Pure pairs. VII. Homogeneous submatrices in 0/1-matrices with a forbidden submatrix

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Abstract

For integer $n > 0$, let $f(n)$ be the number of rows of the largest all-0 or all-1 square submatrix of M, minimized over all $n \times n$ 0/1-matrices M. Thus $f(n) = O(\log n)$. But let us fix a matrix H, and define $f_H(n)$ to be the same, minimized over over all $n \times n$ 0/1-matrices M such that neither M nor its complement (that is, change all 0's to 1's and vice versa) contains H as a submatrix. It is known that $f_H(n) \geq \varepsilon n^c$, where $c, \varepsilon > 0$ are constants depending on H.

When can we take $c = 1$? If so, then one of H and its complement must be an acyclic matrix (that is, the corresponding bipartite graph is a forest). Korándi, Pach, and Tomon [6] conjectured the converse, that $f_H(n)$ is linear in n for every acyclic matrix H; and they proved it for certain matrices H with only two rows.

Their conjecture remains open, but we show $f_H(n) = n^{1-o(1)}$ for every acyclic matrix H; and indeed there is a 0/1-submatrix that is either $\Omega(n) \times n^{1-o(1)}$ or $n^{1-o(1)} \times \Omega(n)$.

1 Introduction

A 0/1-matrix can be regarded as a bipartite graph, with a distinguished bipartition (V_1, V_2) say, in which there are linear orders imposed on V_1 and on V_2. Submatrix containment corresponds, in graph theory terms, to induced subgraph containment, respecting the two bipartitions and preserving the linear orders. In two earlier papers [3, 8] (one with with Maria Chudnovsky), we proved some results about excluding induced subgraphs, in a general graph and in a bipartite graph respectively. Now we impose orders on the vertex sets, and only consider induced subgraph containment that respects the orders; and we ask how far our earlier theorems remain true under this much weaker hypothesis.

In this paper, all graphs are finite and with no loops or parallel edges. Two disjoint sets are *complete* to each other if every vertex of the first is adjacent to every vertex of the second, and *anticomplete* if there are no edges between them. A pair (Z_1, Z_2) of subsets of $V(G)$ is *pure* if Z_1 is either complete or anticomplete to Z_2. Let us state the earlier theorems that we want to extend to ordered graphs. First, we proved the following, with Chudnovsky [3]:

1.1 *For every forest T, there exists $\varepsilon > 0$ such that if G is a graph with $n \geq 2$ vertices, and no induced subgraph is isomorphic to T or its complement, then there is a pure pair (Z_1, Z_2) of subsets of $V(G)$ with $|Z_1|, |Z_2| \geq \varepsilon n$.*

This theorem characterizes forests: if T is a graph that is not a forest or the complement of one, then there is no $\varepsilon > 0$ as in 1.1.

Second, we proved a similar theorem about bipartite graphs, but for this we need some more definitions. A *bigraph* is a graph together with a bipartition $(V_1(G), V_2(G))$ of G. A bigraph G *contains* a bigraph H if there is an isomorphism from H to an induced subgraph of G that maps $V_i(H)$ into $V_i(G)$ for $i = 1, 2$. The *bicomplement* of a bigraph H is the bigraph obtained by reversing the adjacency of v_1, v_2 for all $v_i \in V_i(G)$ $(i = 1, 2)$. We proved the following in [8]:

1.2 *For every forest bigraph T, there exists $\varepsilon > 0$ such that if G is a bigraph that does not contain T or its bicomplement, then there is a pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq \varepsilon |V_i(G)|$ for $i = 1, 2$.*

Again, this characterizes forests, in that if H is a bigraph that is not a forest or the bicomplement of a forest, there is no $\varepsilon > 0$ as in 1.2. Axenovich, Tompkins and Weber [2] and Alecu, Atminas, Lozin and Zamaraev [1] give further discussion on excluding bigraphs.

