Clarifications concerning our problems with [DHRT]

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This is a complement to the note published on Sergiu Klainerman's web page[1] regarding the ongoing conflict with the authors of [DHRT]. Many of those who have read our note [K-S:remarks] are under the impression that the issue is merely one of priority. The present note is meant to clarify that, beyond the issue of priority, our main complaint is the use by the authors of [DHRT] of crucial ideas in our earlier works [K-S18] [GCM1] [GCM2] without appropriate references. Despite of our repeated requests, more than 30 months after the appearance of [DHRT] on arXiv, no corrections have been made.

1 Important past contributions recognized in our work

As we have acknowledged in our papers and repeated many time in lectures, our work builds on ideas developed in the last 6-7 decades by a large number of physicists and mathematicians. Here is a short list of the main papers which have been directly influential in our work:

1. The derivation of the Teukolsky equations, in [Teuk], based on the Newmann-Penrose null frame formalism [NP].
2. The proof of the nonlinear stability of the Minkowski space- Christodoulou- Klainerman [CK93].
3. Chandrasekhar transformation, see [Chan], which, in the case of the linearized Einstein equations near Schwarzschild, connects the Teukolsky equation to the Regge-Wheeler one.

[1]See https://web.math.princeton.edu/ seri/homepage/.

4. Morawetz estimates for the scalar wave equation in a fixed Schwarzschild background; works of Blue-Soffer [B-S1], Blue-Sterbenz [B-St], Marzuola-Metcalfe-Tataru-Tohaneanu [Ma-Me-Ta-To].

5. Morawetz estimates for the scalar wave equation for slowly rotating Kerr; work of Anderson- Blue [A-B], Tataru-Tohaneanu [Ta-Toh].

6. The red shift, r^p-weighted and decay of energy- flux estimates; works of Dafermos-Rodnianski, [Da-Ro1] and [Da-Ro2].

7. Morawetz estimates for the Teukolsky equations for Schwarzschild (based on the Chandrasekhar transformation); work of Dafermos-Holzegel-Rodnianski [D-H-R].

8. Morawetz estimates for the Teukolsky equations for slowly rotating Kerr; work of S. Ma [Ma] and [D-H-R-Kerr].

2 Main new ideas in our work

Here is a list of the main new ideas introduced in our works, see more details in section 4 of [K-S:brief]:

I. Idea of constructing and making use of GCM spheres S_* and spacelike hypersurfaces Σ_* as a way to initialize the outgoing geodesic foliation of our GCM admissible spacetimes. The idea, which requires the construction of co-dimension 2 spheres not directly related to the initial data, lies at the heart of our entire approach. It was already present in [K-S18] in the simpler polarized case, extended to the general case of perturbations of Kerr in [GCM1], [GCM2] and [Shen] and then used in the proof of the nonlinear stability of slowly rotating Kerr, see [K-S:Kerr].

II. Definition of the angular momentum based on the GCM "last sphere" S_*, see [GCM2] and [K-S:Kerr].

III. Extension of the geometric framework based on S-foliations, first used in the nonlinear stability of the Minkowski space [CK93], to deal with non-integrable horizontal structures typical to Kerr, see [GKS1] and [GKS-2022].

IV. In [K-S18] the GCM admissible spacetime $\mathcal{M}$ was divided into an exterior region ${}^{(ext)}\mathcal{M}$, foliated by an outgoing geodesic foliation which initiates at the GCM boundary Σ_* and terminating at a timelike boundary $\mathcal{T}$, and an interior region ${}^{(int)}\mathcal{M}$ foliated by an incoming geodesic foliation which initiates at $\mathcal{T}$. This was necessary in view of the fact that the outgoing foliation of ${}^{(ext)}\mathcal{M}$ is uncontrollable near the event horizon. In [K-S:Kerr] we need to introduce an additional region ${}^{(top)}\mathcal{M}$, such that $\mathcal{M} = {}^{(int)}\mathcal{M} \cup {}^{(top)}\mathcal{M} \cup {}^{(ext)}\mathcal{M}$ is causal.

