A counterexample to a conjecture of Schwartz

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August 25, 2011; revised December 2, 2011

[1]Supported by DFG grants BR 2312/7-1 and BR 2312/10-1.
[2]Supported by NSF grants DMS-1001091 and IIS-1117631.
[3]Supported by ONR grant N00014-10-1-0680 and NSF grant DMS-0070912.

Abstract

In 1990, motivated by applications in the social sciences, Thomas Schwartz made a conjecture about tournaments which would have had numerous attractive consequences. In particular, it implied that there is no tournament with a partition A, B of its vertex set, such that every transitive subset of A is in the out-neighbour set of some vertex in B, and vice versa. But in fact there is such a tournament, as we show in this paper, and so Schwartz' conjecture is false. Our proof is non-constructive and uses the probabilistic method.

1 Introduction

The goal of this paper is to disprove a popular conjecture of Schwartz [12], but before that we need to introduce some terminology and notation. If G is a tournament, let $V(G)$ and $N_G^-(v) = N^-(v)$ denote respectively the set of vertices of G and the set of in-neighbours of vertex $v \in V(G)$. Suppose that ϕ is a function such that $\phi(H)$ is defined and satisfies $\phi(H) \subseteq V(H)$ for every non-null proper subtournament H of G. We say a subset $A \subseteq V(G)$ is *ϕ-retentive* if $A \neq \emptyset$ and $\phi(G|N^-(a)) \subseteq A$ for each $a \in A$.

Let $\mathcal{G}$ be the class of all non-null finite tournaments. A *tournament solution* is a function ϕ with domain $\mathcal{G}$, and with $\emptyset \neq \phi(G) \subseteq V(G)$ for each $G \in \mathcal{G}$. Let τ be the tournament solution defined inductively as follows. Assume that $\tau(G)$ is defined for all non-null proper subtournaments of G. Then $\tau(G)$ is the union of all minimal τ-retentive subsets of $V(G)$. (We see that $\tau(G)$ is nonempty, since $V(G)$ is τ-retentive.) $\tau(G)$ is called the *tournament equilibrium set*.

In 1990, Thomas Schwartz [12] proposed the following conjecture.

1.1 (Schwartz' conjecture.) *In every non-null tournament there is a unique minimal τ-retentive set.*

In this paper, we give a counterexample to Schwartz' conjecture (with about 10^{136} vertices). Indeed, we give a series of weakenings of Schwartz' conjecture, and disprove the weakest.

2 Background

Tournament solutions are of great interest in social choice theory, where tournaments are induced by pairwise majority comparisons and various tournament solutions have been proposed in the literature [11]. Schwartz—a political scientist—motivated τ using a well-defined cooperative recontracting process.

Over the years, Schwartz' conjecture has been extensively studied [6, 10, 11, 8, 2, 3]. For instance, it is known that Schwartz' conjecture is equivalent to τ having any one of several desirable properties of tournament solutions, including monotonicity, independence of unchosen alternatives, and the "strong superset property". These equivalences were shown by induction on the tournament order and imply that if Schwartz' conjecture holds for all tournaments with at most n vertices, then τ has all the above-mentioned properties in these tournaments.

A strengthening of Schwartz' conjecture was disproved by Houy, who found a counterexample with 11 vertices [8]. By means of an exhaustive computer analysis, this counterexample was later shown to be of minimum cardinality [3]. The same analysis did not yield a counterexample to Schwartz' conjecture itself in all tournaments with less than 13 vertices and billions of random tournaments with up to 50 vertices. (Counterexamples to Houy's strengthening were encountered quite frequently during this random search.) Brandt et al. studied retentiveness for different underlying tournament solutions and proved a weaker variant of Schwartz' conjecture [4].

