# Cycles in dense digraphs

Maria Chudnovsky[1]
Columbia University, New York, NY 10027

Paul Seymour[2]
Princeton University, Princeton, NJ 08544

Blair Sullivan[3]
Princeton University, Princeton, NJ 08544

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**Abstract**

Let $G$ be a digraph (without parallel edges) such that every directed cycle has length at least four; let $\beta(G)$ denote the size of the smallest subset $X \subseteq E(G)$ such that $G \setminus X$ has no directed cycles, and let $\gamma(G)$ be the number of unordered pairs $\{u, v\}$ of vertices such that $u, v$ are nonadjacent in $G$. It is easy to see that if $\gamma(G) = 0$ then $\beta(G) = 0$; what can we say about $\beta(G)$ if $\gamma(G)$ is bounded?

We prove that in general $\beta(G) \leq \gamma(G)$. We conjecture that in fact $\beta(G) \leq \frac{1}{2}\gamma(G)$ (this would be best possible if true), and prove this conjecture in two special cases:

- when $V(G)$ is the union of two cliques,
- when the vertices of $G$ can be arranged in a circle such that if distinct $u, v, w$ are in clockwise order and $uw$ is a (directed) edge, then so are both $uv, vw$.

## 1 Introduction

We begin with some terminology. All digraphs in this paper are finite and have no parallel edges; and for a digraph $G$, $V(G)$ and $E(G)$ denote its vertex- and edge-sets. The members of $E(G)$ are ordered pairs of vertices, and we abbreviate $(u,v)$ by $uv$. For integer $k \geq 0$, let us say a digraph $G$ is *$k$-free* if there is no directed cycle of $G$ with length at most $k$. A digraph is *acyclic* if it has no directed cycle.

We are concerned here with 3-free digraphs. It is easy to see that every 3-free tournament is acyclic, and one might hope that every 3-free digraph that is "almost" a tournament is "almost" acyclic. That is the topic of this paper.

More exactly, for a digraph $G$, let $\gamma(G)$ be the number of unordered pairs $\{u,v\}$ of distinct vertices $u,v$ that are nonadjacent in $G$ (that is, both $uv, vu \notin E(G)$). Thus, every 2-free digraph $G$ can be obtained from a tournament by deleting $\gamma(G)$ edges. Let $\beta(G)$ denote the minimum cardinality of a set $X \subseteq E(G)$ such that $G \setminus X$ is acyclic. We already observed that every 3-free digraph with $\gamma(G) = 0$ satisfies $\beta(G) = 0$, and our first result is an extension of this.

**1.1** *If $G$ is a 3-free digraph then $\beta(G) \leq \gamma(G)$.*

**Proof.** We proceed by induction on $|V(G)|$, and we may assume that $V(G) \neq \emptyset$. Let us say a *2-path* is a triple $(x,y,z)$ such that $x,y,z \in V(G)$ are distinct, and $xy, yz \in E(G)$, and $x,z$ are nonadjacent. For each vertex $v$, let $f(v)$ denote the number of 2-paths $(x,y,z)$ with $x = v$, and let $g(v)$ be the number of 2-paths $(x,y,z)$ with $y = v$. Since $V(G) \neq \emptyset$ and $\sum_{v \in V(G)} f(v) = \sum_{v \in V(G)} g(v)$, there exists $v \in V(G)$ such that $f(v) \leq g(v)$. Choose some such vertex $v$, and let $A,B,C$ be the set of all vertices $u \neq v$ such that $vu \in E(G)$, $uv \in E(G)$, and $uv, vu \notin E(G)$ respectively. Thus the four sets $A,B,C,\{v\}$ are pairwise disjoint and have union $V(G)$. Let $G_1, G_2$ be the subdigraphs of $G$ induced on $A$ and on $B \cup C$ respectively. Since $g(v)$ is the number of pairs $(a,b)$ with $a \in A$ and $b \in B$ such that $a,b$ are nonadjacent, it follows that $\gamma(G) \geq \gamma(G_1) + \gamma(G_2) + g(v)$. From the inductive hypothesis, $\beta(G_1) \leq \gamma(G_1)$ and $\beta(G_2) \leq \gamma(G_2)$; for $i = 1,2$, choose $X_i \subseteq E(G_i)$ with $|X_i| \leq \beta(G_i)$ such that $G_i \setminus X_i$ is acyclic. Let $X_3$ be the set of all edges $ac \in E(G)$ with $a \in A$ and $c \in C$; thus $|X_3| = f(v)$. Since there is no edge $xy \in E(G)$ with $x \in A$ and $y \in B$ (because $G$ is 3-free), it follows that every edge $xy$ with $x \in A$ and $y \in \{v\} \cup B \cup C$ belongs to $X_3$, and so $G \setminus X$ is acyclic, where $X = X_1 \cup X_2 \cup X_3$. Hence

$$\beta(G) \leq |X| = |X_1| + |X_2| + |X_3| = \beta(G_1) + \beta(G_2) + f(v) \leq \gamma(G_1) + \gamma(G_2) + g(v) \leq \gamma(G).$$

This proves 1.1. ∎

But 1.1 does not seem to be sharp, and we believe that the following holds.

**1.2 Conjecture.** *If $G$ is a 3-free digraph then $\beta(G) \leq \frac{1}{2}\gamma(G)$.*

If true, this is best possible for infinitely many values of $\gamma(G)$. For instance, let $G$ be the digraph with vertex set $\{v_1, \ldots, v_{4n}\}$, and with edge set as follows (reading subscripts modulo $4n$):

- $v_i v_j \in E(G)$ for all $i,j,k$ with $1 \leq k \leq 4$ and $(k-1)n < i < j \leq kn$
- $v_i v_j \in E(G)$ for all $i,j,k$ with $1 \leq k \leq 4$ and $(k-1)n < i \leq kn < j \leq (k+1)n$.

It is easy to see that this digraph $G$ is 3-free, and satisfies $\beta(G) = n^2$ (certainly $\beta(G) \geq n^2$ since $G$ has $n^2$ directed cycles that are pairwise edge-disjoint), and $\gamma(G) = 2n^2$.

The reason for our interest in 1.2 was originally its application to the Caccetta-Häggkvist conjecture [2]. A special case of that conjecture asserts the following:

**1.3 Conjecture.** *If $G$ is a 3-free digraph with $n$ vertices, then some vertex has outdegree less than $n/3$.*

This is a challenging open question and has received a great deal of attention. Any counterexample to 1.3 satisfies $\gamma(G) \leq \frac{1}{2}|E(G)|$, so our conjecture 1.2 would tell us that $\beta(G) \leq \frac{1}{4}|E(G)|$, and this would perhaps be useful information towards solving 1.3. Indeed, 1.1 itself has already been used to prove new approximations for 1.3, by Hamburger, Haxell and Kostochka [3], and by Shen [5].

