required C_i. It will be proved after the proof of (2.2.2).

2.2.3 Lemma. *Let X and C be as in* (2.2.1)*. Let $D = \sum_{i=1}^{s} d_i C_i = D^+ - D^-$, where D^+ and D^- are the positive and negative parts of D. Assume, that $D \neq 0$ and* $\operatorname{Supp} D^+ \neq C$*. Then*

$$\text{(2.2.3.1)} \qquad \textit{If } D^- \neq 0, \textit{ then } \exists\, C_i \subset \operatorname{Supp} D^- \textit{ such that } D \cdot C_i > 0.$$

$$\text{(2.2.3.2)} \qquad \textit{If } D^- = 0, \textit{ then } \exists\, C_i \not\subset \operatorname{Supp} D^+ \textit{ such that } D \cdot C_i > 0.$$

In particular, if $D \cdot C_i \leq 0$ for all i, then D is effective and if there exists also a j such that $D \cdot C_j < 0$ then D is effective and $\operatorname{Supp} D = C$*.*

If there exists an i such that $r_i > b_i$, then (2.2.3.1) implies (2.2.1). If $b_i = r_i = 1$, for all i, such that $r_i \neq 0$, then $\operatorname{Supp} Z = C \setminus \operatorname{Supp}(\sum_{j=1}^{s} b_j C_j - Z)$ by (2.2.1.2). If $Z \neq C$, then (2.2.3.2) provides the required C_i. If $Z = C$, then there exists a j such that $N \cdot C_j > 0$ by (2.2.1.2). Thus in any case we find a C_i, such that

$$N \cdot C_i + (\sum_{j=1}^{s} b_j C_j - Z) \cdot C_i > 0.$$

This proves (2.2.2) as well as (2.2.1). □

2.2.2.1 Remark. If we replace (2.2.1.2) with $0 \leq b_i \leq 1$ and further assume, that $r_i \geq b_i$, then the same proof gives, that

$$H^1(Z, L \otimes \mathcal{O}_Z) \to H^1(\sum \ulcorner b_i \urcorner C_i, L \otimes \mathcal{O}_{\sum \ulcorner b_i \urcorner C_i})$$

is an isomorphism.

Proof of 2.2.3. First let $D^- \neq 0$. Then $(D^-)^2 < 0 \leq D^+ \cdot D^-$, so

$$0 < D \cdot D^- = \sum_{d_i < 0} (-d_i) D \cdot C_i.$$

Therefore (2.2.3.1) holds.

Next consider the case $D^- = 0$. Let i be such an index that $d_i = 0$. Then

$$D \cdot C_i = \sum_{C_j \cdot C_i > 0} d_j C_j \cdot C_i \geq 0.$$

Now if there exists a $d_j \neq 0$ among these coefficients, then we are done, otherwise we can repeat the same with any C_j instead of the original C_i. C is connected, so this process will end. □

2.2.4 Corollary. *Let X and C be as in* (2.2.1)*. Let H be an ample divisor on X. The there exists a set of natural numbers $\{r, r_i \mid i = 1, 2, \dots, s\}$ such that r divides* $\det(C_i \cdot C_j)$ *and if $Z = \sum_{i=1}^{s} r_i C_i$, then $-rH \cdot C_i = Z \cdot C_i$ for all i.*

Proof. $(C_i \cdot C_j)$ is negative definite, so in particular invertible. Hence there exist rational numbers q_i, $i = 1, 2, \dots, s$ that satisfy the condition. They are positive by (2.2.3). Now let r be their common denominator and $r_i = q_i r$. □

The following relative vanishing result is a straightforward consequence of (2.2.1):

2.2.5 Corollary. *Let $f : S \to S'$ be a proper, birational morphism between normal surfaces with S smooth and with exceptional locus $E = \cup_{i=1}^{s} E_i$. Let L be a line bundle on S, $\{b_i \mid i = 1, 2, \dots, s\}$ a set of rational numbers satisfying the condition (2.2.1.2), N an f-nef $\mathbb{Q}$-divisor and assume that*

$$L \equiv K_S + \sum_{i=1}^{s} b_i E_i + N.$$

Then

$$R^1 f_* L = 0.$$

Proof. The conditions of (2.2.1) are satisfied, so the Theorem on Formal Functions implies that $(R^1 f_* L)\hat{} = 0$ and then $R^1 f_* L = 0$.

To successfully run the Minimal Model Program one needs additional restrictions on the possible singularities.

2.2.6 Definition. Let (X, B) be a log variety. $B = \sum b_i B_i$, $0 \le b_i \le 1$. If $\dim X \ge 3$, assume, that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $Y \to X$ be a log-resolution and E_i the irreducible components of the exceptional locus of f. By (2.2.4) there exists a unique collection $a_i \in \mathbb{Q}$ for $i = 1, 2, \dots, s$ such that

$$K_Y + f_*^{-1} B \equiv f^*(K_X + B) + \sum_{i=1}^{s} a_i E_i.$$

a_i is called the *discrepancy* of E_i with respect to (X, B). The pair (X, B) is said to be

$$\left.\begin{array}{ll} \textit{log canonical} & \textit{lc} \\ \textit{purely log terminal} & \textit{plt} \\ \textit{Kawamata log terminal} & \textit{klt} \end{array}\right\} \quad \text{if} \quad \left\{\begin{array}{ll} a_i \ge -1 & \text{for all } i \\ a_i > -1 & \text{for all } i \\ a_i > -1 \text{ and } b_j < 1 & \text{for all } i, j \end{array}\right.$$

