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Lectures on Heegaard Floer Homology

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These are notes for the second lecture course on Heegaard Floer homology in the Clay Mathematics Institute Budapest Summer School in June 2004, taught by the first author.[1] Although some of the topics covered in that course did not make it into these notes (specifically, the discussion of "knot Floer homology" which instead is described in the lecture notes for the first course, cf. [**44**]), the central aim has remained largely the same: we have attempted to give a fairly direct path towards some topological applications of the surgery long exact sequence in Heegaard Floer homology. Specifically, the goal was to sketch with the minimum amount of machinery necessary a proof of the Dehn surgery characterization of the unknot, first established in a collaboration with Peter Kronheimer, Tomasz Mrowka, and Zoltán Szabó. (This problem was first solved in [**29**] using Seiberg-Witten gauge theory, rather than Heegaard Floer homology; the approach outlined here can be found in [**39**].)

In Lecture 1, the surgery exact triangle is stated, and some of its immediate applications are given. In Lecture 2, it is proved. Lecture 3 concerns the maps induced by smooth cobordisms between three-manifolds. This is the lecture containing the fewest technical details – though most of those can be found in [**34**]. In Lecture 4, we show how the exact triangle, together with properties of the maps appearing in it, lead to a proof of the Dehn surgery classification of the unknot.

An attempt has been made to keep the discussion as simple as possible. For example, in these notes we avoid the use of "twisted coefficients". This comes at a price: as a result, we do not develop the necessary machinery required to handle knots with genus one. It is hoped that the reader's interest will be sufficiently piqued to study the original papers to fill in this gap. There are also a number of exercises scattered throughout the text, in topics ranging from homological algebra and elementary conformal mapping to low-dimensional topology. The reader is strongly encouraged to think through these exercises; some of the proofs in the text rely on them. At the conclusion of each lecture, there is a discussion on further reading on the material.

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1. Introduction to the surgery exact triangle

The exact triangle is a key calculational tool in Heegaard Floer homology. It relates the Heegaard Floer homology groups of three-manifolds obtained by surgeries along a framed knot in a closed, oriented three-manifold. Before stating the result precisely, we review some aspects of Heegaard Floer homology briefly, and then some of the topological constructions involved.

1.1. Background on Heegaard Floer groups: notation. Recall that Heegaard Floer homology is an Abelian group associated to a three-manifold, equipped with a Spin^c structure $\mathfrak{t} \in \mathrm{Spin}^c(Y)$. It comes in several variants.

Let $(\Sigma, \{\alpha_1, ...\alpha_g\}, \{\beta_1, ..., \beta_g\}, z)$ be a Heegaard diagram for Y, where here $\boldsymbol{\alpha} = \{\alpha_1, ..., \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, ..., \beta_g\}$ are attaching circles for two handlebodies bounded by Σ, and $z \in \Sigma - \alpha_1 - ... - \alpha_g - \beta_1 - ... - \beta_g$ is a reference point.

Form the g-fold symmetric product $\mathrm{Sym}^g(\Sigma)$, and let $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ be the tori

$$\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_g \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times ... \times \beta_g.$$

The simplest version of Heegaard Floer homology is the homology groups of a chain complex generated by the intersection points of $\mathbb{T}_\alpha$ with $\mathbb{T}_\beta$: $\widehat{CF}(Y) = \bigoplus_{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{Z}\mathbf{x}$. This is endowed with a differential

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \big| \mu(\phi)=1, n_z(\phi)=0\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \mathbf{y}.$$

Here, $\pi_2(\mathbf{x}, \mathbf{y})$ denotes the space of homology classes of Whitney disks connecting $\mathbf{x}$ and $\mathbf{y}$ [2], $n_z(\phi)$ denotes the algebraic intersection number of a representative of ϕ with the codimension-two submanifold $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^g(\Sigma)$, $\mathcal{M}(\phi)$ denotes the moduli space of pseudo-holomorphic representatives of ϕ, and $\mu(\phi)$ denotes the expected dimension of that moduli space, its Maslov index. Also, $\# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right)$ is an appropriately signed count of points in the quotient of $\mathcal{M}(\phi)$ by the natural $\mathbb{R}$ action defined by automorphisms of the domain. To avoid a distracting discussion of signs, we sometimes change to the base ring $\mathbb{Z}/2\mathbb{Z}$, where now this coefficient is simply the parity of the number of points in $\mathcal{M}(\phi)/\mathbb{R}$. The loss of generality coming with this procedure is irrelevant for the topological applications appearing later in these lecture notes.

There is an obstruction to connecting $\mathbf{x}$ and $\mathbf{y}$ by a Whitney disk, which leads to a splitting of the above chain complex according to Spin^c structures over Y, induced from a partitioning of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ according to Spin^c structures, $\widehat{CF}(Y) = \bigoplus_{\mathfrak{t} \in \mathrm{Spin}^c(Y)} \widehat{CF}(Y, \mathfrak{t})$. The homology groups of $\widehat{CF}(Y, \mathfrak{t})$, $\widehat{HF}(Y, \mathfrak{t})$, are topological invariants of Y and the Spin^c structure $\mathfrak{t}$.

[2] In the case where $g(\Sigma) > 2$, we have that $\pi_2(\mathrm{Sym}^g(\Sigma)) \cong \mathbb{Z}$, and hence the distinction between homotopy and homology classes of Whitney disks disappears.

There are other versions of these groups, taking into account more of the homology classes $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. Specifically, we consider the boundary operator

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \big| \mu(\phi)=1\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \cdot U^{n_z(\phi)} \mathbf{y},$$

where U is a formal variable. This can be thought of as acting on either the free $\mathbb{Z}[U]$-module generated by intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ ($CF^-(Y, \mathfrak{t})$), or the free $\mathbb{Z}[U, U^{-1}]$-module generated by these same intersection points ($CF^\infty(Y, \mathfrak{t})$), or the module with one copy of $\mathcal{T}^+ = \mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$ for each intersection point ($CF^+(Y, \mathfrak{t})$). Note also that when the first Betti number of Y, $b_1(Y)$, is non-zero, special "admissible" Heegaard diagrams must be used to ensure the necessary finiteness properties for the sums defining the boundary maps. Once this is done, the homology groups of the chain complexes $HF^-(Y, \mathfrak{t})$, $HF^\infty(Y, \mathfrak{t})$, and $HF^+(Y, \mathfrak{t})$ are topological invariants of Y equipped with its Spin^c structure $\mathfrak{t}$.

For instance, when working with $\widehat{HF}$ and HF^+ for a three-manifold with $b_1(Y) > 0$, we need the following notions.

Definition 1.1. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a pointed Heegaard diagram. The attaching curves divide Σ into a collection of components $\{\mathcal{D}_i\}_{i=1}^n$, one of which contains the distinguished point z. Let $P = \sum_i n_i \cdot \mathcal{D}_i$ be a two-chain in Σ. Its boundary can be written as a sum of subarcs of the α_i and the β_j. The two-chain P is called a *periodic domain* its local multiplicity at z vanishes and if for each i the segments of α_i appear with the same multiplicity. (More informally, we express this condition by saying that the boundary of P can be represented as a sum of the α_i and the β_j.) A Heegaard diagram is said to be *weakly admissible* if all the non-trivial periodic domains have both positive and negative local multiplicities.

Exercise 1.2. Identify the group of periodic domains (where the group law is given by addition of two-chains) with $H_2(Y; \mathbb{Z})$.

Weakly admissible Heegaard diagrams can be found for any three-manifold, and the groups $\widehat{HF}(Y, \mathfrak{t})$ and $HF^+(Y, \mathfrak{t})$ are the homology groups of the chain complexes $\widehat{CF}(Y, \mathfrak{t})$ and $CF^+(Y, \mathfrak{t})$ associated to such a diagram. For more details, and also a stronger notion of admissibility which gives HF^- and HF^∞, see for example Subsection 4.2.2 of [**41**]).

Exercise 1.3. Show that, with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, $\widehat{HF}(S^1 \times S^2) \cong \mathbb{F} \oplus \mathbb{F}$. Note that there is a also a Heegaard diagram for $S^1 \times S^2$ for which $\mathbb{T}_\alpha \cap \mathbb{T}_\beta = \emptyset$ (but of course this diagram is not weakly admissible). *Hint:* draw a genus one Heegaard diagram for $S^2 \times S^1$.

Exercise 1.4. Let M be a module over the ring $\mathbb{Z}[U]$. Let M_U denote its localization $M_U = M \otimes_{\mathbb{Z}[U]} \mathbb{Z}[U, U^{-1}]$.

(1) Show that the kernel of the natural map $M \longrightarrow M_U$ consists of the submodule of $m \in M$ such that there is an $n \geq 0$ with $U^n \cdot m = 0$.

(2) Let C be a chain complex of free modules over the ring $\mathbb{Z}[U]$. Show that there is a natural isomorphism $H_*(C_U) \cong H_*(C)_U$.

If C is a chain complex of free $\mathbb{Z}[U]$-modules, we have natural short exact sequences

$$0 \longrightarrow C \longrightarrow C_U \longrightarrow C_U/C \longrightarrow 0,$$

and

$$0 \longrightarrow C/UC \longrightarrow C_U/C \xrightarrow{U} C_U/C \longrightarrow 0,$$

both of which are functorial under chain maps between complexes over $\mathbb{Z}[U]$.

(3) Show that if a chain map $f\colon C \longrightarrow C'$ of free $\mathbb{Z}[U]$-modules induces an isomorphism on $H_*(C/UC) \longrightarrow H_*(C'/UC')$, then it induces isomorphisms

$$H_*(C) \cong H_*(C'), \qquad H_*(C_U) \cong H_*(C'_U), \qquad H_*(C_U/C) \cong H_*(C'_U/C')$$

as well. Indeed, if $g\colon C_U/C \longrightarrow C'/C'_U$ is a map of $\mathbb{Z}[U]$-complexes (not necessarily induced from a map from C to C'), then there is an induced map $\widehat{g}\colon C/UC \longrightarrow C'/UC'$, and if $\widehat{g}$ induces an isomorphism on homology, then so does g.

(4) Suppose that there is some d so that Ker U^d = Ker U^{d+1} on $H_*(C)$ (as is the case, for example, if C is a finitely generated complex of $\mathbb{Z}[U]$ modules). Show then that $H_*(C_U) \longrightarrow H_*(C_U/C)$ is surjective if and only if the map $U\colon H_*(C_U/C) \longrightarrow H_*(C_U/C)$ is.

(5) Show that $H_*(C_U/C) \neq 0$ if and only if $H_*(C/UC) \neq 0$.

The relevance of the above exercises is the following: $CF^-(Y,\mathfrak{t})$ is a chain complex of free $\mathbb{Z}[U]$-modules, and $CF^\infty(Y,\mathfrak{t})$ $CF^+(Y,\mathfrak{t})$, and $\widehat{CF}(Y,\mathfrak{t})$ are the associated complexes $CF^-(Y,\mathfrak{t})_U$, $CF^-(Y,\mathfrak{t})_U/CF^-(Y,\mathfrak{t})$, and $CF^-(Y,\mathfrak{t})/UCF^-(Y,\mathfrak{t})$ respectively. In particular, we have two functorially assigned long exact sequences

$$(1) \qquad ... \longrightarrow HF^-(Y,\mathfrak{t}) \xrightarrow{\ell_*} HF^\infty(Y,\mathfrak{t}) \xrightarrow{q_*} HF^+(Y,\mathfrak{t}) \longrightarrow ...$$

and

$$(2) \qquad ... \longrightarrow \widehat{HF}(Y,\mathfrak{t}) \longrightarrow HF^+(Y,\mathfrak{t}) \xrightarrow{U} HF^+(Y,\mathfrak{t}) \longrightarrow ...$$

(both of which are natural under chain maps $CF^-(Y,\mathfrak{t}) \longrightarrow CF^-(Y',\mathfrak{t}')$).

1.2. Background: $\mathbb{Z}/2\mathbb{Z}$ gradings. Heegaard Floer homology is a relatively $\mathbb{Z}/2\mathbb{Z}$-graded group. To describe this, fix arbitrary orientations on $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, and give $\mathrm{Sym}^g(\Sigma)$ its induced orientation from Σ. At each intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we can then define a local intersection number $\iota(\mathbf{x})$ by the rule that the complex orientation on $T_\mathbf{x}\mathrm{Sym}^g(\Sigma)$ is $\iota(\mathbf{x}) \in \{\pm 1\}$ times the induced orientation from $T_\mathbf{x}\mathbb{T}_\alpha \oplus T_\mathbf{x}\mathbb{T}_\beta$. As is familiar in differential topology (compare [**33**]), we can define the algebraic intersection number of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ by the formula

$$\#(\mathbb{T}_\alpha \cap \mathbb{T}_\beta) = \sum_{\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \iota(\mathbf{x}).$$

The overall sign of this depends on the choice of orientations of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, but once this is decided, the intersection number depends only on the induced homology classes of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$.

We can think about the intersection number directly in terms of the Heegaard surface as follows. Fix orientations on all the curves $\{\alpha_i\}_{i=1}^g$ and $\{\beta_i\}_{i=1}^g$ (these in turn induce orientations on the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$). In this case, $\#(\mathbb{T}_\alpha \cap \mathbb{T}_\beta)$ is the determinant of the $g \times g$ matrix formed from the algebraic intersection of α_i and β_j (with $i,j \in \{1,...,g\}$).

EXERCISE 1.5. Let $(\Sigma, \{\alpha_1, ..., \alpha_g\}, \{\beta_1, ..., \beta_g\})$ be a Heegaard diagram for a closed, oriented three-manifold Y. Show that there is a corresponding CW-complex structure on Y with one zero-cell, one three-cell, g one-cells $\{a_i\}_{i=1}^g$, and g two-cells $\{b_i\}_{i=1}^g$. Show that the only non-trivial boundary operator $\partial\colon C_2 \longrightarrow C_1$ has the form

$$\partial b_i = \sum_{i=1}^{g} \#(\alpha_i \cap \beta_j) a_j.$$

Choose orientations for the $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ as above. Then there is a corresponding splitting of $\widehat{CF}(Y)$ into two summands,

$$\widehat{CF}(Y) = \bigoplus_{i \in \mathbb{Z}/2\mathbb{Z}} \widehat{CF}_i(Y), \tag{3}$$

where here $\widehat{CF}_i(Y)$ is generated by intersection points $\mathbf{x}$ with $\iota(\mathbf{x}) = (-1)^i$. Note that although $\iota(\mathbf{x})$ depends on the (arbitrarily chosen) orientations of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, if $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ are two different intersection points, it is easy to see that the product $\iota(\mathbf{x}) \cdot \iota(\mathbf{y})$ is independent of this choice. In fact, according to standard properties of the Maslov index (see for example [**46**]), if $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$

$$\iota(\mathbf{x}) \cdot \iota(\mathbf{y}) = (-1)^{\mu(\phi)},$$

where $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is any homology class of Whitney disk. Thus, the boundary map reverses the splitting from Equation (3), i.e. we have that

$$\partial\colon \widehat{CF}_i(Y) \longrightarrow \widehat{CF}_{i+1}(Y)$$

(thinking of $i \in \mathbb{Z}/2\mathbb{Z}$). It is a straightforward consequence of this that there is also a $\mathbb{Z}/2\mathbb{Z}$ splitting of the homology:

$$\widehat{HF}(Y) = \bigoplus_{i \in \mathbb{Z}/2\mathbb{Z}} \widehat{HF}_i(Y),$$

where here $\widehat{HF}_i(Y)$ is represented by cycles supported in $\widehat{CF}_i(Y)$. An element of $\widehat{HF}(Y)$ which is supported in $\widehat{HF}_i(Y)$ for some $i \in \mathbb{Z}/2\mathbb{Z}$ is said to be *homogeneous.*

Now according to standard properties of the Euler characteristic, we have that

$$\chi(\widehat{HF}_*(Y)) = \mathrm{rk}(\widehat{HF}_0(Y)) - \mathrm{rk}(\widehat{HF}_1(Y)) = \mathrm{rk}(\widehat{CF}_0(Y)) - \mathrm{rk}(\widehat{CF}_1(Y));$$

and it is also clear from the definitions that

$$\chi(\widehat{CF}_*(Y)) = \#(\mathbb{T}_\alpha \cap \mathbb{T}_\beta).$$

Indeed, the latter intersection number can also be interpreted in terms of homological data, as follows.

LEMMA 1.6. *Given a three-manifold Y, let $|H_1(Y;\mathbb{Z})|$ denote the integer defined as follows. If the number of elements n in $H_1(Y;\mathbb{Z})$ is finite, then $|H_1(Y;\mathbb{Z})| = n$; otherwise, $|H_1(Y;\mathbb{Z})| = 0$. Then,*

$$\chi(\widehat{HF}(Y)) = \pm |H_1(Y;\mathbb{Z})|.$$

In fact, if Y is a three-manifold and $\mathfrak{t} \in \mathrm{Spin}^c(Y)$, then

$$\chi(\widehat{HF}(Y, \mathfrak{t})) = \begin{cases} \pm 1 & \text{if } H_1(Y;\mathbb{Z}) \text{ is finite} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

PROOF. The identification of $\chi(\widehat{HF}(Y))$ with $|H_1(Y;\mathbb{Z})|$ is a direct consequence of the above discussion and Exercise 1.5. Now, Equation (4) amounts to the fact that $\chi(\widehat{HF}(Y,\mathfrak{t}))$ is independent of the choice of $\mathfrak{t}$. This is a consequence of the fact that $\chi(\widehat{HF}(Y,\mathfrak{t}))$ is independent of the choice of basepoint (i.e. by varying the basepoint, the generators of the chain complex and their $\mathbb{Z}/2\mathbb{Z}$ gradings remain the same), whereas the Spin^c depends on this choice. $\square$

We can use Lemma 1.6 to lift the relative $\mathbb{Z}/2\mathbb{Z}$ grading on $\widehat{HF}(Y)$ to an absolute grading, provided that $H_1(Y;\mathbb{Z})$ is finite: the $\mathbb{Z}/2\mathbb{Z}$ grading is pinned down by the convention that $\chi(\widehat{HF}(Y))$ is positive. (In fact, this $\mathbb{Z}/2\mathbb{Z}$ grading can be naturally generalized to all closed three-manifolds, cf. Section 10.4 of [**40**].)

There are refinements of this $\mathbb{Z}/2\mathbb{Z}$ grading in the presence of additional structure. For example, a rational homology three-sphere is a three-manifold with finite $H_1(Y;\mathbb{Z})$ (equivalently, $H_*(Y;\mathbb{Q})\cong H_*(S^3;\mathbb{Q})$). For a rational homology three-sphere, if $\mathbf{x},\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ can be connected by some $\phi\in\pi_2(\mathbf{x},\mathbf{y})$, then in fact the quantity

$$\mathrm{gr}(\mathbf{x},\mathbf{y})=\mu(\phi)-2n_z(\phi) \tag{5}$$

is independent of the choice of ϕ (depending only on $\mathbf{x}$ and $\mathbf{y}$). Correspondingly, we can use $\mathrm{gr}(\mathbf{x},\mathbf{y})$ to define relative $\mathbb{Z}$ gradings on the Heegaard Floer homology groups, by defining the grading of the generator $U^i\cdot\mathbf{x}$ minus the grading of $U^j\cdot\mathbf{y}$ to be $\mathrm{gr}(\mathbf{x},\mathbf{y})-2(i-j)$. This relative $\mathbb{Z}$ grading can be lifted to an absolute $\mathbb{Q}$ grading, as discussed in Lecture 3.

There is one additional basic property of Heegaard Floer homology which we will need, and that is the conjugation symmetry. The set of Spin^c structures over Y admits an involution, written $\mathfrak{t}\mapsto\overline{\mathfrak{t}}$. It is always true that

$$HF^\circ(Y,\mathfrak{t})\cong HF^\circ(Y,\overline{\mathfrak{t}}) \tag{6}$$

(for any of the variants $HF^\circ=\widehat{HF}$, HF^-, HF^∞, or HF^+).

1.3. L-spaces. An L-space is a rational homology three-sphere whose Heegaard Floer homology is as simple as possible.

EXERCISE 1.7. Prove that the following conditions on Y are equivalent:

- $\widehat{HF}(Y)$ is a free Abelian group with rank $|H^2(Y;\mathbb{Z})|$
- $HF^-(Y)$ is a free $\mathbb{Z}[U]$-module with rank $|H^2(Y;\mathbb{Z})|$
- $HF^\infty(Y)$ is a free $\mathbb{Z}[U,U^{-1}]$ module of rank $|H^2(Y;\mathbb{Z})|$, and the map
$$U\colon HF^+(Y)\longrightarrow HF^+(Y)$$
is surjective.