We have not been able to prove 1.2 in general, and in this paper we prove two partial results, that 1.2 holds for every 3-free digraph $G$ such that either

- $V(G)$ is the union of two cliques, or
- the vertices of $G$ can be arranged in a circle such that if distinct $u, v, w$ are in clockwise order and $uw \in E(G)$, then $uv, vw \in E(G)$.

The first result is proved in 3.1, and the second in 5.1. Incidentally, Kostochka and Stiebitz [4] proved that in any minimal counterexample to 1.2, every vertex is nonadjacent to at least three other vertices, and the conjecture is true for all digraphs with at most eight vertices.

In the proof of 1.1 we find a partition of the vertex set of $G$ into two nonempty sets $X, Y$, with the property that the number of edges with tail in $X$ and head in $Y$ is at most the number of nonadjacent pairs $(x, y)$ with $x \in X$ and $y \in Y$; and given such a partition, the result follows by applying the inductive hypothesis to $G|X$ and $G|Y$. Bruce Reed (private communication) asked whether the analogous strengthening of 1.2 was true, that is:

**1.4 Conjecture.** *If $G$ is a 3-free digraph with $|V(G)| \geq 2$, then there is a partition $(X, Y)$ of $V(G)$ with $X, Y \neq \emptyset$, such that the number of edges with tail in $X$ and head in $Y$ is at most half the number of nonadjacent pairs $(x, y)$ with $x \in X$ and $y \in Y$.*

We have not been able to decide this, even in the two cases when we can prove 1.2.

## 2 A distant relative of the four functions theorem

In this section we prove a result that we apply in the next section. We begin with an elementary lemma. ($\mathbf{R}_+$ denotes the set of nonnegative real numbers.)

**2.1** *If $a_1, a_2, c_1, c_2, d_1, d_2 \in \mathbf{R}_+$ and $a_k^2 \leq c_k d_k$ for $k = 1, 2$, then $(a_1 + a_2)^2 \leq (c_1 + d_1)(c_2 + d_2)$.*

**Proof.** If say $c_1 = 0$, then since $a_1^2 \leq c_1 d_1$, it follows that $a_1 = 0$, and so

$$(a_1 + a_2)^2 = a_2^2 \leq c_2 d_2 \leq (c_1 + c_2)(d_1 + d_2)$$

as required. We may therefore assume that $c_1, c_2$ are nonzero. Now

$$\begin{aligned}(c_1+c_2)(d_1+d_2) &= c_1d_1+c_1d_2+c_2d_1+c_2d_2\\ &\geq a_1^2+c_1(a_2^2/c_2)+c_2(a_1^2/c_1)+a_2^2\\ &= (a_1+a_2)^2+c_1c_2(a_2/c_2-a_1/c_1)^2\\ &\geq (a_1+a_2)^2.\end{aligned}$$

This proves 2.1. ∎

Before the main result of this section we must set up some notation. Let $m, n \geq 1$ be integers, and let $P$ denote the set of all pairs $(i, j)$ of integers with $1 \leq i \leq m$ and $1 \leq j \leq n$. If $f : P \to \mathbf{R}_+$, and $X \subseteq P$, we define $f(X)$ to mean $\sum_{x\in X} f(x)$. For $(i, j), (i', j') \in P$, we say that $(i', j')$ *dominates* $(i, j)$ if $i < i'$ and $j < j'$. Let $a, b : P \to \mathbf{R}_+$ be functions. We say that $b$ *dominates* $a$ if

- $a(P) = b(P)$
- for all $X, Y \subseteq P$, if $a(X) + b(Y) > a(P)$ then there exist $x \in X$ and $y \in Y$ such that $y$ dominates $x$.

The main result of this section is the following. (It is reminiscent of the "four functions" theorem of Ahlswede and Daykin [1], but we were not able to derive it from that theorem.)

**2.2** *Let $m, n \geq 1$ be integers, let $P$ be as above, and let $a, b, c, d$ be functions from $P$ to $\mathbf{R}_+$, satisfying the following:*

1. *$a(i, j)b(i', j') \leq c(i', j)d(i, j')$ for $1 \leq i < i' \leq m$ and $1 \leq j < j' \leq n$, and*
2. *$b$ dominates $a$.*

*Then $a(P)b(P) \leq c(P)d(P)$.*

**Proof.** We proceed by induction on $m+n$. Let $Q$ be the set of all quadruples $(a, b, c, d)$ of functions from $P$ to $\mathbf{R}_+$ that satisfy conditions 1 and 2 above. We say that $(a, b, c, d) \in Q$ is *good* if

$$a(P)b(P) \leq c(P)d(P).$$

Thus, we need to show that every member of $Q$ is good. Certainly if $m = 1$ or $n = 1$ then condition 2 implies that $a(P) = b(P) = 0$, and therefore $(a, b, c, d)$ is good; so we may assume that $m, n \geq 2$.

(1) *If $(a, b, c, d) \in Q$ then $b(i, 1) = 0$ for $1 \leq i \leq m$, and $a(m, j) = 0$ for $1 \leq j \leq n$.*

For let $X = P$, and let $Y$ be the set of all pairs $(i, 1)$ with $1 \leq i \leq m$. There do not exist $x \in X$ and $y \in Y$ such that $y$ dominates $x$, and since $b$ dominates $a$ it follows that $a(X) + b(Y) \leq a(P)$. Since $a(X) = a(P)$ we deduce that $b(Y) = 0$. This proves the first statement, and the second follows similarly. This proves (1).

(2) *If $(a, b, c, d) \in Q$ and $a(i, 1) = 0$ for all $i \in \{1, \ldots, m\}$ then $(a, b, c, d)$ is good.*

This follows from (1) and the inductive hypothesis applied to the restriction of $a, b, c, d$ to the set of all $(i, j) \in P$ with $j > 1$ (relabeling appropriately).

For $(a, b, c, d) \in Q$, let us define its *margin* to be the number of pairs $(i, j)$ such that either $j = 1$ and $a(i, j) > 0$, or $i = m$ and $b(i, j) > 0$. For fixed $m, n$ we proceed by induction on the margin. Thus, we assume that $t \geq 0$ is an integer, and every $(a, b, c, d) \in Q$ with margin smaller than $t$ is good. We must show that every $(a, b, c, d) \in Q$ with margin $t$ is good.