2.2.7 Exercise. Prove that the discrepancies do not depend on the log-resolution chosen.

2.2.8 Exercise. Let $\dim X = 2$, $f : Y \to X$ the minimal resolution. Prove that either all the discrepancies of exceptional divisors with respect to $(X, \emptyset)$ are zero or all of them are negative. [Hint: Use (2.2.3)]

2.2.9 Exercise. Let $\dim X = 2$, $f : Y \to X$ the minimal resolution. Assume, that every exceptional curve over x has discrepancy -1. Prove, that the exceptional locus consists of either an irreducible curve with arithmetic genus 1 or a cycle of smooth rational curves (i.e. a set of smooth rational curves such that each of them meets two others at one point. We also include the degenerate case when two curves meet at two separate points.) or three smooth rational curves meeting at one point or two rational curves tangent to each other at one point. Note, that these last two cases and when the exceptional locus is a rational curve with a cusp are not log canonical.

2.2.10 Definition. If the exceptional locus of a minimal resolution of a surface singularity is a smooth elliptic curve, then it is called a *simple elliptic singularity*. If the exceptional locus is a rational curve with a node or a cycle of smooth rational curves, then it is called a *cusp singularity*.

2.2.11 Corollary. *Let (X, B) be a log canonical surface, $f : Y \to X$ a minimal resolution and $x \in X$ a point. Then*

$$(R^1 f_* \mathcal{O}_Y)_x = 0,$$

unless $x \notin \operatorname{Supp} B$ and (x, X) is a simple elliptic or cusp singularity.

Proof. Let $\cup_{i=1}^s E_i$ be the irreducible decomposition of the exceptional locus of f. By definition there exists a collection $a_i \in \mathbb{Q}$ for $i = 1, 2, \dots, s$ such that

$$K_Y + f_*^{-1} B \equiv f^*(K_X + B) + \sum_{i=1}^s a_i E_i,$$

so

$$\mathcal{O}_Y \equiv K_Y + \sum_{i=1}^s (-a_i) E_i + \underbrace{f_*^{-1} B - f^*(K_X + B)}_{N}.$$

Y is a minimal resolution, hence $K_Y \cdot E_i \geq 0$ for all i. Therefore by (2.2.3) either $a_i = 0$ for $i = 1, \dots, s$ or $0 < -a_i \leq 1$ for all i. If $x \in \operatorname{Supp} B$, then $N = f_*^{-1} B - f^*(K_X + B)$ is not f-numerically trivial and there exists a j such that $N \cdot E_j > 0$. If $N \cdot E_j = 0$ for all i, then (2.2.9) and the assumption on the singularity imply, that there exists a j such that $-a_j \neq 1$. Therefore (2.2.1.2) is satisfied in any case, so by (2.2.5)

$$(R^1 f_* \mathcal{O}_Y)_x = 0. \quad \square$$

§2.3 Minimal Model Program for Log Surfaces

The aim of this section is to establish the Minimal Model Program for log surfaces. The first step toward this is a contractibility result.

2.3.1 Theorem. *Let X be a 2-dimensional regular quasi-projective scheme, $C = \cup_{i=1}^s C_i$ a proper connected curve such that $(C_i \cdot C_j)$ is a negative definite matrix. Let $\{a_i \mid 1 \leq i \leq s\}$ be a set of rational numbers such that $a_i \geq -1$ and*

$$C_j \cdot K_X = C_j \cdot \sum_{i=1}^s a_i C_i, \qquad \textit{for all } j.$$

Further assume that $C_j \cdot K_X \geq 0$ *for all* j *and not all the* a_j*'s are equal to* -1. *Then* C *can be contracted to a log terminal quasi-projective scheme.*

2.3.1.1 Remark. If there are -1-curves on X, then by contracting those curves the discrepancies remain unchanged, so we can apply the theorem in this case, too.

2.3.1.2 Remark. By a result of [Artin62], (2.2.1) implies (2.3.1). In our case, some steps of Artin's proof can be simplified and we get a shorter proof.

Proof. By (2.2.3) either $a_i = 0$ for all i or $0 < -a_i \leq 1$ for all i. Then by (2.2.1) $H^1(Z, \mathcal{O}_Z) = 0$ for every $Z = \sum_{i=1}^s r_i C_i$ with $r_i \in \mathbb{N}$. In particular every $C_i \simeq \mathbb{P}^1$. Now choose a very ample line bundle H on X such that $H^1(X, H) = 0$. Changing H to $\det(C_i \cdot C_j)H$ if necessary, one can find a set of natural numbers $\{r_i \mid i = 1, 2, \ldots, s\}$ such that if $Z = \sum_{i=1}^s r_i C_i$, then $-H \cdot C_i = Z \cdot C_i$ for all i (cf. (2.2.4)). This implies, that

$$H\left(\sum_{i=1}^s r_i C_i\right) \otimes \mathcal{O}_{C_i} \simeq \mathcal{O}_{C_i}.$$