In fact, the hypothesis that $HF^\infty(Y)$ is a free $\mathbb{Z}[U,U^{-1}]$-module of rank $|H^2(Y;\mathbb{Z})|$ holds for any rational homology three-sphere (cf. Theorem 10.1 of [**40**]); but we do not require this result for our present purposes.

A three-manifold satisfying any of the hypotheses of Exercise 1.7 is called an *L-space*. Note that any lens space is an L-space. (This can be seen by drawing a genus one Heegaard diagram for $L(p,q)$, for which the two circles α and β meet transversally in p points.)

1.4. Statement of the surgery exact triangle. Let K be a knot in a closed, oriented three-manifold Y. Let $\mathrm{nd}(K)$ denote a tubular neighborhood of K, so that $M = Y - \mathrm{nd}(K)$ is a three-manifold with torus boundary. The *meridian* for K in Y is a primitive homology class in ∂M which lies in the kernel of the natural map

$$H_1(\partial M) = H_1(\partial \mathrm{nd}(K)) \longrightarrow H_1(\mathrm{nd}(K)).$$

Such a homology class can be represented by a homotopically non-trivial, simple, closed curve in the boundary of M which bounds a disk in $\mathrm{nd}(K)$. The homology (or isotopy) class of the meridian is uniquely specified up to multiplication by ± 1 by this property. A *longitude* for K is a homology class in $H_1(\partial M)$, with the property that the algebraic intersection number of $\#(\mu \cap \lambda) = -1$, where here ∂M is oriented as the boundary of M. Unlike the meridian, the homology (or isotopy) class of a longitude is not uniquely determined by this property. In fact, the set of longitudes for K is of the form $\{\lambda + n \cdot \mu\}_{n \in \mathbb{Z}}$. A *framed knot* $K \subset Y$ is a knot, together with a choice of longitude λ. When $K \subset Y$ is a knot with framing λ, we can form the new three-manifold $Y_\lambda(K)$ obtained by attaching a solid torus to M, in such a way that λ bounds a disk in the new solid torus. This three-manifold is said to be obtained from Y by λ-framed surgery along K.

It might seem arbitrary to restrict attention to longitudes. After all, if γ is any homotopically non-trivial, simple closed curve in ∂M, we can form a three-manifold which is a union of M and a solid torus, attached so that γ bounds a disk in the solid torus. (This more general operation is called *Dehn filling.*) However, if we restrict attention to longitudes, then there is not only a three-manifold, but also a canonical four-manifold $W_\lambda(K)$ consisting of a single two-handle attached to $[0,1] \times Y$ along $\{1\} \times Y$ with the framing specified by λ, giving a cobordism from Y to $Y_\lambda(K)$.

Exercise 1.8. Note that if $K \subset Y$ is a null-homologous knot (e.g. any knot in S^3), then there is a unique longitude λ for K which is null-homologous in $Y - \mathrm{nd}(K)$. This longitude is called the *Seifert framing* for $K \subset Y$. Show that for this choice of framing, the first Betti number of $Y_\lambda(K)$ is one; more generally,

$$H_1(Y_{p\cdot\mu + q\cdot\lambda}(K); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}.$$

The three-manifold $Y_{p\cdot\mu+q\cdot\lambda}(K)$ is typically denoted $Y_{p/q}(K)$, where here $p/q \in \mathbb{Q}$.

Exercise 1.9. Let $K \subset S^3$ be a knot, equipped with its Seifert framing, and let $r \in \mathbb{Q}$ be any rational number. Show that $S^3_r(K) \cong -S^3_{-r}(\overline{K})$, where here $\overline{K}$ denotes the mirror of K (i.e. given a knot projection of K, $\overline{K}$ has a knot projection where all the over-crossings have been replaced by under-crossings), and the orientation on $S^3_r(K)$ is taken to be the one it inherits from S^3.

Fix a closed, oriented three-manifold Y, and let K be a framed knot in Y (i.e. a knot with a choice of longitude λ). Let $Y_0 = Y_0(K)$ denote the three-manifold obtained from λ-framed surgery on Y along K, and let $Y_1 = Y_1(K)$ denote the three-manifold obtained from $(\mu + \lambda)$-framed surgery on Y along K. We call the ordered triple (Y, Y_0, Y_1) a *triad* of three-manifolds.

This relationship between Y, Y_0, and Y_1 is symmetric under a cyclic permutation of the three three-manifolds. Indeed, it is not difficult to see that (Y, Y_0, Y_1) fit into a triad if and only if there is a single oriented three-manifold M with torus boundary, and three simple, closed curves γ, γ_0, and γ_1 in ∂M with

$$\#(\gamma \cap \gamma_0) = \#(\gamma_0 \cap \gamma_1) = \#(\gamma_1 \cap \gamma) = -1, \tag{7}$$

so that Y resp. Y_0 resp. Y_1 are obtained from M by attaching a solid torus along the boundary with meridian γ resp. γ_0 resp. γ_1.

EXAMPLE 1.10. Let $K \subset S^3$ be a knot in S^3 equipped with its Seifert framing, cf. Exercise 1.8. Then the three-manifolds S^3, $S^3_p(K)$ and $S^3_{p+1}(K)$ form a triad for any integer p. More generally, given relatively prime integers p_1 and q_1, we can find p_2 and q_2 so that $p_1q_2 - q_1p_2 = 1$. Then, writing $p_3 = p_1 + p_2$ and $q_3 = q_1 + q_2$, we have that $S^3_{p_1/q_1}(K)$, $S^3_{p_2/q_2}(K)$, and $S^3_{p_3/q_3}(K)$ fit into a triad.

Another natural example of triads appears in skein theory for links.

Let $L \subset S^3$ be a link. The *branched double-cover of* L, $\Sigma(L)$ is the three-manifold which admits an orientation-preserving involution whose quotient is S^3, so that the fixed point set of the involution is identified with $L \subset S^3$. The three-manifold $\Sigma(L)$ is uniquely determined by L.

Fix a generic planar projection of L, and let x denote a crossing for this planar projection. There are two naturally associated links L_0 and L_1 which are obtained by resolving the crossing x. These two resolutions are pictured in Figure 1. Note that if we begin with a knot, and fix a crossing, then one of its resolutions will also be a knot, but the other will be a two-component link.

EXERCISE 1.11. Show that the three-manifolds $\Sigma(L)$, $\Sigma(L_0)$, and $\Sigma(L_1)$ form a triad. *Hint:* Use the fact that the branched double-cover of the three-ball branched along two disjoint arcs is a solid torus.

We have set up the relevant topology necessary to state the surgery exact triangle:

THEOREM 1.12. *(Theorem 9.12 of* [**40**]*) Let Y, Y_0, and Y_1 be three three-manifolds which fit into a triad then there are long exact sequences which relate their Heegaard Floer homologies (thought of as modules over $\mathbb{Z}[U]$):*

$$\ldots \longrightarrow \widehat{HF}(Y) \xrightarrow{\widehat{F}} \widehat{HF}(Y_0) \xrightarrow{\widehat{F}_0} \widehat{HF}(Y_1) \xrightarrow{\widehat{F}_1} \ldots$$

and

$$\ldots \longrightarrow HF^+(Y) \xrightarrow{F^+} HF^+(Y_0) \xrightarrow{F_0^+} HF^+(Y_1) \xrightarrow{F_1^+} \ldots$$

All of the above maps respect the relative $\mathbb{Z}/2\mathbb{Z}$ gradings, in the sense that each map carries homogeneous elements to homogeneous elements.

We return to the proof of Theorem 1.12 in Lecture 2. In Lecture 3, we interpret the maps appearing in the long exact sequences as maps induced by the canonical

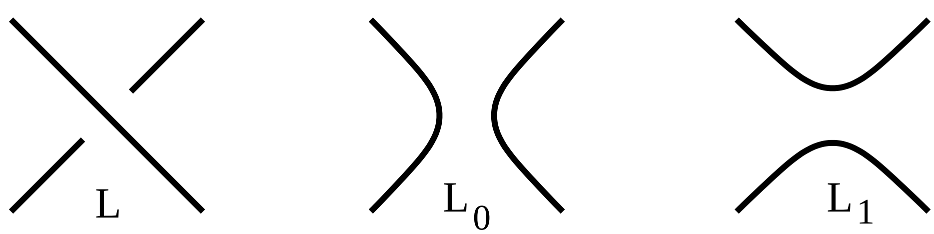

FIGURE 1. Resolutions. Given a link with a crossing as labeled in L above, we have two "resolutions" L_0 and L_1, obtained by replacing the crossing by the two simplifications pictured above.

two-handle cobordisms from Y to Y_0, Y_0 to Y_1, and Y_1 to Y. We focus now on some immediate applications. First, we use Theorem 1.12 to find examples of L-spaces.

EXERCISE 1.13. Suppose that Y, Y_0, and Y_1 are three three-manifolds which fit into a triad. For some cyclic reordering (Y, Y_0, Y_1), we can arrange that

$$|H_1(Y)| = |H_1(Y_0)| + |H_1(Y_1)|,$$

in the notation of Lemma 1.6.

The following is a quick application of Theorem 1.12 for $\widehat{HF}$:

EXERCISE 1.14. Let (Y, Y_0, Y_1) be a triad of rational homology three-spheres, ordered so that

$$|H_1(Y)| = |H_1(Y_0)| + |H_1(Y_1)|.$$

If Y_0 and Y_1 are L-spaces, then so is Y. *Hint:* Apply Theorem 1.12 and Lemma 1.6.

Exercise 1.14 provides a large number of examples of L-spaces.

For example, if $K \subset S^3$ is a knot in S^3 with the property that $S^3_r(K)$ is an L-space for some rational number $r > 0$ (with respect to the Seifert framing), then $S^3_s(K)$ is also an L-space for all $s > r$. This follows from Exercise 1.14, combined with Example 1.10. Concretely, if K is the (p, q) torus knot, then $S^3_{pq-1}(K)$ is a lens space, and hence, applying this principle, we see that in fact $S^3_r(K)$ is an L-space for all $r \geq pq - 1$.

There are other knots which admit lens space surgeries, which give rise to infinitely many interesting L-spaces. For example, if K is the $(-2, 3, 7)$ pretzel knot (cf. Figure 2), then $S^3_{18}(K) \cong L(18, 5)$ and $S^3_{19}(K) \cong L(19, 8)$ (cf. [**13**]).

Let $L \subset S^3$ be a link, and fix a generic projection of L. This projection gives a four-valent planar graph, which divides the plane into regions. These regions can be given a checkerboard coloring: we color them black and white so that two regions with the same color never meet along an edge. Thus, at each vertex there are always two (not necessarily distinct) black regions which meet. The *black graph*

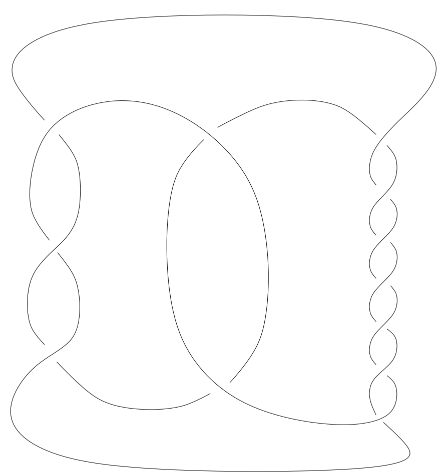

FIGURE 2. The $(-2, 3, 7)$ pretzel knot. Surgery on this knot with coefficients 18 and 19 give the lens spaces $L(18, 5)$ and $L(19, 8)$ respectively.

$\mathcal{B}(L)$ is the graph whose vertices correspond to the black regions, and whose edges correspond to crossings for the original projection, connecting the two black regions which meet at the corresponding crossing. (Strictly speaking, the black graph $\mathcal{B}(L)$ depends on a projection of L, but we do not record this dependence in the notation.) See Figure 3 for an illustration.

A knot or link projection is called *alternating* if, as we traverse each component of the link, the crossings of the projections alternate between over- and under-crossings. A knot which admits an alternating projection is simply called an *alternating knot*.

PROPOSITION 1.15. *Let K be an alternating knot or, more generally, a link which admits an alternating, connected projection; then its branched double-cover $\Sigma(K)$ is an L-space.*

PROOF. We claim that if K is an alternating link with connected, alternating projection, and we can choose a crossing with the property that K_0 and K_1 both have connected projections, then the projections of K_0 and K_1 remain alternating, and moreover

$$|H_1(\Sigma(K))| = |H_1(\Sigma(K_0))| + |H_1(\Sigma(K_1))|. \tag{8}$$

This follows from two observations: first, it is a standard result in knot theory (see for example Chapter 9 of [**31**]) that for any link K, $|H_1(\Sigma(K))| = |\Delta_K(-1)|$. Second, if K is an alternating link with a connected, alternating projection, then $|\Delta_K(-1)|$ is the number of maximal subtrees of the black graph of that projection, cf. [**2**]. (Note that $|\Delta_K(-1)|$ for an arbitrary link can be interpreted as a signed count of maximal subtrees of $\mathcal{B}(L)$; but for an alternating projection, the signs are all $+1$.)

Returning to Equation (8), note that the black graph of K_0 and K_1 can be obtained from the black graph of K by either deleting or contracting the edge e corresponding to the given crossing; thus, the maximal subtrees of $\mathcal{B}(K_0)$ correspond to the maximal subtrees of $\mathcal{B}(K)$ which contain e, while the maximal subtrees of $\mathcal{B}(K_1)$ correspond to the maximal subtrees of $\mathcal{B}(K)$ which do not contain e. Equation (8) now follows at once from the expression of $|H_1(\Sigma(K))|$ for an alternating link (with connected projection) in terms of the number of maximal subtrees.

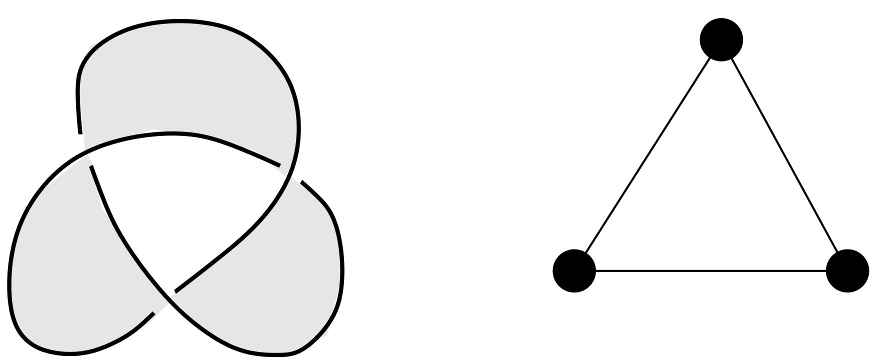

FIGURE 3. BlackGraph. We have illustrated at the left a checkerboard coloring of a projection of the trefoil; at the right, we have illustrated its corresponding “black graph”.

Recall also that a connected, alternating projection for link is called *reduced*, if for each crossing, either resolution is connected. If a connected, alternating projection of a link is not reduced, then we can always find a reduced projection as well. This is constructed inductively: if there is a crossing one of whose resolutions disconnected the projection, then it can be eliminated by twisting half the projection (to obtain a new connected, alternating projection with one fewer crossing).

The proposition now follows from induction on $|H_1(\Sigma(K))|$. Take a reduced projection of K. Either K represents the unknot, whose branched double-cover is S^3 (this is the basic case), or there is a crossing, neither of whose resolutions disconnects the connected, alternating projection. Thus, Equation (8) holds, and in particular, $0 < |H_1(\Sigma(K_i))| < |H_1(\Sigma(K))|$ for $i = 1, 2$. Thus, in view of the inductive hypothesis, we verify the inductive step by applying Exercise 1.14. $\square$

1.5. An application to Dehn surgery on knots in S^3. Note that for $K \subset S^3$, $H^2(S^3_0(K);\mathbb{Z}) \cong \mathbb{Z}$, and hence we can identify $\mathrm{Spin}^c(S^3_0(K)) \cong \mathbb{Z}$ (this is done by taking the first Chern class of $\mathfrak{s} \in \mathrm{Spin}^c(S^3_0(K))$, dividing it by two, and using a fixed isomorphism $H^2(S^3_0(K);\mathbb{Z}) \cong \mathbb{Z}$). We will correspondingly think of the decomposition of $HF^+(S^3_0(K))$ as indexed by integers,

$$HF^+(S^3_0(K)) = \bigoplus_{i\in\mathbb{Z}} HF^+(S^3_0(K), i).$$

COROLLARY 1.16. *Suppose that $K \subset S^3$ is a knot with $\widehat{HF}(S^3_{+1}(K)) \cong \widehat{HF}(S^3)$ (as $\mathbb{Z}/2\mathbb{Z}$-graded Abelian groups). Then $\widehat{HF}(S^3_0(K), i) = 0$ for all $i \neq 0$.*

PROOF. The long exact sequence from Theorem 1.12 ensures that $\widehat{HF}(S^3_0(K))$ must be either $\mathbb{Z}/m\mathbb{Z}$ for some m (which can be ruled out by other properties of Heegaard Floer homology, but this is is not necessary for our present purposes) or $\widehat{HF}(S^3_0(K)) \cong \mathbb{Z}^2$. Our goal is to understand in which Spin^c structure this group is supported. In order to be consistent with the Euler characteristic calculation, (Equation (4)) we must have that $\widehat{HF}(S^3_0(K), s) = 0$ for all but at most one s. But the conjugation symmetry $\widehat{HF}(S^3_0(K), t) \cong \widehat{HF}(S^3_0(K), -t)$ for all t ensures that in fact $\widehat{HF}(S^3_0(K), t) = 0$ for all $t \neq 0$. $\square$

The above corollary is particularly powerful when it is combined with a theorem from [**39**] (sketched in the proof of Theorem 4.2 below), according to which Heegaard Floer homology of the zero-surgery detects the genus of K. Combining these results, we get that, if K is a knot with $\widehat{HF}(S^3) \cong \widehat{HF}(S^3_{+1}(K))$ as $\mathbb{Z}/2\mathbb{Z}$-graded Abelian groups, then either K is the unknot, or the Seifert genus of K is one. This claim should be compared with a theorem of Gordon and Luecke [**24**] which states that if $S^3 \cong S^3_{+1}(K)$, then K is the unknot. It is not a strict consequence of that result, since there are three-manifolds $Y \not\cong S^3$ with $\widehat{HF}(S^3) \cong \widehat{HF}(Y)$, such as the Poincaré homology three-sphere P, cf. [**37**]. Note that $+1$ surgery on the right-handed trefoil gives this three-manifold.

Note that any three-manifold Y which is a connected sum of several copies of P (with either orientation) has $\widehat{HF}(S^3) \cong \widehat{HF}(Y)$ (as $\mathbb{Z}/2\mathbb{Z}$-graded Abelian groups), and it is a very interesting question whether there are any other three-manifolds with this property. We return to generalizations and refinements of Corollary 1.16 in Lecture 4.

1.6. Further remarks. Heegaard Floer homology fits into a general framework of a (3+1)-dimensional topological quantum field theory. The first non-trivial theory which appears to possess this kind of structure is the instanton theory for four-manifolds, defined by Simon Donaldson [**8**], coupled with its associated three-manifold invariant, defined by Andreas Floer [**15**], [**7**]. Floer's instanton homology has not yet been constructed for all three-manifolds, but it can be defined for three-manifolds with some additional algebro-topological assumptions. For instance, it is defined in the case where $H_1(Y;\mathbb{Z}) = 0$. In a correspondingly more restricted setting, Floer noticed the existence of an exact triangle, see [**16**] and also [**1**].

A number of other instances of exact triangles have since appeared in several other variants of Floer homology, including Seidel's exact sequence for Lagrangian Floer homology, cf. [**48**], and another exact triangle [**29**] which holds for the Seiberg-Witten monopole Floer homology defined by Kronheimer and Mrowka, cf. [**26**].

L-spaces are of interest to three-manifold topologists, since these are three-manifolds which admit no taut foliations, cf. [**29**], [**39**]. Hyperbolic three-manifolds which admit no taut foliations were first constructed in [**47**], see also [**3**].

2. Proof of of the exact triangle.

We sketch here a proof of Theorem 1.12. To avoid issues with signs and orientations, we will restrict attention to coefficients in $\mathbb{Z}/2\mathbb{Z}$. We also focus on the case of $\widehat{HF}$ for simplicity, returning to HF^+ in Subsection 2.5.