(3) *Let $(a, b, c, d) \in Q$ with margin $t$, and suppose that there exist $X, Y \subseteq P$ such that*

- $a(X) + b(Y) = a(P)$
- *there do not exist $x \in X$ and $y \in Y$ such that $y$ dominates $x$*
- *there exists $i \in \{1, \ldots, m\}$ such that $(i, 1) \notin X$ and $a(i, 1) > 0$, and there exists $j \in \{1, \ldots, n\}$ such that $(m, j) \notin Y$ and $b(m, j) > 0$.*

*Then $(a, b, c, d)$ is good.*

Let $A_1 = X$ and $A_2 = P \setminus X$. Let $B_1 = P \setminus Y$ and $B_2 = Y$. For $k = 1, 2$, let $C_k$ be the set of all pairs $(i', j) \in P$ such that there exist $i, j'$ with $i < i'$ and $j < j'$ and $(i, j) \in A_k$ and $(i', j') \in B_k$; and let $D_k$ be the set of all pairs $(i, j')$ such that there exist $i', j$ with $i < i'$ and $j < j'$ and $(i, j) \in A_k$ and $(i', j') \in B_k$. We observe first that $C_1 \cap C_2 = \emptyset$; for suppose that $(i', j) \in C_1 \cap C_2$. Since $(i', j) \in C_1$, there exists $i < i'$ such that $(i, j) \in X$; and since $(i', j) \in C_2$, there exists $j' > j$ such that $(i', j') \in Y$. But then $(i', j') \in Y$ dominates $(i, j) \in X$, contradicting the second hypothesis about $X, Y$. This proves that $C_1 \cap C_2 = \emptyset$, and similarly $D_1 \cap D_2 = \emptyset$. For $k = 1, 2$, and $x \in P$, define $a_k(x) = a(x)$ if $x \in A_k$, and $a_k(x) = 0$ otherwise. Define $b_k(x), c_k(x), d_k(x)$ similarly. Since $a_1(P) + a_2(P), b_1(P) + b_2(P)$ and $a_1(P) + b_2(P)$ all equal $a(P)$, it follows that $a_1(P) = b_1(P)$ and $a_2(P) = b_2(P)$. We claim that $(a_k, b_k, c_k, d_k) \in Q$ for $k = 1, 2$. To see this, let $i < i'$ and $j < j'$; we must show first that $a_k(i, j) b_k(i', j') \leq c_k(i', j) d_k(i, j')$. Hence we may assume that $a_k(i, j)$ and $b_k(i', j') \neq 0$, and therefore $(i, j) \in A_k$ and $(i', j') \in B_k$. From the definition of $C_k, D_k$ it follows that $(i', j) \in C_k$ and $(i, j') \in D_k$. Hence $a_k(i, j) = a(i, j)$, and $b_k(i', j') = b(i', j')$, and $c_k(i', j) = c(i', j)$, and $d_k(i, j') = d(i, j')$; and since $a(i, j) b(i', j') \leq c(i', j) d(i, j')$, this proves the claim. Second, we must show that $b_k$ dominates $a_k$. We have already seen that $a_k(P) = b_k(P)$. Let $X', Y' \subseteq P$ with $a_k(X') + b_k(Y') > a_k(P)$; we must show that there exist $x \in X'$ and $y \in Y'$ such that $y$ dominates $x$. From the symmetry we may assume that $k = 1$. Now $a(X \cap X') = a_k(X')$, and $b(Y \cup Y') = b(Y) + b_k(Y')$, and so

$$a(X \cap X') + b(Y \cup Y') = a_k(X') + b(Y) + b_k(Y') > a_k(P) + b(Y) = a(X) + b(Y) = a(P).$$

Since $b$ dominates $a$, there exist $x \in X \cap X'$ and $y \in Y \cup Y'$ such that $y$ dominates $x$. No vertex in $Y$ dominates a vertex in $X$, from the choice of $X, Y$, and it follows that $y \in Y'$, as required. This proves that $b_k$ dominates $a_k$, and consequently $(a_k, b_k, c_k, d_k) \in Q$, for $k = 1, 2$.

We claim that for $k = 1, 2$, the margin of $(a_k, b_k, c_k, d_k)$ is less than $t$. For from the third hypothesis about $X, Y$, there exists $i \in \{1, \ldots, m\}$ such that $a(i, 1) > 0$ and $(i, 1) \notin X$ (and hence $a_1(i, 1) = 0$); this shows that the margin of $(a_1, b_1, c_1, d_1)$ is less than that of $(a, b, c, d)$, and so less than $t$. Also, there exists $j \in \{1, \ldots, n\}$ such that $b(m, j) > 0$ and $(m, j) \notin Y$; and so similarly the margin of $(a_2, b_2, c_2, d_2)$ is less than $t$. Hence from the second inductive hypothesis, we deduce that $a_k(P)b_k(P) \le c_k(P)d_k(P)$ for $k = 1, 2$. But $a_k(P) = b_k(P)$ for $k = 1, 2$; thus $a_k(P)^2 \le c_k(P)d_k(P)$ for $k = 1, 2$. Since $a_1(P)+a_2(P) = a(P) = b(P)$ and since $c(P) \ge c_1(P)+c_2(P)$ (because $C_1 \cap C_2 = \emptyset$), and similarly $d(P) \ge d_1(P) + d_2(P)$, it suffices to show that

$$(a_1(P) + a_2(P))^2 \le (c_1(P) + c_2(P))(d_1(P) + d_2(P)),$$

and this follows from 2.1. This proves (3).

(4) *If $(a, b, c, d) \in Q$ with margin $t$, and there exists $j \ge 3$ such that $b(m, j) > 0$, then $(a, b, c, d)$ is good.*

For let $\epsilon$ satisfy $0 \le \epsilon \le 1$. For $1 \le i \le m$, define

$$\begin{aligned}
a_1(i,1) &= (1-\epsilon)a(i,1)\\
a_1(i,2) &= \epsilon a(i,1) + a(i,2)\\
a_1(i,j) &= a(i,j) \text{ for } 3 \le j \le n\\
c_1(i,1) &= (1-\epsilon)c(i,1)\\
c_1(i,2) &= \epsilon c(i,1) + c(i,2)\\
c_1(i,j) &= c(i,j) \text{ for } 3 \le j \le n.
\end{aligned}$$

Since $b$ dominates $a$, by compactness we may choose $\epsilon \le 1$ maximum such that $b$ dominates $a_1$. We claim that $(a_1, b, c_1, d) \in Q$; for let $i < i'$ and $j < j'$. We must check that $a_1(i, j)b(i', j') \le c_1(i', j)d(i, j')$. If $j = 1$, then

$$a_1(i, j)b(i', j') = (1 - \epsilon)a(i, 1)b(i', j')$$

and

$$c_1(i', j)d(i, j') = (1 - \epsilon)c(i, 1)d(i, j'),$$

and since $a(i, 1)b(i', j') \le c(i, 1)d(i, j')$ it follows that $a_1(i, j)b(i', j') \le c_1(i', j)d(i, j')$ as required. If $j = 2$, then

$$a_1(i, j)b(i', j') = (\epsilon a(i, 1) + a(i, 2))b(i', j')$$

and

$$c_1(i', j)d(i, j') = (\epsilon c(i, 1) + c(i, 2))d(i, j'),$$

and since $a(i, 1)b(i', j') \le c(i, 1)d(i', j')$ and $a(i, 2)b(i', j') \le c(i, 2)d(i', j')$, it follows that $a_1(i, j)b(i', j') \le c_1(i', j)d(i, j')$ as required. Finally, if $j > 2$ the claim is clear, since $a_1(i, j) = a(i, j)$ and $c_1(i', j) = c(i', j)$. This proves that $(a_1, b, c_1, d) \in Q$.

