Testing Branch-width

Sang-il Oum[*†]
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia USA

Paul Seymour[‡§]
Department of Mathematics
Princeton University
Princeton, New Jersey USA

September 25, 2013

Abstract

An integer-valued function f on the set 2^V of all subsets of a finite set V is a *connectivity function* if it satisfies the following conditions: (1) $f(X)+f(Y) \geq f(X\cap Y)+f(X\cup Y)$ for all subsets X, Y of V, (2) $f(X)=f(V\setminus X)$ for all $X \subseteq V$, and (3) $f(\emptyset)=0$. Branch-width is defined for graphs, matroids, and more generally, connectivity functions. We show that for each constant k, there is a polynomial-time (in $|V|$) algorithm to decide whether the branch-width of a connectivity function f is at most k, if f is given by an oracle. This algorithm can be applied to branch-width, carving-width, and rank-width of graphs. In particular, we can recognize matroids $\mathcal{M}$ of branch-width at most k in polynomial (in $|E(\mathcal{M})|$) time if the matroid is given by an independence oracle.

1 Introduction

Branch-width (for graphs) was defined by Robertson and Seymour [5]. We will define the more general *branch-width* of *connectivity functions* later in Section 2. One natural question is the following.

> Let k be a constant and let V be a finite set. Can we decide in polynomial time whether the branch-width of a connectivity function $f : 2^V \to \mathbb{Z}$ is at most k?

[*]sangil@math.gatech.edu
[†]The first author was partially supported by NSF grant 0354742.
[‡]pds@math.princeton.edu
[§]The second author was supported by ONR grant N00014-04-1-0062 and NSF grant DMS03-54465.

(We assume that f is presented by an oracle.)

We answer this question completely. We show that, for fixed k, there is a polynomial-time (in $|V|$) algorithm to decide whether the branch-width of a connectivity function f is at most k. If γ is the time to compute $f(X)$ for any set X, then our algorithm runs in time $O(\gamma n^{8k+6} \log n)$.

There have been answers for our problem for a few connectivity functions separately. We summarize them in Table 1. Our result unifies all algorithms listed in Table 1, but our algorithm is slightly weaker because it is not fixed parameter tractable.

In particular, it was open whether there exists a polynomial-time algorithm that decides whether a matroid (given by an independence oracle) has branch-width at most k for fixed k. Hliněný [2] showed an $O(|E(\mathcal{M})|^3)$-time algorithm to decide whether branch-width is at most k for matroids represented over a fixed finite field.

In Section 6, we provide a polynomial-time algorithm to output a branch-decomposition of width at most k if one exists. We use the above algorithm as a subroutine. We remark that no such algorithms were known for rank-decompositions of graphs or branch-decompositions of matroids.

Object	Results
Branch-width of graphs	Linear time [1]
Carving-width of graphs	Linear time [7]
Branch-width of matroids $\mathcal{M}$ represented over a fixed finite field	$O(\lvert E(\mathcal{M})\rvert^3)$-time[1] [2]
Rank-width of graphs G	$O(\lvert V(G)\rvert^3)$-time [4]

Table 1: Algorithms for deciding branch-width $\leq k$ for fixed k

2 Definitions

Let us write $\mathbb{Z}$ to denote the set of integers. Let V be a finite set. We write 2^V to denote the set of all subsets of V. If a function $f : 2^V \to \mathbb{Z}$ satisfies

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

for all $X, Y \subseteq V$, then f is said to be *submodular*. If f satisfies $f(X) = f(V \setminus X)$ for all $X \subseteq V$, then f is said to be *symmetric*. An integer-valued symmetric submodular function f is called a *connectivity function* if $f(\emptyset) = 0$.

[1] The input is given by the matrix representation of matroids.