V. Robust Morawetz estimates for generalized Regge-Wheeler (gRW) equations in perturbations of slowly rotating Kerr, see [GKS-2022], based on an appropriate geometric construction of an approximate second order symmetry operator[2]. The idea was first used in [A-B] to derive a Morawetz estimate for the scalar wave equation in a fixed slowly rotating Kerr background.

3 Ideas in our work used in [DHRT] without attribution

Below are comments[3] on how our ideas, in **I**, **II** and **IV** are used without any attribution in [DHRT].

Issue concerning I. The introduction of [DHRT] contains no specific reference to the GCM type construction on which the work is based until page 22 of the introduction, under the heading "The bootstrap region and the teleological normalization of the null gauges". There, with minor modifications, our GCM conditions appear out of the blue on a last sphere denoted $S(u_f, v_\infty)$, the precise analogue the GCM last sphere S_* in [K-S18]. There is however no reference to where the idea originates from. There is indeed no mention whatsoever (!) to the GCM constructions developed in [K-S18], [GCM1], [GCM2]. The two GCM papers, which extend the construction of GCM spheres to the full setting of general perturbations of Kerr, are in fact not mentioned at all in the bibliography! See section 2, item 1b in [K-S:remarks] for more details about this.

There can be no possible excuse for this omission. Both of us have been given plenty of lectures on our work on the stability of Schwarzschild, many of them in Princeton[4] in which we have expressly emphasized the importance of GCM spheres. The authors of [DHRT] were also well aware of our two GCM papers. We have strong reasons to believe that they were in fact instrumental in having our papers rejected from JAMS and Acta where we have first submitted them, more than 18 months before the release

[2]Which commutes with the scalar wave operator in Kerr. The existence of such operators was first discovered by Carter.

[3]Only **I**, **II** and **IV** are relevant to the case of [K-S18] and [DHRT].

[4]Starting with my November 2017 Weyl lectures, attended by both Dafermos and Rodnianski. I also gave a lecture at at the inaugural meeting (6–8th March 2019) of the Gravity Initiative. Jeremie gave a talk at the Princeton Analysis seminar on October 22, 2018.

of [DHRT]. One may reasonably wonder if it is possible that they had arrived at the same concept independently, before the release of our Schwarzschild paper [K-S18]. The following incidents show that in fact the authors of [DHRT] were not originally aware, at least by early 2019, of the GCM approach which they make full use of in their paper.

Klainerman: *In my graduate course of the Spring semester 2019, I gave a full presentation of the proof of the nonlinear stability of Schwarzschild following the second arXiv version of Dec. 2018 of [K-S18]. Martin Taylor, the fourth author of [DHRT], attended the first 2-3 lectures in Feb.2019 when I explicitly emphasized and explained our GCM approach. At some point, knowing that Martin was already giving talks*[5] *on the not yet released [DHRT], three years before the paper appeared!, I asked him how he and his co-authors solve the same issue, that is how they fix the gauge at null infinity. The answer given by Martin was that " we proceed as in the proof of the stability of the Minkowski space" [CK93]. Yet in the proof of the stability of the Minkowski space one constructs, on a given spacelike hypersurface*[6] *Σ_*, a specific co-dimension 1 foliation initialized at spacelike infinity, in reference to a fixed, given, foliation on the initial data hypersurface*[7]*. By contrast in [DHRT], as in [K-S18], the authors start from the far away co-dimension 2 GCM sphere, the last sphere mentioned above, with no direct reference to the initial data.*

Later that Spring Taylor gave a very well attended one hour lecture at the Gravity Initiative inaugural meeting (6–8th March 2019) at the end of which I asked him the same question. This time Igor Rodnianski intervened and gave the same answer, "as in the proof of the stability of the Minkowski space".