We claim that $(a_1, b, c_1, d)$ is good. If $\epsilon = 1$, then $a_1(i, 1) = 0$ for $1 \le j \le m$, and therefore $(a_1, b, c_1, d)$ is good by (2). We may therefore assume that $\epsilon < 1$. From the maximality of $\epsilon$, there exist $X, Y \subseteq P$ such that

- there does not exist $x \in X$ and $y \in Y$ such that $y$ dominates $x$
- $a_1(X) + b(Y) = a_1(P)$

- for some $i$ with $1 \le i \le m$, $(i,1) \notin X$ and and $(i,2) \in X$ and $a(i,1) > 0$.

(The third statement follows from the fact that increasing $\epsilon$ will cause $a_1(X)$ strictly to increase.) Now we recall that there exists $j \ge 3$ such that $b(m,j) > 0$. Since $(i,2) \in X$ is dominated by $(m,j)$ (for $i < m$ by (1), since $a(i,2) > 0$), it follows that $(m,j) \notin Y$. But then $(a_1, b, c_1, d)$ satisfies the hypotheses of (3), and therefore $(a_1, b, c_1, d)$ is good. This proves the claim.

Since $a_1(P) = a(P)$ and $c_1(P) = c(P)$, we deduce that $(a,b,c,d)$ is good. This proves (4).

Now let $(a,b,c,d) \in Q$ with margin $t$; we shall prove that it is good. By (4) we may assume that $b(m,j) = 0$ for $3 \le j \le m$, and similarly that $a(i,1) = 0$ for $1 \le i \le m-2$. Since $a(m,1) = b(m,1) = 0$ by (1), it follows that $a(i,1) \neq 0$ only if $i = m-1$, and $b(m,j) \neq 0$ only if $j = 2$. Let $X = \{(m-1,1)\}$ and let $Y$ be the set of all $(i,j) \in P$ with $i < m$; then there do not exist $x \in X$ and $y \in Y$ such that $y$ dominates $x$. Consequently $a(X) + b(Y) \le a(P)$. But $a(X) = a(m-1,1)$ and $b(Y) \ge a(P) - b(m,2)$, and so $a(m-1,1) \le b(m,2)$. Similarly the reverse inequality holds, and so $a(m-1,1) = b(m,2)$. For $(i,j) \in P$, if either $i = m$ or $j = 1$, define

$$a_1(i,j) = b_1(i,j) = c_1(i,j) = d_1(i,j) = 0.$$

If $i < m$ and $j > 1$ let $a_1(i,j) = a(i,j), b_1(i,j) = b(i,j)$, and $c_1(i,j) = c(i,j)$; and let $d_1(i,j) = d(i,j)$ except that $d_1(m-1,2) = 0$. We claim that $(a_1, b_1, c_1, d_1) \in Q$. For let $i < i'$ and $j < j'$. We must check that $a_1(i,j)b_1(i',j') \le c_1(i',j)d_1(i,j')$. If $i' < m$ and $j > 1$ then $a_1(i,j) = a(i,j)$ and so on, and the claim is clear. If $i' = m$ or $j = 1$ then $a_1(i,j)b_1(i',j') = 0$ and again the claim is clear. Thus $a_1(i,j)b_1(i',j') \le c_1(i',j)d_1(i,j')$. Next we must check that $b_1$ dominates $a_1$. Certainly

$$a_1(P) = a(P) - a(m-1,1) = b(P) - b(m,2) = b_1(P).$$

Let $X, Y \subseteq P$ such that $a_1(X) + b_1(Y) > a_1(P)$. We must show that there exist $x \in X$ and $y \in Y$ such that $y$ dominates $x$. We may therefore assume that $a_1(x) > 0$ for all $x \in X$, and $b_1(y) > 0$ for all $y \in Y$. In particular, since $a_1(m-1,1) = b_1(m,2) = 0$, it follows that $(m-1,1) \notin X$ and $(m,2) \notin Y$. Let $X' = X \cup \{(m-1,1)\}$. Then $a(X') = a_1(X) + a(m-1,1)$, and so

$$a(X') + b(Y) = a_1(X) + a(m-1,1) + b(Y) > a_1(P) + a(m-1,1) = a(P).$$

Hence there exist $x \in X'$ and $y \in Y$ such that $y$ dominates $x$. If $x = (m-1,1)$, then $y = (m,j)$ for some $j > 1$, and therefore $b_1(y) = 0$, a contradiction, since $b_1(y) > 0$ for all $y \in Y$. Thus $x \neq (m-1,1)$, and so $x \in X$, as required. This proves that $b_1$ dominates $a_1$.

By (2), $(a_1, b_1, c_1, d_1)$ is good, and so $a_1(P)b_1(P) \le c_1(P)d_1(P)$. Moreover,

$$a(m-1,1)b(m,2) \le c(m,1)d(m-1,2)$$

and $b(m,2) = a(m-1,1)$, and so $a(m-1,1)^2 \le c(m,1)d(m-1,2)$. Hence 2.1 implies that

$$(a_1(P) + a(m-1,1))^2 \le (c_1(P) + c(m,1))(d_1(P) + d(m-1,2)).$$

But $a(P) = a_1(P) + a(m-1,1) = b(P)$, and $c(P) \ge c_1(P) + c(m,1)$, and $d(P) \ge d_1(P) + d(m-1,2)$; and it follows that $(a,b,c,d)$ is good. This completes the inductive proof that every member of $Q$ is good, and so proves 2.2. ∎

## 3 The two cliques result

In this section we prove the following.

**3.1** *Let $G$ be a 3-free digraph and let $M, N$ be a partition of $V(G)$ such that $M, N$ are both cliques of $G$. Then there is a set $X \subseteq E(G)$ such that every member of $X$ has one end in $M$ and one end in $N$, and $|X| \le \frac{1}{2}\gamma(G)$, and $G \setminus X$ is acyclic. In particular, $\beta(G) \le \frac{1}{2}\gamma(G)$.*

**Proof.** The second assertion follows immediately from the first, so we just prove the first. Since the restriction of $G$ to $M$ is a 3-free tournament, we can number $M = \{u_1, \ldots, u_m\}$ such that $u_i u_{i'} \in E(G)$ for $1 \le i < i' \le m$. The same holds for $N$, but it is convenient to number its members in reverse order; thus we assume that $N = \{v_1, \ldots, v_n\}$, where $v_{j'} v_j \in E(G)$ for $1 \le j < j' \le n$. Let $P$ be the set of all pairs $(i, j)$ with $1 \le i \le m$ and $1 \le j \le n$. For $a = (i, j) \in P$ and $b = (i', j') \in P$, let us say that $(a, b)$ is a *cross* if $v_j u_i, u_{i'} v_{j'} \in E(G)$ and $1 \le i < i' \le m$ and $1 \le j < j' \le n$. Let $A_0$ be the set of all edges of $G$ from $N$ to $M$, and $B_0$ the set of all edges from $M$ to $N$. Let $k$ be the minimum cardinality of a subset $X \subseteq A_0 \cup B_0$ such that $G \setminus X$ is acyclic. (Such a number exists since $G \setminus (A_0 \cup B_0)$ is acyclic.)