A *subcubic tree* is a tree with at least two vertices such that every vertex is incident with at most three edges. A *leaf* of a tree is a vertex incident with exactly one edge. We call $(T, \mathcal{L})$ a *branch-decomposition* of a symmetric submodular function f if T is a subcubic tree and $\mathcal{L} : V \to \{t : t \text{ is a leaf of } T\}$ is a bijective function. (If $|V| \leq 1$ then f admits no branch-decomposition.)

For an edge e of T, the connected components of $T \setminus e$ induce a partition (X, Y) of the set of leaves of T. The *width* of an edge e of a branch-decomposition $(T, \mathcal{L})$ is $f(\mathcal{L}^{-1}(X))$. The *width* of $(T, \mathcal{L})$ is the maximum width of all edges of T. The *branch-width* of f is the minimum width of a branch-decomposition of f. (If $|V| \leq 1$, we define that the branch-width of f is $f(\emptyset)$.)

For a connectivity function f on 2^V and disjoint subsets A, B of V, we define

$$f_{\min}(A, B) = \min_{A \subseteq Z \subseteq V \setminus B} f(Z).$$

We present several lemmas on connectivity functions, which will be used later.

Lemma 1. *Let A, B, C, D be subsets of V such that $A \cap B = C \cap D = \emptyset$. For a connectivity function f on 2^V,*

$$f_{\min}(A, B) + f_{\min}(C, D) \geq f_{\min}(A \cap C, B \cup D) + f_{\min}(A \cup C, B \cap D).$$

Proof. Let S be a subset of V such that $A \subseteq S \subseteq V \setminus B$ and $f(S) = f_{\min}(A, B)$. Let T be a subset of V such that $C \subseteq T \subseteq V \setminus D$ and $f(T) = f_{\min}(C, D)$. By the submodularity of f, we deduce

$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$$

and moreover $f(S \cap T) \geq f_{\min}(A \cap C, B \cup D)$ and $f(S \cup T) \geq f_{\min}(A \cup C, B \cap D)$. $\square$

Lemma 2. *Let $g : 2^V \to \mathbb{Z}$ be a submodular function such that $g(\emptyset) = 0$ and $g(X) \leq g(Y)$ if $X \subseteq Y$. For all $X \subseteq V$, there exists a subset A of X such that $|A| \leq g(X)$ and $g(A) = g(X)$.*

Proof. We proceed by induction on $|X|$. If $X = \emptyset$, then it is trivial.

Suppose $|X| = k > 0$. We assume that this lemma is true when $|X| < k$. Let A be the minimal subset of X such that $g(A) = g(X)$. Since $g(\emptyset) = 0$, $A \neq \emptyset$. Let v be an element of A maximizing $g(A \setminus \{v\})$. By our assumption, $g(A \setminus \{v\}) \leq k - 1$.

By the induction hypothesis, there exists a subset B of $A \setminus \{v\}$ such that $|B| \leq k - 1$ and $g(B) = g(A \setminus \{v\})$. If $B = A \setminus \{v\}$, then $|A| \leq k$ and therefore we are done. Thus we may assume that $B \neq A \setminus \{v\}$ and thus there exists $w \in (A \setminus \{v\}) \setminus B$. By the choice of v, we know that $g(A \setminus \{w\}) \leq g(A \setminus \{v\})$. Since $B \subseteq A \setminus \{w\}$, we deduce that $g(A \setminus \{v\}) = g(B) \leq g(A \setminus \{w\})$. Therefore

$$g(A \setminus \{v\}) = g(A \setminus \{w\}).$$

Moreover, $g(A \setminus \{v, w\}) = g(A \setminus \{v\})$ because $g(B) \leq g(A \setminus \{v, w\}) \leq g(A \setminus \{v\})$. Now let us apply the submodular inequality.

$$g(A \setminus \{v\}) + g(A \setminus \{w\}) \geq g(A \setminus \{v, w\}) + g(A) \geq g(A \setminus \{v\}) + k.$$

We deduce that $g(A \setminus \{v\}) \geq k$, a contradiction. $\square$

Lemma 3. *For a connectivity function f on 2^V and a subset Z of V, there exist a subset A of Z and a subset B of $V \setminus Z$ such that* $\max(|A|, |B|) \leq f_{\min}(A, B) = f(Z)$.