Let $K \subset Y$ be a knot with framing λ. Then, we can find a compatible Heegaard diagram. Specifically, we can assume that K is an unknotted knot in the β-handlebody, meeting the attaching disk belonging to β_1 transversally in one point, and disjoint from all the other attaching disks for the β_i with $i > 1$. Thus, $(\Sigma, \{\alpha_1, ..., \alpha_g\}, \{\beta_1, ..., \beta_g\}, z)$ is a pointed Heegaard diagram for Y, and β_1 is a meridian for K. There is also a curve γ_1 which represents the framing λ for K, so that if we replace β_1 by γ_1, and let γ_i be an isotopic translate of β_i for $i > 1$, then $(\Sigma, \{\alpha_1, ..., \alpha_g\}, \{\gamma_1, ..., \gamma_g\}, z)$ is a pointed Heegaard diagram for $Y_\lambda(K)$. Similarly, we can find an embedded curve δ_g representing $\mu + \lambda$, so that if we let δ_i be an isotopic translate of β_i for $i > 1$, then $(\Sigma, \{\alpha_1, ..., \alpha_g\}, \{\delta_1, ..., \delta_g\}, z)$ is a pointed Heegaard diagram representing $Y_{\mu+\lambda}(K)$.

With this understood, we choose a more symmetrical notation $Y_{\alpha\beta}$, $Y_{\alpha\gamma}$, $Y_{\alpha\delta}$ to represent the three-manifolds Y, $Y_\lambda(K)$, and $Y_{\mu+\lambda}(K)$ respectively. Also, $Y_{\beta\gamma}$ denotes the three-manifold described by the Heegaard diagram $(\Sigma, \{\beta_1, ..., \beta_g\}, \{\gamma_1, ..., \gamma_g\})$, and $Y_{\gamma\delta}$ and $Y_{\delta\beta}$ are defined similarly. The reason we chose isotopic translates of the β_i to be our γ_i and δ_i (when $i > 1$) was to ensure that all the tori $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$, $\mathbb{T}_\gamma$, $\mathbb{T}_\delta$ meet transversally in $\mathrm{Sym}^g(\Sigma)$.

Exercise 2.1. Show that $Y_{\beta\gamma} \cong Y_{\gamma\delta} \cong Y_{\delta\beta} \cong \#^{g-1}(S^1 \times S^2)$.

Before defining the maps appearing in the exact triangle, we allow ourselves a digression on holomorphic triangles. Counts of holomorphic triangles play a prominent role in Lagrangian Floer homology, cf. [**32**], [**5**], [**19**].

2.1. Holomorphic triangles. A *pointed Heegaard triple*

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$$

is an oriented two-manifold Σ, together with three g-tuples of attaching circles $\boldsymbol{\alpha} = \{\alpha_1, ..., \alpha_g\}$,$\boldsymbol{\beta} = \{\beta_1, ..., \beta_g\}$,$\boldsymbol{\gamma} = \{\gamma_1, ..., \gamma_g\}$ for handlebodies, and a choice of reference point

$$z \in \Sigma - \alpha_1 - ... - \alpha_g - \beta_1 - ... - \beta_g - \gamma_1 - ... - \gamma_g.$$

In the preceding discussion, we constructed the Heegaard triple of a framed link.

Let Δ denote the two-simplex, with vertices $v_\alpha, v_\beta, v_\gamma$ labeled clockwise, and let e_i denote the edge v_j to v_k, where $\{i, j, k\} = \{\alpha, \beta, \gamma\}$. Fix $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, $\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$. Consider the map

$$u \colon \Delta \longrightarrow \mathrm{Sym}^g(\Sigma)$$

with the boundary conditions that $u(v_\gamma) = \mathbf{x}$, $u(v_\alpha) = \mathbf{y}$, and $u(v_\beta) = \mathbf{w}$, and $u(e_\alpha) \subset \mathbb{T}_\alpha$, $u(e_\beta) \subset \mathbb{T}_\beta$, $u(e_\gamma) \subset \mathbb{T}_\gamma$. Such a map is called a *Whitney triangle connecting* $\mathbf{x}$*,* $\mathbf{y}$*, and* $\mathbf{w}$. We let $\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ denote the space of homology classes of Whitney triangles connecting $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{w}$.

Given $z \in \Sigma - \alpha_1 - ... - \alpha_g - \beta_1 - ... - \beta_g - \gamma_1 - ... - \gamma_g$, the algebraic intersection of a Whitney triangle with $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$ descends to a well-defined map on homology classes

$$n_z \colon \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w}) \longrightarrow \mathbb{Z}.$$

This intersection number is additive in the following sense. Letting $\mathbf{x}' \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\phi \in \pi_2(\mathbf{x}', \mathbf{x})$ and $\psi \in \pi_2(\mathbf{x}', \mathbf{y}, \mathbf{w})$, we can juxtapose ϕ and ψ to construct a new Whitney triangle $\psi * \phi \in \pi_2(\mathbf{x}', \mathbf{y}, \mathbf{w})$. Clearly,

$$n_z(\psi * \phi) = n_z(\psi) + n_z(\phi).$$

Also, if $n_z(\psi)$ is negative, then the homology class ψ supports no pseudo-holomorphic representative (for suitably chosen almost-complex structures).

Indeed, the homology class of a Whitney triangle $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ determines a two-chain in Σ, just as homology classes of Whitney disks give rise to two-chains. The two-chain can be thought of as a sum of closures of the regions in

$$\Sigma - \alpha_1 - ... - \alpha_g - \beta_1 - ... - \beta_g - \gamma_1 - ... - \gamma_g,$$

where the multiplicity assigned to some region R is $n_x(\psi)$, where $x \in R$ is an interior point. We can generalize the notions of periodic domain and weak admissibility to this context:

Definition 2.2. A *triply-periodic domain* P for $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ is a two-chain whose local multiplicity at z is zero and whose boundary is a linear combination of one-cycles chosen from the α_i, β_j, and γ_k. The set of triply periodic domains is naturally an Abelian group, denoted $\mathcal{P}$.

Definition 2.3. A triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ is called *weakly admissible* if all the non-zero triply-periodic domains have both positive and negative local multiplicities.

Exercise 2.4. Suppose that Y is a rational homology three-sphere, and $K \subset Y$ is a knot with framing λ with the property that $Y_{\alpha\gamma}$ is a rational homology three-sphere. For the corresponding Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$, find the dimension of the space of triply-periodic domains (in terms of the genus of Σ). Show that, after a sequence of isotopies, we can always arrange that this Heegaard triple is weakly admissible.

Suppose that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ is weakly admissible. Then, we construct a map

$$\widehat{f}_{\alpha\beta\gamma} \colon \widehat{CF}(Y_{\alpha\beta}) \otimes \widehat{CF}(Y_{\beta\gamma}) \longrightarrow \widehat{CF}(Y_{\alpha\gamma})$$

by the formula

$$\widehat{f}_{\alpha\beta\gamma}(\mathbf{x}\otimes\mathbf{y}) = \sum_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\{\psi\in\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})\big| n_z(\psi)=0=\mu(\psi)\}} \#(\mathcal{M}(\psi))\cdot\mathbf{w}. \tag{9}$$

Note that if $\psi,\psi'\in\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$ both have $n_z(\psi)=n_z(\psi')=0$, then $\mathcal{D}(\psi)-\mathcal{D}(\psi')$ can be thought of as a triply-periodic domain. In fact, if $\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$ is non-empty, we can fix some $\psi_0\in\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$ with $n_z(\psi)=0$; then there is an isomorphism

$$\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})\cong\mathbb{Z}\oplus\mathcal{P},$$

defined by

$$\psi\mapsto n_z(\psi)\oplus(\mathcal{D}(\psi)-\mathcal{D}(\psi_0)).$$

It is not difficult to see that weak admissibility ensures that for any fixed $\mathbf{x},\mathbf{y},\mathbf{w}$, there are only finitely many $\psi\in\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$ with $n_z(\psi)=0$ and $\mathcal{D}(\psi)\geq 0$. In particular this guarantees finiteness of the sum appearing in Equation (9).

Modifying the usual proof that $\partial^2=0$, we have the following:

Proposition 2.5. *The map $\widehat{f}_{\alpha\beta\gamma}$ defined above determines a chain map, where the tensor product appearing in the domain of Equation* (9) *is given its usual differential*

$$\partial(\mathbf{x}\otimes\mathbf{y}) = (\partial\mathbf{x})\otimes\mathbf{y}+\mathbf{x}\otimes(\partial\mathbf{y}).$$

Sketch of Proof. The idea is to consider ends of one-dimensional moduli spaces of pseudo-holomorphic representatives of $\psi\in\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$. Such moduli spaces have three types of ends. For example, there is an end where a pseudo-holomorphic Whitney disk connecting $\mathbf{x}$ to $\mathbf{x}'$ (for some other $\mathbf{x}'\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$) is juxtaposed with a pseudo-holomorphic Whitney triangle connecting $\mathbf{x}',\mathbf{y},\mathbf{w}$. The number of such ends corresponds to the $\mathbf{w}$-component of $\widehat{f}_{\alpha\beta\gamma}((\partial\mathbf{x})\otimes\mathbf{y})$. There are two other types of ends, where Whitney disks bubble off at the $\mathbb{T}_\beta\cap\mathbb{T}_\gamma$ (representing the $\mathbf{w}$-component of $\widehat{f}_{\alpha\beta\gamma}(\mathbf{x}\otimes(\partial\mathbf{y}))$) resp. $\mathbb{T}_\alpha\cap\mathbb{T}_\gamma$ corner (representing the $\mathbf{w}$ component of $\partial\widehat{f}_{\alpha\beta\gamma}(\mathbf{x}\otimes\mathbf{y})$. □

In particular, we obtain an induced map on homology

$$\widehat{F}_{\alpha\beta\gamma}\colon \widehat{HF}(Y_{\alpha\beta})\otimes\widehat{HF}(Y_{\beta\gamma})\longrightarrow\widehat{HF}(Y_{\alpha\gamma}).$$

The maps induced by counting holomorphic triangles satisfy an associativity law, stating that if we start with four g-tuples of attaching circles $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$, then

$$\widehat{F}_{\alpha\gamma\delta}(\widehat{F}_{\alpha\beta\gamma}(\cdot\otimes\cdot)\otimes\cdot) = \widehat{F}_{\alpha\beta\delta}(\cdot\otimes\widehat{F}_{\beta\gamma\delta}(\cdot\otimes\cdot)) \tag{10}$$

as maps

$$\widehat{HF}(Y_{\alpha\beta})\otimes\widehat{HF}(Y_{\beta\gamma})\otimes\widehat{HF}(Y_{\gamma\delta})\longrightarrow\widehat{HF}(Y_{\alpha\delta}).$$

We give a more precise version presently. Let $\square$ denote the "rectangle": unit disk whose boundary is divided into four arcs (topologically closed intervals) labeled e_α, e_β, e_γ, and e_δ (in clockwise order). The justification for calling this a rectangle is given in the following:

Exercise 2.6. Let $\square$ be any rectangle in the above sense, with the conformal structure induced from $\mathbb{C}$. Show that there is a pair of real numbers w and h and a unique holomorphic identification

$$\theta\colon\square\longrightarrow[0,w]\times[0,h]$$

carrying e_α to $[0,w]\times\{h\}$, e_β to $\{0\}\times[0,h]$, e_γ to $[0,w]\times\{0\}$, and e_δ to $\{w\}\times[0,h]$. Indeed, the ratio w/h is uniquely determined by the conformal structure on $\square$.

The above exercise can be interpreted in the following manner: the space of conformal structures on $\square$ is identified with $\mathbb{R}$ under the map which takes a fixed conformal structure to the real number $\log(w)-\log(h)$ in the above uniformization.

A map $\varphi\colon \square \longrightarrow \mathrm{Sym}^g(\Sigma)$ which carries e_α, e_β, e_γ, and e_δ to $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$, $\mathbb{T}_\gamma$, and $\mathbb{T}_\delta$ respectively is called a *Whitney rectangle.* For fixed $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, $\mathbf{y}\in\mathbb{T}_\beta\cap\mathbb{T}_\gamma$, $\mathbf{w}\in\mathbb{T}_\gamma\cap\mathbb{T}_\delta$, and $\mathbf{p}\in\mathbb{T}_\alpha\cap\mathbb{T}_\delta$, spaces of Whitney rectangles can be collected into homology classes $\pi_2(\mathbf{x},\mathbf{y},\mathbf{w},\mathbf{p})$. Let $\mathcal{M}(\varphi)$ denote the moduli space of pseudo-holomorphic representatives of φ, with respect to any conformal structure on the domain $\square$.

Given a pointed Heegaard quadruple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta},z)$, define a map

$$\widehat{h}_{\alpha\beta\gamma\delta}\colon \widehat{CF}(Y_{\alpha\beta})\otimes\widehat{CF}(Y_{\beta\gamma})\otimes\widehat{CF}(Y_{\gamma\delta})\longrightarrow\widehat{CF}(Y_{\alpha\delta})$$

by the formula

$$\widehat{h}_{\alpha\beta\gamma\delta}(\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{w})=\sum_{\mathbf{p}\in\mathbb{T}_\alpha\cap\mathbb{T}_\delta}\;\sum_{\{\varphi\in\pi_2(\mathbf{x},\mathbf{y},\mathbf{w},\mathbf{p})\big|\mu(\varphi)=-1,n_z(\varphi)=0\}}\#\mathcal{M}(\varphi)\cdot\mathbf{p}.$$

Again, to ensure that this formula has the required finiteness properties, we need the quadruple to satisfy a weak admissibility hypothesis analogous to Definition 2.3.

THEOREM 2.7. *The map $\widehat{h}_{\alpha\beta\gamma\delta}$ determines a chain homotopy between the maps*

$$\widehat{f}_{\alpha\gamma\delta}(\widehat{f}_{\alpha\beta\gamma}(\cdot\otimes\cdot)\otimes\cdot)\quad\text{and}\quad\widehat{f}_{\alpha\beta\delta}(\cdot\otimes\widehat{f}_{\beta\gamma\delta}(\cdot\otimes\cdot));$$

i.e. for all $\xi\in\widehat{CF}(Y_{\alpha\beta})$, $\eta\in\widehat{CF}(Y_{\beta\gamma})$, $\zeta\in\widehat{CF}(Y_{\gamma\delta})$, we have that

$$\partial\widehat{h}_{\alpha\beta\gamma\delta}(\xi\otimes\eta\otimes\zeta)+\widehat{h}_{\alpha\beta\gamma\delta}(\partial(\xi\otimes\eta\otimes\zeta))=\widehat{f}_{\alpha\gamma\delta}(\widehat{f}_{\alpha\beta\gamma}(\xi\otimes\eta)\otimes\zeta)+\widehat{f}_{\alpha\beta\delta}(\xi\otimes\widehat{f}_{\beta\gamma\delta}(\eta\otimes\zeta)).$$

Sketch of Proof. We wish to consider moduli spaces of pseudo-holomorphic Whitney rectangles with formal dimension one. Some ends of these moduli spaces are modeled on flowlines breaking off at the corners, but there is another type of end not encountered before in the counts of trianges, arising from the non-compactness of $\mathcal{M}(\square)\cong\mathbb{R}$. As this parameter goes to $\pm\infty$, the corresponding rectangle breaks up conformally into a pair of triangles meeting at a vertex (in two different ways, depending on which end we are considering), as illustrated in Figure 4. This is how a count of holomorphic squares induces a chain homotopy between two different compositions of holomorphic triangle counts. $\square$

For more details on the associativity in Lagrangian Floer homology, compare [**32**] [**5**] [**19**].

EXERCISE 2.8. Deduce Equation (10) from Theorem 2.7.

2.2. Maps in the exact sequence. We are now ready to define the maps appearing in the exact sequence. Since $Y_{\beta\gamma}\cong\#^{g-1}(S^2\times S^1)$, we have that $\widehat{HF}(Y_{\beta\gamma})\cong\Lambda^*H^1(\#^{g-1}(S^2\times S^1);\mathbb{Z}/2\mathbb{Z})$. As such, its top-dimensional group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let $\widehat{\Theta}_{\beta\gamma}$ denote this generator. The map $\widehat{F}$ is defined by

$$\widehat{F}(\xi)=\widehat{F}_{\alpha\beta\gamma}(\xi\otimes\widehat{\Theta}_{\beta\gamma}).$$

In fact, we can exhibit a Heegaard diagram for $\#^{g-1}(S^2\times S^1)$ for which all the differentials are trivial, and hence $\widehat{\Theta}_{\beta\gamma}$ is represented by an intersection point of

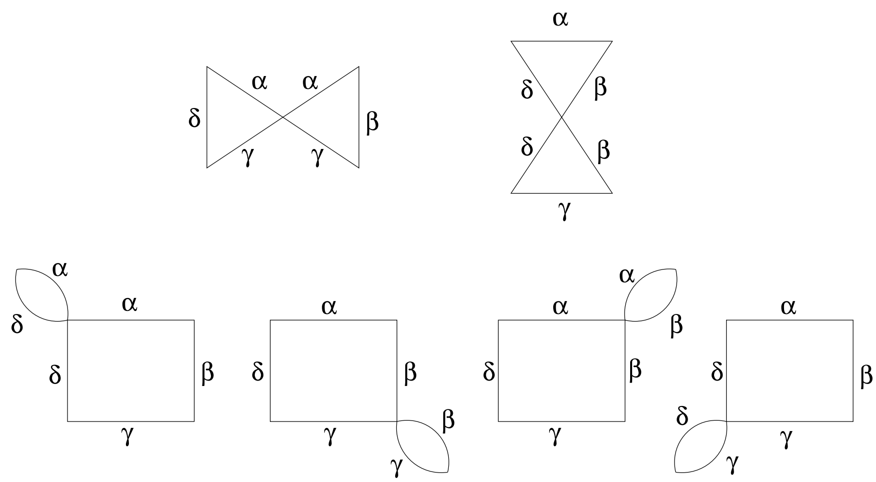

FIGURE 4. Degenerate rectangles. We have illustrated here a schematic diagram for the degenerations of pseudo-holomorphic rectangles. Edges are marked with the corresponding torus they are mapped to. (Conformal moduli for rectangles appearing in this figure are arbitrary.)

$\mathbb{T}_\beta \cap \mathbb{T}_\gamma$. By a slight abuse of notation, we also denote this intersection point by the symbol $\widehat{\Theta}_{\beta\gamma}$.

We can define the maps $\widehat{F}_0$ and $\widehat{F}_1$ analogously;

$$\widehat{F}_0(\eta) = \widehat{F}_{\alpha\gamma\delta}(\eta \otimes \widehat{\Theta}_{\gamma\delta}), \quad \text{and} \quad \widehat{F}_1(\zeta) = \widehat{F}_{\alpha\delta\beta}(\zeta \otimes \widehat{\Theta}_{\delta\beta}),$$

where here $\widehat{\Theta}_{\gamma\delta}$ and $\widehat{\Theta}_{\delta\beta}$ are generators for the top-dimensional non-trivial groups in $\widehat{HF}(Y_{\gamma\delta}) \cong \widehat{HF}(Y_{\delta\beta}) \cong \Lambda^* H^1(S^2 \times S^1)$.

To prove Theorem 1.12, we must verify that Ker $\widehat{F}_0 = \mathrm{Im}\ \widehat{F}$. As a first step, we would like to prove that Im $\widehat{F}_0 \subseteq \mathrm{Ker}\ \widehat{F}$, i.e. $\widehat{F}_0 \circ \widehat{F} = 0$. To this end, note that

$$\widehat{F}_0 \circ \widehat{F}(\xi) = \widehat{F}_{\alpha\gamma\delta}(\widehat{F}_{\alpha\beta\gamma}(\xi \otimes \widehat{\Theta}_{\beta\gamma}) \otimes \widehat{\Theta}_{\gamma\delta}) = \widehat{F}_{\alpha\beta\delta}(\xi \otimes \widehat{F}_{\beta\gamma\delta}(\widehat{\Theta}_{\beta\gamma} \otimes \widehat{\Theta}_{\gamma\delta})), \tag{11}$$

so it suffices to prove that

$$\widehat{F}_{\beta\gamma\delta}(\widehat{\Theta}_{\beta\gamma} \otimes \widehat{\Theta}_{\gamma\delta}) = 0,$$

which in turn follows from a model calculation.

EXERCISE 2.9. Let β, γ, δ be three straight curves in the torus Σ as above, and let $\widehat{\Theta}_{\beta\gamma}$, $\widehat{\Theta}_{\gamma\delta}$, $\widehat{\Theta}_{\delta\beta}$ denote the three intersection points. Prove that $\pi_2(\widehat{\Theta}_{\beta\gamma}, \widehat{\Theta}_{\gamma\delta}, \widehat{\Theta}_{\delta\beta}) = \{\psi_k^\pm\}_{k=1}^\infty$, where $\mu(\psi_k^\pm) = 0$, $n_z(\psi_k^\pm) = \frac{k(k-1)}{2}$, and each $\psi_k^\pm$ has a unique holomorhpic representative. *Hint:* Lift to the universal cover of Σ.

