Orbit Parametrizations of Curves

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Mathematics
Adviser: Manjul Bhargava

September 2009

Abstract

We investigate orbits of certain representations of reductive groups and their parametrizations of algebraic curves and vector bundles on those curves. Our results can be viewed as a higher-dimensional analogue of Bhargava's parametrizations of low rank rings and modules over them by integral orbits of prehomogeneous vector spaces.

The first spaces we study are the standard representations of $\mathrm{GL}_3 \times \mathrm{GL}_n \times \mathrm{GL}_n$ for $n \geq 3$, also called the spaces of $3 \times n \times n$ boxes. The associated orbit space is shown to be a moduli space for plane curves of degree n with specified line bundles, which is closely related to work of Beauville, Cook-Thomas, and Ng. We also derive parametrizations for related "symmetrized" representations, such as ternary cubic forms and triples of n-ary quadratic forms. Specializing to small n gives theorems on unirationality of certain universal Picard varieties, and for odd $n \geq 5$, the symmetrized theorem reinterprets work of Reid and Tjurin on intersections of quadrics.

We next examine the orbits of $2 \times 2 \times m \times m$ boxes for $m \geq 2$, which are shown to correspond to bidegree (m, m) curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with extra data. Again, we also consider the symmetrized orbit problem, for which there are additional conditions on the line bundles. Exploiting the numerous symmetries for a box when $m = 2$ produces parametrizations of many other spaces, including pairs of binary cubic forms and binary quartic forms. For $m = 3$, linking two different geometric interpretations of symmetrized boxes explicitly recovers a special case of the trigonal construction of Recillas.

Finally, we consider ternary quadratic forms taking values in a line bundle over an arbitrary base. The correspondence between ternary quadratic forms and quaternion algebras in this generality was given recently by Gross-Lucianovic and Voight, and we relate each of those categories to moduli spaces of vector bundles over genus zero curves.

As most of these orbit spaces are just affine spaces modulo fairly standard group actions, we obtain some geometric consequences. For example, all the moduli spaces thus obtained are unirational, and in cases where the invariant ring is free, even rational. Furthermore, the constructions we detail are completely explicit, and these descriptions of the moduli spaces lend themselves to computations.

Acknowledgements

I would like to sincerely thank my adviser Manjul Bhargava, whose own doctoral thesis inspired many of the ideas here. Not only has he suggested interesting problems and fruitful directions for research throughout graduate school, he has patiently answered my questions, both mathematical and meta-mathematical. Manjul has been much more than a mentor and a teacher these past years; he has been a friend, a role model, and an inspiration.

I am deeply grateful to Benedict Gross, Rahul Pandharipande, Peter Sarnak, and Andrew Wiles for discussing my research, introducing me to related ideas, and helping me develop as a mathematician. In addition, I wish to thank Max Lieblich, for reading this thesis as well as for answering many other questions during the past years.

My work has also benefited from conversations with other mathematicians, and I am especially grateful to John Cremona, Sam Grushevsky, Cathy O'Neil, and Ravi Vakil for their time and help.

Many graduate students at Princeton have contributed to my years here. I would like to specifically acknowledge Ben Bakker, Davesh Maulik, Nitin Saksena, Andrew Snowden, Melanie Wood, and Andrew Young, and especially Bhargav Bhatt for answering innumerable algebraic geometry questions.

I am thankful for the opportunities afforded me by the generous support of the National Science Foundation, the National Defense Science and Engineering Graduate Fellowship, and the Princeton University Centennial Fellowship during graduate school.

Finally, I would like to thank my family, for their encouragement and support, always.

To my father.

Contents

Chapter 1

Introduction

The underlying philosophy of this thesis is the investigation of links between moduli spaces from representation theory, algebra, and geometry. In particular, we study orbits of representations of reductive groups and their parametrizations of algebraic and geometric objects.

In 1801, Gauss gave the first example of such a parametrization in his celebrated *Disquisitiones Arithmeticae*. He studied integral binary quadratic forms under a certain action of the group $\mathrm{GL}_2(\mathbb{Z})$. Although the space of binary quadratic forms is a *prehomogeneous vector space*, meaning that it has an open orbit over $\mathbb{C}$, the integral orbits of binary quadratic forms are in bijection with ideal classes in quadratic rings. One of the key concepts used in understanding this correspondence is the discriminant of such a form, which is invariant under the group action.

By the late 1800s, classical invariant theory became one of the most active and productive areas of mathematics. Although most of the focus was on the computation of invariants of representations, which we will also use in our analyses, some orbit problems were also considered. For example, for two vector spaces V_1 and V_2 over an algebraically closed field, the product $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ acting on the tensor product $V_1 \otimes V_2$ of their standard representations is a prehomogeneous vector space; the same is true for a threefold product if one of the vector spaces has dimension one or two.

More recently, Bhargava, in a series of papers [Bha04a, Bha04b, Bha04c, Bha08], has studied the orbit problems for essentially all prehomogeneous vector spaces. In these cases, which are a generalization of Gauss's theorem, he proves that the *integral* orbits parametrize objects such as low rank rings and ideal classes of such rings or modules over them. From a geometric point of view, these objects generally correspond to zero-dimensional schemes and vector bundles over them, and their moduli spaces—although nontrivial—essentially have only one $\mathbb{C}$-point.[1]

Not all products of reductive groups and their standard representations are prehomogeneous, however. In fact, if all the vector spaces have dimension greater than 3, threefold products have continuous families of orbits even over an algebraically closed field, which of course increases the difficulty of understanding the orbits over any base, including $\mathbb{C}$. In these cases, even the nullcone[2] has infinitely many orbits. In this thesis, we primarily study the orbits of certain threefold and fourfold products and determine the algebro-geometric objects they parametrize. These representations have continuous families of orbits corresponding to higher-dimensional schemes.

In particular, we will study a number of representations that parametrize curves and vector bundles on these curves. For the most part (Chapters 2 to 5), the algebraic objects parametrized will visibly be equivalent to the geometric objects; that is, we could use either the language of curves and vector bundles on them, or that of the corresponding coordinate rings and modules over them. We will use the former, since we may then make use of the theory of moduli of curves, especially Brill-Noether theory, to understand their moduli spaces. In Chapter 6, however, the algebraic objects parametrized are noncommutative rings, so we investigate the links between all three types of objects: orbits of the representation, quaternion algebras, and vector bundles over genus zero curves.

In Chapters 2 to 5, the basic representations we consider are standard tensor products of products of general linear groups. For any length k vector $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$, let $G(\mathbf{a})$ be the reductive group $\mathrm{GL}_{a_1} \times \cdots \times \mathrm{GL}_{a_k}$, and let $V(\mathbf{a})$ be the tensor product $V(a_1) \otimes \cdots \otimes V(a_k)$, where $V(a_i)$ is the standard representation of GL_{a_i} for $1 \leq i \leq k$. We call the representation $V(\mathbf{a})$, with the action of $G(\mathbf{a})$, the space of $a_1 \times \cdots \times a_k$ *boxes*. In our cases, the $G(\mathbf{a})$-orbits of an open set of $V(\mathbf{a})$ parametrize curves with line bundles. In general, it will be clear that

[1]To be precise, the definition of prehomogeneous vector space implies that a dense open substack of the quotient stack, and hence the corresponding moduli stack given by Bhargava's parametrizations, has only one $\mathbb{C}$-valued point.

[2]The nullcone of a representation is the set of the most "degenerate" elements, specifically those whose orbits contain 0 in their closure.

the dimension of the orbit space $V(\mathbf{a})/G(\mathbf{a})$ matches the dimension of the corresponding moduli space of geometric objects. Also, since $k-1$ copies of $\mathbb{G}_m$ lie in the stabilizer of the action of $G(\mathbf{a})$ on $V(\mathbf{a})$, the projectivized orbit space $\mathbb{P}(V(\mathbf{a}))/(\mathrm{PGL}_{a_1} \times \cdots \times \mathrm{PGL}_{a_k})$ has the same coarse moduli space. For our purposes, however, the affine representation $V(\mathbf{a})$ with the action of $G(\mathbf{a})$ is the more natural space to study.

We begin in Chapter 2 by examining the case of $3 \times 3 \times 3$ boxes or, in other words, the tensor product of three three-dimensional vector spaces with the standard action of GL_3^3. First studied in [Ng95], the orbits of this space of trilinear forms correspond to genus one curves with degree 3 line bundles. Using the symmetry among the vector spaces, we rewrite such geometric data as genus one curves and points on their Jacobians. Although the bijection is initially proved over an algebraically closed field, we find that with the right definitions, the parametrizations hold over an arbitrary base scheme. The bijection then becomes an equivalence of moduli stacks. We also derive parametrizations for related "symmetrized" spaces, including triples of ternary quadratic forms and ternary cubic forms. In the appendix, we reinterpret the moduli space for $3 \times 3 \times 3$ boxes in terms of torsors of elliptic curves, which is closely related to unpublished work of Bhargava and O'Neil on this moduli problem over $\mathbb{Q}$

In Chapter 3, we study a fourfold tensor product of two-dimensional spaces, that is, $2 \times 2 \times 2 \times 2$ boxes or, more descriptively, *hypercubes*, which were also initially considered by Bhargava and O'Neil. The orbits of hypercubes also parametrize genus one curves with line bundles, this time of degree 2, as they naturally give rise to genus one curves embedded in $\mathbb{P}^1 \times \mathbb{P}^1$. In the same way as for $3 \times 3 \times 3$ boxes, we may reformulate the geometric data in several different ways, and the theorems extend to equivalences of the moduli stacks. Finally, the process of symmetrization applied to hypercubes gives parametrizations of spaces such as pairs of binary cubic forms and binary quartic forms, among others.

The analysis of $3 \times 3 \times 3$ boxes depends heavily on understanding smooth genus one curves as the zero locus of cubic forms in the projective plane. Ternary forms of degree n give higher genus curves with embeddings in $\mathbb{P}^2$, and many others have studied determinantal representations of plane curves [CT79, Bea00]. A related way to obtain such curves is from $3 \times n \times n$ boxes.

In Chapter 4, we investigate the orbits of $3 \times n \times n$ boxes using similar techniques to those of Chapter 2. Each orbit of these boxes correspond to a genus $\frac{1}{2}(n-1)(n-2)$ curve with certain line bundles, including a line bundle inducing a plane embedding of the curve. Orbits of the symmetrized space of triples of n-ary quadratic forms are in bijection with the same sorts of curves and line bundles, but with additional restrictions on the line bundles. Specializing to $n = 3$ recovers the results of Chapter 2, and to $n = 4$ gives a description of (an open subspace of) the universal degree 6 Picard variety over the moduli space $\mathscr{M}_3$ of genus 3 curves. For $n = 5$, these theorems imply that there is a close relationship between the space of plane quintics and the moduli space $\mathscr{M}_5$ of genus 5 curves. More generally, for $n \geq 5$ odd, the symmetrized $3 \times n \times n$ boxes also may be interpreted as nets of quadrics in $\mathbb{P}^{n-1}$; the bijection then exhibits the relationship, first proved in [Rei72] and [Tju75], between a certain Prym variety associated to the degree n plane curve and the intermediate Jacobian of the base locus of the net of quadrics.

Just as $3 \times n \times n$ boxes are a generalization of $3 \times 3 \times 3$ boxes, the space of $2 \times 2 \times m \times m$ boxes is one way to generalize hypercubes. In Chapter 5, using ideas from Chapters 3 and 4, we examine the orbits of $2 \times 2 \times m \times m$ boxes, which correspond to bidegree (m, m) curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with some extra data. Again, we look at the symmetrized orbit problem, where there are additional conditions on the line bundles. For $m = 3$, the symmetrized boxes also correspond to plane quartics with a noncanonical degree 4 map to $\mathbb{P}^1$. Combining these two interpretations of this space explicitly recovers a special case of the trigonal construction of Recillas, relating degree 3 and degree 4 covers of $\mathbb{P}^1$ [Rec74]. Our methods give an easy proof of the relationship between the space of Prym varieties of genus 4 curves and the universal Jacobian over $\mathscr{M}_3$, previously studied in [Rec93].

Finally, in Chapter 6, we study the space of ternary quadratic forms, not only over an arbitrary base, but also taking values in a line bundle. The realization that such forms correspond to quaternion algebras over the integers goes back to [Lat37, Bra43, Pal46], and an explicit correspondence over local rings and principal ideal domains is given in the recent work of Gross and Lucianovic [GL09]. Voight gives the most general construction relating ternary quadratic forms with their Clifford algebras [Voi09], and we relate each of those spaces to genus zero curves. We find that the category of smooth genus zero curves and that of quaternion algebras are equivalent, and ternary quadratic forms are closely related to both of those types of objects, along with a line bundle. The goal in this chapter is to connect the geometric moduli space to both the algebraic objects and the orbit problem in a consistent manner.

We should make a few disclaimers, however. Throughout the thesis, we work with a "nondegenerate" open subset of the tensor spaces, or an open substack of the natural quotient stacks, where the curves we obtain are smooth and irreducible. The condition for nondegeneracy in each case turns out to be determined locally by the nonvanishing of a certain polynomial that is invariant under the group action, which we will call the *discriminant*. Also, except in Chapter 6, the group actions under consideration generally give the standard representations. Changing this group action can significantly change the moduli spaces that these orbits parametrize. For example, even in the case of ternary cubics or binary quartics, a twisted action induces different parametrizations, as in [Fis06] and [CF09], respectively.

In the end, we hope these parametrizations are not only worthwhile for their own sakes, but also because they have useful applications to the study of curves. Because the orbit spaces are just affine spaces modulo fairly standard group actions, their invariant theory is often understood. As a result, we obtain geometric consequences; for example, all the moduli spaces thus obtained are unirational, and in cases where the invariant ring is free, rational. Furthermore, the constructions we detail are completely explicit, in that we can compute the equations of the curves with given embeddings. We hope that these descriptions of the moduli spaces will lend themselves to computations. In future work, we hope to carefully study the integral orbits of these representations and apply counting techniques, such as from [Bha05, Bha09], to obtain asymptotics for the geometric data with bounded invariants.

And now... let us step out into the night and pursue that flighty temptress, adventure.

—Albus Dumbledore, in *Harry Potter and the Half-Blood Prince* by J.K. Rowling

We turn the Cube and it twists us.

—Erno Rubik

Chapter 2

Rubik's Cubes and Curves of Genus One

In this chapter, we introduce the space of $3 \times 3 \times 3$ *boxes*, which we also call *Rubik's cubes*. The relationship between this space of trilinear forms and genus one curves, introduced in [Ng95], is our first example of the heuristic connecting certain orbits of representations with geometric data. The spaces of "symmetrized" Rubik's cubes also parametrize related geometric information.

From a Rubik's cube, one may naturally construct three ternary cubic forms. We first discuss the orbit problem for ternary cubic forms as an example, and then follow a similar method to understand orbits of Rubik's cubes. In the Appendix, we relate the moduli spaces of ternary cubic forms and Rubik's cubes over more general base schemes to torsors of elliptic curves.

Preliminaries. Throughout this chapter, let F be an algebraically closed field of characteristic not 2 or 3. In this chapter, we use the convention that the projectivization of a vector space parametrizes lines instead of hyperplanes. For example, a basepoint-free line bundle L on a variety X over F induces a natural map $\phi_L : X \to \mathbb{P}(H^0(X, L)^\vee)$. For this chapter, unless stated otherwise, a *genus* 1 *curve* means a proper, smooth, geometrically connected curve with arithmetic genus 1. An *elliptic curve* is such a genus one curve equipped with a base point.

2.1 Orbits of Ternary Cubic Forms

We first examine the orbits of ternary cubic forms over the field F. A *ternary cubic form* over F is a three-dimensional vector space U, a basis $\mathfrak{B} = \{w_1, w_2, w_3\}$ for U, and an element f of $\mathrm{Sym}^3 U$, represented as a polynomial

$$f(w_1, w_2, w_3) = aw_1^3 + bw_2^3 + cw_3^3 + a_2 w_1^2 w_2 + a_3 w_1^2 w_3 \\ + b_1 w_1 w_2^2 + b_3 w_2^2 w_3 + c_1 w_1 w_3^2 + c_2 w_2 w_3^2 + m w_1 w_2 w_3. \tag{2.1}$$

Then there is a natural action of $\mathrm{GL}(U) = \mathrm{GL}_3(F)$ on the space of all ternary cubic forms by the standard action of $\mathrm{GL}(U)$ on U.[1] We usually refer to the polynomial f as the ternary cubic form, with the vector space U and its basis understood. The ring of SL_3-invariants of the space of ternary cubic forms is a polynomial ring generated by a degree 4 invariant S and a degree 6 invariant T, and they may be computed by classical formulas (see [AKM$^+$01, Sil92], for example). Thus, the space of orbits $\mathrm{Sym}^3 U/\mathrm{SL}(U)$ is birational to the affine plane $\mathbb{A}^2$.

We claim that the "nondegenerate" locus of ternary cubic forms, up to linear transformations, parametrizes genus one curves with degree 3 line bundles, up to isomorphisms. In particular, a ternary cubic form f defines a curve $\iota : C := \{f = 0\} \hookrightarrow \mathbb{P}(U^\vee)$. We call f a *nondegenerate* ternary cubic form if C is smooth, which occurs if and only if the degree 12 discriminant $\Delta(f) := S^3 - T^2$ of f is nonzero. In this case, the curve C has genus one, and the pullback $\iota^* \mathcal{O}_{\mathbb{P}(U^\vee)}(1)$ is a degree 3 line bundle on C.

On the other hand, given a genus one curve C and a degree 3 line bundle L on C, the embedding of C into $\mathbb{P}(H^0(C, L)^\vee) = \mathbb{P}^2$ gives rise to the exact sequence of sheaves

$$0 \longrightarrow I_C \longrightarrow \mathcal{O}_{\mathbb{P}(H^0(C,L)^\vee)} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

on $\mathbb{P}(H^0(C, L)^\vee)$, where I_C is the ideal defining the curve C. Tensoring the sequence with

[1] The choice of the $\mathbb{G}_m$-action here is important. If we consider a twisted action of GL_3 on the space of ternary cubic forms, e.g., $\mathrm{GL}(U)$ acting on $\mathrm{Sym}^3 U \otimes (\det U)^k$ for some integer k, the orbits may be different (see [Fis06] for the case $m = -1$).

$\mathcal{O}_{\mathbb{P}(H^0(C,L)^\vee)}(3)$ and taking cohomology produces

$$0 \longrightarrow F \longrightarrow H^0(\mathbb{P}(H^0(C,L)^\vee), \mathcal{O}(3)) \longrightarrow H^0(C, L^{\otimes 3}) \longrightarrow 0,$$

where the image of $1 \in F$ is an element of $H^0(\mathbb{P}(H^0(C,L)^\vee), \mathcal{O}(3)) = \operatorname{Sym}^3(H^0(C,L))$, i.e., a ternary cubic form with $U := H^0(C,L)$. These two functors between ternary cubic forms and pairs (C,L) are inverse to one another, as long as a basis for $H^0(C,L)$ is specified.

Therefore, there is a bijection

$$\left\{\begin{array}{l}\text{nondegenerate}\\ \text{ternary cubic}\\ \text{forms over } F\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{isomorphism classes of triples } (C, L, \mathfrak{B}) \text{ where } C \text{ is}\\ \text{a genus one curve, } L \text{ is a degree 3 line bundle on}\\ C\text{, and } \mathfrak{B} \text{ is a basis for } H^0(C,L)\end{array}\right\}. \tag{2.2}$$

Here, triples $(C, L, \mathfrak{B})$ and $(C', L', \mathfrak{B}')$ are isomorphic if there is an isomorphism $\sigma : C \to C'$ such that $\sigma^* L' = L$ and $\sigma^* : H^0(C', L') \to H^0(C, L)$ is an isomorphism taking the basis $\mathfrak{B}'$ to the basis $\mathfrak{B}$.

The action of $\mathrm{GL}(U)$ on ternary cubic forms simply changes the basis for U, which via the bijection, corresponds to changing the basis of $H^0(C,L)$ on the right side. Also, the action of $\mathrm{GL}(U)$ preserves nondegeneracy of ternary cubic forms, since the condition is the nonvanishing of a $\mathrm{SL}(U)$-invariant. The bijection (2.2) thus descends to a bijection of the quotient spaces

$$\left\{\begin{array}{l}\text{nondegenerate}\\ \mathrm{GL}(U)\text{-equivalence}\\ \text{classes of } \operatorname{Sym}^3 U\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{isomorphism classes of pairs } (C, L) \text{ where } C\\ \text{is a genus one curve and } L \text{ is a degree 3 line}\\ \text{bundle on } C\end{array}\right\}. \tag{2.3}$$

The right side of this bijection (2.3), though, may be rewritten without the line bundle, since there exists a translation $\sigma : C \to C$ such that the pullback of a degree 3 line bundle L is any other degree 3 line bundle. In other words, the automorphism group of C, which includes $\operatorname{Pic}^0(C)$, acts transitively on the group of degree 3 line bundles $\operatorname{Pic}^3(C)$. Forgetting the bundle L, though, is dependent on working over the algebraically closed field F. More generally, over a field k, such translations by $P \in \operatorname{Pic}^0(C)$ shift L by $3P$, so given the curve C, the set of isomorphism classes of pairs (C, L) is visibly a quotient of $\operatorname{Pic}^0(C)/3\operatorname{Pic}^0(C)$.

We will analyze this phenomenon more carefully in Appendix 2.A when describing this bijection over an arbitrary base scheme.

Also, as the nondegenerate subspaces of $\mathrm{Sym}^3 U/\mathrm{GL}(U)$ and $\mathbb{P}(\mathrm{Sym}^3 U)/\mathrm{PGL}(U)$ are the same, we may also think of the left side of the bijection (2.3) as ternary cubic forms up to scaling and linear transformations, by separating the action of $\mathrm{GL}(U)$ into the actions of $\mathbb{G}_m$ and $\mathrm{PGL}(U)$.

Remark 2.1. It is possible to generalize the notion of ternary cubic form over an arbitrary base scheme and taking values in a line bundle over that scheme, similarly to the definition of a ternary quadratic form in Chapter 6. That is, one may define a ternary cubic form $(\mathcal{W}, L_S, f)$ over a scheme S taking values in the line bundle L_S over S as a vector bundle $\mathcal{W}$ over S, a line bundle L_S over S, and a section f of $\mathrm{Sym}^3 \mathcal{W} \otimes L_S$. In Appendix 2.A.2, we study ternary cubic forms over $\mathbb{Z}[\frac{1}{6}]$-schemes S and relate them to elements of the fppf cohomology group $H^1_{\mathrm{f}}(S, E[3])$ for elliptic curves E.

The bijection described above, between equivalence classes of ternary cubic forms over F and genus one curves over F with a degree 3 line bundle, is classical, but it illustrates some of the techniques we will use to obtain similar bijections between orbits of other representations and geometric data.

2.2 Rubik's Cubes

Let U_1, U_2, and U_3 be three-dimensional vector spaces over F. Then the reductive group $G := \mathrm{GL}(U_1) \times \mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$ acts on the tensor space $U_1 \otimes U_2 \otimes U_3$ by the natural action on each factor. With choices of bases for the vector spaces U_1, U_2, and U_3, we may represent an element of $U_1 \otimes U_2 \otimes U_3$ as a $3 \times 3 \times 3$ *box* or *Rubik's cube*

$$\mathcal{A} = (a_{rst})_{1 \leq r,s,t \leq 3}.$$

The group G acts by row, column, and "other direction" operations on the space of Rubik's cubes. In the sequel, we will refer to both the 3-dimensional array and the trilinear form as the Rubik's cube, with the vector spaces U_i and bases for each U_i understood.

We use the notation $\mathcal{A}(\cdot,\cdot,\cdot)$ to denote the trilinear form, where the dots may be replaced by substituting elements of the respective $U_i^\vee$. For example, given an element $w \in U_1^\vee$, the notation $\mathcal{A}(w,\cdot,\cdot)$ will refer to the 3×3 matrix $\mathcal{A}\lrcorner w \in U_2 \otimes U_3$. By a slight abuse of notation, we will also use this notation to specify whether $\mathcal{A}(w,\cdot,\cdot)$ vanishes for $w \in \mathbb{P}(U_1^\vee)$, for example.

2.2.1 Genus One Curves and Degree 3 Line Bundles

Let $\mathcal{A} = (a_{rst}) \in U_1 \otimes U_2 \otimes U_3$ be a Rubik's cube. Then the vanishing of the ternary cubic form, if nonzero,

$$f_1(w_1, w_2, w_3) := \det \mathcal{A}(w,\cdot,\cdot) \in \operatorname{Sym}^3 U_1$$

defines a degree 3 curve $C_1 \subset \mathbb{P}(U_1^\vee) = \mathbb{P}^2$. In other words, the curve C_1 is a determinantal variety, given by the determinant of a matrix of linear forms on $\mathbb{P}(U_1^\vee)$. We define analogous ternary cubic forms $f_2 \in \operatorname{Sym}^3 U_2$ and $f_3 \in \operatorname{Sym}^3 U_3$, which (if nonzero) give rise to degree 3 curves $C_2 \subset \mathbb{P}(U_2^\vee)$ and $C_3 \subset \mathbb{P}(U_3^\vee)$. We show below that these curves are isomorphic for a nondegenerate Rubik's cube.

We call a Rubik's cube $\mathcal{A}$ *nondegenerate* if the variety C_1 (equivalently, C_2 or C_3) thus defined is smooth and one-dimensional, which corresponds to the nonvanishing of a degree 36 polynomial in a_{rst}. This polynomial is called the *discriminant* of the Rubik's cube $\mathcal{A}$, and it coincides with the usual degree 12 discriminant $\Delta(f_1)$ of the ternary cubic form f_1.[2] If $\mathcal{A}$ is nondegenerate, the degree 3 plane curve C_1 is smooth of genus one. Then for all points $w^\dagger \in C_1$, we claim that the singular matrix $\mathcal{A}(w^\dagger,\cdot,\cdot)$ has exactly rank 2. If not, then the 2×2 minors of $\mathcal{A}(w,\cdot,\cdot)$ would vanish on $w^\dagger$, and so would all the partial derivatives

$$\left.\frac{\partial f}{\partial w_i}\right|_{w=w^\dagger} = \sum_{s,t} a_{ist} A^*_{ij}(w^\dagger)$$

where $A^*_{ij}(w^\dagger)$ is the (i,j)th 2×2 minor of $\mathcal{A}(w^\dagger,\cdot,\cdot)$. Since C_1 was assumed to be smooth, however, the rank of the matrix $\mathcal{A}(w^\dagger,\cdot,\cdot)$ cannot drop by two. In the sequel, we will assume $\mathcal{A}$ is nondegenerate.

[2]We will see later that the nonvanishing of the discriminants $\Delta(f_i)$ for any of the ternary cubic forms f_i, $1 \le i \le 3$, are all equivalent conditions.

Given a nondegenerate Rubik's cube $\mathcal{A}$, define the variety

$$C_{12} := \{(w,x) \in \mathbb{P}(U_1^\vee) \times \mathbb{P}(U_2^\vee) : \mathcal{A}(w,x,\cdot) = 0\} \subset \mathbb{P}(U_1^\vee) \times \mathbb{P}(U_2^\vee).$$

Because $\mathcal{A}$ is a trilinear form and the locus on which it vanishes in $U_1 \times U_2$ is invariant under scaling, this is a well-defined locus in $\mathbb{P}(U_1^\vee) \times \mathbb{P}(U_2^\vee)$. Since $\mathcal{A}$ is nondegenerate, the projection

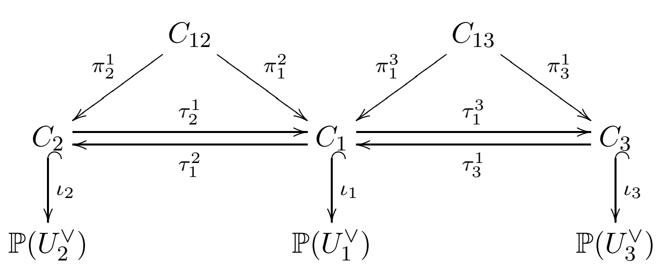
$$C_{12} \longrightarrow \mathbb{P}(U_1^\vee)$$

is an isomorphism onto C_1. The inverse map takes a point $w \in C_1 \subset \mathbb{P}(U_1^\vee)$ to the pair $(w,x) \in \mathbb{P}(U_1^\vee) \times \mathbb{P}(U_2^\vee)$, where x corresponds to the exactly one-dimensional kernel of the linear map $\mathcal{A}(w,\cdot,\cdot) \in U_2 \otimes U_3 \cong \mathrm{Hom}(U_2^\vee, U_3)$. This map $C_1 \to C_{12}$ is algebraic, as this kernel is given as a regular map by the 2×2 minors of the matrix $\mathcal{A} \lrcorner w$. Therefore, by dimension considerations, the curve C_{12} is the complete intersection of 3 bidegree $(1,1)$ forms on $\mathbb{P}(U_1^\vee) \times \mathbb{P}(U_2^\vee) = \mathbb{P}^2 \times \mathbb{P}^2$. Similarly, the projection from C_{12} to $\mathbb{P}(U_2^\vee)$ is an isomorphism onto C_2, which shows that C_1 and C_2 are isomorphic.

We may also consider the curve

$$C_{13} := \{(w,y) \in \mathbb{P}(U_1^\vee) \times \mathbb{P}(U_3^\vee) : \mathcal{A}(w,\cdot,y) = 0\},$$

and the analogous maps between C_1, C_3, and C_{13} are also isomorphisms. Thus, all the curves C_1, C_2, C_3, C_{12}, and C_{13} are isomorphic, and the nondegeneracy condition is equivalent to the smoothness of any or all of these curves. The diagram



summarizes the relationships between these curves. By construction, the maps τ_i^j and τ_j^i are inverses to one another. These maps from the curve C_1 to each projective space give the degree 3 line bundles

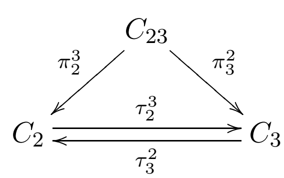
$$
\begin{aligned}
L_1 &:= \iota_1^* \mathcal{O}_{\mathbb{P}(U_1^\vee)}(1) \\
L_2 &:= (\iota_2 \circ \tau_1^2)^* \mathcal{O}_{\mathbb{P}(U_2^\vee)}(1) \\
L_3 &:= (\iota_3 \circ \tau_1^3)^* \mathcal{O}_{\mathbb{P}(U_3^\vee)}(1)
\end{aligned}
$$

on C_1. For $1 \leq i \leq 3$, all 3 dimensions of sections of the degree 3 bundle L_i arise from pulling back sections from $\mathcal{O}_{\mathbb{P}(U_i^\vee)}(1)$.

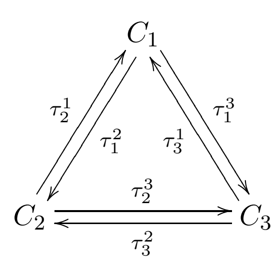
Lemma 2.2. *The degree* 3 *line bundle* L_1 *on* C_1 *is not isomorphic to either of the line bundles* L_2 *or* L_3*.*

Proof. It suffices, without loss of generality, to show that L_1 and L_2 are not isomorphic line bundles. If $L_1 \cong L_2$, then the curve C_{12} would lie on a diagonal of $\mathbb{P}^2 \times \mathbb{P}^2 = \mathbb{P}(U_1^\vee) \times \mathbb{P}(U_2^\vee)$, and with an identification of the bases for U_1 and U_2, we have $\mathcal{A}(w, w, \cdot) = 0$ for all $w \in C_1$, for example. Because C_1 spans $\mathbb{P}(U_1^\vee)$, we must have that $\mathcal{A}(\cdot, \cdot, y)$ is a skew-symmetric 3×3 matrix for any $y \in \mathbb{P}(U_3^\vee)$. Since odd-dimensional skew-symmetric matrices have determinant zero, we would have $C_3 = \mathbb{P}(U_3^\vee)$, which is a contradiction. ☐

Of course, there also exists a curve $C_{23} \subset \mathbb{P}(U_2^\vee) \times \mathbb{P}(U_3^\vee)$ and corresponding maps



so we have the diagram



(2.4)

relating the three plane cubics. Generically, the composition maps such as

$$\alpha_{132} := \tau_2^1 \circ \tau_3^2 \circ \tau_1^3 : C_1 \longrightarrow C_1$$

are not the identity map. In particular, the following relation among the line bundles L_i and the symmetry of the constructions will show that the composition maps are automorphisms of the curves given as translations by points on their Jacobians.

Lemma 2.3. *The line bundles L_1, L_2, L_3 on C_1 defined above satisfy the relation*

$$L_1 \otimes L_1 \cong L_2 \otimes L_3. \tag{2.5}$$

Proof. For $w \in C_1 \subset \mathbb{P}(U_1^\vee)$, each coordinate of $\tau_1^2(w) \in \mathbb{P}(U_2^\vee)$ is given by the 2×2 minors $A_{ij}^*(w)$ of $\mathcal{A}(w, \cdot, \cdot)$ for some fixed j where not all $A_{ij}^*(w)$ vanish. Let D_2 be an effective degree 3 divisor on C_1 such that $\mathcal{O}(D_2) \cong L_2$. Then the points of D_2 are the preimage on C_1 of the intersection of a hyperplane with the image of the curve C_{12} in $\mathbb{P}(U_2^\vee)$; in particular, we may choose D_2, without loss of generality, to be the divisor defined by the locus where a particular minor, say $A_{11}^*(w)$, vanishes on the curve C_1 but at least one $A_{i1}^*(w)$ is nonzero. Similarly, we may choose a divisor D_3 such that $\mathcal{O}(D_3) \cong L_3$ to be the points $w \in C_1$ where $A_{11}^*(w) = 0$ but not all other $A_{j1}^*(w)$ vanish. Then the points of the degree 6 effective divisor $D_2 + D_3$ are exactly the intersection of the curve C_1 and $A_{11}^*(w) = 0$, and the corresponding line bundle is isomorphic to the pullback of $\mathcal{O}_{\mathbb{P}(U_1^\vee)}(2)$ to C_1. □

Because the construction of the curves C_1, C_2, and C_3 are entirely symmetric, the line bundle relation of Lemma 2.3 can be, without loss of generality, also applied to line bundles

$$M_2 := \iota_2^* \mathcal{O}_{\mathbb{P}(U_2^\vee)}(1)$$
$$M_1 := (\iota_1 \circ \tau_2^1)^* \mathcal{O}_{\mathbb{P}(U_1^\vee)}(1)$$
$$M_3 := (\iota_3 \circ \tau_2^3)^* \mathcal{O}_{\mathbb{P}(U_3^\vee)}(1)$$

on C_2. That is, the relation $M_2 \otimes M_2 \cong M_1 \otimes M_3$ holds, and pulling back these bundles through ι_1 to C_1 gives the relation

$$L_2 \otimes L_2 \cong L_1 \otimes L_3',$$

where $L_3' := \iota_1^* M_3$. Similarly, if $L_1' := (\iota_1 \circ \tau_2^1 \circ \tau_3^2 \circ \tau_1^3)^* \mathcal{O}_{\mathbb{P}(U_1^\vee)}(1) = (\iota_1 \circ \alpha_{132})^* \mathcal{O}_{\mathbb{P}(U_1^\vee)}(1)$, then

$$L_3' \otimes L_3' \cong L_2 \otimes L_1'.$$

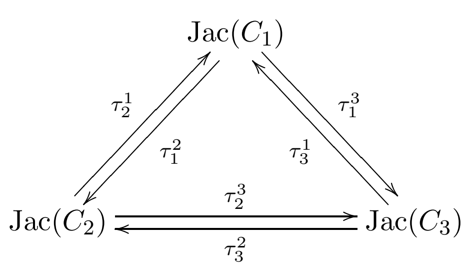
A straightforward calculation shows that

$$L_3' \otimes L_3^{-1} \cong L_1' \otimes L_1^{-1} \cong (L_1 \otimes L_2^{-1})^{\otimes 3},$$

i.e., the automorphism α_{132} of C_1 is given by the action of the point in $\mathrm{Jac}(C_1)$ corresponding to the line bundle $P_{132} := L_1' \otimes L_1^{-1} \in \mathrm{Pic}^0(C_1)$. If $Q_{12} := L_1 \otimes L_2^{-1}$ in $\mathrm{Pic}^0(C_1)$, we have $P_{132} = (Q_{12})^{\otimes 3}$.

All of the other composition maps $\alpha_{ijk} : C_i \to C_i$, for $\{i, j, k\} = \{1, 2, 3\}$, given by following the appropriate maps around the diagram (2.4), are similarly given by translating C_i by the corresponding line bundles P_{ijk} of the degree 0 Picard group of C_i. Clearly $P_{ijk} = -P_{ikj}$, since reversing the three-cycle in diagram (2.4) is the inverse map.

Recall that $\mathrm{Pic}^0(C)$ and $\mathrm{Jac}(C)$ are naturally dual to one another for any curve C, and the principal polarization on $\mathrm{Jac}(C)$ gives a natural isomorphism between the two. So the degree 0 line bundles Q_{ij} and P_{ijk} may also be thought of as points of the Jacobian $\mathrm{Jac}(C)$ of C. Furthermore, note that the Jacobians of C_1, C_2, and C_3 are canonically isomorphic, and the induced diagram on Jacobians



is commutative, so the composition map $\mathrm{Jac}(C_i) \to \mathrm{Jac}(C_i)$ is the identity for $1 \leq i \leq 3$.

2.2.2 The Group Action

We saw in Section 2.2.1 that a Rubik's cube $\mathcal{A}$ gives rise to a genus one curve, up to isomorphism, and certain degree 3 line bundles. This data is, in fact, determined up to isomorphism under the action of the group G on the space of Rubik's cubes.

Let $\mathcal{Q}$ be the space of quadruples (C, L_1, L_2, L_3), where C is a genus one curve and L_1, L_2, and L_3 are degree 3 line bundles on C. We call two quadruples (C, L_1, L_2, L_3) and (C', L_1', L_2', L_3') *equivalent* if there exists an isomorphism $\sigma : C \to C'$ such that $\sigma^* L_i' \cong L_i$ for $1 \leq i \leq 3$.

It is evident that we actually have a map from Rubik's cubes to $\mathcal{Q}$ up to equivalence. Note that scaling visibly does not change the curves in $\mathbb{P}(U_i^\vee)$ for $1 \leq i \leq 3$ defined by the ternary cubic forms arising from a Rubik's cubes, nor the maps between them, so this map factors through $\mathbb{P}(U_1 \otimes U_2 \otimes U_3)$. Those cubics in $\mathbb{P}^2$ and the maps between them give rise to all the geometric data in the quadruple.

Lemma 2.4. *The map Φ from the nondegenerate open subscheme of $U_1 \otimes U_2 \otimes U_3$ to the equivalence classes of $\mathcal{Q}$ is G-invariant.*

Proof. We only need to show that this map is well-defined on nondegenerate G-orbits. First, note that the G-action preserves the nondegeneracy of a Rubik's cube. In fact, the degree 36 discriminant of a Rubik's cube is an invariant of the action of $\mathrm{SL}(U_1) \times \mathrm{SL}(U_2) \times \mathrm{SL}(U_3)$. By the construction of the curve $C_1 \subset \mathbb{P}(U_1^\vee)$, the condition for smoothness of C_1 implies that the nonvanishing of the discriminant is invariant under the action of $\mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$, and because the discriminant of the Rubik's cube is the same as the discriminant of any of the curves $C_i \subset \mathbb{P}(U_i^\vee)$ for $1 \leq i \leq 3$, its nonvanishing is invariant under the natural action of $\mathrm{GL}(U_1) \times \mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$ on $U_1 \otimes U_2 \otimes U_3$.

Moreover, given an element $g = (g_1, g_2, g_3) \in G$, we show that the Rubik's cubes $\mathcal{A}$ and $g(\mathcal{A})$ produce equivalent quadruples (C, L_1, L_2, L_3) and (C', L_1', L_2', L_3'), respectively. First, for $1 \leq i \leq 3$, the ternary cubic forms f_i and f_i' differ only by the change of basis $g_i \in \mathrm{GL}(U_i)$, so the curves C_i and C_i' may be taken to one another via the image $\overline{g_i}$ of g_i in $\mathrm{PGL}(U_i)$. Similarly, the curves C_{ij} and C_{ij}' also differ by the action of $(\overline{g_i}, \overline{g_j})$ on the bases of $\mathbb{P}(U_i^\vee)$ and $\mathbb{P}(U_j^\vee)$, so each of the squares

$$\begin{array}{ccc} C_i & \xrightarrow{\overline{g_i}} & C_i' \\ {\scriptstyle \tau_i^j}\downarrow & & \downarrow{\scriptstyle (\tau_i^j)'} \\ C_j & \xrightarrow[\overline{g_j}]{} & C_j' \end{array}$$

commutes, as desired. ☐

Therefore, from a nondegenerate G-orbit of Rubik's cubes, we have produced a genus one curve with these three line bundles, which is the same data as the curve with three embeddings into $\mathbb{P}^2$, with a relation among the bundles.

2.3 The Moduli Problem for Rubik's Cubes

In fact, the geometric data as described in Section 2.2.1 is exactly enough to recover a nondegenerate G-orbit of Rubik's cubes. We will also show that Rubik's cubes parametrize a genus one curve with line bundles and bases for related vector spaces, up to the most natural notion of isomorphism; when the action of the group G is taken into account, the bases will disappear from this set of data. This parametrization holds not only over the base field F but for families of Rubik's cubes as well.

2.3.1 Preliminary Bijection

A version of the following theorem appears in [Ng95], but we include slightly modified statements and proofs to set up subsequent generalizations.

Theorem 2.5. *There exists a bijection*

$$\left\{\begin{array}{l} G\text{-equivalence} \\ \text{classes of} \\ \text{nondegenerate} \\ \text{Rubik's cubes} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \text{equivalence classes of } (C, L_1, L_2, L_3) \text{ with } C \text{ a} \\ \text{genus } 1 \text{ curve and } L_1, L_2, L_3 \text{ degree } 3 \text{ line bun-} \\ \text{dles on } C \text{ with } L_1 \otimes L_1 \cong L_2 \otimes L_3 \text{ and } L_1 \text{ not} \\ \text{isomorphic to } L_2 \text{ or } L_3 \end{array}\right\} \tag{2.6}$$

where the right arrow is given by Φ in Lemma 2.4.

Proof. We have already shown that there is a well-defined map Φ from G-orbits of non-degenerate Rubik's cubes to the listed geometric data. In the other direction, given such a quadruple (C, L_1, L_2, L_3), we consider the multiplication map (i.e., the cup product on cohomology)

$$\mu_{12} : H^0(C, L_1) \otimes H^0(C, L_2) \longrightarrow H^0(C, L_1 \otimes L_2). \tag{2.7}$$

A simple case of a theorem of [Mum70] shows that μ is surjective. Thus, by Riemann-Roch, the kernel of μ_{12} has dimension $9 - 6 = 3$. Now let $U_1 := H^0(C, L_1), U_2 := H^0(C, L_2)$, and $U_3 := (\ker(\mu_{12}))^\vee$, and we obtain a Rubik's cube by the injection

$$\ker(\mu_{12}) \hookrightarrow H^0(C, L_1) \otimes H^0(C, L_2),$$

which is an element of $\mathrm{Hom}(\ker(\mu_{12}), H^0(C, L_1) \otimes H^0(C, L_2)) \cong H^0(C, L_1) \otimes H^0(C, L_2) \otimes (\ker(\mu_{12}))^\vee$. The actual $3 \times 3 \times 3$ box requires a choice of basis for each of the vector spaces, and here the box we recover as an element of $H^0(C, L_1) \otimes H^0(C, L_2) \otimes (\ker(\mu_{12}))^\vee$ is only defined up to linear transformations of those vector spaces, so we have constructed a Rubik's cube up to G-equivalence. Note that with the choice of bases for each of the vector spaces $H^0(C, L_1), H^0(C, L_2)$, and $(\ker(\mu_{12}))^\vee$, however, the Rubik's cube is uniquely specified.

If the quadruples (C, L_1, L_2, L_3) and (C', L'_1, L'_2, L'_3) are equivalent, then there exists an isomorphism $\sigma : C \to C'$ such that $\sigma^* L'_i \cong L_i$ for $1 \leq i \leq 3$. The isomorphisms induced on the spaces of sections, e.g., $H^0(C, L_1) \xrightarrow{\cong} H^0(C', L'_1)$, commute with the multiplication maps, so the Rubik's cubes constructed by their kernels differ only by choices of bases.

We check that the two functors between orbits of Rubik's cubes and the equivalence classes of quadruples are inverse to one another. Given a quadruple (C, L_1, L_2, L_3) of the appropriate type, define the images of the natural embeddings

$$\begin{aligned}
C_1 &:= \phi_{L_1}(C) && \subset \mathbb{P}(H^0(C, L_1)^\vee) \\
C_2 &:= \phi_{L_2}(C) && \subset \mathbb{P}(H^0(C, L_2)^\vee) \\
C_{12} &:= (\phi_{L_1}, \phi_{L_2})(C) && \subset \mathbb{P}(H^0(C, L_1)^\vee) \times \mathbb{P}(H^0(C, L_2)^\vee).
\end{aligned}$$

We construct the Rubik's cube $\mathcal{A} \in H^0(C, L_1) \otimes H^0(C, L_2) \otimes (\ker \mu_{12})^\vee$ as above. Now let

$$\begin{aligned} D_1 &:= \{w \in \mathbb{P}(H^0(C, L_1)^\vee) : \det \mathcal{A}(w, \cdot, \cdot) = 0\} \qquad \subset \mathbb{P}(H^0(C, L_1)^\vee) \\ D_2 &:= \{x \in \mathbb{P}(H^0(C, L_2)^\vee) : \det \mathcal{A}(\cdot, x, \cdot) = 0\} \qquad \subset \mathbb{P}(H^0(C, L_2)^\vee) \\ D_{12} &:= \{(w, x) \in \mathbb{P}(H^0(C, L_1)^\vee) \times \mathbb{P}(H^0(C, L_2)^\vee) : \mathcal{A}(w, x, \cdot) = 0\} \\ &\qquad\qquad\qquad\qquad \subset \mathbb{P}(H^0(C, L_1)^\vee) \times \mathbb{P}(H^0(C, L_2)^\vee) \end{aligned}$$

be the varieties cut out by the trilinear form $\mathcal{A}(\cdot, \cdot, \cdot)$.

We claim that $C_1 = D_1, C_2 = D_2$, and $C_{12} = D_{12}$ as sets and thus as varieties, which implies that the curve D_1 is isomorphic to C and that the line bundles on D_1 defined as pullbacks of $\mathcal{O}(1)$ on $\mathbb{P}(H^0(C, L_1)^\vee)$ and $\mathbb{P}(H^0(C, L_2)^\vee)$ are isomorphic to the pullbacks of L_1 and L_2, respectively, via the isomorphism $C \xrightarrow{\cong} D_1$. For all $(w^\dagger, x^\dagger) \in C_{12}$, the construction of the Rubik's cube $\mathcal{A}$ as the kernel of μ_{12} implies that $\mathcal{A}(w^\dagger, x^\dagger, \cdot) = 0$, so D_{12} contains C_{12} and also $D_1 \supset C_1$ and $D_2 \supset C_2$.

Now either the polynomial $\det \mathcal{A}(w, \cdot, \cdot)$ or $\det \mathcal{A}(\cdot, x, \cdot)$ is not identically 0. If they both were identically 0, then $\mathcal{A}(w, x, \cdot) = 0$ for all $(w, x) \in \mathbb{P}(H^0(C, L_1)^\vee) \times \mathbb{P}(H^0(C, L_2)^\vee)$, which contradicts the fact that $\mathcal{A}$ must have nonzero tensor rank by construction. Without loss of generality, assume $\det \mathcal{A}(w, \cdot, \cdot)$ is not identically zero. Then both D_1 and C_1 are given by nonzero degree 3 polynomials and thus define the same variety, so D_1 is a smooth irreducible genus one curve in $\mathbb{P}^2 = \mathbb{P}(H^0(C, L_1)^\vee)$. Because D_1 is smooth, the Rubik's cube $\mathcal{A}$ is nondegenerate, and D_{12} is also smooth and irreducible, hence exactly the same set of points as C_{12}.

Note that the above argument shows that the Rubik's cube constructed from a smooth irreducible genus one curve C and two nonisomorphic line bundles L_1 and L_2 is nondegenerate.

It remains to show that the geometric data coming from a Rubik's cube produces the same cube again. Given a nondegenerate Rubik's cube $\mathcal{A} \in U_1 \otimes U_2 \otimes U_3$, where U_i are three-dimensional vector spaces for $1 \leq i \leq 3$, we have described the associated quadruple (C, L_1, L_2, L_3) as the image of Φ. Then the vector spaces U_i and $H^0(C, L_i)$ are naturally isomorphic for $i = 1, 2$, and $U_3^\vee$ can be identified with the kernel of the multiplication map

μ_{12} as above. With these identifications, the Rubik's cube constructed from this quadruple is well-defined and G-equivalent to the original cube $\mathcal{A}$. If we also identify the bases for each of these vector spaces, then in fact, the Rubik's cube obtained by this contruction will be the same as $\mathcal{A}$. □

Remark 2.6. Note that the line bundle L_3 is not used directly in the proof of Theorem 2.5. The bundle L_3 does not need to be included in the data on the right side of bijection (2.6), since L_3 and the condition that $L_1 \not\cong L_3$ can be completely recovered from L_1 and L_2. It is not a priori clear that the construction of (the G-orbit of) a Rubik's cube from L_1 and L_2 gives the same orbit as the analogous construction from L_1 and L_3. Because the same proof works if we switch the roles of L_2 and L_3 throughout, the bijection above shows that from a quadruple (C, L_1, L_2, L_3), either construction would produce G-equivalent Rubik's cubes. As a result, there exists a natural identification between the space of sections of the bundle $L_1 \otimes L_1 \otimes L_2^{-1}$ on the curve C and the dual of $\ker \mu_{12}$ (and, likewise, $H^0(C, L_1 \otimes L_1 \otimes L_3^{-1})$ and the dual of $\ker \mu_{13}$).

Remark 2.7. Like in the case of ternary cubic forms, as we are working over the algebraically closed field F, one of the line bundles on the genus one curve C that arises is superfluous under the notion of equivalence of (C, L_1, L_2, L_3). The automorphism $\sigma_{-P} : C \to C$ given by translation by a point $-P \in \mathrm{Pic}^0(C)$ will shift each of the line bundles L_i on C by $3P$ under the pullback, i.e., $\sigma_{-P}^* L_i \cong L_i \otimes P^{\otimes 3}$. So the line bundle L_1, for example, can be taken to be any degree 3 line bundle on C. The differences $L_1 \otimes L_2^{-1}$ and $L_1 \otimes L_3^{-1}$ are not changed, however, by these translations. The only other automorphisms of a generic genus one curve C are "flips" around a point $x \in C$, in which case the differences $L_1 \otimes L_2^{-1}$ and $L_1 \otimes L_3^{-1}$ are taken to their duals under pullback. There are, of course, other ways for quadruples to be equivalent.

There also is an algebraic proof of Theorem 2.5, which is detailed in more generality (for $3 \times n \times n$ boxes) in Section 4.2.3. Given a Rubik's cube, computing the corresponding ternary cubic forms is of course straightforward, but the algebraic proof gives a very explicit method for computing the box, given a genus one curve and line bundles (or, as we will see, a point on the Jacobian of the curve). We will provide an example in the next section after formulating the geometric data in a more easily presentable manner.

2.3.2 Reformulations

Theorem 2.5 may be restated or modified in several ways, such as by changing the categories on either side to equivalent information or by rigidifying the data. In this section, we discuss several of these reformulations.