(1) *There are $k$ crosses $(a_1, b_1), \ldots, (a_k, b_k)$ such that $a_1, \ldots, a_k, b_1, \ldots, b_k$ are all distinct.*

For suppose not. Let $H$ be the bipartite graph with vertex set $A_0 \cup B_0$, in which $v_j u_i \in A_0$ and $u_{i'} v_{j'} \in B_0$ are adjacent if $((i, j), (i', j'))$ is a cross. Then $H$ has no $k$-edge matching, and so by König's theorem, there exists $X \subseteq A_0 \cup B_0$ with $|X| < k$ meeting every edge of $H$; that is, such that for every cross $((i, j), (i', j'))$, $X$ contains at least one of the edges $v_j u_i, u_{i'} v_{j'}$. We claim that $G \setminus X$ is acyclic. For suppose that $C$ is a directed cycle of $G \setminus X$, with vertices $c_1, \ldots, c_t$ in order, say. We shall show that some two edges of $C$ correspond to a cross, contradicting the choice of $X$. We may assume that $c_t = v_j$ say, and none of $v_1, \ldots, v_{j-1}$ are vertices of $C$. Thus $c_1 \in M$, say $c_1 = u_i$. If $c_2 \in N$, say $c_2 = v_{j'}$, then $j' > j$ and so $c_2 c_t \in E(G)$; but then the vertices $c_t, c_1, c_2$ are the vertices of a directed cycle of $G$, contradicting that $G$ is 3-free. Thus $c_2 \in M$. Since $c_t \notin M$, we may choose $s$ with $3 \le s \le t$, minimum such that $c_s \in N$. Let $c_s = v_{j'}$, and $c_{s-1} = u_{i'}$ say. Since $c_2, \ldots, c_{s-1} \in M$ and form a directed path in this order, and the restriction of $G$ to $M$ is acyclic, it follows that $i' > i$. Also, since none of $v_1, \ldots, v_{j-1}$ are vertices of $C$, it follows that $j' \ge j$. If $j' = j$ then $s = t$ and $c_{t-1}, c_t, c_1$ are the vertices of a directed cycle, a contradiction; so $j' > j$. Hence $((i, j), (i', j'))$ is a cross, and $X$ contains neither of the edges $v_j u_i, u_{i'} v_{j'}$, a contradiction. Thus $G \setminus X$ is acyclic. This proves (1).

Let $(a_1, b_1), \ldots, (a_k, b_k)$ be crosses as in (1). Let $A = \{a_1, \ldots, a_k\}$, and $B = \{b_1, \ldots, b_k\}$. Let $C$ be the set of all $(i', j) \in P$ such that there exist $i, j'$ with $1 \le i < i' \le m$ and $1 \le j < j' \le n$ satisfying $(i, j) \in A$ and $(i', j') \in B$; and let $D$ be the set of all $(i, j') \in P$ such that there exist $i', j$ with $1 \le i < i' \le m$ and $1 \le j < j' \le n$ satisfying $(i, j) \in A$ and $(i', j') \in B$.

(2) $C \cap D = \emptyset$, *and* $|C| + |D| \le \gamma(G)$.

For suppose first that $(i, j) \in C \cap D$. Since $(i, j) \in C$, there exists $j' > j$ such that $(i, j') \in B$; and since $(i, j) \in D$, there exists $j'' < j$ such that $(i, j'') \in A$. But then $v_{j'} v_{j''} \in E(G)$ since $j'' < j < j'$, and $v_{j''} u_i \in E(G)$ since $(i, j'') \in A$; and $u_i v_{j'} \in E(G)$ since $(i, j') \in B$, contradicting that $G$ is 3-free. This proves that $C \cap D = \emptyset$. Moreover, if $(i', j) \in C$, we claim that $u_{i'}, v_j$ are nonadjacent. For choose $i, j'$ with $1 \leq i < i' \leq m$ and $1 \leq j < j' \leq n$ such that $(i, j) \in A$ and $(i', j') \in B$. Since $\{v_j, u_i, u_{i'}\}$ is not the vertex set of a directied cycle, it follows that $u_{i'}v_j \notin E(G)$; and since $\{u_{i'}, v_{j'}, v_j\}$ is also not the vertex set of a directed cycle, $v_j u_{i'} \notin E(G)$. This proves that $u_{i'}, v_j$ are nonadjacent. Similarly $u_i, v_{j'}$ are nonadjacent for all $(i, j') \in D$. Since $C \cap D = \emptyset$, it follows that $|C| + |D| \leq \gamma(G)$. This proves (2).

Let $a : P \to \mathbf{R}_+$ be defined by $a(x) = 1$ if $x \in A$, and $a(x) = 0$ if $x \in P \setminus A$; thus, $a$ is the characteristic function of $A$. Similarly let $b, c, d$ be the characteristic functions of $B, C, D$ respectively. We claim that the hypotheses of 2.2 are satisfied. For if $1 \leq i < i' \leq m$ and $1 \leq j < j' \leq n$, and $a(i, j)b(i', j') > 0$, then $(i, j) \in A$ and $(i', j') \in B$; hence $v_j u_i, u_{i'} v_{j'} \in E(G)$, and so $(i', j) \in C$ and $(i, j') \in D$ from the definitions of $C, D$; and therefore condition 1 of 2.2 holds. For condition 2, note first that $a(P) = k = b(P)$. Let $X, Y \subseteq P$ with $a(X) + b(Y) > a(P) = k$. We recall that $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ where $(a_i, b_i)$ is a cross for $1 \leq i \leq k$. Thus, $a(X) = |A \cap X|$ is the number of values of $h \in \{1, \dots, k\}$ such that $a_h \in X$, and similarly $b(Y)$ is the number of $h$ with $b_h \in Y$. Since $a(X) + b(Y) > k$, there exists $h$ such that $a_h \in X$ and $b_h \in Y$, and so $b_h$ dominates $a_h$. This proves that $b$ dominates $a$, and therefore the hypotheses of 2.2 are satisfied.