What if we impose an order on the vertex set, and ask for the induced subgraph containment to respect the order? Let us say an *ordered graph* is a graph with a linear order on its vertex set. Every induced subgraph inherits an order on its vertex set in the natural way: let us say an ordered graph G *contains* an ordered graph H if H is isomorphic to an induced subgraph H' of G, where the isomorphism carries the order on $V(H)$ to the inherited order on $V(H')$. One could hope for an analogue of 1.1 for ordered graphs, but it is false. Fox [5] showed:

1.3 *Let H be the three-vertex path with vertices h_1, h_2, h_3 in order, and make H an ordered graph using the same order. For all sufficiently large n, there is an ordered graph G with n vertices, that does not contain H, and such that there do not exist two disjoint subsets of $V(G)$, both of size at least $n/\log(n)$, and complete or anticomplete.*

To deduce that 1.1 does not extend to ordered graphs, let T be an ordered tree such that both T and its complement contain H, and use the construction from 1.3. Pach and Tomon [7] give further discussion of this question. But something like 1.1 *is* true: we proved in [9] that:

1.4 *For every ordered forest T and all $c > 0$, there exists $\varepsilon > 0$ such that if G is an ordered graph with $|G| \geq 2$ that does not contain T or its complement, there is a pure pair (Z_1, Z_2) in G with $|Z_1|, |Z_2| \geq \varepsilon |G|^{1-c}$.*

For ordered bipartite graphs, perhaps the situation is better: certainly we are not so well-supplied with counterexamples, and there are some positive results, proved recently by Korándi, Pach, and Tomon [6]. Let us say an *ordered bigraph* is a bigraph with linear orders on $V_1(G)$ and on $V_2(G)$. This is just a 0/1 matrix in disguise, but graph theory language is convenient for us. (Note that we are not giving a linear order of $V(G)$: that is much too strong and trivially does not work.) An ordered bigraph G *contains* an ordered bigraph H if there is an induced subgraph H' of G and an isomorphism from H to H' mapping $V_i(H)$ to $V_i(H')$ and mapping the order on $V_i(H)$ to the inherited order on $V_i(H')$, for $i = 1, 2$. (In matrix language, this is just submatrix containment.) Korándi, Pach, and Tomon [6] showed:

1.5 *Let H be an ordered bigraph with $|V_1(H)| \leq 2$, such that either*

- *$|V_2(H)| \leq 2$ and both H and its bicomplement are forests, or*
- *every vertex in $V_2(H)$ has degree exactly one.*

Then there exists $\varepsilon > 0$ with the following property. Let G be an ordered bigraph that does not contain H, with $|V_1(G)|, |V_2(G)| \geq n$; then there is a pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq \varepsilon n$ for $i = 1, 2$.

In both cases of 1.5, the bigraph H is a forest and so is its bicomplement. Korándi, Pach, and Tomon asked which other ordered bigraphs H satisfy the conclusion of 1.5. They observed that every such bigraph must be a forest and the bicomplement of a forest, and conjectured that this was sufficient as well as necessary, that is:

1.6 Conjecture: *Let H be an ordered bigraph such that both H and its bicomplement are forests. Then there exists $\varepsilon > 0$ with the following property. Let G be an ordered bigraph that does not contain H, with $|V_1(G)|, |V_2(G)| \geq n$; then there is a pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq \varepsilon n$ for $i = 1, 2$.*

We have not been able to decide this conjecture, and indeed have not even been able to prove it for the forest H consisting of a five-vertex path and an isolated vertex with $|V_1(H)| = |V_2(H)|$ (under any ordering of $V_1(H)$ and $V_2(H)$). But we will prove in 1.12 that, for a much more general class of ordered bigraphs, it is possible to find pairs of almost linear size.

Korándi, Pach, and Tomon also proposed an even stronger conjecture (to see that it implies 1.6, let H be as in 1.6, let H' be an ordered forest that contains both H and its bicomplement, and apply 1.7 for H'):

1.7 Conjecture: *For every ordered forest bigraph H, there exists $\varepsilon > 0$ with the following property. Let G be an ordered bigraph that does not contain H or its bicomplement, with $|V_1(G)|, |V_2(G)| \geq n$: then there is a pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq \varepsilon n$ for $i = 1, 2$.*

This seems a natural extension of 1.6, in analogy with 1.1 and 1.2; but, for what it is worth, our guess is that 1.7 is false. Perhaps $n/\operatorname{polylog}(n)$ might be true?

There is another result of Korándi, Pach, and Tomon, in the same paper [6]:

1.8 *Let H be an ordered forest bigraph such that $|V_1(H)| = 2$ and $|V_2(H)| = k$. For every $\tau > 0$, there exists $\delta > 0$ with the following property. Let G be an ordered bigraph that does not contain H, with $|V_1(G)| = |V_2(G)| = n$, and such that its bicomplement has at least τn^2 edges. Then there are subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$ with $|Z_1| \geq \delta n 2^{-(1+o(1))(\log\log(\delta n))^k}$ and $|Z_2| \geq \delta n$, such that Z_1, Z_2 are anticomplete.*

So, $|Z_1|$ is not quite linear, but there is more of significance. There is nothing here about forbidding G to contain the bicomplement of a forest, since the bicomplement of H need not be a forest; and the "Z_1 complete to Z_2" outcome is gone. In compensation they have the assumption that the bicomplement of G is not too sparse.