An equivalent way to view a quadruple (C, L_1, L_2, L_3) is via the points Q_{ij} or P_{ijk} on the Jacobian of C, as in Section 2.2.1. That is, from the data (C, L_1, L_2, L_3), we obtain the nonzero point $Q_{12} \in \mathrm{Pic}^0(C)$, say, as the difference $L_1 \otimes L_2^{-1}$. From the genus one curve C, a degree 3 line bundle L_1, and a nonzero point $Q_{12} \in \mathrm{Pic}^0(C) \cong \mathrm{Jac}(C)$, we may recover the line bundles $L_2 = L_1 \otimes (Q_{12})^{-1}$ and $L_3 = L_1 \otimes Q_{12}$. Putting together these observations, we have a simpler way to state Theorem 2.5:

Corollary 2.8. *There exists a bijection*

$$\left\{ \begin{array}{l} G\text{-equivalence classes of} \\ \text{nondegenerate Rubik's} \\ \text{cubes over } F \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{equivalence classes of } (C, L, Q) \text{ where } C \\ \text{is a genus one curve over } F,\ L \text{ is a degree} \\ 3 \text{ line bundle on } C, \text{ and } 0 \neq Q \in \mathrm{Jac}(C) \end{array} \right\}. \tag{2.8}$$

As mentioned in Section 2.1, over F there are automorphisms σ of C such that $\sigma^* L$ is any degree 3 line bundle, but if σ is a translation, it fixes any degree 0 line bundle. Otherwise, σ is a "flip" that preserves L, which sends other line bundles to their negative, and in this case, Q_{12} to $-Q_{12}$. Consequently, over F, the geometric data corresponding to G-orbits of nondegenerate Rubik's cubes is just isomorphism classes of genus one curves with a nonzero point $Q \in \mathrm{Jac}(C)/\{\pm 1\}$.

Example 2.9. As F is algebraically closed by assumption, a genus 1 curve C always has a point over F, so without loss of generality, we may take C to be isomorphic to its Jacobian $\mathrm{Jac}(C) =: E$. If $C \cong E$ is given in Weierstrass form as

$$Y^2 = X^3 + aX^2 + bX + c$$

and (x, y) is a point on the Jacobian E, then by Corollary 2.8, there is an associated Rubik's cube, up to G-equivalence. We may use the sections of the line bundles $\mathcal{O}(3 \cdot O)$ and $\mathcal{O}(2 \cdot O + (x, \pm y))$, where O is the identity point of E, to build a Rubik's cube up to G-equivalence. We write an element $\mathcal{A}$ of the G-orbit below as three 3×3 "slices" of $\mathcal{A}$, namely $\mathcal{A}((1,0,0),\cdot,\cdot), \mathcal{A}((0,1,0),\cdot,\cdot)$, and $\mathcal{A}((0,0,1),\cdot,\cdot)$:

$$\begin{pmatrix} -a-3x & 0 & -b-ax \\ 0 & -1 & y \\ -b-ax & -y & -3c-bx \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & -y \\ -1 & 0 & x \\ y & x & 0 \end{pmatrix} \quad \begin{pmatrix} -b-ax & y & -3c-bx \\ -y & x & 0 \\ -3c-bx & 0 & b^2-4ac-3cx \end{pmatrix}$$

The three associated ternary cubic forms to $\mathcal{A}$ all can be written

$$(b^2 - 4ac - 12cx - 6bx^2 - 4ax^3 - 3x^4)(X^3 + aX^2Z - Y^2Z + bXZ^2 + cZ^3)$$

which clearly has Jacobian isomorphic to E.

We next rewrite Theorem 2.5 with bases for all the vector spaces in question. With these bases, the proof shows what Rubik's cubes, not just their G-orbits, exactly parametrize. Recall that a Rubik's cube by definition is not just an element of $U_1 \otimes U_2 \otimes U_3$ but also the information of bases for U_i for $1 \leq i \leq 3$. On the other hand, let $\mathcal{D}$ be the data of (C, L_1, L_2, L_3), where C is a curve of genus one and each L_i for $1 \leq i \leq 3$ is a degree 3 line bundle on C with $L_1 \otimes L_1 \cong L_2 \otimes L_3$ and $L_1 \ncong L_2$ and $L_1 \ncong L_3$, along with bases $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$ for the spaces of sections $H^0(C, L_1), H^0(C, L_2), H^0(C, L_3)$, respectively. Then two data $\mathcal{D}$ and $\mathcal{D}'$ are *equivalent* if there exists an isomorphism $\sigma : C \to C'$ such that for $1 \leq i \leq 3$, we have both $\sigma^* L_i' \cong L_i$ and that $\sigma^* : H^0(C', L_i') \to H^0(C, L_i)$ is an isomorphism taking $\mathfrak{B}_i'$ to $\mathfrak{B}_i$. Then each set of such data, up to equivalence, corresponds to a nondegenerate Rubik's cube. The proposition below follows directly from the proof of Theorem 2.5 and the a posteriori identifications of bases for $H^0(C, L_3)$ and $(\ker \mu_{12})^\vee$ (and for $H^0(C, L_2)$ and $(\ker \mu_{13})^\vee$) discussed in Remark 2.6.

Proposition 2.10. *There exists a bijection*

$$\left\{\begin{matrix}\textit{nondegenerate}\\ \textit{Rubik's cubes}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{equivalence classes of } (C, L_1, L_2, L_3, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)\textit{: } C\\ \textit{a genus } 1 \textit{ curve; } L_1, L_2, L_3 \textit{ degree } 3 \textit{ line bundles on}\\ C \textit{ with } L_1 \otimes L_1 \cong L_2 \otimes L_3 \textit{ and } L_1 \textit{ not isomorphic to}\\ L_2 \textit{ or } L_3\textit{; and } \mathfrak{B}_i \textit{ a basis for } H^0(C, L_i) \textit{ for } 1 \leq i \leq 3\end{matrix}\right\}. \tag{2.9}$$

There are no automorphisms of the data on either side of the bijection (2.9), since we have rigidified the data completely. On the left side, there are clearly no automorphisms, and on the right, an automorphism of $(C, L_1, L_2, L_3, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)$ would be an automorphism of the curve C that fixes all the other data. An automorphism σ of the curve C that fixes all three line bundles must be a translation of the curve by a 3-torsion point of $\mathrm{Pic}^0(C) \cong \mathrm{Jac}(C)$. The image of the curve C into each $\mathbb{P}(H^0(C, L_i)^\vee)$ via ϕ_{L_i} would be fixed setwise but not pointwise under a nonzero σ, and the automorphism σ extends to a nontrivial linear transformation of the projective space $\mathbb{P}(H^0(C, L_i)^\vee)$. Thus, the bases $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$ are not fixed by σ, so there are no automorphisms on the right side of bijection (2.9).

However, both the bijections in the original Theorem 2.5 and Corollary 2.8 have nontrivial automorphisms on both sides. In order for Theorem 2.5 to generalize to an equivalence of moduli stacks, we determine what geometric data has the same automorphisms as those of a nondegenerate Rubik's cube, up to the action of G.

On the right side of bijection (2.6), the stabilizer of a quadruple (C, L_1, L_2, L_3) also includes copies of $\mathbb{G}_m$: each of the three line bundles has automorphism group isomorphic to $\mathbb{G}_m$, and we may use the relation (2.5) to reduce those to two copies of $\mathbb{G}_m$. That is, if we include the actual isomorphism $\varphi : L_1^{\otimes 2} \xrightarrow{\cong} L_2 \otimes L_3$ with the other data, there are only two copies of $\mathbb{G}_m$ (the kernel of $\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$) in the automorphism group. (Another way to reduce the number of copies of $\mathbb{G}_m$ would be to only include the data of two line bundles, say L_1 and L_2.) There are also automorphisms of C that fix all three line bundles, namely translations by 3-torsion points of $\mathrm{Pic}^0(C)$. Since the action of these 3-torsion points of the elliptic curve $\mathrm{Jac}(C) \cong \mathrm{Pic}^0(C)$ on the genus one curve C extends to a linear transformation of $\mathbb{P}(H^0(C, L_i)^\vee)$ for $1 \leq i \leq 3$, there is a natural map

$$\operatorname{Jac}(C)[3] \longrightarrow \operatorname{PGL}(H^0(C, L_1)) \times \operatorname{PGL}(H^0(C, L_2)) \times \operatorname{PGL}(H^0(C, L_3)).$$

Then the automorphism group of our data is an extension of $\operatorname{Jac}(C)[3]$ by $\mathbb{G}_m^2$, and it has a natural map to $\operatorname{GL}(H^0(C, L_1)) \times \operatorname{GL}(H^0(C, L_2)) \times \operatorname{GL}(H^0(C, L_3))$.

The stabilizer of the group G acting on a nondegenerate Rubik's cube contains a group of order 9 (see [Nur00]) as well as the product of two copies of $\mathbb{G}_m$, which is the kernel of the multiplication map

$$\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m \longrightarrow \mathbb{G}_m$$

for the factors of $\mathbb{G}_m$ in each of the three copies of GL_3. Given the equivariant action of GL_3^3 on each side of bijection (2.9), these automorphism groups for G-orbits of nondegenerate Rubik's cubes and for equivalence classes of $(C, L_1, L_2, L_3, \varphi)$ are the same groups.

2.3.3 Families

In this section, we show that the bijections described in Theorem 2.5 and Proposition 2.10 are stronger, namely that they hold in families. We thus get an isomorphism of the corresponding moduli stacks. All of the schemes in this section are defined over $\mathbb{Z}[\frac{1}{6}]$ (so the moduli stacks are also over $\mathbb{Z}[\frac{1}{6}]$). First, we describe each set of data over a scheme S.

Recall that specifying a Rubik's cube over F is the same as giving three 3-dimensional vector spaces U_1, U_2, U_3 with bases along with an element of $U_1 \otimes U_2 \otimes U_3$. As we define a Rubik's cube over a scheme, we will make a distinction between those with and those without bases. In particular, we say that a *based* $3 \times 3 \times 3$ *box* or *based Rubik's cube over a scheme* S consists of three free rank 3 $\mathcal{O}_S$-modules $\mathcal{U}_i$ with isomorphisms $\psi_i : \mathcal{U}_i \xrightarrow{\cong} \mathcal{O}_S^{\oplus 3}$ for $1 \leq i \leq 3$ and a section $\mathcal{A}$ of the rank 27 $\mathcal{O}_S$-algebra $\mathcal{U}_1 \otimes \mathcal{U}_2 \otimes \mathcal{U}_3$. An isomorphism of based Rubik's cubes $(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \psi_1, \psi_2, \psi_3, \mathcal{A})$ and $(\mathcal{U}_1', \mathcal{U}_2', \mathcal{U}_3', \psi_1', \psi_2', \psi_3', \mathcal{A}')$ consists of isomorphisms $\sigma_i : \mathcal{U}_i \xrightarrow{\cong} \mathcal{U}_i'$ with $\psi_i = \psi_i' \circ \sigma_i$ for $1 \leq i \leq 3$ and taking $\mathcal{A}$ to $\mathcal{A}'$. A based Rubik's cube is *nondegenerate* if it is locally nondegenerate.

In contrast, we define a $3 \times 3 \times 3$ *box* or *Rubik's cube over* S as three locally free rank 3 $\mathcal{O}_S$-modules $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ and a section $\mathcal{A}$ of $\mathcal{U}_1 \otimes \mathcal{U}_2 \otimes \mathcal{U}_3$. Two such Rubik's cubes $(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{A})$ and $(\mathcal{U}_1', \mathcal{U}_2', \mathcal{U}_3', \mathcal{A}')$ are isomorphic if there are isomorphisms $\sigma_i : \mathcal{U}_i \xrightarrow{\cong} \mathcal{U}_i'$ taking $\mathcal{A}$ to $\mathcal{A}'$.

Next, we define the geometric data over S. A *genus one curve C over S* is a proper smooth morphism $\pi : C \to S$ with relative dimension 1 such that $\mathrm{R}^0\pi_*(\mathcal{O}_C) = \mathcal{O}_S$ and $\mathrm{R}^1\pi_*(\mathcal{O}_C)$ is a line bundle over S, i.e., the fibers are connected and have arithmetic genus one. We define a *rigidified tuple* over S to be a genus one curve $\pi : C \to S$ and three degree 3 line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ on C with isomorphisms $\chi_i : \mathrm{R}^0\pi_*(\mathcal{L}_i) \xrightarrow{\cong} \mathcal{O}_S^{\oplus 3}$ for $1 \leq i \leq 3$. Note that this definition forces the sections of $\mathcal{L}_i$ to be free rank 3 vector bundles over S.

A *balanced* rigidified tuple also includes an isomorphism $\varphi : \mathcal{L}_1^{\otimes 2} \xrightarrow{\cong} \mathcal{L}_2 \otimes \mathcal{L}_3 \otimes \pi^* L_S$ for some line bundle L_S on S. Such a quadruple is *nondegenerate* if $\mathrm{R}^0\pi_*(\mathcal{L}_1^\vee \otimes \mathcal{L}_i) = 0$ for $i = 2$ or 3; because the bundle $\mathcal{L}_1^\vee \otimes \mathcal{L}_i$ has degree 0, this condition is equivalent to requiring that fiberwise the line bundles are not isomorphic. An isomorphism of balanced rigidified tuples is the usual notion, requiring an isomorphism of the curves that commutes with the line bundles and isomorphisms.

Theorem 2.11. *Over a scheme S, there is an equivalence between the category of nondegenerate based Rubik's cubes over S and the category of nondegenerate balanced rigidified tuples $(C, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \varphi)$ over S as defined above.*

Proof. This relative version of Proposition 2.10 is essentially a direct consequence of that proposition. The functors in both directions are as before. That is, given a nondegenerate based Rubik's cube $(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \psi_1, \psi_2, \psi_3, \mathcal{A})$ over S, we define the curves $C_i \hookrightarrow \mathbb{P}(\mathcal{U}_i^\vee)$ by the vanishing of the corresponding ternary cubic form[3] over S, which is a section of $\mathrm{Sym}^3 \mathcal{U}_i \otimes \wedge^3 \mathcal{U}_j \otimes \wedge^3 \mathcal{U}_k$, for $\{i, j, k\} = \{1, 2, 3\}$. By the nondegeneracy assumption, locally on S the curves C_i are genus one curves, so each C_i is a genus one curve. In addition, there are isomorphisms $\tau_i^j : C_i \to C_j$, defined in the same way as before, either by taking kernels of $\mathcal{A}$ evaluated on sections of C_i or by defining a curve $C_{ij} \in \mathbb{P}(\mathcal{U}_i^\vee) \times \mathbb{P}(\mathcal{U}_j^\vee)$ with each projection onto C_i and C_j. Finally, there exist line bundles on C_i from pulling back $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{U}_i^\vee)$; pulling back the corresponding bundles via the identity map, τ_1^2, and τ_1^3

[3]See Appendix 2.A for more about ternary cubic forms over a base scheme S.

to C_1 gives the bundles we call $\mathcal{L}_1, \mathcal{L}_2$, and $\mathcal{L}_3$, respectively. For $1 \leq i \leq 3$, the space of sections $\mathrm{R}^0\pi_*(\mathcal{L}_i)$ is naturally isomorphic to $\mathcal{U}_i$, so composing those with the isomorphisms ψ_i give isomorphisms $\mathrm{R}^0\pi_*(\mathcal{L}_i) \xrightarrow{\cong} \mathcal{O}_S^{\oplus 3}$, where $\pi : C_1 \to S$. Also, the same argument as Lemma 2.3 shows that $\mathcal{L}_1^{\otimes 2}$ and $\mathcal{L}_2 \otimes \mathcal{L}_3$ are the same element in the group $\mathrm{Pic}(C/S)$, which induces an isomorphism $\varphi : \mathcal{L}_1^{\otimes 2} \to \mathcal{L}_2 \otimes \mathcal{L}_3 \otimes \pi^* L_S$ for some line bundle L_S on S. We have produced a nondegenerate balanced rigidified tuple.

In the other direction, let $\pi : C \to S$ be a genus 1 curve over S and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ degree 3 line bundles on C with isomorphisms $\chi_i : \mathrm{R}^0\pi_*(\mathcal{L}_i) \xrightarrow{\cong} \mathcal{O}_S^{\oplus 3}$ for $1 \leq i \leq 3$ and an isomorphism $\varphi : \mathcal{L}_1^{\otimes 2} \to \mathcal{L}_2 \otimes \mathcal{L}_3$. The kernel of the surjective map

$$\mu_{12} : \mathrm{R}^0\pi_*(\mathcal{L}_1) \otimes \mathrm{R}^0\pi_*(\mathcal{L}_2) \longrightarrow \mathrm{R}^0\pi_*(\mathcal{L}_1 \otimes \mathcal{L}_2)$$

is a free rank 3 $\mathcal{O}_S$-module. We find an *unbased* Rubik's cube as a section of $\mathrm{R}^0\pi_*(\mathcal{L}_1) \otimes \mathrm{R}^0\pi_*(\mathcal{L}_2) \otimes (\ker \mu_{12})^\vee$; we only need a trivialization for $(\ker \mu_{12})^\vee$ to produce a based Rubik's cube. Just as Remark 2.6 is used to show Proposition 2.10, repeating the construction for $\mathcal{L}_3$ in place of $\mathcal{L}_2$ and using the isomorphism φ shows that the trivialization χ_3 induces a trivialization for $(\ker \mu_{12})^\vee$. Therefore, we have a nondegenerate based Rubik's cube.

These two constructions are locally inverse to one another, as a result of Proposition 2.10, so they are inverse, and we are done. □

In fact, the space of based Rubik's cubes over S is simply the scheme $\mathbb{A}^{27}$ over S, since there are no automorphisms of a based Rubik's cube. We have thus shown that the moduli space of nondegenerate balanced rigidified tuples over S is isomorphic to an open subscheme of $\mathbb{A}^{27}$ over S; in other words, the stack of nondegenerate balanced rigidified quadruples is equivalent to an open substack of $\mathbb{A}^{27}$.

There is a natural action of the group $G = \mathrm{GL}_3^3$ on the space of based Rubik's cubes, essentially by removing the choice of trivializations $\psi_i : \mathcal{U}_i \xrightarrow{\cong} \mathcal{O}_S^{\oplus 3}$, and this action preserves nondegeneracy. On the other hand, the group G also acts on balanced rigidified tuples by acting on the trivializations $\chi_i : \mathrm{R}^0\pi_*(\mathcal{L}_i) \xrightarrow{\cong} \mathcal{O}_S^{\oplus 3}$. By quotienting both sides of the bijection in Theorem 2.11 by G, we find an equivalence of the corresponding quotient stacks, since the functors of Theorem 2.11 are G-equivariant.

Quotienting the stack of based Rubik's cubes by G is simply the quotient stack $[\mathbb{A}^{27}/G]$. An S-point of this stack corresponds to what we have called an (unbased) Rubik's cube, i.e., three rank 3 vector bundles $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ over S and a section $\mathcal{A}$ of $\mathcal{U}_1 \otimes \mathcal{U}_2 \otimes \mathcal{U}_3$. Because nondegeneracy is defined locally and preserved under the action of G, there is an open substack of $[\mathbb{A}^{27}/G]$ corresponding to the nondegenerate Rubik's cubes.

On the other hand, quotienting the space of nondegenerate balanced rigidified tuples by the action of the group G produces the stack $\mathcal{Y}_{333}$ whose S-points consist of tuples $(C, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \varphi)$, where $\pi : C \to S$ is a genus one curve over S, and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are degree 3 line bundles over S, with an isomorphism $\varphi : \mathcal{L}_1^{\otimes 2} \xrightarrow{\cong} \mathcal{L}_2 \otimes \mathcal{L}_3 \otimes \pi^* L_S$ for some line bundle L_S on S, and with the condition that $\mathrm{R}^0\pi_*(\mathcal{L}_1^\vee \otimes \mathcal{L}_i) = 0$ for $i = 2$ or 3. Note that $\mathcal{Y}_{333}$ is a substack of the fiber product $\mathrm{Pic}_1^3 \times_{\mathscr{M}_1} \mathrm{Pic}_1^3 \times_{\mathscr{M}_1} \mathrm{Pic}_1^3$ of three copies of the degree 3 universal Picard stack Pic_1^3 over the moduli space $\mathscr{M}_1$ of genus 1 curves.

Theorem 2.12. *The nondegenerate open substack of* $[\mathbb{A}^{27}/\mathrm{GL}_3 \times \mathrm{GL}_3 \times \mathrm{GL}_3]$ *is equivalent to the stack* $\mathcal{Y}_{333}$ *of nondegenerate balanced tuples as defined above.*

In Appendix 2.A.3, we provide another interpretation of the moduli stack of nondegenerate balanced tuples $(C, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \varphi)$, by relating pairs $(C, \mathcal{L}_i)$ to elements of the fppf cohomology group $H^1_{\mathrm{f}}(S, \mathrm{Jac}(C)[3])$.

2.4 Symmetrized Rubik's Cubes

We may use the bijections for the space of Rubik's cubes to analyze other related spaces, what we call *symmetrized Rubik's cubes.* We will produce similar bijections for these spaces, with bases, as well as when quotiented by an appropriate group action.

For a vector space U over F, we use the notation $\mathrm{Sym}_n U$ to denote the subspace of $U^{\otimes n}$ given by symmetric tensors, which is isomorphic to $(\mathrm{Sym}^n(U^\vee))^\vee$. We distinguish the space $\mathrm{Sym}_n U$ from that of $\mathrm{Sym}^n U$, which is naturally a quotient of $U^{\otimes n}$, although over any field not of characteristic dividing n, the two are isomorphic. For example, for $n = 2$ and a 3-dimensional vector space U with dual basis x, y, z, the space $\mathrm{Sym}^2 U$ may be thought of as ternary quadratic forms

$$ax^2 + by^2 + cz^2 + uyz + vxz + wxy,$$

where $a, b, c, u, v, w \in F$. On the other hand, the space $\mathrm{Sym}_2 U \subset U \otimes U$ consists of ternary quadratic forms of the form

$$ax^2 + by^2 + cz^2 + 2uyz + 2vxz + 2wxy,$$

which may also be represented by a symmetric 3×3 matrix

$$\begin{pmatrix} a & w & v \\ w & b & u \\ v & u & c \end{pmatrix}.$$

Over a ring, say $\mathbb{Z}$, these are clearly different spaces, since the latter only has ternary quadratic forms with even cross terms.

A *doubly symmetrized Rubik's cube* is an element of the vector space $U_1 \otimes \mathrm{Sym}_2 U_2$, for 3-dimensional F-vector spaces U_1 and U_2 with a choice of bases. An element of this space may be thought of as three symmetric matrices, for example, and there is an inclusion of doubly symmetrized Rubik's cubes into the space of Rubik's cubes with two of the vector spaces identified:

$$U_1 \otimes \mathrm{Sym}_2 U_2 \hookrightarrow U_1 \otimes U_2 \otimes U_2. \tag{2.10}$$

There is a natural action of $\mathrm{GL}(U_1) \times \mathrm{GL}(U_2)$ on the space of doubly symmetrized Rubik's cubes.

Likewise, we define a *triply symmetrized Rubik's cube* as an element of $\mathrm{Sym}_3 U$ for a 3-dimensional F-vector space U with a choice of basis. To such a triply symmetric $3 \times 3 \times 3$ box, one may also associate a ternary cubic of the form

$$ax^3 + by^3 + cz^3 + 3a_2x^2y + 3a_3x^2z + 3b_1xy^2 + 3b_3y^2z + 3c_1xz^2 + 3c_2yz^2 + 6mxyz.$$

Again, triply symmetrized Rubik's cubes are also Rubik's cubes by the injection

$$\mathrm{Sym}_3 U \hookrightarrow U \otimes U \otimes U,$$

and the group $\mathrm{GL}(U)$ acts on triply symmetrized Rubik's cubes.

Both of the spaces of symmetrized Rubik's cubes lend themselves to moduli interpretations. As they both are subspaces of the space of Rubik's cubes, we may use the same geometric constructions for their intersection with the nondegenerate locus of the space of Rubik's cubes. These symmetrized spaces will correspond to subsets of the space of tuples $(C, L_1, L_2, L_3, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)$ where C is a genus 1 curve; L_1, L_2, L_3 are degree 3 line bundles on C with $L_1 \otimes L_1 \cong L_2 \otimes L_3$, $L_1 \not\cong L_2$, and $L_1 \not\cong L_3$; and $\mathfrak{B}_i$ is a basis for $H^0(C, L_i)$ for $1 \leq i \leq 3$. In particular, the symmetry implies that some of the line bundles will be isomorphic.

Proposition 2.13. *The restriction of bijection* (2.9) *to doubly symmetrized Rubik's cubes gives the bijection*

$$\left\{\begin{array}{l} \textit{nondegenerate} \\ \textit{doubly symmetrized} \\ \textit{Rubik's cubes} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \textit{equivalence classes of tuples } (C, L_1, L_2, \mathfrak{B}_1, \mathfrak{B}_2) \\ \textit{where } C \textit{ is an irreducible genus } 1 \textit{ curve, } L_1 \textit{ and} \\ L_2 \textit{ are nonisomorphic degree } 3 \textit{ line bundles on} \\ C \textit{ such that } L_1^{\otimes 2} \cong L_2^{\otimes 2}, \textit{ and } \mathfrak{B}_i \textit{ are bases for} \\ H^0(C, L_i) \textit{ for } i = 1, 2 \end{array}\right\}.$$

Proof. If we think of a nondegenerate doubly symmetrized Rubik's cube $\mathcal{A} \in U_1 \otimes \mathrm{Sym}_2 U_2$ as a Rubik's cube in $U_1 \otimes U_2 \otimes U_2$ by the injection (2.10), then it is clear that the curves C_2 and C_3 associated to $\mathcal{A}$ are identical curves in $\mathbb{P}(U_2^\vee)$. Not only are the curves given by the same ternary cubic forms, but the isomorphisms $\tau_1^2 : C_1 \to C_2$ and $\tau_1^3 : C_1 \to C_3$ are also the same maps. Thus, the line bundles that we call L_2 and L_3 (defined as pullbacks of $\mathcal{O}(1)$ from $\mathbb{P}(U_2^\vee)$ to C_1) are canonically isomorphic and their spaces of sections are, of course, canonically isomorphic. Thus, $\mathcal{A}$ gives rise to the quadruple (C, L_1, L_2, L_2) with the relation $L_1 \otimes L_1 \cong L_2 \otimes L_2$; in other words, the bundles L_1 and L_2 differ by a 2-torsion point of $\mathrm{Pic}^0(C)$.

In the reverse direction, we know how to build a Rubik's cube from the "symmetrized" data $(C, L_1, L_2, L_3, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)$ where L_2 and L_3 are the same line bundle, and the bases $\mathfrak{B}_2$ and $\mathfrak{B}_3$ are canonically identified. If $\mathfrak{B}_1 = \{b_1, b_2, b_3\}$, then the construction using L_1 and L_2 gives a Rubik's cube $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, where for $1 \leq i \leq 3$ we define $\mathcal{A}_i$ to be the 3×3 matrix $\mathcal{A}(b_i^\vee, \cdot, \cdot)$ in $H^0(C, L_2) \otimes (\ker \mu_{12})^\vee$. This latter vector space may be, as explained in Remark 2.6, identified with $H^0(C, L_2) \otimes H^0(C, L_3)$. The analogous construction using L_1 and L_3 gives the Rubik's cube $\mathcal{A}' = (\mathcal{A}'_1, \mathcal{A}'_2, \mathcal{A}'_3)$, with $\mathcal{A}'_i \in (\ker \mu_{13})^\vee \otimes H^0(C, L_3) \cong H^0(C, L_2) \otimes H^0(C, L_3)$. But since L_2 and L_3 are the same, with the identifications between $H^0(C, L_j)$ and $(\ker \mu_{1j})^\vee$ the same for $j = 2$ or 3, we have $\mathcal{A}_i = (\mathcal{A}'_i)^t$ for $1 \leq i \leq 3$. Therefore, each $\mathcal{A}_i$ is in fact a symmetric matrix, and the Rubik's cube constructed lies in the space $H^0(C, L_1) \otimes \mathrm{Sym}_2(H^0(C, L_2))$. □

If we take $\mathrm{GL}_3 \times \mathrm{GL}_3$-equivalence classes of each side, we obtain a bijection between nondegenerate elements of $U_1 \otimes \mathrm{Sym}_2 U_2$ (without a choice of bases for U_1 or U_2) and the geometric data (C, L_1, L_2). In the same way as in Corollary 2.8, because of the relation $L_1^{\otimes 2} \cong L_2^{\otimes 2}$, this geometric data is equivalent to just the curve C and the 2-torsion point $L_1 \otimes L_2^{-1}$ on $\mathrm{Pic}^0(C) \cong \mathrm{Jac}(C)$.

Corollary 2.14. *The restriction of the bijection in Corollary 2.8 to doubly symmetrized Rubik's cubes produces the bijection*

$$\left\{\begin{array}{c}\mathrm{GL}_3 \times \mathrm{GL}_3\text{-}\textit{equivalence classes} \\ \textit{of nondegenerate doubly sym-} \\ \textit{metrized Rubik's cubes over } F\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{equivalence classes of } (C, L, Q) \textit{ where } C \\ \textit{is a genus } 1 \textit{ curve over } F,\ L \textit{ is a degree } 3 \\ \textit{line bundle on } C, \textit{ and } 0 \neq Q \in \mathrm{Jac}(C)[2]\end{array}\right\}.$$

The same techniques applied to triply symmetrized Rubik's cubes, as a subset of the space of Rubik's cubes, give similar bijections. Although it is tempting to guess that all three line bundles L_1, L_2, L_3 coming from a triply symmetrized Rubik's cube are the same, Lemma 2.2 shows that for any nondegenerate Rubik's cube, the line bundle L_1 cannot be the same as the other two.

Let $\mathcal{A} \in \mathrm{Sym}_3 U \subset U \otimes U \otimes U$ be a nondegenerate Rubik's cube with a basis $\mathfrak{B}$ of U, and recall the geometric construction of Section 2.2.1. The three ternary cubics $f_i \in \mathrm{Sym}^3 U$ are the same because of the symmetry of $\mathcal{A}$, so the three cubic curves $C_i \in \mathbb{P}(U^\vee)$ are identical. However, none of the maps τ_i^j between the curves C_i and C_j is the identity map.

Those maps may be computed by the difference $L_1 \otimes L_2^{-1}$ in the line bundles on C_1. Just as for a doubly symmetrized Rubik's cube, the line bundles L_2 and L_3 are clearly isomorphic, since the maps $\tau_1^2 : C_1 \to C_2$ and $\tau_1^3 : C_1 \to C_3$ are the same. Therefore, $L_1 \otimes L_2^{-1}$ is again a nonzero 2-torsion point Q of $\mathrm{Pic}^0(C)$, and the map $\tau_1^2 : C_1 \to C_2$ is an automorphism of the curve $C_1 \subset \mathbb{P}(U^\vee)$ given by translation by Q. Similarly, each τ_i^j is the same translation of the curve $C_i \subset \mathbb{P}(U^\vee)$, and the composition $\tau_i^j \circ \tau_k^i$ is the identity map $C_k \to C_j$ as varieties in $\mathbb{P}(U^\vee)$. The triple composition $\alpha_{132} : \tau_2^1 \circ \tau_3^2 \circ \tau_1^3$ is also just translation by Q.

Since U may be identified with both $H^0(C_1, L_1)$ and $H^0(C_1, L_2)$, the basis $\mathfrak{B}$ of U gives a basis for $H^0(C_1, L_1)$ and $H^0(C_1, L_2)$. Since the embeddings of the curve by ϕ_{L_1} and by ϕ_{L_2} span the projective plane, we may recover a basis for $H^0(C_1, L_2)$ from a basis for $H^0(C_1, L_1)$ by requiring that the ternary cubic forms are the same, via bijection (2.3).

Therefore, from a nondegenerate triply symmetrized Rubik's cube, we have obtained a genus 1 curve C, a degree 3 line bundle L_1 on C, a nonzero 2-torsion point of $\mathrm{Jac}(C)$, and a basis for $H^0(C, L_1)$. This is exactly enough information to recover the cube itself.

Proposition 2.15. *The restriction of bijection* (2.9) *to triply symmetrized Rubik's cubes gives the bijection*

$$\left\{ \begin{array}{l} \textit{nondegenerate} \\ \textit{triply} \\ \textit{symmetrized} \\ \textit{Rubik's cubes} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{equivalence classes of quadruples } (C, L, Q, \mathfrak{B}) \textit{ where} \\ C \textit{ is a genus } 1 \textit{ curve, } L \textit{ is a degree } 3 \textit{ line bundle on} \\ C,\ Q \textit{ is a nonzero } 2\textit{-torsion point of } \mathrm{Jac}(C), \textit{ and } \mathfrak{B} \\ \textit{is a basis of } H^0(C, L) \end{array} \right\}. \qquad (2.11)$$

Proof. We have already shown that a nondegenerate triply symmetrized Rubik's cube gives the data $(C, L, Q, \mathfrak{B})$ as in the bijection above. On the other hand, given $(C, L, Q, \mathfrak{B})$, we recover a tuple $(C, L_1, L_2, L_3, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)$ where $L_1 := L, L_2 := L \otimes Q, L_3 := L \otimes Q$, and $\mathfrak{B}_1 = \mathfrak{B}$. The basis $\mathfrak{B}_2 = \mathfrak{B}_3$ of $H^0(C, L_2) \equiv H^0(C, L_3)$ is recovered from the basis for $H^0(C, L_1)$ by requiring that the corresponding ternary cubic forms (via bijection (2.3)) are the same.

Let $U = H^0(C, L_1)$, which is also naturally identified with the other spaces of sections. Applying the argument in the proof of Proposition 2.13 shows that the Rubik's cube $\mathcal{A}$ constructed from this data lies in $U \otimes \mathrm{Sym}_2 U \subset U \otimes U \otimes U$, i.e., it is invariant under the transposition (23) of the symmetric group $\mathbb{S}_3$ acting on the three copies of U. A similar argument shows invariance under the transpositions (12) and (13); in other words, $\mathcal{A}$ lies in the images of all three natural inclusions $U \otimes \mathrm{Sym}_2 U \hookrightarrow U \otimes U \otimes U$. Because $\mathbb{S}_3$ is generated by transpositions, the cube $\mathcal{A}$ is invariant under all of $\mathbb{S}_3$ and thus lies in $\mathrm{Sym}_3 U$. ☐

There is a natural action of GL_3 on each side of bijection (2.11). Quotienting each side by this action gives a parametrization of the geometric data by the GL_3-orbits of triply symmetrized Rubik's cubes. Recall that the degree 3 line bundle in the geometric data may be "forgotten" over an algebraically closed field.

Corollary 2.16. *The restriction of the bijection in Corollary 2.8 to triply symmetrized Rubik's cubes produces the bijection*

$$\left\{\begin{array}{l}\mathrm{GL}_3\textit{-equivalence classes of non-}\\ \textit{degenerate triply symmetrized}\\ \textit{Rubik's cubes over } F\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{equivalence classes of } (C, L, Q) \textit{ where } C\\ \textit{is a genus } 1 \textit{ curve over } F,\ L \textit{ is a degree } 3\\ \textit{line bundle on } C, \textit{ and } 0 \neq Q \in \mathrm{Jac}(C)[2]\end{array}\right\}.$$

Thus, doubly and triply symmetrized Rubik's cubes, without bases for the corresponding vector spaces, parametrize exactly the same geometric data: a genus one curve C, a degree 3 line bundle on C, and a nonzero 2-torsion point on $\mathrm{Jac}(C)$.

Remark 2.17. For a 3-dimensional vector space U over the field F of characteristic not 3, the spaces $\mathrm{Sym}^3 U$ and $\mathrm{Sym}_3 U$ are isomorphic. The GL_3-orbits of the former, by bijection (2.3), parametrizes genus one curves and a degree 3 line bundle, and the orbits of the latter parametrize genus one curves with a degree 3 line bundle and a nonzero 2-torsion point on the Jacobian. In other words, the orbits of $\mathrm{Sym}_3 U$ are naturally a degree 3 cover of the orbits of $\mathrm{Sym}^3 U$. Note that even if a ternary cubic form can be interpreted as an element of both $\mathrm{Sym}^3 U$ and $\mathrm{Sym}_3 U$, it gives rise to different genus one curves via the two moduli interpretations. In particular, a ternary cubic form f of $\mathrm{Sym}_3 U$ is associated to the genus one curve given by the Hessian of f.

2.A Appendix: Torsors for Elliptic Curves and Line Bundles

In this appendix, we describe geometric data, such as genus one curves with line bundles, in terms of torsors for elliptic curves. This viewpoint comes from ideas of [O'N02, CFO$^+$08] in the case of a base field k, and it has applications to descent on elliptic curves. We obtain a generalization of bijection (2.3) for ternary cubic forms over a base scheme, as well as a better understanding of the moduli spaces for Rubik's cubes.

Preliminaries. Let k be a field not of characteristic 2 or 3, and let $n \geq 2$ be an integer invertible in k. We will specialize to the case of $n = 3$ for the applications of the theory. Let S be a $\mathbb{Z}[\frac{1}{6n}]$-scheme, and denote the Brauer group of S by $\mathrm{Br}(S)$.

As before, a *genus one curve C over S* is a proper, smooth morphism $\pi : C \to S$ with relative dimension 1 such that $\mathrm{R}^0\pi_*(\mathcal{O}_C) = \mathcal{O}_S$ and $\mathrm{R}^1\pi_*(\mathcal{O}_C)$ is a line bundle over S. An *elliptic curve over S* is such a genus one curve over S equipped with a base point, i.e., a section $S \to C$.

Recall that we use the convention that $\mathbb{P}(V)$ denotes the rank one subbundles, not the quotients, of a vector bundle V over S.[4]

2.A.1 Torsors and Obstruction Maps

Let E be an elliptic curve over k, and let $E[n]$ denote the n-torsion of E. Then as n is invertible in k, the Kummer sequence

$$0 \longrightarrow E[n] \longrightarrow E \stackrel{n}{\longrightarrow} E \longrightarrow 0$$

given by multiplication by n induces the sequence of Galois cohomology

$$0 \longrightarrow E(k)/nE(k) \longrightarrow H^1(k, E[n]) \stackrel{\alpha}{\longrightarrow} H^1(k, E) \tag{2.12}$$

Elements of the group $H^1(k, E)$, also known as the Weil-Châtelet group $\mathrm{WC}(E/k)$, may be thought of as isomorphism classes of torsors for the elliptic curve E, namely genus one

[4]Warning: this convention is the opposite of the one used in Chapter 6.

curves C over k with a specified isomorphism between E and the connected component $\mathrm{Aut}^0(C)$ of the automorphism group scheme of C. Two such torsors are isomorphic if there exists an isomorphism between the curves that respects the action of E. This identification of $H^1(k, E)$ with E-torsors C is an example of the phenomenon that $H^1(k, G)$ for any group G parametrizes G-torsors.

On the other hand, the group $H^1(k, G)$ may be identified with $\mathrm{Gal}(\overline{k}/k)$-sets whose automorphism group is isomorphic to G. By this principle, elements of the group $H^1(k, E[n])$ are in correspondence with twists of objects with automorphism group $E[n]$. As explained in [CFO$^+$08], there are many interpretations for these twists. For example, the pair $(C, [D])$, where C is a torsor for E and $[D]$ is a k-rational[5] divisor class on C of degree n, is a twist of $(E, [n \cdot O])$ where O is the identity point of E. The group $H^1(k, E[n])$ parametrizes isomorphism classes of such pairs $(C, [D])$, where two such pairs $(C, [D])$ and $(C', [D'])$ are isomorphic if there is an isomorphism $\sigma : C \to C'$ such that $\sigma^* D'$ is linearly equivalent to D. Under this interpretation, the map $\alpha : H^1(k, E[n]) \to H^1(k, E)$ from (2.12) simply sends the pair $(C, [D])$ to the curve C.

Pairs $(C, [D])$ are equivalent to so-called Brauer-Severi diagrams $[C \to \mathbb{P}]$, where $\mathbb{P}$ is a dimension $n - 1$ Brauer-Severi variety. Given a pair $(C, [D])$, the k-rationality of the divisor class $[D]$ gives rise to a k-rational structure on the embedding of $C_{\overline{k}} := C \otimes_k \overline{k}$ into $\mathbb{P}(H^0(C_{\overline{k}}, \mathcal{O}(D_{\overline{k}}))^\vee)$. The resulting closed immersion $[C \to \mathbb{P}]$ is the Brauer-Severi diagram representing $(C, [D])$ in $H^1(k, E[n])$. These Brauer-Severi diagrams are twists of the diagram $[E \to \mathbb{P}^{n-1}]$, and they are (up to isomorphism) another way to represent elements of $H^1(k, E[n])$.

There is an *obstruction map* (defined in [O'N02])

$$\mathrm{Ob} : H^1(k, E[n]) \longrightarrow \mathrm{Br}(k)$$

that sends a Brauer-Severi diagram $[C \to \mathbb{P}]$ to the Brauer class of $\mathbb{P}$. The obstruction map is not a group homomorphism in general; although the kernel is not a group, it contains the identity of $H^1(k, E[n])$ and is closed under inverses. For our purposes, the key point is that

[5]The divisor class $[D]$ being k-rational means that D is linearly equivalent to all of its Galois conjugates; the divisor D itself may not be k-rational.

the kernel of the obstruction map consists of pairs $(C, [D])$ for which there actually exists a k-rational divisor D representing the class $[D]$ (equivalently, such that $|D|$ is isomorphic to $\mathbb{P}_k^{n-1}$).

The obstruction map may also be given in terms of natural cohomological maps coming from the elliptic curve E, using a related group $\Theta_{E,n}$. If $n = 3$ and k is algebraically closed, we have previously seen that the automorphism group of a genus one curve with a degree 3 divisor class is exactly the 3-torsion of $E \cong \mathrm{Jac}(C)$, since the action of $E[3]$ on C extends to $\mathbb{P}^2$. In general, the action of $E[n]$ on E by translation extends to a linear automorphism of $\mathbb{P}(H^0(E, n \cdot O)^\vee) = \mathbb{P}^{n-1}$, so there exists a map

$$E[n] \longrightarrow \mathrm{PGL}_n,$$

in other words, a projective representation of $E[n]$. The inverse image $\Theta_{E,n}$ of $E[n]$ in $\mathrm{GL}_n \to \mathrm{PGL}_n$ is a central extension of $E[n]$ by $\mathbb{G}_m$:

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \Theta_{E,n} \longrightarrow E[n] \longrightarrow 0 \tag{2.13}$$

with commutator given by the Weil pairing [Mum08]. As proved in [CFO$^+$08], the obstruction map is just the coboundary map

$$\mathrm{Ob} : H^1(k, E[n]) \longrightarrow H^2(k, \mathbb{G}_m)$$

from taking non-abelian cohomology of the exact sequence (2.13). Thus, elements of $\ker(\mathrm{Ob})$ may be identified with $H^1(C, \Theta_{E,n})$ by the exact sequence of pointed sets

$$0 = H^1(k, \mathbb{G}_m) \longrightarrow H^1(k, \Theta_{E,n}) \xrightarrow{\gamma} H^1(k, E[n]) \xrightarrow{\mathrm{Ob}} H^2(k, \mathbb{G}_m).$$

The elements of $H^1(C, \Theta_{E,n})$ may be viewed as isomorphism classes of torsors for $\Theta_{E,n}$, namely pairs (C, L) where C is an E-torsor and L is a degree n line bundle on C. Two pairs (C, L) and (C', L') are isomorphic if there is an isomorphism $\sigma : C \xrightarrow{\cong} C'$ such that $\sigma^* L' \cong L$. The action of $E[n]$ on C fixes the degree n line bundle L, and the line bundle

L itself has automorphism group $\mathbb{G}_m$. This viewpoint agrees with the interpretation of $\ker(\mathrm{Ob})$ as pairs $(C, [D])$ where D is a k-rational degree n divisor on the E-torsor C, since the divisor D (up to equivalence) gives rise to a line bundle $\mathcal{O}(D)$.

This theory of $E[n]$-torsors and obstruction maps may be extended from fields k to schemes S (even without the assumption that n is invertible on S). Instead of Galois cohomology groups, we work with the fppf cohomology groups $H^1_{\mathrm{f}}(S, E)$ and $H^1_{\mathrm{f}}(S, E[n])$ for an elliptic curve E over S. That is, the group $H^1_{\mathrm{f}}(S, E)$ still parametrizes fppf E-torsors, which are genus one curves C over S with a specified isomorphism $E \xrightarrow{\cong} \mathrm{Aut}^0(C)$. Likewise, the group $H^1_{\mathrm{f}}(S, E[n])$ parametrizes isomorphism classes of Brauer-Severi diagrams $[C \to \mathbb{P}]$, where C is a fppf E-torsor and $\mathbb{P}$ is a Brauer-Severi scheme over S. There is the usual forgetful map

$$\alpha : H^1_{\mathrm{f}}(S, E[n]) \longrightarrow H^1_{\mathrm{f}}(S, E)$$

induced by the Kummer sequence. Just as before, we may define an S-group scheme $\Theta_{E,n}$ as the automorphism group of the pair $(E, \mathcal{O}(n \cdot O))$, where O is the zero section $S \to E$. In other words, for an S-scheme T, the group of T-points of $\Theta_{E,n}$ is $\mathrm{Aut}_T(E_T, \mathcal{O}_{E_T}(n \cdot O))$. Then $\Theta_{E,n}$ is a central extension of $E[n]$ by $\mathbb{G}_m$ as in (2.13). The long exact sequence of (fppf) cohomology of the sequence (2.13) gives

$$H^1_{\mathrm{f}}(S, \mathbb{G}_m) \longrightarrow H^1_{\mathrm{f}}(S, \Theta_{E,n}) \xrightarrow{\gamma} H^1_{\mathrm{f}}(S, E[n]) \xrightarrow{\mathrm{Ob}} H^2_{\mathrm{f}}(S, \mathbb{G}_m)$$

which defines the obstruction map Ob in this case. Now the kernel of Ob is identified with a possibly nontrivial quotient of $H^1_{\mathrm{f}}(S, \Theta_{E,n})$, since $H^1_{\mathrm{f}}(S, \mathbb{G}_m) \cong \mathrm{Pic}(S)$ may be nontrivial, and $\ker(\mathrm{Ob})$ corresponds to isomorphism classes of Brauer-Severi diagrams $[C \to \mathbb{P}]$ where $\mathbb{P}$ is the projectivization of a rank n vector bundle on S, i.e., a split Brauer-Severi scheme. The elements of $H^1_{\mathrm{f}}(S, \Theta_{E,n})$ correspond to isomorphism classes of pairs (C, L) where C is a fppf E-torsor and L is a degree n line bundle on C.

Using these descriptions of the cohomology groups and the fact that they naturally correspond to S-points of certain stacks, we may reinterpret our previous orbit space parametrizations.

2.A.2 Orbits of Ternary Cubic Forms Redux

A *ternary cubic form* $(\mathcal{W}, L_S, f)$ *over a scheme* S is a rank 3 vector bundle $\mathcal{W}$ over S, a line bundle L_S on S, and a section f of $\operatorname{Sym}^3 \mathcal{W} \otimes (\wedge^3 \mathcal{W})^\vee \otimes L_S$. This ternary cubic form takes values in the line bundle $(\wedge^3 \mathcal{W})^\vee \otimes L_S$ over S, since f "evaluated" on sections of $\mathcal{W}^\vee$ produces a section of $(\wedge^3 \mathcal{W})^\vee \otimes L_S$. In this section, we show that the bijection (2.3), given over the algebraically closed field F, generalizes to ternary cubic forms over S.

Remark 2.18. Throughout this section, if there exist no nontrivial vector bundles on S, the theorems and constructions simplify considerably; most of the complications arise from adding the action of $\operatorname{Pic}(S)$. These simplifications will occur, for example, when $S = \operatorname{Spec} R$ for R a field or a local ring. In these cases, a ternary cubic form $(\mathcal{W}, L_S, f)$ over S is the usual notion of a ternary cubic form since $\mathcal{W} \cong \mathcal{O}_S^{\oplus 3}$ and $L_S \cong \mathcal{O}_S$. That is, we may represent the form as the polynomial (2.1) with coefficients in R.

From a ternary cubic form $(\mathcal{W}, L_S, f)$, let C be the zero set of the section f in $\mathbb{P}(\mathcal{W}^\vee)$. We call the ternary cubic *nondegenerate* if C is smooth. Locally, nondegeneracy corresponds to the nonvanishing of the discriminant Δ of the form f, and the ternary cubic f is nondegenerate if it is locally so.

Given a nondegenerate ternary cubic form $(\mathcal{W}, L_S, f)$, the variety C in $\mathbb{P}(\mathcal{W}^\vee)$ has codimension 1 and is a curve over S. Each fiber is a smooth proper curve of genus one, so by cohomology and base change, the curve $\pi : C \to S$ is a genus one curve over S. Let L be the pullback of the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(1)$ via $\iota : C \to \mathbb{P}(\mathcal{W}^\vee)$. Then L is a degree 3 line bundle on C. Two pairs (C, L) and (C', L') are isomorphic if there is an isomorphism $\sigma : C \to C'$ such that $\sigma^* L' \cong L$.

Theorem 2.19. *There is an equivalence of categories between nondegenerate ternary cubic forms* $(\mathcal{W}, L_S, f)$ *over* S *and pairs* (C, L)*, where* C *is a genus one curve over* S *and* L *is a degree* 3 *line bundle on* C*. The reverse functor takes the pair* (C, L) *to the ternary cubic form* $(\pi_* L, \pi_* \Omega^1_{C/S}, f)$*.*

Proof. We have already described the functor from ternary cubic forms to genus one curves with degree 3 line bundles. On the other hand, given a genus one curve $\pi : C \to S$ and a degree 3 line bundle L on C, we obtain a ternary cubic form in the following manner. Let $\mathcal{W}$ be the rank 3 vector bundle $\pi_* L$ over S, with $p : \mathbb{P}(\mathcal{W}^\vee) \to S$, so that there is a natural embedding $\iota : C \to \mathbb{P}((\pi_* L)^\vee) = \mathbb{P}(\mathcal{W}^\vee)$. If I_C is the ideal defining the curve C, there is an exact sequence

$$0 \longrightarrow I_C(3) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(3) \longrightarrow \mathcal{O}_C(3) \longrightarrow 0$$

which gives the exact sequence of cohomology

$$0 \longrightarrow p_* I_C(3) \longrightarrow p_* \mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(3) \longrightarrow p_* \mathcal{O}_C(3) \longrightarrow 0. \tag{2.14}$$

Now $I_C(3)$ is a degree 0 line bundle on $\mathbb{P}(\mathcal{W}^\vee)$, so $p_* I_C(3)$ is a line bundle on S. We thus obtain, by tensoring (2.14) by $(p_* I_C(3))^\vee$, the sequence

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{f} p_* \mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(3) \otimes (p_* I_C(3))^\vee \longrightarrow p_* \mathcal{O}_C(3) \otimes (p_* I_C(3))^\vee \longrightarrow 0$$

where the injection is a section f of

$$p_* \mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(3) \otimes (p_* I_C(3))^\vee = \operatorname{Sym}^3 \mathcal{W} \otimes (p_* I_C(3))^\vee,$$

in other words, a ternary cubic form taking values in $(p_* I_C(3))^\vee$. Using the vanishing of $\mathrm{R}^0 \pi_*(I_C \otimes I_C(3))$ and $\mathrm{R}^1 \pi_*(I_C \otimes I_C(3))$, the adjunction sequence for ι, the Euler sequence on $\mathbb{P}(\mathcal{W}^\vee)$, and the projection formula, we compute

$$\begin{aligned} p_* I_C(3) &= p_*(\mathcal{O}_C \otimes I_C(3)) \\ &= \pi_* \iota^* I_C(3) \\ &= \pi_*(I_C / I_C^2 \otimes L^{\otimes 3}) \\ &= \pi_* \left(((L^\vee)^{\otimes 3} \otimes \pi^*(\wedge^3 \mathcal{W}) \otimes (\Omega^1_{C/S})^\vee) \otimes L^{\otimes 3} \right) \\ &= \wedge^3 \mathcal{W} \otimes \pi_*((\Omega^1_{C/S})^\vee) = \wedge^3 \mathcal{W} \otimes (\pi_* \Omega^1_{C/S})^\vee. \end{aligned}$$

Therefore, the ternary cubic form associated to (C, L) is $(\pi_* L, \pi_* \Omega^1_{C/S}, f)$, as desired. These functors are locally inverse, and are thus inverse. $\square$

Note that the line bundle $\pi_*\Omega^1_{C/S}$ on S appearing in Theorem 2.19 is the Hodge bundle of C over S. That is, a pair (C, L) corresponds to a ternary quadratic form taking values in the Hodge bundle twisted by the determinant of the ambient vector bundle $\pi_* L$.

We may interpret the moduli space of pairs (C, L) in Theorem 2.19 in terms of the fppf cohomology groups introduced in Section 2.A.1.