Proof. For a subset X of Z, let $g_1(X) = f_{\min}(X, V \setminus Z)$. By Lemma 1, $g_1(X) + g_1(Y) \geq g_1(X \cap Y) + g_1(X \cup Y)$ for two subsets X, Y of Z. In addition, $0 \leq g_1(\emptyset) \leq f(\emptyset) = 0$ and $g_1(X) \leq g_1(Y)$ if $X \subseteq Y \subseteq Z$. By Lemma 2, there exists a subset A of Z such that

$$|A| \leq g_1(Z) = f(Z) \text{ and } g_1(A) = f_{\min}(A, V \setminus Z) = f(Z).$$

For a subset X of $V \setminus Z$, let $g_2(X) = f_{\min}(A, X)$. It is again routine to show that g_2 satisfies all conditions of Lemma 2. Therefore there exists a subset B of $V \setminus Z$ such that

$$|B| \leq g_2(V \setminus Z) = f_{\min}(A, V \setminus Z) \text{ and } g_2(B) = f_{\min}(A, B) = f_{\min}(A, V \setminus Z) = f(Z).$$

Therefore $\max(|A|, |B|) \leq f_{\min}(A, B) = f(Z)$. $\square$

3 Loose Tangles

Let f be a connectivity function on 2^V. We wish to test whether the branch-width of f is at most k, but instead of searching for a branch-decomposition of small width directly, we search for a dual object called a *tangle*, introduced by Robertson and Seymour [5].

A set $\mathcal{T}$ of subsets of V is called an *f-tangle* of order $k+1$ if it satisfies the following three axioms.

(T1) For all $A \subseteq V$, if $f(A) \leq k$, then either $A \in \mathcal{T}$ or $V \setminus A \in \mathcal{T}$.

(T2) If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \neq V$.

(T3) For all $v \in V$, we have $V \setminus \{v\} \notin \mathcal{T}$.

Robertson and Seymour [5] showed that tangles are related to branch-width.

Theorem 4 (Robertson and Seymour [5])**.** *Let f be a connectivity function on 2^V. There is no f-tangle of order $k+1$ if and only if the branch-width of f is at most k.*

We introduce a relaxed notion of tangles, which we will call *loose tangles*. A *loose f-tangle* of order $k+1$ is a set $\mathcal{T}$ of subsets of V satisfying the following three axioms.

(L1) For a subset X of V, if $|X| \leq 1$ and $f(X) \leq k$, then $X \in \mathcal{T}$.

(L2) If $A, B \in \mathcal{T}$, $C \subseteq A \cup B$, and $f(C) \leq k$, then $C \in \mathcal{T}$.

(L3) $V \notin \mathcal{T}$.

Even though the definition of loose tangles looks weaker than that of tangles, we show that a loose tangle exists if and only if a tangle exists. We present a direct proof.

Theorem 5. *Let f be a connectivity function on 2^V. Then, no loose f-tangle of order $k+1$ exists if and only if the branch-width of f is at most k.*

Proof. A set $X \subseteq V$ is called *k-branched* if the connectivity system obtained from f by identifying $V \setminus X$ has branch-width at most k. (We assume that V is k-branched if and only if f has branch-width at most k.) Let $\mathcal{B}$ be the set of all k-branched subsets of V and let $\mathcal{B}' = \{X : X \subseteq Y, Y \in \mathcal{B}, f(X) \leq k\}$.