2.3.2 Lemma. *Let* X *and* C *be as in* (2.3.1), $W = \sum_{i=1}^s t_i C_i$ *with* $t_i \in \mathbb{N}$ *and* L *a line bundle such that* $L \otimes \mathcal{O}_{C_i} \simeq \mathcal{O}_{C_i}$ *for all* i. *Then*

$$L \otimes \mathcal{O}_W \simeq \mathcal{O}_W.$$

Proof. Let C_i be contained in W (i.e. $t_i > 0$), $W_i = W - C_i$ and consider the short exact sequence:

$$0 \to L \otimes \mathcal{O}_{C_i}(C_i - W) \to L \otimes \mathcal{O}_W \to L \otimes \mathcal{O}_{W_i} \to 0.$$

By (2.2.2) there exists an i such that $t_i > 0$ and $H^1(C_i, L \otimes \mathcal{O}_{C_i}(C_i - W)) = 0$ and by induction on $\sum t_i$, $L \otimes \mathcal{O}_{W_i} \simeq \mathcal{O}_{W_i}$. Therefore there exists a section of $L \otimes \mathcal{O}_W$ mapping to a nowhere vanishing section on W_i. This defines a nonzero morphism $\mathcal{O}_W \to L$ which is an isomorphism since both of these line bundles have degree zero on every component of $\operatorname{Supp} W$. $\square$

Next consider the short exact sequence:

$$0 \to H \to H\left(\sum_{i=1}^s r_i C_i\right) \to H\left(\sum_{i=1}^s r_i C_i\right) \otimes \mathcal{O}_Z \to 0.$$

$H\left(\sum_{i=1}^s r_i C_i\right)$ is generated by global sections, since $H\left(\sum_{i=1}^s r_i C_i\right) \otimes \mathcal{O}_Z \simeq \mathcal{O}_Z$ and $H^1(X, H) = 0$. Furthermore it is very ample outside Z and trivial on Z, so it defines a morphism that contracts Z. Let $f : X \to Y$ be its Stein factorization, so Y is normal, f contracts Z and by (2.2.2) Y has rational singularities.

Let D be a Weil divisor on Y and $m = \det(C_i \cdot C_j)$. Choose $\{d_i \mid i = 1, 2, \ldots, s\} \subset \mathbb{N}$ such that for all i

$$f_*^{-1}(mD) \cdot C_i = -\sum_{j=1}^s d_j C_j \cdot C_i.$$

Let $L = \mathcal{O}_X(f_*^{-1}mD + \sum_{j=1}^s d_jC_j)$. Then by (2.3.2) $L \otimes \mathcal{O}_{\sum_{j=1}^s \gamma_j C_j} \simeq \mathcal{O}_{\sum_{j=1}^s \gamma_j C_j}$ for every collection of natural numbers $\{\gamma_j \mid j = 1, 2, \dots, s\}$. Therefore L is trivial in every infinitesimal neighborhood of the exceptional divisor of f, so f_*L is a line bundle. Finally this line bundle is isomorphic to $\mathcal{O}_Y(mD)$ outside the singular points by construction, hence they are isomorphic. Therefore mD is a Cartier divisor. $\square$

Next we recall the steps of the Minimal Model Program in this setting.

2.3.3 Minimal Model Program for Log Surfaces.

Let (S, B) be a log surface, $f : S' \to S$ the minimal resolution. Then by (2.1.1)

$$\overline{NE}(S) = \overline{NE}(S)_{K_S+B+\varepsilon H \geq 0} + \sum_{C \text{ smooth rational curve on } S'} \mathbb{R}^+ [\, f(C) \,].$$

Let C be a $(K_S + B + \varepsilon H)$-negative extremal ray on S.

Case I. $C^2 > 0$. Then $\rho(S) = 1$ by [CKM88, 4.4] and $-(K_S + B)$ is ample.

Case II. $C^2 = 0$. Let m be an integer such that $D = mf^*C$ is Cartier. $D \cdot K_{S'} = mC \cdot K_S \leq mC \cdot (K_S + B + \varepsilon H) < 0$, so by the Riemann-Roch Theorem

$$h^0(nD) \geq \frac{1}{2} nD(nD - K_{S'}) + \chi(\mathcal{O}_X) > 1 \qquad \text{for } n \gg 1.$$

Then $|\, nD \,|$ is base point free and it defines a morphism $h : S \to T$ to a curve T. This case will be analyzed in more details.

Case III. $C^2 < 0$. Then $(f_*^{-1}C)^2 < 0$, so $f_*^{-1}C \simeq \mathbb{P}^1$. However it is not clear whether $C \simeq \mathbb{P}^1$ and whether it can be contracted projectively or not. We will give affirmative answers for both questions in the log canonical case.

2.3.4 Lemma. *Let $h : S \to T$ be the morphism defined in Case II. Then*
(2.3.4.1) *the general fiber of h is isomorphic to $\mathbb{P}^1$,*
(2.3.4.2) *S is $\mathbb{Q}$-factorial and has rational singularities,*
(2.3.4.3) *the fibers of h are irreducible.*

Proof. Let C_t be a fiber of h such that S is smooth along C_t. $C_t \equiv aC$ for some $a > 0$, so

$$2p_a(C) - 2 = C_t \cdot (K_S + C_t) = C_t \cdot K_S \leq C_t \cdot (K_S + B) < 0.$$

Therefore $C_t \simeq \mathbb{P}^1$.

Next let $h' = h \circ f : S' \to T$. S' is a (birationally) ruled surface over T, so $R^1h'_*\mathcal{O}_{S'} = 0$. Then by the Leray spectral sequence $R^1f_*\mathcal{O}_S = 0$, so S has rational singularities and the rest of (2.3.4.2) follows by the work of [Artin62].