Example 2.20. If $S = \text{Spec } k$ is a field, then the pairs (C, L) in the correspondence are torsors for the elliptic curve $\text{Jac}(C)$ and the standard degree 3 line bundle $\mathcal{O}(3 \cdot O)$ where O represents $\mathcal{O}_C$ in $\text{Jac}(C)$. In other words, given an elliptic curve E over k, the k-rational pairs (C, L) with $\text{Aut}^0(C) \cong E$ are exactly parametrized up to isomorphism by $H^1(k, \Theta_{E,3})$ (or the kernel of the obstruction map). It is more natural to think of the pair (C, L) as corresponding to an element of $H^1(k, \Theta_{E,3})$ since the automorphism group of (C, L) is $\Theta_{E,3}$.

Ranging over all possible elliptic curves as Jacobians shows that nondegenerate ternary cubic forms over k correspond to k-points of the quotient stack $[\mathscr{M}_{1,1}/\Theta_{E^{\text{univ}},3}]$, where $\mathscr{M}_{1,1}$ is the moduli space of elliptic curves and $\Theta_{E^{\text{univ}},3}$ is the theta group scheme for the universal elliptic curve E^{univ} over $\mathscr{M}_{1,1}$.

The argument of Example 2.20 works in general for any $\mathbb{Z}[\frac{1}{6}]$ scheme S. For an elliptic curve E over S, recall that isomorphism classes of the pairs (C, L), where C is a genus one curve over S with an isomorphism $\text{Aut}^0(C) \xrightarrow{\cong} E$ and L is a degree 3 line bundle on S, are in bijection with elements of $H^1_{\text{f}}(S, \Theta_{E,3})$. Thus, the nondegenerate ternary cubic forms over S correspond to S-points of the quotient stack $[\mathscr{M}_{1,1}/\Theta_{E^{\text{univ}},3}]$.

Corollary 2.21. *The stack of nondegenerate ternary cubic forms is equivalent to the quotient stack* $[\mathscr{M}_{1,1}/\Theta_{E^{\text{univ}},3}]$.

Remark 2.22. Recall that the kernel of the obstruction map is a quotient of $H^1_{\text{f}}(S, \Theta_{E,3})$. Here, the elements of $\ker(\text{Ob})$ correspond to nondegenerate ternary cubic forms $(\mathcal{W}, L_S, f)$ under the equivalence of forgetting the line bundle L_S.

2.A.3 Rubik's Cubes as Torsors

In this section, we reinterpret the moduli stack $\mathcal{Y}_{333}$ of Theorem 2.12 in terms of torsors of elliptic curves. This moduli stack is equivalent to the nondegenerate substack of the quotient stack $[\mathbb{A}^{27}/\mathrm{GL}_3^3]$ of Rubik's cubes.

Recall that $\mathcal{Y}_{333}(S)$ parametrizes quintuples $(C, L_1, L_2, L_3, \varphi)$, where $\pi : C \to S$ is a genus one curve over S, and L_1, L_2, and L_3 are degree 3 line bundles on C, with an isomorphism $\varphi : L_1^{\otimes 2} \xrightarrow{\cong} L_2 \otimes L_3 \otimes \pi^* L_S$ for some line bundle L_S on S, and with the condition that $R^0\pi_*(L_1^\vee \otimes L_i) = 0$ for $i = 2$ or 3. Rewriting these restrictions in terms of torsors in $H^1_{\mathrm{f}}(S, \Theta_{\mathrm{Jac}(C),3})$ will give more symmetric conditions.

We first describe $\mathcal{Y}_{333}(k)$ for a field k. The general case is similar but requires recording the action of $\mathrm{Pic}(S)$. Just as for ternary cubics, each pair (C, L_i) may be viewed as an element of $H^1(k, \mathrm{Jac}(C)[3])$ for $1 \leq i \leq 3$, and since L_i is an actual line bundle on C (not just a k-rational divisor class), the pair (C, L_i) lies in $\ker(\mathrm{Ob})$ and is more naturally viewed as an element $\eta_i \in H^1(k, \Theta_{\mathrm{Jac}(C),3})$. A quadruple (C, L_1, L_2, L_3) gives rise to three such elements (C_i, L_i) of $H^1(k, \Theta_{\mathrm{Jac}(C),3})$, with the conditions that the curves C_i are all isomorphic to C. In other words, for a specified elliptic curve E over k, all such quadruples (C, L_1, L_2, L_3) with $\mathrm{Jac}(C) \cong E$ (and no other conditions) are parametrized up to isomorphism by the preimage of the diagonal of $H^1(k, E)^3$ in the natural map

$$H^1(k, \Theta_{E,3})^3 \xrightarrow{(\gamma,\gamma,\gamma)} H^1(k, E[3])^3 \xrightarrow{(\alpha,\alpha,\alpha)} H^1(k, E)^3.$$

The existence of an isomorphism such as φ for an element of $\mathcal{Y}_{333}(k)$ is equivalent to the requirement that the sum $\gamma(\eta_1) + \gamma(\eta_2) + \gamma(\eta_3)$ in $H^1(k, \mathrm{Jac}(C)[3])$ is the identity. Finally, the condition that the line bundles $L_1^\vee \otimes L_2$ and $L_1^\vee \otimes L_3$ have no sections (that is, are not isomorphic to a pullback of a line bundle from S) translates into $\eta_i \neq \eta_j$ in $H^1(k, \Theta_{E,3})$ for all $1 \leq i \neq j \leq 3$, so that (η_1, η_2, η_3) does not lie in the (big) diagonal of $H^1(k, \Theta_{E,3})^3$.

Therefore, by ranging over all elliptic curves E over k, we may describe $\mathcal{Y}_{333}(k)$ as the k-points of a substack of

$$[\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},3}] \times_{\mathscr{M}_1} [\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},3}] \times_{\mathscr{M}_1} [\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},3}] \setminus \Delta$$

where Δ denotes the diagonal. There is a natural addition map

$$[\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},3}] \times_{\mathscr{M}_1} [\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},3}] \times_{\mathscr{M}_1} [\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},3}] \setminus \Delta \longrightarrow [\mathscr{M}_{1,1}/E^{\mathrm{univ}}[3]],$$

and we are interested in the kernel substack $\mathcal{Z}$, i.e., the fiber over the identity section $\mathscr{M}_{1,1} \to [\mathscr{M}_{1,1}/E^{\mathrm{univ}}[3]]$.

More generally, over a $\mathbb{Z}[\frac{1}{6}]$-scheme S, the quadruple (C, L_1, L_2, L_3) again corresponds to a triple $(\eta_1, \eta_2, \eta_3) \in H^1_{\mathrm{f}}(S, \Theta_{\mathrm{Jac}(C),3})^3$, which is an S-point of

$$[\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},3}] \times_{\mathscr{M}_1} [\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},3}] \times_{\mathscr{M}_1} [\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},3}].$$

Note that in this case, the data of a pair (C, L_i) is more than that of an element of the kernel of the obstruction map, since the pairs (C, L_i) and $(C, L_i \otimes \pi^* L_S)$ for any line bundle L_S on S correspond to the same element of $H^1_{\mathrm{f}}(S, \mathrm{Jac}(C)[3])$.

As before, the isomorphism φ translates into an isomorphism of $\gamma(\eta_1)+\gamma(\eta_2)+\gamma(\eta_3)$ with the identity element of $[S/\operatorname{Jac}(C)[3]](S)$. Also, the condition on the line bundles $L_1^\vee \otimes L_2$ and $L_1^\vee \otimes L_3$ is again equivalent to not being an element of the big diagonal of $H^1_{\mathrm{f}}(S, \Theta_{\mathrm{Jac}(C),3})^3$. Thus, the quadruple (C, L_1, L_2, L_3) corresponds to a triple $(\eta_1, \eta_2, \eta_3) \in H^1_{\mathrm{f}}(S, \Theta_{\mathrm{Jac}(C),3})^3$, with the symmetric conditions that the triple does not lie in the (big) diagonal, $\alpha(\gamma(\eta_i))$ is the same for all $1 \leq i \leq 3$, and $\sum_{i=1}^3 \gamma(\eta_i) = 0$. Therefore, the S-points of $\mathcal{Y}_{333}$ are the S-points of the stack $\mathcal{Z}$ described above.

Corollary 2.23. *The stack of nondegenerate Rubik's cubes is equivalent to the stack* $\mathcal{Y}_{333}$*, which is equivalent to the kernel substack of the addition map*

$$[\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},3}] \times_{\mathscr{M}_1} [\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},3}] \times_{\mathscr{M}_1} [\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},3}] \setminus \Delta \longrightarrow [\mathscr{M}_{1,1}/E^{\mathrm{univ}}[3]].$$

This place changes your perception about what's possible.

—Kate, in *Cube 2: Hypercube*

Chapter 3

Hypercubes and Curves of Genus One

In this chapter, we study the space of $2 \times 2 \times 2 \times 2$ boxes, also called *hypercubes*. Similarly to Chapter 2, the points of this space correspond to genus one curves with some extra geometric data, and this correspondence is equivariant under the natural action of a related reductive group, for which the space of hypercubes is a representation.

We will first introduce two ways to represent genus one curves, as binary quartics and as bidegree $(2,2)$ forms in $\mathbb{P}^1 \times \mathbb{P}^1$. A hypercube naturally gives rise to both of these types of polynomials, as well as other geometric objects, and we will show exactly what geometric data is needed to recover a hypercube or its orbit. Imposing various sorts of symmetry conditions on hypercubes will restrict the geometric data obtained.

Preliminaries. Let F be an algebraically closed field, of characteristic not 2 or 3. As in Chapter 2 we use the convention that the projectivization of a vector space parametrizes lines instead of hyperplanes, so a basepoint-free line bundle L on a scheme X induces $\phi_L : X \to \mathbb{P}(H^0(X, L)^\vee)$. Unless stated otherwise, a genus 1 curve means a proper, smooth, geometrically connected curve with arithmetic genus 1. In addition, the notation $\mathrm{Sym}^n V$ for a vector space V will refer to the symmetric tensor space as a quotient of $V^{\otimes n}$, while $\mathrm{Sym}_n V \cong (\mathrm{Sym}^n(V^\vee))^\vee$ is the subspace of $V^{\otimes n}$ of symmetric tensors.

3.1 Orbits of Binary Quartic Forms and $(2,2)$ Forms

In this section, we show that two types of polynomials, like ternary cubic forms in Section 2.1, give rise to genus one curves with certain types of line bundles. In both cases, a genus one curve and the same data is also enough to recover the form. Both types of forms will naturally arise in our analysis of hypercubes.

3.1.1 Binary Quartic Forms

Let k be a (not necessarily algebraically closed) field, not of characteristic 2 or 3. A *binary quartic form over* k is a two-dimensional vector space V over k, a basis $\{w_1, w_2\}$ for V, and an element q of $\operatorname{Sym}^4 V$, which may be represented as a polynomial

$$q(w_1, w_2) = aw_1^4 + bw_1^3 w_2 + cw_1^2 w_2^2 + dw_1 w_2^3 + ew_2^4, \tag{3.1}$$

for $a, b, c, d, e \in k$. The group $\mathrm{GL}(V)$ acts on $\operatorname{Sym}^4 V$ by acting on V in the standard way. The ring of $\mathrm{SL}(V)$-invariants of a binary quartic form q as in equation (3.1) is a polynomial ring, generated by the two invariants

$$I(q) = 12ae - 3bd + c^2 \qquad \text{and} \qquad J(q) = 72ace + 9bcd - 27ad^2 - 27eb^2 - 2c^3.$$

The naive orbit space $\operatorname{Sym}^4 V/\mathrm{SL}_2(V)$ is thus birational to the affine plane. There is also a natural notion of the discriminant $\Delta(q) = 4I(q)^3 - J(q)^2$ of a binary quartic. The nonvanishing of the discriminant $\Delta(q)$ corresponds to q having four distinct roots over the algebraic closure $\overline{k}$ of k; such binary quartic forms are called *nondegenerate*.

A nondegenerate binary quartic form q defines a (singular) genus one curve C in $\mathbb{P}^2$ by the equation

$$t^2 w_2^2 = q(w_1, w_2) = aw_1^4 + bw_1^3 w_2 + cw_1^2 w_2^2 + dw_1 w_2^3 + ew_2^4,$$

as a degree 2 cover of $\mathbb{P}(V^\vee)$. In particular, if k is algebraically closed, the four roots of the binary quartic are the four points of $\mathbb{P}(V^\vee)$ over which C ramifies. From a nondegenerate binary quartic, then, we naturally obtain a smooth irreducible genus one curve, i.e., the normalization $\widetilde{C}$ of C, as well as a degree 2 line bundle L on $\widetilde{C}$, which is the pullback of $\mathcal{O}_{\mathbb{P}(V^\vee)}(1)$ to $\widetilde{C}$. Then the space of sections $H^0(\widetilde{C}, L)$ may be identified with the vector space V.

On the other hand, given a smooth irreducible genus one curve C over k and a degree 2 line bundle L, the hyperelliptic map $\phi_L : C \to \mathbb{P}(H^0(C,L)^\vee)$ is tamely ramified at four points over $\overline{k}$, by Riemann-Hurwitz. The ramification locus is a degree 4 subscheme of $\mathbb{P}(H^0(C,L)^\vee)$ defined over k, which recovers a binary quartic form over k.

3.1.2 Bidegree $(2,2)$ Curves in $\mathbb{P}^1 \times \mathbb{P}^1$

Let V_1 and V_2 be two-dimensional vector spaces over F. A $(2,2)$ *form* f over F is an element of $\operatorname{Sym}^2 V_1 \otimes \operatorname{Sym}^2 V_2$, with a choice of basis for V_1 and V_2. Such a form f may be represented as a polynomial

$$\begin{aligned} f(w_1,w_2,x_1,x_2) = a_{22}w_1^2x_1^2 + a_{32}w_1w_2x_1^2 + a_{42}w_2^2x_1^2 + a_{23}w_1^2x_1x_2 + a_{33}w_1w_2x_1x_2 \\ + a_{43}w_2^2x_1x_2 + a_{24}w_1^2x_2^2 + a_{34}w_1w_2x_2^2 + a_{44}w_2^2x_2^2 \end{aligned}$$

for bases $\{w_1,w_2\}$ and $\{x_1,x_2\}$ of V_1 and V_2, respectively. The group $\mathrm{GL}(V_1)\times\mathrm{GL}(V_2)$ acts on the space of $(2,2)$ forms by the standard action on each factor.

The $(2,2)$ form f cuts out a bidegree $(2,2)$ curve C in $\mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)$. If the curve C is smooth, then a standard computation shows that C has genus $(2-1)(2-1)=1$. Pulling back line bundles via the embedding $\iota : C \hookrightarrow \mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)$ gives two degree 2 line bundles on C,

$$L_1 := \iota^*\mathcal{O}_{\mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)}(1,0) \qquad \text{and} \qquad L_2 := \iota^*\mathcal{O}_{\mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)}(0,1).$$

Each of the projection maps $\mathrm{pr}_i : C \to \mathbb{P}(V_i^\vee)$, for $i=1$ or 2, is a degree 2 cover of $\mathbb{P}(V_i^\vee)$, ramified at four points. A binary quartic q_1 on V_1 associated to the ramification locus in

$\mathbb{P}(V_1^\vee)$ may be computed by taking the discriminant of f as a quadratic polynomial on V_2:

$$q_1(w_1,w_2) := \text{disc}\,(f(x_1,x_2)) = (a_{23}w_1^2 + a_{33}w_1w_2 + a_{43}w_2^2)^2 - (a_{22}w_1^2 + a_{32}w_1w_2 + a_{42}w_2^2)(a_{24}w_1^2 + a_{34}w_1w_2 + a_{44}w_2^2),$$

and similarly for $q_2(x_1,x_2)$ as a binary quartic form on V_2. The smooth genus one curve obtained from each of these binary quartics is isomorphic to the curve C. Moreover, the line bundles L_1 and L_2 are essentially the degree 2 line bundles given by these binary quartics on their associated genus one curves. (That is, for $i = 1$ or 2, the line bundle L_i on C is isomorphic to the pullback of that line bundle from q_i, via the isomorphism sending C to the smooth curve associated to q_i.)

We call a $(2,2)$ form f or its associated curve C *nondegenerate* if both of the associated binary quartics are nondegenerate, i.e., have four distinct roots over an algebraic closure. For each of the binary quartics, this condition is given by the nonvanishing of the discriminant $\Delta(q_i)$. As the binary quartic q_i is invariant under the action of $\mathrm{SL}(V_j)$ on f, the discriminant $\Delta(q_i)$ is a degree 12 $\mathrm{SL}(V_i) \times \mathrm{SL}(V_j)$-invariant for f. Moreover, it is easy to check that $I(q_1) = I(q_2)$ and $J(q_1) = J(q_2)$, so $\Delta(q_1) = \Delta(q_2)$. Thus, the polynomials $I(f) := I(q_i)$ and $J(f) := J(q_i)$ for $i = 1$ or 2 are degree 4 and 6 $\mathrm{SL}(V_1)\times\mathrm{SL}(V_2)$-invariants, respectively. The *discriminant* $\Delta(f) = \Delta(q_i)$ of the $(2,2)$ form f is a degree 12 invariant, and a *nondegenerate* $(2,2)$ form is one with nonzero discriminant.[1] The nonvanishing of this discriminant is also equivalent to the condition that the curve C cut out by f be nonsingular.

Thus, from a nondegenerate $(2,2)$ form f, we have constructed a genus one curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Conversely, given a genus one curve C and two degree 2 line bundles L_1 and L_2 on C, there are natural degree 2 maps $\phi_{L_i} : C \to \mathbb{P}(H^0(C,L_i)^\vee) = \mathbb{P}^1$ and the product map

$$(\phi_{L_1},\phi_{L_2}) : C \longrightarrow \mathbb{P}(H^0(C,L_1)^\vee) \times \mathbb{P}(H^0(C,L_2)^\vee)\,.$$

If $L_1 \cong L_2$, then (ϕ_{L_1},ϕ_{L_2}) is a degree 2 cover of a diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., the image of this map is isomorphic to $\mathbb{P}^1$. Otherwise, we claim that this map is a closed immersion.

[1] Note that the discriminant $\Delta(f)$ is not a generator for the ring of $\mathrm{SL}(V_1)\times\mathrm{SL}(V_2)$-invariants of $\mathrm{Sym}^2 V_1 \otimes \mathrm{Sym}^2 V_2$. The invariant ring is a polynomial ring with generators in degrees $2, 3,$ and 4.

Lemma 3.1. *For a smooth irreducible genus one curve C and non-isomorphic degree 2 line bundles L_1 and L_2 on C, the composition*

$$\kappa : C \xrightarrow{(\phi_{L_1}, \phi_{L_2})} \mathbb{P}(H^0(C, L_1)^\vee) \times \mathbb{P}(H^0(C, L_2)^\vee) \xhookrightarrow{\text{Segre}} \mathbb{P}(H^0(C, L_1)^\vee \otimes H^0(C, L_2)^\vee)$$

is a closed immersion.

Proof. By Riemann-Roch, the spaces of sections $H^0(C, L_1), H^0(C, L_2)$, and $H^0(C, L_1 \otimes L_2)$ have dimensions $2, 2$, and 4, respectively. We claim that the multiplication map

$$\mu_{12} : H^0(C, L_1) \otimes H^0(C, L_2) \longrightarrow H^0(C, L_1 \otimes L_2)$$

is an isomorphism. Because of the assumption that $L_1 \ncong L_2$, this follows easily from Castelnuovo's basepoint-free pencil trick (see [ACGH85, p. 126] or [Eis95, Exercise 17.18]). In particular, since $h^0(C, L_1) = 2$, we have the exact sequence

$$0 \longrightarrow L_1^{-1} \longrightarrow H^0(C, L_1) \otimes \mathcal{O}_C \longrightarrow L_1 \longrightarrow 0, \tag{3.2}$$

where the surjective map is the natural adjunction map associated to the map $C \to \{\text{pt}\}$ and the sheaf L_1. Tensoring (3.2) with L_2 and taking cohomology gives the exact sequence

$$H^0(C, L_1^{-1} \otimes L_2) \longrightarrow H^0(C, L_1) \otimes H^0(C, L_2) \xrightarrow{\mu_{12}} H^0(C, L_1 \otimes L_2) \longrightarrow H^1(C, L_1^{-1} \otimes L_2).$$

Since $L_1^{-1} \otimes L_2$ is a degree 0 line bundle not isomorphic to $\mathcal{O}_C$, the middle map μ_{12} is an isomorphism.

Since $\deg(L_1 \otimes L_2) = 4$, the curve C is isomorphic to its image in $\mathbb{P}(H^0(C, L_1 \otimes L_2)^\vee) = \mathbb{P}^3$ under the map $\phi_{L_1 \otimes L_2}$. The diagram

$$\begin{array}{ccc} C & \xrightarrow{\phi_{L_1 \otimes L_2}} & \mathbb{P}(H^0(C, L_1 \otimes L_2)^\vee) \\ & \searrow^{\kappa} & \downarrow \cong \; \mathbb{P}(\mu_{12}^\vee) \\ & & \mathbb{P}(H^0(C, L_1)^\vee \otimes H^0(C, L_2)^\vee) \end{array}$$

commutes, by the definitions of the multiplication map μ_{12} and the maps given by linear systems. Therefore, the desired map κ is a closed immersion. $\square$

The image C_{12} of the curve C in $\mathbb{P}(H^0(C, L_1)^\vee) \times \mathbb{P}(H^0(C, L_2)^\vee)$ is cut out by a $(2,2)$ form, via the exact sequence defining C_{12}. Tensoring with $I_{C_{12}}^{-1}$, where $I_{C_{12}}$ is the ideal defining C_{12}, and taking cohomology gives the exact sequence

$$0 \longrightarrow F \longrightarrow H^0(\mathbb{P}(H^0(C, L_1)^\vee) \times \mathbb{P}(H^0(C, L_2)^\vee), \mathcal{O}(2,2)) \longrightarrow H^0(C, L_1^{\otimes 2} \otimes L_2^{\otimes 2}) \longrightarrow 0,$$

thereby defining a degree $(2,2)$ form in the middle term, which is naturally isomorphic to $\mathrm{Sym}^2(H^0(C, L_1)) \otimes \mathrm{Sym}^2(H^0(C, L_2))$.

Thus, a genus one curve and two nonisomorphic degree 2 line bundles L_1 and L_2, along with bases $\mathfrak{B}_i$ for the two-dimensional space $H^0(C, L_i)$ for $1 \leq i \leq 2$, give rise to a $(2,2)$ form. Call $(C, L_1, L_2, \mathfrak{B}_1, \mathfrak{B}_2)$ and $(C', L_1', L_2', \mathfrak{B}_1', \mathfrak{B}_2')$ *equivalent* if there exists an isomorphism $\sigma : C \to C'$, such that $\sigma^* L_i' \cong L_i$ and the induced map $\sigma^* : H^0(C, L_i') \to H^0(C, L_i)$ sends $\mathfrak{B}_i'$ to $\mathfrak{B}_i$ for $i = 1$ and 2. Similarly, triples (C, L_1, L_2) and (C', L_1', L_2') are equivalent if there exists an isomorphism $\sigma : C \to C'$ with $\sigma^* L_i' \cong L_i$ for $1 \leq i \leq 2$.

Proposition 3.2. *Over F, there exists a bijection between nondegenerate $(2,2)$ forms and equivalence classes of triples $(C, L_1, L_2, \mathfrak{B}_1, \mathfrak{B}_2)$, where C is a genus one curve, and L_1 and L_2 are nonisomorphic degree 2 line bundles on C, and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are bases for $H^0(C, L_1)$ and $H^0(C, L_2)$, respectively. The $\mathrm{GL}_2 \times \mathrm{GL}_2$-orbits of nondegenerate $(2,2)$ forms, i.e., nondegenerate elements of $\mathrm{Sym}^2 V_1 \otimes \mathrm{Sym}^2 V_2$ for two-dimensional vector spaces V_1 and V_2, are in bijection with equivalence classes of triples (C, L_1, L_2).*

3.2 Hypercubes

Let V_1, V_2, V_3, V_4 be two-dimensional vector spaces over the field F. Then the reductive group $G := \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_3) \times \mathrm{GL}(V_4)$ has a natural action on the vector space $V_1 \otimes V_2 \otimes V_3 \otimes V_4$, the space of quadrilinear forms. With a choice of basis for the vector spaces V_1, V_2, V_3, V_4, say $\{e_i^{(1)}, e_i^{(2)}\}$ for V_i, we may represent an element

$$\sum_{r,s,t,u=1}^{2} a_{rstu} e_1^{(r)} \otimes e_2^{(s)} \otimes e_3^{(t)} \otimes e_4^{(u)} \in V_1 \otimes V_2 \otimes V_3 \otimes V_4$$

as a $2 \times 2 \times 2 \times 2$ *box* or *hypercube* $H = (a_{rstu})_{1 \leq r,s,t,u \leq 2}$. In the sequel, we will refer to both the array and the quadrilinear form as the hypercube, with the vector spaces V_i and their bases understood. The group G acts on the space of hypercubes by the analogue of row and column operations.

As in Chapter 2, we also use the notation $H(\cdot, \cdot, \cdot, \cdot)$ to denote the quadrilinear form associated to the hypercube, where the dots may be replaced by elements of the respective $V_i^\vee$. For example, for $w \in V_1^\vee, x \in V_2^\vee$, the notation $H(w, x, \cdot, \cdot)$ refers to the 2×2 matrix $H \lrcorner (w \otimes x) \in V_3 \otimes V_4$. We will also use this notation for evaluating the hypercube H at points in the projective spaces $\mathbb{P}(V_i^\vee)$, but only to say whether the tensor or a determinant vanishes.

In this section, we will describe the geometric data that naturally arises from hypercubes (in particular, *nondegenerate* hypercubes).

3.2.1 Varieties Associated to Hypercubes

Given a hypercube $H = (a_{rstu}) \in V_1 \otimes V_2 \otimes V_3 \otimes V_4$, we construct three types of varieties associated to H. Throughout this section, let $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

(i) Define the $(2, 2)$ form

$$f_{12}(w_1, w_2, x_1, x_2) = \det(H(w, x, \cdot, \cdot)) \in \operatorname{Sym}^2 V_1 \otimes \operatorname{Sym}^2 V_2.$$

The vanishing of f_{12} defines a variety in $\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee)$, namely

$$C_{12} := \{(w, x) \in \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) : f_{12}(w, x) = 0\} \subset \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) \hookrightarrow \mathbb{P}(V_1^\vee \otimes V_2^\vee)$$

where the last inclusion is given by the Segre embedding. Similarly, we define the $(2, 2)$ form $f_{ij} \in \operatorname{Sym}^2 V_i \otimes \operatorname{Sym}^2 V_j$, and the varieties $C_{ij} \subset \mathbb{P}(V_i^\vee) \times \mathbb{P}(V_j^\vee)$ as the vanishing of f_{ij}, for each pair $i \neq j \in \{1,2,3,4\}$. (The symbols f_{ij} and f_{ji}, as well as C_{ij} and C_{ji}, will refer to the same forms, and varieties.) By construction, the $(2,2)$ form f_{ij} is invariant under the action of $\mathrm{SL}(V_k) \times \mathrm{SL}(V_l)$.

(ii) From a $(2,2)$ form, recall from Section 3.1.2 that we may naturally define two binary quartics, corresponding to the ramification locus of each of the projection maps to $\mathbb{P}^1$. Define the binary quartic $q_{12}(w_1, w_2) \in \mathrm{Sym}^4 V_1$ as the discriminant of the $(2,2)$ form f_{12}, considered as a quadratic polynomial in x_1 and x_2. We may similarly define $q_{ij} \in \mathrm{Sym}^4 V_i$ as the discriminant of the $(2,2)$ form f_{ij} as a quadratic form on V_j. Since f_{ij} is invariant under the action of $\mathrm{SL}(V_k) \times \mathrm{SL}(V_l)$, the binary quartic $q_{ij} \in \mathrm{Sym}^4 V_i$ is invariant under the action of $\mathrm{SL}(V_j) \times \mathrm{SL}(V_k) \times \mathrm{SL}(V_l)$.

We claim that the binary quartic q_{ij} is independent of the choice of j, that is, $q_{ij} = q_{ik}$ for all $j \neq k$. Without loss of generality, we show this statement for $i = 1$. The key is that the quartic q_{1j} for $j = 2, 3$, or 4 is invariant under the action of $\mathrm{SL}(V_2) \times \mathrm{SL}(V_3) \times \mathrm{SL}(V_4)$ on the $2 \times 2 \times 2$ cube $H(w, \cdot, \cdot, \cdot) \in V_2 \otimes V_3 \otimes V_4$. The ring of invariants of this prehomogeneous vector space of $2 \times 2 \times 2$ cubes is generated by one degree 4 invariant, called the discriminant. By symmetry or direct computation, we see that each of these quartics is exactly the discriminant of the cube, so they are all equal, and we may call $q_i := q_{ij}$ the quartic in $\mathrm{Sym}^4 V_i$ coming from H.

As in Section 3.1.1, there is a natural curve associated to each binary quartic. Let C_i denote the normalization of the variety associated to q_i; for example, C_1 is the normalization of the curve $t^2 w_2^2 = q_1(w_1, w_2)$ in $\mathbb{P}^2$. When the quartic q_i has no repeated roots, the variety C_i is a smooth irreducible genus one curve.

(iii) Finally, we define the variety

$$\begin{aligned} C_{123} :&= \{(w,x,y) \in \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) \times \mathbb{P}(V_3^\vee) : H(w,x,y,\cdot) = 0\} \\ &\subset \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) \times \mathbb{P}(V_3^\vee) \hookrightarrow \mathbb{P}(V_1^\vee \otimes V_2^\vee \otimes V_3^\vee) \end{aligned} \tag{3.3}$$

where the last inclusion is given by the Segre embedding, and the analogous varieties C_{ijk} in $\mathbb{P}(V_i^\vee) \times \mathbb{P}(V_j^\vee) \times \mathbb{P}(V_k^\vee)$ for $\{i,j,k\} \in \{1,2,3,4\}$. (For a permutation $\sigma \in \mathbb{S}_3$, the notation C_{ijk} and $C_{\sigma(i)\sigma(j)\sigma(k)}$ will refer to the same variety, of course.) Let $\mathcal{H}_{ijk}$ denote the map in

$$\mathrm{Hom}(V_i^\vee \otimes V_j^\vee \otimes V_k^\vee, V_l) \cong V_i \otimes V_j \otimes V_k \otimes V_l$$

corresponding to the hypercube H. Then, as a set, we have

$$C_{ijk} = \mathbb{P}(\ker \mathcal{H}_{ijk}) \cap \left(\mathbb{P}(V_i^\vee) \times \mathbb{P}(V_j^\vee) \times \mathbb{P}(V_k^\vee)\right).$$

These three types of varieties coming from a hypercube H are closely related. Since the existence of a kernel for a 2×2 matrix is the same as its determinant vanishing, the images of the projections $\pi_{ij}^k : C_{ijk} \to \mathbb{P}(V_i^\vee) \times \mathbb{P}(V_j^\vee)$ are, by definition, the varieties C_{ij}. Furthermore, as f_{ij} is a bidegree $(2,2)$ form, if C_{ij} is a smooth curve, then the projections $\pi_i^j : C_{ij} \to \mathbb{P}(V_i^\vee)$ and $\pi_j^i : C_{ij} \to \mathbb{P}(V_j^\vee)$ are generically two-to-one with four ramification points; the ramification loci of π_i^j and π_j^i correspond to the vanishing of the binary quartics q_i and q_j, respectively.

3.2.2 Nondegenerate Hypercubes

In this section, we define the condition of nondegeneracy for a hypercube, and show that for a nondegenerate hypercube, the varieties C_{ij} and C_{ijk} are isomorphic genus one curves for any distinct $i, j, k \in \{1, 2, 3, 4\}$. We also describe the various maps between these curves and their projections to each $\mathbb{P}^1$; we will show in Section 3.2.3 that these projections produce natural line bundles on the curves.

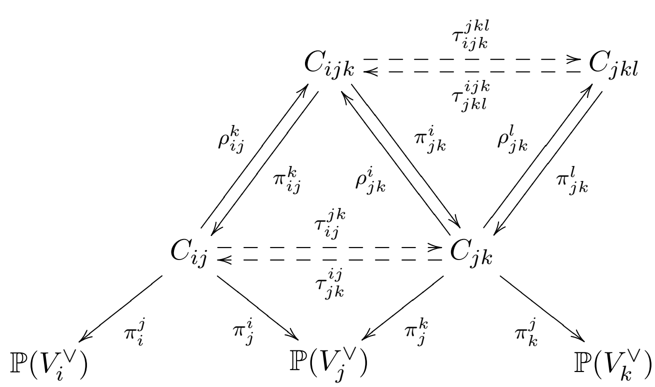
As in Section 3.1.2, we call the $(2,2)$ form f_{ij} *nondegenerate* when both the projections π_i^j and π_j^i have four distinct ramification points, that is, if the discriminants of q_i and of q_j do not vanish. This condition also implies that f_{ij} is irreducible and the variety C_{ij} is a smooth curve. As shown in Section 3.1.2, for $i \neq j$, the nondegeneracy of f_{ij} corresponds to the nonvanishing of $\Delta(f_{ij})$, which is equal to $\Delta(q_i) = \Delta(q_j)$. This condition, therefore, is exactly the same for all $i \neq j$.

We can thus define the *discriminant* $\Delta(H)$ of a hypercube H to be the discriminant of any of the binary quartics q_i or the discriminant of any of the $(2, 2)$ forms f_{ij}, all of which coincide. The discriminant $\Delta(H)$ has degree 24 and is invariant under the action of $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \mathrm{SL}(V_3) \times \mathrm{SL}(V_4)$ on H. We call a hypercube *nondegenerate* if its discriminant $\Delta(H)$ is nonzero, so nondegeneracy is preserved under the action of the group G.

Suppose H is a nondegenerate hypercube. Then each variety C_{ij} is smooth and irreducible, and it has a degree 2 map to $\mathbb{P}(V_i^\vee)$ with four distinct ramification points; thus, C_{ij} is a genus one curve. The smooth genus one curve C_i associated to the binary quartic q_i is isomorphic to C_{ij} for all j. Therefore, all of the curves C_{ij}, for all $i \neq j$, are isomorphic to one another.

As remarked earlier, the image of $\pi_{ij}^k : C_{ijk} \to \mathbb{P}(V_i^\vee) \times \mathbb{P}(V_j^\vee)$ is exactly C_{ij}. On the other hand, there exists an inverse map ρ_{ij}^k, which we define without loss of generality for $(i, j, k) = (1, 2, 3)$. The inverse map ρ_{12}^3 will take a point $(w, x) \in C_{12}$ to $(w, x, y) \in \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) \times \mathbb{P}(V_3^\vee)$, where y is the point in $\mathbb{P}(V_3^\vee)$ representing the kernel of the singular 2×2 matrix $H(w, x, \cdot, \cdot) \in V_3 \otimes V_4$. This kernel in $V_3^\vee$ is exactly one-dimensional; if not, then $H(w, x, \cdot, \cdot) = 0$, which is a contradiction because H was assumed to be nondegenerate (so the zero locus of f_{12} is a smooth irreducible curve). Since the kernel of the singular 2×2 matrix $H(w, x, \cdot, \cdot)$ is given algebraically, the map ρ_{12}^3 is clearly inverse to π_{12}^3. Thus, without loss of generality, the curves C_{ijk} and C_{ij} are isomorphic for all triples i, j, k.

Therefore, we have that all of the curves C_{ijk} and C_{ij} associated to a nondegenerate hypercube H are isomorphic smooth irreducible genus one curves. We summarize their relationships below, where the following diagram gives examples of all of these maps:

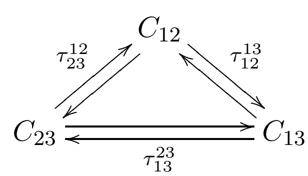


(3.4)

The dotted arrows in (3.4) are isomorphisms, defined to make the diagram commute, e.g.,

$$\tau_{ijk}^{jkl} := \rho_{jk}^{l} \circ \pi_{jk}^{i} : C_{ijk} \longrightarrow C_{jkl} \qquad \text{and} \qquad \tau_{ij}^{jk} := \pi_{jk}^{i} \circ \rho_{ij}^{k} : C_{ij} \longrightarrow C_{jk}.$$

It is clear that τ_{ijk}^{jkl} and τ_{jkl}^{ijk} are inverse maps, as are τ_{ij}^{jk} and τ_{jk}^{ij}. We will show later that, although these maps τ_{ijk}^{jkl} and τ_{ij}^{jk} are all isomorphisms, composing more than two such maps in sequence will not always give identity maps on these curves. For three-cycles such as the diagram

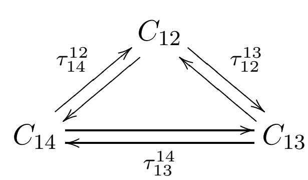


(3.5)

each of the τ maps factors through C_{123}, so the composition map

$$\tau_{23}^{12} \circ \tau_{13}^{23} \circ \tau_{12}^{13} = (\pi_{12}^{3} \circ \rho_{23}^{1}) \circ (\pi_{23}^{1} \circ \rho_{13}^{2}) \circ (\pi_{13}^{2} \circ \rho_{12}^{3}) = \pi_{12}^{3} \circ (\rho_{23}^{1} \circ \pi_{23}^{1}) \circ (\rho_{13}^{2} \circ \pi_{13}^{2}) \circ \rho_{12}^{3}$$

is the identity on C_{12}. However, for three-cycles of the form



(3.6)

the composition map $\alpha_{234} : \tau_{14}^{12} \circ \tau_{13}^{14} \circ \tau_{12}^{13}$ is not the identity map on C_{12}.[2] Given a point $w \in \mathbb{P}(V_1^\vee)$ not in the ramification locus of the projection from C_{12}, there are two distinct points $x, x' \in \mathbb{P}(V_2^\vee)$ such that $\det H(w, x, \cdot, \cdot) = 0$. Then $H(w, x, y, \cdot) = 0$ for exactly one point $y \in \mathbb{P}(V_3^\vee)$, and $H(w, x', y, z') = 0$ for some $z' \in \mathbb{P}(V_4^\vee)$. Then the linear form $H(w, \cdot, y, z')$ vanishes when evaluated at both x and x', so it is identically 0. So $\tau_{13}^{14}(w, y) = (w, z')$. In other words, the composition map α_{234} is given by

$$\alpha_{234} : \qquad \begin{array}{ccccccc} C_{12} & \xrightarrow{\tau_{12}^{13}} & C_{13} & \xrightarrow{\tau_{13}^{14}} & C_{14} & \xrightarrow{\tau_{14}^{12}} & C_{12} \\ (w, x) & \longmapsto & (w, y) & \longmapsto & (w, z') & \longmapsto & (w, x'). \end{array}$$

[2]This argument comes from an idea of John Cremona for $2 \times 2 \times 2$ boxes [Cre].

The analogous composition maps α_{ijk} for all three-cycles of this sort also flip the two sheets of the double covers of $\mathbb{P}(V_i^\vee), \mathbb{P}(V_j^\vee)$, and $\mathbb{P}(V_k^\vee)$.

Lemma 3.3. *For any permutation $\{i, j, k, l\} = \{1, 2, 3, 4\}$, the composition map*

$$\alpha_{ijk}: \qquad C_{li} \xrightarrow{\tau_{li}^{lj}} C_{lj} \xrightarrow{\tau_{lj}^{lk}} C_{lk} \xrightarrow{\tau_{lk}^{li}} C_{li}$$

is not the identity map. For $w \in \mathbb{P}(V_l^\vee)$, if x_1 and x_2 are the two (not necessarily distinct) solutions for $x \in \mathbb{P}(V_i^\vee)$ such that $H \lrcorner (w \otimes x) = 0$, then the points (w, x_1) and (w, x_2) lie on $C_{li} \subset \mathbb{P}(V_l^\vee) \times \mathbb{P}(V_i^\vee)$, and $\alpha_{ijk}(w, x_1) = (w, x_2)$.

We will show in the next section that four-cycles of maps τ_{ijk}^{jkl} are also not the identity.

3.2.3 Line Bundles and Relations

A nondegenerate hypercube and all the isomorphic curves associated to it also naturally give rise to certain line bundles on those curves. Understanding the relations among them will explain why the four-cycles of isomorphisms of these curves do not commute, for example. These line bundles will also be part of the geometric data that hypercubes parametrize.

For simplicity of notation, choose one curve, say C_{12}, to be the primary curve under consideration. This choice matters in the definitions and constructions we will make in the sequel, but all choices are equivalent. All the constructions for the rest of the chapter are completely symmetric in the indices $\{1, 2, 3, 4\}$.

Define four line bundles L_i on C_{12} by pulling back the line bundle $\mathcal{O}(1)$ from each $\mathbb{P}(V_i^\vee)$. Of course, it is important through which maps we choose to pullback the bundle:

$$\begin{aligned} L_1 &:= (\pi_1^2)^* \mathcal{O}_{\mathbb{P}(V_1^\vee)}(1) \\ L_2 &:= (\pi_2^1)^* \mathcal{O}_{\mathbb{P}(V_2^\vee)}(1) \\ L_3 &:= (\pi_3^2 \circ \pi_{23}^1 \circ \rho_{12}^3)^* \mathcal{O}_{\mathbb{P}(V_3^\vee)}(1) \\ L_4 &:= (\pi_4^2 \circ \pi_{24}^1 \circ \rho_{12}^4)^* \mathcal{O}_{\mathbb{P}(V_4^\vee)}(1). \end{aligned} \tag{3.7}$$

That is, L_1 and L_2 are come directly from the maps $C_{12} \to \mathbb{P}(V_1^\vee)$ and $C_{12} \to \mathbb{P}(V_2^\vee)$, and

L_3 and L_4 are pulled back via the simplest maps $C_{12} \to C_{12i} \to C_{2i} \to \mathbb{P}(V_i^\vee)$ for $i = 3$ or 4. From the commutativity of diagrams like (3.5), the bundles L_3 and L_4 could also be defined by going through the curve C_{1i} instead of C_{2i} for $i = 3$ and 4, respectively. Since all the curves C_{ij} are defined by bidegree $(2, 2)$ equations, all of these line bundles on C_{12} have degree 2.

Moreover, by Lemma 3.1, the line bundles L_1 and L_2 are not isomorphic, since C_{12} is a smooth irreducible genus one curve given by a nondegenerate $(2, 2)$ form. Similarly, since C_{ij} is also a smooth irreducible genus one curve for $i = 1$ or 2 and $j = 3$ or 4, the line bundles $(\tau_{ij}^{12})^* L_i = (\pi_i^j)^* \mathcal{O}_{\mathbb{P}(V_i^\vee)}(1)$ and $(\tau_{ij}^{12})^* L_j = (\pi_j^i)^* \mathcal{O}_{\mathbb{P}(V_j^\vee)}(1)$ on C_{ij} are not isomorphic, so L_i and L_j are not isomorphic bundles on C_{12}. Thus, the four line bundles defined in (3.7) are all pairwise nonisomorphic, except possibly L_3 and L_4.

Lemma 3.4. *We have the relation*

$$L_1 \otimes L_2 \cong L_3 \otimes L_4 \tag{3.8}$$

on the line bundles on C_{12} defined above.

Proof. We will prove this lemma in the language of divisors and with explicit choices of points for simplicity. With the choice of a basis for V_i, points of the projective spaces $\mathbb{P}(V_i^\vee)$ may be represented as $[a : b]$. Let $\mathcal{D}(L)$ be the linear equivalence class of divisors corresponding to a line bundle L. A representative D_3 of $\mathcal{D}(L_3)$ is (the sum of the points in) the preimage of a fixed point, say $[1 : 0]$, in $\mathbb{P}(V_3^\vee)$, and similarly, we may choose a divisor D_4 in the class of $\mathcal{D}(L_4)$ as the preimage of $[1 : 0] \in \mathbb{P}(V_4^\vee)$. Let $H(w, x, \cdot, \cdot)$ be denoted by the matrix

$$\begin{pmatrix} H_{11}(w,x) & H_{12}(w,x) \\ H_{21}(w,x) & H_{22}(w,x) \end{pmatrix} \in V_3 \otimes V_4.$$

Then $D_3 + D_4$ is the sum of the four points that are solutions (counted up to multiplicity) of the system

$$\begin{Bmatrix} H_{11}(w,x) = 0 \\ \det H(w,x,\cdot,\cdot) = 0 \end{Bmatrix}.$$

Interpreted in another way, these four points of intersection are exactly the points of intersection of C_{12} and the bidegree $(1,1)$ curve given by H_{11} in $\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee)$. Thus, the line bundle corresponding to the sum of these four points is just the pullback of $\mathcal{O}_{\mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)}(1,1)$ to C_{12}. In other words, $\mathcal{O}(D_3 + D_4) \cong L_1 \otimes L_2$, which is the desired relation. $\square$

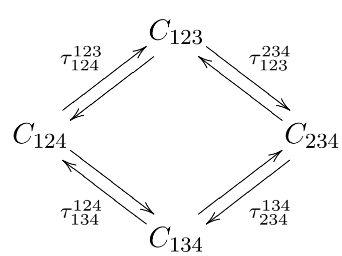
Similarly, for $\{i, j, k, l\} = \{1, 2, 3, 4\}$, if we define line bundles M_1, M_2, M_3, M_4 on the curve C_{ij} in the analogous way, these degree 2 line bundles would satisfy the relation

$$M_i \otimes M_j \cong M_k \otimes M_l.$$

Using this relation among the line bundles, we will show that composition maps such as

$$\alpha_{4123} := \tau^{123}_{124} \circ \tau^{124}_{134} \circ \tau^{134}_{234} \circ \tau^{234}_{123} : C_{123} \longrightarrow C_{123}$$

are nontrivial automorphisms; this one may be given as a translation by a point on the Jacobian of C_{123}. In other words, the four-cycle



(3.9)

and the two others like it are not commutative.

Let L_1, L_2, L_3, L_4 be line bundles defined on C_{12} as in (3.7). We would like to compare line bundles, pulled back from each C_{ijk}, on the curve C_{12}, to understand the composition α_{4123}. On each curve C_{ijk} in the four-cycle (3.9), we may define three line bundles, attained by pulling back $\mathcal{O}(1)$ from the natural projections to $\mathbb{P}(V_i^\vee), \mathbb{P}(V_j^\vee)$, and $\mathbb{P}(V_k^\vee)$. Call these line bundles $M_i[C_{ijk}]$. We pull back each of these $M_i[C_{ijk}]$ to C_{12} via the labeled τ maps in diagram (3.9) and the isomorphism $\rho^3_{12} : C_{12} \to C_{123}$. The diagram below shows the different line bundles coming from each C_{ijk} as well as the relations among them, due to the analogues of Lemma 3.4:

$$\begin{array}{ccccc} & & (L_1, L_2, L_3) & & \\ & & C_{123} & \xrightarrow{L_2\otimes L_3 \cong L_1 \otimes L_4'} & \\ (L_1', L_2', L_4') \quad C_{124} & \nearrow & & \searrow & C_{234} \quad (L_2, L_3, L_4') \\ & \nwarrow_{L_1'\otimes L_4' \cong L_3 \otimes L_2'} & & \swarrow_{L_3\otimes L_4' \cong L_2 \otimes L_1'} & \\ & & C_{134} & & \\ & & (L_1', L_3, L_4') & & \end{array} \tag{3.10}$$

For instance, we have the line bundles

$$\begin{aligned} L_2 &\cong (\rho_{12}^3 \circ \tau_{123}^{234})^* M_2[C_{234}] \\ L_3 &\cong (\rho_{12}^3 \circ \tau_{123}^{234})^* M_3[C_{234}] \cong (\rho_{12}^3 \circ \tau_{123}^{234} \circ \tau_{234}^{134})^* M_3[C_{134}] \\ L_4' &:= (\rho_{12}^3 \circ \tau_{123}^{234})^* M_4[C_{234}] \cong (\rho_{12}^3 \circ \tau_{123}^{234} \circ \tau_{234}^{134})^* M_4[C_{134}] \\ &\qquad\qquad \cong (\rho_{12}^3 \circ \tau_{123}^{234} \circ \tau_{234}^{134} \circ \tau_{134}^{124})^* M_4[C_{124}]. \end{aligned}$$

Finally, pulling back the line bundles $(M_1[C_{123}], M_2[C_{123}], M_3[C_{123}])$ via the entire four-cycle α_{4123} and then ρ_{12}^3 gives the line bundles (L_1', L_2', L_3') for some new line bundle $L_3' := (\rho_{12}^3 \circ \alpha_{4123})^* M_3[C_{123}]$, with the relation

$$L_1' \otimes L_2' \cong L_3' \otimes L_4'. \tag{3.11}$$

Using all the relations from diagram (3.10), along with (3.8) and (3.11), shows that

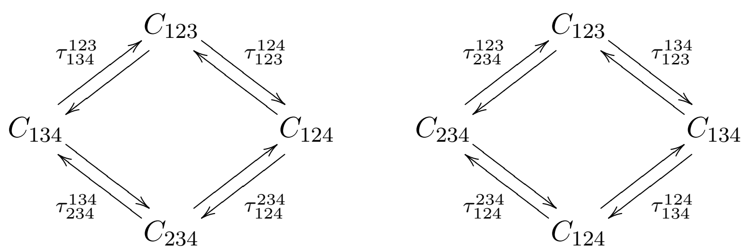
$$\begin{aligned} L_1' \otimes L_1^{-1} &\cong L_2' \otimes L_2^{-1} \cong L_3' \otimes L_3^{-1} \\ &\cong (L_3 \otimes L_1^{-1})^{\otimes 2} \cong (L_2 \otimes L_4^{-1})^{\otimes 2}. \end{aligned}$$

In other words, the automorphism α_{4123} is essentially given by the differences in the line bundles L_i' and L_i for $i = 1, 2, 3$; this automorphism of C_{123} is given by translation by a point

$$P_{4123} := (L_3 \otimes L_1^{-1})^{\otimes 2} \in \mathrm{Pic}^0(C_{12}) \cong \mathrm{Jac}(C_{12}) \cong \mathrm{Jac}(C_{123}),$$

where the two isomorphisms are entirely canonical, so P_{4123} can be thought of as a point on $\mathrm{Jac}(C_{123})$. Note that the composition map α_{4321}, by traversing diagram (3.9) in the counterclockwise direction, is the inverse map to α_{4123} and is given by translation by $P_{4321} := -P_{4123}$.

Similarly, all other four-cycles of the curves C_{ijk} are of the same form, and they also do not commute. For example, consider the automorphisms α_{4312} and α_{4231} of C_{123} given by the following two four-cycles:



The points associated to each of these four-cycles may also be computed with the relations among the line bundles, and we obtain

$$P_{4312} := (L_2 \otimes L_3^{-1})^{\otimes 2} \qquad\qquad P_{4231} := (L_1 \otimes L_2^{-1})^{\otimes 2},$$

where P_{4ijk} is the point on $\mathrm{Pic}^0(C_{12}) \cong \mathrm{Jac}(C_{12}) \cong \mathrm{Jac}(C_{123})$ by which the curve C_{123} is translated via the automorphism α_{4ijk}. Now the sum of the three points P_{4123}, P_{4312}, and P_{4231} associated to the three four-cycles is

$$(L_3 \otimes L_1^{-1})^{\otimes 2} \otimes (L_2 \otimes L_3^{-1})^{\otimes 2} \otimes (L_1 \otimes L_2^{-1})^{\otimes 2} = \mathcal{O}.$$

In other words, the composition of the three maps $\alpha_{4123}, \alpha_{4312}$, and α_{4231} (in any order) is the identity on C_{123}. We summarize these results, for any permutation of the indices, below:

Proposition 3.5. *Given a nondegenerate hypercube H and the associated curves and maps as in diagram* (3.4)*, we have the following statements, for any permutation of* $\{i, j, k, l\} = \{1, 2, 3, 4\}$*:*

(i) The composition map

$$\alpha_{ijkl} := \tau^{jkl}_{ijk} \circ \tau^{ijk}_{ijl} \circ \tau^{ijl}_{ikl} \circ \tau^{ikl}_{jkl} : C_{jkl} \longrightarrow C_{jkl}$$

is the automorphism of C_{jkl} given by translation by the point

$$P_{ijkl} := (M_l \otimes M_j^{-1})^{\otimes 2} \in \mathrm{Pic}^0(C_{jl}) \cong \mathrm{Jac}(C_{jl}) \cong \mathrm{Jac}(C_{jkl})$$

where $M_j = (\pi^l_j)^* \mathcal{O}_{\mathbb{P}(V_j^\vee)}(1)$ and $M_l = (\pi^j_l)^* \mathcal{O}_{\mathbb{P}(V_l^\vee)}(1)$ are degree 2 line bundles on C_{jl}.

(ii) We have $P_{ijkl} = -P_{ilkj}$, since $\alpha_{ijkl} \circ \alpha_{ilkj}$ is the identity map on C_{jkl}.