We claim that $\mathcal{B}'$ satisfies (L1) and (L2). (L1) is obvious. To see (L2), suppose that $A, B \in \mathcal{B}$ and $C \subseteq A \cup B$ such that $f(C) \leq k$. Pick Z such that $A \setminus B \subseteq Z \subseteq A$ and $f(Z)$ is minimum. We claim that Z and $B \setminus Z$ are k-branched. It is enough to show that for each subset Y of A (or B), if $f(Y) \leq k$ then $f(Y \cap Z) \leq k$ (or $f(Y \setminus Z) \leq k$ respectively). This follows from the submodular inequalities:

$$\begin{aligned} f(Y) + f(Z) &\geq f(Y \cap Z) + f(Y \cup Z) \geq f(Y \cap Z) + f(Z) && \text{if } Y \subseteq A, \text{ and} \\ f(Y) + f(Z) &\geq f(Y \setminus Z) + f(Z \setminus Y) \geq f(Y \setminus Z) + f(Z) && \text{if } Y \subseteq B. \end{aligned}$$

So Z and $B \setminus Z$ are both k-branched and therefore $Z \cup (B \setminus Z) = A \cup B$ is k-branched and we deduce $C \in \mathcal{B}'$.

Now let us prove our theorem. If the branch-width of f is greater than k, then $V \notin \mathcal{B}'$ and so $\mathcal{B}'$ is a loose f-tangle.

If the branch-width of f is at most k, then V is k-branched. It is easy to see that every k-branched set having at least two elements is a union of two proper subsets that are k-branched. By (L1) and (L2), every loose f-tangle should contain all k-branched sets. Since V is k-branched, there is no loose f-tangle. □

4 Loose Tangle Kits

We introduce *loose tangle kits*. A pair (P, μ) is called a *loose f-tangle kit* of order $k+1$ if

$$P = \{(A, B) : A, B \subseteq V, A \cap B = \emptyset, \max(|A|, |B|) \leq f_{\min}(A, B) \leq k\}$$

and $\mu : P \to 2^V$ is a function satisfying the following three axioms.

(M1) If $|X| \le 1$ and $f(X) \le k$, then there exists $(A, B) \in P$ such that $A \subseteq X \subseteq V \setminus B$, $f(X) = f_{\min}(A, B)$, and $X \subseteq \mu(A, B)$.

(M2) If $(A, B), (C, D), (E, F) \in P$, $E \subseteq X \subseteq (\mu(A, B) \cup \mu(C, D)) \setminus F$, and $f(X) = f_{\min}(E, F)$, then $X \subseteq \mu(E, F)$.

(M3) $\mu(\emptyset, \emptyset) \neq V$ if $(\emptyset, \emptyset) \in P$.

We will show that a loose f-tangle exists if and only if a loose f-tangle kit exists.

Theorem 6. *Let f be a connectivity function on 2^V. Then, a loose f-tangle of order $k+1$ exists if and only if a loose f-tangle kit of order $k+1$ exists.*

Proof. Suppose that $\mathcal{T}$ is a loose f-tangle of order $k+1$. We construct a loose f-tangle kit of order $k+1$ as follows. Let

$$P = \{(A, B) : A, B \subseteq V, A \cap B = \emptyset, \max(|A|, |B|) \le f_{\min}(A, B) \le k\}.$$

For each $(A, B) \in P$, let

$$\mathcal{T}_{A,B} = \{X : A \subseteq X \subseteq V \setminus B,\ f_{\min}(A, B) = f(X),\ \text{and } X \in \mathcal{T}\},$$
$$\mu(A, B) = \bigcup_{X \in \mathcal{T}_{A,B}} X. \qquad (\text{If } \mathcal{T}_{A,B} = \emptyset, \text{ then let } \mu(A, B) = \emptyset.)$$

Notice that $\mu(A, B)$ may be different from $\mu(B, A)$, even though f is symmetric.