From 2.2, it follows that $a(P)b(P) \leq c(P)d(P)$, and so $|A||B| \leq |C||D|$. But $|A| = |B| = k$, and so $|C||D| \geq k^2$. Consequently $|C| + |D| \geq 2k$, and hence by (2), $k \leq \frac{1}{2}\gamma(G)$. This proves 3.1. ∎

# 4 A lemma for the second theorem

Now we turn to the second special case of 1.2 that we can prove. The proof is in the next section, and in this section we prove a lemma which is the main step of the proof. First we need some notation. Let $t \geq 1$ be an integer and let $s = 3t + 1$. If $n$ is an integer, $n \bmod s$ denotes the integer $n'$ with $0 \leq n' < s$ such that $n - n'$ is a multiple of $s$. If $0 \leq i, j < s$ and $i, j$ are distinct, let $q > 0$ be minimum such that $(i + q) \bmod s = j$ (so $q = j - i$ if $j > i$, and $q = j - i + s$ if $j < i$). We define $D_s(ij) = \{(i + p) \bmod s : 0 \leq p < q\}$. Let $E_s$ denote the set of all ordered pairs $ij$ with $0 \leq i, j < s$ and $j \neq i$ such that $|D_s(ij)| \leq t$, and let $F_s$ be the set of all unordered pairs $\{i, j\}$ such that $0 \leq i, j < s$ and $j \neq i$ and $ij, ji \notin E_s$. For $0 \leq k < s$, let $C_s(k)$ be the set of all pairs $ij \in E_s$ such that $k \in D_s(ij)$.

The lemma asserts the following.

**4.1** *Let $t > 0$ be an integer, let $s = 3t + 1$, and for $0 \leq i < s$ let $n_i \in \mathbf{R}_+$. Then there exists $k$ with $0 \leq k < s$ such that*

$$\sum_{ij \in C_s(k)} n_i n_j \leq \frac{1}{2} \sum_{\{i,j\} \in F_s} n_i n_j.$$

**Proof.** Let $Q_s$ be the set of all sequences $(n_0, \dots, n_{s-1})$ of members of $\mathbf{R}_+$. We say that $(n_0, \dots, n_{s-1}) \in Q_s$ is *good* if there exists $k$ with $0 \leq k < s$ such that

$$\sum_{ij \in C_s(k)} n_i n_j \leq \frac{1}{2} \sum_{\{i,j\} \in F_s} n_i n_j.$$

Thus we must show that every member of $Q_s$ is good. We prove this by induction on $t$.

(1) *If $t = 1$ then every member of $Q_s$ is good.*

For suppose that $t = 1$. Let $(n_0, n_1, n_2, n_3) \in Q_s$; we must show that there exists $k$ with $0 \le k \le 3$ such that $n_k n_{k+1} \le \frac{1}{2}(n_0 n_2 + n_1 n_3)$. But

$$\min(n_0 n_1, n_2 n_3)^2 \le n_0 n_1 n_2 n_3 \le n_0 n_1 n_2 n_3 + \frac{1}{4}(n_0 n_2 - n_1 n_3)^2 = \frac{1}{4}(n_0 n_2 + n_1 n_3)^2$$

and the claim follows. This proves (1).

Henceforth we assume that $t > 1$.

(2) *If $(n_0, \ldots, n_{s-1}) \in Q_s$ and some $n_i = 0$ then $(n_0, \ldots, n_{s-1})$ is good.*

For we may assume that $n_0 = 0$, from the symmetry. Define $m_i$ for $0 \le i \le 3t - 3$ as follows.

$$\begin{aligned}
m_0 &= n_{3t}; \\
m_i &= n_i \text{ for } 1 \le i \le t-1; \\
m_t &= n_t + n_{t+1}; \\
m_i &= n_{i+1} \text{ for } t+1 \le i \le 2t-2; \\
m_{2t-1} &= n_{2t} + n_{2t+1}; \\
m_i &= n_{i+2} \text{ for } 2t \le i \le 3t-3.
\end{aligned}$$

From the inductive hypothesis and since $t > 1$, the sequence $(m_0, \ldots, m_{3t-3}) \in Q_{s-3}$ satisfies the theorem, and so there exists $k'$ with $0 \le k' < s - 3$ such that

$$\sum_{ij \in C_{s-3}(k')} m_i m_j \le \frac{1}{2} \sum_{\{i,j\} \in F_{s-3}} m_i m_j.$$

If $0 \le k' < t$, let $k = k'$; if $t \le k' < 2t - 1$, let $k = k' + 1$; and if $2t - 1 \le k' \le 3t - 3$, let $k = k' + 2$. Since $n_0 = 0$, in each case it follows easily (we leave checking this to the reader) that

$$\sum_{ij \in C_s(k)} n_i n_j \le \sum_{ij \in C_{s-3}(k')} m_i m_j.$$

But

$$\sum_{\{i,j\} \in F_{s-3}} m_i m_j = \sum_{\{i,j\} \in F_s} n_i n_j - n_t n_{2t+1} \le \sum_{\{i,j\} \in F_s} n_i n_j,$$

as we can check by rewriting the left side in terms of the $n_i$'s and expanding and using that $n_0 = 0$. Consequently,

$$\sum_{ij \in C_s(k)} n_i n_j \le \sum_{ij \in C_{s-3}(k')} m_i m_j \le \frac{1}{2} \sum_{\{i,j\} \in F_{s-3}} m_i m_j \le \frac{1}{2} \sum_{\{i,j\} \in F_s} n_i n_j,$$

and so $(n_0, \dots, n_{s-1})$ is good. This proves (2).

(3) *Let* $(n_0, \dots, n_{s-1}) \in Q_s$*, such that*

$$\sum_{ij \in C_s(3t)} n_i n_j \le \sum_{ij \in C_s(k)} n_i n_j$$

*for all* $k$ *with* $0 \le k \le 3t$*. Then*

$$\sum_{0 \le i < t} (t-i)(n_{3t-i} + n_i) \le \frac{1}{2} t \sum_{0 \le i < s} n_i.$$

For let $0 \le k \le t-1$. For $0 \le i \le k$, define

$$a_i = \sum_{k+1 \le j \le i+t} n_j - \sum_{i+2t+1 \le j \le 3t} n_j.$$

Then

$$\sum_{ij \in C_s(k)} n_i n_j - \sum_{ij \in C_s(3t)} n_i n_j = \sum_{1 \le i \le k} a_i n_i.$$

Since the left side of this is nonnegative, and $a_0 \le a_1 \le \cdots \le a_k$, it follows that $a_k \ge 0$, that is,

$$\sum_{k+1 \le j \le k+t} n_j - \sum_{k+2t+1 \le j \le 3t} n_j \ge 0.$$

Similarly, for $2t+1+k \le i \le 3t$ let

$$b_i = \sum_{i-t \le j \le 2t+k} n_j - \sum_{0 \le j \le i-2t-1} n_j;$$

then

$$\sum_{ij \in C_s(2t+k)} n_i n_j - \sum_{ij \in C_s(3t)} n_i n_j = \sum_{2t+1+k \le i \le 3t} b_i n_i.$$

Since $b_{3t} \le b_{3t-1} \le \cdots \le b_{2t+1+k}$, we deduce similarly that $b_{2t+1+k} \ge 0$, that is,

$$\sum_{t+1+k \le j \le 2t+k} n_j - \sum_{0 \le j \le k} n_j \ge 0.$$

Hence

$$\sum_{k+1 \le j \le k+t} n_j - \sum_{k+2t+1 \le j \le 3t} n_j + \sum_{k+t+1 \le j \le k+2t} n_j - \sum_{0 \le j \le k} n_j \ge 0,$$

that is,

$$\sum_{k+2t+1 \le j \le 3t} n_j + \sum_{0 \le j \le k} n_j \le \sum_{k+1 \le j \le k+2t} n_j.$$

But the sum of the left and right sides of this inequality equals $N$, where $N = \sum_{0 \le i \le 3t} n_i$, and so the left side is at most $\frac{1}{2}N$. Summing over all $k$ with $0 \le k \le t-1$, we deduce that

$$\sum_{0 \le i < t} (t-i)(n_{3t-i} + n_i) \le \frac{1}{2} N t.$$

This proves (3).