Hence every Weil divisor is $\mathbb{Q}$-Cartier. Let F be a fiber and suppose it is reducible, i.e. $F = A_1 \cup A_2$. C is extremal, so

$$C \equiv \alpha_1 A_1 \equiv \alpha_2 A_2,$$

for some $\alpha_1, \alpha_2 \in \mathbb{Q}$. Then $A_1 \cdot A_2 > 0$ and $C^2 = 0$ are contradictory, so F must be irreducible. □

2.3.4.4 Remark. This lemma provides the missing piece of the proof of (2.1.1), namely if there exists a $(K_X + B + \varepsilon H)$-negative extremal ray with nonnegative self-intersection, then $\rho(Y) \leq 2$.

2.3.5 Lemma. *Let (S, B) be a log canonical surface and $C \subset S$ a curve with $C^2 < 0$ and $C \cdot (K_S + B) < 0$. Then $C \simeq \mathbb{P}^1$ and it can be contracted to a plt point.*

Proof. Let $f : S' \to S$ be the minimal resolution of the singularities lying on C with exceptional divisor $\sum_{i=1}^{s} C_i$. We have $a_i \geq -1$ such that

$$K_{S'} + f_*^{-1}B \equiv f^*(K_S + B) + \sum_{i=1}^{s} a_i C_i.$$

Let $b \geq 0$ be such that $B = bC + B'$ and C is not contained in B'. Define

$$a_0 = \frac{(K_S + B) \cdot C}{C^2} > 0$$

and let $N = K_S + B - a_0 C$. Then $N \cdot C = 0$ and

$$K_{S'} + f_*^{-1}B \equiv f^*N + a_0 f^*C + \sum_{i=1}^{s} a_i C_i$$

Let $f^*C = f_*^{-1}C + \sum c_i C_i$, $a'_0 = a_0 - b$ and $a'_i = a_i + a_0 c_i$. Then

$$K_{S'} + f_*^{-1}B' \equiv f^*N + a'_0 f_*^{-1}C + \sum_{i=1}^{s} a'_i C_i \qquad a'_i > -1.$$

Let $C_0 = f_*^{-1}C$ and choose $\{r_i \mid i = 0, 1, \dots, s\} \subset \mathbb{Q}^+$ such that for all i

$$f_*^{-1}B' \cdot C_i = -\sum_{j=0}^{s} r_j C_j \cdot C_i.$$

Now let $a''_i = a'_i + r_i$ and then

$$K_{S'} \equiv f^*N + \sum_{i=0}^{s} a''_i C_i \qquad a''_i > -1.$$

By (2.3.1) there exists a $g : S' \to T$ that contracts $\cup_{i=0}^{s} C_i$. By construction this g factors through S, so there exists an $h : S \to T$ that contracts C and (T, h_*B) is log terminal along h_*B. By (2.2.11)

$$(R^1 g_* \mathcal{O}_{S'})_x = 0,$$

so T has rational singularities and by the Leray spectral sequence

$$(R^1 h_* \mathcal{O}_S)_x = 0.$$

Then $H^1(C, \mathcal{O}_C) = 0$, hence $C \simeq \mathbb{P}^1$. Finally $\mathbb{Q}$-factoriality follows from (2.3.1). $\square$

The following theorem summarizes the results in this section.

2.3.6 Theorem (log MMP for log surfaces). *Let (S, B) be a log canonical surface. There exists a sequence of contractions $f : S \to S_1 \to \ldots \to S_n = S'$ such that S' is log canonical (even plt at every point where f^{-1} is not an isomorphism) and satisfies exactly one of the following conditions:*

(2.3.6.1) *$K_{S'} + f_*B$ is nef.*

(2.3.6.2) *There exists a $g : S' \to T$ morphism, such that S' is a birationaly ruled surface over the curve T.*

(2.3.6.3) *(S', f_*B) is a log Del Pezzo surface.*

Proof. Repeatedly using (2.3.5) we find a sequence of contractions $f : S \to S'$ such that there is no curve $C \subset S'$ such that $C^2 < 0$ and $C \cdot (K_{S'} + f_*B) < 0$. Hence either $K_{S'} + f_*B$ is nef or we have Case I or Case II of (2.3.3). These in turn correspond to (2.3.6.3) and (2.3.6.2) respectively. $\square$

§2.4 Log Canonical Surface Singularities

The aim of this section is to get a rough classification of the log canonical surface singularities.

2.4.1 Notation. Let X be a surface and B a $\mathbb{Q}$-divisor on X. $f : Y \to X$ a resolution of singularities of X and E_i, $i \in I$ the set of f-exceptional prime divisors. We define the resolution graph Γ_f of f in the following way: Let $E_i,\ i \in I$ be the set of vertices of Γ_f, with $n_i = -E_i^2$, $p_i = p_a(E_i)$, $\beta_i = E_i \cdot f_*^{-1}B$ and let $(E_i, E_j),\ i \neq j$ be an edge of weight $w_{i,j} = E_i \cdot E_j$ if E_i and E_j meet.

If $f : Y \to X$ is a minimal resolution, then $n_i \geq 2$ for each i such that $p_i = 0$. Motivated by this example we make the following abstract definition.