PROPOSITION 2.10. *The are exactly two homology classes of* $\psi \in \pi_2(\widehat{\Theta}_{\beta\gamma}, \widehat{\Theta}_{\gamma\delta}, \widehat{\Theta}_{\delta\beta})$ *with* $\mathcal{D}(\psi) \geq 0$, $n_z(\psi) = 0$, *and* $\mu(\psi) = 0$. *For either homology class* ψ,

$$\#\mathcal{M}(\psi) \equiv 1 \pmod 2.$$

PROOF. In the case where $g(\Sigma) = 1$, we appeal to Exercise 2.9.

In the general case, we can decompose the Heegaard surface $\Sigma = E_1 \# \ldots \# E_g$ as a connected sum of g tori, with each β_i, γ_i, and δ_i supported inside E_i. For

$i > 1$, the summand E_i with its three curves is homeomorphic to the one pictured in Figure 5, while for E_1, it is the case considered in Exercise 2.9. In this case, any homology class $\psi \in \pi_2(\widehat{\Theta}_{\beta\gamma}, \widehat{\Theta}_{\gamma\delta}, \widehat{\Theta}_{\delta\beta})$ with $n_z(\psi) = 0$ decomposes as a product of homology classes $\psi_i \in \pi_2(\beta_i, \gamma_i, \delta_i)$ for E_i. It is easy to see that there are precisely two homology class ψ in $\pi_2(\widehat{\Theta}_{\beta\gamma}, \widehat{\Theta}_{\gamma\delta}, \widehat{\Theta}_{\delta\beta})$ with $\mathcal{D}(\psi) \geq 0$ and $n_z(\psi) = 0$. These homology classes are obtained by taking the product of $g-1$ copies of the distinguished homology classes from Figure 5 in the E_i summand for $i > 1$, and one copy of either ψ_1^+ or ψ_1^- from Exercise 2.9 in the E_1 summand. $\square$

The fact that

$$\widehat{f}_{\beta\gamma\delta}(\widehat{\Theta}_{\beta\gamma} \otimes \widehat{\Theta}_{\gamma\delta}) = 0 \tag{12}$$

is a quick consequence of Proposition 2.10. Thus, from the associativity of the maps induced by holomorphic triangles (cf. Equation (11) above), it follows that $\widehat{F}_0 \circ \widehat{F} = 0$. The other double composites $\widehat{F}_1 \circ \widehat{F}_0$ and $\widehat{F} \circ \widehat{F}_1$ also vanish by a symmetrical argument.

Thus, we have verified that the sequence of maps on $\widehat{HF}$ appearing in the statement of Theorem 1.12 form a chain complex. It remains to verify that the chain complex has trivial homology. To this end, we find it useful to make a digression into some homological algebra.

2.3. Some homological algebra. We begin with some terminology.

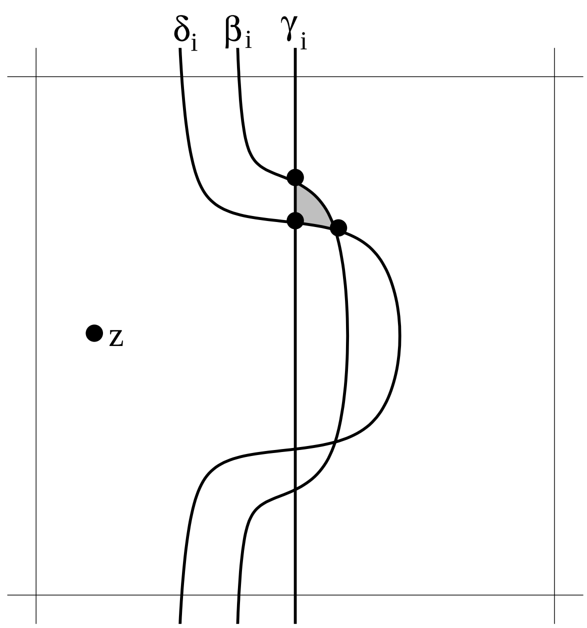

FIGURE 5. Other factors of the holomorphic triangle. We have illustrated here a Heegaard triple, where γ_i, β_i and δ_i are small isotopic translates of one another. The unique homology class of triangles $\pi_2(\widehat{\Theta}_{\beta\gamma}, \widehat{\Theta}_{\gamma\delta}, \widehat{\Theta}_{\delta\beta})$ with $n_z(\psi) = 0$ and $\mathcal{D}(\psi) \geq 0$ is indicated by the shading.

Let A_1 and A_2 be a pair of chain complexes of vector spaces over the field $\mathbb{Z}/2\mathbb{Z}$ (though the discussion here could again be given over $\mathbb{Z}$, with more attention paid to signs). A chain map

$$\phi\colon A_1 \longrightarrow A_2$$

is called a *quasi-isomorphism* if the induced map on homology is an isomorphism.

Recall that if we have a chain map between chain complexes $f_1\colon A_1 \longrightarrow A_2$, we can form its mapping cone $M(f_1)$, whose underlying module is the direct sum $A_1 \oplus A_2$, endowed with the differential

$$\partial = \left(\begin{array}{cc} \partial_1 & 0 \\ f_1 & \partial_2 \end{array}\right),$$

where here ∂_i denotes the differential for the chain complex A_i. Recall that there is a short exact sequence of chain complexes

$$0 \longrightarrow A_2 \xrightarrow{\iota} M(f_1) \xrightarrow{\pi} A_1 \longrightarrow 0. \tag{13}$$

Exercise 2.11. Show that the short exact sequence from Equation (13) induces a long exact sequence in homology, for which the connecting homomorphism is the map on homology induced by f_1.

Exercise 2.12. Verify naturality of the mapping cylinder in the following sense. Suppose that we have a diagram of chain complexes

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "\psi_1"'] & A_2 \arrow[d, "\psi_2"] \\ B_1 \arrow[r, "g_1"] & B_2 \end{tikzcd}$$

which commutes up to homotopy; then there is an induced map

$$m(\psi_1,\psi_2)\colon M(f_1) \longrightarrow M(g_1)$$

which fits into the following diagram, where the rows are exact and the squares are homotopy-commutative:

$$\begin{tikzcd} 0 \arrow[r] & A_2 \arrow[r] \arrow[d, "\psi_2"'] & M(f_1) \arrow[r] \arrow[d, "m(\psi_1,\psi_2)"'] & A_1 \arrow[r] \arrow[d, "\psi_1"] & 0 \\ 0 \arrow[r] & B_2 \arrow[r] & M(g_1) \arrow[r] & B_1 \arrow[r] & 0. \end{tikzcd}$$

Lemma 2.13. *Let $\{A_i\}_{i=1}^{\infty}$ be a collection of chain maps and let*

$$\{f_i\colon A_i \longrightarrow A_{i+1}\}_{i\in\mathbb{Z}}$$

be a collection of chain maps satisfying the following two properties:

(1) *$f_{i+1}\circ f_i$ is chain homotopically trivial, by a chain homotopy*

$$H_i\colon A_i \longrightarrow A_{i+2}$$

(2) *the map*

$$\psi_i = f_{i+2}\circ H_i + H_{i+1}\circ f_i \colon A_i \longrightarrow A_{i+3}$$

is a quasi-isomorphism.

Then, $H_*(M(f_i)) \cong H_*(A_{i+2})$.

EXERCISE 2.14. Show that the hypotheses of Lemma 2.13 imply that ψ_i is a chain map. Then supply a proof of Lemma 2.13. *Hint:* Construct chain maps $M(f_i) \longrightarrow A_{i+2}$ and $A_i \longrightarrow M(f_{i+1})$, and use the five-lemma to prove that they induce isomorphisms on homology.

2.4. Completion of the proof of Theorem 1.12 for $\widehat{HF}$ with $\mathbb{Z}/2\mathbb{Z}$ coefficents. Continuing notation from before, let $Y_{\alpha\beta}$, $Y_{\alpha\gamma}$ $Y_{\alpha\delta}$ describe Y, Y_0, and Y_1 respectively, and so that the remaining three-manifolds $Y_{\beta\gamma}$, $Y_{\gamma\delta}$, $Y_{\delta\beta}$ describe $\#^{g-1}(S^2\times S^1)$. Indeed, to fit precisely with the hypotheses of Lemma 2.13, we choose infinitely many copies of the g-tuples $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ (denoted $\boldsymbol{\beta}^{(i)}$, $\boldsymbol{\gamma}^{(i)}$, $\boldsymbol{\delta}^{(i)}$ for $i\in\mathbb{Z}$), all of which are generic exact Hamiltonian perturbations of one another.

Let A_{3i+1}, A_{3i+2} and A_{3i+3} represent $\widehat{CF}(Y_0)$, $\widehat{CF}(Y_1)$ and $\widehat{CF}(Y)$ respectively, only now we use the various translates of the $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, and $\boldsymbol{\beta}$; in particular A_{3i+1} is the Floer complex $\widehat{CF}(Y_{\alpha\gamma^{(i)}})$.

We have already verified Hypothesis (1) in the discussion of Subsection 2.2. For example, the null-homotopy $H_i\colon A_{3i}\longrightarrow A_{3i+2}$ is given by the map

$$H_i(\xi)=\widehat{h}_{\alpha,\beta^{(i)},\gamma^{(i)},\delta^{(i)}}(\xi\otimes\widehat{\Theta}_{\beta^{(i)}\gamma^{(i)}}\otimes\widehat{\Theta}_{\gamma^{(i)}\delta^{(i)}})$$

gotten by counting pseudo-holomorphic rectangles.

It remains to verify Hypothesis (2) of Lemma 2.13. It is useful to have the following:

DEFINITION 2.15. An $\mathbb{R}$-filtration of a group G is a sequence of subgroups indexed by $r\in\mathbb{R}$, so that

- $G_r\subseteq G_s$ if $r\leq s$ and
- $G=\cup_{r\in\mathbb{R}}G_r$.

This induces a partial ordering on G. If $x,y\in G$, we say $x<y$ if $x\in G_r$, while $y\notin G_r$.

DEFINITION 2.16. The *area filtration* on $\widehat{CF}(Y_{\alpha\beta})$ is the $\mathbb{R}$-filtration defined as follows. Fix $\mathbf{x}_0\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, and define a function

$$\mathcal{F}\colon\mathbb{T}_\alpha\cap\mathbb{T}_\beta\longrightarrow\mathbb{R}$$

gotten by taking

$$\mathcal{F}(\mathbf{x})=\mathcal{A}(\mathcal{D}(\phi))-2n_z(\phi)\cdot\mathcal{A}(\Sigma),$$

where here $\phi\in\pi_2(\mathbf{x}_0,\mathbf{x})$ is any homotopy class connecting $\mathbf{x}$ and $\mathbf{y}$, $\mathcal{A}(R)$ denotes the signed area of some region R in Σ, with respect to a fixed area form over Σ.

In the case where $b_1(Y_{\alpha\beta})>0$, in order for the area filtration to be well-defined, we must use an area form over Σ with the property that $\mathcal{A}(P)=0$ for each periodic domain. Such an area form can be found for any weakly admissible diagram.

LEMMA 2.17. *If $\boldsymbol{\beta}'$ is a sufficiently small perturbation of $\boldsymbol{\beta}$, and $\widehat{\Theta}_{\beta\beta'}$ denotes the canonical top-dimensional homology class in $\widehat{HF}(Y_{\beta\beta'})$, then the chain map*

$$\widehat{CF}(Y_{\alpha\beta})\longrightarrow\widehat{CF}(Y_{\alpha\beta'})$$

defined by

$$\xi\mapsto\widehat{f}_{\alpha\beta\beta'}(\xi\otimes\widehat{\Theta}_{\beta\beta'})$$

induces an isomorphism in homology.

Proof. We perform the perturbation so that β_i' and β_i intersect transversally in two points, and indeed, so that the signed area of the region between β_i and β_i' is zero.

If each β_i' is sufficiently close to the corresponding β_i, then for each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, there is a corresponding closest point $\mathbf{x}' \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}$. This closest point map induces a group isomorphism

$$\iota\colon \widehat{CF}(Y_{\alpha\beta}) \longrightarrow \widehat{CF}(Y_{\alpha\beta'}).$$

Note that for each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, there is a canonical smallest triangle $\psi \in \pi_2(\mathbf{x}, \widehat{\Theta}_{\beta\beta'}, \mathbf{x}')$ which admits a unique holomorphic representative (by the Riemann mapping theorem). By taking sufficiently nearby translates β_i' of the β_i, we can arrange for the area of this triangle to be smaller than the areas of any homotopy classes of $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ for any $\mathbf{x}$ and $\mathbf{y}$ either in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ or in $\mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}$.

This map perhaps does not quite agree with the chain map

$$f(\xi) = \widehat{f}_{\alpha\beta\beta'}(\xi \otimes \widehat{\Theta}_{\beta\beta'}).$$

However, for each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the element $f(\mathbf{x}) - \iota(\mathbf{x})$ can be written as a linear combination of $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}$, with $\mathcal{F}(\iota(\mathbf{x})) < \mathcal{F}(\mathbf{y})$ with respect to the area filtration of $\widehat{CF}(Y_{\alpha\beta'})$. Since ι induces an isomorphism on the group level, it is easy to see that f induces an isomorphism on the group level as well. Since f is also a chain map, it follows that it induces an isomorphism of chain complexes. □

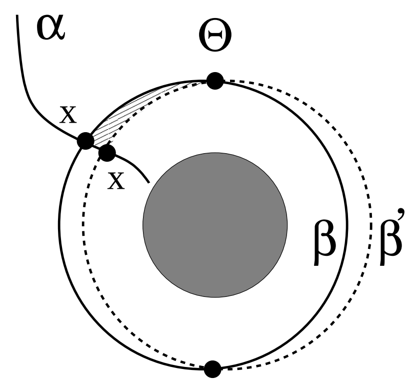

Figure 6. Small triangles. In the proof of Lemma 2.17, we let β_i' be a nearby isotopic translate of β_i, arranged so that the two curves meet transversally in two points. The top-dimensional generator of $\mathbb{T}_\beta \cap \mathbb{T}_{\beta'}$ is represented by the product of intersection points $\Theta_1 \times ... \times \Theta_g = \widehat{\Theta}$. Any intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ has a nearest intersection point $\mathbf{x}' \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}$; and there is a canonical homology class of smallest triangle $\psi \in \pi_2(\mathbf{x}, \widehat{\Theta}, \mathbf{x}')$ which supports a unique holomorphic representative. We have illustrated here an annular region in Σ (i.e. delete the shaded circle from the picture) which is a neighborhood of β_i, though we have dropped the subscripts. Θ, x, and x' represent the corresponding factor of $\widehat{\Theta}$, $\mathbf{x}$, and $\mathbf{x}'$ respectively, and the hatched region illustrates part of the region of the canonical smallest triangle.

Let $\theta_i\colon A_i \longrightarrow A_{i+3}$ be the quasi-isomorphisms defined as in the above lemma; e.g. θ_{3i+1} is the chain map $\widehat{CF}(Y_{\alpha\gamma^{(i)}}) \longrightarrow \widehat{CF}(Y_{\alpha\gamma^{(i+1)}})$ obtained by product with the canonical generator $\widehat{\Theta}_{\gamma^{(i)}\gamma^{(i+1)}}$.

We claim that

$$f_3 \circ H_1 + H_2 \circ f_1 \colon A_1 \longrightarrow A_4$$

is chain homotopic to θ_1. More precisely, counting pseudo-holomorphic pentagons with edges on $\mathbb{T}_\alpha$, $\mathbb{T}_\gamma$, $\mathbb{T}_\delta$ $\mathbb{T}_\beta$, $\mathbb{T}_\gamma^{(1)}$ can be used to give a homotopy to prove a generalized associativity law analogous to Theorem 2.7; i.e. looking at ends of one-dimensional moduli spaces of pseudo-holomorphic pentagons, we get a null-homotopy of the sum of composite maps:

$$\begin{aligned}
(14) \qquad & \widehat{f}_{\alpha\beta,\gamma^{(1)}}(\widehat{h}_{\alpha\gamma\delta\beta}(\xi\otimes\widehat{\Theta}_{\gamma\delta}\otimes\widehat{\Theta}_{\delta\beta})\otimes\widehat{\Theta}_{\beta\gamma^{(1)}}) \\
+ \quad & \widehat{h}_{\alpha\gamma\delta\gamma^{(1)}}(\xi\otimes\widehat{\Theta}_{\gamma\delta}\otimes\widehat{f}_{\delta\beta\gamma^{(1)}}(\widehat{\Theta}_{\delta\beta}\otimes\widehat{\Theta}_{\beta\gamma^{(1)}})) \\
+ \quad & \widehat{h}_{\alpha\gamma\beta\gamma^{(1)}}(\xi\otimes\widehat{f}_{\gamma\delta\beta}(\widehat{\Theta}_{\gamma\delta}\otimes\widehat{\Theta}_{\delta\beta})\otimes\widehat{\Theta}_{\beta\gamma^{(1)}}) \\
+ \quad & \widehat{h}_{\alpha\delta\beta\gamma^{(1)}}(\widehat{f}_{\alpha\gamma\delta}(\xi\otimes\widehat{\Theta}_{\gamma\delta})\otimes\widehat{\Theta}_{\delta\beta}\otimes\widehat{\Theta}_{\beta\gamma^{(1)}}) \\
+ \quad & \widehat{f}_{\alpha\gamma\gamma^{(1)}}(\xi\otimes\widehat{h}_{\gamma\delta\beta\gamma^{(1)}}(\widehat{\Theta}_{\gamma\delta}\otimes\widehat{\Theta}_{\delta\beta}\otimes\widehat{\Theta}_{\beta\gamma^{(1)}})).
\end{aligned}$$

This sum is more graphically illustrated in Figure 7. Two of these terms vanish, since

$$\widehat{f}_{\delta\beta\gamma^{(1)}}(\widehat{\Theta}_{\delta\beta}\otimes\widehat{\Theta}_{\beta\gamma^{(1)}}) = 0 = \widehat{f}_{\gamma\delta\beta}(\widehat{\Theta}_{\gamma\delta}\otimes\widehat{\Theta}_{\delta\beta}).$$

The first and fourth terms are identified with $f_3 \circ H_1 + H_2 \circ f_1$. To see that the final term is identified with θ_1, it suffices to show that

$$\widehat{h}_{\gamma\delta\beta\gamma^{(1)}}(\widehat{\Theta}_{\gamma\delta}\otimes\widehat{\Theta}_{\delta\beta}\otimes\widehat{\Theta}_{\beta\gamma^{(1)}}) = \widehat{\Theta}_{\gamma\gamma^{(1)}}. \tag{15}$$

This latter equality follows from a direct inspection of the Heegaard diagram for the quadruple $(\Sigma, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\gamma}^{(1)}, z)$. (i.e. the count of pseudo-holomorphic quadrilaterals), as illustrated in Figures 8 and 9.

In Figure 8, we consider the special case where the genus $g = 1$. In the picture, and in the following discussion, $\gamma_1^{(1)}$ is denoted γ_1'. The four corners of the shaded quadrilateral are the canonical generators $\widehat{\Theta}_{\gamma_1,\delta_1}$, $\widehat{\Theta}_{\delta_1,\beta_1}$, $\widehat{\Theta}_{\beta_1,\gamma_1'}$, and $\widehat{\Theta}_{\gamma_1',\gamma_1}$ (read in clockwise order). Indeed, it is straightforward to see (by passing to the universal cover), that the shaded quadrilateral represents the only homology class φ_1 of Whitney quadrilaterals connecting these four points with $n_z(\varphi_1) = 0$ and all of whose local multiplicities are non-negative. By the Riemann mapping theorem,

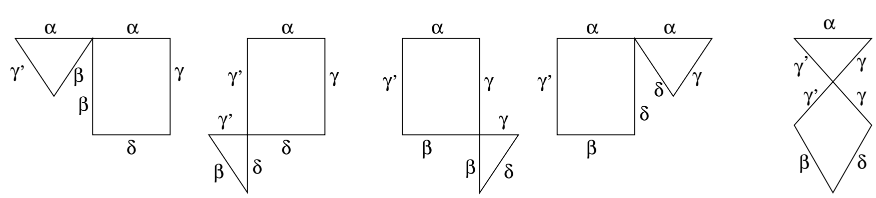

Figure 7. Degenerate rectangles. We have illustrated here a schematic diagram for the degenerations of pseudo-holomorphic pentagons. We have dropped the five additional degenerations, where a Whitney disk bubbles off the vertex of any pentagon.