Now to complete the proof, let $(n_0, \ldots, n_{s-1}) \in Q_s$. Choose $h$ with $0 \leq h < s$ such that $n_h \leq n_i$ for all $i$ with $0 \leq i < s$. Let $n_h = x$, and for $0 \leq i < s$, define $m_i = n_i - x$. Thus $(m_0, \ldots, m_{s-1}) \in Q_s$. We may assume that

$$\sum_{ij \in C_s(3t)} m_i m_j \leq \sum_{ij \in C_s(k)} m_i m_j$$

for all $k$ with $0 \leq k \leq 3t$, by cyclically permuting $n_0, \ldots, n_{3t}$. By (2), $(m_0, \ldots, m_{s-1})$ is good, since $m_h = 0$. Hence

$$\sum_{ij \in C_s(3t)} m_i m_j \leq \frac{1}{2} \sum_{\{i,j\} \in F_s} m_i m_j.$$

But

$$\begin{aligned}
\sum_{ij \in C_s(3t)} n_i n_j &= \sum_{ij \in C_s(3t)} (m_i + x)(m_j + x) \\
&= \sum_{ij \in C_s(3t)} m_i m_j + \sum_{0 \leq k < t} x(t-k)(m_{3t-k} + m_k) + |C_s(3t)| x^2 \\
&\leq \sum_{ij \in C_s(3t)} m_i m_j + \frac{1}{2} xtM + \frac{1}{2} t(t+1) x^2,
\end{aligned}$$

by (3), where $M = \sum_{0 \leq i \leq 3t} m_i$. Moreover,

$$\begin{aligned}
\frac{1}{2} \sum_{\{i,j\} \in F_s} n_i n_j &= \frac{1}{2} \sum_{\{i,j\} \in F_s} (m_i + x)(m_j + x) \\
&= \frac{1}{2} \sum_{\{i,j\} \in F_s} m_i m_j + \frac{1}{2} xtM + \frac{1}{4} stx^2 \\
&\geq \sum_{ij \in C_s(3t)} m_i m_j + \frac{1}{2} xtM + \frac{1}{4} stx^2 \\
&\geq \sum_{ij \in C_s(3t)} n_i n_j - (\frac{1}{2} xtM + \frac{1}{2} t(t+1) x^2) + (\frac{1}{2} xtM + \frac{1}{4} stx^2) \\
&\geq \sum_{ij \in C_s(3t)} n_i n_j.
\end{aligned}$$

It follows that $(n_0, \ldots, n_{3t})$ is good. This completes the proof of 4.1. ∎

# 5 Circular interval digraphs

We say that a digraph $G$ is a *circular interval digraph* if its vertices can be arranged in a circle such that for every triple $u, v, w$ of distinct vertices, if $u, v, w$ are in clockwise order and $uw \in E(G)$, then $uv, vw \in E(G)$. This is equivalent to saying that the vertex set of $G$ can be numbered as $v_1, \ldots, v_n$

such that for $1 \le i \le n$, the set of outneighbours of $v_i$ is $\{v_{i+1}, \dots, v_{i+a}\}$ for some $a \ge 0$, and the set of inneighbours of $v_i$ is $\{v_{i-b}, \dots, v_{i-1}\}$ for some $b \ge 0$, reading subscripts modulo $n$. The examples given earlier to show that conjecture 1.2 is tight infinitely often are circular interval digraphs. The main result of this section is:

**5.1** *$\beta(G) \le \frac{1}{2}\gamma(G)$ for every 3-free circular interval digraph.*

First we need a couple of lemmas. Here is a special kind of circular interval graph. Let $t \ge 1$ be an integer, let $n_0, \dots, n_{3t} \ge 0$ be integers, and let $n = \sum_{0 \le k \le 3t} n_i$. Let $N_0, \dots, N_{3t}$ be disjoint sets of cardinalities $n_0, \dots, n_{3t}$ respectively, and let $N = N_0 \cup \dots \cup N_{3t}$. Let $N = \{v_1, \dots, v_n\}$, where

$$N_i = \{v_j \ : \ n_0 + n_1 + \dots + n_{i-1} < j \le n_0 + n_1 + \dots + n_{i-1} + n_i\}.$$

Let $G$ be a digraph with vertex set $N$ and adjacency as follows.

- for $0 \le k \le 3t$, if $i < j$ and $v_i, v_j \in N_k$ then $v_i v_j \in E(G)$
- for $0 \le h \le 3t$ and $k \in \{(h+i) \bmod n \,; 1 \le i \le t\}$, every vertex in $N_h$ is adjacent to every vertex in $N_k$.

In this case $G$ is a circular interval graph, and we denote it by $G(n_0, \dots, n_{3t})$. We observe

**5.2** *For all $t \ge 1$ and all choices of $n_0, \dots, n_{3t} \ge 0$, if $G = G(n_0, \dots, n_{3t})$ then $\beta(G) \le \frac{1}{2}\gamma(G)$.*

**Proof.** By 4.1, there exists $k$ with $0 \le k \le 3t$ such that

$$\sum_{ij \in C_s(k)} n_i n_j \le \frac{1}{2} \sum_{\{i,j\} \in F_s} n_i n_j,$$

with notation as in 4.1. But the left side of this is at least $\beta(G)$, since every directed cycle of $G$ contains an edge $uv$ with $u \in N_i$ and $v \in N_j$ for some $ij \in C_s(k)$; and the right side equals $\frac{1}{2}\gamma(G)$. This proves 5.2. ∎

Let us say a 3-free circular interval digraph is *maximal* if there is no pair $u, v$ of nonadjacent distinct vertices such that adding the edge $uv$ results in a 3-free circular interval digraph.

**5.3** *Let $G$ be a maximal 3-free circular interval graph. Then either $G$ is a transitive tournament, or $G$ is isomorphic to $G(n_0, \dots, n_{3t})$ for some choice of $t, n_0, \dots, n_{3t}$.*

**Proof.** Let the vertices of $G$ be $v_1, \dots, v_n$, numbered as in the definition of a circular interval digraph, and throughout we read these subscripts modulo $n$. For each vertex $v$, let $N^+(v), N^-(v)$ denote the set of outneighbours and inneighbours of $v$, respectively.