(iii) The points P_{ijkl}, P_{iklj}, and P_{iljk} sum to 0 on the Jacobian of C_{jkl}, so the composition of the automorphisms α_{ijkl}, α_{iklj}, and α_{iljk} in any order is the identity map on C_{jkl}.

Using the line bundles and relations also gives a better description of three-cycles like (3.6). Let $L_1 = (\pi^2_1)^* \mathcal{O}_{\mathbb{P}(V_1^\vee)}$ and $L_2 = (\pi^1_2)^* \mathcal{O}_{\mathbb{P}(V_2^\vee)}$ be line bundles on C_{12}. Then pulling back L_2 through the composition map $\alpha_{234} : C_{12} \to C_{13} \to C_{14} \to C_{12}$ gives the line bundle $L_2' := L_1^{\otimes 2} \otimes L_2^{-1}$, and the bundle $\alpha^*_{234} L_1$ is just L_1. So the automorphism α_{234} of C_{12} can be described as a "flip" around L_1, sending (L_1, L_2) to $(L_1, L_1^{\otimes 2} \otimes L_2^{-1})$.

3.3 The Moduli Problem for Hypercubes

In this section, we describe the moduli problem for hypercubes. We will show that (the GL_2^4-orbits of) nondegenerate hypercubes correspond bijectively to some of the geometric data given in Section 3.2. This geometric data may be formulated in several different ways, and the constructions also work over arbitrary base schemes, leading to an equivalence of moduli stacks.

3.3.1 Constructing Hypercubes

We have seen that a nondegenerate hypercube gives rise to a genus one curve up to isomorphism. In fact, we may construct a nondegenerate hypercube from such a curve, along with some line bundles. Let C be a genus one curve and let L_1, L_2, L_3 be nonisomorphic line bundles of degree 2 on C.

Lemma 3.6. *Given a genus one curve C and three non-isomorphic degree 2 line bundles L_1, L_2, L_3 on C, the multiplication map (i.e., the cup product on cohomology)*

$$\mu_{123} : H^0(C, L_1) \otimes H^0(C, L_2) \otimes H^0(C, L_3) \longrightarrow H^0(C, L_1 \otimes L_2 \otimes L_3)$$

is surjective, and its kernel may be naturally identified with the space of global sections $H^0(C, L_i^{-1} \otimes L_j \otimes L_k)$ for any permutation $\{i, j, k\}$ of $\{1, 2, 3\}$.

Proof. Recall from the proof of Lemma 3.1 that the multiplication map

$$\mu_{ij} : H^0(C, L_i) \otimes H^0(C, L_j) \longrightarrow H^0(C, L_i \otimes L_j)$$

for two such line bundles is an isomorphism, due to the basepoint-free pencil trick. We apply the same trick again here: for any permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, we tensor the sequence $0 \to L_i^{-1} \to H^0(C, L_i) \otimes \mathcal{O}_C \to L_i \to 0$ with $L_j \otimes L_k$ and take cohomology to obtain the exact sequence

$$\begin{aligned} 0 \to H^0(C, L_i^{-1} \otimes L_j \otimes L_k) \to H^0(C, L_i) \otimes H^0(C, L_j \otimes L_k) \to H^0(C, L_i \otimes L_j \otimes L_k) \\ \to H^1(C, L_i^{-1} \otimes L_j \otimes L_k) = 0. \end{aligned} \tag{3.12}$$

As the map μ_{123} factors through the isomorphism

$$(\mathrm{id}, \mu_{jk}) : H^0(C, L_i) \otimes H^0(C, L_j \otimes L_k) \to H^0(C, L_i \otimes L_j \otimes L_k),$$

the sequence (3.12) shows that μ_{123} is surjective and its kernel may be naturally identified with $H^0(C, L_i^{-1} \otimes L_j \otimes L_k)$. □

From Riemann-Roch, each of the vector spaces $H^0(C, L_i)$ for $1 \leq i \leq 3$ has dimension 2, and $H^0(C, L_1 \otimes L_2 \otimes L_3)$ has dimension 6. The kernel of μ_{123} has dimension 2, and we may use the inclusion of this kernel into the domain to specify a hypercube

$$\begin{aligned} H &\in H^0(C, L_1) \otimes H^0(C, L_2) \otimes H^0(C, L_3) \otimes (\ker \mu_{123})^\vee \\ &\cong \mathrm{Hom}(\ker \mu_{123}, H^0(C, L_1) \otimes H^0(C, L_2) \otimes H^0(C, L_3)). \end{aligned}$$

Recall that our definition of hypercube also requires a choice of basis for each of the vector spaces, so here to obtain a well-defined hypercube, we must also specify a basis for each of the vector spaces $H^0(C, L_i)$ for $1 \leq i \leq 3$ and $(\ker \mu_{123})^\vee$. In the remainder of this section, we will show that the hypercube H thus constructed is nondegenerate.

Let C'_{ij} be the image of the map $\phi_{L_i} \times \phi_{L_j}$; by Lemma 3.1, C'_{ij} is isomorphic to C since $L_i \ncong L_j$. Let C_{ij} be constructed from H as in Section 3.2.1, that is,

$$C_{ij} := \{(w, x) \in \mathbb{P}(V_i^\vee) \times \mathbb{P}(V_j^\vee) : \det(H \lrcorner (w \otimes x)) = 0\} \subset \mathbb{P}(V_i^\vee) \times \mathbb{P}(V_j^\vee).$$

We will show that these two varieties are the same for all $i \neq j$, but first for *some* $i \neq j$.

Claim 3.7. *For some $1 \leq i \neq j \leq 3$, we have $C_{ij} = C'_{ij}$ as sets.*

Proof. For all $i \neq j$, the inclusion $C'_{ij} \subseteq C_{ij}$ is easy: for $1 \leq k \leq 3$, let $\{r_{k1}, r_{k2}\}$ be the basis for each $H^0(C, L_k)$. Then the definition of H implies that

$$H \lrcorner ([r_{i1}(p) : r_{i2}(p)] \otimes [r_{j1}(p) : r_{j2}(p)] \otimes [r_{k1}(p) : r_{k2}(p)]) = 0$$

for all points $p \in C$, so $([r_{i1}(p) : r_{i2}(p)], [r_{j1}(p) : r_{j2}(p)])$ lies in C_{ij}. Since C_{ij} is defined by a bidegree $(2, 2)$ equation f_{ij} in $\mathbb{P}(V_i^\vee) \times \mathbb{P}(V_j^\vee)$, if f_{ij} is nonzero and irreducible, then C_{ij} is a smooth irreducible genus one curve and thus $C_{ij} = C'_{ij}$.

An irreducible bidegree (d_1, d_2) equation in $\mathbb{P}^1 \times \mathbb{P}^1$ defines a genus $(d_1 - 1)(d_2 - 1)$ curve. So if f_{ij} is nonzero and factors nontrivially, then no irreducible component can be a smooth irreducible genus one curve. However, since C_{ij} contains the smooth irreducible genus one curve C'_{ij}, the polynomial f_{ij} must be either zero or irreducible for each pair $i \neq j$. If $f_{ij} = 0$ identically, then C_{ij} is all of $\mathbb{P}(V_i^*) \times \mathbb{P}(V_j^*)$.

We consider the variety

$$C_{123} := \{(w,x,y) \in \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) \times \mathbb{P}(V_3^\vee) : H(w,x,y,\cdot) = 0\}.$$

The projection of C_{123} to any $\mathbb{P}(V_i^\vee) \times \mathbb{P}(V_j^\vee)$ is exactly C_{ij} by definition, and we will show that at least one of these projections is not two-dimensional.

Let f and g be the two tridegree $(1,1,1)$ equations defining C_{123}. Because H is defined by two linearly independent elements of $\ker \mu_{123}$, we have that f and g are nonzero and not multiples of one another. If $\gcd(f,g) = 1$, then C_{123} is a complete intersection and thus a one-dimensional variety. Otherwise, suppose without loss of generality that $\gcd(f,g)$ has tridegree $(1,1,0)$ or $(1,0,0)$. In either case, the projection to $\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee)$ is still one-dimensional. Therefore, there exists *some* $i \neq j$ such that C_{ij} is not two-dimensional, and thus must be exactly C'_{ij}. □

Since f_{ij} cuts out a smooth irreducible genus one curve, we have disc $f_{ij} \neq 0$. Thus, the hypercube H has nonzero discriminant and is nondegenerate.

As disc $(H) \neq 0$, the polynomials f_{kl} do not vanish for any $k \neq l$, and the C_{kl} are smooth irreducible genus one curves. It follows from the proof of Claim 3.7 that all of the C_{kl} are in fact set-theoretically equal to C'_{kl}. Moreover, C_{123} is set-theoretically equal to the image C'_{123} of the embedding $\phi_{L_1} \times \phi_{L_2} \times \phi_{L_3}$ of C into the triple product space $\mathbb{P}(H^0(C,L_1)^\vee) \times \mathbb{P}(H^0(C,L_2)^\vee) \times \mathbb{P}(H^0(C,L_3)^\vee)$. Because there is a canonical isomorphism $C'_{123} \to C_{123}$, the pullbacks of $\mathcal{O}_{\mathbb{P}(H^0(C,L_i)^\vee)}(1)$ to C_{123} and then to C are exactly the line bundles L_1, L_2, and L_3.

From a genus one curve and three nonisomorphic degree 2 line bundles on this curve, we have constructed a nondegenerate hypercube. Call this functor Ψ. This hypercube, in turn, produces—via the constructions of Section 3.2—an isomorphic curve and the same line bundles.

This functor Ψ is a well-defined map to G-orbits of nondegenerate hypercubes, in that Ψ sends isomorphic geometric data to the same orbit. Call two quadruples (C, L_1, L_2, L_3) and (C', L'_1, L'_2, L'_3) *equivalent* if there exists an isomorphism $\sigma : C \to C'$ with $\sigma^* L'_i \cong L_i$ for $1 \leq i \leq 3$. Then the isomorphism σ gives an identification of $H^0(C, L_i)$ and $H^0(C', L'_i)$, which commutes with the corresponding multiplication maps μ_{123} and μ'_{123}. The hypercubes constructed will thus be the same, up to a change of bases for each vector space. That is, equivalent quadruples gives rise to G-equivalent hypercubes. Therefore, we have

Lemma 3.8. *There exists a well-defined map Ψ from equivalence classes of quadruples (C, L_1, L_2, L_3), where C is a genus one curve and L_1, L_2, L_3 are pairwise nonisomorphic degree 2 line bundles on C, to G-orbits of nondegenerate hypercubes.*

3.3.2 Preliminary Bijection

We have described a natural way to construct a hypercube from a curve and line bundles, and to construct a curve from a hypercube. By keeping track of the G-action on hypercubes and isomorphisms between curves with line bundles, we obtain a bijection between these two spaces.

Theorem 3.9. *There exists a bijection*

$$\left\{\begin{array}{l}\textit{G-orbits of}\\ \textit{nondegenerate}\\ \textit{hypercubes}\end{array}\right\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \left\{\begin{array}{l}\textit{equivalence classes of quadruples } (C, M_1, M_2, M_3)\\ \textit{with } C \textit{ a genus } 1 \textit{ curve and } M_1, M_2, M_3 \textit{ pairwise}\\ \textit{nonisomorphic degree } 2 \textit{ line bundles on } C\end{array}\right\}. \tag{3.13}$$

Proof. Lemma 3.8 shows that Ψ is well-defined. We need to show that Φ is also a well-defined map, and that Φ and Ψ are inverse to one another.

First, let $H \in V_1 \otimes V_2 \otimes V_3 \otimes V_4$ be a nondegenerate hypercube. Then the curve $C_{123} \in \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) \times \mathbb{P}(V_3^\vee)$ associated to H is a genus 1 curve. For $\{i, j, k\} = \{1, 2, 3\}$, we may define line bundles $M_i := (\pi_i^j \circ \pi_{ij}^k)^* \mathcal{O}_{\mathbb{P}(V_i^\vee)}(1)$ on C_{123}. Note that the definition of these line bundles does not depend on the permutation of $\{j, k\}$. So we need to show that the G-action on nondegenerate hypercubes does not change the equivalence class of the tuple (C_{123}, M_1, M_2, M_3) thus constructed.

The action of $\mathrm{GL}(V_4)$ on H fixes the curve C_{123}, and as the projections to $\mathbb{P}(V_i^\vee)$ for $i = 1, 2, 3$ are the same under this action, the line bundles M_i are fixed (up to isomorphisms of the curve). Therefore, the equivalence class of (C_{123}, M_1, M_2, M_3) is fixed under $\mathrm{GL}(V_4)$. For $1 \le i \le 3$, let $g \in \mathrm{GL}(V_i)$ act on H, giving the tuple (C', M_1', M_2', M_3'). Then C' differs from C_{123} in $\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) \times \mathbb{P}(V_3^\vee)$ only by the automorphism of $\mathbb{P}(V_i^\vee)$ induced by the action of g on V_i. If π and π' are the projections from C_{123} and C' to $\mathbb{P}(V_i^\vee)$, respectively, there exists a map $\gamma : C_{123} \to C'$ such that the diagram

$$\begin{array}{ccc} C_{123} & \xrightarrow{\ \gamma\ } & C' \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ \mathbb{P}(V_i^\vee) & \xrightarrow[\ g\ ]{} & \mathbb{P}(V_i^\vee) \end{array}$$

commutes, and therefore, we compute

$$\gamma^* M_i' = (\pi' \circ \gamma)^* \mathcal{O}_{\mathbb{P}(V_i^\vee)}(1) = (g \circ \pi)^* \mathcal{O}_{\mathbb{P}(V_i^\vee)}(1) \cong \pi^* \mathcal{O}_{\mathbb{P}(V_i^\vee)}(1) = M_i.$$

The other two bundles are unchanged. Consequently, Φ is well-defined.

We have already shown, after Claim 3.7 in Section 3.3.1, that $\Phi \circ \Psi$ is the identity map, i.e., that the hypercube constructed from (C, M_1, M_2, M_3) returns an isomorphic curve and line bundles that pull back to the M_i under the isomorphism. On the other hand, suppose Φ sends a nondegenerate hypercube $H \in V_1 \otimes V_2 \otimes V_3 \otimes V_4$, with a choice of bases for each two-dimensional vector space V_i for $1 \leq i \leq 4$, to (C, M_1, M_2, M_3). Then the vector spaces V_i and $H^0(C, L_i)$ are naturally isomorphic for $1 \leq i \leq 3$, and the space V_4 may be identified with $(\ker \mu_{123})^\vee$. Thus, the hypercube constructed from (C, M_1, M_2, M_3) will be G-equivalent to H, and if we identify the bases of V_i with bases for the spaces $H^0(C, L_i)$ and $(\ker \mu_{123})^\vee$, then this hypercube is H. □

Remark 3.10. In the proof of Theorem 3.9, if a hypercube H is in $V_1 \otimes V_2 \otimes V_3 \otimes V_4$, we have chosen three of the vector spaces V_1, V_2, and V_3 to construct the line bundles. The theorem and proof are identical for any choice of these three, however, and in each case, the curve constructed is the same (up to isomorphism). The line bundles, however, will not be the same.

3.3.3 Reformulations

In this section, we rewrite the bijection in Theorem 3.9 in several ways that are more useful for understanding the data or for generalizing the theorem. We will, for example, rigidify the data on each side, as well as slightly change the geometric data.

First, the right side of bijection (3.13) contains equivalence classes of curves with three line bundles. In order to obtain the same automorphism groups on each side of the bijection, it is more natural to consider genus 1 curves C with four line bundles L_1, L_2, L_3, L_4 with a relation, say $L_1 \otimes L_2 \cong L_3 \otimes L_4$ as in Lemma 3.4. We also require that the L_i are pairwise nonisomorphic, except possibly L_3 and L_4. This set of four line bundles is more natural to geometric data given by a nondegenerate hypercube, such as in (3.7). Note that there is no obvious way to simultaneously and symmetrically define four line bundles on the curve coming from a hypercube, due to the nontrivial automorphisms given by four-cycles like (3.9). Therefore, we may rewrite Theorem 3.9 as

Proposition 3.11. *There exists a bijection*

$$\left\{\begin{array}{l} G\text{-orbits of} \\ \text{nondegenerate} \\ \text{hypercubes} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \text{equivalence classes of quintuples } (C, L_1, L_2, L_3, L_4) \\ \text{with } C \text{ a genus } 1 \text{ curve and } L_1, L_2, L_3, L_4 \text{ degree } 2 \\ \text{line bundles on } C \text{ such that } L_1 \otimes L_2 \cong L_3 \otimes L_4 \text{ and} \\ L_i \not\cong L_j \text{ for distinct } i \in \{1,2\} \text{ and } j \in \{1,2,3,4\} \end{array}\right\}. \tag{3.14}$$

Remark 3.12. Of course, in the proof of Theorem 3.9, only three of the line bundles are used to construct the hypercube. With the formulation of Proposition 3.11, using (L_1, L_2, L_3) or (L_1, L_2, L_4) to construct the hypercube—by identifying each space of sections $H^0(C, L_i)$ with V_i for $1 \leq i \leq 4$—will give the same hypercube, up to the action of G. We will call these two constructions Ψ_{123} and Ψ_{124}, respectively.

Recall that Lemma 3.6 gives an identification of $\ker \mu_{12i}$ with the space of sections $H^0(C, L_1 \otimes L_2 \otimes L_i^{-1})$ for $i = 3$ or 4. Therefore, with a choice of basis, we may identify a space with its dual, so there is an identification of $H^0(C, L_4)$ with $(\ker \mu_{123})^\vee$, and of $H^0(C, L_3)$ with $(\ker \mu_{124})^\vee$. Then, with these identifications and choices of bases, the functors Ψ_{123} and Ψ_{124} produce the same hypercube.

We may also use the points on the Jacobian of the curve, as in Proposition 3.5 instead of the line bundles to rewrite the geometric data of the bijection. Recall that the line bundles on the curve C_{12} constructed from the hypercube H give rise to three points $P_{4123}, P_{4312}, P_{4231}$ on the Jacobian of C_{12} that sum to 0. If we instead take "half" of these points, since they are defined as squares, the sum is still 0, not only a 2-torsion point. That is, given a hypercube and the line bundles L_1, L_2, L_3, L_4 on C_{12} defined as in (3.7), let $Q_{4123} := L_3 \otimes L_1^{-1}, Q_{4312} := L_2 \otimes L_3^{-1}$, and $Q_{4231} := L_1 \otimes L_2^{-1}$ on $\mathrm{Pic}^0(C_{12}) \cong \mathrm{Jac}(C_{12})$. From one of these line bundles, say L_1, and these three points, we may recover the other line bundles L_2, L_3, L_4.

Proposition 3.13. *There exists a bijection between*

$$\left\{\begin{array}{l} G\text{-orbits of} \\ \text{nondegenerate} \\ \text{hypercubes} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \text{equivalence classes of } (C, L, Q_1, Q_2, Q_3) \text{ with } C \text{ a} \\ \text{genus 1 curve, } L \text{ a degree 2 line bundle on } C, \text{ and} \\ 0 \neq Q_i \in \mathrm{Jac}(C) \text{ for } 1 \leq i \leq 3 \text{ with } Q_1 + Q_2 + Q_3 = 0 \end{array}\right\}. \quad (3.15)$$

As always, (C, L, Q_1, Q_2, Q_3) and $(C', L', Q_1', Q_2', Q_3')$ are equivalent if there exists an isomorphism $\sigma : C \to C'$ such that the line bundles match under pullback, i.e., $\sigma^* L' \cong L$ and $\sigma^* Q_i' \cong Q_i$ when viewed as line bundles in the degree 0 Picard variety for $1 \leq i \leq 3$. Over an algebraically closed field, there is always a translation of C such that L can be taken to L', since translating C by $P \in \mathrm{Pic}^0(C)$ sends L to $L \otimes P^{\otimes 2}$. Such an automorphism would also preserve the points Q_i above, so it is not necessary to include the line bundle L in bijection (3.15). Note that other automorphisms of C, such as so-called flips, will not usually preserve all the points Q_i.

Finally, we may add bases to each side of bijection (3.14) to obtain a rigidified theorem. A hypercube $H \in V_1 \otimes V_2 \otimes V_3 \otimes V_4$ comes with a basis for the vector spaces V_i for $1 \leq i \leq 4$. On the other side, let $\mathfrak{D}$ be the rigidified data (C, L_1, L_2, L_3, L_4) and bases $\mathfrak{B}_i$ for $H^0(C, L_i)$ for $1 \leq i \leq 4$, with the conditions as in (3.14). Two such data $\mathfrak{D}$ and $\mathfrak{D}'$ are *equivalent* if there exists an isomorphism $\sigma : C \to C'$ such that for $1 \leq i \leq 4$, we have $\sigma^* L_i' \cong L_i$ and that $\sigma* : H^0(C', L_i') \to H^0(C, L_i)$ is an isomorphism taking $\mathfrak{B}_i'$ to $\mathfrak{B}_i$. Then the proof of Theorem 3.9, along with the identification of $(\ker \mu_{123})^\vee$ and $H^0(C, L_4)$ discussed in Remark 3.12, gives the following theorem:

Proposition 3.14. *There exists a bijection between nondegenerate hypercubes and equivalence classes of tuples $(C, L_1, L_2, L_3, L_4, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4)$, where C is a genus 1 curve; L_i is a degree 2 line bundle on C for $1 \leq i \leq 4$, with $\mathfrak{B}_i$ a basis for $H^0(C, L_i)$; and $L_1 \otimes L_2 \cong L_3 \otimes L_4$ and $L_i \ncong L_j$ for distinct $i \in \{1,2\}$ and $j \in \{1,2,3,4\}$.*

3.3.4 Families

In this section, we work exclusively with schemes over $\mathbb{Z}[\frac{1}{2}]$. We show that the bijections described in Proposition 3.14 and Theorem 3.9 hold in families, which implies that there are equivalences of the corresponding moduli stacks over $\mathbb{Z}[\frac{1}{2}]$. We first generalize the data on each side of the bijections to families over schemes S.

A *based hypercube over a scheme S* is four free rank 2 $\mathcal{O}_S$-modules $\mathcal{V}_i$ with isomorphisms $\psi_i : \mathcal{V}_i \xrightarrow{\cong} \mathcal{O}_S^{\oplus 2}$ for $1 \leq i \leq 4$ and a section $\mathcal{H}$ of $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \otimes \mathcal{V}_4$. An isomorphism of based hypercubes $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \psi_1, \psi_2, \psi_3, \psi_4, \mathcal{H})$ and $(\mathcal{V}_1', \mathcal{V}_2', \mathcal{V}_3', \mathcal{V}_4', \psi_1', \psi_2', \psi_3', \psi_4', \mathcal{H}')$ consists of isomorphisms $\sigma_i : \mathcal{V}_i \xrightarrow{\cong} \mathcal{V}_i'$ with $\psi_i = \psi_i' \circ \sigma_i$ for $1 \leq i \leq 4$ and taking $\mathcal{H}$ to $\mathcal{H}'$. A based hypercube is *nondegenerate* if it is locally nondegenerate.

On the other side, we defined in Section 2.3.3 a genus one curve C over S as a proper smooth morphism $\pi : C \to S$ with relative dimension 1 such that $\mathrm{R}^0\pi_*(\mathcal{O}_C) = \mathcal{O}_S$ and $\mathrm{R}^1\pi_*(\mathcal{O}_C)$ is a line bundle over S. Let the *rigidified quintuple* $\mathcal{D}$ over S consist of a genus one curve $\pi : C \to S$ and four degree 2 line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ on C, along with isomorphisms $\chi_i : \mathrm{R}^0\pi_*(\mathcal{L}_i) \xrightarrow{\cong} \mathcal{O}_S^{\oplus 2}$ for $1 \leq i \leq 4$. A *balanced* rigidified quintuple includes an isomorphism $\varphi : \mathcal{L}_1 \otimes \mathcal{L}_2 \xrightarrow{\cong} \mathcal{L}_3 \otimes \mathcal{L}_4 \otimes \pi^* L_S$ for some line bundle L_S on S. A *nondegenerate* balanced rigidified quintuple also satisfies $\mathrm{R}^0\pi_*(\mathcal{L}_i^\vee \otimes \mathcal{L}_j) = 0$ for distinct $i \in \{1,2\}$ and $j \in \{1,2,3,4\}$, which is equivalent to $\mathcal{L}_i$ and $\mathcal{L}_j$ being fiberwise nonisomorphic.

Theorem 3.15. *Over a scheme S, there is an equivalence between the category of nondegenerate based hypercubes over S and the category of nondegenerate balanced rigidified quintuples $(C, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \varphi)$ over S as defined above.*

Proof. Just like in Chapter 2, this relative version of the based bijection follows in a very straightforward manner from Proposition 3.14. The functors in both directions are the relative analogues of the ones before:

Suppose $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \psi_1, \psi_2, \psi_3, \psi_4, \mathcal{H})$ is a nondegenerate based hypercube over S. Then for $\{i, j, k, l\} = \{1, 2, 3, 4\}$, define curves $C_{ij} \in \mathbb{P}(\mathcal{V}_i^\vee) \times \mathbb{P}(\mathcal{V}_j^\vee)$ by the vanishing of the corresponding bidegree $(2, 2)$ form[3] over S, which is an element of $\operatorname{Sym}^2 \mathcal{V}_i \otimes \operatorname{Sym}^2 \mathcal{V}_j \otimes \wedge^2 \mathcal{V}_k \otimes \wedge^2 \mathcal{V}_l$. Locally on S, the curves C_{ij} are smooth irreducible genus one curves by assumption, so cohomology and base change implies that each C_{ij} is a genus one curve over S. We may similarly define the curves C_{ijk} over S and all the maps between these isomorphic curves. The pullback of $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{V}_i^\vee)$ to C_{12} in the most natural way, as in (3.7), gives the four degree 2 line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ on C_{12}.

By an analogous argument to Lemma 3.4, these bundles locally satisfy $\mathcal{L}_1 \otimes \mathcal{L}_2 \cong \mathcal{L}_3 \otimes \mathcal{L}_4$, that is, as elements of $\operatorname{Pic}(C_{12}/S)$, and we thus have an isomorphism $\varphi : \mathcal{L}_1 \otimes \mathcal{L}_2 \cong \mathcal{L}_3 \otimes \mathcal{L}_4 \otimes \pi^* L_S$ for some line bundle L_S on S, where $\pi : C_{12} \to S$. For $1 \leq i \leq 4$, the sections $\mathrm{R}^0 \pi_*(\mathcal{L}_i)$ are naturally isomorphic to $\mathcal{V}_i$, so composing these isomorphisms with the isomorphisms ψ_i gives isomorphisms $\mathrm{R}^0 \pi_*(\mathcal{L}_i) \xrightarrow{\cong} \mathcal{O}_S^{\oplus 2}$. In other words, from the based hypercube $\mathcal{H}$, we have produced a nondegenerate balanced rigidified quintuple.

Conversely, suppose $\pi : C \to S$ is a genus one curve over S and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ are degree 2 line bundles on C with $\chi_i : \mathrm{R}^0 \pi_*(\mathcal{L}_i) \xrightarrow{\cong} \mathcal{O}_S^{\oplus 2}$ for $1 \leq i \leq 4$. The map

$$\mu_{123} : \mathrm{R}^0 \pi_*(\mathcal{L}_1) \otimes \mathrm{R}^0 \pi_*(\mathcal{L}_2) \otimes \mathrm{R}^0 \pi_*(\mathcal{L}_3) \longrightarrow \mathrm{R}^0 \pi_*(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_3)$$

is surjective, and the kernel is a rank 2 free $\mathcal{O}_S$-module. Because a trivialization χ_4 of $\mathrm{R}^0 \pi_*(\mathcal{L}_4)$ gives a trivialization of the dual of the kernel of μ_{123} (by the same reasoning as Remark 3.12), we obtain a based hypercube in $\mathrm{R}^0 \pi_*(\mathcal{L}_1) \otimes \mathrm{R}^0 \pi_*(\mathcal{L}_2) \otimes \mathrm{R}^0 \pi_*(\mathcal{L}_3) \otimes (\ker \mu_{123})^\vee$.

These two constructions are locally inverse by Proposition 3.14, so they are inverse. □

Since there are no automorphisms of a based hypercube, the space of based hypercubes is just the scheme $\mathbb{A}^{16}$. So the moduli space of nondegenerate balanced rigidified quintuples over S is an open subscheme of $\mathbb{A}^{16}$ over S. Similarly to the case of Rubik's cubes, the equivalence above is equivariant for the action of $G = \mathrm{GL}_2^4$, which induces an equivalence of the quotient stacks.

[3] We have not defined $(2, 2)$ forms over an arbitrary base scheme and taking values in a line bundle, but the theory is entirely analogous to that of other forms, e.g., for ternary cubic forms as in Appendix 2.A.2.

A *hypercube over* S is four locally free rank 2 $\mathcal{O}_S$-modules $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4$ and a section $\mathcal{H}$ of the tensor product $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \otimes \mathcal{V}_4$. Such a hypercube (up to the obvious notion of isomorphism) is an S-point of the quotient stack $[\mathbb{A}^{16}/\mathrm{GL}_2^4]$, and we are interested in its nondegenerate open substack.

The geometric data may be described as before, without trivializations for the sections of the line bundles. Let $\mathcal{Y}_{2222}$ be the stack whose S-points consists of $(C, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \varphi)$, where $\pi : C \to S$ is a genus one curve over S; $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ are degree 2 line bundles on C; and $\varphi : \mathcal{L}_1 \otimes \mathcal{L}_2 \xrightarrow{\cong} \mathcal{L}_3 \otimes \mathcal{L}_4 \otimes \pi^* L_S$ is an isomorphism for some line bundle L_S on S, with the condition that $\mathrm{R}^0\pi_*(\mathcal{L}_i^\vee \otimes \mathcal{L}_j) = 0$ for distinct $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$.

Theorem 3.16. *The nondegenerate open substack of* $[\mathbb{A}^{16}/\mathrm{GL}_2^4]$ *is equivalent to the stack* $\mathcal{Y}_{2222}$ *of nondegenerate balanced quintuples as defined above.*

The analysis of Appendix 2.A.3 applied to the moduli stack $\mathcal{Y}_{2222}$ gives an interpretation in terms of torsors for elliptic curves and line bundles. For an elliptic curve E, recall that the elements of $H^1_{\mathrm{f}}(S, \Theta_{E,2})$ are isomorphism classes of pairs (C, L) for a genus one curve C over S with an isomorphism $E \xrightarrow{\cong} \mathrm{Aut}^0(C)$ and L a degree 2 line bundle on C. A quintuple $(C, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4)$ over S corresponds to four elements $\eta_1, \eta_2, \eta_3, \eta_4 \in H^1_{\mathrm{f}}(S, \Theta_{\mathrm{Jac}(C),2})$ with $\alpha(\gamma(\eta_i)) \in H^1_{\mathrm{f}}(S, \mathrm{Jac}(C))$ the same element for all $1 \leq i \leq 4$. In other words, the quintuple, up to equivalence, is an S-point of the fiber product

$$\mathcal{X} := [\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},2}] \times_{\mathscr{M}_1} [\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},2}] \times_{\mathscr{M}_1} [\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},2}] \times_{\mathscr{M}_1} [\mathscr{M}_{1,1}/\Theta_{E^{\mathrm{univ}},2}].$$

The isomorphism φ translates into requiring that $\sum_{i=1}^4 \gamma(\eta_i) = 0$ in $H^1_{\mathrm{f}}(S, \mathrm{Jac}(C)[2])$, and the nondegeneracy condition is equivalent to requiring that $\eta_i \neq \eta_j$ for $1 \leq i \neq j \leq 4$.

Corollary 3.17. *The stack* $\mathcal{Y}_{2222}$*, and by Theorem 3.16, the stack of nondegenerate hypercubes, is equivalent to the kernel substack of the natural addition map*

$$\mathcal{X} \setminus \Delta \to [\mathscr{M}_{1,1}/E^{\mathrm{univ}}[2]],$$

i.e., the fiber over the identity section $\mathscr{M}_{1,1} \to [\mathscr{M}_{1,1}/E^{\mathrm{univ}}[2]]$*, where* Δ *denotes the big diagonal of* $\mathcal{X}$.

3.4 Symmetrized Hypercubes

The bijections for the space of hypercubes may be used to construct bijections for related spaces of *symmetrized hypercubes.* Just as in Section 2.4, these spaces analogously parametrize curves and line bundles. The geometric data will include additional restrictions, especially on the line bundles, as the symmetrized spaces are naturally subsets of the space of hypercubes.

The first example is the space of *doubly symmetrized hypercubes*, which is the vector space $V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_3$ for 2-dimensional F-vector spaces V_1, V_2, and V_3 with specified bases. An element of this space may be thought of as a 2×2 matrix of binary quadratic forms of the form $ax^2 + 2bxy + cy^2$, and the group $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$ acts in the standard way on this space. Because there is an inclusion

$$V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_3 \hookrightarrow V_1 \otimes V_2 \otimes V_3 \otimes V_3, \tag{3.16}$$

nondegenerate doubly symmetrized hypercubes are in bijection with a subset of the space of tuples $(C, L_1, L_2, L_3, L_4, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4)$ from Proposition 3.14. The additional condition will be that the line bundles L_3 and L_4 are the same.

Proposition 3.18. *The restriction of Proposition 3.14 to doubly symmetrized hypercubes gives a bijection between nondegenerate doubly symmetrized hypercubes and equivalence classes of tuples $(C, L_1, L_2, L_3, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)$ where C is a genus one curve; L_i is a degree 2 line bundle on C for $1 \leq i \leq 3$, with $\mathfrak{B}_i$ a basis for $H^0(C, L_i)$; and $L_1 \otimes L_2 \cong L_3^{\otimes 2}$ and L_3 is not isomorphic to L_1 or L_2.*

Proof. A doubly symmetrized hypercube in $V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_3$ may be viewed as a hypercube in $V_1 \otimes V_2 \otimes V_3 \otimes V_3$ by the injection (3.16). Because the line bundles L_3 and L_4, defined in (3.7) as pullbacks from $\mathbb{P}(V_3^\vee)$ to the curve C_{12}, are visibly the same, a doubly symmetrized hypercube gives rise to the usual data (C, L_1, L_2, L_3, L_3), subject to the relation $L_1 \otimes L_2 \cong L_3 \otimes L_3$.

Starting with the "symmetrized" data $(C, L_1, L_2, L_3, L_4, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4)$, where $L_3 = L_4$ and $\mathfrak{B}_3 = \mathfrak{B}_4$, we may construct a nondegenerate hypercube in two different ways, using either the line bundles (L_1, L_2, L_3) or (L_1, L_2, L_4). These two constructions Ψ_{123} and Ψ_{124} produce hypercubes $H \in H^0(C, L_1) \otimes H^0(C, L_2) \otimes H^0(C, L_3) \otimes (\ker \mu_{123})^\vee$ and $H' \in H^0(C, L_1) \otimes H^0(C, L_2) \otimes H^0(C, L_4) \otimes (\ker \mu_{124})^\vee$, respectively. Since the spaces $H^0(C, L_i)$ and $(\ker \mu_{12j})^\vee$ may be identified for $\{i, j\} = \{3, 4\}$ by Remark 3.12, the 2×2 matrices $H(w, x, \cdot, \cdot)$ and $H'(w, x, \cdot, \cdot)$ are transpose to one another.

These two matrices are also equal, since by assumption, the line bundles L_3 and L_4 are the same, with their bases identified. Therefore, all such matrices $H(w, x, \cdot, \cdot)$ are symmetric matrices, i.e., the hypercube is doubly symmetrized and an element of $H^0(C, L_1) \otimes H^0(C, L_2) \otimes \mathrm{Sym}_2 H^0(C, L_3)$. □

Taking $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$-equivalence classes gives a bijection between nondegenerate unbased hypercubes (elements of $V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_3$) and the geometric data (C, L_1, L_2, L_3). In terms of the points on the Jacobian, as in Proposition 3.13, the points $Q_{4123} := L_3 \otimes L_1^{-1}$ and $Q_{4312} := L_2 \otimes L_3^{-1}$ are the same. So we have

Corollary 3.19. *The restriction of bijection* (3.15) *to doubly symmetrized hypercubes produces the bijection*

$$\left\{\begin{matrix} \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2\textit{-orbits} \\ \textit{of nondegenerate doubly} \\ \textit{symmetrized hypercubes} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{equivalence classes of } (C, L, Q_1, Q_2) \textit{ with } C \textit{ a} \\ \textit{genus } 1 \textit{ curve, } L \textit{ a degree } 2 \textit{ line bundle on } C, \textit{ and} \\ 0 \neq Q_i \in \mathrm{Jac}(C) \textit{ for } 1 \leq i \leq 2 \textit{ with } Q_1 + 2Q_2 = 0 \end{matrix}\right\}.$$

A similar analysis will show that other symmetrizations also parametrize the same sort of curves and line bundles. A *triply symmetrized hypercube* is an element of $V_1 \otimes \mathrm{Sym}_3 V_2$, where V_1 and V_2 are two-dimensional F-vector spaces with specified bases. Again, the space of triply symmetrized hypercubes embeds into the space of hypercubes, so the geometric data parametrized will be similar to that for hypercubes.

Proposition 3.20. *Restricting Proposition 3.14 to triply symmetrized hypercubes gives a bijection between nondegenerate triply symmetrized hypercubes and equivalence classes of quintuples $(C, L_1, L_2, \mathfrak{B}_1, \mathfrak{B}_2)$, where C is a genus one curve, L_i is a degree 2 line bundle on C with basis $\mathfrak{B}_i$ for $H^0(C, L_i)$ for $1 \leq i \leq 2$, and $L_1 \otimes L_2^{-1}$ is a nonzero 3-torsion line bundle in $\mathrm{Pic}^0(C)$.*

Proof. From a nondegenerate triply symmetrized hypercube

$$H \in V_1 \otimes \mathrm{Sym}_3 V_2 \hookrightarrow V_1 \otimes V_2 \otimes V_2 \otimes V_2,$$

we obtain the usual curve C_{12} and line bundles L_1, L_2, L_3, L_4 on it as in (3.7). We think of V_3 and V_4 as equal to V_2. It is immediately clear that $L_3 \cong L_4$ and their spaces of sections may be identified. We consider the line bundles and relations in the four-cycle (3.10). In particular, we have the extra relation $M_3[C_{134}] \cong M_2[C_{124}]$ because of the symmetry of the hypercube; this relation reflects the symmetry in pulling back $\mathcal{O}(1)$ from $\mathbb{P}(V_3^\vee)$ and $\mathbb{P}(V_2^\vee)$, respectively, to C_{14} as directly as possible. This additional relation gives the isomorphism $L_1^{\otimes 2} \cong L_2 \otimes L_3$, which implies

$$L_1^{\otimes 3} \cong L_2^{\otimes 3} \cong L_3^{\otimes 3}.$$

Therefore, we have a genus one curve C_{12} with two line bundles L_1 and L_2 that differ by a 3-torsion point of $\mathrm{Pic}^0(C_{12})$.

On the other hand, suppose we have $(C, L_1, L_2, \mathfrak{B}_1, \mathfrak{B}_2)$ as in the proposition. By Proposition 3.2, this data gives a $(2,2)$ form $f \in \mathrm{Sym}^2(H^0(C, L_1)) \otimes \mathrm{Sym}^2(H^0(C, L_2))$. Let $L_i := L_1^{\otimes 2} \otimes L_2^{-1} \cong L_2^{\otimes 2} \otimes L_1^{-1}$ for $i = 3$ or 4. To obtain a natural basis for $H^0(C, L_3)$, we use the $(2,2)$ form f. Because $L_1^{\otimes 2} \cong L_2 \otimes L_3$, there is an automorphism σ of C fixing L_1 and sending L_2 to L_3 (we have called this automorphism a "flip" around L_1). Thus, the triple (C, L_1, L_3) corresponds to the $\mathrm{GL}_2 \times \mathrm{GL}_2$-orbit of f, so there exists a basis $\mathfrak{B}_3$ of $H^0(C, L_3)$ such that $(C, L_1, L_3, \mathfrak{B}_1, \mathfrak{B}_3)$ gives the $(2,2)$ form f as well, with the identification of $H^0(C, L_2)$ and $H^0(C, L_3)$ via σ^*. Then we may construct a hypercube H from the data $(C, L_1, L_2, L_3, L_4, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4)$, where $\mathfrak{B}_3 = \mathfrak{B}_4$.

Let $V_i := H^0(C, L_i)$ with the basis $\mathfrak{B}_i$, for $1 \leq i \leq 4$. Then there is a natural identification $V_3 = V_4$ that respects the bases, and by construction, σ induces an isomorphism of V_2 with V_3 and V_4 such that the bases coincide. By the argument in the proof of Proposition 3.18, the hypercube

$$H \in V_1 \otimes V_2 \otimes V_3 \otimes V_4 \cong V_1 \otimes V_2 \otimes V_2 \otimes V_2$$

is doubly symmetrized; that is, $H \in V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_2$ is invariant under the transposition (23) of $\mathbb{S}_3$ acting on the three factors of V_2.

We may use a similar argument to show that H is invariant under the transpositions (12) and (13) of $\mathbb{S}_3$ as well. The key idea is to construct other curves $C_{ij} \in \mathbb{P}(V_i^\vee) \times \mathbb{P}(V_j^\vee)$ from the hypercube H, and recover the same hypercube by the functor Ψ_{ijk} for some $k \neq i, j$. Using the argument from Proposition 3.18 for bundles on C_{13} (resp., C_{14}) instead of C_{12} shows that the hypercube H is doubly symmetrized with respect to V_2 and V_4 (resp., V_3). Therefore, the hypercube H is invariant under all of $\mathbb{S}_3$ acting on the three factors of V_2, so it is a triply symmetrized hypercube in $V_1 \otimes \mathrm{Sym}_3 V_2$. $\square$

Quotienting each side of the bijection in Proposition 3.20 by the natural action of $\mathrm{GL}_2 \times \mathrm{GL}_2$ gives a bijection between nondegenerate triply symmetrized hypercubes, up to changes of bases, and (C, L_1, L_2) with $L_1 \otimes L_2^{-1}$ a nonzero 3-torsion point of $\mathrm{Pic}^0(C) \cong \mathrm{Jac}(C)$. This geometric data is also just (C, L_1, Q) where $Q := L_1 \otimes L_2^{-1}$. Note that each of the three points on the Jacobian that sum to zero in Proposition 3.13 is the same 3-torsion point Q here, since the line bundles arising from a triply symmetrized hypercube satisfy

$$L_3 \otimes L_1^{-1} \cong L_2 \otimes L_3^{-1} \cong L_1 \otimes L_2^{-1}.$$

Corollary 3.21. *The restriction of the bijection in Proposition 3.13 to triply symmetrized hypercubes gives the bijection*

$$\left\{\begin{array}{l}\mathrm{GL}_2 \times \mathrm{GL}_2\text{-}orbits\ of \\ nondegenerate\ triply \\ symmetrized\ hypercubes\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}equivalence\ classes\ of\ (C, L, Q)\ with\ C\ a \\ genus\ 1\ curve,\ L\ a\ degree\ 2\ line\ bundle \\ on\ C,\ and\ 0 \neq Q \in \mathrm{Jac}(C)[3]\end{array}\right\}.$$

Another obvious symmetrization is to consider the space $\mathrm{Sym}_4 V$ for a 2-dimensional vector space V over F with basis $\{X, Y\}$. A *quadruply symmetrized hypercube* may be represented as a binary quartic form of the form

$$aX^4 + 4bX^3Y + 6cX^2Y^2 + 4dXY^3 + eY^4,$$

with the obvious GL_2 action on the basis of V. For such a hypercube, the $(2,2)$ forms $f_{ij} \in \mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 V$ are all identical, but the most direct maps between the curves C_{ij} are not the identity maps.

Example 3.22. Let H be the quadruply symmetrized hypercube represented by the quartic $h = aX^4 + 4bX^3Y + 6cX^2Y^2 + 4dXY^3 + eY^4$. All the binary quartics q_i coming from H are

$$
\begin{aligned}
(-3b^2c^2 + 4ac^3 + 4b^3d - 6abcd + a^2d^2)X^4 \\
&+ (-2bc^3 + 6ac^2d - 4abd^2 + 4b^3e - 6abce + 2a^2de)X^3Y \\
&+ (-3c^4 + 6bc^2d - 8b^2d^2 + 6acd^2 + 6b^2ce - 6ac^2e - 2abde + a^2e^2)X^2Y^2 \\
&+ (-2c^3d + 4ad^3 + 6bc^2e - 4b^2de - 6acde + 2abe^2)XY^3 \\
&+ (-3c^2d^2 + 4bd^3 + 4c^3e - 6bcde + b^2e^2)XY^4,
\end{aligned}
$$

which is a linear combination of the quartic h and the Hessian of h:

$$q_i = \frac{I(h)}{12}\mathrm{Hessian}(h) + \frac{J(h)}{432}h.$$

Moreover, since the invariants $I(q_i)$ and $J(q_i)$ of q_i may be written in terms of the invariants $I(h)$ and $J(h)$ of h, the discriminant of the hypercube H is a multiple of the discriminant of q, specifically

$$\Delta(H) = \Delta(q_i) = \frac{J(h)^6\Delta(h)}{2^{24}3^{18}}.$$

So the hypercube H is nondegenerate if and only if h has no repeated roots and $J(h) \neq 0$.

From the hypercube H, we may construct the usual $(2,2)$ forms f_{ij} and curves $C_{ij} \in \mathbb{P}(V_i^\vee) \times \mathbb{P}(V_j^\vee)$. As noted earlier, the forms and the curves are all exactly the same, but the maps between the curves will not be the identity. The same argument as for Lemma 3.3 shows, for example, if $(w, x_1) \in C_{12}$, then τ_{12}^{13} sends (w, x_1) to (w, x_2), where x_1 and x_2 are the solutions for x in $f_{12}(w, x) = f_{13}(w, x) = 0$. It is evident, then, that the line bundles $L_2 := (\pi_2^1)^*\mathcal{O}_{\mathbb{P}(V_2^\vee)}(1)$ and $L_3 := (\pi_3^2 \circ \pi_{23}^1 \circ \rho_{12}^3)^*\mathcal{O}_{\mathbb{P}(V_3^\vee)}(1)$ on C_{12} will not be the same bundles, despite the supposed symmetry. In particular, since the map between C_{12} and C_{13} flips the sheets covering $\mathbb{P}(V_1^\vee)$, the line bundles L_2 and L_3 will be symmetric around L_1, i.e., $L_1^{\otimes 2} \cong L_2 \otimes L_3$. (Since a quadruply symmetrized hypercube is triply symmetrized, this relation is already known by the proof of Proposition 3.20.)

Proposition 3.23. *Restricting Proposition 3.14 to quadruply symmetrized hypercubes gives a bijection between nondegenerate quadruply symmetrized hypercubes and equivalence classes of quadruples $(C, L, Q, \mathfrak{B})$, where C is a genus one curve, L is a degree 2 line bundle on C, Q is a nonzero 3-torsion point on $\mathrm{Jac}(C)$, and $\mathfrak{B}$ is a basis for $H^0(C, L)$.*

Proof. Because a quadruply symmetrized hypercube H is also triply symmetrized, Proposition 3.20 shows that H gives rise to $(C, L, Q, \mathfrak{B})$. In particular, let L be the line bundle we usually call L_1 and Q correspond to $L_1 \otimes L_2^{-1} \in \mathrm{Pic}^0(C) \cong \mathrm{Jac}(C)$.

Conversely, given $(C, L, Q, \mathfrak{B})$, define the line bundles $L_1 := L, L_2 := L \otimes Q$, and $L_3 = L_4 := L \otimes Q^{-1}$, where Q is thought of as a degree 0 line bundle in $\mathrm{Pic}^0(C)$. These satisfy the conditions to construct a hypercube; we only need bases for each of the spaces of sections. For the moment, choose any basis $\mathfrak{B}_3'$ for $H^0(C, L_3)$. Then there exists a basis $\mathfrak{B}_2$ for $H^0(C, L_2)$ such that the $(2,2)$ form f_{13} (by Proposition 3.2, corresponding to $(C, L_1, L_3, \mathfrak{B}, \mathfrak{B}_3'))$ and the $(2,2)$ form f_{23} (from $(C, L_2, L_3, \mathfrak{B}_2, \mathfrak{B}_3'))$ are the same polynomials. This follows from Proposition 3.2 and the existence of an automorphism of C sending (L_1, L_3) to (L_2, L_3), namely translation by Q followed by a flip around the line bundle L_1. As in the proof of Proposition 3.20, we may apply an analogous argument to obtain bases $\mathfrak{B}_3$ and $\mathfrak{B}_4$ for $H^0(C, L_3)$ and $H^0(C, L_4)$, respectively, such that the $(2,2)$ forms f_{1i} (corresponding to (C, L_1, L_i)) for $2 \leq i \leq 4$ are all the same. The vector spaces $H^0(C, L_i)$ for $2 \leq i \leq 4$ are all identified with these bases. Then the hypercube H constructed is triply symmetrized in the indices $\{2, 3, 4\}$, that is, it can be viewed as an element of $H^0(C, L_1) \otimes \mathrm{Sym}_3 H^0(C, L_2)$.

Since the roles of L_1 and L_2 may be switched in this construction by an automorphism of C, there also exists an identification of $H^0(C, L_1)$ with $H^0(C, L_3)$ that respects the bases $\mathfrak{B}$ and $\mathfrak{B}_3$, such that the hypercube H constructed is triply symmetrized in the indices $\{1, 3, 4\}$. We may thus identify all $H^0(C, L_i)$ for $1 \leq i \leq 4$ with respect to the chosen bases. As the permutation group $\mathbb{S}_4$ is generated by two conjugate subgroups $\mathbb{S}_3$, the hypercube H is actually quadruply symmetrized. In other words, for $V = H^0(C, L_1)$ with basis $\mathfrak{B}$, the hypercube H may be considered as an element of $V \otimes V \otimes V \otimes V$ that is invariant under the symmetric group $\mathbb{S}_4$ acting on the four factors of V. $\square$

Quotienting by GL_3 in Proposition 3.23 gives the following correspondence:

Corollary 3.24. *The restriction of the bijection in Proposition 3.13 to quadruply symmetrized hypercubes gives the bijection*

$$\left\{\begin{array}{l}\mathrm{GL}_2\textit{-orbits of nondegener-}\\ \textit{ate quadruply symmetrized}\\ \textit{hypercubes}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{equivalence classes of } (C, L, Q) \textit{ with } C \textit{ a}\\ \textit{genus } 1 \textit{ curve, } L \textit{ a degree } 2 \textit{ line bundle}\\ \textit{on } C, \textit{ and } 0 \neq Q \in \mathrm{Jac}(C)[3]\end{array}\right\}.$$

Consequently, triply and quadruply symmetrized hypercubes, without bases for the corresponding vector spaces, parametrize exactly the same geometric data: a genus one curve C, a degree 2 line bundle L on C (which may be "forgotten" over an algebraically closed field), and a nonzero 3-torsion point on $\mathrm{Jac}(C)$. So a triply symmetrized hypercube may be taken to a quadruply symmetrized one by a change of basis!

Finally, we describe one more type of symmetrization. A $(2,2)$-*symmetrized hypercube* is an element of $\mathrm{Sym}_2 V_1 \otimes \mathrm{Sym}_2 V_3$, where V_1 and V_3 are 2-dimensional vector spaces over F with specified bases. These $(2,2)$-symmetrized hypercubes may be seen as a subset of all $(2,2)$ forms, specifically those polynomials of the form

$$\begin{aligned} f(w_1, w_2, x_1, x_2) = a_{22} w_1^2 x_1^2 &+ 2a_{32} w_1 w_2 x_1^2 + a_{42} w_2^2 x_1^2 \\ &+ 2(a_{23} w_1^2 x_1 x_2 + 2a_{33} w_1 w_2 x_1 x_2 + a_{43} w_2^2 x_1 x_2) \\ &+ a_{24} w_1^2 x_2^2 + 2a_{34} w_1 w_2 x_2^2 + a_{44} w_2^2 x_2^2. \end{aligned}$$

There is a natural action of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_3)$ on these hypercubes, which are a subset of hypercubes in $V_1 \otimes V_1 \otimes V_3 \otimes V_3$.