2.4.2 Definition. A *resolution graph* is a graph $\Gamma = (E_i, n_i, p_i, \beta_i, w_{i,j}, i, j \in I)$ where $I \subset \mathbb{N}$ is a finite set, E_i are the vertices of the graph with *weight* n_i, *p-weight* p_i and *β-weight* β_i, and with edges of *weight* $w_{i,j}$. We define $w_{i,i} = -n_i$ and $w_{i,j} = 0$ if (E_i, E_j) is not an edge of Γ. In addition we assume that the matrix $(w_{i,j})$ is negative definite, $n_i > 0$, $p_i \geq 0$, $\beta_i \geq 0$, for all i and $w_{i,j} \geq 0$ for all $i \neq j$.

For a resolution graph Γ we define $d_i = 2p_i - 2 + n_i + \beta_i$. Also, we define a_i – the *discrepancies* of Γ – as the solutions of the system of linear equations:

$$\sum_{i\in I} a_i w_{i,j} = d_j, \qquad j \in I. \tag{2.4.2.1}$$

$\delta(\Gamma) = \det(w_{i,j}) \neq 0$ since $(w_{i,j})$ is negative definite, hence a_i exist. Furthermore, they are rational numbers such that their denominator divides $\delta(\Gamma)$.

Γ is called a *minimal resolution graph* if $d_i \geq 0$ for all i. (This is slightly more general than being the graph of a minimal resolution.)

If Γ is a minimal resolution graph, then by (2.2.3) $a_i \leq 0$ for all i, with strict inequalities unless $d_i = 0$ (i.e. $n_i = 2$, $p_i = \beta_i = 0$ or $n_i = \beta_i = 1$, $p_i = 0$ for all i). Such a Γ is called a *Du Val graph*. In this case $a_i = 0$ for all i.

Γ is called *log canonical* (resp. *log terminal*) if $a_i \geq -1$ (resp. $a_i > -1$) for all i.

Γ' is a *resolution subgraph* of Γ if it is a subgraph which is a resolution graph $\Gamma' = (E'_i, n'_i, p'_i, \beta'_i, w'_{i,j}, i, j \in I' \subset I)$ such that $n'_i \leq n_i$, $p'_i \leq p_i$, $\beta'_i \leq \beta_i$ for all i and $w'_{i,j} \leq w_{i,j}$ for all $i \neq j$. Note that $(w'_{i,j})$ is negative definite by assumption.

If Γ is a minimal resolution graph, then by a *subgraph* we will mean a resolution subgraph that itself is a minimal resolution graph, i.e. $d'_i \geq 0$.

For a subgraph we define $n'_i = p'_i = \beta'_i = w'_{i,j} = w'_{j,i} = 0$ for all $i \in I \setminus I'$ and $j \in I$. A subgraph is called *proper* if there exists an $i \in I$ such that $p'_i < p_i$ or $\beta'_i < \beta_i$ or $a_i \neq -1$ and $n'_i < n_i$ or there exists a $j \neq i$ such that $w'_{i,j} < w_{i,j}$. In other words we call a subgraph proper either if it is a proper subgraph as a graph or one of the weights is strictly smaller than the corresponding weight of the ambient graph except that a subgraph is not considered proper if only the weights of those vertices decrease which have discrepancy -1.

The following lemma will be our main tool in order to classify the log canonical minimal resolution graphs.

2.4.3 Lemma. *Let $\Gamma = (E_i, n_i, p_i, \beta_i, w_{i,j}, i, j \in I)$ be a minimal resolution graph.*

(2.4.3.1) *Let α_i be such that $\sum_{i\in I} \alpha_i w_{i,j} \leq d_j$ for all j.*

Then $a_i \leq \alpha_i$ with strict inequalities unless $a_i = \alpha_i$ for all i.

(2.4.3.2) *Let Γ' be a proper subgraph of Γ and assume that one of the following holds:*

(2.4.3.2.1) *$n'_i = n_i$ for all $i \in I'$,*

(2.4.3.2.2) *$a_i \geq -1$ for all $i \in I$,*

(2.4.3.2.3) *$a'_i \geq -1$ for all $i \in I'$,*

Then $a_i \leq a'_i$ for all $i \in I'$ with strict inequalities unless Γ is a Du Val graph

Proof. (2.4.3.1) implies that $\sum_{i\in I}(a_i - \alpha_i)w_{i,j} \geq 0$, so $a_i \leq \alpha_i$ by (2.2.3).

In cases (2.4.3.2.1) and (2.4.3.2.2) let $W_{i,j} = w_{i,j}$ and in case (2.4.3.2.3) $W_{i,j} = w'_{i,j}$. Also let $N_i = -W_{i,i}$. Then $w'_{i,j} \leq W_{i,j} \leq w_{i,j}$ and $n'_i \leq N_i \leq n_i$. In general we have

$$
\begin{aligned}
\sum_{i,j\in I'}(a_i-a_i')W_{i,j} &= \sum_{i,j\in I',i\neq j} a_iW_{i,j}-a_jN_j-\sum_{i,j\in I',i\neq j} a_i'W_{i,j}+a_j'N_j\\
&\geq \sum_{i,j\in I',i\neq j} a_iw_{i,j}-a_jN_j-\sum_{i,j\in I',i\neq j} a_i'w_{i,j}'+a_j'N_j\\
&= d_j-d_j'+a_j(n_j-N_j)-a_j'(n_j'-N_j)\\
&= 2(p_j-p_j')+n_j-n_j'+\beta_j-\beta_j'+a_j(n_j-N_j)-a_j'(n_j'-N_j)\\
&\geq (1+a_j)(n_j-N_j)+(1+a_j')(N_j-n_j').
\end{aligned}
$$