First we show that if $(A, B) \in P$, then $\mu(A, B) \in \mathcal{T}$. Since $(A, B) \in P$, we have $f(\emptyset) = 0 \le f_{\min}(A, B) \le k$ and therefore $\emptyset \in \mathcal{T}$. So we may assume that $\mathcal{T}_{A,B} \neq \emptyset$. We claim that if $X, Y \in \mathcal{T}_{A,B}$, then $X \cup Y \in \mathcal{T}_{A,B}$. Since $2f_{\min}(A, B) = f(X) + f(Y) \ge f(X \cap Y) + f(X \cup Y)$ and $f(X \cap Y) \ge f_{\min}(A, B)$, $f(X \cup Y) \ge f_{\min}(A, B)$, we have $f(X \cup Y) = f_{\min}(A, B)$. By (L2), $X \cup Y \in \mathcal{T}_{A,B}$. We conclude that $\mu(A, B) \in \mathcal{T}_{A,B} \subseteq \mathcal{T}$.

We claim that (P, μ) is a loose f-tangle kit of order $k+1$. (M3) is trivial by (L3). To show (M2), suppose that $(A, B), (C, D), (E, F) \in P$, $E \subseteq X \subseteq (\mu(A, B) \cup \mu(C, D)) \setminus F$, and $f(X) = f_{\min}(E, F) \le k$. By (L2), $X \in \mathcal{T}$ and therefore $X \in \mathcal{T}_{E,F}$. So $X \subseteq \mu(E, F)$. Finally, to show (M1), let us assume that $|X| \le 1$ and $f(X) \le k$. By Lemma 3, there exists $(A, B) \in P$ such that $f_{\min}(A, B) = f(X)$ and $A \subseteq X \subseteq V \setminus B$. By (L1), $X \in \mathcal{T}$ and therefore $X \in \mathcal{T}_{A,B}$. Thus, $X \subseteq \mu(A, B)$. We conclude that (P, μ) is a loose f-tangle kit of order $k+1$.

Conversely, suppose that (P, μ) is a loose f-tangle kit of order $k+1$. We define

$$\mathcal{T} = \{X : \text{ there exists } (A, B) \in P \text{ such that } A \subseteq X \subseteq V \setminus B,\ f_{\min}(A, B) = f(X),\ \text{and } X \subseteq \mu(A, B)\}.$$

We claim that $\mathcal{T}$ is a loose f-tangle of order $k+1$. (L3) is trivial by (M3). To show (L2), suppose that $X, Y \in \mathcal{T}$, $Z \subseteq X \cup Y$, and $f(Z) \leq k$. By Lemma 3, there exists $(E, F) \in P$ such that $E \subseteq Z \subseteq V \setminus F$ and $f(Z) = f_{\min}(E, F)$. By the construction of $\mathcal{T}$, there are $(A, B), (C, D) \in P$ such that $X \subseteq \mu(A, B)$ and $Y \subseteq \mu(C, D)$. Then $E \subseteq Z \subseteq (\mu(A, B) \cup \mu(C, D)) \setminus F$ and therefore $Z \subseteq \mu(E, F)$. We conclude that $Z \in \mathcal{T}$. Now it remains to show (L1). Suppose that $|X| \leq 1$ and $f(X) \leq k$. By (M1), there exists $(A, B) \in P$ such that $A \subseteq X \subseteq V \setminus B$, $f(X) = f_{\min}(A, B)$, and $X \subseteq \mu(A, B)$. By the construction of $\mathcal{T}$, $X \in \mathcal{T}$. We conclude that $\mathcal{T}$ is indeed a loose f-tangle of order $k+1$. $\square$

5 Algorithms

Let f be a connectivity function on 2^V. We want to find a polynomial-time (in $|V|$) algorithm to decide whether the branch-width of f is at most k for fixed k, when f is given by an oracle. Instead of searching directly for a branch-decomposition of width at most k, we will search for a loose f-tangle kit of order $k+1$.

Algorithm 1. *Decide whether branch-width of f is at most k.*

(A1) *Construct* $P = \{(A, B) : A, B \subseteq V,\ A \cap B = \emptyset,\ \max(|A|, |B|) \leq f_{\min}(A, B) \leq k\}$.