Proposition 3.25. *Restricting Proposition 3.14 to* $(2,2)$*-symmetrized hypercubes gives a bijection between nondegenerate* $(2,2)$*-symmetrized hypercubes and equivalence classes of quintuples* $(C, L_1, L_3, \mathfrak{B}_1, \mathfrak{B}_3)$ *with* C *a genus one curve,* L_1 *and* L_3 *degree* 2 *line bundles on* C *with* $L_1 \not\cong L_3$*, and* $\mathfrak{B}_1$ *and* $\mathfrak{B}_3$ *bases for* $H^0(C, L_1)$ *and* $H^0(C, L_3)$*, respectively.*

Proof. A $(2,2)$-symmetrized hypercube is a doubly symmetrized hypercube, say in $V_1 \otimes V_1 \otimes \mathrm{Sym}_2 V_3$. By Proposition 3.18, such a hypercube gives rise to the data $(C, L_1, L_3, \mathfrak{B}_1, \mathfrak{B}_3)$.

On the other hand, suppose we have $(C, L_1, L_3, \mathfrak{B}_1, \mathfrak{B}_3)$ as in the proposition. Then let $L_2 := L_3^{\otimes 2} \otimes L_1^{-1}$ and $L_4 := L_3$, and let $\mathfrak{B}_4$ be the same basis as $\mathfrak{B}_3$. By Proposition 3.18, with any choice of basis for $H^0(C, L_2)$, we obtain a doubly symmetrized hypercube, invariant with respect to switching the indices 3 and 4. To produce the other symmetry, choose a basis $\mathfrak{B}_2$ for $H^0(C, L_2)$ by requiring that $(C, L_1, L_3, \mathfrak{B}_1, \mathfrak{B}_3)$ and $(C, L_2, L_3, \mathfrak{B}_2, \mathfrak{B}_3)$ give the same $(2,2)$ forms f_{13} and f_{23} via Proposition 3.2. In other words, because there is an automorphism of C taking L_1 to L_2 and fixing L_3, the data (C, L_1, L_3) and (C, L_2, L_3) corresponds to the same $(2,2)$-form up to a change of basis. The spaces of sections $H^0(C, L_1)$ and $H^0(C, L_2)$ may be identified with respect to the bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Swapping the roles of L_1 and L_2 in the construction of the hypercube also gives the same hypercube, so it is invariant under switching the indices 1 and 2. Thus, we have constructed a hypercube invariant under the subgroup $\langle (12), (34) \rangle$ of $\mathbb{S}_4$, which is a $(2,2)$-symmetrized hypercube. $\square$

Without choices of bases, the nondegenerate $(2,2)$-symmetrized hypercubes correspond to the same data as doubly symmetrized hypercubes.

Corollary 3.26. *The restriction of bijection 3.15 to $(2,2)$-symmetrized hypercubes gives a bijection*

$$\left\{ \begin{array}{l} \mathrm{GL}_2 \times \mathrm{GL}_2\text{-orbits of nonde-} \\ \text{generate } (2,2)\text{-symmetrized} \\ \text{hypercubes} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{equivalence classes of } (C, L, Q_1, Q_2) \text{ with } C \text{ a} \\ \text{genus } 1 \text{ curve, } L \text{ a degree } 2 \text{ line bundle on } C\text{, and} \\ 0 \neq Q_i \in \mathrm{Jac}(C) \text{ for } 1 \leq i \leq 2 \text{ with } Q_1 + 2Q_2 = 0 \end{array} \right\}.$$

In this section, we investigated the moduli problems corresponding to hypercubes invariant under certain subgroups of the permutation group $\mathbb{S}_4$ acting on the four factors of $V^{\otimes 4}$ for a 2-dimensional vector space V over F. In particular, the subgroups in this section are all conjugates of the subgroups $\mathbb{S}_2$, $\mathbb{S}_3$, $\mathbb{S}_4$, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in $\mathbb{S}_4$. Invariance under other subgroups of $\mathbb{S}_4$ gives rise to reducible representations. While their moduli problems can be considered a product of those already considered and trivial ones, we plan to treat other aspects of these sorts of symmetrized hypercubes in future work.

A box without hinges, key, or lid,
Yet golden treasure inside is hid.

—Bilbo Baggins, in *The Hobbit*
by J.R.R. Tolkein

Chapter 4

Moduli of Plane Curves

To describe the moduli problem for Rubik's cubes in Chapter 2, we study the three ternary cubic forms arising from each such $3 \times 3 \times 3$ box. More generally, a $3 \times n \times n$ box naturally produces a ternary degree n form, whose locus in $\mathbb{P}^2$ is a genus $\frac{1}{2}(n-1)(n-2)$ curve, if smooth. In this chapter, we show that generically the space of $3 \times n \times n$ boxes over F, up to linear transformations, parametrizes plane curves with certain types of line bundles. Although $3 \times n \times n$ boxes are closely related to determinantal representations of plane curves, which have been studied classically (see [CT79, Bea00] for more recent work), our geometric techniques are different and generalize to curves in $\mathbb{P}^1 \times \mathbb{P}^1$ in the next chapter. In addition, all of these results generalize to families of curves as well as to the corresponding moduli stacks. We also obtain a parametrization of certain curves and line bundles by the space of symmetrized $3 \times n \times n$ boxes. For $n \geq 5$ odd, the bijection for symmetrized boxes is related to the isomorphism, studied in [Tju75, Rei72], between the Prym variety of a certain étale double cover of the degree n plane curve and the intermediate Jacobian of a related $(n-4)$-dimensional variety.

Preliminaries. Let $n \geq 3$ be an integer. Let F be an algebraically closed field of characteristic not 3 and not dividing n or $n-1$. In this chapter, we use the convention that the projectivization of a vector bundle parametrizes lines, not hyperplanes, and for a basepoint-free line bundle L on a scheme X, the map $\phi_L : X \to \mathbb{P}(H^0(X, L)^\vee)$ is the natural map given by the complete linear system $|L|$. Also, unless stated otherwise, a genus g curve means a proper, smooth, geometrically connected curve with arithmetic genus g.

4.1 Trilinear Forms and Associated Curves

Let U_1, U_2, and U_3 be vector spaces over F of dimensions 3, n, and n, respectively. The reductive group $G := \mathrm{GL}(U_1) \times \mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$ acts on the tensor product $U_1 \otimes U_2 \otimes U_3$, the space of trilinear forms, by the natural action on each factor.

With choices of bases for the vector spaces U_1, U_2, U_3, we may represent an element of $U_1 \otimes U_2 \otimes U_3$ as a $3 \times n \times n$ box $\mathcal{A} = (a_{rst})_{1 \le r \le 3, 1 \le s,t \le n}$ for $a_{rst} \in F$. The box $\mathcal{A}$ is like a three-dimensional matrix, and the group G acts by row, column, and "other direction" operations on the space of boxes. In the sequel, we will refer to both the array and the trilinear form as the box, with the vector spaces U_i and their bases understood.

We use the notation $\mathcal{A}(\cdot,\cdot,\cdot)$ to denote the trilinear form, where the dots may be replaced by substituting elements of the respective $U_i^\vee$. For example, given an element $w \in U_1^\vee$, the notation $\mathcal{A}(w,\cdot,\cdot)$ will refer to the $n \times n$ matrix $\mathcal{A} \lrcorner\, w \in U_2 \otimes U_3$. The notation $\mathcal{A}(w,x,\cdot)$ for $x \in U_2^\vee$ refers to the vector $\mathcal{A} \lrcorner\, (w \otimes x) \in U_3$. We also, by a slight abuse of notation, will use conditions such as whether $\mathcal{A}(w,\cdot,\cdot)$ vanishes or not for $w \in \mathbb{P}(V_1^\vee)$.

Let $\mathcal{A} = (a_{rst})$ be a $3 \times n \times n$ box. Then the vanishing of the degree n polynomial

$$f(w_1, w_2, w_3) := \det \mathcal{A}(w,\cdot,\cdot) \in \mathrm{Sym}^n U_1$$

defines a variety $C_1 \subset \mathbb{P}(U_1^\vee) = \mathbb{P}^2$. In other words, the variety C_1 is a determinantal variety, given by the determinant of a matrix of linear forms on $U_1^\vee$. By definition, the zero locus of f (and thus the curve C_1) is fixed under the action of $\mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$, and the group $\mathrm{GL}(U_1)$ acts as linear transformations on $\mathbb{P}(U_1^\vee)$ by the standard action on U_1.

We call a $3 \times n \times n$ box $\mathcal{A}$ *nondegenerate* if the variety C_1 thus defined is smooth and one-dimensional. Note that the nondegeneracy of the box is a single algebraic condition on the entries a_{rst}, which we will call the *discriminant*, and degenerate boxes form a codimension one subvariety given by the vanishing of this discriminant. For $n = 3$, this notion of nondegeneracy coincides with that for Rubik's cubes defined in Section 2.2.1, i.e., the discriminant here is the degree 36 discriminant of a Rubik's cube. In the sequel, we will only consider nondegenerate boxes.

If $\mathcal{A}$ is nondegenerate, the degree n curve C_1 is smooth of genus $g = \frac{1}{2}(n-1)(n-2)$. For all points $w^\dagger \in C_1$, we claim that the singular matrix $\mathcal{A}(w^\dagger, \cdot, \cdot)$ has exactly rank $n-1$. If not, then the $(n-1) \times (n-1)$ minors of $\mathcal{A}(w, \cdot, \cdot)$ would vanish on $w^\dagger$, and so would all the partial derivatives

$$\left.\frac{\partial f}{\partial w_i}\right|_{w=w^\dagger} = \sum_{s,t} a_{ist} A^*_{ij}(w^\dagger)$$

where $A^*_{ij}(w^\dagger)$ is the (i,j)th $(n-1) \times (n-1)$ minor of $\mathcal{A}(w^\dagger, \cdot, \cdot)$. Since C_1 was assumed to be smooth, however, the rank of the matrix $\mathcal{A}(w^\dagger, \cdot, \cdot)$ cannot drop by two.

Define the variety

$$C_{12} := \{(w, x) \in \mathbb{P}(U_1^\vee) \times \mathbb{P}(U_2^\vee) : \mathcal{A}(w, x, \cdot) = 0\}.$$

The nondegeneracy of $\mathcal{A}$ implies that the projection

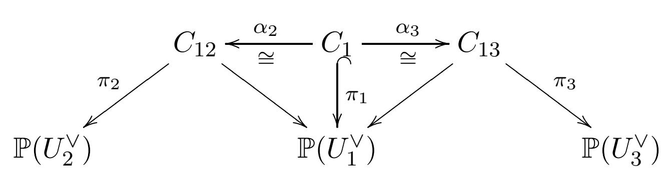
$$C_{12} \longrightarrow \mathbb{P}(U_1^\vee)$$

is an isomorphism onto C_1. The inverse map takes a point $w \in C_1 \subset \mathbb{P}(U_1^\vee)$ to the pair $(w, x) \in \mathbb{P}(U_1^\vee) \times \mathbb{P}(U_2^\vee)$, where x corresponds to the exactly one-dimensional kernel of the linear map $\mathcal{A}(w, \cdot, \cdot) \in U_2 \otimes U_3 \cong \mathrm{Hom}(U_2^\vee, U_3)$. This map $C_1 \to C_{12}$ is algebraic, as the kernel is given by the minors of the matrix $\mathcal{A}(w, \cdot, \cdot)$. Therefore, by dimension considerations, the curve C_{12} is the complete intersection of n bidegree $(1,1)$ forms on $\mathbb{P}(U_1^\vee) \times \mathbb{P}(U_2^\vee) = \mathbb{P}^2 \times \mathbb{P}^{n-1}$.

We may similarly define

$$C_{13} := \{(w, y) \in \mathbb{P}(U_1^\vee) \times \mathbb{P}(U_3^\vee) : \mathcal{A}(w, \cdot, y) = 0\},$$

and the analogous maps between C_{13} and C_1 are also isomorphisms. The diagram



summarizes the relationships between these curves. These maps from the curve C_1 to each projective space give the line bundles

$$\begin{aligned} L_1 &:= \pi_1^* \mathcal{O}_{\mathbb{P}(U_1^\vee)}(1) \\ L_2 &:= (\pi_2 \circ \alpha_2)^* \mathcal{O}_{\mathbb{P}(U_2^\vee)}(1) \\ L_3 &:= (\pi_3 \circ \alpha_3)^* \mathcal{O}_{\mathbb{P}(U_3^\vee)}(1) \end{aligned} \tag{4.1}$$

on C_1. The line bundle L_1 clearly has degree n, and L_2 and L_3 have degree $\frac{1}{2}n(n-1)$. For $1 \leq i \leq 3$, we claim that all the sections of the bundle L_i arise from pulling back sections from $\mathbb{P}(U_i^\vee)$. This is a small generalization of the idea that a complete intersection in projective space is linearly normal; here, the curve C_{1i} for $i = 2$ or 3 is a complete intersection in the product of two projective spaces, and the image of projection to either of those is linearly normal.

Lemma 4.1. *Let $\mathcal{A}$ be a $3 \times n \times n$ box and L_1, L_2, and L_3 the line bundles on the curve C_1 defined in* (4.1)*. Then*

(i) $h^0(C_1, L_1) = 3$,

(ii) $h^0(C_1, L_i) = n$ *and* $h^1(C_1, L_i) = 0$ *for* $i = 2$ *or* 3,

(iii) $h^1(C_1, L_1 \otimes L_i) = 0$ *for* $i = 2$ *or* 3, *and*

(iv) $h^1(C_1, L_1^{-1} \otimes L_i) = 0$ *for* $i = 2$ *or* 3.

Proof. The first part (i) follows directly from taking cohomology of the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(U_1^\vee)}(-C_1)(1) \longrightarrow \mathcal{O}_{\mathbb{P}(U_1)^\vee}(1) \longrightarrow \mathcal{O}_{C_1}(1) \longrightarrow 0,$$

since $H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1-n)) = 0$ for $p = 0, 1$.

For $i = 2$ or 3, because the box is nondegenerate, recall that the curve C_{1i} is a complete intersection in $\mathbb{P} := \mathbb{P}(U_1^\vee) \otimes \mathbb{P}(U_i^\vee)$, and we will prove the vanishing of H^1 for the pullback of the line bundles $\mathcal{O}_\mathbb{P}(0,1)$, $\mathcal{O}_\mathbb{P}(1,1)$, and $\mathcal{O}_\mathbb{P}(-1,1)$ to C_{1i}.

Let $H_1, \dots, H_n$ be n hypersurfaces of bidegree $(1,1)$ in $\mathbb{P}$ that exactly cut out C_{1i} (the polynomials defining the H_j come directly from the box, viewed as n bilinear forms). For $1 \leq d \leq n$, let $Y_d := \cap_{j=1}^{n-d+1} H_j$, which has dimension d, so that there exists a flag

$$C_{1i} = Y_1 \subset Y_2 \subset \cdots \subset Y_n = H_1 \subset \mathbb{P}.$$

For $1 \leq d \leq n-1$, the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-1,-1) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{H_{n-d+1}} \longrightarrow 0$$

is exact, and tensoring with $\mathcal{O}_{Y_{d+1}}$ gives the exact sequence

$$\mathcal{O}_{Y_{d+1}}(-1,-1) \longrightarrow \mathcal{O}_{Y_{d+1}} \longrightarrow \mathcal{O}_{Y_d} \longrightarrow 0.$$

The first map, given by the polynomial defining H_{n-d+1}, is injective, because that polynomial is a regular element of the coordinate ring, by assumption. Now tensoring with $\mathcal{O}_{\mathbb{P}}(m_1, m_2)$ for $m_1, m_2 \in \mathbb{Z}$ and taking cohomology gives the exact sequence

$$H^p(Y_{d+1}, \mathcal{O}_{Y_{d+1}}(m_1, m_2)) \longrightarrow H^p(Y_d, \mathcal{O}_{Y_d}(m_1, m_2)) \longrightarrow H^{p+1}(Y_{d+1}, \mathcal{O}_{Y_{d+1}}(m_1 - 1, m_2 - 1)) \tag{4.2}$$

for $p \geq 0$. By the Kunneth formula, if $k_1 \geq -1$ and $k_2 \geq 1$ are integers, then the cohomology group $H^p(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k_1 - p + 1, k_2 - p + 1))$ vanishes for $1 \leq p \leq n+1$. Applying (4.2) inductively implies that for $1 \leq p \leq d \leq n$,

$$H^p(Y_d, \mathcal{O}_{Y_d}(k_1 - p + 1, k_2 - p + 1)) = 0.$$

Thus, the case $p = 1$ and $d = 1$ implies the desired result: $H^1(C_{1i}, \mathcal{O}_{C_{1i}}(k_1, k_2)) = 0$ for $k_1 \geq -1$ and $k_2 \geq 1$. □

By Riemann-Roch, a non-exceptional line bundle of degree $\frac{1}{2}n(n-1)$ on a curve of genus $g = \frac{1}{2}(n-1)(n-2)$ is such a bundle with exactly n linearly independent sections, so Lemma 4.1 implies that L_2 and L_3 are non-exceptional. Moreover, there exists a nontrivial relation among these line bundles, which is crucial in the parametrization of the orbits of $3 \times n \times n$ boxes. This relation is a generalization of Lemma 2.3 for Rubik's cubes.

Lemma 4.2. *On the curve C_1, we have the relation*

$$L_2 \otimes L_3 \cong \pi_1^* \mathcal{O}_{\mathbb{P}(U_1^\vee)}(n-1) \cong L_1^{\otimes(n-1)}.$$

Proof. We prove this lemma in the language of divisors. For $w \in C_1 \subset \mathbb{P}(U_1^\vee)$, each coordinate of $\pi_2(\alpha_2(w)) \in \mathbb{P}(U_2^\vee)$ is given by the $(n-1) \times (n-1)$ minors $A_{ij}^*(w)$ of $\mathcal{A}(w, \cdot, \cdot)$ for some fixed j where not all $A_{ij}^*(w)$ vanish. Let D_2 be an effective degree $\frac{1}{2}n(n-1)$ divisor on C_1 such that $\mathcal{O}(D_2) \cong L_2$. Then the points of D_2 are the preimage on C_1 of the intersection of a hyperplane with the image of the curve C_{12} in $\mathbb{P}(U_2^\vee)$; in particular, we may choose D_2, without loss of generality, to be the divisor defined by the locus where a particular minor, say $A_{11}^*(w)$, vanishes on the curve C_1 but at least one $A_{i1}^*(w)$ is nonzero. Similarly, we may choose a divisor D_3 such that $\mathcal{O}(D_3) \cong L_3$ to be the sum of the points $w \in C_1$ where $A_{11}^*(w) = 0$ but not all other $A_{j1}^*(w)$ vanish. Then the points of the degree $n(n-1)$ effective divisor $D_2 + D_3$ are exactly the intersection of the curve C_1 and $A_{11}^*(w) = 0$, which is linearly equivalent to the pullback of $\mathcal{O}_{\mathbb{P}(U_1^\vee)}(n-1)$ to C_1. □

Also, for the curve C_1, we can relate its canonical bundle to the embedding into $\mathbb{P}(U_1^\vee)$. In particular, the following lemma will show that

$$\omega_{C_1} \cong \pi_1^* \mathcal{O}_{\mathbb{P}(U_1^\vee)}(n-3),$$

where ω_{C_1} is the canonical line bundle on C_1.

Lemma 4.3. *Let $\iota : X \hookrightarrow \mathbb{P}(V)$ be a smooth hypersurface of degree n, where V is a N-dimensional vector space. Then if ω_X denotes the canonical bundle of X, we have*

$$\omega_X \cong \iota^* \mathcal{O}_{\mathbb{P}(V)}(n-N).$$

Proof. Adjunction produces the exact sequence

$$0 \longrightarrow \iota^* I_X \longrightarrow \iota^* \Omega^1_{\mathbb{P}(V)} \longrightarrow \Omega^1_X \longrightarrow 0$$

where $I_X \cong \mathcal{O}_{\mathbb{P}(V)}(-n)$ is the ideal defining X in $\mathbb{P}(V)$. Then taking determinants gives

$$\begin{aligned}
\omega_X &= \det(\Omega^1_X) \\
&= \det(\iota^* I_X)^{-1} \otimes \det(\iota^* \Omega^1_{\mathbb{P}(V)}) \\
&= \iota^* \mathcal{O}_{\mathbb{P}(V)}(n) \otimes \iota^* \mathcal{O}_{\mathbb{P}(V)}(-N) = \iota^* \mathcal{O}_{\mathbb{P}(V)}(n-N).
\end{aligned}$$

□

Recall that the group $G = \mathrm{GL}(U_1) \times \mathrm{GL}(U_2) \times \mathrm{GL}(U_3)$ acts on the space of $3 \times n \times n$ boxes by the standard action on each factor. These transformations do not affect the isomorphism class of the curve, as remarked earlier, or of the line bundles arising from the box.

4.2 The Moduli Problem for $3 \times n \times n$ Boxes

From a nondegenerate $3 \times n \times n$ box $\mathcal{A}$, we obtain a genus $\frac{1}{2}(n-1)(n-2)$ curve C_1 with an embedding π_1 into $\mathbb{P}^2$, along with two "balanced" degree $\frac{1}{2}n(n-1)$ line bundles L_2 and L_3 such that $L_2 \otimes L_3 \cong \pi_1^* \mathcal{O}_{\mathbb{P}^2}(n-1)$. In fact, this geometric data is essentially enough to recover the box, up to G-equivalence, as well. We show that the G-orbits of $3 \times n \times n$ boxes parametrize the data (C_1, L_1, L_2, L_3), up to equivalence. This bijection of sets may be rewritten in several stronger ways, including as an equivalence of moduli stacks.

4.2.1 A Bijection

In this section, we describe the bijection between orbits of $3 \times n \times n$ boxes and degree n plane curves with line bundles. The reverse functor, from the geometric data to the boxes, is given geometrically first. The geometric data under consideration are quadruples (C, L_1, L_2, L_3), subject to the following conditions:

(a) C is a smooth irreducible genus $\frac{1}{2}(n-1)(n-2)$ curve, L_1 is a degree 3 line bundle on C, and L_2 and L_3 are degree $\frac{1}{2}n(n-1)$ line bundles on C.

(b) $h^0(C, L_1) = 3$.

(c) $h^0(C, L_i) = n$ and $h^0(C, L_1^{-1} \otimes L_i) = 0$ for $i = 2$ or 3. (4.3)

(d) $L_1^{\otimes(n-1)} \cong L_2 \otimes L_3$.

Condition (b) implies that $|L_1| \in \hat{W}_n^2(C)$, where $\hat{W}_d^r(C)$ denotes the open subscheme parametrizing complete linear series on the curve C with degree d and dimension r, i.e., degree d line bundles with exactly $r+1$ independent sections.[1] Likewise, the first part of condition (c) says that $|L_2|, |L_3| \in \hat{W}^{n-1}_{n(n-1)/2}(C)$. The conditions (4.3) are quite strong, as we will see in the next lemmas.

Lemma 4.4. *Let (C, L_1, L_2, L_3) be a quadruple satisfying conditions (a), (b), and (d) of* (4.3). *Then condition (c) for $i = 2$ only is equivalent to condition (c) for $i = 3$ only.*

Proof. By Serre duality, Lemma 4.3, condition (d), and Riemann-Roch, we have

$$\begin{aligned} h^0(C, L_1^{-1} \otimes L_3) &= h^1(C, \omega_C \otimes L_1 \otimes L_3^{-1}) \\ &= h^1(C, (L_1^{\otimes(n-3)}) \otimes L_1 \otimes L_3^{-1}) \\ &= h^1(C, L_1^{-1} \otimes L_2) \\ &= h^0(C, L_1^{-1} \otimes L_2). \end{aligned}$$

Similarly, if $h^0(C, L_2) = n$, we have

$$h^1(C, L_3) = h^0(C, \omega_C \otimes L_3^{-1}) = h^0(L_2 \otimes L_1^{-2}),$$

and taking cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_C(-L_1) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{L_1} \longrightarrow 0$$

tensored with $L_2 \otimes L_1^{-1}$ shows that $h^0(L_2 \otimes L_1^{-2})$ vanishes. By Riemann-Roch, the vanishing of $h^1(C, L_3)$ implies that $h^0(C, L_3) = n$. □

[1] With respect to the standard notation $W_d^r(C)$ parametrizing complete linear series with degree d and dimension at least r, we have $\hat{W}_d^r(C) = W_d^r(C) \setminus W_d^{r+1}(C)$.

Lemma 4.5. *Given a quadruple (C, L_1, L_2, L_3) satisfying (4.3), the multiplication map*

$$\mu_{1i} : H^0(C, L_1) \otimes H^0(C, L_i) \longrightarrow H^0(C, L_1 \otimes L_i), \tag{4.4}$$

which is the cup product on cohomology, is surjective for $i = 2$ or 3.

Proof. This lemma is an application of the basepoint-free pencil trick [Eis95, Exercise 17.18]). Let V be a basepoint-free pencil of $H^0(C, L_1)$, in other words, a two-dimensional subspace of the three-dimensional space $H^0(C, L_1)$ such that for every point $x \in C$, there exists a section in V that is nonzero on x. Almost all pencils of $H^0(C, L_1)$ will be basepoint-free, since choosing V is equivalent to choosing a point in $\mathbb{P}(H^0(C, L_1)^\vee)$ from which to project the image of the curve C to a $\mathbb{P}^1$. There is a surjective map

$$V \otimes \mathcal{O}_C \twoheadrightarrow L_1 \ ,$$

and the kernel is the sheaf L_1^{-1}. Tensoring this exact sequence with L_i and taking cohomology gives the exact sequence

$$H^0(C, L_1^{-1} \otimes L_i) \longrightarrow V \otimes H^0(C, L_i) \longrightarrow H^0(C, L_1 \otimes L_i) \longrightarrow H^1(C, L_1^{-1} \otimes L_i).$$

By condition (4.3)(c) and Riemann-Roch, both $H^0(C, L_1^{-1} \otimes L_i)$ and $H^1(C, L_1^{-1} \otimes L_i)$ vanish. So this sequence gives an isomorphism between $V \otimes H^0(C, L_i)$ and $H^0(C, L_1 \otimes L_i)$, both of which are vector spaces of dimension $2n$ by Riemann-Roch. Thus, μ_{1i} is surjective. $\square$

Note that all the conditions (4.3) are satisfied by quadruples arising from nondegenerate $3 \times n \times n$ boxes. We say that two such quadruples (C, L_1, L_2, L_3) and (C', L_1', L_2', L_3') are *equivalent* if there exists an isomorphism $\lambda : C \to C'$ such that $\lambda^* L_i' \cong L_i$ for $1 \leq i \leq 3$.

Theorem 4.6. *Let $n \geq 3$ be an integer. There exists a bijection*

$$\left\{ \begin{array}{l} G\textit{-equivalence classes of} \\ \textit{nondegenerate } 3 \times n \times n \textit{ boxes} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{equivalence classes of quadruples} \\ (C, L_1, L_2, L_3) \textit{ satisfying } (4.3) \end{array} \right\}. \tag{4.5}$$

Proof. Given a nondegenerate $3 \times n \times n$ box, we have in Section 4.1 constructed the geometric data on the right side of the bijection and have shown that the G-action on the box preserves the equivalence class of this data.

Now let C be an irreducible curve of genus $\frac{1}{2}(n-1)(n-2)$ with line bundles L_1, L_2, L_3 satisfying (4.3). The main idea of the proof is to identify the spaces of sections $H^0(C, L_i)$ with the vector spaces U_i associated to the box.

By Lemma 4.5, the multiplication map $\mu_{12} : H^0(C, L_1) \otimes H^0(C, L_2) \to H^0(C, L_1 \otimes L_2)$ is surjective. We are interested in the inclusion of the kernel $\ker \mu_{12}$ into $H^0(C, L_1) \otimes H^0(C, L_2)$. The line bundle $L_1 \otimes L_2$ has degree $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$. Tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}_C(-L_1) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{L_1} \longrightarrow 0$$

by $L_1 \otimes L_2$ and taking cohomology shows that $H^1(C, L_1 \otimes L_2) = 0$. Therefore, by Riemann-Roch, the dimension of $H^0(C, L_1 \otimes L_2)$ is $2n$, and as μ_{12} is surjective, the kernel is a vector space of dimension n. Define the $3 \times n \times n$ box

$$\mathcal{A} \in H^0(C, L_1) \otimes H^0(C, L_2) \otimes (\ker \mu_{12})^\vee \cong \operatorname{Hom}(\ker \mu_{12}, H^0(C, L_1) \otimes H^0(C, L_2))$$

as the inclusion of $\ker \mu_{12}$ into the domain of μ_{12}. Note that that the box, as a $3 \times n \times n$ array, is defined up to a choice of basis for each of these vector spaces.

If (C', L_1', L_2', L_3') is another equivalent quadruple, then the spaces of global sections are isomorphic, and the isomorphisms commute with the corresponding multiplication maps μ_{12} and μ_{12}'. The box constructed can differ only by changes of bases for $H^0(C, L_1) \cong H^0(C', L_1'), H^0(C, L_2) \cong H^0(C', L_2')$, and $\ker \mu_{12} \cong \ker \mu_{12}'$, in other words, by an element of the group G.

It remains to check that these constructions are inverse to one another. Given a quadruple (C, L_1, L_2, L_3), we construct the box $\mathcal{A} \in H^0(C, L_1) \otimes H^0(C, L_2) \otimes (\ker \mu_{12})^\vee$ as above. Let C_1 and C_{12} be the natural images of the curve C in $\mathbb{P}(H^0(C, L_1)^\vee)$ and in $\mathbb{P}(H^0(C, L_1)^\vee) \times \mathbb{P}(H^0(C, L_2)^\vee)$ via ϕ_{L_1} and (ϕ_{L_1}, ϕ_{L_2}). Now let C_1' be the variety cut out by the degree n polynomial $f(w) := \det \mathcal{A}(w, \cdot, \cdot)$ and let C_{12}' be the variety defined as $\{(w, x) \in \mathbb{P}(H^0(C, L_1)^\vee) \times \mathbb{P}(H^0(C, L_2)^\vee) : \mathcal{A}(w, x, \cdot) = 0\}$.

It suffices to show that $C_1 = C_1'$ and $C_{12} = C_{12}'$. For all $(w^\dagger, x^\dagger) \in C_{12}$, we have $\mathcal{A}(w^\dagger, x^\dagger, \cdot) = 0$ by the construction of $\mathcal{A}$, and thus $\det \mathcal{A}(w^\dagger, \cdot, \cdot) = 0$. Thus, the curve C_1' contains C_1 as a variety in $\mathbb{P}(H^0(C, L_1)^\vee) = \mathbb{P}^2$; since both are given by degree n polynomials, they must in fact be equal as varieties, so C_1' is a smooth irreducible genus $\frac{1}{2}(n-1)(n-2)$ plane curve. We also have that $C_{12} \subseteq C_{12}'$, and as both are irreducible curves, they must be equal. Note that this also implies that the box $\mathcal{A}$ constructed from (C, L_1, L_2, L_3) is nondegenerate.

That the geometric data coming from a box produce the same box again is clear. If $\mathcal{A} \in U_1 \otimes U_2 \otimes U_3$ is a $3 \times n \times n$ is a nondegenerate projective box, and (C, L_1, L_2, L_3) the associated quadruple, then the vector spaces U_i and $H^0(C, L_i)$ are naturally isomorphic for $i = 1, 2$, and $U_3^\vee$ can be identified with $\ker \mu_{12}$. The $3 \times n \times n$ box constructed from the geometric data is thus the same as the original box $\mathcal{A}$, up to a change in bases for each of the vector spaces, so it is G-equivalent to $\mathcal{A}$. $\square$

Remark 4.7. The proof only uses one of the line bundles L_2 and L_3, and from Lemma 4.4, the hypotheses in (4.3) for L_2 are equivalent to those for L_3. The line bundles L_2 and L_3 may be interchanged for the construction of the box from the quadruple (C, L_1, L_2, L_3) satisfying (4.3). Using either constructs the same box, since the boxes from L_2 and from L_3 both recover the full quadruple (C, L_1, L_2, L_3). Thus, there is a posteriori an identification of $H^0(C, L_3)$ with the dual of the kernel of the multiplication map $\mu_{12} : H^0(C, L_1) \otimes H^0(C, L_2) \to H^0(C, L_1 \otimes L_2)$ (and respectively, $H^0(C, L_2)$ with $(\ker \mu_{13})^\vee$).

Theorem 4.6 specifies the conditions for the line bundles involved, but as noted before, for $n \geq 3$, every non-exceptional divisor of degree $\frac{1}{2}n(n-1)$ has a space of sections of dimension n and Remark 4.7 implies that only one degree $\frac{1}{2}n(n-1)$ line bundle needs to be specified. In addition, it is well-known [ACGH85, p. 56] that for a curve of degree ≥ 4, any g^2_d is unique. In other words, for such a curve, there exists only one line bundle that gives a plane embedding. The theorem can thus be rewritten as

Proposition 4.8. *Let $n \geq 4$ be an integer and $g = \frac{1}{2}(n-1)(n-2)$. There exists a bijection*

$$\left\{\begin{array}{l}G\text{-equivalence}\\ \text{classes of}\\ \text{nondegenerate}\\ 3\times n\times n \text{ boxes}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{equivalence classes of triples } (C,L)\text{, where } C \text{ is a}\\ \text{genus } g \text{ curve with a degree } n \text{ plane embedding } \iota,\\ \text{and } L \text{ is a degree } \tfrac{1}{2}n(n-1) \text{ non-exceptional line}\\ \text{bundle on } C \text{ with } L\otimes \iota^*\mathcal{O}_{\mathbb{P}^2}(-1) \text{ non-exceptional}\end{array}\right\}.$$

In the proposition, the triple (C, L) is equivalent to (C', L') if there exists an isomorphism $\sigma : C \to C'$ such that $\sigma^* L' \cong L$. For $n \geq 4$ and $g = \frac{1}{2}(n-1)(n-2)$, this bijection shows that as the coarse moduli space of the degree $\frac{1}{2}n(n-1)$ Picard stack $\mathrm{Pic}_g^{n(n-1)/2}$ over the moduli space $\mathscr{M}_g^{\mathrm{plane}}$ of plane genus g curves is birational to the orbit space $\mathbb{A}^{3n^2}/\mathrm{GL}_3 \times \mathrm{GL}_n \times \mathrm{GL}_n$ of $3 \times n \times n$ boxes.

Another related theorem involves rigidifying the data on each side of bijection (4.5) to use $3 \times n \times n$ boxes, not their orbits, to parametrize curves and line bundles. Recall that the definition of a $3 \times n \times n$ box $\mathcal{A} \in U_1 \otimes U_2 \otimes U_3$ includes the bases for the vector spaces U_1, U_2, and U_3. On the other hand, let $\mathcal{D}$ be the data of (C, L_1, L_2, L_3) satisfying conditions (4.3) along with bases $\mathfrak{B}_i$ for $H^0(C, L_i)$ for $1 \leq i \leq 3$. Then two such rigidified quadruples $\mathcal{D}$ and $\mathcal{D}'$ are *equivalent* if there exists an isomorphism $\sigma : C \to C'$ such that for $1 \leq i \leq 3$, we have both $\sigma^* L_i' \cong L_i$ and that $\sigma^* : H^0(C', L_i') \to H^0(C, L_i)$ is an isomorphism taking $\mathfrak{B}_i'$ to $\mathfrak{B}_i$. The proposition below follows from the proof of Theorem 4.6 and the identification of $H^0(C, L_3)$ and $(\ker \mu_{12})^\vee$ (and of $H^0(C, L_2)$ and $(\ker \mu_{13})^\vee$) by Remark 4.7:

Proposition 4.9. *There exists a bijection*

$$\left\{\begin{array}{l}\text{nondegenerate}\\ 3\times n\times n \text{ boxes}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{equivalence classes of } (C, L_1, L_2, L_3, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)\\ \text{where } (C, L_1, L_2, L_3) \text{ satisfies conditions (4.3)}\\ \text{and } \mathfrak{B}_i \text{ is a basis for } H^0(C, L_i) \text{ for } 1\leq i\leq 3\end{array}\right\}, \qquad (4.6)$$

and quotienting each side by $\mathrm{GL}_3 \times \mathrm{GL}_n \times \mathrm{GL}_n$ *recovers Theorem 4.6.*

4.2.2 Moduli Stack Formulation

In this section, we work exclusively with $\mathbb{Z}[\frac{1}{N}]$-schemes S, where $N = 3n(n-1)$. We now prove that the results hold in families, and thus the bijections become equivalences of moduli stacks over $\mathbb{Z}[\frac{1}{N}]$. We first reformulate the data on each side of the correspondences.

Just as for Rubik's cubes and hypercubes, we distinguish between $3\times n\times n$ boxes with or without bases over a $\mathbb{Z}[\frac{1}{N}]$-scheme S. A *based* $3\times n\times n$ *box over* S consists of a free rank 3 $\mathcal{O}_S$-module $\mathcal{U}_1$ with an isomorphism $\psi_1 : \mathcal{U}_1 \xrightarrow{\cong} \mathcal{O}_S^{\oplus 3}$; for $i = 2$ and 3, a free rank n $\mathcal{O}_S$-module $\mathcal{U}_i$ with isomorphisms $\psi_i : \mathcal{U}_i \xrightarrow{\cong} \mathcal{O}_S^{\oplus n}$; and a section $\mathcal{A}$ of $\mathcal{U}_1 \otimes \mathcal{U}_2 \otimes \mathcal{U}_3$. An isomorphism of based $3 \times n \times n$ boxes $(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \psi_1, \psi_2, \psi_3, \mathcal{A})$ and $(\mathcal{U}_1', \mathcal{U}_2', \mathcal{U}_3', \psi_1', \psi_2', \psi_3', \mathcal{A}')$ consists of isomorphisms $\sigma_i : \mathcal{U}_i \xrightarrow{\cong} \mathcal{U}_i'$ with $\psi_i = \psi_i' \circ \sigma_i$ for $1 \leq i \leq 3$ and taking $\mathcal{A}$ to $\mathcal{A}'$. A based $3 \times n \times n$ box is *nondegenerate* if it is locally nondegenerate.

In contrast, a $3 \times n \times n$ *box over* S is a section $\mathcal{A}$ of $\mathcal{U}_1 \otimes \mathcal{U}_2 \otimes \mathcal{U}_3$, where $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ are locally free $\mathcal{O}_S$-modules of rank $3, n, n$, respectively. An isomorphism of $3 \times n \times n$ boxes $(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{A})$ and $(\mathcal{U}_1', \mathcal{U}_2', \mathcal{U}_3', \mathcal{A}')$ consists of isomorphisms $\sigma_i : \mathcal{U}_i \xrightarrow{\cong} \mathcal{U}_i'$ of $\mathcal{O}_S$-modules for $1 \leq i \leq 3$ that send $\mathcal{A}$ to $\mathcal{A}'$.

To describe the geometric data over S, define a *genus* g *curve* C *over* S as a proper smooth morphism $\pi : C \to S$ with relative dimension 1 such that $\mathrm{R}^0\pi_*(\mathcal{O}_C) = \mathcal{O}_S$ and $\mathrm{R}^1\pi_*(\mathcal{O}_C)$ is a rank g vector bundle over S. Define the *rigidified degree* n *quadruple* $\mathcal{D}$ over S to consist of a genus $\frac{1}{2}(n-1)(n-2)$ curve $\pi : C \to S$; a degree n line bundle $\mathcal{L}_1$ on C with an isomorphism $\chi_1 : \mathrm{R}^0\pi_*(\mathcal{L}_1) \xrightarrow{\cong} \mathcal{O}_S^{\oplus 3}$; and two degree $\frac{1}{2}n(n-1)$ line bundles $\mathcal{L}_2$ and $\mathcal{L}_3$ on C with isomorphisms $\chi_i : \mathrm{R}^0\pi_*(\mathcal{L}_i) \xrightarrow{\cong} \mathcal{O}_S^{\oplus n}$ for $i = 2$ or 3. A *balanced* rigidified degree n quadruple also includes an isomorphism $\varphi : \mathcal{L}_1^{\otimes(n-1)} \xrightarrow{\cong} \mathcal{L}_2 \otimes \mathcal{L}_3 \otimes \pi^* L_S$ for some line bundle L_S on S. Such a quadruple is *nondegenerate* if $\mathrm{R}^0\pi_*(\mathcal{L}_1^\vee \otimes \mathcal{L}_i) = 0$ for $i = 2$ and 3. Note that this last condition is equivalent to requiring that fiberwise $H^0(C_s, (\mathcal{L}_1^\vee \otimes \mathcal{L}_i)_s)$ vanishes for all points $s \to S$, because the Euler characteristic of $\mathcal{L}_1^\vee \otimes \mathcal{L}_i$ is 0.

Theorem 4.10. *Over a scheme S, there is an equivalence between the category of nondegenerate based $3 \times n \times n$ boxes over S and the category of nondegenerate balanced rigidified degree n quadruples $(C, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \varphi)$ over S as defined above.*

Proof. The functors in each direction essentially come from those functors defined for $S =$ Spec F. As $\mathcal{U}_i$ is free for $1 \leq i \leq 3$, the construction of the corresponding curves over S and line bundles are as before. (These constructions are entirely analogous to those given in the proof of Theorem 2.11.)

On the other hand, suppose we have a nondegenerate balanced rigidified degree n quadruple, i.e., a genus $\frac{1}{2}(n-1)(n-2)$ curve $\pi : C \to S$ and invertible sheaves $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ on C with the appropriate isomorphisms χ_i and φ. Then the kernel of the surjective multiplication map

$$\mu_{12} : \mathrm{R}^0\pi_*(\mathcal{L}_1) \otimes \mathrm{R}^0\pi_*(\mathcal{L}_2) \longrightarrow \mathrm{R}^0\pi_*(\mathcal{L}_1 \otimes \mathcal{L}_2)$$

is a rank n free $\mathcal{O}_S$-module. Just as before, we recover a $3 \times n \times n$ box over S as the corresponding section in the sheaf $\mathrm{R}^0\pi_*(\mathcal{L}_1) \otimes \mathrm{R}^0\pi_*(\mathcal{L}_2) \otimes (\ker \mu_{12})^\vee$. In order to produce a based $3 \times n \times n$ box, though, we require a trivialization of $(\ker \mu_{12})^\vee$; by Remark 4.7, repeating the construction for $\mathcal{L}_3$ in place of $\mathcal{L}_2$ gives the same (unbased) box, hence the map χ_3 gives a trivialization of $(\ker \mu_{12})^\vee$. Thus, we have constructed a nondegenerate based $3 \times n \times n$ box.

These functors are locally inverse, as shown in Theorem 4.6, and thus are inverse. □

Since the space of based $3 \times n \times n$ boxes is just the scheme $\mathbb{A}^{3n^2}$, we have shown that the moduli space of nondegenerate balanced rigidified degree n quadruples over S is an open subscheme of $\mathbb{A}^{3n^2}$ over S. That is, the stack of these quadruples is equivalent to an open substack of $\mathbb{A}^{3n^2}$.

If we consider unbased boxes, we find that the stack of (unbased) $3 \times n \times n$ boxes is equivalent to the quotient stack $[\mathbb{A}^{3n^2}/\mathrm{GL}_3 \times \mathrm{GL}_n \times \mathrm{GL}_n]$, and we are interested in the nondegenerate open substack of $[\mathbb{A}^{3n^2}/\mathrm{GL}_3 \times \mathrm{GL}_n \times \mathrm{GL}_n]$, which is given locally by the nonvanishing of the discriminant.

Unrigidifying the geometric data is straightforward: let $\pi : C \to S$ be a genus $\frac{1}{2}(n-1)(n-2)$ curve over S. Let $\mathcal{L}_1$ be a degree 3 line bundle on C such that $\mathrm{R}^0\pi_*(\mathcal{L}_i)$ has rank 3, and let $\mathcal{L}_2$ and $\mathcal{L}_3$ be degree $\frac{1}{2}n(n-1)$ line bundles on C such that $\mathrm{R}^0\pi_*(\mathcal{L}_2)$ and $\mathrm{R}^0\pi_*(\mathcal{L}_3)$ have rank n. If we also have an isomorphism $\varphi : \mathcal{L}_1^{\otimes(n-1)} \xrightarrow{\cong} \mathcal{L}_2 \otimes \mathcal{L}_3 \otimes \pi^* L_S$ for some line bundle L_S on S, and the condition that $\mathrm{R}^0\pi_*(\mathcal{L}_1^\vee \otimes \mathcal{L}_i) = 0$ for $i = 2$ and 3, we call $(C, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \varphi)$ a *nondegenerate balanced degree n quadruple*.

Because the isomorphism of Theorem 4.10 is $\mathrm{GL}_3 \times \mathrm{GL}_n \times \mathrm{GL}_n$-equivariant, we obtain an equivalence of the respective quotient stacks. The nondegenerate balanced degree n quadruples form the quotient stack for such rigidified quadruples.

Corollary 4.11. *For $n \geq 3$, the nondegenerate open substack of* $[\mathbb{A}^{3n^2}/\mathrm{GL}_3 \times \mathrm{GL}_n \times \mathrm{GL}_n]$ *is equivalent to the stack of nondegenerate balanced degree n quadruples.*

Note that the stack of nondegenerate balanced degree n quadruples is visibly a substack of the fiber product

$$\mathrm{Pic}_g^n \times_{\mathscr{M}_g} \mathrm{Pic}_g^{n(n-1)/2} \times_{\mathscr{M}_g} \mathrm{Pic}_g^{n(n-1)/2},$$

where $g = \frac{1}{2}(n-1)(n-2)$ and Pic_g^d denotes the universal degree d Picard stack over the moduli space $\mathscr{M}_g$ of genus g curves.

4.2.3 Explicit Algebraic Construction

Given a smooth irreducible degree n plane curve $\iota : C \to \mathbb{P}^2$ (i.e., a genus $\frac{1}{2}(n-1)(n-2)$ curve and a degree n line bundle L_1 with $h^0(C, L_1) = 3$) and degree $\frac{1}{2}n(n-1)$ line bundles L_2 and L_3 such that $L_2 \otimes L_3 \cong L_1^{\otimes(n-1)}$, there is also an algebraic construction of the related $3 \times n \times n$ box (see [Dix02], for example, for the main ideas in the symmetric case). We sketch the construction below, since it makes the construction very explicit and computable.

Suppose f is the degree n polynomial in $S := F[w_1, w_2, w_3]$ defining the plane curve C, and let $R := S/(f)$ be the graded ring corresponding to the curve C itself. Let S^i and R^i denote the ith graded pieces of S and R, respectively. Let D_2 be a degree $\frac{1}{2}n(n-1)$ effective divisor on C such that $\mathcal{O}(D_2) \cong L_2$ (in other words, D_2 is the divisor of some nonzero holomorphic section of L_2). We will also, by a slight abuse of notation, refer to the subscheme associated to the divisor by the same symbol. Tensoring the exact sequence

$$0 \longrightarrow L_2^{-1} \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{D_2} \longrightarrow 0$$

with $\iota^*\mathcal{O}_{\mathbb{P}^2}(n-1)$ gives the exact sequence

$$0 \longrightarrow L_3 \longrightarrow \iota^*\mathcal{O}_{\mathbb{P}^2}(n-1) \longrightarrow \mathcal{O}_{D_2} \otimes \iota^*\mathcal{O}_{\mathbb{P}^2}(n-1) \longrightarrow 0. \tag{4.7}$$

By taking cohomology, we view sections of L_3 as degree $n-1$ polynomials on $\mathbb{P}^2$, which all vanish on the divisor D_2. A dimension count shows that these sections are all the degree $n-1$ polynomials that vanish on the degree $\frac{1}{2}n(n-1)$ divisor D_2. Choose a basis

$s_1 =: q, \ldots, s_n$ of $H^0(C, L_3)$, where each $s_i \in S^{n-1}$. Now the variety defined by $\{q = 0\}$ intersects the curve C in $D_2 + D_3$, where D_3 is an effective divisor of degree $\frac{1}{2}n(n-1)$. Because of the condition on the line bundles L_2 and L_3, we have that $L_3 \cong \mathcal{O}(D_3)$. A similar argument as above shows that the sections of L_2 can also be viewed as degree $n-1$ polynomials on $\mathbb{P}^2$, and they are exactly all such polynomials that vanish on D_3. Choose a basis $r_1 = q, r_2, \ldots, r_n$ of degree $n-1$ polynomials for $H^0(C, L_2)$.[2]

Now the ideal sheaf $I_{D_2} \cong \mathcal{O}(-D_2)$ is generated by the images of $s_1, \ldots, s_n$ in R, and similarly, the ideal sheaf I_{D_3} is generated by the images of $r_1, \ldots, r_n$. Since $I_{D_2} I_{D_3} = I_{D_2+D_3}$ is generated as an ideal of R by the image of q, we have, for $1 \leq i, j \leq n$,

$$r_i s_j = b_{ij} q + c_{ij} f \tag{4.8}$$

for some $b_{ij} \in S^{n-1}$ and $c_{ij} \in S^{n-2}$. For example, we have $b_{1j} = s_j$ and $b_{i1} = r_i$. Since each $(n-1) \times (n-1)$ minor b^*_{ij} of the $n \times n$ matrix $B = (b_{ij})$ is divisible by f^{n-2}, there exist $a_{kij} \in F$ for $1 \leq k \leq 3$ and $1 \leq i, j \leq n$ such that

$$b^*_{ij}/f^{n-2} = a_{1ij} w_1 + a_{2ij} w_2 + a_{3ij} w_3, \tag{4.9}$$

which defines the $3 \times n \times n$ box $\mathcal{A} = (a_{kij})$. We omit the computations showing that $\mathcal{A}$ is G-equivalent to the box given by the geometric construction in the proof of Theorem 4.6.

4.3 Symmetrizations

Just as for Rubik's cubes and hypercubes, there are related subspaces to which bijection 4.5 may be restricted. In this section, we consider *symmetrized* $3 \times n \times n$ *boxes*, which are elements of $U_1 \otimes \mathrm{Sym}_2 U_2$, where U_1 and U_2 are vector spaces of dimension 3 and n, respectively, with specified bases. Recall that $\mathrm{Sym}_2 U_2$ is the subspace of symmetric tensors of $U_2 \otimes U_2$, so symmetrized $3 \times n \times n$ boxes form a subspace of $3 \times n \times n$ boxes in $U_1 \otimes U_2 \otimes U_2$.

[2] The bundle L_3 does not have any base points on C, and a basis of $H^0(C, L_3)$ is actually given by $\{1, s_2/q, \ldots, s_n/q\}$, i.e., meromorphic functions on C with at worst poles at the divisor $(s_1) - D_2 = D_3$. Similarly, a basis for $H^0(C, L_2)$ is $\{1, r_2/q, \ldots, r_n/q\}$, which generates the space of meromorphic functions on C with at worst poles at the divisor D_2.

There is a natural action of $\mathrm{GL}(U_1) \times \mathrm{GL}(U_2)$ on the space of symmetrized boxes, and a symmetrized box may be thought of as a triple of quadratic forms in n variables.

From a nondegenerate symmetrized $3 \times n \times n$ box, we may obtain in exactly the same way a smooth plane curve of genus $\frac{1}{2}(n-1)(n-2)$ with degree $\frac{1}{2}n(n-1)$ line bundles L_2 and L_3 on the curve. Because of the symmetry of the box, however, it is clear that $L_2 \cong L_3$, and thus $L_2^{\otimes 2}$ must be isomorphic to the pullback of $\mathcal{O}_{\mathbb{P}^2}(n-1)$ to the curve.

On the other hand, if $\iota : C \to \mathbb{P}^2$ is a closed immersion of a genus $\frac{1}{2}(n-1)(n-2)$ curve C and L_2 is a degree $\frac{1}{2}n(n-1)$ line bundle on C satisfying (4.3)(c) with $L_2^{\otimes 2} \cong \iota^* \mathcal{O}_{\mathbb{P}^2}(n-1)$, then slightly modifying the algebraic construction in Section 4.2.3 produces a symmetrized $3 \times n \times n$ projective box [Dix02]. Taking $L_2 \cong L_3$ and $D_2 = D_3$, while identifying sections $r_i = s_i$ for $1 \leq i \leq n$, gives a $3 \times n \times n$ box with $a_{kij} = a_{kji}$ for $1 \leq k \leq 3$ and $1 \leq i, j \leq n$.

To prove the bijection for symmetrized boxes, however, we modify the geometric construction. We consider triples (C, L_1, L_2) where C is a genus $\frac{1}{2}(n-1)(n-2)$ curve, L_1 is a degree 3 line bundle on C with $h^0(C, L_1) = 3$, and L_2 is a degree $\frac{1}{2}n(n-1)$ line bundle on C with $h^0(C, L_2) = 0$, satisfying the conditions $h^0(C, L_1^{-1} \otimes L_2) = 0$ and $L_1^{\otimes(n-1)} \cong L_2^{\otimes 2}$. We sometimes include bases $\mathfrak{B}_i$ for $H^0(C, L_i)$ for $i = 1$ or 2. Then, two such triples are equivalent if there is an isomorphism of the curves preserving the line bundles (and bases).