Our objective in this paper is essentially to generalize 1.8 to all ordered forest bigraphs H. We will give two results. Both prove the existence of anticomplete sets Z_1, Z_2 of cardinalities at least $n^{1-o(1)}$, but neither implies the other. One result (the second) gives a linear lower bound for one of the sets and a sublinear bound for the other; and the other result (the first) gives a sublinear (but better) bound for both sets.

Every forest is an induced subgraph of a tree, so we will assume H is a tree, for convenience. The *radius* of a tree T is the minimum r such that for some vertex v, every vertex of T can be joined to v by a path with at most r edges. In the first half of the paper, we will show:

1.9 *Let T be an ordered tree bigraph, of radius r, and with $t \geq 2$ vertices. Let $n \geq 1$, and let G be an ordered bigraph with $|V_1(G)|, |V_2(G)| \geq n$, that does not contain T, and such that every vertex has degree at most $n/(2t^2)$. Then there are two anticomplete subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, with $|Z_1|, |Z_2| \geq nt^{-5K^{r-1}}$, where $t^{K^r} = n$. (Consequently $|Z_1|, |Z_2| \geq ne^{-O((\log n)^{1-\frac{1}{r}})}$.)*

In the second half of the paper we will show:

1.10 *Let T be an ordered tree bigraph. For all $c > 0$ there exists $\varepsilon > 0$ with the following property. Let G be an ordered bigraph not containing T, such that every vertex in $V_1(G)$ has degree less than $\varepsilon|V_2(G)|$, and every vertex in $V_2(G)$ has degree less than $\varepsilon|V_1(G)|$. Then there are subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, either with $|Z_1| \geq \varepsilon|V_1(G)|$ and $|Z_2| \geq \varepsilon|V_2(G)|^{1-c}$, or with $|Z_1| \geq \varepsilon|V_1(G)|^{1-c}$ and $|Z_2| \geq \varepsilon|V_2(G)|$, such that Z_1, Z_2 are anticomplete.*

The first result, 1.9, implies:

1.11 *Let T be an ordered tree bigraph, of radius r, and with $t \geq 2$ vertices. For all $\tau > 0$ there exists $\delta > 0$ with the following property. Let $n > 0$, and let G be an ordered bigraph not containing T, with $|V_1(G)|, |V_2(G)| \geq n$, and with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges. Then there are two anticomplete subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, with $|Z_1|, |Z_2| \geq \delta nt^{-5K^{r-1}}$, where $t^{K^r} = n$.*

It also implies:

1.12 *Let T be an ordered tree bigraph, of radius r, and with $t \geq 2$ vertices. Let $n > 0$, and let G be an ordered bigraph not containing T or its bicomplement, with $|V_1(G)|, |V_2(G)| \geq n$. Then there is a pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq nt^{-2t-5K^{r-1}}$ for $i = 1, 2$, where $t^{K^r} = n$.*

Similarly, the second result, 1.10, implies:

1.13 *Let T be an ordered tree bigraph. For all $c, \tau > 0$ there exists $\delta > 0$ with the following property. Let G be an ordered bigraph not containing T, with $|V_1(G)|, |V_2(G)| \ge n$, and with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges. Then there are subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, either with $|Z_1| \ge \delta n$ and $|Z_2| \ge \delta n^{1-c}$, or with $|Z_1| \ge \delta n^{1-c}$ and $|Z_2| \ge \delta n$, such that Z_1, Z_2 are anticomplete.*

It also implies:

1.14 *Let T be an ordered tree bigraph. For all $c > 0$ there exists $\delta > 0$ with the following property. Let G be an ordered bigraph not containing T or its bicomplement, with $|V_1(G)|, |V_2(G)| \ge n$. Then there is a pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ for $i = 1, 2$, either with $|Z_1| \ge \delta n$ and $|Z_2| \ge \delta n^{1-c}$, or with $|Z_1| \ge \delta n^{1-c}$ and $|Z_2| \ge \delta n$.*