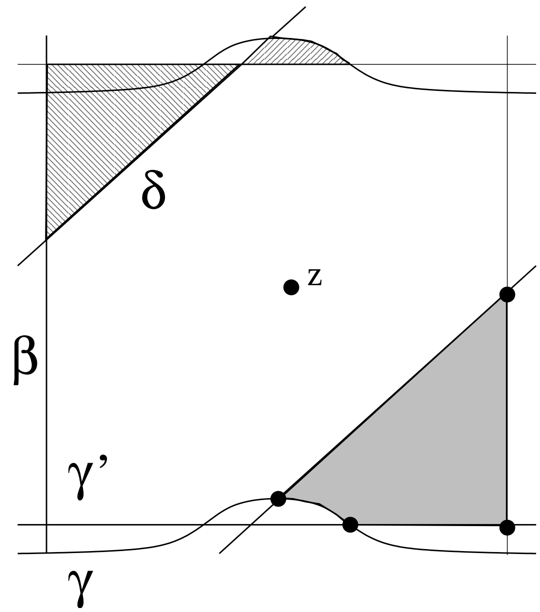

FIGURE 8. A holomorphic quadrilateral. The shaded quadrilateral has a unique holomorphic representative (by the Riemann mapping theorem), while the one indicated with the hatching does not, as it has both positive and negative local multiplicities, as indicated by the two directions in the hatching.

now, this homology class φ_1 has a unique holomorphic representative u_1. (By contrast, we have also pictured here another Whitney quadrilateral with hatchings, whose local multiplicities are all 0, +1, and −1; +1 at the region where the hatchings go in one direction and −1 where they go in the other.)

For the general case ($g > 1$), we take the connected sum of the case illustrated in Figure 8 with $g-1$ copies of the torus illustrated in Figure 9. In this picture, we have illustrated the four curves γ_i, δ_i, β_i, γ_i' for $i > 1$, which are Hamiltonian translates of one another. Now, there is a homology class of quadrilateral $\varphi_i \in \pi_2(\widehat{\Theta}_{\gamma_i\delta_i}, \widehat{\Theta}_{\delta_i\beta_i}, \widehat{\Theta}_{\beta_i\gamma_i'}, \widehat{\Theta}_{\gamma_i'\gamma_i})$, and a forgetful map $\mathcal{M}(\varphi) \longrightarrow \mathcal{M}(\Box)$ which remembers only the conformal class of the domain (where here $\mathcal{M}(\Box)$ denotes the moduli space of rectangles, cf. Exercise 2.6). Both moduli spaces are one-dimensional (the first moduli space is parameterized by the length of the cut into the region, while the second is parameterized by the ratio of the length to the width, after the quadrilateral is uniformized to a rectangle, as in Exercise 2.6). By Gromov's compactness theorem, the forgetful map is proper; and it is easy to see that it has degree one, and hence for some generic conformal class of quadrilateral, there is an odd number of pseudo-holomorphic quadrilaterals appearing in this family whose domain has the specified conformal class. Then the holomorphic quadrilaterals in $\pi_2(\widehat{\Theta}_{\gamma\delta}, \widehat{\Theta}_{\delta\beta}, \widehat{\Theta}_{\beta\gamma'}, \widehat{\Theta}_{\gamma'\gamma})$ are easily seen to be those quadrilaterals of the form $u_1 \times ... \times u_g \in \varphi_1 \times ... \times \varphi_g$ where u_1 is the pseudo-holomorphic representative of the homology class φ_1 described in the previous paragraph, and u_i for $i > 1$ are pseudo-holomorphic representatives for φ_i whose domain supports

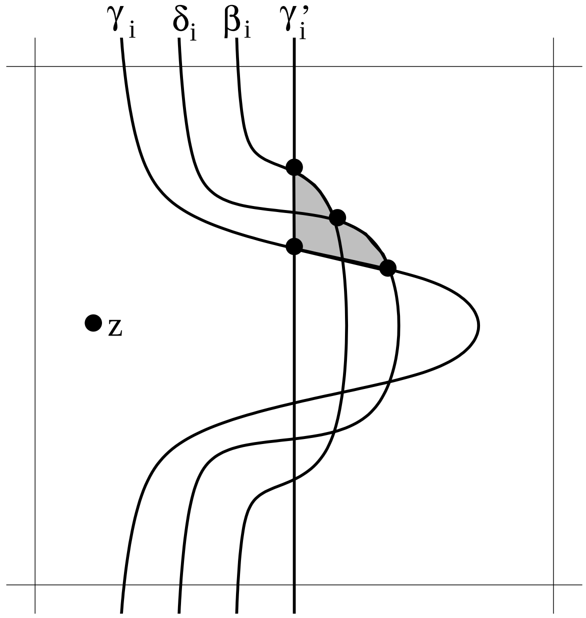

FIGURE 9. Other factors of the holomorphic quadrilateral. We have illustrated here a Heegaard quadruple (in a genus one surface) whose four boundary components are $S^2 \times S^1$. In the homology class indicated by the shaded quadrilateral $\varphi_i \in \pi_2(\widehat{\Theta}_{\gamma\delta}, \widehat{\Theta}_{\delta\beta}, \widehat{\Theta}_{\beta\gamma'}, \widehat{\Theta}_{\gamma'\gamma})$, there is a moduli space of pseudo-holomorphic quadrilaterals which is clearly one-dimensional, parameterized by a cut at the vertex where γ_i and δ_i meet. We take the connected sum of $g-1$ copies of this picture (at the reference point z) with the picture illustrated in Figure 8 to obtain the general case of the quadrilateral considered in the proof of Theorem 1.12.

the same conformal class. This proves Equation (15) which, in turn, yields Hypothesis (2) of Lemma 2.13. The surgery exact triangle for $\widehat{HF}$ (calculated with coefficients in $\mathbb{Z}/2\mathbb{Z}$) stated in Theorem 1.12 now follows directly from Lemma 2.13.

2.5. The case of HF^+. We outline here the modification necessary to adapt the above discussion to the case of HF^+ rather than $\widehat{HF}$.

First we define a map

$$f^+_{\alpha\beta\gamma}\colon CF^+(Y_{\alpha\beta}) \otimes_{\mathbb{F}[U]} CF^-(Y_{\beta\gamma}) \longrightarrow CF^+(Y_{\alpha\gamma})$$

by extending the following map to be U-equivariant:

$$f^+_{\alpha\beta\gamma}(U^{-i}\mathbf{x} \otimes \mathbf{y}) = \sum_{\mathbf{w}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\{\psi\in\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})\,\big|\,0=\mu(\psi)\}} \#(\mathcal{M}(\psi))U^{n_z(\psi)-i}\cdot\mathbf{w}. \tag{16}$$

The fact that f^+ determines a chain map follows from a suitable adaptation of the proof of Proposition 2.5, together with the additivity of n_z under juxtapositions.

Finiteness of the sum is a consequence of the admissibility condition: given any integer i, there are only finitely many $\psi \in \pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$ with $\mathcal{D}(\psi) \geq 0$ and $n_z(\psi) \leq i$.

To define the maps appearing in the exact sequence, we use the fact that $HF^-(\#^{g-1}(S^2 \times S^1)) \cong \Lambda^* H^1(\#^{g-1}(S^2 \times S^1)) \otimes \mathbb{F}[U]$. Again, we take top-dimensional generators $\Theta_{\beta\gamma}$, $\Theta_{\gamma\delta}$ and $\Theta_{\delta\beta}$ for these groups. Now, define

$$f^+(\xi) = f^+_{\alpha\beta\gamma}(\xi \otimes \Theta_{\beta\gamma}), \qquad f^+_0(\xi) = f^+_{\alpha\beta\gamma}(\xi \otimes \Theta_{\gamma\delta}), \qquad f^+_1(\xi) = f^+_{\alpha\beta\gamma}(\xi \otimes \Theta_{\delta\beta}).$$

It is not difficult to see that these maps are consistent with the earlier maps defined on $\widehat{CF}$, in the sense that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{CF}(Y) \arrow[r] \arrow[d, "\widehat{f}"'] & CF^+(Y) \arrow[r, "U"] \arrow[d, "f^+"'] & CF^+(Y) \arrow[r] \arrow[d, "f^+"] & 0 \\
0 \arrow[r] & \widehat{CF}(Y_0) \arrow[r] & CF^+(Y_0) \arrow[r, "U"] & CF^+(Y_0) \arrow[r] & 0
\end{tikzcd}$$

As before, we have an associativity law, according to which

$$f^+_{\alpha\gamma\delta}(f^+_{\alpha\beta\gamma}(\xi \otimes \Theta_{\beta\gamma})) \simeq f^+_{\alpha\beta\delta}(\xi \otimes f^-_{\beta\gamma\delta}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta})), \tag{17}$$

where here $f^-_{\beta\gamma\delta}$ is also obtained by counting holomorphic triangles; e.g.

$$f^-_{\alpha\beta\gamma}(\mathbf{x} \otimes \mathbf{y}) = \sum_{\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\psi \in \pi_2(\mathbf{x},\mathbf{y},\mathbf{w}) \big| \mu(\psi)=0\}} \#(\mathcal{M}(\psi)) \cdot U^{n_z(\psi)} \cdot \mathbf{w}. \tag{18}$$

However, unlike the case of f^+, there is no longer an *a priori* finiteness statement for the number of terms on the right-hand-side (even in the presence of weak admissibility). One way of coping with this issue is to consider yet another variant of Heegaard Floer homology $CF^{--}(Y_{\beta\delta})$, where we take our coefficient ring to be formal power series in U, $\mathbb{F}[[U]]$. The map f^+ defined in Equation (16) readily extends to a map

$$CF^+(Y_{\alpha\beta}) \otimes CF^{--}(Y_{\beta\gamma}) \longrightarrow CF^+(Y_{\alpha\gamma}),$$

and now the map $f^-_{\alpha\beta\gamma}$ as defined in Equation (18) gives a well-defined map

$$CF^{--}(Y_{\alpha\beta}) \otimes CF^{--}(Y_{\beta\gamma}) \longrightarrow CF^{--}(Y_{\alpha\gamma})$$

(since the sum is no longer required to be finite). In this setting the desired homotopy associativity stated in Equation (10) holds.

In view of the above remarks, in order to verify that $F^+_{W_0} \circ F^+_W = 0$, we must prove that

$$F^-_{\beta\gamma\delta}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta}) = 0,$$

which in turn hinges on a generalization of Proposition 2.10. In turn, this generalization relies on a "stretching the neck" argument familiar from gauge theory and symplectic geometry. In the context of symplectic geometry, this means that in order to analyze holomorphic curves in a symplectic manifold, it is sometimes useful to degenerate the almost-complex structure, so that the space becomes singular, and the holomorphic curves localize into strata which are easier to understand. Such an argument has already appeared in the proof of stabilization invariance of HF^+ (cf. [**44**]). We cannot treat this discussion in any detail here, but rather refer the interested reader to Section 10 of [**41**], cf. also Section 6 of [**40**]. For other arguments of this type in symplectic geometry, see [**25**], [**30**]:

PROPOSITION 2.18. *The homology classes* $\psi \in \pi_2(\widehat{\Theta}_{\beta\gamma}, \widehat{\Theta}_{\gamma\delta}, \widehat{\Theta}_{\delta\beta})$ *with* $\#(\mathcal{M}(\psi)) \neq 0$ *and* $\mu(\psi) = 0$. *are of the form* $\{\Psi^{\pm}_k\}_{k=1}^{\infty}$, *where* $n_z(\Psi^{\pm}_k) = \frac{k(k-1)}{2}$.

Sketch of Proof. Of course, if we were dealing with the genus one case, then this is a consequence of Exercise 2.9. For the general case, however, we need to stretch the neck. Specifically, suppose that ψ is a homology class with the property that $\#\mathcal{M}(\psi) = 1$. This in particular means that for any choice of conformal structure on Σ, there is at least one representative for ψ. Take conformal structures on Σ which converge to the nodal Riemann surface consisting of a torus E_1 (which contains β_1, γ_1, and δ_1) meeting a Riemann surface Σ_0 (which contains the remaining curves) at a point p. In this sequence of conformal structures, one can think of $\Sigma = E_1 \# \Sigma_0$ as developing an ever-longer connected sum neck. The sequence of holomorphic representatives for ψ converges to a union of a holomorphic triangle in $E_1 \times \mathrm{Sym}^{g-1}(\Sigma)$ with spheres in $\mathrm{Sym}^g(\Sigma_0)$. According to Exercise 2.9, the projection of the holomorphic triangle into E_1 must be one of the $\{\psi^{\pm}_k\}_{k=1}^{\infty}$. Moreover, the projection onto the other factor is constrained by dimension considerations to be a product of triangles as pictured in Figure 5. These requirements uniquely determine the homology class of ψ: indeed, the possible homology classes are in one-to-one correspondence with the homology classes of $\{\psi^{\pm}_k\}_{k=1}^{\infty}$ appearing in the genus one surface E_1, and $n_z(\psi)$ coincides with n_z for the corresponding triangle in the genus one surface. Conversely, a gluing argument shows that for each homology class arising in this way, the number of holomorphic representatives (counted with sign) agrees with the number of holomorphic representatives for the corresponding $\psi^{\pm}_k$. □

To complete the argument, we must prove that

$$f^{+}_{\alpha\beta\beta'}(\cdot \otimes h^{--}_{\beta\gamma\delta\beta'}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta} \otimes \Theta_{\delta\beta'}))\colon CF^+(Y_{\alpha\beta}) \longrightarrow CF^+(Y_{\alpha\beta'})$$

induces an isomorphism in homology. To this end, it suffices to observe that the restriction of the above map to $\widehat{CF}(Y_{\alpha\beta})$ coincides with the map

$$\widehat{f}_{\alpha\beta\beta'}(\cdot \otimes \widehat{h}_{\beta\gamma\delta\beta'}(\widehat{\Theta}_{\beta\gamma} \otimes \widehat{\Theta}_{\gamma\delta} \otimes \widehat{\Theta}_{\delta\beta'})),$$

which we have already proved induces an isomorphism from $\widehat{HF}(Y_{\alpha\beta})$ to $\widehat{HF}(Y_{\alpha\beta'})$ (cf. Exercise 1.4 part (3)).

EXERCISE 2.19. Suppose that Σ has genus one, and consider the curves β, γ, δ, β'. Show that in this case

$$H^{--}_{\beta\gamma\delta\beta'}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta} \otimes \Theta_{\delta\beta'}) \equiv \left(\sum_{k=0}^{\infty} U^{\frac{k(k+1)}{2}}\right) \Theta_{\beta\beta'} \pmod 2 \tag{19}$$

Hint: Generalizing the picture from Figure 8, show that for each k there are $2k+1$ homology classes of rectangles φ with $\mathcal{D}(\varphi) \geq 0$ and $n_z(\varphi) = \frac{k(k+1)}{2}$.

Note that this Equation (19) also holds in the case where $g(\Sigma) > 1$, by another neck-stretching argument.

2.6. Other variations. There are several other variants of the long exact sequence for surgeries. One which requires the minimum additional machinery to state is the following integer surgeries exact sequence, which we will use later.

Theorem 2.20. *(Theorem 9.19 of* [**40**]*) Consider a knot $K \subset Y$ where Y is a three-manifold with $H_1(Y;\mathbb{Z}) = 0$, and give K its canonical Seifert framing λ. Let Y_n denote the three-manifold obtained by surgery along $K \subset Y$ with framing $n \cdot \mu + \lambda$. There are affine isomorphisms $\mathbb{Z} \cong \mathrm{Spin}^c(Y_0)$ and $\mathbb{Z}/p\mathbb{Z} \cong \mathrm{Spin}^c(Y_p)$ (cf. Exercise 1.8) so that for each $i \in \mathbb{Z}/p\mathbb{Z}$, there are exact sequences*

(20)
$$... \longrightarrow \widehat{HF}(Y) \xrightarrow{\widehat{F}_{;i}} \bigoplus_{j\equiv i \pmod p} \widehat{HF}(Y_0, j) \xrightarrow{\widehat{F}_{0;i}} \widehat{HF}(Y_p, i) \xrightarrow{\widehat{F}_{p;i}} ...$$

and

(21)
$$... \to HF^+(Y) \xrightarrow{F^+_{;i}} \bigoplus_{j\equiv i \pmod p} HF^+(Y_0, j) \xrightarrow{F^+_{0;i}} HF^+(Y_p, i) \xrightarrow{F^+_{p;i}} ...$$

The proof is a slight variation of the proof of Theorem 1.12.

2.7. References and remarks. Proposition 2.5, Theorem 2.7, the null-homotopy of Equation (14), and indeed the fact that $\partial^2 = 0$ are all special cases of a generalized associativity law satisfied by counting pseudo-holomorphic m-gons, compare [**32**], [**5**], [**19**].

The above proof of Theorem 1.12 can be found in [**38**] (for the case of $\widehat{HF}$); a different proof is given in [**40**]. In fact, in [**38**], the exact triangle from Theorem 1.12 is generalized to address the following question: suppose we have a framed link L in Y with n components, and we know the Floer homology groups of the 2^n three-manifolds which are obtained by performing 0 or 1 surgery on each of the components on the link; then what can be said about the Floer homology of Y? Of course, when $n = 1$, we have a long exact sequence relating these three groups. In the general case, there is a spectral sequence whose E_2 term consists of the direct sum $\widehat{HF}$ of all of these 2^n different three-manifolds, and whose E^∞ term calculates $\widehat{HF}(Y)$. The proof involves a generalized associativity law which is gotten by counting pseudo-holomorphic m-gons.

3. Maps from cobordisms

Recall that the maps appearing in the exact triangle are defined by counting pseudo-holomorphic triangles. These maps respect the $\mathbb{Z}/2\mathbb{Z}$ grading of Floer homology, but in general they do not respect the splitting of the groups according to Spin^c structures, or the relative $\mathbb{Z}$ gradings (in the case where the three-manifolds are rational homology spheres). However, by decomposing the maps according to (suitable equivalence classes of) homology classes of triangles, we obtain a decomposition of the maps as a sum of components which preserve these extra structures. To explain this properly, it is useful to digress to the four-dimensional interpretation of these maps.

Let W be a compact, connected, smooth, four-manifold with two boundary components, which we write as $\partial W = -Y_1 \cup Y_2$ (where here Y_1 and Y_2 are a pair of closed, oriented three-manifolds). Such a four-manifold is called a *cobordism* from Y_1 to Y_2, and we write it sometimes as $W\colon Y_1 \longrightarrow Y_2$.

Let $W\colon Y_1 \longrightarrow Y_2$ be a cobordism equipped with a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(W)$; then there are induced maps on Heegaard Floer homology

$$F^\circ_{W,\mathfrak{s}}\colon HF^\circ(Y_1, \mathfrak{s}|_{Y_1}) \longrightarrow HF^\circ(Y_2, \mathfrak{s}|_{Y_2}),$$

where here HF° denotes any of the variants of Heegaard Floer homology $\widehat{HF}$, HF^-, HF^∞, or HF^+, which we take throughout to be calculated with coefficients in the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. The maps $F^\circ_{W,\mathfrak{s}}$ depend only on W (as a smooth four-manifold) and the Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(W)$.

For $\widehat{HF}$ this map is non-trivial for only finitely many $\mathfrak{s} \in \mathrm{Spin}^c(W)$, and hence we can form a map

$$\widehat{F}_W \colon \widehat{HF}(Y_1) \longrightarrow \widehat{HF}(Y_2)$$

defined by

$$\widehat{F}_W = \sum_{\mathfrak{s} \in \mathrm{Spin}^c(W)} \widehat{F}_{W,\mathfrak{s}}.$$

The same construction can be made using HF^+; in this case, although there might be infinitely many $\mathfrak{s} \in \mathrm{Spin}^c(W)$ for which $F^+_{W,\mathfrak{s}}$ is non-trivial, it is still the case that for a fixed $\xi \in HF^+(Y_1)$, there are only finitely many $\mathfrak{s}$ with the property that $F^+_{W,\mathfrak{s}}(\xi)$ is non-zero. Thus, we can define

$$F^+_W \colon HF^+(Y_1) \longrightarrow HF^+(Y_2)$$

by the possibly infinite sum

$$F^+_W = \sum_{\mathfrak{s} \in \mathrm{Spin}^c(W)} F^+_{W,\mathfrak{s}}.$$

These maps are functorial under composition of cobordisms. Specifically, if $W_1 \colon Y_1 \longrightarrow Y_2$ and $W_2 \colon Y_2 \longrightarrow Y_3$ are two cobordisms, we can form their composition $W_1 \#_{Y_2} W_2 \colon Y_1 \longrightarrow Y_3$. Functoriality states that

$$\widehat{F}_{W_1 \#_{Y_2} W_2} = \widehat{F}_{W_2} \circ \widehat{F}_{W_1} \quad \text{and} \quad F^+_{W_1 \#_{Y_2} W_2} = F^+_{W_2} \circ F^+_{W_1}$$

These formulas can be decomposed according to Spin^c structures: assume that $b_1(Y_2) = 0$; then for Spin^c structures $\mathfrak{s}_1 \in \mathrm{Spin}^c(W_1)$, $\mathfrak{s}_2 \in \mathrm{Spin}^c(W_2)$ which agree over Y_2, we have that

$$\widehat{F}_{W_1 \#_{Y_2} W_2, \mathfrak{s}_1 \# \mathfrak{s}_2} = \widehat{F}_{W_2,\mathfrak{s}_2} \circ \widehat{F}_{W_1,\mathfrak{s}_1} \quad \text{and} \quad F^+_{W_1 \#_{Y_2} W_2, \mathfrak{s}_1 \# \mathfrak{s}_2} = F^+_{W_2,\mathfrak{s}_2} \circ F^+_{W_1,\mathfrak{s}_1},$$

where here $\mathfrak{s}_1 \# \mathfrak{s}_2$ denotes the unique Spin^c structure over $W_1 \#_{Y_2} W_2$ whose restriction to W_i is $\mathfrak{s}_i$ (for $i = 1, 2$).