Theorem 4.12. *For $n \geq 3$, the restriction of Proposition 4.9 to symmetrized $3 \times n \times n$ boxes induces a bijection of the rigidified data*

$$\left\{\begin{array}{l} \text{nondegenerate symmetrized} \\ 3 \times n \times n \text{ boxes} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \text{equivalence classes of quintuples} \\ (C, L_1, L_2, \mathfrak{B}_1, \mathfrak{B}_2) \text{ as above} \end{array}\right\}, \tag{4.10}$$

and restricting Theorem 4.6 to symmetrized boxes gives a bijection of the quotients

$$\left\{\begin{array}{l} \mathrm{GL}_3 \times \mathrm{GL}_n\text{-equivalence classes of non-} \\ \text{degenerate symmetrized } 3 \times n \times n \text{ boxes} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \text{equivalence classes of} \\ \text{triples } (C, L_1, L_2) \text{ as above} \end{array}\right\}. \tag{4.11}$$

Proof. Given a nondegenerate symmetrized $3 \times n \times n$ box $\mathcal{A} \in U_1 \otimes \mathrm{Sym}_2 U_2$, Proposition 4.9 gives the data $(C, L_1, L_2, L_3, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)$. Because of the symmetry, the isomorphisms from $C_1 := \{\det(\mathcal{A}(w, \cdot, \cdot)) = 0\} \subset \mathbb{P}(U_1^\vee)$ to the curves $C_{12} \subset \mathbb{P}(U_1^\vee) \times \mathbb{P}(U_2^\vee)$ and $C_{13} \subset \mathbb{P}(U_1^\vee) \times \mathbb{P}(U_2^\vee)$ are the same. Thus, the line bundles L_2 and L_3 are naturally isomorphic, and the bases $\mathfrak{B}_2$ and $\mathfrak{B}_3$ may be identified. The box $\mathcal{A}$ thus gives rise to $(C, L_1, L_2, \mathfrak{B}_1, \mathfrak{B}_2)$ with the appropriate cohomological conditions and the relation $L_1^{\otimes(n-1)} \cong L_2 \otimes L_2$.

On the other hand, given $(C, L_1, L_2, \mathfrak{B}_1, \mathfrak{B}_2)$ with the relevant conditions, Proposition 4.9 produces a $3 \times n \times n$ box from the symmetrized data $(C, L_1, L_2, L_3, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)$, where $L_3 := L_2$ and $\mathfrak{B}_2$ and $\mathfrak{B}_3$ are identified by the canonical isomorphism $H^0(C, L_2) \cong H^0(C, L_2)$. Suppose the construction of the box, using L_1 and L_2, gives the box $\mathcal{A}$, and the analogous construction using L_1 and L_3 gives $\mathcal{A}'$. With the identification of the basis $\mathfrak{B}_3$ for $H^0(C, L_3)$ with a basis for $(\ker \mu_{12})^\vee$, and of $\mathfrak{B}_2$ with a basis for $(\ker \mu_{13})^\vee$, we have that $\mathcal{A}(w, \cdot, \cdot)$ and $\mathcal{A}'(w, \cdot, \cdot)$ are transposes of one another for any $w \in U_1^\vee$. Since they are also equal, for all $w \in U_1^\vee$, the $n \times n$ matrix $\mathcal{A}(w, \cdot, \cdot) \in H^0(C, L_2) \otimes H^0(C, L_2)$ is a symmetric matrix, so the $3 \times n \times n$ box $\mathcal{A}$ lies in $H^0(C, L_1) \otimes \mathrm{Sym}_2(H^0(C, L_2))$. □

4.4 Special Cases

For low values of n, the theorems simplify and lead to corollaries concerning universal Picard varieties. Clearly for $n = 3$, we recover Theorem 2.5. If $n = 4$ or 5, line bundles of degree $\frac{1}{2}n(n-1)$ are almost all non-exceptional, so Theorem 4.6 can be written even more succinctly. Moreover, for $n = 5$, the space of symmetrized boxes is related to the moduli space $\mathscr{M}_5$ of genus 5 curves.

When $n = 4$, the curve C has genus 3 and is nonhyperelliptic, in which case there exists exactly one line bundle L with degree $n = 4$ and $h^0(C, L) = 3$, namely the canonical bundle ω_C. Furthermore, as there are no nonexceptional degree 6 line bundles, the bijection becomes the following:

Corollary 4.13. *There exists a bijection between orbits of nondegenerate* $3 \times 4 \times 4$ *projective boxes and isomorphism class of pairs* (C, L) *with* C *a nonhyperelliptic smooth genus* 3 *curve and* L *a degree* 6 *line bundle on* C *with* $h^0(C, L \otimes \omega_C^{-1}) = 0$.

Because the condition that $h^0(C, L \otimes \omega_C^{-1})$ holds for a general degree 6 line bundle L, we have the following corollary, which implies that the coarse moduli space of the universal Picard stack Pic_3^6, parametrizing degree 6 line bundles over the moduli space $\mathscr{M}_3$ of genus 3 curves, is unirational.

Corollary 4.14. *There is an equivalence between the nondegenerate open substack of the quotient stack* $[\mathbb{A}^{48}/\mathrm{GL}_3 \times \mathrm{GL}_4 \times \mathrm{GL}_4]$ *and an open substack of* Pic^6_3 *over the nonhyperelliptic substack of* $\mathscr{M}_3$.

For $n = 5$, Theorem 4.6 and Lemma 4.3 imply that $\mathrm{GL}_3 \times \mathrm{GL}_5 \times \mathrm{GL}_5$-orbits of $3 \times 5 \times 5$ nondegenerate boxes correspond exactly to smooth plane quintics C with any two degree 10 line bundles L_2, L_3, such that $L_2 \otimes L_3$ is isomorphic to $\omega_C^{\otimes 2}$; L_2 and L_3 are not isomorphic to ω_C; and $h^0(C, L_2 \otimes L_1^{-1})$ and $h^0(C, L_3 \otimes L_1^{-1})$ vanish. As a smooth plane quintic has exactly one g^2_5 (see [ACGH85, p. 209]), this geometric data is equivalent to giving a smooth plane quintic $\iota : C \to \mathbb{P}^2$ and a point P on its Jacobian $\mathrm{Jac}(C)$ (corresponding to, without loss of generality, the degree 0 line bundle $L_2 \otimes \omega_C^{-1} \cong L_3^{-1} \otimes \omega_C$) such that the bundle $\iota^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes P$ has no sections. This last condition holds for a general point P, since a general degree 5 line bundle has no sections.

Corollary 4.15. *The space of* $\mathrm{GL}_3 \times \mathrm{GL}_5 \times \mathrm{GL}_5$*-orbits of nondegenerate* $3 \times 5 \times 5$ *boxes is isomorphic to an open subspace of the universal Jacobian* Jac_6 *over* $\mathscr{M}_6^{\mathrm{plane}}$*, where* $\mathscr{M}_6^{\mathrm{plane}}$ *is the moduli space of genus* 6 *curves with a closed immersion into* $\mathbb{P}^2$.

For symmetrized $3 \times 5 \times 5$ boxes, since the two degree 10 line bundles are isomorphic, the point P must be a 2-torsion point on $\mathrm{Jac}(C)$. We also require that $h^0(C, \iota^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes P) = 0$, where P is thought of as a degree 0 line bundle in $\mathrm{Pic}^0(C) \cong \mathrm{Jac}(C)$. In addition, the $\mathrm{GL}_3 \times \mathrm{GL}_5$-orbit of a nondegenerate symmetrized $3 \times 5 \times 5$ box corresponds to the complete intersection of three quadrics in $\mathbb{P}^4$, which is a canonically embedded genus 5 curve in $\mathbb{P}^4$. It is easy to check that nondegeneracy implies that the three quinary quadratic forms arising from a nondegenerate symmetric $3 \times 5 \times 5$ box give hypersurfaces that intersect transversally.

On the other hand, any three quinary quadratic forms that intersect transversally are given by a symmetrized $3 \times 5 \times 5$ box, not necessarily nondegenerate. Nondegeneracy imposes an extra open condition, which we will specify in the next section. Since the canonical embeddings of all nonhyperelliptic and non-trigonal genus 5 curves can be given as the complete intersection of three quadrics in $\mathbb{P}^4$, nondegenerate symmetrized $3 \times 5 \times 5$ boxes exactly correspond to genus 5 curves that are nonhyperelliptic, non-trigonal, and have this extra condition. We thus have the following corollary:

Corollary 4.16. *An open subspace of the moduli space of nonhyperelliptic, non-trigonal genus* 5 *curves is isomorphic to the moduli space of pairs* (C, P) *where* $\iota : C \to \mathbb{P}^2$ *is a smooth plane quintic and* P *is a nonzero* 2*-torsion point on* $\mathrm{Jac}(C)$ *with* $h^0(C, \iota^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes P) = 0$.

In the next section, we improve this corollary by describing both the domain and the image of this map more precisely. This comparison between genus 5 curves, as the intersection of three quadrics in $\mathbb{P}^4$, and plane quintics with a 2-torsion point on the Jacobian may also be extended to higher dimensions. Symmetrized $3 \times n \times n$ boxes, where n is odd, give a way to relate the intersection of three quadrics in $\mathbb{P}^{n-1}$ and degree n plane curves.

4.4.1 Symmetrized Boxes and Nets of Quadrics

In this section, we show how results of [Tju75] and [Rei72] on the intersection of quadrics relate to symmetrized $3 \times n \times n$ boxes, for $n \geq 5$ odd. In particular, three quadrics in $\mathbb{P}^{n-1}$ generally intersect in a $(n-4)$-dimensional variety, and Reid shows in [Rei72] that the intermediate Jacobian of this intersection is isomorphic to a certain Prym variety, which we obtain from rewriting Theorem 4.12.

For the rest of the section, let $n \geq 5$ be an odd integer. Then the $\mathrm{GL}_3 \times \mathrm{GL}_n$-orbit of a nondegenerate symmetrized $3 \times n \times n$ box, by Theorem 4.12, gives rise to a genus $g = \frac{1}{2}(n-1)(n-2)$ curve C, with a degree n line bundle L_1 and a degree $\frac{1}{2}n(n-1)$ line bundle L_2, such that $h^0(C, L_1) = 3$, the bundles L_2 and $L_2 \otimes L_1^{-1}$ are non-exceptional, and $L_1^{\otimes(n-1)} \cong L_2^{\otimes 2}$. By Lemma 4.3, this last relation implies that

$$(L_2 \otimes L_1^{-1})^{\otimes 2} \cong L_1^{\otimes(n-3)} \cong \omega_C,$$

so $\kappa := L_2 \otimes L_1^{-1} \in \mathrm{Pic}^{g-1}(C)$ is a *theta-characteristic* of C. Furthermore, since n is odd,

$$P := \kappa \otimes L_1^{\otimes(\frac{3-n}{2})} \cong L_2 \otimes L_1^{\otimes(\frac{1-n}{2})} \in \mathrm{Pic}^0(C) \cong \mathrm{Jac}(C) \tag{4.12}$$

is a 2-torsion line bundle on C and can be thought of as a point in $\mathrm{Jac}(C)[2]$. Then the 2-torsion point P naturally produces a degree 2 étale cover $\widetilde{C}$ of C. By Riemann-Hurwitz, the curve $\widetilde{C}$ has genus $2g-1$, so the associated Prym variety (the connected component of the kernel of the induced map $\mathrm{Jac}(\widetilde{C}) \to \mathrm{Jac}(C)$) has dimension $g-1$.

Of course, as in the case of $n = 5$, we do not obtain all 2-torsion points of $\mathrm{Jac}(C)$ in this way, since the symmetrized box produces exactly those theta-characteristics κ such that $h^0(C, \kappa) = 0$. By definition, this condition implies that κ is an even theta-characteristic, but we may express this condition completely in other ways. For example, by Riemann's singularity theorem [BL04, chap. 11], the vanishing of $h^0(C, \kappa)$ is equivalent to the nonvanishing of the theta-constant associated to κ.[3]

Note that from the line bundle L_1 and the theta-characteristic $\kappa = L_2 \otimes L_1^{-1}$ with nonvanishing theta-constant, we recover $L_2 = \kappa \otimes L_1$ and the condition that $h^1(C, L_2) = 0$. Therefore, orbits of nondegenerate symmetrized $3 \times n \times n$ boxes are in bijection with (C, L_1, κ) such that $h^0(C, L_1) = 3$ and the theta-constant of κ is nonzero.

Example 4.17. For $n = 5$, the genus 6 curve C is embedded by L_1 as a smooth plane quintic. We claim that even theta-characteristics κ on C have no nonzero sections. By Clifford's Theorem, it suffices to check that $h^0(C, \kappa) \neq 2$. It is not hard to show (see [ACGH85, p. 211]) that any g^1_5 on a plane quintic C is of the form $|L_1(p-q)|$ where p and q are distinct points on C. But $\mathcal{O}(p-q)$ on C is not 2-torsion; if it were, then the triviality of $\mathcal{O}(2p-2q)$ implies that there exists a function f from C to $\mathbb{P}^1$ with a double pole at ∞ and a double zero at 0. Then this degree 2 map to $\mathbb{P}^1$ implies that C is a hyperelliptic curve, but no smooth plane quintic is hyperelliptic. Thus, on the curve C, no theta-characteristic κ has $h^0(C, \kappa) = 2$, as desired.

Thus, the orbits of nondegenerate symmetrized $3 \times 5 \times 5$ boxes are exactly in bijection with plane quintics with an even theta-characteristic.

Another interpretation for symmetrized $3 \times n \times n$ boxes involves nets of quadrics, which are studied extensively in [Tju75]. If $\mathcal{A} \in V_1 \otimes \mathrm{Sym}_2 V_2$ is a symmetrized $3 \times n \times n$ box, then we may view $\mathcal{A}$ as a triple of symmetric $n \times n$ matrices, which corresponds to a triple (q_1, q_2, q_3) of quadratic forms[4] on $V_2^\vee$. For $1 \leq i \leq 3$, each quadratic form q_i defines a

[3]The theta-constants of odd theta-characteristics are all zero, so the requirement that the theta-constant does not vanish implies that κ is even.

[4]Because we are working over an algebraically closed field not of characteristic 2, the spaces $\mathrm{Sym}_2 V_2$ and $\mathrm{Sym}^2 V_2$ are canonically isomorphic and we will not for the moment distinguish between the quadratic forms in each space.

quadric hypersurface in $\mathbb{P}(V_2^\vee) = \mathbb{P}^{n-1}$, and their intersection $X_\mathcal{A}$ is generically a $(n-4)$-dimensional variety. Note that $\mathrm{GL}(V_1)$ acting on the box does not change the variety $X_\mathcal{A}$, and $\mathrm{GL}(V_2)$ acts as linear transformations on $\mathbb{P}(V_2^\vee)$. In fact, the $\mathrm{GL}(V_1)$-orbit of the box $\mathcal{A}$ produces a well-defined "net" of quadrics, spanned by these q_i, whose base locus is $X_\mathcal{A}$. This net is a projective plane $\mathbb{P}^2$ in the space of all projective quadrics $\mathbb{P}(\mathrm{Sym}_2 V_2)$ on $V_2^\vee$; in the language of boxes, the net is just the projectivization of the inclusion $V_1^\vee \hookrightarrow \mathrm{Sym}_2 V_2$ given by the box $\mathcal{A}$.

From such a net of quadrics, [Tju75] defines the *Hesse curve* H to be the singular quadrics in the net. This Hesse curve exactly corresponds to the curve we have called C_1 in $\mathbb{P}(V_1^\vee)$, and the nondegeneracy of the box (i.e., the smoothness of C_1) is exactly the condition of *regularity* of a net. For a regular net, the *Steiner* embedding of the Hesse curve H into $\mathbb{P}(V_2^\vee)$ sends a quadric q to its singular point; this Steiner embedding is the same as our map to $\mathbb{P}(V_2^\vee)$ given by taking the kernel of $\mathcal{A}(w, \cdot, \cdot)$ for a point $w \in C_1$.

Reid proves in [Rei72] that for such a regular net of quadrics, the intermediate Jacobian of $X_\mathcal{A}$ is exactly the Prym variety associated to the double cover of C_1 given by the 2-torsion point $P \in \mathrm{Jac}(C_1)$ as in (4.12). Therefore, from bijection (4.11) and the fact that the line bundle L_1 is the unique g^2_d on the plane curve C_1, we have

Corollary 4.18. *For odd $n \geq 5$ and $g = \frac{1}{2}(n-1)(n-2)$, there exists a bijection*

$$\left\{\begin{array}{l}\mathrm{GL}_3 \times \mathrm{GL}_n\text{-}\textit{orbits of non-}\\ \textit{degenerate symmetrized } 3\times\\ n \times n \textit{ boxes } \mathcal{A}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{equivalence classes of pairs } (C,P), \textit{ where } C \textit{ is}\\ \textit{a genus } g \textit{ curve with a plane embedding } \iota \textit{ and}\\ 0 \neq P \in \mathrm{Jac}(C)[2] \textit{ with } h^0(C, \iota^*\mathcal{O}(\frac{n-3}{2}) \otimes P) = 0\end{array}\right\}.$$

If $X_\mathcal{A} \subset \mathbb{P}^{n-1}$ is the base locus of the net of quadrics associated to the box $\mathcal{A}$, then the intermediate Jacobian of X is the Prym variety of the étale double cover of C given by P.

Recall that the points P above are exactly those associated to the theta-characteristics $\kappa = \iota^*\mathcal{O}(\frac{n-3}{2}) \otimes P$ with nonvanishing theta-constants.

Example 4.19. For $n = 5$, the variety X is the intersection of three quadrics in $\mathbb{P}(V_2^\vee) = \mathbb{P}^4$. As mentioned in Section 4.4, such a variety X is a genus 5 curve that is nonhyperelliptic and nontrigonal. If we also require that the box is nondegenerate, then the theta-constants of X must vanish at most of order 1. This follows from the fact that singularities of C_1 correspond to quadrics in $\mathbb{P}^4$ of rank 3 containing X, which exist if and only if there exist theta-characteristics κ_X on X with $h^0(X, \kappa_X) \geq 2$. Thus, specializing Corollary 4.18 to $n = 5$ and using Example 4.17 shows that the following sets are in bijection:

(i) $\mathrm{GL}_3 \times \mathrm{GL}_5$-orbits of nondegenerate symmetrized $3 \times 5 \times 5$ boxes $\mathcal{A}$

(ii) pairs (C, κ), for $\iota : C \to \mathbb{P}^2$ a smooth plane quintic and κ an even theta-characteristic of C

(iii) nonhyperelliptic, non-trigonal genus 5 curves X such that $h^0(X, \kappa_X) \leq 1$ for all theta-characteristics κ_X on X

Moreover, if $\widetilde{C}$ is the degree 2 étale cover of C given by the 2-torsion line bundle $\kappa \otimes \iota^*\mathcal{O}_{\mathbb{P}^2}(-1)$, then $\widetilde{C}$ is a genus 11 curve. Via this bijection, the Prym variety of $\widetilde{C} \to C$ is isomorphic to the Jacobian of X.

We will see in Section 5.4.1 that a special case of symmetrized $2 \times 2 \times m \times m$ boxes also describes the Prym variety arising naturally from a box as the Jacobian of a related curve.

You know my methods. Apply them, and it will be instructive to compare results.

—Sherlock Holmes, in *The Sign of Four* by Arthur Conan Doyle

Chapter 5

Moduli of Curves in $\mathbb{P}^1 \times \mathbb{P}^1$

In this chapter, we use similar techniques to prove statements about the moduli of curves in $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, we study curves of bidegree (m, m) in $\mathbb{P}^1 \times \mathbb{P}^1$, which may be represented as the complete intersection of a degree m hypersurface with the Segre quadric in $\mathbb{P}^3$. These curves, along with certain line bundles, will be parametrized by the orbits of $2 \times 2 \times m \times m$ boxes. As before, these bijections will give isomorphisms of the corresponding coarse moduli spaces as well as the moduli stacks.

We also explore the space of symmetrized $2 \times 2 \times m \times m$ boxes, the orbits of which also parametrize bidegree (m, m) curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with more restrictive line bundle data. For the case $m = 3$, our bijections will give an explicit method for a construction of Recillas [Rec74], which relates genus 4 curves C with an étale double cover $\widetilde{C}$ and genus 3 curves X with degree 4 line bundles, where the Jacobian of X is isomorphic to the Prym variety of the cover $\widetilde{C} \to C$.

Preliminaries. Let $m \geq 2$ be an integer. Let F be an algebraically closed field of characteristic not dividing m or $m - 1$. In this chapter, we continue to use the convention that the projectivization of a vector bundle parametrizes lines, not hyperplanes, and for a basepoint-free line bundle L on a scheme X, the map $\phi_L : X \to \mathbb{P}(H^0(X, L)^\vee)$ is the natural map given by the complete linear system $|L|$.. Also, unless stated otherwise, a genus g curve means a proper, smooth, geometrically connected curve with arithmetic genus g.

5.1 Quadrilinear Forms and Associated Curves

In Chapter 4, we constructed geometric information from the moduli space of certain trilinear forms, up to a natural group action. Each trilinear form gives rise to a degree n curve in $\mathbb{P}^2$. Here, we study a space of quadrilinear forms, where each form gives rise to a curve in $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $m \geq 2$ be an integer. Let V_1 and V_2 be vector spaces of dimension 2 over F, and V_3 and V_4 vector spaces of dimension m over F. There is a natural action of the group $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_3) \times \mathrm{GL}(V_4)$ on the tensor product $V_1 \otimes V_2 \otimes V_3 \otimes V_4$, a space of quadrilinear forms. For $1 \leq i \leq 4$, each $\mathrm{GL}(V_i)$ acts in the standard way on the factor V_i. As before, with choices of bases for the vector spaces V_1, V_2, V_3, and V_4, we may represent an element of $V_1 \otimes V_2 \otimes V_3 \otimes V_4$ as a $2 \times 2 \times m \times m$ box $\mathcal{B} = (b_{qrst})_{1 \leq q,r \leq 2, 1 \leq s,t \leq m}$.

The notation will be analogous to that of previous chapters, where $\mathcal{B}(\cdot,\cdot,\cdot,\cdot)$ denotes the quadrilinear form in $V_1 \otimes V_2 \otimes V_3 \otimes V_4$, and the dots may be replaced with elements of the appropriate $V_i^\vee$. That is, if $w \in V_1^\vee$ and $x \in V_2^\vee$, then $\mathcal{B}(w, x, \cdot, \cdot) \in V_3 \otimes V_4$ is the $m \times m$ matrix $\mathcal{B} \lrcorner (w \otimes x)$.

Let $\mathcal{B} = (b_{rstu})$ be a $2 \times 2 \times m \times m$ box. Then the polynomial

$$f(w_1, w_2, x_1, x_2) := \det \mathcal{B}(w, x, \cdot, \cdot) \in \mathrm{Sym}^m V_1 \otimes \mathrm{Sym}^m V_2$$

is a bidegree (m, m) form. The vanishing of f defines a bidegree (m, m) variety $C_{12} \subset \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) = \mathbb{P}^1 \times \mathbb{P}^1$. We call a box $\mathcal{B}$ *nondegenerate* if this variety C_{12} is a smooth irreducible curve. The condition is entirely algebraic, since it simply requires that the partial derivatives do not all vanish at a point on the curve, which translates into the nonvanishing of a single polynomial in the entries of the box. Thus, the nondegenerate $2 \times 2 \times m \times m$ boxes form an open subset of the affine space of all $2 \times 2 \times m \times m$ boxes, and the smooth curves C_{12} arising from nondegenerate $2 \times 2 \times m \times m$ boxes have genus $(m-1)^2$.

If $\mathcal{B}$ is a nondegenerate box, then for a point $(w^\dagger, x^\dagger) \in C_{12}$, the matrix $\mathcal{B}(w^\dagger, x^\dagger, \cdot, \cdot) \in V_3 \otimes V_4$ has exactly rank $m - 1$. Otherwise, if the rank were strictly smaller than $m - 1$, then all of the $(m-1) \times (m-1)$ minors of $\mathcal{B}(w^\dagger, x^\dagger, \cdot, \cdot)$ would vanish. If $B_{ij}^*(w^\dagger, x^\dagger)$ is the

(i,j)th $(m-1)\times(m-1)$ minor of $\mathcal{B}(w^\dagger, x^\dagger, \cdot, \cdot)$, then we may write the partial derivatives as

$$\left.\frac{\partial f}{\partial w_i}\right|_{(w,x)=(w^\dagger,x^\dagger)} = \sum_{r,s,t} b_{irst} x_r B^*_{st}(w^\dagger, x^\dagger) \qquad \text{and}$$
$$\left.\frac{\partial f}{\partial x_i}\right|_{(w,x)=(w^\dagger,x^\dagger)} = \sum_{q,s,t} b_{qist} w_q B^*_{st}(w^\dagger, x^\dagger).$$

If all the minors vanish, then these partials would also vanish, contradicting the nondegeneracy assumption.

Hence, for a nondegenerate box $\mathcal{B}$, the $m\times m$ matrix $\mathcal{B}(w,x,\cdot,\cdot)$ as an element of $\mathrm{Hom}(V_3^\vee, V_4) \cong V_3 \otimes V_4$ has a one-dimensional kernel, which corresponds to a point $y \in \mathbb{P}(V_3^\vee)$. Note that this $y \in \mathbb{P}(V_3^\vee)$ is given by an algebraic map; given bases for the vector spaces V_i, the coordinates of the element y are given by minors of the matrix $\mathcal{B}(w,x,\cdot,\cdot)$. We thus obtain a rational map

$$C_{12} \longrightarrow \mathbb{P}(V_3^\vee).$$

The corresponding graph is given by the curve

$$C_{123} := \left\{(w,x,y) \in \mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)\times\mathbb{P}(V_3^\vee) : \mathcal{B}(w,x,y,\cdot) = 0\right\}.$$

By dimension considerations, the variety C_{123} is the complete intersection of m tridegree $(1,1,1)$ equations on $\mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)\times\mathbb{P}(V_3^\vee) = \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^{m-1}$, and thus C_{123} is a curve. Clearly the projection onto the first two factors

$$\pi_{12}^3 : C_{123} \longrightarrow \mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)$$

is an isomorphism onto C_{12}. We may similarly define the curve $C_{124} \subset \mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)\times\mathbb{P}(V_4^\vee)$, with an isomorphic projection $\pi_{12}^4 : C_{124} \to C_{12}$ as well.

Then for $1 \le i \le 4$, there exist natural maps $\rho_i : C_{12} \to \mathbb{P}(V_i^\vee)$, e.g., for $i = 1$ or 2, the map ρ_i is just the projection from C_{12} onto $\mathbb{P}(V_i^\vee)$. For $i = 3$ or 4, the map ρ_i is the composition of the isomorphism $C_{12} \to C_{12i}$ and the natural projection $C_{12i} \to \mathbb{P}(V_i^\vee)$.

Clearly ρ_1 and ρ_2 are degree m covers of $\mathbb{P}^1$. Let

$$M_i := \rho_i^* \mathcal{O}_{\mathbb{P}(V_i^\vee)}(1)$$

be line bundles defined on the curve C_{12}; then M_1 and M_2 have degree m and M_3 and M_4 have degree $m(m-1)$. For all M_i, all of the global sections arise from pulling back sections from $\mathbb{P}(V_i^\vee)$, just like in Lemma 4.1.

Lemma 5.1. *Let $\mathcal{B}$ be a $2 \times 2 \times m \times m$ box and M_1, M_2, M_3, and M_4 the line bundles defined above. For $i = 3$ or 4, we have*

(i) $h^0(C_{12}, M_1) = 2$ and $h^0(C_{12}, M_2) = 2$,

(ii) $h^0(C_{12}, M_i) = m$ and $h^1(C_{12}, M_i) = 0$, and

(iii) $h^1(C_{12}, M_1^{\otimes k_1} \otimes M_2^{\otimes k_2} \otimes M_i) = 0$ for $(k_1, k_2) = (\pm 1, 0), (0, \pm 1), (-1, 1), (1, -1)$.

Proof. It suffices to prove part (i) for M_1, which follows from tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)}(-C_{12}) \longrightarrow \mathcal{O}_{\mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)} \longrightarrow \mathcal{O}_{C_{12}} \longrightarrow 0$$

with $\mathcal{O}_{\mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)}(1,0)$ and taking cohomology. Using the Kunneth formula to compute the cohomology of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\ell_1, \ell_2)$ shows that

$$H^0(\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee), \mathcal{O}_{\mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)}(1,0)) \cong H^0(C_{12}, \mathcal{O}_{C_{12}}(1,0))$$

has dimension 2, as desired.

For the other parts, the proof is similar to Lemma 4.1. Recall that nondegeneracy implies that C_{123} is a complete intersection in $\mathbb{P} := \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) \times \mathbb{P}(V_i^\vee)$, and without loss of generality, it suffices to show that the first cohomology group vanishes for the pullback of $\mathcal{O}_{\mathbb{P}}(k_1, k_2, 1)$ to C_{12i} for the appropriate pairs (k_1, k_2). Let $H_1, \ldots, H_m$ be hypersurfaces of tridegree $(1,1,1)$ given by the m trilinear forms in the box $\mathcal{B}$ representing a basis of V_4. In other words, the locus of these H_j is the curve C_{12i} in $\mathbb{P}$. For $1 \leq d \leq m$, the intersection

$Y_d := \cap_{j=1}^{m-d+1} H_j$ has dimension d and we have the flag

$$C_{12i} = Y_1 \subset Y_2 \subset \cdots \subset Y_m = H_1 \subset \mathbb{P}.$$

Tensoring the exact sequence defining H_{m-d+1} with $\mathcal{O}_{Y_{d+1}}$ produces the exact sequence

$$\mathcal{O}_{Y_{d+1}}(-1,-1,-1) \longrightarrow \mathcal{O}_{Y_{d+1}} \longrightarrow \mathcal{O}_{Y_d} \longrightarrow 0,$$

where the first map is injective by the nondegeneracy assumption. Taking cohomology of the sequence tensored with $\mathcal{O}_{\mathbb{P}}(\ell_1, \ell_2, \ell_3)$ gives the exact sequence

$$\begin{aligned} H^p(Y_{d+1}, \mathcal{O}_{Y_{d+1}}(\ell_1, \ell_2, \ell_3)) \longrightarrow H^p(Y_d, \mathcal{O}_{Y_d}(\ell_1, \ell_2, \ell_3)) \\ \longrightarrow H^{p+1}(Y_{d+1}, \mathcal{O}_{Y_{d+1}}(\ell_1 - 1, \ell_2 - 1, \ell_3 - 1)). \end{aligned} \tag{5.1}$$

If $k_1, k_2 \geq 0$ and $k_3 \geq 1$, or if $(k_1, k_2, k_3) = (-1, 0, 1), (0, -1, 1), (-1, 1, 1), (1, -1, 1)$, we find that $H^p(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k_1 - p + 1, k_2 - p + 1, k_3 - p + 1))$ vanishes for $1 \leq p \leq m + 1$ by the Kunneth formula, so applying (5.1) inductively shows that

$$H^p(Y_d, \mathcal{O}_{Y_d}(k_1 - p + 1, k_2 - p + 1, k_3 - p + 1)) = 0$$

for $1 \leq p \leq d \leq m$. Therefore, $H^1(C_{12i}, \mathcal{O}_{C_{12i}}(k_1, k_2, k_3))$ vanishes if $k_1, k_2 \geq 0$ and $k_3 \geq 1$, or if $(k_1, k_2, k_3) = (-1, 0, 1), (0, -1, 1), (-1, 1, 1), (1, -1, 1)$. □

By Riemann-Roch, a non-exceptional line bundle of degree $m(m-1)$ on a curve of genus $(m-1)^2$ is one that has exactly m linearly independent sections, so Lemma 5.1 implies that the line bundles L_3 and L_4 on the curve C_{12} are non-exceptional. In addition, the line bundles arising from the box $\mathcal{B}$ satisfy a nontrivial relation, analogous to Lemma 4.2 for $3 \times n \times n$ boxes and generalizing Lemma 3.4 for hypercubes.

Lemma 5.2. *On the curve C_{12}, the following relation holds:*

$$M_3 \otimes M_4 \cong (M_1 \otimes M_2)^{\otimes (m-1)}.$$

Proof. The proof, similar to that of Lemma 4.2, exploits the definition of the maps ρ_3 and ρ_4 as minors of a matrix. Choose a basis for each of the vector spaces V_3 and V_4. Then for a point $(w, x) \in C_{12} \subset \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee)$, each coordinate of $\rho_3((w, x)) \in \mathbb{P}(V_3^\vee)$ is given by the $(n-1) \times (n-1)$ minors $B^*_{ij}(w, x)$ of $\mathcal{B}(w, x, \cdot, \cdot)$, for some fixed j where there exists at least one i for which $B^*_{ij}(w, x)$ is nonzero. Let D_3 be an effective degree $m(m-1)$ divisor on C_{12} such that $\mathcal{O}(D_3) \cong L_3$; without loss of generality, we may choose D_3 to be the sum of the points $(w, x) \in C_{12}$ where a particular minor, say $B^*_{11}(w, x)$, vanishes but not all other B_{i1} vanish. We may also choose an effective divisor D_3 with $\mathcal{O}(D_3) \cong L_3$ whose points are given by the vanishing of B^*_{11} and the nonvanishing of at least one B^*_{1j}. Then $D_3 + D_4$ is an effective degree $m(m-1)$ divisor given by the intersection of $B^*_{11} = 0$ and C_{12}, and is thus linearly equivalent to the pullback of $\mathcal{O}_{\mathbb{P}(V_1^\vee)\times\mathbb{P}(V_2^\vee)}(m-1, m-1)$ to the curve C_{12}. □

Like in Lemma 4.3 for hypersurfaces in projective spaces, the canonical bundle of a hypersurface in the product of projective spaces is the pullback of a line bundle from the ambient space:

Lemma 5.3. *Let $\iota : X \hookrightarrow \mathbb{P}(U) \times \mathbb{P}(V)$ be a smooth hypersurface of bidegree (m, n), where U and V are vector spaces of dimension M and N, respectively. Then*

$$\omega_X \cong \iota^* \mathcal{O}_{\mathbb{P}(U)\times\mathbb{P}(V)}(m - M, n - N).$$

Proof. If $I_X \cong \mathcal{O}_{\mathbb{P}(U)\times\mathbb{P}(V)}(-m, -n)$ denotes the ideal defining X in $\mathbb{P}(U) \times \mathbb{P}(V)$, the adjunction exact sequence for ι is

$$0 \longrightarrow \iota^* I_X \longrightarrow \iota^* \Omega^1_{\mathbb{P}(U)\times\mathbb{P}(V)} \longrightarrow \Omega^1_X \longrightarrow 0.$$

Taking determinants, we compute

$$\begin{aligned}
\omega_X &= \det(\Omega^1_X) \\
&= \det(\iota^* I_X)^{-1} \otimes \det(\iota^* \Omega^1_{\mathbb{P}(U)\times\mathbb{P}(V)}) \\
&= \iota^* \mathcal{O}_{\mathbb{P}(U)\times\mathbb{P}(V)}(m, n) \otimes \iota^* \mathcal{O}_{\mathbb{P}(U)\times\mathbb{P}(V)}(-M, -N) = \iota^* \mathcal{O}_{\mathbb{P}(U)\times\mathbb{P}(V)}(m - M, n - N). \quad \square
\end{aligned}$$

Thus, on the curve C_{12}, we have that

$$\omega_{C_{12}} \cong (M_1 \otimes M_2)^{\otimes(m-2)}.$$

Composing the maps ρ_1 and ρ_2 to $\mathbb{P}(V_1^\vee)$ and $\mathbb{P}(V_2^\vee)$ with the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ gives an embedding of the curve C_{12} into $\mathbb{P}(V_1^\vee \otimes V_2^\vee) = \mathbb{P}^3$, and the line bundle $M_1 \otimes M_2$ is isomorphic to the pullback of $\mathcal{O}_{\mathbb{P}^3}(1)$ to the curve.

The group $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_3) \times \mathrm{GL}(V_4)$ acts on the space of $2 \times 2 \times m \times m$ boxes by the standard action on each factor. By definition, the curve C_{12} is fixed by the action of $\mathrm{GL}(V_3) \times \mathrm{GL}(V_4)$ and is transformed linearly by $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. Thus, the isomorphism classes of the curve and the line bundles M_i for $1 \leq i \leq 4$ coming from the box $\mathcal{B}$ are fixed by the action of the group G. Each factor $\mathrm{GL}(V_i)$ of G acts on the basis for $H^0(C_{12}, M_i)$ for $1 \leq i \leq 4$.

5.2 The Moduli Problem for $2 \times 2 \times m \times m$ Boxes

Just like for $3 \times n \times n$ boxes, the curve and line bundles arising from a nondegenerate $2 \times 2 \times m \times m$ box essentially specify the box. In this section, we show that the G-orbits of $2 \times 2 \times m \times m$ boxes parametrize the data of a genus $(m-1)^2$ curve with four line bundles subject to certain conditions, up to equivalence.

5.2.1 A Bijection

In order to recover a $2 \times 2 \times m \times m$ box from the data of curves and line bundles, we first study quintuples (C, M_1, M_2, M_3, M_4) subject to the following conditions:

(a) C is a smooth irreducible genus $(m-1)^2$ curve.

(b) M_1 and M_2 are nonisomorphic degree m line bundles on C, with $h^0(C, M_1) = 2$ and $h^0(C, M_2) = 2$, such that $(\phi_{M_1}, \phi_{M_2}) : C \to \mathbb{P}(H^0(C, M_1)^\vee) \times \mathbb{P}(H^0(C, M_2)^\vee)$ is a closed immersion.

(c) M_3 and M_4 are degree $m(m-1)$ line bundles on C, with $h^0(C, M_3) = m$ and $h^0(C, M_4) = m$.

(d) $h^0(C, M_i^{-1} \otimes M_j) = 0$ for $i = 1$ or 2 and $j = 3$ or 4.

(5.2)

(e) $M_3 \otimes M_4 \cong (M_1 \otimes M_2)^{\otimes(m-1)}$.

Conditions (b) and (c) imply that $|M_1|, |M_2| \in \hat{W}^1_m(C)$ and $|M_3|, |M_4| \in \hat{W}^{m-1}_{m(m-1)}(C)$, where $\hat{W}^r_d(C)$ parametrizes r-dimensional complete linear systems of degree d on the curve C. Also, in condition (d), note that hypotheses on only one of M_3 or M_4 suffice, since by Serre duality and Riemann-Roch, the vanishing of $H^0(C, M_i^{-1} \otimes M_3)$ for $i = 1$ and 2 is equivalent to the vanishing of $H^0(C, M_i^{-1} \otimes M_4)$ for $i = 1$ and 2.

We say two such quintuples (C, M_1, M_2, M_3, M_4) and $(C', M_1', M_2', M_3', M_4')$ are *equivalent* if there exists an isomorphism $\sigma : C \to C'$ such that $\sigma^* M_i' \cong M_i$ for $1 \leq i \leq 4$.

Theorem 5.4. *Let $m \geq 2$ be an integer. There exists a bijection*

$$\left\{\begin{array}{l} G\textit{-equivalence classes of non-} \\ \textit{degenerate } 2 \times 2 \times m \times m \textit{ boxes} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \textit{equivalence classes of quintuples} \\ (C, M_1, M_2, M_3, M_4) \textit{ satisfying } (5.2) \end{array}\right\}. \quad (5.3)$$

Proof. Given a nondegenerate $2 \times 2 \times m \times m$ box, we described in Section 5.1 how to produce a smooth genus $(m-1)^2$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$ and the corresponding four line bundles, and the G-action on the box preserves the equivalence class of this data. Lemma 5.1 shows that the quintuple thus constructed will satisfy conditions (5.2).

We claim that for (C, M_1, M_2, M_3, M_4) satisfying (5.2), the multiplication map

$$\begin{array}{rcl} \mu_{123} : H^0(C, M_1) \otimes H^0(C, M_2) \otimes H^0(C, M_3) & \xrightarrow{\mu_{23}} & H^0(C, M_1) \otimes H^0(C, M_2 \otimes M_3) \\ & & \downarrow \\ & & H^0(C, M_1 \otimes M_2 \otimes M_3) \end{array}$$

is surjective. In fact, the first map μ_{23} is an isomorphism, from the basepoint-free pencil trick and the fact that $H^0(M_2^{-1} \otimes M_3)$ and $H^1(M_2^{-1} \otimes M_3)$ both vanish, from condition (5.2)(d). Applying the basepoint-free pencil trick again reduces the problem to showing that $H^1(M_1^{-1} \otimes M_2 \otimes M_3) = 0$, which follows easily from the assumption that $H^1(M_1^{-1} \otimes M_3) = 0$.

We will construct the box by looking at the kernel of the map μ_{123}. First, we calculate the dimension of the image. Tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}_C(-M_1 - M_2) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{M_1 \otimes M_2} \longrightarrow 0$$

by $M_1 \otimes M_2 \otimes M_3$ and then taking cohomology shows that $H^1(C, M_1 \otimes M_2 \otimes M_3) = 0$, so $H^0(C, M_1 \otimes M_2 \otimes M_3) = 3m$ by Riemann-Roch. Therefore, the kernel of μ_{123} is a vector space of dimension m, whose inclusion into the domain of μ_{123} gives the $2 \times 2 \times m \times m$ box

$$\begin{aligned} \mathcal{B} &\in \operatorname{Hom}(\ker \mu_{123}, H^0(C, M_1) \otimes H^0(C, M_2) \otimes H^0(C, M_3)) \\ &\cong H^0(C, M_1) \otimes H^0(C, M_2) \otimes H^0(C, M_3) \otimes (\ker \mu_{123})^\vee. \end{aligned}$$

The box is well-defined up to changes of bases for $H^0(C, M_1), H^0(C, M_2), H^0(C, M_3)$, and $\ker \mu_{123}$. Furthermore, another isomorphic tuple will produce the same box, up to linear transformations in G, since all the vector spaces involved are naturally isomorphic and the multiplication maps commute with the isomorphisms.

It remains to check that the two constructions are inverse to one another. Given (C, M_1, M_2, M_3, M_4) satisfying (5.2), we obtain a box

$$\mathcal{B} \in H^0(C, M_1) \otimes H^0(C, M_2) \otimes H^0(C, M_3) \otimes (\ker \mu_{123})^\vee$$

up to choices of bases for each of those vector spaces. Let C_{12} and C_{123} be the natural images of C given by the products of linear systems (ϕ_{M_1}, ϕ_{M_2}) and $(\phi_{M_1}, \phi_{M_2}, \phi_{M_3})$, respectively. Let C'_{12} be the variety in $\mathbb{P}(H^0(C, M_1)^\vee) \times \mathbb{P}(H^0(C, M_2)^\vee)$ defined by the bidegree (m, m) polynomial equation

$$\det \mathcal{B}(w, x, \cdot, \cdot) = 0$$

and let C'_{123} in $\mathbb{P}(H^0(C, M_1)^\vee) \times \mathbb{P}(H^0(C, M_2)^\vee) \times \mathbb{P}(H^0(C, M_3)^\vee)$ be defined by

$$\mathcal{B}(w, x, y, \cdot) = 0$$

where $w \in \mathbb{P}(H^0(C, M_1)^\vee), x \in \mathbb{P}(H^0(C, M_2)^\vee)$, and $y \in \mathbb{P}(H^0(C, M_3)^\vee)$. It suffices to show, without loss of generality, that $C_{12} = C'_{12}$ and $C_{123} = C'_{123}$ as varieties. For all $(w^\dagger, x^\dagger, y^\dagger) \in C_{123}$, the construction of $\mathcal{B}$ implies that $\mathcal{B}(w^\dagger, x^\dagger, y^\dagger, \cdot) = 0$. So $C_{12} \subset C'_{12}$, but since both are given by bidegree (m, m) equations in $\mathbb{P}^1 \times \mathbb{P}^1$, they are equal. Similarly, we have $C_{123} \subset C'_{123}$, and their equality follows from their irreducibility.

On the other hand, a box $\mathcal{B} \in V_1 \otimes V_2 \otimes V_3 \otimes V_4$ produces (C, M_1, M_2, M_3, M_4) satisfying (5.2). Then for $1 \leq i \leq 3$, the vector spaces V_i and $H^0(C, M_i)$ are naturally isomorphic, and $V_4^\vee$ may be identified with $\ker \mu_{123}$; hence the box constructed from the quintuple via the inclusion of $\ker \mu_{123}$ into $H^0(C, M_1) \otimes H^0(C, M_2) \otimes H^0(C, M_3)$ is G-equivalent to $\mathcal{B}$. $\square$

Remark 5.5. As in Remark 4.7, the proof only uses one of the line bundles M_3 and M_4, and they may be interchanged. We may naturally identify $H^0(C, M_4)$, for example, with the dual of the kernel of the full multiplication map μ_{123}, which by the basepoint-free pencil trick, is isomorphic to both $H^0(C, M_1^{-1} \otimes M_2 \otimes M_3)$ and $H^0(C, M_1 \otimes M_2^{-1} \otimes M_3)$. We need not include the bundle M_4 in the geometric data, as the assumptions on M_1, M_2, and M_3 imply that the bundle $(M_1 \otimes M_2)^{\otimes(m-1)} \otimes M_3^{-1}$ has the correct degree and number of sections, and satisfies the cohomological conditions required. Likewise, $H^0(C, M_3)$ may be identified with the dual of the kernel of $\mu_{124} : H^0(C, M_1) \otimes H^0(C, M_2) \otimes H^0(C, M_4) \to H^0(C, M_1 \otimes M_2 \otimes M_4)$.

The conditions (5.2)(c) on the line bundles M_3 and M_4 translate into requiring that they are non-exceptional; condition (d) is equivalent to $M_i^{-1} \otimes M_j$ being non-exceptional for $i = 1$ and 2 and $j = 3$ and 4. By Remark 5.5, we also only need to record one of the line bundles M_3 and M_4 in the geometric data, so Theorem 5.4 simplifies to

Proposition 5.6. *Let $m \geq 2$ be an integer. There is a bijection*

$$\left\{ \begin{array}{l} G\text{-equivalence} \\ \text{classes of} \\ \text{nondegenerate} \\ 2 \times 2 \times m \times m \\ \text{boxes} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{equivalence classes of triples } (C, \iota, M), \text{ with } C \text{ a genus} \\ (m-1)^2 \text{ curve, } \iota \text{ a bidegree } (m,m) \text{ embedding of} \\ C \text{ into } \mathbb{P}^1 \times \mathbb{P}^1 \text{ and } M \text{ a degree } m(m-1) \text{ non-} \\ \text{exceptional line bundle on } C \text{ with } M \otimes \iota^*\mathcal{O}(-1,0) \text{ and} \\ M \otimes \iota^*\mathcal{O}(0,-1) \text{ non-exceptional} \end{array} \right\}. \quad (5.4)$$

The triples (C, ι, M) and (C', ι', M') are equivalent, as usual, if there exists an isomorphism $\sigma : C \to C'$ with $\iota'\sigma = \iota$ and $\sigma^* M' = M$.

We may also rigidify each side of the bijection in Theorem 5.4 to obtain a parametrization of curves and line bundles by $2 \times 2 \times m \times m$ boxes, not just their G-orbits. A $2 \times 2 \times m \times m$ box includes not just an element $\mathcal{B} \in V_1 \otimes V_2 \otimes V_3 \otimes V_4$ but also specified bases for each vector space V_i for $1 \leq i \leq 4$. For the curve and line bundles (C, M_1, M_2, M_3, M_4) satisfying (5.2), let $\mathfrak{B}_i$ be a basis for $H^0(C, M_i)$ for $1 \leq i \leq 4$. Two such *rigidified quintuples* $(C, (M_i, \mathfrak{B}_i)_{i=1}^4)$ and $(C', (M'_i, \mathfrak{B}'_i)_{i=1}^4)$ are equivalent if there exists an isomorphism $\sigma : C \to C'$ such that $\sigma^* M'_i \cong M_i$ and the isomorphism $\sigma^* : H^0(C', M'_i) \to H^0(C, M_i)$ takes $\mathfrak{B}'_i$ to $\mathfrak{B}_i$ for $1 \leq i \leq 4$. Because of the identifications $H^0(C, M_4) \cong (\ker \mu_{123})^\vee$ and $H^0(C, M_3) \cong (\ker \mu_{124})^\vee$ by Remark 5.5, the proposition below follows from the proof of Theorem 5.4:

Proposition 5.7. *There exists a bijection*

$$\left\{ \begin{array}{l} \textit{nondegenerate} \\ 2 \times 2 \times m \times m \\ \textit{boxes} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{equivalence classes of } (C, M_1, M_2, M_3, M_4, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4) \\ \textit{where } (C, M_1, M_2, M_3, M_4) \textit{ satisfies conditions } (5.2) \textit{ and} \\ \mathfrak{B}_i \textit{ is a basis for } H^0(C, M_i) \textit{ for } 1 \leq i \leq 4 \end{array} \right\},$$

and quotienting each side by $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_m \times \mathrm{GL}_m$ *recovers Theorem 5.4.*

5.2.2 Moduli Stack Formulation

Like in Section 4.2.2, we prove that the constructions for $2 \times 2 \times m \times m$ boxes work in families over arbitrary $\mathbb{Z}[\frac{1}{N}]$-schemes, where $N = m(m-1)$. Proposition 5.7 can then be rewritten as an equivalence of moduli stacks (in fact, schemes), and taking quotients by the natural $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_m \times \mathrm{GL}_m$-action gives an equivalence of the quotient stacks. The proofs are essentially formal once the relative data is defined. We work exclusively with $\mathbb{Z}[\frac{1}{N}]$-schemes in this section, so the moduli stacks are also defined over $\mathbb{Z}[\frac{1}{N}]$.

A *based* $2 \times 2 \times m \times m$ *box over a scheme* S consists of the following: for $i = 1$ and 2, a free rank 2 $\mathcal{O}_S$-module $\mathcal{V}_i$ with an isomorphism $\psi_i : \mathcal{V}_i \to \mathcal{O}_S^{\oplus 2}$; for $j = 3$ and 4, a free rank m $\mathcal{O}_S$-module $\mathcal{V}_j$ with an isomorphism $\psi_j : \mathcal{V}_j \to \mathcal{O}_S^{\oplus m}$; and a section $\mathcal{B}$ of $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \otimes \mathcal{V}_4$. An isomorphism of based $2 \times 2 \times m \times m$ boxes $((\mathcal{V}_i, \psi_i)_{i=1}^4, \mathcal{B})$ and $((\mathcal{V}'_i, \psi'_i)_{i=1}^4, \mathcal{B}')$ consists of isomorphisms $\sigma_i : \mathcal{V}_i \xrightarrow{\cong} \mathcal{V}'_i$ with $\psi_i = \psi'_i \circ \sigma_i$ for $1 \leq i \leq 4$ and taking $\mathcal{B}$ to $\mathcal{B}'$. A based $2 \times 2 \times m \times m$ box is nondegenerate if it is locally nondegenerate.

Without bases, a $2 \times 2 \times m \times m$ *box over* S is a section $\mathcal{B}$ of $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \otimes \mathcal{V}_4$, where

$\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4$ are locally free $\mathcal{O}_S$-modules of rank $2, 2, m, m$, respectively. An isomorphism of boxes $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{B})$ and $(\mathcal{V}_1', \mathcal{V}_2', \mathcal{V}_3', \mathcal{V}_4', \mathcal{B}')$ consists of isomorphisms $\sigma_i : \mathcal{V}_i \xrightarrow{\cong} \mathcal{V}_i'$ of $\mathcal{O}_S$-modules for $1 \leq i \leq 4$ that take $\mathcal{B}$ to $\mathcal{B}'$.