The last six theorems all imply that if $|V_1(G)|, |V_2(G)| = n$ then $|Z_1|, |Z_2| = n^{1-o(1)}$. It is easy to show, with a random graph argument, that this characterizes forests and their bicomplements. Indeed, if T is an (unordered) bigraph such that neither T nor its bicomplement are forests, then the conclusions of 1.12 and 1.14 (with "ordered" deleted) are far from true; and therefore for ordered bigraphs they are at least as far from true. More exactly, here is a standard example:

1.15 *Let T be a bigraph, such that both T and its bicomplement have a cycle of length at most g, and let $c > 1 - 1/g$. Then there is a bigraph G, with $|V_1(G)|, |V_2(G)| = n$, which contains neither T nor its bicomplement, and such that* $\min(|Z_1|, |Z_2|) \le n^c$ *for every pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ for $i = 1, 2$.*

Proof. Take n large, and let V_1, V_2 be disjoint sets of cardinality $2n$; and for each $v_1 \in V_1$ and $v_2 \in V_2$, make v_1, v_2 adjacent independently, with probability $\frac{1}{2}n^{1/g-1}$. Then with high probability, there are fewer than $n/2$ cycles of length at most g, and no pure pair Z_1, Z_2 with $Z_i \subseteq V_i$ and $|Z_i| \ge n^c$ for $i = 1, 2$. By deleting half of V_1, V_2 appropriately, we obtain a bigraph G with girth more than g, which therefore does not contain T or its bicomplement. This proves 1.15. ∎

2 Reduction to the sparse case

In this section we do two things. First, we deduce 1.12 assuming 1.9, and will prove the latter in the next section (the proof of 1.14 is similar); and second, we deduce 1.11 and 1.13 from 1.9 and 1.10.

We need the following lemma, a version of a theorem of Erdős, Hajnal and Pach [4] adapted for ordered bipartite graphs. (It is similar to a result of [8] but with different parameters).

2.1 *Let H be an ordered bigraph, let $|V_i(H)| = h_i$ for $i = 1, 2$, let $0 < \varepsilon < 1/8$, let $p = \lceil 1/(4\varepsilon) \rceil$, and let $m_1, m_2 > 0$ be integers. Let G be an ordered bigraph not containing H, with $|V_1(G)| \ge h_1 p^{h_2} m_1$ and $|V_2(G)| \ge 2h_1h_2m_2$. Then there are subsets $Y_i \subseteq V_i(G)$ with $|Y_i| = m_i$ for $i = 1, 2$, such that either*

- *every vertex in Y_1 has at most $\varepsilon|Y_2|$ neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ neighbours in Y_1, or*

- *every vertex in Y_1 has at most $\varepsilon|Y_2|$ non-neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ non-neighbours in Y_1.*

Proof. Divide $V_1(G)$ into h_1 disjoint intervals, each of cardinality at least $m_1p^{h_2}$, numbered B_u ($u \in V_1(H)$) in order. Divide $V_2(G)$ into disjoint intervals B_u ($u \in V_2(H)$) of cardinality at least m_2h_1. Choose $W \subseteq V_2(H)$ maximal such that for each $v \in W$ there exists $x_v \in B_v$, and for each $u \in V_1(H)$ there exist $Q_u \subseteq B_u$, with the following properties:

- $|Q_u| \geq m_1p^{h_2-|W|}$ for each $u \in V_1(H)$;
- for each $u \in V_1(H)$ and each $v \in W$, if u, v are H-adjacent then x_v is complete to Q_u, and if u, v are not H-adjacent then x_v is anticomplete to Q_u.

This is possible since we may take $W = \emptyset$ and $Q_u = B_u$ for each $u \in V_1(H)$. Since G does not contain H, it follows that $W \neq V_2(H)$. Choose $v \in V_2(H) \setminus W$. Say $u \in V_1(H)$ is a *problem* for $x \in B_v$ if either u, v are H-adjacent and x has fewer than $|Q_u|/p$ neighbours in Q_u, or u, v are not H-adjacent and x has fewer than $|Q_u|/p$ non-neighbours in Q_u. From the maximality of W, for each $x \in B_v$ there exists $u \in V_1(H)$ that is a problem for x. Since there are only h_1 possible problems, there exist $u \in V_1(H)$, and $C \subseteq B_v$ with $|C| \geq |B_v|/h_1$, such that for every $x \in C$, u is a problem for x. By moving to the bicomplement if necessary, we may assume that u, v are H-adjacent; and so every vertex in C has fewer than $|Q_u|/p$ neighbours in Q_u. Since $|Q_u| \geq m_1p^{h_2-|W|} \geq m_1p \geq 2m_1$ and $|C| \geq |B_v|/h_1 \geq 2m_2$, it follows by averaging that there are subsets $X_1 \subseteq Q_u$ and $X_2 \subseteq C$, of cardinality exactly $2m_1, 2m_2$ respectively, such that there are at most $|X_1| \cdot |X_2|/p = 4m_1m_2/p$ edges joining them. Let Y_1 be the set of the m_1 vertices in X_1 that have fewest neighbours in X_2; then they each have at most $4m_2/p$ neighbours in X_2. Define Y_2 similarly; then $|Y_i| = m_i$ for $i = 1, 2$, and every vertex in Y_1 has at most $4m_2/p \leq \varepsilon m_2$ neighbours in Y_2 and vice versa. This proves 2.1. ∎