Here $(1+a_j)(n_j-N_j)+(1+a_j')(N_j-n_j')\geq 0$ by the choice of N_j, therefore $\sum_{i,j\in I'}(a_i-a_i')W_{i,j}\geq 0$, and then $a_i-a_i'\leq 0$ for all i by (2.2.3). Here we get strict inequalities unless Γ is a Du Val graph assuming that Γ' is a proper subgraph. $\square$

2.4.4 Corollary. *A proper subgraph of a non Du Val, log canonical minimal resolution graph is log terminal.* $\square$

2.4.5 Corollary. *Let Γ' be a subgraph of a log canonical minimal resolution graph Γ and assume that one of the discrepancies of Γ' is -1. Then $\Gamma=\Gamma'$.* $\square$

We want to classify the log canonical minimal resolution graphs. First we consider special subgraphs which are log canonical but not log terminal, hence no log canonical minimal resolution graph can contain them as a proper subgraph. Also, in these examples we assume, that $B=0$. Once we have the classification of those, using (2.4.3), it is easy to find the graphs which have $B\neq 0$ (cf. [Kollár et al.92, §3]).

2.4.6. Let Γ' be a subgraph of a log canonical minimal resolution graph Γ. Consider the following cases.

2.4.6.1. $\Gamma'=(E',n',p'\geq 1)$. Then $d'\geq n'$, so $n'\geq -a'n'=d'\geq n'$. Therefore $a'=-1$, $p'=1$ and $\Gamma=\Gamma'$.

2.4.6.2. Γ' is a circle of r-vertices such that $p_i'=0$ and $w_{i,i+1}=1$ (including $w_{r,1}=1$) for all edge. Then easy computation shows that $a_i=-1$ and hence $\Gamma=\Gamma'$.

2.4.6.3. $\Gamma'=(E_1,E_2,n_1,n_2,w'=2)$. Then $a_1=a_2=-1$, and again $\Gamma=\Gamma'$. Note that if the weight of an edge is larger than one, then at least one of the end vertices must have weight larger than two, for it would violate the negative definiteness of the matrix $(w_{i,j})$.

2.4.6.4. Γ' is a chain with two forks (vertices with more than two neighbors) at the ends: Let the vertices be $E_1,E_2,E_3,\dots,E_m,E_1',E_2'$, such that $E_3,\dots,E_m$ is a chain and the remaining edges are (E_i,E_3) and (E_i',E_m) for $i=1,2$. Let all the edges be of weight 1 and define $\alpha_i=-1$ for $3\leq i\leq r$ and $\alpha_i=\alpha_i'=-1/2$ for $i=1,2$. Then by (2.4.3.1) $a_i=\alpha_i$ for all i and $\Gamma=\Gamma'$. Also follows that $n_i=n_i'=2$ for $i=1,2$.

2.4.6.5. Γ' is a fork E_0 with neighbors $E_1, \dots, E_k$, $k \geq 4$. Let all the edges be of weight 1 and define $\alpha_0 = -1$ and $\alpha_i = -1/2$ for $i = 1, \dots, k$. Then by (2.4.3.1) $a_i \leq \alpha_i$ for all i. It can be log canonical only if $k = 4$, $a_i = \alpha_i$, $n_i = n'_i = 2$ for $i = 1, \dots, 4$ and $\Gamma = \Gamma'$.

2.4.7. Now let Γ be non Du Val such that it does not have any subgraphs of the types considered in (2.4.6). Then by (2.4.3) $p_i = 0$ for all i and Γ is a tree with edges of weight 1 and it is a chain or has one fork with three neighbors. Here we do not need the precise classification. (It was first done by [Kawamata88] following a different way. Arithmetical proofs were given by [Sakai87] for the case $B = \emptyset$, by S. Nakamura in an appendix to [Kobayashi90] and by V. Alexeev in [Kollár et al.92, §3].) In the latter case leaving out the fork, three chains are left corresponding to quotient singularities of type $\mathbb{C}^2/\mathbb{Z}_{n_i}(1, q_i)$ for $i = 1, 2, 3$. Then Γ is log canonical (resp. log terminal) if $\dfrac{1}{n_1} + \dfrac{1}{n_2} + \dfrac{1}{n_3} \geq 1$ (resp. $\dfrac{1}{n_1} + \dfrac{1}{n_2} + \dfrac{1}{n_3} > 1$).

A lot can be said about the discrepancies occurring on the minimal resolution of lc singularities. (cf. [Shokurov88,93], [Alexeev89], [Kollár et al.92, §18]). Here we prove some simpler results that are needed later.

2.4.8 Proposition. *If Γ is a non Du Val graph, then there exists an i such that $a_i \leq -1/3$.*

Proof. Let E be a vertex of weight $n > 2$. Then $-na \geq d \geq n - 2$, so $a \leq -1 + 2/n \leq -1/3$. $\square$

A quantitative version of the above result is:

2.4.9 Proposition. *Let Γ be a non Du Val minimal log canonical resolution graph. Then*

$$\#\left\{ i \,\middle|\, a_i \leq -\frac{1}{4} \right\} \geq \frac{|\Gamma|}{2}.$$

Proof. The statement is clear if Γ is one of the types considered in (2.4.6), so we may assume that Γ is a chain or has one fork with three neighbors. By (2.4.3.2) it is sufficient to consider a subgraph Γ' of Γ in order to give an upper bound for a_i.