On the other hand, let the *rigidified degree m quintuple* $\mathcal{D}$ over S be a genus $(m-1)^2$ curve $\pi : C \to S$; for $i = 1$ and 2, a degree m line bundle $\mathcal{M}_i$ on C with an isomorphism $\chi_i : \mathrm{R}^0\pi_*(\mathcal{M}_i) \xrightarrow{\cong} \mathcal{O}_S^{\oplus 2}$; and for $j = 3$ and 4, a degree $m(m-1)$ line bundle $\mathcal{M}_j$ on C with an isomorphism $\chi_j : \mathrm{R}^0\pi_*(\mathcal{M}_j) \xrightarrow{\cong} \mathcal{O}_S^{\oplus m}$. A *balanced* rigidified degree m quintuple also includes an isomorphism $\varphi : \mathcal{M}_1^{\otimes(m-1)} \otimes \mathcal{M}_2^{\otimes(m-1)} \xrightarrow{\cong} \mathcal{M}_3 \otimes \mathcal{M}_4 \otimes \pi^* L_S$ for some line bundle L_S on S. Such a quintuple is *nondegenerate* if $\mathcal{M}_1$ and $\mathcal{M}_2$ are not isomorphic fiberwise and if $\mathrm{R}^0\pi_*(\mathcal{M}_i^\vee \otimes \mathcal{M}_j) = 0$ for $i = 1$ and 2 and $j = 3$ and 4. This latter nondegeneracy condition is equivalent to the analogous fiberwise statement, since the Euler characteristic of $\mathcal{M}_i^\vee \otimes \mathcal{M}_j$ is 0.

Theorem 5.8. *Over a scheme S, there is an equivalence between the category of nondegenerate based $2 \times 2 \times m \times m$ boxes over S and the category of nondegenerate balanced rigidified degree m quintuples $(C, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \varphi)$ over S.*

Proof. Both the functors are essentially the same as those over Spec F. As $\mathcal{V}_i$ is free for $1 \leq i \leq 4$, we construct the curve and line bundles much as in Theorem 5.4. (See the proof of Theorem 3.15 for details on how to generalize the construction over S.)

Given a curve $\pi : C \to S$ and invertible sheaves $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$ with the maps χ_i and φ, forming a nondegenerate balanced rigidified degree m quintuple, the kernel of the surjective multiplication map

$$\mu_{123} : \mathrm{R}^0\pi_*(\mathcal{M}_1) \otimes \mathrm{R}^0\pi_*(\mathcal{M}_2) \otimes \mathrm{R}^0\pi_*(\mathcal{M}_3) \longrightarrow \mathrm{R}^0\pi_*(\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathcal{M}_3)$$

is a rank m free $\mathcal{O}_S$-module. The dual of this kernel can be identified with $\mathrm{R}^0\pi_*(\mathcal{M}_4)$ by Remark 5.5, which gives a trivialization $(\ker \mu_{123})^\vee \xrightarrow{\cong} \mathcal{O}_S^{\oplus m}$. Thus, we have a recovered a based $2 \times 2 \times m \times m$ box as a section in $\mathrm{R}^0\pi_*(\mathcal{M}_1) \otimes \mathrm{R}^0\pi_*(\mathcal{M}_2) \otimes \mathrm{R}^0\pi_*(\mathcal{M}_3) \otimes (\ker \mu_{123})^\vee$ with trivializations for each of these vector bundles.

Because these two functors are locally inverse from Theorem 5.4, they are inverse. $\square$

The space of based $2 \times 2 \times m \times m$ boxes is just the affine space $\mathbb{A}^{4m^2}$, so the moduli space of nondegenerate balanced rigidified degree m quintuples over S is an open subscheme of $\mathbb{A}^{4m^2}$ over S.

For $2 \times 2 \times m \times m$ boxes without bases, the corresponding stack is equivalent to the quotient stack $[\mathbb{A}^{4m^2}/\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_m \times \mathrm{GL}_m]$, where we have described the group action as linear transformations in each direction of the $2 \times 2 \times m \times m$ box. We are interested in the nondegenerate open substack of this quotient stack, given locally by the nonvanishing of the discriminant.

To unrigidify the geometric data, we consider genus $(m-1)^2$ curves $\pi : C \to S$ with line bundles $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$, and $\mathcal{M}_4$ of degrees $m, m, m(m-1)$, and $m(m-1)$, respectively, such that $\mathrm{R}^0\pi_*(\mathcal{M}_1)$ and $\mathrm{R}^0\pi_*(\mathcal{M}_2)$ have rank 2 and $\mathrm{R}^0\pi_*(\mathcal{M}_3)$ and $\mathrm{R}^0\pi_*(\mathcal{M}_4)$ have rank m. With an isomorphism $\varphi : \mathcal{M}_1^{\otimes(m-1)} \otimes \mathcal{M}_2^{\otimes(m-1)} \xrightarrow{\cong} \mathcal{M}_3 \otimes \mathcal{M}_4 \otimes \pi^* L_S$ for some line bundle L_S on S, and the conditions that $\mathcal{M}_1$ and $\mathcal{M}_2$ are nonisomorphic fiberwise and $\mathrm{R}^0\pi_*(\mathcal{M}_i^\vee \otimes \mathcal{M}_j) = 0$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$, we call $(C, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \varphi)$ a *nondegenerate balanced degree m quintuple.*

The bijection of Proposition 5.7 is $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_m \times \mathrm{GL}_m$-equivariant, giving the bijection (5.3). In the same way, the functors in Theorem 5.8 are also $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_m \times \mathrm{GL}_m$-equivariant, so we obtain an equivalence of the respective quotient stacks over $\mathbb{Z}[\frac{1}{N}]$.

Corollary 5.9. *For $m \geq 2$, the nondegenerate substack of $[\mathbb{A}^{4m^2}/\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_m \times \mathrm{GL}_m]$ is equivalent to the stack of nondegenerate balanced degree m quintuples.*

Note that the stack of nondegenerate balanced degree m quintuples is visibly a substack of the fiber product

$$\mathrm{Pic}_g^m \times_{\mathscr{M}_g} \mathrm{Pic}_g^m \times_{\mathscr{M}_g} \mathrm{Pic}_g^{m(m-1)} \times_{\mathscr{M}_g} \mathrm{Pic}_g^{m(m-1)},$$

where $g = (m-1)^2$ and Pic_g^d denotes the universal degree d Picard stack over the moduli space $\mathscr{M}_g$ of genus g curves.

5.2.3 Explicit Algebraic Construction

Given the geometric data of a bidegree (m, m) embedding ι of a smooth projective genus $(m-1)^2$ curve C into $\mathbb{P}^1 \times \mathbb{P}^1$ and degree $m(m-1)$ non-exceptional line bundles M_3 and M_4 with $M_3 \otimes M_4 \cong \iota^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m-1, m-1)$ and the usual cohomological conditions, we may also construct the box using the explicit method of Section 4.2.3.

Suppose f is the bidegree (m, m) polynomial in the bigraded ring $S := F[w_1, w_2, x_1, x_2]$ defining the curve C in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $R := S/(f)$ be the coordinate ring of the curve C. Let $S^{i,j}$ and $R^{i,j}$ denote the (i, j)th graded pieces of S and R, respectively. Let D_3 be an effective degree $m(m-1)$ divisor on C such that $\mathcal{O}(D_3) \cong M_3$. Tensoring the exact sequence $0 \to M_3^{-1} \to \mathcal{O}_C \to \mathcal{O}_{D_3} \to 0$ with $\iota^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m-1, m-1)$ produces

$$0 \longrightarrow M_4 \longrightarrow \iota^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m-1, m-1) \longrightarrow \mathcal{O}_{D_3} \otimes \iota^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m-1, m-1) \longrightarrow 0.$$

Global sections of M_4 can be viewed as bidegree $(m-1, m-1)$ polynomials, all of which vanish on the points of the divisor D_3; in fact, the sections of M_4 are exactly the bidegree $(m-1, m-1)$ polynomials that vanish on D_3. Let $s_1 =: q, s_2, \ldots, s_m$ be a basis for $H^0(C, M_4)$, where each $s_i \in S^{m-1,m-1}$. The variety $\{q = 0\}$ intersects the curve in $2m(m-1)$ points (with multiplicity); that is, it determines the divisor $D_3 + D_4$, where D_4 is an effective degree $m(m-1)$ divisor with $M_4 \cong \mathcal{O}(D_4)$ from the relation $M_3 \otimes M_4 \cong \iota^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m-1, m-1)$. Let $r_1 = q, r_2, \ldots, r_m$ be a basis for $H^0(C, M_3)$, where each $r_i \in S^{m-1,m-1}$ is viewed as a bidegree $(m-1, m-1)$ polynomial.[1]

The ideal sheaf $I_{D_3} \cong \mathcal{O}(-D_3)$ is generated by the images of $s_1, \ldots, s_m$ in R, and similarly, the ideal sheaf I_{D_4} is generated by the images of $r_1, \ldots, r_m$. Since $I_{D_3} I_{D_4} = I_{D_3+D_4}$ is generated as an ideal of R by the image of q, we may write, for $1 \leq i, j \leq m$,

$$r_i s_j = d_{ij} q + c_{ij} f \tag{5.5}$$

for some $d_{ij} \in S^{m-1,m-1}$ and $c_{ij} \in S^{m-2,m-2}$. We have, for example, $d_{1j} = s_j$ and $d_{i1} = r_i$.

[1] The bases for $H^0(C, M_3)$ and $H^0(C, M_4)$ are actually given by $\{1, r_2/q, \ldots, r_m/q\}$ and $\{1, s_2/q, \ldots, s_m/q\}$, respectively.

Since each $(m-1) \times (m-1)$ minor d^*_{ij} of the $m \times m$ matrix $D = (d_{ij})$ is divisible by f^{m-2}, there exist $b_{klij} \in F$ for $1 \leq k \leq 3$ and $1 \leq i, j \leq n$ such that

$$d^*_{ij}/f^{m-2} = b_{11ij}w_1x_1 + b_{12ij}w_1x_2 + b_{21ij}w_2x_1 + b_{22ij}w_2x_2, \tag{5.6}$$

which determines the $2 \times 2 \times m \times m$ box $\mathcal{B} = (b_{klij})$. It is easy to check algebraically that this box $\mathcal{B}$ is G-equivalent to the box constructed geometrically in the proof of Theorem 5.4.

5.3 Symmetrizations

Just as for all other boxes studied, the moduli problem for $2 \times 2 \times m \times m$ boxes may be modified to apply to symmetrized $2 \times 2 \times m \times m$ boxes, specifically $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$-orbits of $V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_3$, where V_1, V_2, and V_3 are vector spaces of dimension 2,2, and m, respectively. By interpreting an element of the space

$$V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_3 \hookrightarrow V_1 \otimes V_2 \otimes V_3 \otimes V_3$$

as a $2 \times 2 \times m \times m$ box $\mathcal{B} = (b_{klij})$ with $b_{klij} = b_{klji}$ for $1 \leq k, l \leq 2$ and $1 \leq i, j \leq m$, we see that a symmetrized box produces a smooth genus $(m-1)^2$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$, and the symmetry implies that the two resulting degree $m(m-1)$ bundles L_3 and L_4 on the curve are isomorphic.

Suppose $C \overset{\iota}{\hookrightarrow} \mathbb{P}^1 \times \mathbb{P}^1$ is a genus $(m-1)^2$ curve. Let M_3 be a degree $m(m-1)$ line bundle on C with $M_3^{\otimes 2} \cong \iota^*\mathcal{O}(m-1, m-1)$ such that all three of the line bundles $M_3, M_3 \otimes \iota^*\mathcal{O}(-1, 0)$, and $M_3 \otimes \iota^*\mathcal{O}(0, -1)$ are non-exceptional. Then the explicit algebraic construction in Section 5.2.3 produces a $2 \times 2 \times m \times m$ box in $H^0(C, \iota^*\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0)) \otimes H^0(C, \iota^*\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1)) \otimes \mathrm{Sym}^2 H^0(C, M_3)$. In particular, taking $M_3 \cong M_4$ and $D_3 = D_4$ in the construction allows the identification of sections $r_i = s_i$ for $1 \leq i \leq m$. Because $d_{ij} = d_{ji}$ for $1 \leq i, j \leq m$, the box constructed will be symmetric, that is, $b_{klij} = b_{klji}$ for $1 \leq k, l \leq 2$ and $1 \leq i, j \leq m$.

More specifically, we may describe both symmetrized $2 \times 2 \times m \times m$ boxes as well as their orbits in terms of geometric data, where equivalence of such data is the usual notion. As the proof of the following theorem is almost identical to that of Theorem 4.12, we omit the details.

Theorem 5.10. *Let $m \geq 2$ be an integer. Then the restriction of Proposition 5.7 to symmetrized $2 \times 2 \times m \times m$ boxes induces a bijection of the rigidified data*

$$\left\{\begin{matrix}\text{nondegenerate symmetrized} \\ 2 \times 2 \times m \times m \text{ boxes}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{equivalence classes of} \\ (C, M_1, M_2, M_3, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3)\end{matrix}\right\} \tag{5.7}$$

and restricting Theorem 5.4 to symmetrized boxes gives a bijection of the quotients

$$\left\{\begin{matrix}\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_m\text{-equivalence classes of non-} \\ \text{degenerate symmetrized } 2 \times 2 \times m \times m \text{ boxes}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{equivalence classes of} \\ (C, M_1, M_2, M_3)\end{matrix}\right\}$$

where C is a genus $(m-1)^2$ curve; M_1 and M_2 are nonisomorphic degree m line bundles on C with $h^0(C, M_1) = 2$ and $h^0(C, M_2) = 2$; and M_3 is a non-exceptional degree $m(m-1)$ line bundle on C such that $(M_1 \otimes M_2)^{\otimes(m-1)} \cong M_3^{\otimes 2}$ and both $M_3 \otimes M_1^{-1}$ and $M_3 \otimes M_2^{-1}$ are non-exceptional line bundles. In bijection (5.7), the symbol $\mathfrak{B}_i$ for $1 \leq i \leq 3$ denotes a basis for $H^0(C, M_i)$.

5.4 Special Cases

For small values of m, the theorems on $2 \times 2 \times m \times m$ boxes specialize to simpler statements. If $m = 2$, many of the hypotheses are unnecessary, and the degeneracy condition is very simple. In particular, what we have called a nondegenerate $2 \times 2 \times 2 \times 2$ box here coincides with the notion of a nondegenerate hypercube as in Section 3.2. The specialization of Theorem 5.4 to $m = 2$ is exactly Proposition 3.11.

For $m = 3$, recall that the canonical embedding of a nonhyperelliptic genus 4 curve C lies on a unique quadric surface, either smooth or singular; if the quadric is smooth, we call the genus 4 curve of *Type I*. For a genus 4 curve C arising from a $2 \times 2 \times 3 \times 3$ box, it is of Type I since it lies on the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ by construction. In this case, there are exactly two g^1_3's on the curve C, corresponding to the two rulings of the quadric [ACGH85, p. 206]. Let L and L' denote the degree 3 line bundles corresponding to these g^1_3's. Then there are exactly two pairs of line bundles that we call (M_1, M_2), that is, either (L, L') or (L', L), and clearly $L \otimes L' \cong \omega_C$.

In contrast, if a nonhyperelliptic genus 4 curve lies on a singular quadric surface in $\mathbb{P}^3$, then it has exactly one g^1_3. In this case, the square of the corresponding degree 3 line bundle is the canonical bundle. Thus, the condition that a nonhyperelliptic genus 4 curve C is of Type I is equivalent to C not having a theta-characteristic κ with $h^0(C, \kappa) = 2$, that is, C not having a vanishing even theta-constant. It is well-known that the general genus 4 curve is of Type I, since the hyperelliptic locus of the moduli space $\mathscr{M}_4$ of genus 4 curves has dimension 7 and the locus of curves in $\mathscr{M}_4$ whose canonical embedding lies on a singular quadric has dimension 8.

Any noncanonical degree 6 line bundle M on C is non-exceptional, and the degree 6 bundles M_3 and M_4 naturally arising from the box satisfy the relation $M_3 \otimes M_4 \cong \omega_C^{\otimes 2}$. So giving a nonzero point, say, $P = M_3 \otimes \omega_C^{-1}$ in the Jacobian $\mathrm{Jac}(C) \cong \mathrm{Pic}^0(C)$ of C is the same as giving M_3 and M_4, up to isomorphism, and Theorem 5.4 simplifies.

Corollary 5.11. *There exists a bijection*

$$\left\{\begin{array}{l}\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_3 \times \mathrm{GL}_3\text{-}\\ \textit{equivalence classes of non-}\\ \textit{degenerate } 2\times2\times3\times3 \textit{ boxes}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{isomorphism classes of } (C, M_1, P) \textit{ where } C \textit{ is a}\\ \textit{Type I genus } 4 \textit{ curve, } M_1 \textit{ is a degree } 3 \textit{ line bundle}\\ \textit{on } C \textit{ with } h^0(C, M_1) = 2, \textit{ and } 0 \neq P \in \mathrm{Jac}(C)\\ \textit{such that } M_1 \otimes P \textit{ and } M_1 \otimes P^{-1} \textit{ have no nonzero}\\ \textit{sections}\end{array}\right\}.$$

Each side is an étale double cover of the set of isomorphism classes of (C, P) satisfying the conditions on the right hand side of the bijection.

Note that the conditions that the degree 3 line bundles $M_1 \otimes P$ and $M_1 \otimes P^{-1}$ have no nonzero sections are necessary, since there exist degree 3 line bundles L on C with $h^0(C, L) = 1$ and $h^0(C, \omega_C \otimes L^{-1}) = 1$, e.g., odd theta-characteristics L. The general point P on $\mathrm{Jac}(C)$, however, will satisfy these conditions.

For the symmetrized $2 \times 2 \times 3 \times 3$ box, the point on the Jacobian is 2-torsion, since $M_3 \cong M_4$ implies that $(M_3 \otimes \omega_C^{-1})^{\otimes 2} \cong \mathcal{O}_C$. One direction of the following bijection was known classically by Wirtinger in another form [Wir85, Cat83]:

Corollary 5.12. *There exists a bijection*

$$\left\{\begin{array}{l}\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_3\text{-}\textit{orbits}\\ \textit{of nondegenerate sym-}\\ \textit{metrized } 2\times 2\times 3\times 3\\ \textit{boxes}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{isomorphism classes of } (C, M_1, P) \textit{ with } C \textit{ a}\\ \textit{Type I genus } 4 \textit{ curve, } M_1 \textit{ a degree } 3 \textit{ line bun-}\\ \textit{dle on } C \textit{ with } h^0(C, M_1) = 2, \textit{ and } 0 \neq P \in\\ \mathrm{Jac}(C)[2] \textit{ with } h^0(C, M_1 \otimes P) = 0\end{array}\right\}. \quad (5.8)$$

Each side is an étale double cover of the set of isomorphism classes (C, P) satisfying the conditions on the right hand side of the bijection.

5.4.1 Recillas' Trigonal Construction and Pryms

For $m = 3$, another interpretation of the $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$-orbits of the space $V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_3$ of symmetrized boxes recovers a construction of Recillas [Rec74].

By a slight abuse of notation, we will use $\mathcal{B}(\cdot,\cdot,\cdot) \in V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_3$ to denote the tridegree $(1, 1, 2)$ form $\mathcal{B}$, which may be evaluated on $(w, x, y) \in V_1^\vee \times V_2^\vee \times V_3^\vee$. By viewing a nondegenerate element $\mathcal{B} \in V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_3$ as a 2×2 matrix $M_\mathcal{B}$ of ternary quadratics,[2] we see that there is a $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2)$-invariant and $\mathrm{SL}(V_3)$-equivariant map

$$V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_3 \longrightarrow \mathrm{Sym}^4 V_3$$
$$\mathcal{B} \mapsto Q_\mathcal{B} := \det M_\mathcal{B}$$

by taking the determinant of the 2×2 matrix $M_\mathcal{B}$. In other words, this ternary quartic form $Q_\mathcal{B}$ is a covariant of the action of $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \times \mathrm{SL}(V_3)$ on the representation $V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_3$.

We will further restrict ourselves to boxes $\mathcal{B}$ where the locus X of the ternary quartic $Q_\mathcal{B}$ is a smooth plane quartic, that is, when the degree 27 discriminant of $Q_\mathcal{B}$ is nonzero. Call such boxes *smooth*. Furthermore, smoothness of $\mathcal{B}$ implies that the genus 3 nonhyperelliptic curve X also has natural maps to $\mathbb{P}(V_1^\vee)$ and $\mathbb{P}(V_2^\vee)$, sending $y \in X \subset \mathbb{P}(V_3^\vee)$ to the kernels of the matrix $M_\mathcal{B}(y)$, viewed as an element of $\mathrm{Hom}(V_1^\vee, V_2)$ and of $\mathrm{Hom}(V_2^\vee, V_1)$. The graph

[2] Recall that $\mathrm{Sym}_2 V_3$ and $\mathrm{Sym}^2 V_3$ are canonically isomorphic over any $\mathbb{Z}[\frac{1}{2}]$-scheme. Although over a ring like $\mathbb{Z}$ the ternary quadratics in $\mathrm{Sym}_2 V_3$ must have even cross terms, here there is no such restriction.

of this map for $\mathbb{P}(V_1^\vee)$ is the curve

$$X_{13} := \{(w, y) \in \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_3^\vee) : \mathcal{B}(w, \cdot, y) = 0\}$$

cut out by two bidegree $(1, 2)$ equations in $\mathbb{P}^1 \times \mathbb{P}^2$. Then X_{13} is isomorphic to the genus 3 curve X via the projection to $\mathbb{P}(V_3^\vee)$, and the projection to $\mathbb{P}(V_1)^\vee$ is a degree 4 covering. The pullback of $\mathcal{O}_{\mathbb{P}(V_1^\vee)}$ to the curve X_{13} is a degree 4 noncanonical line bundle.

Similarly, we may define X_{23} in $\mathbb{P}(V_2^\vee) \times \mathbb{P}(V_3^\vee)$, and a line bundle as the pullback of $\mathcal{O}_{\mathbb{P}(V_2^\vee)}(1)$ to X_{23}. Pulling back these two degree 4 line bundles to bundles L_1 and L_2 on X, we obtain the relation

$$L_1 \otimes L_2 \cong \omega_X^{\otimes 2}, \tag{5.9}$$

of line bundles on X, where ω_X denotes the canonical bundle on X. Note that since X is a smooth plane quartic, the pullback of $\mathcal{O}(1)$ from $\mathbb{P}(V_3^\vee)$ to X is isomorphic to ω_X. The isomorphism (5.9) follows from an analogous argument to the proof of Lemma 5.2, namely by showing that the locus of any entry of $M_{\mathcal{B}}$ on X sums to the divisor associated both to the pullback of $\mathcal{O}(2)$ from $\mathbb{P}(V_3^\vee)$ and to the tensor product of L_1 and L_2. Also, the relation (5.9) implies that the data of the nonzero point $L_1 \otimes \omega_X^{-1}$ on the Jacobian of X is equivalent to the data of the line bundles L_1 and L_2.

The genus 3 curve X and the line bundles L_1 and L_2 together recover the symmetric projective box $\mathcal{B}$, up to the action of the group $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_3$. In particular, the box will be given by the kernel of the multiplication map

$$\mu : H^0(X, L_1) \otimes \mathrm{Sym}^2 H^0(X, \omega_X) \longrightarrow H^0(X, L_1 \otimes \omega_X^{\otimes 2}).$$

Max Noether's theorem ([ACGH85, chap. III]) and dimension considerations imply that the natural map $\mathrm{Sym}^2 H^0(X, \omega_X) \longrightarrow H^0(X, \omega_X^{\otimes 2})$ is an isomorphism in this case, and applying the basepoint-free pencil trick gives the surjection

$$H^0(X, L_1) \otimes H^0(X, \omega_X^{\otimes 2}) \longrightarrow H^0(X, L_1 \otimes \omega_X^{\otimes 2})$$

where the kernel may be identified with $H^0(X, L_1^{-1} \otimes \omega_X^{\otimes 2})$, which by (5.9), is also naturally isomorphic to $H^0(X, L_2)$. The kernel of the composition μ has dimension 2, and the inclusion of the kernel into $H^0(X, L_1) \otimes \operatorname{Sym}^2 H^0(X, \omega_X)$ is the desired box.

Including bases for each of the vector spaces, we obtain a rigidified bijection between boxes and the appropriate data of curves and line bundles; quotienting each side by the natural group action gives a parametrization of nonhyperelliptic genus 3 curves with nonzero points on their Jacobians by orbits of smooth boxes. In the proposition below, the notion of equivalence on the tuples is the natural one, where two equivalent tuples differ by an isomorphism of the curve that preserves the line bundles and bases. The proof is similar to that of previous such bijections, e.g., Theorem 5.4.

Proposition 5.13. *There exists a bijection*

$$\left\{\begin{array}{l}\textit{smooth}\\ \textit{symmetrized}\\ 2\times 2\times 3\times 3\\ \textit{boxes}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{equivalence classes of } (X, L_1, L_2, L_3, \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3), \textit{ with } X \textit{ a}\\ \textit{nonhyperelliptic genus } 3 \textit{ curve, } L_1 \textit{ and } L_2 \textit{ noncanonical degree}\\ 4 \textit{ line bundles on } X \textit{ with } L_1 \otimes L_2 \cong \omega_X^{\otimes 2}, \ L_3 \textit{ a degree } 4 \textit{ line}\\ \textit{bundle on } X \textit{ with } L_3 \cong \omega_X, \textit{ and } \mathfrak{B}_i \textit{ is a basis for } H^0(X, L_i)\\ \textit{for } 1 \leq i \leq 3\end{array}\right\}.$$

Quotienting each side by the natural action of $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_3$ *gives the correspondence*

$$\left\{\begin{array}{l}\mathrm{GL}_2{\times}\mathrm{GL}_2{\times}\mathrm{GL}_3\textit{-orbits}\\ \textit{of smooth symmetrized}\\ 2\times 2\times 3\times 3 \textit{ boxes}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{isomorphism classes of } (X, L) \textit{ where } X \textit{ is a}\\ \textit{nonhyperelliptic genus } 3 \textit{ curve with a non-}\\ \textit{canonical degree } 4 \textit{ line bundle } L \textit{ on } X\end{array}\right\}. \tag{5.10}$$

Proof. Note that the constructions described above in each direction are equivariant for the action of the group $G = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_3$. In particular, if $g = (g_1, g_2, g_3) \in G$ acts on a box $\mathcal{B}$, then the ternary quartic $Q_{g(\mathcal{B})}$ differs from $Q_{\mathcal{B}}$ by exactly the element g_3 of GL_3, and the points on their respective Jacobians are identified by the isomorphism. In the other direction, the box is constructed exactly up to choices of bases for each of the vector spaces $H^0(X, L_1), H^0(X, \omega_X)$, and $\ker \mu$, where a basis for $\ker \mu$ is given by a basis for $H^0(C, L_2)$.

Now we only need to show that the constructions above are inverse to one another. Let X be a smooth nonhyperelliptic genus 3 curve and L a noncanonical degree 4 line bundle on

X. We construct a symmetrized box $\mathcal{B} \in (\ker \mu)^\vee \otimes H^0(X, L) \otimes \operatorname{Sym}^2 H^0(X, \omega_X^{\otimes 2})$ as above. Let X_3 be the image of X in $\mathbb{P}(H^0(X, \omega_X)^\vee)$ and X_{13} the image of X in $\mathbb{P}(H^0(X, L)^\vee) \times \mathbb{P}(H^0(X, \omega_X^\vee)$. On the other hand, let $X_3' := \{y \in \mathbb{P}(H^0(X, \omega_X)^\vee) : \det \mathcal{B}(\cdot, \cdot, y) = 0\}$ and $X_{13}' := \{(w, y) \in \mathbb{P}(H^0(X, L)^\vee) \times \mathbb{P}(H^0(X, \omega_X)^\vee) : \mathcal{B}(w, \cdot, y) = 0\}$. Then for $(w, y) \in X_{13}$, we have $\mathcal{B}(x, \cdot, y) = 0$, so $X_{13} \subset X_{13}'$ and $X_3 \subset X_3'$. On the other hand, the variety X_3' is given by a degree 4 equation in $\mathbb{P}^2$, and since X_3 is the canonical embedding of a smooth nonhyperelliptic genus 3 curve, we must have $X_3' \subset X_3$. Thus, these curves are all isomorphic as varieties and the corresponding line bundles commute with the isomorphisms.

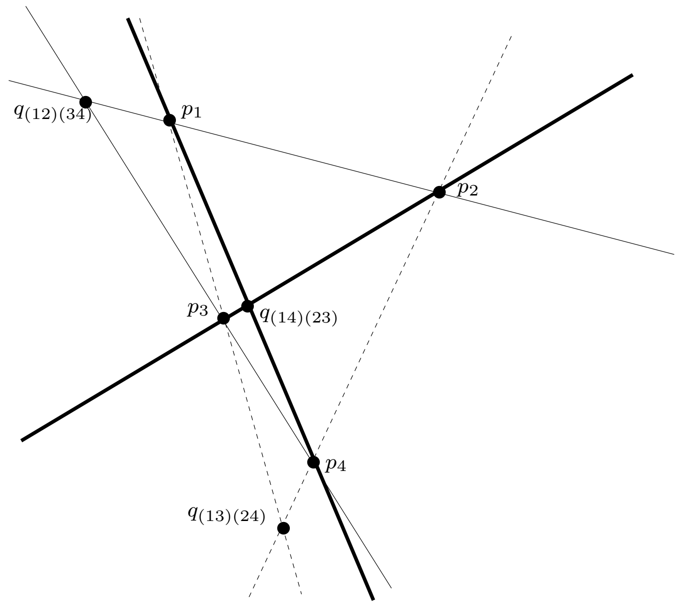
In the other direction, a box $\mathcal{B} \in V_1 \times V_2 \times \operatorname{Sym}^2 V_3$ produces the pair (X, L); identifying the bases for the vector spaces V_1, V_2, and V_3 with $H^0(X, L), (\ker \mu)^\vee$, and $H^0(X, \omega_X)$, respectively, reconstructs the box $\mathcal{B}$. □

Combining Corollary 5.12 and Proposition 5.13 gives a relation between genus 3 and genus 4 curves. In fact, we will show below that our setup satisfies the relationship of [Rec74]: the Jacobian of the genus 3 curve X is naturally isomorphic to the Prym of the étale double cover of the genus 4 curve C given by the 2-torsion point P.

First, we briefly recall the trigonal construction of Recillas, which gives a more general correspondence between connected simply branched degree 4 covers of $\mathbb{P}^1$ and connected unramified double covers of connected simply branched degree 3 covers of $\mathbb{P}^1$ (see [BL04, chap. 12] for an exposition).

Let X be a genus g curve with a degree 4 map $\rho : X \to \mathbb{P}^1$. The basic idea is that the four sheets of X over $\mathbb{P}^1$, labeled $\{1, 2, 3, 4\}$, may be used to construct three sheets, labeled $\{(12)(34), (13)(24), (14)(23)\}$, that form a degree 3 cover C over $\mathbb{P}^1$. More precisely, there is a pencil of planes $\{\mathbb{P}^2_t : t \in \mathbb{P}^1\}$ intersecting the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$ in exactly the linear series given by ρ, that is, the points of $\mathbb{P}^2_t \cap X$ are exactly the preimage of $t \in \mathbb{P}^1$ under ρ. For any given $t \in \mathbb{P}^1$, these four points p_1, p_2, p_3, p_4 determine three points $q_{(12)(34)}, q_{(13)(24)}, q_{(14)(23)}$ as the intersection of the pairs of diagonals, as in Figure 5.4.1. These three points, over $\mathbb{P}^1$, form a trigonal curve C of genus $g+1$, and the diagonals give an unramified double cover of C. The Prym variety of this étale double cover of C is isomorphic to the Jacobian of X.

Figure 5.1: Four noncollinear points in $\mathbb{P}^2$ determine three pairs of diagonals and their intersection points.



Corollary 5.14. *The bijection*

$$\left\{\begin{array}{l}\textit{isomorphism classes of } (C,M,P) \textit{ where}\\ C \textit{ is a Type I genus } 4 \textit{ curve, } M \textit{ a degree } 3\\ \textit{line bundle on } C \textit{ with } h^0(C,M)=2, \textit{ and}\\ 0\neq P\in \operatorname{Jac}(C)[2] \textit{ with } h^0(C,M\otimes P)=0\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\textit{isomorphism classes of}\\ (X,L) \textit{ with } X \textit{ a nonhyper-}\\ \textit{elliptic genus } 3 \textit{ curve with a}\\ \textit{degree } 4 \textit{ line bundle } L\not\cong\omega_X\end{array}\right\} \tag{5.11}$$

is given by Recillas' trigonal construction and factors through the bijections (5.8) *and* (5.10) *of each side with orbits of smooth nondegenerate symmetrized* $2\times 2\times 3\times 3$ *boxes.*

Proof. Corollary 5.12 and Proposition 5.13 show that the two moduli spaces of the theorem and the nondegenerate $\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_3$-orbits of symmetric $2\times 2\times 3\times 3$ boxes can all be identified. We only need to show that the relationship is given by Recillas' construction. Using the two interpretations of the symmetric projective box $\mathcal{B}\in V_1\otimes V_2\otimes \operatorname{Sym}^2 V_3$, we may relate the genus 3 curve X and the genus 4 curve C of the theorem as degree 4 and degree 3 covers, respectively, of $\mathbb{P}(V_1^\vee)$.

For the moment, we work over a particular point $w\in\mathbb{P}(V_1^\vee)$ away from the ramification points of X. Let $B_w=\mathcal{B}(w,\cdot,\cdot)$ be the bidegree $(1,2)$ form on $V_2^\vee\times V_3^\vee$. The form B_w gives rise to a pencil of ternary quadratics, which intersect in a set of four points p_1, p_2, p_3, p_4 in $\mathbb{P}(V_3^\vee) = \mathbb{P}^2$. These are the points of the curve X as a degree 4 cover of $\mathbb{P}(V_1^\vee)$, and L is the line bundle associated to this g^1_4 on X.

On the other hand, this rational conic fibration in $\mathbb{P}(V_2^\vee) \times \mathbb{P}(V_3^\vee)$ defined by $B_w = 0$ has three degenerate fibers, which correspond to those points $x \in \mathbb{P}(V_2^\vee)$ for which the cubic $\det B_w(x, \cdot) = \det \mathcal{B}(w, x, \cdot, \cdot)$ vanishes. Over all of $\mathbb{P}(V_1^\vee)$, they comprise the bidegree $(3, 3)$ curve C in $\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee)$. Each degenerate fiber consists of two lines passing through all four points, and for $x \in \mathbb{P}(V_2^\vee)$ giving a degenerate fiber, the kernel of $B_w(x, \cdot)$ in $\mathbb{P}(V_3^\vee)$ is exactly the point of intersection of those two lines.

In other words, we are thus far in the situation of Recillas' construction. Above any $w \in \mathbb{P}(V_1^\vee)$ that is not a ramification point of X, the fiber of the curve X consists of four points; the three intersection points of the pairs of diagonals through those four points form the fiber of C over $\mathbb{P}(V_1^\vee)$. Furthermore, the ramification loci of $\mathbb{P}(V_1^\vee)$ for the two curves are the same.

Moreover, for each point of $(w, x) \in C \subset \mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee)$, the lines of the degenerate conic in $\mathbb{P}(V_3^\vee)$ correspond to two points in $\mathbb{P}(V_3)$. The set of these points in $\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee) \times \mathbb{P}(V_3)$ gives an étale double cover $\widetilde{C}$ of C. Then the cover $r : \widetilde{C} \to C$ defines a two-torsion point η of the Jacobian $\mathrm{Jac}(C)$ of C as the nonzero element of the kernel of $r^* : \mathrm{Jac}(C) \to \mathrm{Jac}(\widetilde{C})$.

Recall that given a nondegenerate symmetrized $2 \times 2 \times 3 \times 3$ box $\mathcal{B}$, the two-torsion point P is obtained as $P = M_3 \otimes \omega_C^{-1}$, where the degree 6 line bundle M_3 gives the embedding of the curve C into $\mathbb{P}(V_3^\vee)$, that is, sending a point $(w, x) \in C$ to the intersection of the two lines of the degenerate conic. By a construction of [Cob82], this so-called *Steiner embedding* of the curve C is given exactly by the linear series $|\omega_C \otimes \eta|$, so the points η and P are the same points on $\mathrm{Jac}(C)$.

Thus, the pair (C, P) is related to (X, L) by the trigonal construction. □

In bijection (5.11), recall that the right side is equivalent to isomorphism classes of (X, λ), where $\lambda := L \otimes \omega_X^{-1}$ is a nonzero point of $\mathrm{Jac}(X)$. Also, each side is an étale double cover of the space of pairs (C, P), since for each curve C, there are exactly two choices for the line bundle M. Given a choice of M, the other choice of M corresponds to switching the vector spaces V_1 and V_2 for symmetrized $2 \times 2 \times 3 \times 3$ boxes in $V_1 \otimes V_2 \otimes \mathrm{Sym}_2 V_3$. For the genus 3 curve X and the noncanonical degree 4 line bundle L on X, switching V_1 and V_2 is equivalent to changing L to $\omega_X^{\otimes 2} \otimes L^{-1}$, that is, sending λ to $-\lambda$ on $\mathrm{Jac}(X)$. In other words, modulo this $\mathbb{Z}/2\mathbb{Z}$ action, bijection (5.11) becomes

$$\left\{\begin{array}{l}\text{pairs } (C,P) \text{ where } C \text{ is a Type I genus} \\ 4 \text{ curve and } 0 \neq P \in \mathrm{Jac}(C)[2] \text{ with} \\ h^0(C, M \otimes P) = 0 \text{ for } M \text{ a } g^1_3 \text{ on } C\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{pairs } (X,\lambda) \text{ where } X \text{ is a non-} \\ \text{hyperelliptic genus 3 curve and} \\ 0 \neq \lambda \in \mathrm{Jac}(X)/\pm 1\end{array}\right\}$$

where each side is up to isomorphism of the curves.

Remark 5.15. After this work was completed, it was brought to our attention that this explicit correspondence is also used in [Rec93] to show the rationality of certain moduli spaces, but the techniques and proofs are slightly different, as are the statement of the conditions for each side of the bijections.

These correspondences are entirely explicit; given a smooth ternary quartic with a non-canonical degree 4 bundle, we may write down equations for the corresponding genus 4 curve, and even for the appropriate étale double cover (and vice versa). We expect that similar explicit constructions for other cases of the trigonal construction should hold using higher degree $2 \times 2 \times m \times m$ boxes.

What's in a name? That which we call a rose
By any other name would smell as sweet.

—Juliet, in *Romeo and Juliet*
by William Shakespeare

Chapter 6

Curves of Genus Zero

In this chapter, we relate curves of genus zero and quaternion algebras over a base scheme S via a geometric construction. Each of these categories, with some additional geometric data, can also be parametrized by ternary quadratic forms taking values in a line bundle on S; for example, the correspondence between ternary quadratic forms and quaternion algebras has recently been explained in [GL09], or more generally, in [Voi09]. The functors between the three categories commute. In particular, we will show that a ternary quadratic form gives rise to a curve and a line bundle on it, which in turn—via the geometric construction—will determine a quaternion algebra and a line bundle on S matching those constructed directly from the ternary quadratic form by [Voi09].

In the simplest case, where the base scheme $S = \mathrm{Spec}\ \overline{k}$ for an algebraically closed field $\overline{k}$, the correspondences between each pair of these categories is clear. Because all vector bundles over $\overline{k}$ are trivial, there will be no extra line bundle data, and each of the three categories under consideration has just one object:

- The projective line $\mathbb{P}^1_{\overline{k}}$ is the only smooth curve of genus zero, up to isomorphism, over $\overline{k}$, and there is only one degree 2 line bundle on $\mathbb{P}^1_{\overline{k}}$.
- The algebra $\mathrm{Mat}_{2,2}(\overline{k})$ of 2×2 matrices is the only nondegenerate quaternion algebra over $\overline{k}$, as the Brauer group of $\overline{k}$ is trivial.
- Any two nondegenerate ternary quadratic forms (i.e., irreducible elements of $\mathrm{Sym}^2 V$, where V is a 2-dimensional vector space over $\overline{k}$) are $\mathrm{GL}_3(\overline{k})$-equivalent.

The curve $\mathbb{P}^1_{\overline{k}}$ is the Brauer-Severi variety corresponding to the algebra $\mathrm{Mat}_{2,2}(\overline{k})$. Both $\mathbb{P}^1_{\overline{k}}$ and $\mathrm{Mat}_{2,2}(\overline{k})$ have the same automorphism group $\mathrm{PGL}_2(\overline{k})$, which is the units modulo the center of the algebra $\mathrm{Mat}_{2,2}(\overline{k})$. Also, the curve in $\mathbb{P}^2_{\overline{k}}$ defined by the locus of any nondegenerate ternary quadratic form is isomorphic to $\mathbb{P}^1_{\overline{k}}$.

Slightly more generally, if $S = \mathrm{Spec}\ k$ for a field k, not all of these categories will have only one point; the correspondences are still well known, however, especially as there are still no nontrivial vector bundles on S. The smooth curves of genus zero are twisted forms of $\mathbb{P}^1$, and they correspond to the k-valued points of the stack $[\cdot/\mathrm{PGL}_2]$. This stack also parametrizes twisted forms of the algebra $\mathrm{Mat}_{2,2}(k)$ of 2×2 matrices, that is, nondegenerate quaternion algebras, as $[\cdot/\mathrm{PGL}_2](k)$ up to isomorphism is the 2-torsion of the Brauer group of k. The automorphism groups of twisted forms of $\mathbb{P}^1$ and of these quaternion algebras are the corresponding twisted forms of PGL_2. Finally, nondegenerate ternary quadratic forms over k cut out smooth curves of genus zero in $\mathbb{P}^2_k$.

Preliminaries. In this chapter, we assume that the base scheme S is a $\mathbb{Z}[\frac{1}{2}]$-scheme, i.e., the points of S have characteristic not equal to 2. Also, throughout this chapter, we use Grothendieck's definition of a vector bundle's underlying scheme and related objects, such as projective bundles: to a vector bundle $\mathcal{V}$, we associate the scheme $\mathbf{Spec}(\mathrm{Sym}\,\mathcal{V}^\vee)$. For example, given a very ample line bundle L on a scheme X, here we have a natural embedding of X into $\mathbb{P}(H^0(X,L))$, instead of $\mathbb{P}(H^0(X,L)^\vee)$. (This convention is used only in this chapter, not in the rest of this thesis.)

6.1 Genus Zero Curves and Quaternion Algebras

In order to relate genus zero curves and quaternion algebras via a geometric construction, we first describe each category over the base scheme S. A *smooth genus zero curve* C over S is a proper smooth morphism $f : C \to S$ with relative dimension 1 and $\mathrm{R}^0 f_*(\mathcal{O}_C) = \mathcal{O}_S$ and $\mathrm{R}^1 f_*(\mathcal{O}_C) = 0$, i.e., the fibers are connected and have arithmetic genus 0. On the other hand, a *quaternion algebra* $\mathcal{D}$ over S is a rank 4 Azumaya algebra over S, i.e., a rank 4 vector bundle over S with an algebra structure such that étale locally on S the algebra $\mathcal{D}$ is isomorphic to a matrix algebra. Such algebras $\mathcal{D}$ up to Morita equivalence are classified by the two-torsion of the Brauer group of S. Note that because $\mathcal{D}$ is isomorphic to $\mathcal{D}^{\mathrm{op}}$, there is an anti-involution on $\mathcal{D}$.

From a smooth genus zero curve C over S, we construct a quaternion algebra $\mathcal{D}$ as follows. Let L be a degree 2 line bundle on C. Such a line bundle L always exists because the relative tangent bundle $T_{C/S}$ of C over S has degree 2. If S is a point, then $T_{C/S}$ is the only degree 2 line bundle, and in general, all degree 2 line bundles on C may be obtained by twisting $T_{C/S}$ by the pullback of a line bundle on S. For this section, the choice of the line bundle L will not be relevant, since any such L will give the same quaternion algebra under the following construction.[1]

Now we define a rank 2 vector bundle $\mathcal{E} = \mathcal{E}(C)$ on C. The line bundle L gives an embedding of the curve C into $\mathbb{P}(f_*L)$:

$$\begin{array}{ccc} C & \overset{\iota}{\hookrightarrow} & \mathbb{P}(f_*L) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & S & \end{array}$$

and the restriction map

$$\iota^* : \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}(f_*L)}, \Omega^1_{\mathbb{P}(f_*L)}) \longrightarrow \mathrm{Ext}^1(\mathcal{O}_C, \Omega^1_C)$$

gives a rank 2 bundle on C, by applying ι^* to the standard generator. We now explictly construct the vector bundle $\mathcal{E}$. Twisting the Euler sequence on $\mathbb{P}(f_*L)$ by $\mathcal{O}_{\mathbb{P}(f_*L)}(-1)$ gives the sequence of sheaves

$$0 \longrightarrow \Omega^1_{\mathbb{P}(f_*L)} \longrightarrow \mathcal{O}_{\mathbb{P}(f_*L)}(-1) \otimes p^*(f_*L) \longrightarrow \mathcal{O}_{\mathbb{P}(f_*L)} \longrightarrow 0 \tag{6.1}$$

on $\mathbb{P}(f_*L)$. We also have the adjunction sequence of C in $\mathbb{P}(f_*L)$:

$$0 \longrightarrow \mathcal{N}^\vee_{C/\mathbb{P}(f_*L)} \longrightarrow \iota^*\Omega^1_{\mathbb{P}(f_*L)/S} \xrightarrow{\alpha} \Omega^1_{C/S} \longrightarrow 0.$$

[1] When we relate the geometric constructions of this section with ternary quadratic forms in Section 6.2, the choice of the line bundle L will be important.

We restrict the sequence (6.1) to C and pushout along the map α to define the vector bundle $\mathcal{E}$ as below:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \iota^*\Omega^1_{\mathbb{P}(f_*L)/S} & \longrightarrow & \iota^*\mathcal{O}_{\mathbb{P}(f_*L)}(-1)\otimes p^*(f_*L) & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega^1_{C/S} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0. \end{array} \tag{6.2}$$

In the sequel, we will repeatedly use the fact that $\mathcal{E}$ satisfies the exact sequence

$$0 \longrightarrow \Omega^1_{C/S} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O} \longrightarrow 0 \tag{6.3}$$

but is not split. Then $\mathcal{E}nd(\mathcal{E})$ is a rank 4 vector bundle over C that locally looks like a matrix algebra, and we can define $\mathcal{D} = \mathcal{D}(C)$ to be its pushforward $f_*(\mathcal{E}nd(\mathcal{E}))$ to S. Therefore, $\mathcal{D}$ is an Azumaya algebra over S. One can check that the choice of a line bundle L does not change the isomorphism class of the quaternion algebra $\mathcal{D}$.

Example 6.1. In the simplest case, if $S = \text{Spec } \overline{k}$ for an algebraically closed field $\overline{k}$, then C is a genus zero curve in the classical sense, and in fact, isomorphic to $\mathbb{P}^1$. Then the vector bundle $\mathcal{E}(C)$ is isomorphic to $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, and the corresponding division algebra $\mathcal{D}(C)$ is just the ring of global sections of $\mathcal{E}nd(\mathcal{E}(C))$, i.e., the algebra $\text{Mat}_{2,2}(\overline{k})$ of 2×2 matrices.

Example 6.2. If $S = \text{Spec } k$ where k is not necessarily algebraically closed, then C is again a genus zero curve, but not necessarily isomorphic to $\mathbb{P}^1$. Then the k-algebra $H^0(\text{Spec } k, \mathcal{D})$ associated to the $\mathcal{O}_S$-algebra $\mathcal{D}(C)$ is simply a quaternion algebra over k in the usual sense. This algebra will be split—that is, isomorphic to a matrix algebra—if and only if C has a point over k.

Example 6.3. If $S = \text{Spec } k[\epsilon]$, then $\mathcal{D}$ is isomorphic to $\mathcal{D}_0 \otimes_k k[\epsilon]$, where $\mathcal{D}_0$ is the special fiber $\mathcal{D} \otimes_{k[\epsilon]} k$. In other words, the quaternion algebras $\mathcal{D}$ that arise in this way do not admit deformations.

Given a quaternion algebra $\mathcal{D}$ over S, it is straightforward to obtain a smooth genus zero curve over S (first done for an arbitrary base scheme in [Gro68]; see also [Ser79]). Consider the space $C' = C'(\mathcal{D})$ of rank 2 left ideals of $\mathcal{D}$ as a subscheme of the Grassmannian $\mathrm{Gr}(2, \mathcal{D})$ over S. In other words, each rank 2 left ideal of $\mathcal{D}$ is a 2-dimensional subbundle of $\mathcal{D}$ and thus corresponds to a point in the Grassmannian. The set C' is a closed subset of $\mathrm{Gr}(2, \mathcal{D})$, as being an ideal is a closed condition on a subspace of $\mathcal{D}$, and C' has a natural scheme structure arising from the moduli problem. Furthermore, C' is in fact a smooth genus zero curve over S; in the case $\mathcal{D}$ is a matrix algebra, the rank 2 left ideals of $\mathcal{D}$ can be identified with points of $\mathbb{P}^1$ (see Example 6.5 below for details), and the general case follows by étale localization.

Theorem 6.4. *For a scheme S, there is an equivalence of categories, functorial in S,*

$$\{\textit{smooth genus zero curves over } S\} \underset{\mathbf{C}}{\overset{\mathbf{D}}{\rightleftarrows}} \{\textit{quaternion algebras over } S\},$$

where for a smooth genus zero curve C, the functor $\mathbf{D}$ takes C to the quaternion algebra $\mathcal{D}(C) = f_* \mathcal{E}nd(\mathcal{E})$ as described above, and $\mathbf{C}$ takes a quaternion algebra $\mathcal{D}$ to the space $C(\mathcal{D})$ of rank 2 left ideals of $\mathcal{D}$.

The equivalence of categories exists for formal reasons, as both sides may be identified with the S-points of $[\cdot/\mathrm{PGL}_2]$ (see [Art82], for example), but we have given an explicit construction of the functors in each direction. The rest of the section is dedicated to showing that these functors are inverse to one another.

Example 6.5. In the case where $S = \mathrm{Spec}\ \overline{k}$, it is easy to see that these constructions are inverse to one another. As in Example 6.1, we will be relating $\mathbb{P}^1_{\overline{k}}$ with ideals of $\mathrm{Mat}_{2,2}(\overline{k})$. More precisely, let $f : C = \mathbb{P}(\mathcal{V}) \to \mathrm{Spec}\ \overline{k}$ be the projectivization of a trivial rank 2 vector bundle $\mathcal{V} = \mathcal{O}_{\overline{k}} \oplus \mathcal{O}_{\overline{k}}$. Then the vector bundle $\mathcal{E} = \mathcal{E}(C)$ over C is the rank 2 vector bundle $f^*\mathcal{V} \otimes \mathcal{O}_C(-1) = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Then $\mathcal{E}nd(\mathcal{E})$ is the rank 4 vector bundle

$$\mathcal{E}nd(f^*\mathcal{V}) = f^*\mathcal{E}nd(\mathcal{V}) = f^* \mathrm{Mat}_{2,2}(\mathcal{O}_{\overline{k}}) = \mathrm{Mat}_{2,2}(\mathcal{O}_C)$$

over C, and the associated quaternion algebra $\mathcal{D}$ is simply $\mathrm{Mat}_{2,2}(\mathcal{O}_{\overline{k}})$. Now the rank 2 left ideals $C' = C'(\mathcal{D})$ of $\mathcal{D}$ can be naturally identified with $C = \mathbb{P}(\mathcal{V})$: a point $x \in C(T)$ for an arbitrary scheme T over $\overline{k}$ corresponds to a line ℓ in $\mathcal{V}_T$, which then gives the rank

2 left ideal of matrices in $\mathcal{E}nd(\mathcal{V}_T) = \mathrm{Mat}_{2,2}(\mathcal{O}_T)$ that annihilates ℓ. Both C and C' are isomorphic to $\mathbb{P}^1_k$, and this map $C \to C'$ just described is an isomorphism, with the inverse map taking a rank 2 left ideal I of $\mathcal{E}nd(\mathcal{V}_T) = \mathrm{Mat}_{2,2}(\mathcal{O}_T)$ to the one-dimensional subbundle $\mathcal{V}_T[I]$ of $\mathcal{V}_T$ annihilated by I.

On the other hand, we now have C and C' are isomorphic and it is easy to show that $\mathcal{E}$ and $\mathcal{E}' = \mathcal{E}(C')$ are the same bundle, which then implies $\mathcal{D}$ and $\mathcal{D}' = \mathcal{D}(C')$ are isomorphic. In particular, because the only degree 2 line bundle on $C' \cong \mathbb{P}(\mathcal{V})$ is $\mathcal{O}_{C'}(2)$, the vector bundle $\mathcal{E}' = \mathcal{E}(C')$ on C' is $\mathcal{O}_{C'}(-1) \oplus \mathcal{O}_{C'}(-1) = f^*\mathcal{V} \otimes \mathcal{O}_{C'}(-1)$. Then we have, as desired,

$$\mathcal{D}' := f_*\mathcal{E}nd(\mathcal{E}') = f_*\mathcal{E}nd(f^*\mathcal{V}) = f_*f^*\mathcal{E}nd(\mathcal{V}) = \mathcal{E}nd(\mathcal{V}), \tag{6.4}$$

where the last equality follows from the projection formula.