Proof of 1.12, assuming 1.9. Let T be an ordered tree bigraph, of radius r, and with t vertices. Let $\varepsilon = 1/(2t^2)$, and let $c = t^{-2t}$. Let $K > 0$, and let G be an ordered bigraph not containing T or its bicomplement, with $|V_1(G)|, |V_2(G)| \geq n$, where $n = t^{K^r}$. We may assume that $cnt^{-5K^{r-1}} > 1$, for otherwise the result is true, taking $|Z_1| = |Z_2| = 1$. Hence $cn \geq t^{5K^{r-1}} \geq 1$. Let m be the largest integer such that $m \leq 2cn$. Thus $m \geq cn$, since $cn \geq 1$.

Let $|V_i(T)| = h_i$ for $i = 1, 2$, and $p = \lceil 1/(4\varepsilon) \rceil$. Now $n/m \geq 1/(2c) \geq \max(h_1p^{h_2}, 2h_1h_2)$, since $c = t^{-2t}$ and $h_1, h_2 \leq t - 1$. By 2.1, and moving to the bicomplement if necessary, we may assume that there exist $Y_i \subseteq V_i(G)$ with $|Y_i| = m$ for $i = 1, 2$, such that every vertex in Y_1 has at most $\varepsilon|Y_2|$ neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ neighbours in Y_1. Choose J with $m = t^{J^r}$. By 1.9 applied to the ordered bigraph induced on $Y_1 \cup Y_2$, there exist $Z_i \subseteq Y_i$ with $|Z_1|, |Z_2| \geq mt^{-5J^{r-1}}$, such that Z_1, Z_2 are anticomplete. Since $m \geq cn$ and $J \leq K$, it follows that $|Z_1|, |Z_2| \geq ct^{K^r-5K^{r-1}}$. This proves 1.12. ∎

The proof that 1.10 implies 1.14 is similar and we omit it.

The result 1.8 of Korándi, Pach, and Tomon [6] has as a hypothesis that the bicomplement of G has at least τn^2 edges. This is apparently much weaker than the hypothesis that every vertex of G has degree at most εn, but in fact the "not very dense" hypothesis is as good as the "very sparse" hypothesis, because of 2.3 below. To prove this, we need the next result, shown in [8].

2.2 *For all $c, \varepsilon, \tau > 0$ with $\varepsilon < \tau$, there exists $\delta > 0$ with the following property. Let G be a bigraph with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges and with $V_1(G), V_2(G) \neq \emptyset$. Then there exist $Z_i \subseteq V_i(G)$ with $|Z_i| \geq \delta|V_i(G)|$ for $i = 1, 2$, such that there are fewer than $(1-\varepsilon)|Y_1| \cdot |Y_2|$ edges between Y_1, Y_2 for all subsets $Y_i \subseteq Z_i$ with $|Y_i| \geq c|Z_i|$ for $i = 1, 2$.*

We deduce:

2.3 *For every ordered bigraph H, and for all $\varepsilon, \tau > 0$, there exists $\delta > 0$ with the following property. Let G be an ordered bigraph not containing H, with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges. Then there exist $Y_i \subseteq V_i(G)$ with $|Y_i| \geq \delta|V_i(G)|$ for $i = 1, 2$, such that every vertex in Y_1 has at most $\varepsilon|Y_2|$ neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ neighbours in Y_1.*