Recently, Brandt proposed a conjecture on tournaments and showed that it is implied by Schwartz' conjecture [2]. Two weaker variants of this conjecture have been proved by Dutta [5] and Brandt, respectively. The latter statement was shown by reducing it to a large, but finite, number of cases that were checked using a computer. It is easy to see that Brandt's conjecture implies the first weakening we disprove (3.2) and is therefore also false.

3 Results

A subset X of the vertex set of a tournament G is *transitive* if it can be ordered $X = \{x_1, \ldots, x_n\}$ such that $x_i x_j$ is an edge for all i, j with $1 \leq i < j \leq n$; and if so, x_1 is the *source* of X. For a tournament G, let $\beta(G)$ be the set of all vertices v of G such that v is the source of some maximal transitive subset of $V(G)$. Then β is a tournament solution. (This is called the *Banks set*, after Jeffrey S. Banks [1].)

We need the following lemma of Schwartz [12], and we give the proof for the reader's convenience.

3.1 *$\tau(G) \subseteq \beta(G)$, and every β-retentive subset of $V(G)$ is τ-retentive, for every tournament G.*

Proof. We prove the first assertion by induction on $|V(G)|$. Let $x \in \tau(G)$; we must show that $x \in \beta(G)$. If $N^-(x) = \emptyset$, then x belongs to $\beta(G)$ as required, so we may assume that $N^-(x)$ is nonempty. Consequently $\tau(G|N^-(x))$ is nonempty; choose $w \in \tau(G|N^-(x))$. Let A be a minimal τ-retentive set containing x. It follows that $w \in A$, and so $A \backslash \{x\}$ is nonempty. From the minimality of A, it follows that $A \backslash \{x\}$ is not τ-retentive, and so there exists $y \in A \backslash \{x\}$ such that $x \in \tau(G|N^-(y))$.

From the inductive hypothesis, $\tau(G|N^-(y)) \subseteq \beta(G|N^-(y))$, and so there is a maximal transitive subset X_0 of $N^-(y)$ with source x. Thus $X_0 \cup \{y\}$ is transitive; let X be a maximal transitive subset of $V(G)$ including $X_0 \cup \{y\}$. It follows from the maximality of X_0 that no vertex of $X \setminus X_0$ belongs to $N^-(y)$, and so every vertex in $X \setminus X_0$ different from y is an out-neighbour of y and hence of x. Consequently x is the source of X, and so $x \in \beta(G)$. This proves the first assertion.

For the second assertion, let $A \subseteq V(G)$ be β-retentive, and let $a \in A$. From the first assertion, $\tau(G|N^-(a)) \subseteq \beta(G|N^-(a))$; and since A is β-retentive, $\beta(G|N^-(a)) \subseteq A$. Thus $\tau(G|N^-(a)) \subseteq A$, and so A is τ-retentive. This proves the second assertion, and so proves 3.1. ∎

Our first weakening of 1.1 is:

3.2 (First weakening.) *In every tournament G, every two β-retentive sets intersect.*

Proof that 1.1 implies 3.2. Let A_1, A_2 be β-retentive subsets of $V(G)$. By 3.1, A_1, A_2 are both τ-retentive, and hence both include a minimal τ-retentive set. Since there is only one such set by 1.1, and it is nonempty, it follows that $A_1 \cap A_2 \neq \emptyset$. This proves 3.2. ∎

If T is a subset of $V(G)$ where G is a tournament, we say that $v \in V(G) \setminus T$ *dominates* T if $vt \in E(G)$ for every $t \in T$, and if no such a vertex v exists, we say that T is *undominated in* G.

3.3 (Second weakening.) *Let (A, B) be a partition of the vertex set of a tournament G. Then one of A, B includes a transitive subset which is undominated in G.*

Proof that 3.2 implies 3.3. Assume that 3.2 holds, let G be a tournament and let (A, B) be a partition of $V(G)$. Take a second copy G' of G on a disjoint vertex set, and let (A', B') be the corresponding partition. Now make a tournament H from the disjoint union of G, G' as follows; for $v \in V(G)$ and $v' \in V(G')$, let $v'v \in E(H)$ if either $v \in A$ and $v' \in A'$, or $v \in B$ and $v' \in B'$; and otherwise let $vv' \in E(H)$.