These incidents provide conclusive evidence that in 2019 (more than two years before [DHRT] appeared on arXiv on April 2021) the authors were not yet aware of the GCM constructions, introduced in our works, on which their paper relies.

Issue concerning II. This regards the crucial definition of the angular momentum, which was discussed in detail in section 2, item 1d of [K-S:remarks]. The main point is that the [DHRT] authors make the same definition of the angular momentum as in [GCM2] without any reference to that paper. In addition we point out the following highly misleading reference to [K-S18], made at the top of page 3 of the [DHRT] introduction:

" Whereas, as remarked above, the full codimension-3 submanifold of moduli space yield-

[5]Such as the Oberwolfach meeting on Mathematical General Relativity August 5-11, 2018, October 1, 2018 at the Gravity Initiative lunch at Princeton and November 20, 2018 at the University of Kentucky.

[6]Σ_* is the last slice of the maximal time foliation used in the proof.

[7]The more appropriate comparison is with [KN03] where the choice of the foliation is on a fixed incoming null hypersurface initiated on the initial slice. The reference to [CK93] is however reasonable since the main ideas are comparable. As we show in [K-S:remarks], the linear paper [D-H-R] relies on the same type of (linearized version) co-dimension 1 foliations initialized on the initial hypersurface.

ing asymptotic stability (iii) of Schwarzschild, in the absence of symmetry, can only be characterized teleologically, in contrast, in the restrictive setting of symmetry, one can easily identify criteria on initial data which guarantee that their only possible end-state within the Kerr family would be Schwarzschild."

It is correct that our polarized assumption is made on the initial data yet, what the authors call their " teleological" construction, is based on nothing else than a GCM sphere S_* ($S(u_f, v_\infty)$ in their version)! It is true that our polarized assumption implies that the angular momentum is fixed a-priori to be zero on S_*, while in the case of [DHRT] it requires an additional modulation argument to force it to vanish, but the fact remains that the "teleological" GCM spheres appeared already in [K-S18] and were defined in full generality in [K-S:Kerr]. The additional finite modulation argument, of which the authors of [DHRT] are so proud, is in fact rather straightforward[8] once the infinite dimensional modulation argument, implicit in the choice of the last sphere S_*, is implemented.

Issue concerning IV. In the linear paper [D-H-R] the authors work with a global, double null foliation which covers the entire future of the Schwarzschild horizon and, as we pointed out in [K-S:remarks], is initiated on the horizon. Prior to the release of the [DHRT] paper in April 2021, in all their talks on the subject the authors seem to suggest that they planned to work with a family of finite spacetimes, each foliated by a double null foliation, which converge to a spacetime defined to the future of the final event horizon, thus foliated by a unique double null foliation as in the case of the Israel-Pretorius double null foliation of the Kerr exterior. In other words they were originally intending to follow an approach consistent to what was done in the linear paper [D-H-R], where the double null foliation was initiated at the horizon. It is pretty clear that such a procedure cannot work and in fact in [DHRT] the authors adopt a procedure very similar to ours (in [K-S18]), based on two foliations one outgoing (initialized at "null infinity"). and one incoming (initialized on a timelike hypersurface $\mathcal{T}$ using the induced data of the outgoing foliation). No reference to this important fact is made in their paper.

[8] The authors of [DHRT] do themselves provide evidence in this regard. Thus, towards the middle of page 7 in their introduction: " Indeed, given that the fundamental double null gauges already have to be constructed teleologically (in fact, already in linear theory; see Section II), it turns out that the additional problem of constructing $\mathfrak{M}_{stable}$ does not in practice present a major conceptual difficulty (although it gives rise to a number of interesting technical issues; see Section V)". Similarly, in his 10/10/2021 Joint Online Mathematical Relativity Colloquium (JoMaReC) talk (" https://educast.fccn.pt/vod/clips/1jvqoxsa0t/streaming.html?locale=en " -from 37'40 to 39'20- it is clearly stated that "after having done infinite dimensional modulation, doing three extra parameters is " not such a big problem".

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