(A2) *Let* $\mu(\emptyset, \emptyset) = \{v \in V : f(\{v\}) = 0\}$ *if* $(\emptyset, \emptyset) \in P$.

For each $v \in V$, *if* $0 < f(\{v\}) \leq k$, *then find a subset* B *of* $V \setminus \{v\}$ *such that* $|B| \leq f_{\min}(\{v\}, B) = f(\{v\})$. *Let* $\mu(\{v\}, B) = \{v\}$.

For all other $(A, B) \in P$, *let* $\mu(A, B) = \emptyset$.

(A3) *Test (M3).*

If it fails, then there is no loose f-tangle kit of order $k+1$. Stop.

(A4) *Test (M2).*

If it fails, then we have $(A, B), (C, D), (E, F) \in P$ *and* X *such that* $E \subseteq X \subseteq (\mu(A, B) \cup \mu(C, D)) \setminus F$, $f(X) = f_{\min}(E, F)$, *and* $X \not\subseteq \mu(E, F)$. *We make* $\mu(E, F)$ *to be* $\mu(E, F) \cup X$, *thus increasing* $|\mu(E, F)|$ *at least by* 1. *Go back to (A3).*

(A5) (P, μ) *is a loose f-tangle kit of order $k+1$. Stop.*

Let $n = |V|$. We claim that the running time of this algorithm is polynomial in n. We first note that $|P| \leq (\sum_{i=0}^{k} \binom{n}{i})^2 = O(n^{2k})$. (A1) can be done in polynomial (in $|V|$) time because we can evaluate $f_{\min}$ in polynomial time by using submodular function minimization algorithms [3, 6]. For (A2), for each v, we may enumerate all subsets B of $V \setminus \{v\}$ having at most $f(\{v\})$ elements such that $f_{\min}(\{v\}, B) = f(\{v\})$. There are at most $O(n^k)$ subsets of V of size at most k and therefore (A2) can be done in polynomial time. There always exists a set B as in (A2) because of Lemma 3. (A3) is easy.

(A4) is more difficult than the others. For every possible triple $(A, B), (C, D), (E, F) \in P$, we try to find X such that

$$E \subseteq X \subseteq (\mu(A, B) \cup \mu(C, D)) \setminus F, \; f(X) = f_{\min}(E, F), \text{ and } X \not\subseteq \mu(E, F). \tag{1}$$

Let $U = (\mu(A, B) \cup \mu(C, D)) \setminus F$ to simply notation. There is no X satisfying (1) if and only if for every $v \in U \setminus \mu(E, F)$, $f_{\min}(E \cup \{v\}, V \setminus U) > f_{\min}(E, F)$. Therefore, to test (M2), we evaluate $f_{\min}$ for each triple $(A, B), (C, D), (E, F) \in P$ and for all $v \in U \setminus \mu(E, F)$. If the test fails, the submodular function minimization algorithm outputs X such that $f(X) = f_{\min}(E, F)$ and $E \cup \{v\} \subseteq X \subseteq U$. Then we increase $|\mu(E, F)|$ by at least 1. The number of iterations of the loop between (A3) and (A4) is at most $O(n^{2k}) \times O(n) = O(n^{2k+1})$. In the (A4) step of each iteration, we test $O(n^{6k+1})$ choices of triples and elements. Let γ be the time to compute $f(X)$ for any set X. To calculate $f_{\min}$, we use the submodular function minimization algorithm [3], whose running time is $O(n^5 \gamma \log M)$ where M is the maximum value of f and $n = |V|$. We may assume that $f(\{v\}) \leq k$ for all $v \in V$, because otherwise the branch-width of f is larger than k. Then $M \leq nk$. Thus, for each choice of E, U, and v in (A4), we can evaluate $f_{\min}(E \cup \{v\}, V \setminus U)$ in $O(n^5 \gamma \log n)$ time. Thus, our algorithm runs in time $O(n^{2k+1} n^{6k+1} n^5 \gamma \log n) = O(\gamma n^{8k+6} \log n)$.