Γ is non Du Val, so there exists an $r \in I$ such that $n_r > 2$. Let Γ' be a chain $E_1, E_2, \dots, E_r$ and let the weights $n_1, \dots, n_{r-1}$ drop to 2 and n_r to 3. Then $a'_i = i a'_1$ and

$$-3a'_r + a'_{r-1} = -(2r+1)a'_1 = 1.$$

Let $i \geq (r+1)/2$, then we have

$$a'_i \leq -\frac{1}{2}\,\frac{r+1}{2r+1} \leq -\frac{1}{4}.$$

Next assume that E_r has two neighbors and let the other one be E_{r+1} and $\Gamma'' = \Gamma' \cup \{E_{r+1}\}$ be the chain with one more vertex E_{r+1} of weight 2. Then $a''_r = 2a''_{r+1}$ and

$$-3a''_r + a''_{r-1} + a''_{r+1} = -\left(\frac{3}{2}r + 1\right) a''_1 = 1.$$

This time let $i \geq r/2$. Then

$$a''_i \leq -\frac{r}{3r+2} \leq -\frac{1}{4}.$$

It follows that if a vertex E_r of Γ has weight larger than two and has more than one neighbors, then for any chain Γ^* of vertices $E^*_1, \dots, E^*_{s-1}, E_r$ we have

$$\#\left\{ i \;\middle|\; a^*_i \leq -\frac{1}{4} \right\} \geq \frac{s}{2}.$$

Therefore the statement is proved if Γ is a chain or the fork of Γ has weight larger than two.

Again by (2.4.3.2) the only case we have left is a tree, Γ with one fork E_0 with neighbors E_1, E'_1, E''_1 and branches $E_1, \dots, E_r$; $E'_1, \dots, E'_{r'}$; $E''_1, \dots, E''_{r''}$ such that all the weights are 2 except $n_r = 3$.

First consider $r' = r'' = 2$ and let $a = a'_2 = a''_2$. Then $a'_1 = a''_1 = 2a$, $a_0 = 3a$ and then $a_1 = 2a$.

Suppose that $r \geq 2$. Then $a_2 = a$ and $r = 2$ since otherwise a_3 would have to be zero. Also, $1 = -3a_2 + a_1 = -a$ shows that this case is not log canonical. Thus we have $r = 1$ and then $a = -1/3$, $a_1 = a'_1 = a''_1 = -2/3$ and $a_0 = -1$.

Therefore we may assume that $r' = 1$ by (2.4.3.2). Now consider $r'' = 1$. Then $a_i = 2a$ for $i = 0, 1, \dots, r$ where $a = a'_1 = a''_1$. Also, $1 = -3a_r + a_{r-1} = -4a$ implies that $a = -1/4$.

Finally, if $r'' > 1$ we still have $a'_1 \leq -1/4$ and the rest is a chain for which we already know the result. $\square$

2.4.9.1 Remark. Essentially the same proof gives, that if $\varepsilon > 0$, then

$$\#\left\{ i \;\middle|\; a_i \leq -\frac{1}{2} + \varepsilon \right\} \geq 2\varepsilon\,|\,\Gamma\,|-1\,.$$

On the other hand one cannot find a similar estimate for $-1/2$ instead of $-1/2 + \varepsilon$.

In view of the fact that the denominator of the discrepancies divides $\det(C_i \cdot C_j)$, having an estimate for this determinant gives one for the index, that is the common denominator of the discrepancies.

2.4.10 Lemma. *Let Γ be a log canonical minimal resolution graph. Then*

$$|\det{}_{i,j\in I}\, w_{i,j}| \leq \prod_{i\in I} n_i.$$

Proof. Let E_1 be a vertex of Γ with only one neighbor E_2. Let $\delta(\Gamma) = \det_{i,j\in I} w_{i,j}$, $\delta_1 = \det_{i,j\in I\setminus\{1\}} w_{i,j}$ and $\delta_2 = \det_{i,j\in I\setminus\{1,2\}} w_{i,j}$. Then $\delta(\Gamma) = -n_1\delta_1 - \delta_2$.

Since $(w_{i,j})$ is negative definite, the signs of δ_1 and δ_2 are different, so

$$|\delta(\Gamma)| \leq \max\{|n_1\delta_1|, |\delta_2|\}.$$

Now we are done by induction. If Γ is a circle, a similar argument works. $\square$

§2.5 Singularities in the Minimal Model Program

2.5.1 Principle. Run the Minimal Model Program for a log surface (X, B), where X is smooth. If $B = 0$ we get only smooth points. For arbitrary B we may get log terminal points, too. The smaller B, the simpler the singularities of the minimal model of (X, B).