(1) *If $N^-(v) = \emptyset$ or $N^+(v) = \emptyset$ for some vertex $v$, then $G$ is a transitive tournament.*

For suppose that $N^-(v) = \emptyset$ for some vertex $v$, say $v_1$. If $v_k v_j \in E(G)$ for some $j, k$ with $1 \le j < k \le n$, then $j > 1$ and $v_1, v_j, v_k$ are in clockwise order, and therefore $v_k v_1 \in E(G)$, a contradiction. Thus $G$ is acyclic; suppose it is not a tournament. Choose $i, j$ with $1 \le i < j \le n$ with $j - i$ minimum such that $v_i v_j \notin E(G)$, and let $G'$ be obtained from $G$ by adding the edge $v_i v_j$.

Then $G'$ is a 3-free circular interval digraph, a contradiction. Thus $G$ is a tournament, and hence a transitive tournament since it is 3-free. Similarly if $N^+(v) = \emptyset$ for some vertex $v$, then $G$ is a transitive tournament. This proves (1).

We may therefore assume that $v_iv_{i+1} \in E(G)$ for $1 \leq i \leq n$. Let us say that $X \subseteq V(G)$ is a *cluster* if $X$ is nonempty, every two vertices in $X$ are adjacent, $X$ can be written in the form $\{v_a, v_{a+1}, \ldots, v_b\}$ for some $a, b$, and for every vertex $v \notin X$, either $X \subseteq N^+(v)$, or $X \subseteq N^-(v)$, or $X \cap (N^+(v) \cup N^-(v)) = \emptyset$.

(2) *For $1 \leq i \leq n$, if $\{v_i, v_{i+1}\}$ is not a cluster, then $N^+(v_{i+1}) \not\subseteq N^+(v_i)$ and $N^-(v_i) \not\subseteq N^-(v_{i+1})$.*

For certainly $v_iv_{i+1} \in E(G)$. Let $N^+(v_i) = \{v_{i+1}, \ldots, v_{i+a}\}$, where $a \geq 1$. Suppose that $N^+(v_{i+1}) \subseteq N^+(v_i)$. Then $N^+(v_{i+1}) = \{v_{i+2}, \ldots, v_{i+a}\}$. Let the set of inneighbours of $v_i$ be $\{v_{i-b}, \ldots, v_{i-1}\}$, where $b \geq 1$, and let the set of inneighbours of $v_{i+1}$ be $\{v_{i-c}, \ldots, v_i\}$. Thus $c \leq b$; suppose that $c < b$. Then $v_{i-c-1}v_{i+1} \notin E(G)$, and also $v_{i+1}v_{i-c-1} \notin E(G)$ since $G$ is 3-free and $v_{i-c-1}v_i, v_iv_{i+1} \in E(G)$. Since $v_{i-c-1}v_i, v_{i-c}v_{i+1} \in E(G)$, it follows that $v_{i-c-1}v_h, v_hv_{i+1} \in E(G)$ for all $h \in \{(i-k) \bmod n0 \leq k \leq c\}$. Consequently, the digraph $G'$ obtained from $G$ by adding the edge $v_{i-c-1}v_{i+1}$ is a circular interval digraph. From the maximality of $G$, $G'$ is not 3-free, and so there exists $u \in N^+(v_{i+1}) \cap N^-(v_{i-c-1})$; and therefore $u \in N^+(v_i) \cap N^-(v_{i-c-1})$, which is impossible since $G$ is 3-free. This proves that $c = b$, and so $\{v_i, v_{i+1}\}$ is a cluster. Similarly if $N^-(v_i) \subseteq N^-(v_{i+1})$ then $\{v_i, v_{i+1}\}$ is a cluster. This proves (2).

If $X, Y$ are clusters with $X \cap Y \neq \emptyset$, it follows easily that $X \cup Y$ is a cluster. Consequently every two maximal clusters are disjoint. Since $\{v\}$ is a cluster for every vertex $v$, it follows that the maximal clusters form a partition of $V(G)$. Let the maximal clusters be $N_0, \ldots, N_{s-1}$ say, numbered in their natural circular order, and let $|N_i| = n_i$ for $0 \leq i < s$. From the definition of a cluster, if $X, Y$ are disjoint clusters and there exists $xy \in E(G)$ with $x \in X$ and $y \in Y$, then $xy \in E(G)$ for all $x \in X$ and $y \in Y$; we denote this by $X \to Y$. For $0 \leq h < s$, let $T_h$ be the set of all $k \in \{0, \ldots, s-1\} \setminus \{h\}$ such that $N_h \to N_k$; then $T_h = \{(h+i) \bmod s \ : 1 \leq i \leq t_h\}$ say, for some $t_h \geq 0$. Choose $h$ with $0 \leq h < s$, and choose $i$ such that $v_i \in N_h$ and $v_{i+1} \in N_{h+1}$. Since $\{v_i, v_{i+1}\}$ is not a cluster (because maximal clusters are disjoint), it follows from (2) that $N^+(v_{i+1}) \not\subseteq N^+(v_i)$, and so $t_{i+1} \geq t_i$. Since this holds for all choices of $i$, and $t_0 \geq t_{s-1}$, we deduce that $t_0 = t_1 = \cdots = t_{s-1} = t$ say. We claim that $s = 3t + 1$. For $s \geq 3t + 1$ since $G$ is 3-free; let us prove the reverse inequality. Let $i = n_0$ and $j = n_0 + \cdots + n_t + 1$; thus $v_i \in N_0, v_{i+1} \in N_1$, $v_{j-1} \in N_t$ and $v_j \in N_{t+1}$. Since $G$ is maximal and so adding the edge $v_iv_j$ does not result in a 3-free circular interval digraph, it follows that there exists $k$ such that $v_jv_k, v_kv_i \in E(G)$, and therefore there exists $q$ such that $q \in T_{t+1}$ and $0 \in T_q$. Hence $q - (t+1) \leq t$ and $s - q \leq t$; and so $s \leq 3t + 1$. This proves that $s = 3t + 1$, and so $G$ is isomorphic to $G(n_0, \ldots, n_{3t})$. This proves 5.3. ∎

**Proof of 5.1.** We proceed by induction on $\gamma(G)$. Suppose that $G$ is not a maximal 3-free circular interval graph. Then we can add an edge to $G$ forming a 3-free circular interval graph $G'$; and $\gamma(G') = \gamma(G) - 1$, so $\beta(G') \leq \frac{1}{2}\gamma(G')$ from the inductive hypothesis. Then

$$\beta(G) \leq \beta(G') \leq \frac{1}{2}\gamma(G') \leq \frac{1}{2}\gamma(G)$$

as required.

Thus we may assume that $G$ is maximal, and we may assume that $G$ is not a transitive tournament. From 5.3 and 5.2, this proves 5.1. ∎

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