Let us prove that Algorithm 1 is correct. We need a lemma.

Lemma 7. *Let f be a connectivity function on 2^V and (P, μ) be a loose f-tangle kit of order $k + 1$. Suppose that X is a subset of V such that $|X| \leq 1$ and $f(X) \leq k$. For all $(A, B) \in P$, if $A \subseteq X \subseteq V \setminus B$ and $f_{\min}(A, B) = f(X)$, then $X \subseteq \mu(A, B)$.*

Proof. By (M1), there exists $(A', B') \in P$ such that $A' \subseteq X \subseteq V \setminus B'$ and $X \subseteq \mu(A', B')$. Then

$$A \subseteq X \subseteq \mu(A', B') \setminus B \text{ and } f_{\min}(A, B) = f(X).$$

By (M2), $X \subseteq \mu(A, B)$. □

Theorem 8. *Algorithm 1 is correct.*

Proof. If the algorithm stops at (A5), then (P, μ) is clearly a loose f-tangle kit of order $k + 1$, because it satisfies (M1)—(M3).

Now let us assume that the algorithm stops at (A3). We will show that there is no loose f-tangle kit of order $k + 1$. Let μ_i be the function μ after i iterations of (A3).

We claim that if there exists a loose f-tangle kit (P, μ') of order $k + 1$, then for all i, μ_i satisfies (M1) and $\mu_i(A, B) \subseteq \mu'(A, B)$ for all $(A, B) \in P$. If this claim is true, then there exist $(A,B),(C,D)\in P$ such that $\mu(A,B)\cup\mu(C,D)=V$, and therefore there is no loose f-tangle kit of order $k+1$ because of (M3).

We proceed by induction on i. Right after (A2) is done (when $i=0$), (M1) is true. Moreover by Lemma 7, $\mu_0(A,B)\subseteq\mu'(A,B)$ for all $(A,B)\in P$ if $(A,B)\neq(\emptyset,\emptyset)$. If $(\emptyset,\emptyset)\in P$, then by (M1) $\mu_0(\emptyset,\emptyset)\subseteq\mu'(\emptyset,\emptyset)$.

Suppose the induction hypothesis is true when $i=m$. When $i=m+1$, we update $\mu_{m+1}(E,F)=\mu_m(E,F)\cup X$. (M2) implies that $X\subseteq\mu'(E,F)$ and therefore $\mu_{m+1}(E,F)\subseteq\mu'(E,F)$. It is easy to see that (M1) is again true for μ_{m+1}. □

6 Obtaining a Branch-Decomposition

Algorithm 1 decides whether a connectivity function f has branch-width at most k for fixed k by searching for a loose f-tangle kit. But this does not necessarily mean that we can find a branch-decomposition of width at most k when the algorithm outputs that such branch-decompositions exist. The following idea to find a branch-decomposition was suggested by Jim Geelen [personal communication, 2005].

We will use Algorithm 1 as a black box. Let V be a finite set with at least three elements. Let f be a connectivity function on 2^V. For distinct $u,v\in V$, let $V/uv=W\setminus\{u,v\}\cup\{uv\}$ and let f/uv be a connectivity function on $2^{V/uv}$ defined as follows: $(f/uv)(X)=f(X)$ if $uv\notin X$ and $(f/uv)(X)=f((X\setminus\{uv\})\cup\{u,v\})$ if $uv\in X$.