The first illustration of the principle is the following:

2.5.2 Proposition. *Let $(X, B = \sum b_i B_i)$ be a smooth log surface such that $b_i \leq 1/3$ for all i. Then the log minimal model of (X, B) has only Du Val singularities.*

2.5.3 Lemma. *Let Y be a smooth surface, $C = \cup_{i=1}^{s} C_i$ a proper connected curve and $f : Y \to X$ a birational morphism onto a normal surface with exceptional locus C. Define a_i by*

$$C_j \cdot K_Y = C_j \cdot \sum_{i=1}^{s} a_i C_i, \qquad \text{for all } j.$$

(2.5.3.1) *Let $a_i > 0$ for all i. Then X is smooth.*
(2.5.3.2) *Let $a_i \geq 0$ for all i. Then X has Du Val singularities.*
(2.5.3.3) *Let $a_i > -1/3$ for all i. Then X has Du Val singularities.*

Proof. In case (2.5.3.1) by (2.2.3) there exists a j such that $C_j \cdot K_X < 0$, i.e. C_j is a -1-curve, so it can be contracted to a smooth point. Iterating this process we find that X is smooth.

In case (2.5.3.2) after contracting all the -1-curves, we have $a_i = 0$ for the remaining discrepancies by (2.2.3) and then X has Du Val singularities.

In case (2.5.3.3) by (2.4.8) in fact $a_i \geq 0$, so X has Du Val singularities. $\square$

2.5.4 Lemma. *Let (X, B) be a smooth log surface, $f : X \to X'$ the morphism onto the log minimal model of X and $C = \cup_{i=1}^{s} C_i$ the set of curves contracted by f. Let a_i be such that*

$$K_X + B \equiv f^*(K_{X'} + f_* B) + \sum_{i=1}^{s} a_i C_i.$$

Then $a_i > 0$ for $i = 1, \ldots, s$.

Proof. Let $f_i : X_i \to X_{i+1}$ be the step when C_i is contracted and $g_i : X = X_1 \to X_i$ be the composition of $f_1, \ldots, f_{i-1}$. Denote by B_i the birational transform of B on X_i and let α_i be such that

$$K_{X_i} + B_i \equiv f_i^*(K_{X_{i+1}+B_{i+1}}) + \alpha_i C_i.$$

$C_i \cdot (K_{X_i} + B_i) < 0$ shows that $\alpha_i > 0$. Next let γ_j, $j = 1, \dots, i-1$ be such that

$$K_X + B \equiv g_i^*(K_{X_i+B_i}) + \sum_{j=1}^{i-1} \gamma_j C_j.$$

Assume, by induction, that $\gamma_j > 0$. Then

$$K_X + B \equiv g_{i+1}^*(K_{X_{i+1}+B_{i+1}}) + \alpha_i g_i^* C_i + \sum_{j=1}^{i-1} \gamma_j C_j,$$

so the γ's defined for $i+1$ will be positive again and the statement follows. $\square$

2.5.5 Lemma. *Let $(X, B = \sum b_i B_i)$ be a smooth log surface and $C = \cup_{j=1}^s C_j$ the set of curves contracted by the log Minimal Model Program and a_j defined as in (2.5.3). Then $a_j > -b_j$.*

Proof. Let $f : X \to X'$ be the morphism onto the log minimal model of X. Let a_i' be such that

$$K_X + B \equiv f^*(K_{X'} + f_* B) + \sum_{i=1}^{s} a_i' C_i.$$

Now $a_j' > 0$ by (2.5.4). Let $b_i \geq 0$ be such that $B = \sum_{i=1}^s b_i C_i + B'$ and C_i are not contained in B'. Choose $\{r_i \mid i = 1, \dots, s\} \subset \mathbb{Q}^+$ such that

$$C_j \cdot B' = C_j \cdot \sum_{i=1}^{s} (-r_i) C_i, \qquad \text{for all } j.$$

Then

$$C_j \cdot K_X = C_j \cdot \sum_{i=1}^{s} (a_i' - b_i + r_i) C_i,$$

so $a_i = a_i' - b_i + r_i$ and the statement follows. $\square$

Proof of 2.5.2. Follows from (2.5.5) and (2.5.3.3). $\square$

The second example illustrating (2.5.1) becomes important later:

2.5.6 Proposition. *Let $(X, B' + B'')$ be a smooth log surface such that $B' = \sum_{i=1}^{r'} b_i' B_i'$ with $b_i' \leq 1/4$ and $B'' = \sum_{i=1}^{r''} b_i'' B_i''$. Let X' be the log minimal model of X and $g : X'' \to X'$ the minimal resolution of X'. Then the number of curves contracted by g is at most $2r''$.*

Proof. Let $C \subset X''$ be a g-extremal curve. If (the birational transform of) C is not contained in B'', then C has discrepancy at least $-1/4$. By (2.4.9) the number of such curves is at most r''. $\square$

2.5.7 Corollary. *With the same assumptions as in* (2.5.6) *assume further that there exists an* $\varepsilon > 0$ *such that* $b_i'' \leq 1-\varepsilon$ *for all* i. *Then the non Du Val singularities of the log minimal model of* (X, B) *are from a finite list. In particular, the index of the log minimal model of* X *divides* $\llcorner(2/\varepsilon)^{r''}\lrcorner!$.

Proof. Let the number of vertices of the resolution graph be k. k is at most $2r''$ and by (2.4.8) and (2.5.5)

$$-1 - \frac{2}{C_i^2} \geq a_i \geq -b_i \geq -1 + \varepsilon,$$

so $C_i^2 \geq -2/\varepsilon$.

Also, by (2.4.10) $|\det(C_i \cdot C_j)| \leq \prod(-C_i^2) \leq (2/\varepsilon)^{r''}$. This bounds the index by (2.4.2.1). □

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