In the case where $b_1(Y_2) > 0$, a Spin^c structure over $W_1 \#_{Y_2} W_2$ is no longer necessarily determined by its restrictions to the W_i. Rather, if we consider the Poincaré dual M of the image of the map induced by inclusion $H_2(Y_2) \longrightarrow H_2(W)$, this requirement chooses an M-orbit in $\mathrm{Spin}^c(W_1 \#_{Y_2} W_2)$. Now, the left-hand-sides of the equations are replaced by the sum of maps on $W_1 \#_{Y_2} W_2$ induced by all the Spin^c structures in this M-orbit. For example, in the case of $\widehat{HF}$, we have

$$\sum_{\left\{\mathfrak{s} \in \mathrm{Spin}^c(W_1 \#_{Y_2} W_2) \middle| \mathfrak{s}|_{W_i} = \mathfrak{s}_i\right\}} \widehat{F}_{W,\mathfrak{s}} = \widehat{F}_{W_2,\mathfrak{s}_2} \circ \widehat{F}_{W_1,\mathfrak{s}_1}.$$

We will sketch the construction of $F^\circ_{W,\mathfrak{s}}$ in Subsection 3.2, but there are cases of this construction which we have seen already. Suppose that $K \subset Y$ is a knot with framing λ. Then, if $W \colon Y \longrightarrow Y_\lambda(K)$ is the cobordism obtained by attaching a two-handle with framing λ to $[0,1] \times Y$, then the induced maps $\widehat{F}_W$ and F^+_W are the maps constructed in Section 2 which appear in the exact sequence for Theorem 1.12.

Suppose that Y_1 and Y_2 are rational homology three-spheres. Then the Heegaard Floer homology groups of Y_1 and Y_2 can be given a relative $\mathbb{Z}$-grading, cf.

Equation (5). In general, the map $\widehat{F}_W$ need not be homogeneous with respect to this relative grading. However, the terms $\widehat{F}_{W,\mathfrak{s}}$ are homogeneous. We can give a much stronger version of this result, after introducing some notions.

Suppose that M is a compact, oriented four-manifold with the property that $H^2(\partial M;\mathbb{Q})=0$. Then, there is an intersection form

$$Q_M\colon H^2(M;\mathbb{Q})\otimes H^2(M;\mathbb{Q})\longrightarrow\mathbb{Q}$$

defined by

$$Q(\xi\otimes\eta)=\langle\xi\cup\eta,[M]\rangle,$$

where $[M]$ is the fundamental cycle of M. To make sense of the evaluation, implicitly use an identification $H^2(M,\partial M;\mathbb{Q})\cong H^2(M;\mathbb{Q})$ which exists thanks to the hypothesis that $H^2(\partial M;\mathbb{Q})=0$. Let $\sigma(M)$ denote the signature of this intersection form. Sometimes, we write ξ^2 for $Q(\xi,\xi)$. Observe that ξ^2 need not be integral, even if $\xi\in H^2(M;\mathbb{Z})$; however if $\xi\in H^2(M;\mathbb{Z})$ satisfies $n\xi|_{\partial M}=0$, then $n\cdot\xi^2\in\mathbb{Z}$.

Exercise 3.1. Let W be the four-manifold which is the unit disk bundle over a two-sphere with Euler number n. There is an isomorphism

$$\phi\colon\mathbb{Z}\longrightarrow H^2(W;\mathbb{Z}).$$

Find $\phi(i)^2$ for $i\in\mathbb{Z}$.

Exercise 3.2. If Y is a rational homology three-sphere, then there is a $\mathbb{Q}/\mathbb{Z}$-valued linking

$$q\colon H_1(Y;\mathbb{Z})\otimes H_1(Y;\mathbb{Z})\longrightarrow\mathbb{Q}/\mathbb{Z}$$

on $H_1(Y;\mathbb{Z})$ defined as follows. Given $\alpha,\beta\in H_1(Y;\mathbb{Z})$, there is some n with the property that $n\beta=0$ in $H_1(Y;\mathbb{Z})$, and hence $n\beta=\partial F$ for some oriented two-manifold $F\subset Y$. Let $q(\alpha,\beta)=\#(\alpha\cap F)/n$. Show that this is a symmetric bilinear form, which is independent of the choice of F. If $Y=\partial W$, and $\alpha,\beta\in H_2(W,Y;\mathbb{Z})$ then show that

$$Q(\mathrm{PD}[\alpha]\otimes\mathrm{PD}[\beta])\equiv q(\partial\alpha\otimes\partial\beta)\pmod{\mathbb{Z}}.$$

Theorem 3.3. *(Theorem 7.1 of [**34**]) If Y is a rational homology three-sphere, then there is a unique $\mathbb{Q}$-lift of the relative $\mathbb{Z}$ grading on $HF^+(Y,\mathfrak{t})$, which satisfies the following properties:*

- *$\widehat{HF}(S^3)\cong\mathbb{F}$ is supported in degree zero*
- *the inclusion map $\widehat{CF}(Y,\mathfrak{t})\longrightarrow CF^+(Y,\mathfrak{t})$ is degree-preserving*
- *if ξ is a homogeneous element in $CF^+(Y,\mathfrak{t})$, then*

$$\mathrm{gr} f^+_{W,\mathfrak{s}}(\xi)-\mathrm{gr}(\xi)=\frac{c_1(\mathfrak{s})^2-2\chi(W)-3\sigma(W)}{4},\tag{22}$$

where here $f^+_{W,\mathfrak{s}}$ is a chain map inducing $F^+_{W,\mathfrak{s}}$ on homology.

Actually, verifying the existence of this $\mathbb{Q}$-lift is rather more elementary than proving that $F^+_{W,\mathfrak{s}}$ is a topological invariant of the cobordism: Equation (22) uses only the grading of $f^+_{W,\mathfrak{s}}$, not the count of holomorphic disks.

The $\mathbb{Q}$-grading from Theorem 3.3 allows us to define a numerical invariant for rational homology three-spheres from its Heegaard Floer homology.

Definition 3.4. Let Y be a rational homology three-sphere equipped with a Spin^c structure $\mathfrak{t}$. Then its *correction term* $d(Y,\mathfrak{t})$ is the minimal $\mathbb{Q}$-degree of any homogeneous element in $HF^+(Y,\mathfrak{t})$ coming from $HF^\infty(Y,\mathfrak{t})$.

The above correction term is analogous to the invariant defined in gauge theory by Kim Frøyshov, cf. [**17**].

3.1. Whitney triangles and four-manifolds. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ be a pointed Heegaard triple. We can form the identification space

$$X_{\alpha,\beta,\gamma} = \frac{(\Delta \times \Sigma) \coprod (e_\alpha \times U_\alpha) \coprod (e_\beta \times U_\beta) \coprod (e_\gamma \times U_\gamma)}{(e_\alpha \times \Sigma) \sim (e_\alpha \times \partial U_\alpha), (e_\beta \times \Sigma) \sim (e_\beta \times \partial U_\beta), (e_\gamma \times \Sigma) \sim (e_\gamma \times \partial U_\gamma)}.$$

Over the vertices of Δ this space has corners, which can be naturally smoothed out to obtain a smooth, oriented, four-dimensional cobordism between the three-manifolds $Y_{\alpha\beta}$, $Y_{\beta\gamma}$, and $Y_{\alpha\gamma}$ as claimed. More precisely,

$$\partial X_{\alpha,\beta,\gamma} = -Y_{\alpha\beta} - Y_{\beta\gamma} + Y_{\alpha\gamma},$$

with the obvious orientation.

The group of periodic domains for $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ (cf. Definition 2.2) has a natural interpretation in terms of the homology of $X_{\alpha,\beta,\gamma}$.

Exercise 3.5. Show that $\mathcal{P} \cong H_2(X_{\alpha\beta\gamma}; \mathbb{Z})$. Consider the quotient group $\mathcal{Q}$ of $\mathcal{P}$ by the subgroup of elements which can be written as sums of doubly-periodic domains for $Y_{\alpha\beta}$, $Y_{\alpha\gamma}$, and $Y_{\beta\gamma}$. Show that this quotient group is isomorphic to $H^2(X_{\alpha\beta\gamma}; \mathbb{Z})$.

Let $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ and $\psi' \in \pi_2(\mathbf{x}', \mathbf{y}', \mathbf{w}')$ where $\mathbf{x}, \mathbf{x}'$, $\mathbf{y}, \mathbf{y}'$ and $\mathbf{w}, \mathbf{w}'$ are equivalent. We can define a difference $\delta(\psi, \psi') \in H^2(X_{\alpha\beta\gamma})$ which corresponds to $\mathcal{D}(\psi) + \mathcal{D}(\phi_1) + \mathcal{D}(\phi_2) + \mathcal{D}(\phi_3) - \mathcal{D}(\psi')$ in $\mathcal{Q}$.

We say that two homology classes ψ, ψ' are Spin^c-*equivalent* if this difference $\delta(\psi, \psi')$ vanishes. The maps corresponding to counting holomorphic triangles, cf. Equation (9) clearly split into sums of maps which are indexed by Spin^c equivalence classes of triangles.

Example 3.6. Consider the Heegaard triple in the torus obtained by three straight curves β, γ, δ as in Exercise 2.9. Observe that the triangles $\{\psi_k^\pm\}_{k=1}^\infty$ represent distinct Spin^c-equivalence classes. Moreover, the results of that exercise can be interpreted as saying that the map

$$F^{--}_{\beta\gamma\delta,[\psi_k^\pm]} \colon HF^{--}(S^3) \otimes_{\mathbb{F}[U]} HF^{--}(S^3) \longrightarrow HF^{--}(S^3)$$

represents the map $\mathbb{F}[[U]] \longrightarrow \mathbb{F}[[U]]$ given by multiplication by $U^{\frac{k(k-1)}{2}}$.

Recall that a pointed Heegaard diagram for Y, $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ gives rise to a map from equivalence classes of intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to Spin^c structures over Y. In a similar, but somewhat more involved manner, there is a map from Spin^c-equivalence classes of Whitney triangles to Spin^c structures over $X_{\alpha\beta\gamma}$. Moreover, there is a $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ for $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, $\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ if and only if there is $\mathfrak{s} \in \mathrm{Spin}^c(X_{\alpha\beta\gamma})$ such that $\mathfrak{s}|_{Y_{\alpha\beta}} = \mathfrak{s}_z(\mathbf{x})$, $\mathfrak{s}|_{Y_{\beta\gamma}} = \mathfrak{s}_z(\mathbf{y})$, and $\mathfrak{s}|_{Y_{\alpha\gamma}} = \mathfrak{s}_z(\mathbf{w})$. We leave the reader to consult Section 8 of [**41**] for details.

In Example 3.6 above, the four-manifold $X_{\beta\gamma\delta}$ is diffeomorphic to $\overline{\mathbb{CP}}^2$ (i.e. $\mathbb{CP}^2$ given the orientation for which its intersection form is negative definite) with three four-balls removed. The triangle $\psi_k^\pm$ represents the Spin^c structure over $\overline{\mathbb{CP}}^2$ whose first Chern class evaluates as $\pm(2k-1)$ on a fixed generator for $H_2(\overline{\mathbb{CP}}^2; \mathbb{Z})$.

3.2. Construction of the cobordism invariant. Let $W\colon Y_1 \longrightarrow Y_2$ be a cobordism. The induced map

$$\widehat{F}_W\colon \widehat{HF}(Y_1) \longrightarrow \widehat{HF}(Y_2)$$

is defined using a decomposition of W into handles. Specifically, W can be expressed as a union of one-, two-, and three-handles.

Suppose that W consists entirely of one-handles. Then $Y_2 \cong Y_1\#^\ell(S^2\times S^1)$, and a Künneth principle for connected sums ensures that

$$\widehat{HF}(Y_2) \cong \widehat{HF}(Y_1) \otimes \Lambda^* H^1(\#^\ell(S^2\times S^1)).$$

Letting $\widehat{\Theta} \in \Lambda^*H^1(\#^\ell(S^2\times S^1))$ be the a generator of the top-dimensional element of the exterior algebra, the map $\widehat{F}_W$ is defined to be the map $\xi \mapsto \xi\otimes\widehat{\Theta}$ under the above identification.

Suppose that W consists entirely of three-handles. Then, $Y_1 \cong Y_2\times\#^\ell(S^2\times S^1)$. In this case, there is a corresponding map $\widehat{F}_W\colon \widehat{HF}(Y_1) \longrightarrow \widehat{HF}(Y_2)$ which is induced by projection onto the bottom-dimensional element of the exterior algebra $H^1(\#^\ell(S^2\times S^1))$ under the identification $\widehat{HF}(Y_1) \cong \widehat{HF}(Y_2)\otimes\Lambda^*H^1(\#^\ell(S^2\times S^1))$.

The more interesting case is when $W\colon Y_1 \longrightarrow Y_2$ consists of two-handles. In this case, W can be expressed as surgery on an ℓ-component link $L\subset Y_1$. In this case, $\widehat{F}_W$ can be obtained as follows.

Consider a Heegaard decomposition of $Y_1 = U_\alpha \cup_\Sigma U_\beta$ with the property that $L = L_{i=1}^\ell$ is supported entirely inside U_β in a special way: the L_i is dual to the i^{th} attaching disk for U_β (i.e. it is unknotted, disjoint from all but one attaching disk, which it meets transversally in a single intersection point). Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a corresponding Heegaard diagram for Y_1. The framings of the components of L_i provide an alternate set of attaching circles γ_i. For all $i>\ell$, we let γ_i be an isotopic copy of β_i. In this way, we obtain a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$, where $Y_{\alpha\beta}\cong Y_1$, $Y_{\beta\gamma}\cong \#^{g-\ell}(S^2\times S^1)$, and $Y_{\alpha\gamma}\cong Y_2$. The map

$$\widehat{F}_W\colon \widehat{HF}(Y_1) \longrightarrow \widehat{HF}(Y_2)$$

is defined now by

$$\widehat{F}_W(\xi) = \widehat{F}_{\alpha\beta\gamma}(\xi\otimes\widehat{\Theta}_{\beta\gamma}),$$

where as usual $\widehat{\Theta}_{\beta\gamma}$ represents a top-dimensional homology class for $\widehat{HF}(Y_{\beta\gamma})$.

Of course, when the number of link components $\ell = 1$, the map $\widehat{F}_W$ coincides with the construction of the map appearing in an exact sequence which contains $\widehat{HF}(Y_1)$ and $\widehat{HF}(Y_2)$.

In the general case where W has handles of all three types, we decompose $W = W_1\cup W_2\cup W_3$ where W_i consists of i-handles, and define $\widehat{F}_W$ to be the composite of $\widehat{F}_{W_1}$, $\widehat{F}_{W_2}$, $\widehat{F}_{W_3}$ defined as above.

The verification that the above procedure gives rise to a topological invariant of smooth four-manifolds is lengthy: one must show that it is independent of the decomposition of W into handles; and in the case where W consists of two-handles, that it is independent of the particular choice of Heegaard triple. In particular, one shows that the map (on homology) is invariant under handleslides between various handles and stabilizations. Typically, one interprets such a move as a move on the Heegaard diagram. The key technical point used frequently in these arguments is the associativity law, and some model calcuations. We refer the reader to [**34**] for details (see esp. Section 4 of [**34**]).

The decomposition of $\widehat{F}_W$ according to Spin^c structures proceeds as follows. If W consists entirely of one- or three-handles, then this decomposition is canonical: if $W\colon Y_1 \longrightarrow Y_2$ is a union of one- resp. three-handles then each Spin^c structure over Y_1 resp. Y_2 has a unique extension to a Spin^c structure over W. In the case where W consists of two-handles, the decomposition is represented by the decomposition of $\widehat{F}_{\alpha\beta\gamma}$ into the maps induced by the various Spin^c-equivalence classes of triangles over $X_{\alpha\beta\gamma}$. To identify these with Spin^c structures over W, observe that, after filling in the $Y_{\beta\gamma}$ boundary of $X_{\alpha\beta\gamma}$ by $\#^{g-\ell}(B^3 \times S^1)$, we obtain a four-manifold which is diffeomorphic to W, and hence $\mathrm{Spin}^c(X_{\alpha,\beta,\gamma}) \cong \mathrm{Spin}^c(W)$.

Maps $F^-_{W,\mathfrak{s}}$ $F^\infty_{W,\mathfrak{s}}$, and $F^+_{W,\mathfrak{s}}$ can be defined analogously. Indeed, these maps can all be thought of as induced from an $\mathbb{F}[U]$-equivariant chain map from $CF^-(Y_1, \mathfrak{s}|_{Y_1}) \longrightarrow CF^-(Y_2, \mathfrak{s}|_{Y_2})$, and as such, they respect the fundamental exact sequences relating $\widehat{HF}$, HF^-, HF^∞, and HF^+ (cf. Equation (2)).

Example 3.7. The results of Example 3.6 can be interpreted as follows: let W be the cobordism obtained by deleting two four-balls from $\overline{\mathbb{CP}}^2$ (equivalently, this is the cobordism obtained by attaching a two-handle to $S^3 \times [0,1]$ along the unknot with framing -1). Then, for the Spin^c structure $\mathfrak{s}$ whose first Chern class is $\pm(2k-1)$ times a generator of $H^2(W;\mathbb{Z})$, the induced map $F^-_{W,\mathfrak{s}}$ is multiplication by $U^{k(k-1)/2}$. Thus, if $c_1(\mathfrak{s})$ is a generator of $H^2(W;\mathbb{Z})$, then the map

$$\widehat{F}_{W,\mathfrak{s}}\colon \widehat{HF}(S^3) \longrightarrow \widehat{HF}(S^3)$$

is an isomorphism

3.3. Absolute gradings. Let Y be a rational homology three-sphere. The $\mathbb{Q}$-lift of the relative $\mathbb{Z}$ grading on $\widehat{HF}(Y)$ is defined as follows. For any three-manifold Y, there is a cobordism $W\colon S^3 \longrightarrow Y$ consisting entirely of two-handles. As indicated above, this gives a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ with $Y_{\alpha\beta} \cong S^3$, $Y_{\beta\gamma} \cong \#^m(S^2 \times S^1)$, and $Y_{\alpha\gamma} \cong Y$. Indeed, there exists a triangle $\psi \in \pi_2(\widehat{\Theta}_{\alpha\beta}, \widehat{\Theta}_{\beta\gamma}, \mathbf{x})$, where here $\widehat{\Theta}_{\alpha\beta}$, $\Theta_{\beta\gamma}$ are generators representing the canonical (top-dimensional) homology classes of S^3 and $\#^m(S^2 \times S^1)$. We then define

$$\mathrm{gr}(\mathbf{x}) = -\mu(\psi) + 2n_z(\psi) + \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}.$$

The verification that this is well-defined can be found in Theorem 7.1 of [**34**].

Exercise 3.8. Consider $X = \#^n\overline{\mathbb{CP}}^2$. Let $\mathfrak{s}$ be the Spin^c structure with

$$c_1(\mathfrak{s}) = E_1 + ... + E_n,$$

where $E_i \in H_2(\overline{\mathbb{CP}}^2;\mathbb{Z}) \cong \mathbb{Z}$ is a generator. Show that X can be decomposed along $L(n,1)$ as a union $X_1 \#_{L(n,1)} X_2$ in such a way that X_2 is composed of a single zero- and two-handle, and $c_1(\mathfrak{s})|_{X_1} = 0$. Deduce from the composition law that there is a Spin^c structure $\mathfrak{s}$ over $L(n,1)$ with $c_1(\mathfrak{s}) = 0$ and $d(L(n,1),\mathfrak{s}) = \frac{n-1}{4}$. *Hint*: Let $W_1\colon S^3 \longrightarrow L(n,1)$, $W_2\colon L(n,1) \longrightarrow S^3$ denote X_1 and X_2 with two four-balls removed, so that $W = W_1 \#_{L(n,1)} W_2$ is X with two four-balls removed. According to Example 3.7, $\widehat{F}_W\colon \widehat{HF}(S^3) \cong \mathbb{F} \longrightarrow \widehat{HF}(S^3) \cong \mathbb{F}$ is an isomorphism, and hence so is $\widehat{F}_{W_1}\colon \widehat{HF}(S^3) \longrightarrow \widehat{HF}(L(m,1), \mathfrak{s}|_{L(m,1)})$.