Proof. We may assume that $\varepsilon < 1/8$ and $\varepsilon < \tau$, by reducing ε, and $|V_1(H)|, |V_2(H)| \neq \emptyset$, by adding vertices to H. Let $h_i = |V(H_i)|$ for $i = 1, 2$, let $d = \lceil 1/(4\varepsilon) \rceil$, and let $1/c = \max(h_1 d^{h_2}, 2h_1h_2)$. Choose δ' such that 2.2 holds with δ replaced by δ', and let $\delta = c\delta'$. Now let G be an ordered bigraph not containing H, with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges. We may assume that $V_1(G), V_2(G) \neq \emptyset$. By 2.2, there exist $Z_i \subseteq V_i(G)$ with $|Z_i| \geq \delta'|V_i(G)|$ for $i = 1, 2$, such that there are fewer than $(1-\varepsilon)|Y_1| \cdot |Y_2|$ edges between Y_1, Y_2 for all subsets $Y_i \subseteq Z_i$ with $|Y_i| \geq c|Z_i|$ for $i = 1, 2$. By 2.1, applied to the ordered sub-bigraph induced on $Z_1 \cup Z_2$, there exist $Y_i \subseteq Z_i$ with $|Y_i| \geq c|V_i(G)|$ for $i = 1, 2$, such that either

- every vertex in Y_1 has at most $\varepsilon|Y_2|$ neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ neighbours in Y_1, or
- every vertex in Y_1 has at most $\varepsilon|Y_2|$ non-neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ non-neighbours in Y_1.

Since there are fewer than $(1-\varepsilon)|Y_1| \cdot |Y_2|$ edges between Y_1, Y_2, the second is impossible, and so the first holds. Then for $i = 1, 2$, $|Y_i| \geq c|Z_i| \geq c\delta'|V_i(G)| = \delta|V_i(G)|$. This proves 2.3. ∎

Proof of 1.11, assuming 1.9. Let T be an ordered tree bigraph, of radius r, and with $t \geq 2$ vertices, and let $\tau > 0$. Let $\varepsilon = 1/(2t^2)$, and choose $\delta > 0$ as in 2.2, with H replaced by T.

Let $K > 0$, and let G be an ordered bigraph not containing T, with $|V_1(G)|, |V_2(G)| \geq t^{K^r}$, such that the bicomplement of G has at least $\tau|V_1(G)| \cdot |V_2(G)|$ edges. From the choice of δ, there exist $Y_i \subseteq V_i(G)$ with $|Y_i| \geq \delta|V_i(G)|$ for $i = 1, 2$, such that every vertex in Y_1 has at most $\varepsilon|Y_2|$ neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ neighbours in Y_1. Let k satisfy $t^{k^r} = \delta t^{K^r}$. By 1.9 applied to the sub-bigraph $G[Y_1 \cup Y_2]$, there are subsets $Z_i \subseteq Y_i$ for $i = 1, 2$, with $|Z_1|, |Z_2| \geq t^{k^r - 5k^{r-1}}$, such that Z_1, Z_2 are anticomplete. Since $K \geq k$, and $t^{k^r} = \delta t^{K^r}$, it follows that $|Z_1|, |Z_2| \geq \delta t^{K^r - 5K^{r-1}}$. This proves 1.11. ∎

The proof that 1.10 implies 1.13 is similar and we omit it.

3 Proof of the first main theorem

In this section we prove 1.9, which we restate:

3.1 *Let T be an ordered tree bigraph, of radius r, and with $t \ge 2$ vertices. Let $n > 0$, and let G be an ordered bigraph with $|V_1(G)|, |V_2(G)| \ge n$, such that every vertex has degree at most $n/(2t^2)$, and there do not exist two anticomplete subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, with $|Z_1|, |Z_2| \ge nt^{-5K^{r-1}}$, where $n = t^{K^r}$. Then G contains T.*

Proof. Since G is not complete bipartite, and therefore we can choose Z_1, Z_2 anticomplete with cardinality one, it follows that $nt^{-5K^{r-1}} > 1$. Hence $K > 5$.

Suppose first that $r = 1$. Thus T has a vertex v (of degree d say), adjacent to all other vertices of T. Let $v \in V_1(T)$, without loss of generality. Suppose that all vertices in $V_1(G)$ have degree at most $d-1$. Choose a set Z_1 of $\lfloor n/d \rfloor$ vertices in $V_1(G)$; then the set of vertices with neighbours in X has cardinality at most $(d-1)|X| \le (d-1)n/d$, and so there is a set Z_2 of at least n/d vertices in $V_2(G)$ anticomplete to Z_1. Thus, $\lfloor n/d \rfloor < nt^{-5K^{r-1}}$. But $n/d \ge n/t$, since $t = d+1$; and $\lfloor n/t \rfloor \ge n/(2t)$, since $n \ge t$ (because $n \ge t^{5^r}$); so $\lfloor n/d \rfloor \ge n/(2t)$, and therefore $n/(2t) < nt^{-5K^{r-1}}$, a contradiction. Thus there exists a vertex in $V_1(G)$ with degree at least d, and so G contains T, as required.