We apply 3.2 to H, and deduce that one of $V(G), V(G')$ is not β-retentive in H, and from the symmetry we may assume that $V(G)$ is not β-retentive in H. Consequently, there exists $v \in V(G)$, and a maximal transitive subset T of $N^-_H(v)$, with source some $u \in V(G')$. From the symmetry we may assume that $v \in A$. It follows that $T \cap V(G') \subseteq A'$, since every vertex of $T \cap V(G')$ is an in-neighbour of v. In particular, $u \in A'$. Since u is the source of T, similarly every vertex of $T \cap V(G)$ belongs to A. Let $X = (T \cup \{v\}) \cap V(G)$. Suppose that some $x \in V(G) \setminus X$ dominates X. Since $T \cap V(G') \subseteq A'$, either $xy \in E(H)$ for all $y \in T \cap V(G')$, or $yx \in E(H)$ for all $y \in T \cap V(G')$, and in either case $T \cup \{x\}$ is a transitive subset of $N^-_H(v)$, contrary to the maximality of T. Thus X is undominated in G. This proves 3.3. ∎

Now, we give a counterexample to 3.3, which therefore provides a counterexample to all the previous conjectures. The idea is somewhat related to a proof by Laffond and Laslier [9]. We need the following lemma, due to Erdős and Moser [7] (logarithms are to base two), and we include a proof, for the reader's convenience.

3.4 *For every integer $n \geq 2$ there is a tournament with n vertices in which every transitive subset has cardinality less than $1 + 2\log(n)$.*

Proof. Let k be the smallest integer at least $1 + 2\log(n)$. Take a set V of n vertices, and for each pair $\{u, v\}$ of distinct members of V, make one of uv, vu an edge, independently with probability $1/2$, forming a tournament G. Let Q denote the expected number of subsets of V of cardinality k that are transitive in G. For every sequence $x_1, \dots, x_k$ of k distinct members of V, the probability that $x_i x_j$ is an edge for all i, j with $1 \leq i < j \leq k$ is $2^{-k(k-1)/2}$. Since there are fewer than n^k such sequences (because $k > 1$), it follows that

$$Q < n^k 2^{-k(k-1)/2} \leq 1.$$

Consequently there is a positive probability that G has no transitive subset of cardinality k. This proves the lemma. ∎

Now for the counterexample. Let k be a positive even integer large enough that $2^{k/2} > k^3$ (for instance, $k = 30$), and let $n = 2^{k/2}$. By 3.4, there is a tournament G_1 with n vertices, in which every transitive subset has cardinality less than $1 + 2\log(n) = k + 1$, and consequently at most k. Let $A = V(G_1)$. For each transitive subset $X \subseteq A$, let v_X be a new vertex, and let B be the set of all these new vertices. So $|B| \leq n^k$, and therefore $2\log(|B|) \leq 2k\log(n) = k^2$.

By 3.4, there is a tournament G_2 with vertex set B in which every transitive subset has cardinality less than $1 + 2\log|B|$, and hence at most k^2. We construct a tournament G from the disjoint union of G_1 and G_2 as follows. For each $a \in A$ and each $b \in B$, let $ba \in E(G)$ if $a \in X$, where $X \subseteq A$ is the transitive subset of A with $b = v_X$, and let $ab \in E(G)$ otherwise. We observe:

- Every transitive subset X of A is dominated in G; because $v_X \in B$ dominates X.
- Every transitive subset Y of B is dominated in G. To see this, note first that $|Y| \leq k^2$, and since each vertex in Y has at most k out-neighbours in A, it follows that there are at most $k^3 < n$ vertices in A that are adjacent from some vertex in Y. Consequently some vertex in A dominates Y.

It follows that G, A, B do not satisfy 3.3.

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