Example 6.6. Slightly more generally, we now consider the split case, i.e., when $C = \mathbb{P}(\mathcal{V})$ is the projectivization of a (not necessarily trivial) rank 2 vector bundle over a base scheme S. The argument in Example 6.5 to show that the two constructions are inverse essentially holds. The desired vector bundle $\mathcal{E}(C)$ is isomorphic to $f^*\mathcal{V} \otimes \mathcal{O}_C(-1)$, and the quaternion algebra is $\mathcal{D}(C) := f_*\mathcal{E}nd(\mathcal{E}) = f_*f^*\mathcal{E}nd(\mathcal{V}) = \mathcal{E}nd(\mathcal{V})$. As before, given a point in $C(T)$ for any S-scheme T, the annihilator in $\mathcal{E}nd(\mathcal{V}_T)$ of the associated line in $\mathcal{V}_T$ is a rank 2 left ideal. This map is an isomorphism from C to the rank 2 left ideals C' of $\mathcal{D}(C)$. Furthermore, the nonsplit extension of $\mathcal{O}_{C'}$ by a degree -2 line bundle on C' is the rank 2 vector bundle $\mathcal{E}' := \mathcal{E}(C') = f^*\mathcal{V} \otimes \mathcal{O}_{C'}(-1)$, and equation (6.4) shows that the quaternion algebra $\mathcal{D}(C')$ constructed from C' is exactly $\mathcal{D}$.

We now check that these constructions are always inverse to one another; the main ideas of the proof are contained in Example 6.5. Let $f : C \to S$ be a smooth genus curve over S, $\mathcal{E} = \mathcal{E}(C)$ the rank 2 vector bundle over C as defined above, and $\mathcal{D} = \mathcal{D}(C) := \mathrm{R}^0 f_*(\mathcal{E}nd(\mathcal{E}))$ the corresponding quaternion algebra over S. Let $C' = C'(\mathcal{D})$ be the rank 2 left ideals of $\mathcal{D}$. For any S-scheme T, from a T-valued point x of C, we wish to naturally define a point of C', that is, a rank 2 left ideal of the Azumaya algebra $\mathcal{D}$. The fiber $\mathcal{E}_x$ of $\mathcal{E}$ at x is a 2-dimensional vector bundle over T, and the algebra $\mathcal{D}$ naturally acts on $\mathcal{E}_x$. The map $\mathcal{D} \to \mathcal{E}nd(\mathcal{E}_x)$ is an isomorphism because it is true when $\mathcal{D}$ is a matrix algebra. Pulling back the exact sequence (6.3) by x gives

$$0 \longrightarrow x^*(\Omega^1_{C/S}) \longrightarrow x^*(\mathcal{E}) = \mathcal{E}_x \longrightarrow x^*(\mathcal{O}_C) = \mathcal{O}_T \longrightarrow 0$$

as bundles over T. Now $\mathcal{D} \cong \mathcal{E}nd(\mathcal{E}_x)$ acts on the middle term $\mathcal{E}_x$, and the annihilator of $x^*(\Omega^1_{C/T})$ is a rank 2 left ideal in $\mathcal{D}_T$. We have thus constructed a map from C to C' over T. This map is an isomorphism: it is an isomorphism in the split case and it suffices to check that it is an isomorphism étale locally.

On the other hand, let $\mathcal{D}$ be a quaternion algebra over S. Let $f : C' = C'(\mathcal{D}) \to S$ be the space of rank 2 left ideals of $\mathcal{D}$, let $\mathcal{E}' = \mathcal{E}(C')$ be the rank 2 vector bundle that is a nonsplit extension of $\mathcal{O}_{C'}$ by $\Omega^1_{C'/S}$, and let $\mathcal{D}' = \mathcal{D}(C') := \mathrm{R}^0 f_*(\mathcal{E}nd(\mathcal{E}'))$ be the rank 4 Azumaya algebra associated to C. We want to show that $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic.

The curve C' is the moduli space of rank 2 left ideals of $\mathcal{D}$; let $\mathcal{I}$ be the universal left ideal over C' given by the pullback of the universal rank 2 vector bundle over $\mathrm{Gr}(2, \mathcal{D})$ to C'. That is, $\mathcal{I}$ is a left ideal of $f^*\mathcal{D}$. In fact, as we prove below in Lemma 6.7, the vector bundles $\mathcal{E}'$ and $\mathcal{I}$ over C' are isomorphic.

Now given an element $d \in \mathcal{D}$, there is an obvious action of d on a left ideal of D given by multiplication, so $f^*\mathcal{D}$ acts on $\mathcal{I}$. In particular, there is a map $f^*\mathcal{D} \to \mathcal{E}nd(\mathcal{I}) \cong \mathcal{E}nd(\mathcal{E})$, and pushing forward to S gives the map $\mathcal{D} = f_*f^*\mathcal{D} \to f_*(\mathcal{E}nd(\mathcal{E}))$, where the first equality follows from the projection formula and the definition of a smooth genus 0 curve over S. This map is an isomorphism because checking the split case, where it is an isomorphism, suffices. Thus, modulo showing $\mathcal{E}' \cong \mathcal{I}$, we have $\mathcal{D} \cong \mathcal{D}'$.

Lemma 6.7. *The vector bundles $\mathcal{E}'$ and $\mathcal{I}$ over C' are isomorphic.*

Proof. To prove this statement, we use a different moduli interpretation for C'. We define the scheme $\mathbb{P}_{\mathcal{D}}$ by letting $\mathbb{P}_{\mathcal{D}}(T)$ be the category of triples $(\mathcal{V}, \varphi, \nu)$ where $\mathcal{V}$ is a rank 2 vector bundle on T, $\varphi : \mathcal{E}nd(\mathcal{V}) \cong \mathcal{D}_T$ is an isomorphism of algebras, and $\nu : \mathcal{V} \to \mathcal{O}_T$ is a surjection. To see that $\mathbb{P}_{\mathcal{D}}$ is a scheme, note that forgetting ν realizes $\mathbb{P}_{\mathcal{D}}$ as the complement of the zero section of the universal bundle $\mathcal{V}$ over the $\mathbb{G}_m$-gerbe $X_{\mathcal{D}}$ parametrizing pairs $(\mathcal{V}, \varphi)$ as above. In fact, if $(\mathcal{V}_{\mathbb{P}_{\mathcal{D}}}, \nu_{\mathbb{P}_{\mathcal{D}}})$ gives the map $\mathbb{P}_{\mathcal{D}} \to X_{\mathcal{D}}$, then the aforementioned description of $\mathbb{P}_{\mathcal{D}}$ identifies the kernel of $\nu_{\mathbb{P}_{\mathcal{D}}}$ with $\Omega^1_{\mathbb{P}_{\mathcal{D}}}$ to give an exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}_{\mathcal{D}}} \longrightarrow \mathcal{V}_{\mathbb{P}_{\mathcal{D}}} \xrightarrow{\nu} \mathcal{O}_{\mathbb{P}_{\mathcal{D}}} \to 0. \tag{6.5}$$

A splitting of $\mathcal{D}$ gives an identification of $\mathbb{P}_{\mathcal{D}}$ with $\mathbb{P}^1$, under which the sequence (6.5) coincides with the Euler sequence. In particular, the bundle $\mathcal{V}_{\mathbb{P}_{\mathcal{D}}}$ is a nonsplit extension of $\mathcal{O}_{\mathbb{P}_{\mathcal{D}}}$ by $\Omega^1_{\mathbb{P}_{\mathcal{D}}}$.

We now show that $\mathbb{P}_{\mathcal{D}}$ and $C'(\mathcal{D})$ are isomorphic. Given a point $(\mathcal{V}, \varphi, \nu) \in \mathbb{P}_{\mathcal{D}}(T)$, we may define a rank 2 left ideal in $\mathcal{E}nd(\mathcal{V})$ as the annihilator of the kernel of $\nu : \mathcal{V} \to \mathcal{O}$. Under the isomorphism φ, we have a rank 2 left ideal of $\mathcal{D}_T$, which corresponds to a point of $C'(\mathcal{D})(T)$. Thus, we have given a map $a : \mathbb{P}_{\mathcal{D}} \to C'(\mathcal{D})$.

Conversely, given a rank 2 left ideal $I \subset \mathcal{D}_T$, the ideal structure of I defines a natural isomorphism $\varphi_I : \mathcal{E}nd(I) \xrightarrow{\cong} \mathcal{D}_T$, and composing the trace map $\mathcal{D}_T \to \mathcal{O}_T$ with the inclusion $I \subset \mathcal{D}_T$ gives a surjection $\nu_I : I \to \mathcal{O}_T$. In particular, the triple (I, φ_I, ν_I) defines a point of $\mathbb{P}_{\mathcal{D}}(T)$, which produces the promised map $b : C'(\mathcal{D}) \to \mathbb{P}_{\mathcal{D}}$.

We may check that a and b are mutually inverse isomorphisms in the split case, which suffices to show that this moduli space $\mathbb{P}_{\mathcal{D}}$ is isomorphic to our original curve $C'(\mathcal{D})$.

Under these isomorphisms, the universal left ideal $\mathcal{I} \subset \mathcal{D}_{C'(\mathcal{D})}$ corresponds to the universal bundle $\mathcal{V}_{\mathbb{P}_{\mathcal{D}}}$ on $\mathbb{P}_{\mathcal{D}}$ by construction. The latter, as shown in the sequence (6.5), is the unique (up to non-unique isomorphism) non-split extension of $\mathcal{O}_{\mathbb{P}_{\mathcal{D}}}$ by $\Omega^1_{\mathbb{P}_{\mathcal{D}}}$. Thus, the universal left ideal $\mathcal{I}$ may be identified with the unique non-split extension of $\mathcal{O}_{C'(\mathcal{D})}$ by $\Omega^1_{C'(\mathcal{D})}$. Since the the bundle $\mathcal{E}'$ has the same property, the bundles $\mathcal{E}'$ and $\mathcal{I}$ are isomorphic. □

Remark 6.8. The bijection between genus zero curves and quaternion algebras described in this section may be generalized to higher dimensions, giving an equivalence of categories between dimension $n-1$ Brauer-Severi varieties and rank n^2 Azumaya algebras. As both categories are equivalent to $[\cdot/\mathrm{PGL}_n]$, the equivalence exists formally. The geometric constructions detailed in this section describe the functors: given a rank n^2 Azumaya algebra $\mathcal{D}$, the rank n left ideals form a $(n-1)$-dimensional Brauer-Severi variety in $\mathrm{Gr}(n, \mathcal{D})$ [Gro68], and given a $(n-1)$-dimensional Brauer-Severi variety X, an analogous rank n vector bundle $\mathcal{E}$ on X has endomorphism algebra isomorphic to the desired Azumaya algebra.

6.2 Ternary Quadratic Forms

In this section, we show that ternary quadratic forms are closely related to both geometric and algebraic objects. In Section 6.2.1, we relate ternary quadratics and genus zero curves with degree 2 line bundles, and Section 6.2.2 connects ternary quadratic forms and quaternion algebras with a line bundle on the base.

A *ternary quadratic form* $(\mathcal{W}, L_S, Q)$ over a $\mathbb{Z}[\frac{1}{2}]$-scheme S is a rank 3 vector bundle $\mathcal{W}$ over S, a line bundle L_S on S, and a section Q of the vector bundle $\mathcal{Q} = \mathrm{Sym}^2(\mathcal{W}^\vee) \otimes \wedge^3\mathcal{W} \otimes L_S$. Corresponding to the usual notion of a quadratic form as a polynomial, we may "evaluate" the quadratic form Q on sections of $\mathcal{W}$ to obtain a section of $\wedge^3\mathcal{W} \otimes L_S$, or evaluate the bilinear form $\langle \cdot, \cdot \rangle_Q$ corresponding to Q on sections of $\mathcal{W} \otimes \mathcal{W}$.

6.2.1 Ternary Quadratic Forms and Genus Zero Curves

To a ternary quadratic form $(\mathcal{W}, L_S, Q)$ over S, we may associate geometric objects, namely a curve over S and a line bundle on that curve. We will only be considering *nondegenerate* ternary quadratic forms, however.

For some intuition, let us first consider the simplest case, where $S = \mathrm{Spec}\ \overline{k}$ for an algebraically closed field $\overline{k}$. Then the vector bundles $\mathcal{W}$ and L_S are trivial, and a ternary quadratic form defines a curve C in the projective plane $\mathbb{P}(\mathcal{W}^\vee) = \mathbb{P}^2$. Suppose the polynomial representing the section Q is irreducible; then the curve has genus zero and, since the base field is algebraically closed, it is isomorphic to $\mathbb{P}^1$. Pulling back $\mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(1)$ to the curve C gives a degree 2 line bundle L on C, and in this case, the bundle L is isomorphic to $\mathcal{O}(2)$ on $C \cong \mathbb{P}^1$.

On the other hand, given a smooth irreducible genus zero curve and a degree 2 line bundle L on it, the sections of L give an embedding of the curve into the projective plane $\mathbb{P}(H^0(C, L))$. Because $\dim H^0(C, L^{\otimes 2})) = 5$ and $\dim \mathrm{Sym}^2 H^0(C, L) = 6$, the sections of L satisfy a quadratic relation, which gives a quadratic polynomial in the sections of L, i.e., a ternary quadratic form over $S = \mathrm{Spec}\ \overline{k}$. However, this quadratic polynomial is only defined up to scaling via this method; to fix this ambiguity, we will find a more functorial relationship between ternary quadratic forms and genus zero curves.

For any base scheme S, given a ternary quadratic form $(\mathcal{W}, L_S, Q)$, let C be the zero set of the section Q in $\mathbb{P}(\mathcal{W}^\vee)$. We will only consider such ternary quadratic forms that give rise to *smooth* curves C; we will call these *nondegenerate* ternary quadratic forms. Over a field, for example, these correspond to irreducible polynomials Q. More generally, one may define a *discriminant* $\Delta(\mathcal{W}, L_S, Q)$ as a $\mathcal{O}_S$-submodule of S for the ternary quadratic form $(\mathcal{W}, L_S, Q)$, and the nondegeneracy condition corresponds to requiring that we stay away from the vanishing locus of $\Delta(\mathcal{W}, L_S, Q)$ in S. In the case of Example 6.10, the (half-)discriminant $\Delta(Q)$ can be written as the polynomial

$$\Delta(Q) = 4abc + uvw - au^2 - bv^2 - cw^2,$$

and then the discriminant $\Delta(\mathcal{W}, L_S, Q)$ of $(\mathcal{W}, L_S, Q)$ is the ideal of $\mathcal{O}_S$ generated by $\Delta(Q\mid_N)$ for all free rank 3 subsheaves of $\mathcal{W}$. Then the nondegeneracy condition is the requirement that the $\Delta(\mathcal{W}, L_S, Q)$ is in fact an invertible ideal, i.e., a line bundle on S.

The restriction to nondegenerate ternary quadratic forms is an open condition, since locally it is given by the nonvanishing of the discriminant. In the sequel, instead of working with the entire stack of ternary quadratic forms, we will be considering the open substack of nondegenerate ternary quadratic forms. The other space we will be using—smooth curves of genus zero with line bundle data—is also an open substack of the stack of all genus zero curves with the same data.

If C is smooth over S, then C has codimension 1 in $\mathbb{P}(\mathcal{W}^\vee)$ and is a curve; by cohomology and base change, the relative curve C has genus zero since each fiber does. Define the line bundle L to be the pullback of $\mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(1)$ to C. Note that L is a degree 2 line bundle on C.

Given a ternary quadratic form $(\mathcal{W}, L_S, Q)$, there exists a moduli interpretation of the curve C thus constructed. In particular, for an S-scheme T, the T-valued points of C exactly correspond to the Q-isotropic subbundles of $\mathcal{W}_T$.

Conversely, suppose we have a pair (C, L) over S, where $f : C \to S$ is a smooth genus zero curve and L is a degree 2 line bundle on C. Then there is a natural embedding $\iota : C \hookrightarrow \mathbb{P}(f_*L)$, and we will define the vector bundle $\mathcal{W} := (f_*L)^\vee \to S$ with $p : \mathbb{P}(\mathcal{W}^\vee) \to S$.

If I_C is the ideal defining the curve C, then we have an exact sequence

$$0 \longrightarrow I_C(2) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(2) \longrightarrow \mathcal{O}_C(2) \longrightarrow 0$$

and taking cohomology gives

$$0 \longrightarrow p_* I_C(2) \longrightarrow p_* \mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(2) \longrightarrow p_* \mathcal{O}_C(2) \longrightarrow 0. \tag{6.6}$$

Because $I_C(2)$ is a degree 0 line bundle on $\mathbb{P}(\mathcal{W}^\vee)$, it is the pullback of a line bundle M on S and $p_* I_C(2) = M$. Tensoring the sequence (6.6) with $M^\vee$ gives a ternary quadratic form via the sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & p_* \mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(2) \otimes M^\vee & \longrightarrow & p_* \mathcal{O}_C(2) \otimes M^\vee & \longrightarrow & 0. \\ & & & & \| & & & & \\ & & & & \operatorname{Sym}^2 \mathcal{W}^\vee \otimes M^\vee & & & & \end{array} \tag{6.7}$$

The first inclusion in the sequence (6.7) is a section Q of $\operatorname{Sym}^2 \mathcal{W}^\vee \otimes M^\vee$, so we now have produced a ternary quadratic form $(\mathcal{W}, M^\vee \otimes (\wedge^3 \mathcal{W})^\vee, Q)$.

We would like to compare the line bundle $L_S = M^\vee \otimes (\wedge^3 \mathcal{W})^\vee$ on S and the original bundle L on C. The vanishing of the cohomology $\mathrm{R}^0 p_*(I_C \otimes I_C(2))$ and $\mathrm{R}^1 p_*(I_C \otimes I_C(2))$ gives the first equality below:

$$\begin{aligned} M := p_* I_C(2) = p_*(\mathcal{O}_C \otimes I_C(2)) = f_* \iota^* I_C(2) \\ = f_*(\iota^* I_C \otimes \iota^* \mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(2)) = f_*(I_C/I_C^2 \otimes L^{\otimes 2}). \end{aligned}$$

Using the adjunction sequence for $\iota : C \to \mathbb{P}(\mathcal{W}^\vee)$, we compute

$$\det(\iota^* \Omega^1_{\mathbb{P}(\mathcal{W}^\vee)/S}) = \det(\Omega^1_{C/S}) \otimes \det(I_C/I_C^2) = \Omega^1_{C/S} \otimes I_C/I_C^2.$$

On the other hand, twisting the Euler sequence on $\mathbb{P}(\mathcal{W}^\vee)$ by $\mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(-1)$, pulling back to

C, and taking determinants implies

$$\begin{aligned}\det(\iota^*(\Omega^1_{\mathbb{P}(\mathcal{W}^\vee)/S})) &= \det(\iota^*(\mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(-1) \otimes p^*\mathcal{W}^\vee)) \\ &= \det(L^\vee \otimes \iota^* p^* \mathcal{W}^\vee) = (L^\vee)^{\otimes 3} \otimes f^*(\wedge^3(\mathcal{W})^\vee).\end{aligned}$$

Finally, with the aid of the projection formula, we have

$$\begin{aligned}M = f_*(I_C/I_C^2 \otimes L^{\otimes 2}) &= f_*(\det(\iota^*\Omega^1_{\mathbb{P}(\mathcal{W}^\vee)/S}) \otimes (\Omega^1_{C/S})^\vee \otimes L^{\otimes 2}) \\ &= f_*((L^\vee)^{\otimes 3} \otimes f^*(\wedge^3(\mathcal{W})^\vee) \otimes (\Omega^1_{C/S})^\vee \otimes L^{\otimes 2}) \\ &= f_*(f^*(\wedge^3(\mathcal{W})^\vee) \otimes L^\vee \otimes (\Omega^1_{C/S})^\vee) \\ &= \wedge^3(\mathcal{W})^\vee \otimes f_*(L^\vee \otimes (\Omega^1_{C/S})^\vee).\end{aligned}$$

Again using the fact that a degree 0 line bundle on C (this time, $L \otimes \Omega^1_{C/S}$) is the pullback of a line bundle on S, we find that

$$L_S = M^\vee \otimes (\wedge^3\mathcal{W})^\vee = f_*(L \otimes \Omega^1_{C/S}). \tag{6.8}$$

As a sanity check, when $S = \operatorname{Spec} \overline{k}$ and hence all the vector bundles on S are trivial, the only degree 2 line bundle L on a genus 0 curve is the tangent bundle $T_{C/S}$, which gives $L_S = \mathcal{O}_S$, as in Example 6.10.

The functors between nondegenerate ternary quadratic forms and genus zero curves with a degree 2 line bundle are inverse to one another. It suffices to check that the functors are inverse locally, e.g., over a local ring, where the result is straightforward using the moduli interpretation of the curve.

Proposition 6.9. *There is an equivalence of categories between nondegenerate ternary quadratic forms $(\mathcal{W}, L_S, Q)$ over S and pairs (C, L), where C is a smooth genus zero curve over S and L is a degree 2 line bundle on C.*

Note that the data of a degree 2 line bundle L on C is the same data as a line bundle L_S over S because C has genus zero. In the bijection, as we have shown, we have the relationships

$$L = f^*L_S \otimes T_{C/S} \qquad \text{and} \qquad L_S = f_*(L \otimes \Omega^1_{C/S}).$$

The bundle L_S on the base S simply records the difference between L and the tangent bundle. If there are no nontrivial line bundles on S, for example, if S is a point, then the only degree 2 line bundle on the curve will be the tangent bundle. Because of this feature, the category of pairs (C, L) as defined above is equivalent to the category of pairs (C, L_S) where C is a smooth genus zero curve over S, and L_S is a line bundle on S. Thus, the moduli problem under consideration is simply the product $[./\mathrm{PGL}_2] \times [./\mathrm{G}_m]$: the first factor records the curve C, while the second factor records the line bundle L_S.

6.2.2 Clifford Algebras

From a ternary quadratic form over S, one can construct a quaternion algebra over S. In the case where the quadratic form is untwisted, i.e., $\mathcal{W}$ and L_S are both trivial, [GL09] have a clever algebraic construction of the associated quaternion algebra. More generally, the even part of the Clifford algebra of a ternary quadratic form will be a quaternion algebra.

Example 6.10. Assume that $S = \text{Spec } R$ where R is a principal ideal domain or a local ring.[2] In this case, all vector bundles over S are trivial, and thus it is enough to say that a *ternary quadratic form* is just a section Q of the vector bundle $\mathcal{Q} = \mathrm{Sym}^2(\mathcal{W}^\vee) \otimes \wedge^3\mathcal{W} \otimes L_S$, where $\mathcal{W}$ is a trivial rank 3 bundle on S and $L_S = \mathcal{O}_S$. The bundle $\mathcal{Q}$ is also trivial and can be identified with the usual notion of ternary quadratic form, i.e., a section Q can be represented by

$$Q = ax^2 + by^2 + cz^2 + uyz + vxz + wxy \tag{6.9}$$

where $a, b, c, u, v, w \in R$ and x, y, z form a basis for $\mathcal{W}^\vee$. We only need to be careful that the group $\mathrm{Aut}(\mathcal{W})$ acts on Q not by the usual GL_3-action on such a polynomial, but by a twisted action:

$$g \cdot Q(x, y, z) = (\det g)Q(x', y', z')$$

[2] Proving the statements in this section for local rings will in fact suffice to extend the results to any base scheme S, as we will see later.

where $(x', y', z') = g^{-1} \cdot (x, y, z)^t$ for $g \in \mathrm{Aut}(\mathcal{W})$. This twisted action arises exactly because the vector bundle $\mathcal{Q}$ is not just $\mathrm{Sym}^2(\mathcal{W}^\vee)$ but has a twist by $\det \mathcal{W} = \wedge^3 \mathcal{W}$. From the ternary quadratic form Q, Gross and Lucianovic construct a quaternion algebra $A = A(Q)$ over R with basis $1, i, j, k$ satisfying the following relations:

$$
\begin{aligned}
i^2 &= ui - bc & jk &= a\bar{i} & kj &= -vw + ai + wj + vk \\
j^2 &= vj - ac & ki &= b\bar{j} & ik &= -uw + wi + bj + uk \\
k^2 &= wk - ab & ij &= c\bar{k} & ji &= -uv + vi + uj + ck
\end{aligned}
\tag{6.10}
$$

where $\bar{d}$ indicates the conjugate of any $d \in A$ under the anti-involution on A. Call a quaternion algebra that has a basis (with any values of $a, b, c, u, v, w \in R$) a *free* quaternion ring (after [Voi09]); then these exactly correspond to ternary quadratic forms:

Proposition 6.11 ([GL09]). *For $S = \mathrm{Spec}\ R$ where R is a local ring or a principal ideal domain, there is a discriminant-preserving bijection between the set of orbits of $\mathrm{Aut}(\mathcal{W}) \cong \mathrm{GL}_3(R) \times S$ acting on $\mathcal{Q}$ and the set of isomorphism classes of free quaternion rings over S.*

As observed in [Voi09], the Gross-Lucianovic construction of a quaternion algebra in Example 6.10 in fact works for a ternary quadratic form $(\mathcal{W}, L_S, Q)$ over Spec R for any commutative ring R, provided that $L_S = \mathcal{O}_S$.

The quaternion algebra $A(Q)$ of Example 6.10 corresponding to Q can also be constructed via the Clifford algebra of Q, and this construction works for more general base S to produce a quaternion algebra $A(\mathcal{W}, L_S, Q)$ from a ternary quadratic form $(\mathcal{W}, L_S, Q)$ over S. More precisely, the Clifford algebra $\mathrm{Cliff}(\mathcal{W}, L_S, Q)$ of the ternary quadratic form $(\mathcal{W}, L_S, Q)$ is an associative algebra with unit, and in fact, a rank 8 $\mathcal{O}_S$-algebra. It has a natural $\mathbb{Z}/2\mathbb{Z}$-grading, and the even part $\mathcal{C}^+(\mathcal{W}, L_S, Q)$ is the desired central simple $\mathcal{O}_S$-algebra of rank 4 over S.

Example 6.12. If $\mathcal{W} = \mathcal{O}_S^{\oplus 3}$ and $L_S = \mathcal{O}_S$ are trivial, then $\mathrm{Cliff}(\mathcal{O}_S^{\oplus 3}, \mathcal{O}_S, Q)$ can be constructed as the quotient of the tensor algebra of $\mathcal{W}$ by the two-sided ideal $\mathfrak{a}$ generated by elements of the form $\xi \otimes \xi - Q(\xi)$ for all sections $\xi \in p_*(\mathcal{O}_S^{\oplus 3})$. Such elements are clearly in the tensor algebra, as Q takes values in $\wedge^3(\mathcal{O}_S^{\oplus 3}) = \mathcal{O}_S$. All elements $s, t \in \mathrm{Cliff}(\mathcal{O}_S^{\oplus 3}, \mathcal{O}_S, Q)$ satisfy the relation

$$s \cdot t + t \cdot s = 2\langle s, t\rangle_Q,$$

where $\langle s, t\rangle_Q = Q(s+t) - Q(s) - Q(t)$ is the R-bilinear form associated to Q. The even part $\mathcal{C}^+(\mathcal{O}_S^{\oplus 3}, \mathcal{O}_S, Q)$ can be written

$$\mathcal{C}^+(\mathcal{O}_S^{\oplus 3}, \mathcal{O}_S, Q) \cong \frac{R \oplus (\mathcal{W} \otimes \mathcal{W})}{\mathfrak{a}}.$$

Furthemore, as explained in [Voi09], the quaternion algebra $A(Q)$ with relations (6.10) on the basis $1, i, j, k$ is isomorphic to $\mathcal{C}^+(\mathcal{O}_S^{\oplus 3}, \mathcal{O}_S, Q)$. If $\mathcal{W}$ has sections ξ_1, ξ_2, ξ_3, then the isomorphism is given by

$$\begin{aligned} A(Q) &\xrightarrow{\cong} \mathcal{C}^+(\mathcal{O}_S^{\oplus 3}, \mathcal{O}_S, Q) = \mathcal{O}_S \oplus \mathcal{O}_S\xi_2\xi_3 \oplus \mathcal{O}_S\xi_3\xi 1 \oplus \mathcal{O}_S\xi_1\xi_2 \\ i, j, k &\longmapsto \xi_2\xi_3, \xi_3\xi_1, \xi_1\xi_2. \end{aligned}$$

When $\mathcal{W}$ and L_S are not necessarily trivial bundles on S, the section Q takes values in sections of the line bundle $\wedge^3\mathcal{W} \otimes L_S$, so constructing the Clifford algebra requires replacing the tensor algebra of $\mathcal{W}$ in Example 6.12 with an algebra involving this line bundle. In general, however, the Clifford *algebra* is not defined over S but instead over the $\mathbb{G}_m$-torsor associated to the line bundle in which the quadratic form takes values. More explicitly, one may naturally define the even Clifford algebra $\mathcal{C}^+$, as well as the odd part $\mathcal{C}^-$ as a bimodule over $\mathcal{C}^+$, over S. In the general case, the direct sum $\mathcal{C}(Q) = \mathcal{C}^+ \oplus \mathcal{C}^-$ does not have a natural algebra structure. However, the choice of a trivialization $\mathcal{O}_S \cong \wedge^3\mathcal{W} \otimes L_S$ would give $\mathcal{C}(Q)$ an algebra structure in the usual way. As the formation of $\mathcal{C}(Q)$ commutes with base change on S, it follows that $\mathrm{Cliff}(\mathcal{W}, L_S, Q) = \mathcal{C}(Q) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ has a natural algebra structure, where $S' = \big(\mathbf{Spec}\,\mathrm{Sym}((\wedge^3\mathcal{W} \otimes L_S)^\vee)\big) \setminus Z(S)$ is the complement of the zero section $Z(S)$ in the total space of the line bundle $\wedge^3\mathcal{W} \otimes L_S$. In the case $S = \mathrm{Spec}\ R$ is affine, the R-algebra $\mathcal{O}_{S'}$ is often called the *Rees* or the *Laurent algebra* of the invertible R-module $(\wedge^3\mathcal{W} \otimes L_S)^\vee$. Note that $\mathrm{Cliff}(\mathcal{W}, L_S, Q)$ is a rank 8 sheaf over S', not S, and it is the "classical" Clifford algebra associated to the ternary quadratic form $(\mathcal{W} \otimes \mathcal{O}_{S'}, \mathcal{O}_{S'}, Q \otimes 1)$.

Following [BK94], we define the Clifford algebra $\mathrm{Cliff}(\mathcal{W}, L_S, Q)$ as a quotient of the tensor algebra $\mathcal{T}(\mathcal{W})$ of $\mathcal{W}$ and the algebra $\mathcal{O}_{S'}$. Let $\mathfrak{a}$ be the ideal of $\mathcal{T}(\mathcal{W}) \otimes \mathcal{O}_{S'}$ generated by elements of the form $\xi \otimes \xi \otimes 1 - 1 \otimes Q(\xi)$ for all $\xi \in p_* \mathcal{W}$. Then let

$$\mathrm{Cliff}(\mathcal{W}, L_S, Q) = \frac{\mathcal{T}(\mathcal{W}) \otimes \mathcal{O}_{S'}}{\mathfrak{a}}.$$

This algebra is $\mathbb{Z}$-graded, and the degree 0 piece is a subalgebra classically considered the even Clifford algebra $\mathcal{C}^+(\mathcal{W}, L_S, Q)$, which has rank 4 as an $\mathcal{O}_S$-module. We have

$$\mathcal{C}^+(\mathcal{W}, L_S, Q) \cong \frac{\mathcal{O}_S \oplus (\mathcal{W} \otimes \mathcal{W} \otimes (\wedge^3 \mathcal{W} \otimes L_S)^\vee)}{\mathfrak{a}'}$$

where $\mathfrak{a}'$ is generated by elements of the form $\xi \otimes \xi \otimes \lambda - \lambda(Q(\xi))$ for $\xi \in p_* \mathcal{W}$ and λ a section of $(\wedge^3 \mathcal{W} \otimes L_S)^\vee$. Furthermore, the first graded piece of $\mathrm{Cliff}(\mathcal{W}, L_S, Q)$ is isomorphic to the usual odd Clifford algebra, and is a bimodule over $\mathcal{C}^+(\mathcal{W}, L_S, Q)$.

Call two ternary quadratic forms $(\mathcal{W}, L_S, Q)$ and $(\mathcal{W}', L'_S, Q')$ *isomorphic* if there are isomorphisms $\alpha : \mathcal{W} \xrightarrow{\cong} \mathcal{W}'$ and $\beta : L_S \xrightarrow{\cong} L'_S$ such that $Q'(\alpha(\xi)) = \beta(Q(\xi))$ for any section $\xi \in p_* \mathcal{W}$. It is clear that nondegeneracy is preserved under isomorphism.

In [Voi09], Voight proves that in fact, this functor from ternary quadratic forms to the even part of their Clifford algebras is a bijection on isomorphism classes. It is necessary to keep track of what he calls a *parity factorization*; for us, a parity factorization is equivalent to the data of the line bundle L_S on S. Here, we slightly modify Voight's theorem also by restricting to nondegenerate ternary quadratic forms:

Proposition 6.13. *Over a base scheme S, there is a bijection*

$$\left\{ \begin{array}{l} \textit{isomorphism classes of} \\ \textit{nondegenerate ternary quadratic} \\ \textit{forms } (\mathcal{W}, L_S, Q) \textit{ over } S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{isomorphism classes of quaternion} \\ \textit{algebras } A \textit{ over } S \textit{ and line bundles} \\ L_S \textit{ over } S \end{array} \right\}$$

where the right arrow takes a ternary quadratic form $(\mathcal{W}, L_S, Q)$ to the even Clifford algebra $\mathcal{C}^+(\mathcal{W}, L_S, Q)$ and the line bundle L_S.

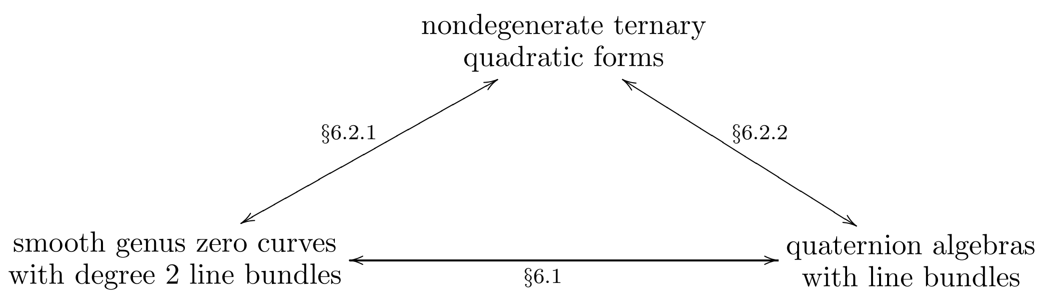
Note that our definition of quaternion algebra is more restrictive than that in [Voi09].

For example, we do not include what [Voi09] calls *free exceptional* quaternion algebras; they would be considered degenerate, so they do not appear in the above proposition.

Remark 6.14. On the right side of the bijection in Proposition 6.13, the line bundle on S is completely unrelated to the quaternion algebra. Instead of carrying the data of this line bundle L_S on S, it may be more natural to include other equivalent data, such as a representation of the quaternion algebra. Such a representation, for example, naturally arises when constructing the quaternion algebra from a genus zero curve with a degree 2 line bundle.

6.3 A Composition of Functors

Now we show that the constructions of Sections 6.1, 6.2.1, and 6.2.2 commute, in the sense that the following heuristic diagram commutes:



(6.11)

We already know that each of these arrows is essentially an equivalence of categories, and we will show that starting with a ternary quadratic, the natural genus zero curve and degree 2 line bundle give rise to (via the geometric construction of Section 6.1) the same quaternion algebra as that of Section 6.2.2, the even part of the Clifford algebra associated to the ternary quadratic form. Our general strategy is to match the unit groups of the two quaternion algebras thus constructed, which then will show that the two algebras give rise to the same Brauer class and thus are isomorphic.

Remark 6.15. In the case where $\mathcal{W}$ and L_S are trivial for a ternary quadratic form, it may also be enlightening to construct an explicit basis for the quaternion algebra given in Section 6.1 and show that the relations in [GL09] hold; we will see, in our proof, that it is at least possible to find an explicit basis for a three-dimensional quotient of this algebra and verify the corresponding relations.

To begin, we will work over a local ring. Let $(\mathcal{W}, \mathcal{O}_S, Q)$ be a ternary quadratic form over $S = \text{Spec } R$ for R a local ring, with Q represented by equation (6.9) and $\mathcal{W}$ a trivial vector bundle. Let $f : C = C(Q) \to S$ be the genus zero curve in $p : \mathbb{P}(\mathcal{W}^\vee) \to S$ cut out by Q. We would like to describe $\mathcal{D} = \mathcal{D}(C(Q)) := \mathrm{R}^0 f_*(\mathcal{E}nd(\mathcal{E}))$ (and its algebra structure) for the vector bundle $\mathcal{E}$, defined by the diagram (6.2) and satisfying the sequence

$$0 \longrightarrow \Omega^1_{C/S} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_C \longrightarrow 0 \tag{6.12}$$

of locally free sheaves on C. Applying $\mathcal{H}om(\cdot, \mathcal{E})$ to the exact sequence (6.12) and taking cohomology gives the exact sequence

$$0 = \mathrm{R}^0 f_*(\mathcal{E}) \longrightarrow \mathrm{R}^0 f_*(\mathcal{E}nd(\mathcal{E})) \xrightarrow{\cong} \mathrm{R}^0 f_*(\mathcal{E} \otimes T_{C/S}) \longrightarrow \mathrm{R}^1 f_*(\mathcal{E}) = 0,$$

which reduces the question to understanding $\mathrm{R}^0 f_*(\mathcal{E} \otimes T_{C/S})$. Then cohomology of the sequence (6.12) tensored with $T_{C/S}$ gives the exact sequence

$$0 \longrightarrow \mathrm{R}^0 f_*(\mathcal{O}_C) \longrightarrow \mathrm{R}^0 f_*(\mathcal{E} \otimes T_{C/S}) \longrightarrow \mathrm{R}^0 f_*(T_{C/S}) \longrightarrow \mathrm{R}^1 f_*(\mathcal{O}_C) = 0, \tag{6.13}$$

where $\mathrm{R}^1 f_*(\mathcal{O}_C)$ vanishes because C has genus 0. Now $\mathrm{R}^0 f_*(\mathcal{O}_C)$ has rank one and can be viewed as the scalars in $\mathrm{R}^0 f_*(\mathcal{E} \otimes T_{C/S})$; we would like to find a basis for the rank three quotient bundle $\mathrm{R}^0 f_*(T_{C/S})$. Taking the cohomology of the adjunction sequence realizes this rank three bundle as a subbundle of $\mathrm{R}^0 f_*(T_{\mathbb{P}(\mathcal{W}^\vee)/S} \mid_C)$:

$$0 \longrightarrow \mathrm{R}^0 f_*(T_{C/S}) \longrightarrow \mathrm{R}^0 f_*(T_{\mathbb{P}(\mathcal{W}^\vee)/S} \mid_C) \xrightarrow{\pi_Q} \mathrm{R}^0 f_*(\mathcal{N}_{C/\mathbb{P}(\mathcal{W}^\vee)}) \longrightarrow \mathrm{R}^1 f_*(T_{C/S}) = 0. \tag{6.14}$$

Tensoring the sequence defining the curve C with $T_{\mathbb{P}(\mathcal{W}^\vee)/S}$ and taking cohomology shows that $\mathrm{R}^0 f_*(T_{\mathbb{P}(\mathcal{W}^\vee)/S} \mid_C)$ can be identified with $\mathrm{R}^0 p_*(T_{\mathbb{P}(\mathcal{W}^\vee)/S})$. This last space is, by the

Euler sequence on $\mathbb{P}(\mathcal{W}^\vee)$, a rank 8 quotient bundle of $\mathcal{E}nd(\mathcal{W})$:

$$0 \longrightarrow \mathrm{R}^0 p_*(\mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}) \longrightarrow \mathrm{R}^0 p_*((p^*(\mathcal{W}^\vee))^\vee \otimes \mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(1)) \longrightarrow \mathrm{R}^0 p_*(T_{\mathbb{P}(\mathcal{W}^\vee)/S}) \longrightarrow 0$$

where the middle term is just $p_*((p^*(\mathcal{W}^\vee))^\vee \otimes \mathcal{O}(1)) \cong \mathcal{W} \otimes \mathcal{W}^\vee \cong \mathcal{E}nd(\mathcal{W})$ via the projection formula. Thus, locally the rank nine bundle $\mathrm{R}^0 p_*((p^*(\mathcal{W}^\vee))^\vee \otimes \mathcal{O}_{\mathbb{P}(\mathcal{W}^\vee)}(1))$ can be thought of as 3×3 matrices, and the quotient $\mathrm{R}^0 p_*(T_{\mathbb{P}(\mathcal{W}^\vee)/S})$ as traceless 3×3 matrices. Note that $\mathcal{E}nd(\mathcal{E})$ has an algebra structure; although the rank 8 quotient, which is isomorphic to $\mathrm{R}^0 f_*(T_{\mathbb{P}(\mathcal{W}^\vee)/S} \mid_C)$, does not have an algebra structure, it does have a well defined Lie bracket and can be thought of as $\mathfrak{sl}_3$.

Furthermore, the image $\mathrm{R}^0 f_*(\mathcal{N}_{C/\mathbb{P}(\mathcal{W}^\vee)})$ of the map π_Q in (6.14) can be thought of as quadratic forms on $\mathbb{P}(\mathcal{W}^\vee)$ with Q vanishing. With these local identifications, the map π_Q simply sends a traceless 3×3 matrix M to $MQ + QM^t$, where (by a slight abuse of notation) here Q is the 3×3 symmetric matrix associated to the bilinear form of the quadratic form $ax^2 + by^2 + cz^2 + uyz + vxz + wxy$. The kernel of π_Q can be identified with $\mathfrak{so}(Q) \subset \mathfrak{sl}_3$ and has a natural Lie algebra structure.

In summary, to write down the algebra structure on $\mathrm{R}^0 f_*(\mathcal{E}nd(\mathcal{E}))$, we would like to first understand $R^0 f_*(T_{C/S})$. By the sequence (6.14), the rank three bundle $R^0 f_*(T_{C/S})$ corresponds to traceless endomorphisms M of $\mathcal{W}$ that fix the ternary quadratic Q (in the sense that $MQ + QM^t = 0$), namely $\mathfrak{so}(Q)$.

Example 6.16. If Q is the standard ternary quadratic form $x^2 + y^2 + z^2$ and its matrix is the 3×3 identity matrix, then we can think of the rank three bundle $R^0 f_*(T_{C/S})$ as the standard $\mathfrak{so}(3)$ in $\mathfrak{sl}(3)$, that is, the skew-symmetric matrices. The quaternion algebra corresponding to this Q has a "good basis" $1, i, j, k$ and the relations (6.10) become

$$\begin{array}{lll} i^2 = -1 & jk = -i & kj = i \\ j^2 = -1 & ki = -j & ik = j \\ k^2 = -1 & ij = -k & ji = k. \end{array} \tag{6.15}$$

This associative algebra gives rise to the standard Lie algebra with brackets $[r, s] = rs - sr$, and the Lie algebra relations derived from (6.15) include

$$[j,k] = -2i \qquad [k,i] = -2j \qquad [i,j] = -2k.$$

It is easy to verify that the standard basis for 3×3 skew-symmetric matrices satisfies these Lie algebra relations. More generally, for any ternary quadratic form Q, one can computationally find a basis for $\mathfrak{so}(Q)$ that satisfies the corresponding Lie algebra relations; note that these relations need to be checked in the three-dimensional quotient sending scalars to zero.

Now recall that using the isomorphism $\mathrm{R}^0 f_*(\mathcal{E}nd(\mathcal{E})) \cong \mathrm{R}^0 f_*(\mathcal{E} \otimes T_{C/S})$ and the sequence (6.13), we have the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{D} \longrightarrow \mathrm{R}^0 f_*(T_{C/S}) \longrightarrow 0,$$

describing our quaternion algebra $\mathcal{D}$, where the last term has been identified with $\mathfrak{so}(Q)$. Because the Lie algebra $\mathfrak{so}(3)$ is self-dual (i.e., the adjoint representation is self-dual), dualizing this sequence gives

$$0 \longrightarrow \mathfrak{so}(Q)^\vee \longrightarrow \mathcal{D}^\vee \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

where the last map is just the trace map. Note furthermore that $\mathcal{D}^\vee = \mathcal{D}^{\mathrm{op}}$ is actually isomorphic to $\mathcal{D}$, since $\mathcal{D}$ gives rise to a two-torsion element of the Brauer group of S. Another way to see this duality is by the definition of $\mathcal{D}$, since $\mathcal{E}nd(\mathcal{E})$ is self-dual:

$$\mathcal{D} = f_*(f^*(\mathcal{D})) = f_*(\mathcal{E}nd(\mathcal{E})) = f_*(\mathcal{E}nd(\mathcal{E})^\vee) = f_*(f^*(\mathcal{D}^{\mathrm{op}})) = \mathcal{D}^{\mathrm{op}}.$$

We thus have the exact sequence

$$0 \longrightarrow \mathfrak{so}(Q) \xrightarrow{\ \tau\ } \mathcal{D} \xrightarrow{\ \mathrm{Tr}\ } \mathcal{O}_S \longrightarrow 0 \tag{6.16}$$

where τ is a map of Lie algebras and Tr is the trace map. We may identify $\mathfrak{so}(Q)$ with the

Lie algebra Lie(Spin(Q)) of the group Spin(Q) and $\mathcal{D}$ with the tangent space Lie($\mathcal{D}^*$) of its units $\mathcal{D}^*$. Using these identifications, the fact that Spin(Q) is simply connected, and the exponential map, we obtain a map $\tilde{\tau} : \mathrm{Spin}(Q) \to \mathcal{D}^*$ such that $\mathrm{Lie}(\tilde{\tau}) = \tau$. Étale localizing on S shows that this map induces an isomorphism of Spin(Q) with the group $G_{\mathcal{D}}$ of norm 1 units.

On the other hand, a nondegenerate ternary quadratic form Q gives rise, via Proposition 6.13, to the Azumaya algebra isomorphic to the even part $\mathcal{C}^+(Q)$ of the Clifford algebra Cliff(Q) to Q. The group Spin(Q) is, by definition, a subgroup of $\mathcal{C}^+(Q)^*$, and for dimension reasons, Spin(Q) coincides with the group $G_{\mathcal{C}^+(Q)}$ of norm 1 units in $\mathcal{C}^+(Q)$.

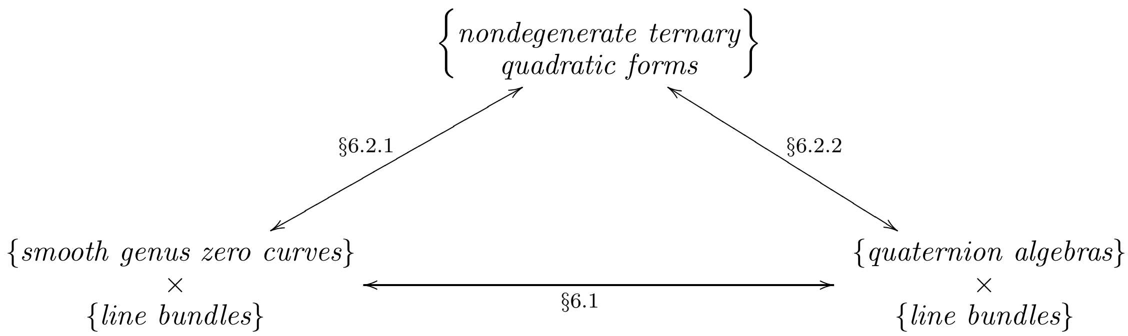
Over a base scheme S, the Brauer-Severi variety underlying an Azumaya algebra A of dimension 4 can be recovered from its group G_A of norm 1 units as the space $B(G_A)$ of Borel subgroups of G_A. Hence, the group G_A determines the Brauer-Severi variety associated to A and thus the Brauer class of A. Because two algebras with the same dimension and same Brauer class are isomorphic, two central simple algebras of dimension 4 with the same group of norm 1 units are isomorphic.

In the case at hand, we have, from a nondegenerate ternary quadratic form Q, constructed two quaternion algebras–one geometrically as $f_*(\mathcal{E}nd(\mathcal{E}))$ and one as the even part of the Clifford algebra associated to Q—and their groups of norm 1 units are isomorphic. Therefore, we have the following proposition:

Proposition 6.17. *Let R be a local ring and $S = \mathrm{Spec}\ R$. Given an irreducible ternary quadratic form $(\mathcal{W}, \mathcal{O}_S, Q)$ over S, the quaternion algebras $\mathcal{D}(C(Q))$ and $\mathcal{C}^+(Q)$ are isomorphic.*

Despite the initial assumption on the base scheme, the result holds more generally. In particular, we have shown that there are morphisms between every pair of the three stacks in diagram (6.11), and *locally* the diagram commutes. Therefore, because each vertex of the diagram (6.11) is a stack, we find that the morphisms commute globally.

Theorem 6.18. *The stacks below are all equivalent:*



and the diagram commutes.

Chapter 7

The Way Ahead

In this final chapter, we suggest a few natural generalizations and applications of the orbit parametrizations discussed in this thesis.

We have described, in Chapters 2 to 5, moduli spaces of geometric objects corresponding to orbits of representations of $3 \times n \times n$ and $2 \times 2 \times m \times m$ boxes. Orbit problems of other representations of reductive groups may be described as geometric moduli spaces using the same techniques. Boxes of other dimensions and sizes also naturally correspond to moduli problems, e.g., orbits of $d \times n \times n$ boxes give rise to degree d hypersurfaces in $\mathbb{P}^{n-1}$ and line bundles, with certain cohomological conditions. Of course, we need not restrict ourselves to standard tensor representations of products of general linear groups. Not only may we change the action of the groups on these tensor spaces, but we also may consider other reductive groups. In fact, the space of $2 \times 2 \times m \times m$ boxes is closely related to the standard tensor representation of $\mathrm{SO}_4 \times \mathrm{GL}_m \times \mathrm{GL}_m$. More generally, representations involving orthogonal, symplectic, or even exceptional groups will produce geometric objects with additional structure.

In our parametrizations, we have not needed to use much of the invariant theory of each of these spaces. Fully understanding the invariant ring of each representation, e.g., the invariants under the action of the product of special linear groups, would produce explicit equations for the moduli spaces. Also, the invariants in each case have geometric interpretations, which is much clearer—and perhaps more interesting—in the cases related to genus one curves. We plan to describe these interpretations in future work.

Certain cases of the orbit problems, studied in Sections 4.4.1 and 5.4.1, use parametrizations by boxes to exhibit an isomorphism between two geometric moduli spaces. In fact, both the Reid-Tjurin correspondence between nets of quadrics and plane curves as well as the trigonal construction of Recillas are special cases of Beauville's theorem on subvarieties of Pryms [Bea82]. It would be worthwhile to investigate whether some more general parametrizations would recover other similar relationships between moduli spaces of geometric objects, or even better, give new ones.

In Chapter 6, we studied the pairwise relationships between ternary quadratic forms, genus zero curves, and quaternion algebras. In Bhargava's classification of integral orbits of prehomogeneous vector spaces, there are several other representations related to ternary quadratic forms and quaternion algebras. In future work, we plan to consider these equivalences over arbitrary base schemes and describe the equivalent geometric objects.

Finally, throughout this thesis, we have avoided the so-called degenerate loci of the representations. A careful analysis of the degenerate orbits, which others have treated in sporadic cases, would have applications such as compactifying the moduli spaces, thereby giving a generalization of the moduli problems to bad characteristic and, in fact, the integers. Fully understanding the integral orbits of these representations will have numerous arithmetic applications, including improving any potential counting results.

The Road goes ever on and on
Down from the door where it began.
Now far ahead the Road has gone,
And I must follow, if I can,
Pursuing it with eager feet,
Until it joins some larger way
Where many paths and errands meet.
And whither then? I cannot say.

—Bilbo Baggins, in *The Lord of the Rings*
by J.R.R. Tolkein

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