Thus we may assume that $r \ge 2$, and so $t \ge 4$. Let $\varepsilon = 1/(2t^2)$, and choose a real number $x \ge 0$ with $x \le K^{r-1}$, maximum such that there exist $A_1 \subseteq V_1(G)$ and $A_2 \subseteq V_2(G)$ with the properties that

- $|A_1|, |A_2| \ge nt^{-x}$; and
- every vertex in A_1 has at most εnt^{-Kx} neighbours in A_2 and vice versa.

This is possible since we may take $x = 0$ and $A_i = V_i(G)$ for $i = 1, 2$.

(Remark: If we permit $x > K^{r-1}$, for instance when $nt^{-x} \le 1$, there are sets A_1, A_2 satisfying the bullets; but there are no two such sets when x is about K^{r-1}, because otherwise we could find the desired anticomplete pair. This motivates the upper bound $x \le K^{r-1}$.)

Let A_1, A_2 be as above, and let $d = \varepsilon nt^{-Kx}$. The remainder of the proof involves only the subgraph induced on $A_1 \cup A_2$. Since $|A_1| \ge nt^{-x} \ge nt^{-5K^{r-1}}$, A_1 is not anticomplete to A_2, so $d \ge 1$. For $1 \le s \le r$, let $k_s = 4(K^{s-1} + K^{s-2} + \cdots + 1)$. We show first that:

(1) *$x < K^{r-1} - k_{r-1}$, and hence $nt^{-x} \ge t^{24}$.*

Since

$$|A_1| \ge nt^{-x} \ge nt^{-K^{r-1}} \ge 2nt^{-5K^{r-1}} \ge nt^{-5K^{r-1}} + 1,$$

there exists a set $X \subseteq A_1$ of cardinality $\lceil nt^{-5K^{r-1}} \rceil$. The union of the neighbours in A_2 of vertices in X has cardinality at most $(2nt^{-5K^{r-1}})(\varepsilon nt^{-Kx})$. Since $|A_2|/2 \ge nt^{-5K^{r-1}}$, there are fewer than $|A_2|/2$ vertices in A_2 anticomplete to X. Consequently $(2nt^{-5K^{r-1}})(\varepsilon nt^{-Kx}) > |A_2|/2$, and so $(4nt^{-5K^{r-1}})(\varepsilon nt^{-Kx}) > nt^{-x}$. This implies that

$$t^{(K-1)x} < 4\varepsilon nt^{-5K^{r-1}} \le nt^{-5K^{r-1}} = t^{K^r - 5K^{r-1}}.$$

Hence

$$(K-1)x < K^r - 5K^{r-1} \le (K-1)(K^{r-1} - k_{r-1})$$

and so $x < K^{r-1} - k_{r-1}$. This proves the first statement of (1). For the second, since $x < K^{r-1} - 4$ (because $k_{r-1} \ge 4$), and $n > t^{5K^{r-1}}$, it follows that

$$nt^{-x} \ge t^{5K^{r-1} - x} \ge t^{4K^{r-1}+4} \ge t^{4K+4} \ge t^{24}.$$

This proves (1).

Since $nt^{-x} \geq t^{24}$, it follows that

$$|A_i| \geq nt^{-x} \geq (t-1)nt^{-x-1} + nt^{-x-1} \geq (t-1)nt^{-x-1} + t$$

for $i = 1, 2$. Since $|V_1(T)| \leq t-1$, it follows that we may choose $|V_1(T)|$ disjoint blocks B_u ($u \in V_1(T)$), all intervals of A_1, and of cardinality $\lceil nt^{-x-1} \rceil$, numbered in order (more exactly, such that for all distinct $u, u' \in V(T)$ and all $v \in B_u$ and $v' \in B_{u'}$, if u occurs before u' in the ordering of $V_1(T)$ given by the ordered bigraph T, then v occurs before v' in the ordering of $V_1(G)$ given by the ordered bigraph G). Partition A_2 into blocks B_v ($v \in V_2(T)$) similarly. (See figure 1.) We will show that for each $v \in V(T)$ there exists $\phi(v) \in B_v$ such that the map sending v to $\phi(v)$ for each v is an isomorphism from the ordered bigraph T to an induced ordered sub-bigraph of G. To show that some choice of the vertices $\phi(v); (v \in V(T))$ works, it is enough to show that the map is an isomorphism of the corresponding *unordered* bigraphs, because of the way we numbered the sets B_v ($v \in V(T)$).