3.4. Construction of the closed four-manifold invariant. If X is a four-manifold, let $b_2^+(X)$ denote the maximal dimension of any subspace of $H^2(X;\mathbb{Z})$ on which the intersection form is positive-definite. Let X be a closed, smooth four-manifold with $b_2^+(X)>1$. Then, the maps associated to cobordisms can be used to construct a smooth invariant for X analogous to the Seiberg-Witten invariant for closed manifolds. Its construction uses the following basic fact about the map induced by cobordisms:

Proposition 3.9. *If $W\colon Y_1 \longrightarrow Y_2$ is a four-manifold with $b_2^+(X)>0$, then $F^\infty_{W,\mathfrak{s}}\equiv 0$.*

The proof can be found in Lemma 8.2 of [**34**]

Deleting two four-balls from X, we obtain a cobordism $W\colon S^3 \longrightarrow S^3$. When $b_2^+(X)>1$, we can always find a separating hypersurface $N\subset W$ which decomposes W as a union of two cobordisms $W=W_1\#_N W_2$ with $b_2^+(W_i)>0$ and the image of $H_2(N;\mathbb{Z})$ in $H_2(W;\mathbb{Z})$ is trivial (so that each Spin^c structure over X is uniquely determined by its restrictions to W_1 and W_2). Such a separating hypersurface N is called an *admissible cut*.

Fix $\mathfrak{s}\in\mathrm{Spin}^c(X)$, and let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ denote its restrictions to W_1 and W_2 respectively. In view of Proposition 3.9, the image of the map

$$F^-_{W_1,\mathfrak{s}_1}\colon HF^-(S^3)\longrightarrow HF^-(N,\mathfrak{s}|_N)$$

is contained in the kernel of the natural map $\ell_*\colon HF^-(N,\mathfrak{s}|_N)\longrightarrow HF^\infty(N,\mathfrak{s}|_N)$ (cf. Equation (1)). Another application of the same proposition shows that the map

$$F^+_{W_2,\mathfrak{s}_2}\colon HF^+(N,\mathfrak{s}|_N)\longrightarrow HF^+(S^3).$$

induces a well-defined map on the cokernel of $q_*\colon HF^\infty(N,\mathfrak{s}|_N)\longrightarrow HF^+(N,\mathfrak{s}|_N)$. Using the canonical identification between the kernel of ℓ_* and the cokernel of q_* (following from exactness in Equation (1)), we can compose the two maps to obtain a map

$$\Phi_{X,\mathfrak{s}}\colon \mathbb{F}[U]\cong HF^-(S^3)\longrightarrow \mathcal{T}^+\cong HF^+(S^3).$$

By U-invariance, we can view $\Phi_{X,\mathfrak{s}}$ as a function from $\mathbb{F}[U]$ to $\mathbb{F}$. In fact, by Equation (22), $\Phi_{X,\mathfrak{s}}$ is a homogeneous function from $\mathbb{F}[U]\longrightarrow\mathbb{F}$, with degree given by

$$\deg(X,\mathfrak{s})=\frac{c_1(\mathfrak{s})^2-2\chi(X)-3\sigma(W)}{4};$$

i.e. $\Phi_{X,\mathfrak{s}}(U^i)=0$ if $2i\neq\deg(X,\mathfrak{s})$. Thus, $\Phi_{X,\mathfrak{s}}$ is determined by the element $\Phi_{X,\mathfrak{s}}(U^{\deg(X,\mathfrak{s})/2})\in\mathbb{F}$ and, of course, the degree $\deg(X,\mathfrak{s})$. (Indeed, with more work, one can lift this to an integer, uniquely determined up to sign.) The following is proved in Section 8 of [**34**]:

Theorem 3.10. *Let X be a smooth four-manifold with $b_2^+(X)>1$. Then the function $\Phi_{X,\mathfrak{s}}$ depends on the diffeomorphism type of X and the choice of $\mathfrak{s}\in\mathrm{Spin}^c(X)$.*

In particular, $\Phi_{X,\mathfrak{s}}$ is independent of the choice of admissible cut used in its definition.

3.5. Properties of the closed four-manifold invariant. The following is a combination of the functoriality of W under cobordisms and the definition of $\Phi_{X,\mathfrak{s}}$:

Proposition 3.11. *Let X be a closed, smooth four-manifold which is separated (smoothly) by a three-manifold Y as $X = X_1 \#_Y X_2$ with $b_2^+(X_i) > 0$, and choose $\mathfrak{s}_i \in \mathrm{Spin}^c(X_i)$ whose restriction to Y is some fixed Spin^c structure $\mathfrak{t} \in \mathrm{Spin}^c(Y)$. Then, if*

$$\sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)\big|\mathfrak{s}|_{X_i}=\mathfrak{s}_i} \Phi_{X,\mathfrak{s}} \neq 0$$

then $HF^+(Y,\mathfrak{t}) \neq 0$. In fact, in this case the natural map $HF^\infty(Y,\mathfrak{t}) \longrightarrow HF^+(Y,\mathfrak{t})$ has non-trivial cokernel.

In particular, it follows that if X is the connected sum of two four-manifolds, each of which has $b_2^+ > 0$, then $\Phi_{X,\mathfrak{s}} \equiv 0$. This is interesting when combined with the following non-vanishing theorem:

Theorem 3.12. *(Theorem 1.1 of* [**42**]*; compare also Taubes* [**52**]*) Let (M,ω) be a symplectic four-manifold with $b_2^+(M) > 1$, and let k represent the canonical Spin^c structure; then $\Phi_{M,k} \neq 0$.*

The above theorem is proved by first constructing a Lefschetz pencil [**6**], and using a naturally induced handle decomposition on a suitable blow-up of M.

3.6. References and remarks. The cobordism invariant, absolute gradings on Heegaard Floer homology, and the closed four-manifold invariant are all defined in [**34**]. Further applications of the absolute grading, and also the correction term $d(Y,\mathfrak{t})$ are given in [**36**].

The closed four-manifold invariant $\Phi_{X,\mathfrak{s}}$ is analogous to the Seiberg-Witten invariant for $(X,\mathfrak{s})$, compare [**53**]. That invariant, too, vanishes for connected sums (of four-manifolds with $b_2^+ > 0$) [**53**], and is non-trivial for symplectic manifolds, according to a theorem of Taubes [**50**], [**51**]. In fact, it is conjectured that $\Phi_{X,\mathfrak{s}}$ agrees with the Seiberg-Witten invariant for four-manifolds. (Note also that Donaldson's theory behaves similarly: Donaldson's invariants for such connected sums vanish, and for Kähler surfaces, they are known not to vanish, cf. [**8**].) Example (3.7) corresponds to the "blow-up" formula for Seiberg-Witten invariants, cf. [**14**]. The correction term $d(Y,\mathfrak{s})$ is analogous to Frøyshov's invariants [**17**], cf. also [**18**] for the corresponding invariants using Donaldson's theory.

4. Dehn surgery characterization of the unknot

Suppose that $K \subset S^3$. For each rational number r, we can construct a new three-manifold $S^3_r(K)$ by Dehn filling. Not every three-manfiold can be obtained as Dehn surgery on a single knot, but for those which are, it is a natural question to ask how much of the Dehn surgery description the three-manifold remembers. There are also many examples of three-manifolds which are obtained as surgery descriptions in more than one way. For example, $+5$ surgery on the right-handed trefoil is the lens space obtained as -5 surgery on the unknot. Note that for this example, the surgery coefficients are opposite in sign.

EXERCISE 4.1. Consider the three-manifold Y obtained as surgery on the Borromean rings with surgery coefficients $+1$, $+1$, and -1. By blowing down the two circles with coefficient $+1$ we obtain a description of Y as -1 surgery on a knot K_1. By blowing down two circles with coefficients $+1$ and -1, we obtain a description of Y as $+1$ surgery on K_2. What are K_1 and K_2? What is Y?

More interesting examples were described by Lickorish [**31**], who gives two distinct knots K_1 and K_2 with the property that $S^3_{-1}(K_1) \cong S^3_{-1}(K_2)$. His examples are constructed from a two-component link $L_1 \cup L_2$, each of whose components is individually unknotted, and hence K_1 is the knot induced from L_1 in $S^3_{-1}(L_2) \cong S^3$, while K_2 is the knot induced from L_2 in $S^3_{-1}(L_1) \cong S^3$, cf. Figure 4.

For suitably simple three-manifolds, though, the phenomenon illustrated above does not occur. Specifically, our aim here is to sketch the proof of the following conjecture of Gordon [**23**], first proved by the authors in collaboration with Peter Kronheimer and Tomasz Mrowka [**29**], using Floer homology for Seiberg-Witten monopoles constructed by Kronheimer and Mrowka (cf. [**26**]).

THEOREM 4.2. *(Kronheimer-Mrowka-Ozsváth-Szabó* [**29**]*) Let $\mathcal{U}$ denote the unknot in S^3, and let K be any knot. If there is an orientation-preserving diffeomorphism $S^3_r(K) \cong S^3_r(\mathcal{U})$ for some rational number r, then $K = \mathcal{U}$.*

Of course, $S^3_{p/q}(\mathcal{U})$ is the lens space $L(p, q)$. (The reader should be warned that this orientation convention on the lens space is opposite to the one adopted by some other authors.) This result has the following immediate application, where one can discard orientations:

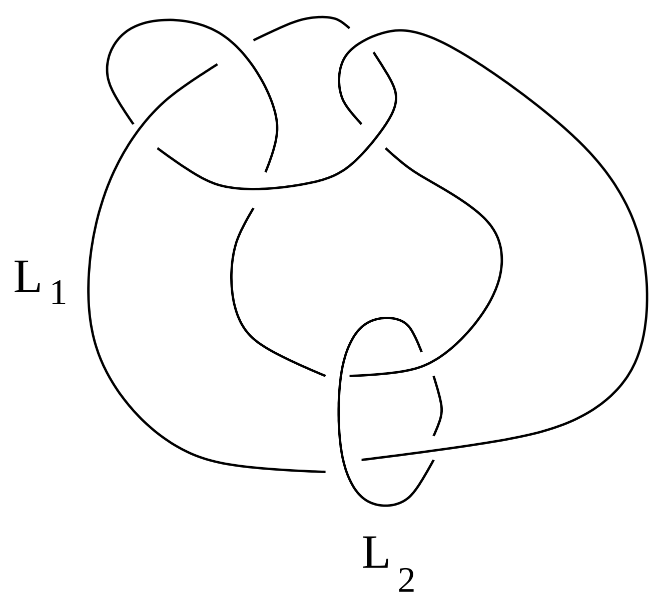

FIGURE 10. A two-component link. Each component is unknot -ted; blowing down either one or the other component gives a pair of distinct knots K_1 and K_2 in S^3 with $S^3_{-1}(K_1) \cong S^3_{-1}(K_2)$.

COROLLARY 4.3. *If K is a knot with the property that some surgery on K is the real projective three-space $\mathbb{RP}^3$, then K is the unknot.*

Many cases of Theorem 4.2 had been known previously. The case where $r=0$ was the "Property R" conjecture proved by Gabai [**21**]; the case where r is non-integral follows from the cyclic surgery theorem of Culler, Gordon, Luecke, and Shalen [**4**], the case where $r=\pm 1$ is a theorem of Gordon and Luecke [**24**].

We outline here the proof for integral $r\neq 0$, using Heegaard Floer homology, though a re-proof of the result for all rational r can be given by adapting the arguments from [**29**]. The Heegaard Floer homology proof is strictly logically independent of the proof using monopole Floer homology, though the two proofs are formally quite analogous. Moreover, to keep the discussion simple, we prove only that $g(K)\leq 1$. To exclude the possibility that $g(K)=1$, we require a little of the theory beyond what has been explained so far: either a discussion of "twisted coefficients" or an extra discussion of knot Floer homology (compare [**35**]). Or, alternatively, one could appeal to an earlier result of Goda and Teragaito [**22**].

The proof can be subdivided into two components: first, one proves that $HF^+(S^3_r(K))\cong HF^+(S^3_r(\mathcal{U}))$ implies a corresponding isomorphism $HF^+(S^3_0(K))\cong HF^+(S^3_0(\mathcal{U}))$. In the second component, one shows that the Heegaard Floer homology of $S^3_0(K)$ distinguishes any non-trivial knot from the unknot. The first component follows from a suitably enhanced application of the long exact sequence for surgeries. The second component rests on fundamental work by a large number of researchers, including the construction of taut foliations by Gabai [**20**], [**21**], Eliashberg and Thurston [**10**], Eliashberg [**9**] and Etnyre [**11**], and Donaldson [**6**].

We describe these two components in more detail in the following two subsections.

4.1. The first component: $HF^+(S^3_p(K))\cong HF^+(S^3_p(\mathcal{U}))\Rightarrow HF^+(S^3_0(K))\cong HF^+(S^3_0(\mathcal{U}))$

In view of our earlier remarks, it will suffice to prove that

$$HF^+(S^3_p(K))\cong HF^+(S^3_p(\mathcal{U}))\Rightarrow HF^+(S^3_0(K),i)=0$$

for all $i\neq 0$. In fact, for simplicity, we always work with Heegaard Floer homology with coefficients in some field $\mathbb{F}$ (which the reader can take to be $\mathbb{Z}/2\mathbb{Z}$), although since the field is generic, the results hold over $\mathbb{Z}$, as well.

The proof hinges on the following application of the exact triangle, combined with absolute gradings. In the following statement (cf. Equation (23)), we fix an identification $\mathbb{Z}/p\mathbb{Z}\cong \mathrm{Spin}^c(L(p,1))$, made explicit later.

THEOREM 4.4. *(Theorem 7.2 of [**36**]) Suppose that $K\subset S^3$ is a knot in S^3 with the property that some integral $p>0$ surgery on K gives the L-space Y; then there is a map $\sigma\colon \mathbb{Z}/p\mathbb{Z}\longrightarrow \mathrm{Spin}^c(Y)$ with the property that for each $i\neq 0$, with $|i|\leq p/2$,*

$$HF^+(S^3_0(K),i)\cong \mathbb{F}[U]/U^{\ell_i}$$

where

$$2\ell_i=-d(Y,\sigma(i))+d(L(p,1),i), \tag{23}$$

while $HF^+(S^3_0(K),i)=0$ for $|i|>p/2$. In particular, each $\ell_i\geq 0$.

The proof of this result uses the integer surgeries long exact sequence, Theorem 2.20, with the understanding that the map appearing there, $F^+_{p;i}\colon HF^+(Y_p,i)$

$\longrightarrow HF^+(Y)$, is the sum of maps induced by the two-handle cobordism $W_p(K)\colon Y_p(K) \longrightarrow Y$, where we sum over all Spin^c structures whose restriction to $Y_p(K)$ corresponds to $i \in \mathbb{Z}/p\mathbb{Z}$. In fact, given $i \in \mathbb{Z}/p\mathbb{Z}$, the set of Spin^c structures over $W_p(K)$ whose restriction to $Y_p(K)$ corresponds to i is the set of Spin^c structures $\mathfrak{s} \in \mathrm{Spin}^c(W_p(K))$, for which

$$c_1(\mathfrak{s}) \equiv 2i + p \pmod{2p}, \tag{24}$$

under an isomorphism $H^2(W_p(K);\mathbb{Z}) \cong \mathbb{Z}$. Indeed, Equation (24) can also be viewed as determining the map $\mathbb{Z}/p\mathbb{Z} \longrightarrow \mathrm{Spin}^c(S^3_p(K))$ (up to an irrelevant overall sign – irrelevant due to the conjugation symmetry of the groups in question) arising in Theorem 2.20: $\mathfrak{s} \in \mathrm{Spin}^c(W_p(K);\mathbb{Z})$ is uniquely determined by its first Chern class, and in turn its equivalence class modulo $2p$ uniquely determines its restriction to $S^3_p(K)$.

Exercise 4.5. Show that if $\mathfrak{t} \in \mathrm{Spin}^c(S^3_p(K))$ is a Spin^c structure which corresponds to $i = 0$ under Equation (24), then $c_1(\mathfrak{t}) = 0$. (Note also that when p is odd, there is only one Spin^c structure over $S^3_p(K)$ with trivial first Chern class; when p is even, there are two.)

It will also be useful to have the following:

Exercise 4.6. Let $\mathcal{T}^+ \cong \mathbb{F}[U,U^{-1}]/U\cdot\mathbb{F}[U]$. Given any formal power series in U, $\sum_{i=0}^{\infty} a_i \cdot U^i$, there is a corresponding endomorphism of $\mathcal{T}^+$, defined by

$$\xi \mapsto \sum_{i=0}^{\infty} a_i U^i \cdot \xi.$$

Show that in fact every endomorphism of $\mathcal{T}^+$ can be described in this manner. In particular, every non-trivial endomorphism of $\mathcal{T}^+$ is surjective, with kernel isomorphic to $\mathbb{F}[U]/U^\ell$, where $\ell = \min\{i \big| a_i \neq 0\}$.

Lemma 4.7. *Let $K \subset S^3$ be a knot. Then for all $i \neq 0$, $HF^+(S^3_0(K), i)$ is a finite-dimensional vector space (over $\mathbb{F}$).*

Proof. Clearly, there are only finitely many integers i for which $HF^+(S^3_0(K), i) \neq 0$. It follows that for sufficiently large N, we can arrange that there is some $i \in \mathbb{Z}/N\mathbb{Z}$ with the property that $\bigoplus_{j\equiv i \pmod N} HF^+(S^3_0, j) = 0$. According to Theorem 2.20, this forces $F^+_{W_N(K)}\colon HF^+(S^3_N(K), j) \longrightarrow HF^+(S^3)$ to be an isomorphism. It follows from this that $F^\infty_{W_N(K),\mathfrak{s}}\colon HF^\infty(S^3_N(K), \mathfrak{s}|_{S^3_N(K)}) \longrightarrow HF^\infty(S^3)$ is an isomorphism for some choice of $\mathfrak{s}$.

We would like to conclude that it holds for all $\mathfrak{s} \in \mathrm{Spin}^c(W_N(K))$. To this end, recall first that for a rational homology three-sphere Y such as $S^3_N(K)$, the group $HF^\infty(Y,\mathfrak{s})$ is indpendent of the choice of $\mathfrak{s}$. This follows readily from the definition of the differential: for two different choices of reference point z_1 and z_2 and a fixed $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the $\mathbf{y}$ component of $\partial\mathbf{x}$ differs only by multiplication by some power of U. It is easy to see that by changing basis for $CF^\infty(Y,\mathfrak{s})$ (multiplying each generator $\mathbf{x}$ for $CF^\infty(Y,\mathfrak{s})$ by $U^{m_\mathbf{x}}$), we get an isomorphism between the chain complex defining $CF^\infty(Y,\mathfrak{s}_1)$ and $CF^\infty(Y,\mathfrak{s}_2)$ (where here $\mathfrak{s}_i = \mathfrak{s}_{z_i}(\mathbf{x})$).

Modifying this argument, we can also see that the induced map

$$F^\infty_{W_N(K),\mathfrak{s}}\colon HF^\infty(S^3_N(K), i) \longrightarrow HF^\infty(S^3),$$

where $\mathfrak{s}$ is any Spin^c structure over $W_N(K)$ whose restriction to $S^3_N(K)$ corresponds to i, depends on $\mathfrak{s}$ only up to an overall multiplication by some U-power. Again, this is seen from the definition of $F^\infty_{W_N(K),\mathfrak{s}}$ as a count of holomorphic triangles in a Heegaard triple representing some fixed Spin^c equivalence class, and then moving the reference point. Moreover, the precise dependence of the U-power on the choice of $\mathfrak{s}$ is determined by $c_1(\mathfrak{s})^2$, according to Equation (22) (which determines the grading of the image of any element).