Suppose that $(T,\mathcal{L})$ is a branch-decomposition of f having width at most k. We may assume that no vertex of T has degree two, otherwise we may contract one of the two incident edges. Then T must have two leaves u_T, v_T of T sharing a common neighbor w_T of degree three. Let $u=\mathcal{L}^{-1}(u_T)$, $v=\mathcal{L}^{-1}(v_T)$. We claim that f/uv has branch-width at most k. To see this, let $T'=T\setminus v_T\setminus u_T$ and let $\mathcal{L}':V/uv\to\{t : t \text{ is a leaf of } T'\}$ be a function such that $\mathcal{L}'(uv)=w_T$ and $\mathcal{L}'(x)=\mathcal{L}(x)$ if $x\in W\setminus\{uv\}$. Then it is obvious that $(T',\mathcal{L}')$ is a branch-decomposition of f/uv having width at most k.

Conversely if we have a branch-decomposition $(T',\mathcal{L}')$ of f/uv of width at most k, then it is trivial to extend $(T',\mathcal{L}')$ to the branch-decomposition $(T,\mathcal{L})$ of f as long as $f(\{u\})\le k$ and $f(\{v\})\le k$: we can attach two leaves u_T and v_T to the leaf $\mathcal{L}'(uv)$ of T' corresponding to uv and then let $\mathcal{L}(u)=u_T$ and $\mathcal{L}(v)=v_T$.

So the algorithm is as follows. The correctness follows easily from the above argument.

Algorithm 2. *Output the branch-decomposition of width at most k if there exists.*

(B1) If $|V|<1$, then no branch-decomposition exists. If $|V|=2$, then there is a unique branch-decomposition. Its width is determined by f. If $f(\{v\})>k$ for $v\in V$, then branch-width is larger than k. Stop.

(B2) Find a pair $\{u,v\}$ of V such that branch-width f/uv is at most k by Algorithm 1.

(B3) If no such pair exists, then the branch-width of f is larger than k. Stop.

(B4) Obtain the branch-decomposition $(T', \mathcal{L}')$ of f/uv of width at most k by calling this algorithm recursively.

(B5) Extend $(T', \mathcal{L}')$ to a branch-decomposition $(T, \mathcal{L})$ of f by attaching two leaves u_T and v_T to the leaf $\mathcal{L}'(uv)$ of T' corresponding to uv and then letting $\mathcal{L}(u) = u_T$ and $\mathcal{L}(v) = v_T$.

It is easy to compute the running time of the above algorithm. If A is the running time of Algorithm 1, then Algorithm 2 runs in time $O(n^3 A)$.

Acknowledgment. The authors would like to thank the anonymous referees for their careful reading and constructive suggestions for better presentation. In particular the direct proof of Theorem 5 is a modified version of the proof suggested by one of the referees. We also thank Jim Geelen for suggesting the idea of Section 6.

References

[1] H. L. Bodlaender and D. M. Thilikos, *Constructive linear time algorithms for branchwidth*, in Automata, languages and programming (Bologna, 1997), vol. 1256 of Lecture Notes in Comput. Sci., Springer, Berlin, 1997, pp. 627–637.

[2] P. Hliněný, *A parametrized algorithm for matroid branch-width*, SIAM J. Comput., 35 (2005), pp. 259–277.

[3] S. Iwata, L. Fleischer, and S. Fujishige, *A combinatorial strongly polynomial algorithm for minimizing submodular functions*, Journal of the ACM (JACM), 48 (2001), pp. 761–777.

[4] S. Oum, *Graphs of bounded rank-width*, PhD thesis, Princeton University, 2005.

[5] N. Robertson and P. Seymour, *Graph minors. X. Obstructions to tree-decomposition*, J. Combin. Theory Ser. B, 52 (1991), pp. 153–190.

[6] A. Schrijver, *A combinatorial algorithm minimizing submodular functions in strongly polynomial time*, J. Combin. Theory Ser. B, 80 (2000), pp. 346–355.

[7] D. M. Thilikos, M. J. Serna, and H. L. Bodlaender, *Constructive linear time algorithms for small cutwidth and carving-width*, in Algorithms and computation (Taipei, 2000), vol. 1969 of Lecture Notes in Comput. Sci., Springer, Berlin, 2000, pp. 192–203.