In view of Equation (24), it is easy to see that for all $i \not\equiv 0 \pmod p$, for all $\mathfrak{s} \in \mathrm{Spin}^c(W_p(K))$ whose restriction to $S^3_p(K)$ corresponds to i, the lengths $c_1(\mathfrak{s})^2$ are all distinct. Thus, the homomorphism

$$\mathcal{T}^+ \cong \mathrm{Im}\left(HF^\infty(S^3_p(K),i) \subset HF^+(S^3_p(K),i)\right) \longrightarrow \mathcal{T}^+ \cong HF^+(S^3)$$

gotten by restricting $F^+_{W_N(K)}$ is non-trivial, and in particular, according to Exercise 4.6, it follows that $HF^+(S^3_0(K),i)$ is a finite-dimensional vector space. □

Proof of Theorem 4.4. We use the integral surgeries long exact sequence, Theorem 2.20.

As a preliminary step, we argue that the only integer $j \equiv 0 \pmod p$ with $HF^+(S^3_0(K),j) \neq 0$ is $j = 0$. This follows easily from the exact sequence in the form of Equation (20). (We leave the details to the reader; it is a straightforward adaptation of the proof of Corollary 1.16, together with the observation that $\widehat{HF}(S^3_0(K),j) \neq 0$ if and only if $HF^+(S^3_0(K),j) \neq 0$, cf. Exercise 1.4.)

Next, we consider $j \not\equiv 0 \pmod p$. If the map

$$F^+_{W_p(K)}|_{HF^+(S^3_p(K),i)} \colon HF^+(S^3_p(K),i) \longrightarrow HF^+(S^3)$$

were trivial, the long exact sequence would would force $HF^+(S^3_0(K),j)$ to be infinitely generated (as an $\mathbb{F}$-vector space) for some $j \neq 0$, contradicting Lemma 4.7. Thus Exercise 4.6, together with the long exact sequence, gives us that

$$\bigoplus_{j\equiv i \pmod{\mathbb{Z}}} HF^+(S^3_0(K),j) \cong \mathbb{F}[U]/U^\ell \tag{25}$$

for some $\ell \geq 0$. In particular, for each $i \in \mathbb{Z}/p\mathbb{Z}$, there is at most one $j \equiv i \pmod p$ with $HF^+(S^3_0(K),j) \neq 0$. Next, we argue that in fact if $HF^+(S^3_0(K),m) \neq 0$, then $|m| \leq p/2$ as follows. If it were not the case, then since $2m \geq p$, $S^3_{2m}(K)$ would also be an L-space (cf. Exercise 1.14); but now both $m \equiv -m \pmod{2m}$ and $HF^+(S^3_0(K),m) \neq 0$ and $HF^+(S^3_0(K),-m) \neq 0$, violating the principle just established.

It remains now to show that the power of U, ℓ, appearing in Equation (25) is the quantity ℓ_i given by Equation (23).

Let $c(p,i)$ be the maximal value of

$$\frac{c_1(\mathfrak{s})^2 + 1}{4}$$

for any $\mathfrak{s} \in \mathrm{Spin}^c(W_p(K))$ with $\mathfrak{s}|_{S^3_p(K)}$ corresponding to $i \in \mathbb{Z}/p\mathbb{Z}$, and let $\mathfrak{s}_0 \in \mathrm{Spin}^c(W_p(K))$ be the Spin^c structure with given restriction to $S^3_p(K)$ which achieves this maximal value. Note that $c(p,i)$ is independent of the choice of $K \subset S^3$. The element of $HF^\infty(Y_p(K),i)$ of degree $-c(p,i)$ is mapped by $F^\infty_{W_p(K);\mathfrak{s}_0}$ to the generator of $HF_0^+(S^3)$, and hence its image in $HF^+(S^3_p(K),i)$ is mapped to the generator of $HF_0^+(S^3)$, in view of the diagram:

$$\begin{tikzcd} HF^\infty_{-c(p,i)}(S^3_p(K),i) \arrow[r, "F^\infty_{W_p(K),\mathfrak{s}_0}"] \arrow[d] & HF_0^\infty(S^3) \arrow[d, "\cong"'] \\ HF^+_{-c(p,i)}(S^3_p(K),i) \arrow[r, "F^+_{W_p(K),\mathfrak{s}_0}"] & HF_0^+(S^3). \end{tikzcd}$$

(where here the subscripts on Heegaard Floer groups denote the summands with specified $\mathbb{Q}$ grading). But since $\mathfrak{s}_0$ is the unique Spin^c structure which maximizes $c_1(\mathfrak{s})^2$ among all $\mathfrak{s}\in \mathrm{Spin}^c(W_p(K))$ with given restriction to $S^3_p(K)$, it follows that $F^+_{W_p(K)}$ carries $HF^+_{-c(p,i)}(S^3_p(K))$ isomorphically to $HF^+_0(S^3)$.

Moreover, it also follows from this formula that all elements in $HF^+(S^3_p(K),i)$ of degree less than $-c(p,i)$ are mapped to zero, and the set of such elements form a vector space of rank

$$-1-\left(\frac{c(p,i)+d(S^3_p(K),i)}{2}\right).$$

We can conclude now that the kernel of $F^+_{W_p(K)}\colon HF^+(S^3_p(K),i)\longrightarrow HF^+(S^3)$ is isomorphic to $\mathbb{F}[U]/U^\ell$, with $2\ell=-c(p,i)-d(S^3_p(K),i)$. By comparing with the unknot $\mathcal{U}$, and recalling that $HF^+(S^3_0(\mathcal{U}),i)=HF^+(S^2\times S^1,i)=0$ for all $i\neq 0$, we conclude that $-c(p,i)=d(L(p,1),i)$. $\square$

Exercise 4.8. Using the above proof (and Equation (24)) calculate $d(L(p,1),i)$ for all $i\neq 0$. As a test, when p is even, you should find that $d(L(p,1),p/2)=-\frac{1}{4}$.

Corollary 4.9. *If $HF^+(S^3_p(K))\cong HF^+(S^3_p(\mathcal{U}))$ as $\mathbb{Q}$-graded Abelian groups, then $HF^+(S^3_0(K),i)=0$ for all $i\neq 0$.*

Proof. The expression $S^3_p(\mathcal{U})\cong L(p,1)$ gives an affine identification $\mathbb{Z}/p\mathbb{Z}\cong \mathrm{Spin}^c(L(p,1))$ (determined by Equation (24)), and hence the affine identification $\mathbb{Z}/p\mathbb{Z}\cong\mathrm{Spin}^c(L(p,1))$ induced from the expression of $S^3_p(K)\cong L(p,1)$ can be viewed as a permutation $\sigma\colon\mathbb{Z}/p\mathbb{Z}\longrightarrow\mathbb{Z}/p\mathbb{Z}$. According to Theorem 4.4, this permutation σ has the property that for all $i\neq 0$, $-d(L(p,1),\sigma(i))+d(L(p,1),i)\geq 0$. Moreover, in the case where p is odd (cf. Exercise 4.5), σ fixes 0, inducing a permutation on the remaining $\{d(L(p,1),i)\}_{i=1}^{p-1}$. It follows readily that $d(L(p,1),\sigma(i))=d(L(p,1),i)$ for all i. From Theorem 4.4, it follows that $HF^+(S^3_0(K),i)=0$ for all $i\neq 0$. In the case where p is even, write $p=2n$, and observe that σ either fixes 0 or it permutes 0 and n (Exercise 4.5); if it fixes both, the previous argument applies. We must rule out the possibility that $\sigma(n)=0$. Observe, however, that $d(L(2n,1),0)=\frac{2n-1}{4}$ according to Exercise 3.8, while $d(L(2n,1),n)=-\frac{1}{4}$ according to Exercise 4.8, so in this case, it would not be possible for $-d(L(2n,1),\sigma(n))+d(L(2n,1),n)\geq 0$, as required by Theorem 4.4. $\square$

4.2. The second component: $HF^+(S^3_0(K))\cong HF^+(S^3_0(\mathcal{U}))\Rightarrow K=\mathcal{U}$

Again, we set slightly more modest goals in this article, sketching the proof that $HF^+(S^3_0(K),i)=0$ for all $i\neq 0$ implies that $g(K)\leq 1$.

We rely on the following fundamental result of Gabai. For our purposes, an oriented foliation $\mathcal{F}$ of an oriented three-manifold Y is *taut* if there is a closed two-form ω_0 over Y whose restriction to the tangent space to $\mathcal{F}$ is always non-degenerate.

THEOREM 4.10. *(Gabai [**21**]) If K is a knot with Seifert genus $g(K) > 1$, then there is a smooth taut foliation over $S^3_0(K)$ whose first Chern class is $2g-2$ times a generator of $H^2(S^3_0(K);\mathbb{Z})$.*

Gabai's taut foliation can be interpreted as an infinitesimal symplectic structure, according to the following result:

THEOREM 4.11. *(Eliashberg-Thurston [**10**]) Let Y be a three-manifold which admits a taut foliation $\mathcal{F}$, and ω_0 be a two-form positive on the leaves. Then there is a symplectic two-form ω over $[-1,1]\times Y$ which is convex at the boundary, and whose restriction to $\{0\}\times Y$ agrees with ω_0.*

We use here the usual notion of convexity from symplectic geometry (see for example [**49**] or [**12**]). This in turn can be extended to a symplectic structure over a closed manifold according to the following convex filling result:

THEOREM 4.12. *(Eliashberg [**9**] and Etnyre [**11**]) If (X,ω) is a symplectic manifold with convex boundary, then there is a closed symplectic four-manifold $(\widetilde{X},\widetilde{\omega})$ which contains (X,ω) as a submanifold.*

There is considerable flexibility in constructing $\widetilde{X}$; in particular, it is technically useful to note that one can always arrange that $b_2^+(\widetilde{X}) > 1$.

In sum, the above three theorems say the following: if $K\subset S^3$ is a knot with Seifert genus $g(K) > 1$, then there is a closed symplectic four-manifold (M,ω) which is divided in two by $S^3_0(K)$, in such a way that $c_1(k)|_{S^3_0(K)} \neq 0$, where here k is the canonical Spin^c structure of the symplectic form specified by ω, and hence $c_1(k)$ restricts to $2g-2$ times a generator of $H^2(S^3_0(K);\mathbb{Z})$.

Proof of Theorem 4.2. As explained in the discussion preceding the statement, it suffices to consider the case where the Seifert genus g of K is greater than one (according to [**22**]), r integral (according to [**4**]) and $|r| > 1$ (according to [**24**]). After reflecting K if necessary (cf. Exercise 1.9), we can assume that $r > 1$. Since $g(K) > 1$, as explained in the above discussion (combining Theorems 4.10, 4.11, and 4.12) we obtain a symplectic four-manifold (M,ω) which is divided in two by $S^3_0(K)$ in such a way that $c_1(k)|_{S^3_0(K)} \neq 0$. According to Theorem 3.12, $\Phi_{M,k}\neq 0$, and hence, according to Proposition 3.11, $HF^+(S^3_0(K), g-1)\neq 0$. (Note that Proposition 3.11 requires the non-vanishing of a sum of invariants associated to $\{k+n\mathrm{PD}[\Sigma]\}_{n\in\mathbb{Z}}$; but since each has distinct $c_1(\mathfrak{s})^2$ and hence $\deg(X,\mathfrak{s})$, these terms are linearly independent.) But this now contradicts the conclusion of Theorem 4.4. $\square$

4.3. Comparison with Seiberg-Witten theory. The original proof of Theorem 4.2 was obtained using the monopole Floer homology for Seiberg-Witten monopoles, cf. [**29**]. The basic components of the proof are analogous: an exact triangle argument reduces the problem to showing that the monopole Floer homologies of $S^3_0(K))$ and $S^3_0(\mathcal{U})$ coincide, and a second component proves that if this holds, then $K=\mathcal{U}$. This second component had been established by Kronheimer and Mrowka [**27**] and [**28**], shortly after the discovery of the Seiberg-Witten equations. More specifically, combining Gabai's foliation with the Eliashberg-Thurston filling, one obtains a symplectic structure on $[-1,1]\times S^3_0(K)$ with convex boundary.

For four-manifolds with symplectically convex boundary, Kronheimer and Mrowka construct an invariant analogous to the invariant for closed symplectic manifolds. Using the symplectic form as a perturbation for the Seiberg-Witten equations (as Taubes did in the case of closed four-manifolds, cf. [**50**]), Kronheimer and Mrowka show that their invariant for $[-1,1]\times S^3_0(K)$ is non-trivial. It follows that the Seiberg-Witten monopoloe Floer homology of $S^3_0(K)$ is non-trivial.

4.4. Further remarks. Theorem 4.4, including an analysis of the case where $i=0$, was first proved in Theorem 7.2 [**36**]. This result can be used to give bounds on genera of knots admitting lens space surgeries. Further bounds on the genera of these knots have been obtained by Rasmussen [**45**].

See also [**43**] for a generalization of Theorem 4.4 to the case of knots which admit rational L-space surgeries.

References

[1] P. Braam and S. K. Donaldson. Floer's work on instanton homology, knots, and surgery. In H. Hofer, C. H. Taubes, A. Weinstein, and E. Zehnder, editors, *The Floer Memorial Volume*, number 133 in Progress in Mathematics, pages 195–256. Birkhäuser, 1995.

[2] G. Burde and H. Zieschang. *Knots*. Number 5 in de Gruyter Studies in Mathematics. Walter de Gruyter & Co., 1985.

[3] D. Calegari and N. M. Dunfield. Laminations and groups of homeomorphisms of the circle. *Invent. Math.*, 152(1):149–204, 2003.

[4] M. Culler, C. McA. Gordon, J. Luecke, and P. B. Shalen. Dehn surgery on knots. *Ann. of Math*, 125(2):237–300, 1987.

[5] V. de Silva. *Products in the symplectic Floer homology of Lagrangian intersections*. PhD thesis, Oxford University, 1999.

[6] S. K. Donaldson. Lefschetz pencils on symplectic manifolds. *J. Differential Geom.*, 53(2):205–236, 1999.

[7] S. K. Donaldson. *Floer homology groups in Yang-Mills theory*, volume 147 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2002. With the assistance of M. Furuta and D. Kotschick.

[8] S. K. Donaldson and P. B. Kronheimer. *The Geometry of Four-Manifolds*. Oxford Mathematical Monographs. Oxford University Press, 1990.

[9] Y. Eliashberg. A few remarks about symplectic filling. *Geom. Topol.*, 8:277–293 (electronic), 2004.

[10] Y. M. Eliashberg and W. P. Thurston. *Confoliations*. Number 13 in University Lecture Series. American Mathematical Society, 1998.

[11] J. B. Etnyre. On symplectic fillings. *Algebr. Geom. Topol.*, 4:73–80, 2004.

[12] J. B. Etnyre. Lectures on open book decompositions and contact structures. In *Floer homology, gauge theory, and low-dimensional topology*, 2006.

[13] R. Fintushel and R. J. Stern. Instanton homology of Seifert fibered homology three spheres. *Proc. of the London Math. Soc.*, 61:109–137, 1990.

[14] R. Fintushel and R. J. Stern. Immersed spheres in 4-manifolds and the immersed Thom conjecture. *Turkish J. Math*, 19(2):145–157, 1995.

[15] A. Floer. An instanton-invariant for 3-manifolds. *Comm. Math. Phys.*, 119:215–240, 1988.

[16] A. Floer. Instanton homology and Dehn surgery. In H. Hofer, C. H. Taubes, A. Weinstein, and E. Zehnder, editors, *The Floer Memorial Volume*, number 133 in Progress in Mathematics, pages 77–97. Birkhäuser, 1995.

[17] K. A. Frøyshov. The Seiberg-Witten equations and four-manifolds with boundary. *Math. Res. Lett*, 3:373–390, 1996.

[18] K. A. Frøyshov. Equivariant aspects of Yang-Mills Floer theory. *Topology*, 41(3):525–552, 2002.

[19] K. Fukaya, Y-G. Oh, K. Ono, and H. Ohta. *Lagrangian intersection Floer theory—anomaly and obstruction*. Kyoto University, 2000.

[20] D. Gabai. Foliations and the topology of 3-manifolds. *J. Differential Geom.*, 18(3):445–503, 1983.

[21] D. Gabai. Foliations and the topology of 3-manifolds III. *J. Differential Geom.*, 26(3):479–536, 1987.

[22] H. Goda and M. Teragaito. Dehn surgeries on knots which yield lens spaces and genera of knots. *Math. Proc. Cambridge Philos. Soc.*, 129(3):501–515, 2000.

[23] C. McA. Gordon. *Some aspects of classical knot theory*, pages pp. 1–60. Number 685 in Lecture Notes in Math. Springer-Verlag, 1978.

[24] C. McA. Gordon and J. Luecke. Knots are determined by their complements. *J. Amer. Math. Soc.*, 2(2):371–415, 1989.

[25] E-N. Ionel and T. H. Parker. Relative Gromov-Witten invariants. *Ann. of Math. (2)*, 157(1):45–96, 2003.

[26] P. B. Kronheimer and T. S. Mrowka. Floer homology for Seiberg-Witten Monopoles. Preprint.

[27] P. B. Kronheimer and T. S. Mrowka. Monopoles and contact structures. *Invent. Math.*, 130(2):209–255, 1997.

[28] P. B. Kronheimer and T. S. Mrowka. Scalar curvature and the Thurston norm. *Math. Res. Lett.*, 4(6):931–937, 1997.

[29] P. B. Kronheimer, T. S. Mrowka, P. S. Ozsváth, and Z. Szabó. Monopoles and lens space surgeries. math.GT/0310164.

[30] A-M. Li and Y. Ruan. Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds. *Invent. Math.*, 145(1):151–218, 2001.

[31] W. B. R. Lickorish. *An introduction to knot theory*, volume 175 of *Graduate Texts in Mathematics*. Springer-Verlag, 1997.

[32] D. McDuff and D. Salamon. *J-holomorphic curves and quantum cohomology*. Number 6 in University Lecture Series. American Mathematical Society, 1994.

[33] J. W. Milnor. *Topology from the differentiable viewpoint*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Based on notes by David W. Weaver, Revised reprint of the 1965 original.

[34] P. S. Ozsváth and Z. Szabó. Holomorphic triangles and invariants for smooth four-manifolds. math.SG/0110169.

[35] P. S. Ozsváth and Z. Szabó. On knot Floer homology and lens space surgeries. math.GT/0303017, to appear in *Topology*.

[36] P. S. Ozsváth and Z. Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Advances in Mathematics*, 173(2):179–261, 2003.

[37] P. S. Ozsváth and Z. Szabó. On the Floer homology of plumbed three-manifolds. *Geometry and Topology*, 7:185–224, 2003.

[38] P. S. Ozsváth and Z. Szabó. On the Heegaard Floer homology of branched double-covers. math.GT/0309170, 2003.

[39] P. S. Ozsváth and Z. Szabó. Holomorphic disks and genus bounds. *Geom. Topol.*, 8:311–334 (electronic), 2004.

[40] P. S. Ozsváth and Z. Szabó. Holomorphic disks and three-manifold invariants: Properties and applications. *Ann. of Math.*, 159(3):1159–1245, 2004.

[41] P. S. Ozsváth and Z. Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math.*, 159(3):1027–1158, 2004.

[42] P. S. Ozsváth and Z. Szabó. Holomorphic triangle invariants and the topology of symplectic four-manifolds. *Duke Math. J.*, 121(1):1–34, 2004.

[43] P. S. Ozsváth and Z. Szabó. Knot Floer homology and rational surgeries. In preparation, 2005.

[44] P. S. Ozsváth and Z. Szabó. An introduction to Heegaard Floer homology. In *Floer homology, gauge theory, and low-dimensional topology*, 2006.

[45] J. Rasmussen. Lens space surgeries and a conjecture of Goda and Teragaito. *Geometry and Topology*, 8:1013–1031, 2004.

[46] J. Robbin and D. Salamon. The Maslov index for paths. *Topology*, 32(4):827–844, 1993.

[47] R. Roberts, J. Shareshian, and M. Stein. Infinitely many hyperbolic 3-manifolds which contain no Reebless foliation. *J. Amer. Math. Soc.*, 16(3):639–679 (electronic), 2003.

[48] P. Seidel. A long exact sequence for symplectic Floer cohomology. *Topology*, 42(5):1003–1063, 2003.

[49] A. I. Stipsicz. Contact surgery and Heegaard Floer theory. In *Floer homology, gauge theory, and low-dimensional topology*, 2006.
[50] C. H. Taubes. The Seiberg-Witten invariants and symplectic forms. *Math. Research Letters*, 1(6):809–822, 1994.
[51] C. H. Taubes. More constraints on symplectic forms from Seiberg-Witten invariants. *Math. Research Letters*, 2(1):9–13, 1995.
[52] C. H. Taubes. *Metrics, connections and gluing theorems*. Number 89 in CBMS Regional Conference Series in Mathematics. AMS, 1996.
[53] E. Witten. Monopoles and four-manifolds. *Math. Research Letters*, 1:769–796, 1994.

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