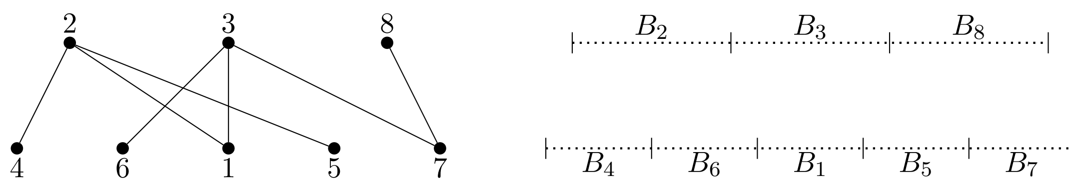

Figure 1: An ordered tree bigraph, and the corresponding intervals of $V(G)$.

Since T has radius at most r, there is a vertex $v_0 \in V(T)$ such that every vertex of T can be joined to v_0 by a path of length at most r. If $uv \in E(T)$, and u belongs to the path of T between v_0 and v, we say that u is the *parent* of v, and v is a *child* of u. For $0 \leq s \leq r$ let T_s be the subtree of T induced on the vertices with distance at most s from v_0. So $V(T_0) = \{v_0\}$, and $T_r = T$. Let L_s be the set of vertices of T with distance exactly s from v_0.

To select the vertices $\phi(v) \in B_v$ ($v \in V(T)$) that give the isomorphism we require, we will choose them in rounds, choosing all the vertices $\phi(v)$ ($v \in L_s$) simultaneously, for $s = 0, 1, 2, \ldots, r$ in turn. But we must choose the vertices $\phi(v)$ ($v \in L_s$) carefully. Before we go on with the proof, let us sketch what we need. Suppose that we have selected $\phi(u)$ for each $u \in V(T_{s-1})$; how should we choose $\phi(v)$ for $v \in L_s$? Let $u \in L_{s-1}$ be the parent of v. Certainly, the vertex we choose to be $\phi(v)$ must be adjacent to $\phi(u)$, and nonadjacent to all the vertices $\phi(u')$ for $u' \in V(T_{s-1}) \setminus \{u\}$. Let us call such a vertex a $\phi(v)$-*candidate*. Prima facie, there might not be any $\phi(v)$-candidates: so in the previous round, we must be careful to choose $\phi(u)$ in such a way that there will be $\phi(v)$-candidates when we need them. And similarly, we need to choose $\phi(v)$ in such a way that there will be $\phi(w)$-candidates when we need them, for each $w \in L_{s+1}$ that is T-adjacent to v. This can be arranged, by making sure that for each s and each $v \in L_s$, there are at least p_s $\phi(v)$-candidates, where p_s is some function of s that drops rapidly as s increases.

Let us see this in more detail. Let v, u, s be as before, and let P_u be the set of all $\phi(u)$-candidates. (For this sketch, we assume that u has only one child v in T; the general case is not much different.) Let $C \subseteq B_v$ be the set of vertices in B_v that have a neighbour in $\{\phi(u') : u' \in V(T_{s-2})\}$. Then C contains only a small fraction of B_v, and our task is to find a vertex in P_u that has at least p_s

neighbours in $B_v \setminus C$. Suppose that there is no such vertex; then one might hope we could obtain a contradiction to the choice of x, replacing the pair A_1, A_2 with the pair P_u, Q, where Q is obtained from B_v by deleting C and the few vertices in $B_v \setminus C$ that have many neighbours in P_u. But this does not work, because the set P_u turns out to be too small. Here is a trick to overcome this difficulty. If this situation occurs, and we find a set $P_u \subseteq B_u$ and a set $Q \subseteq B_v$ such that every vertex in P_u has fewer than p_s neighbours in Q, and $|P_u|, |B_v \setminus Q|$ are appropriately small, put the vertices in P_u aside, and start over again the process of choosing the vertices $\phi(v)$ $(v \in V(T))$, not using any vertices that have been put aside. Now we might find more vertices that need to be put aside, and repeat; but for each $u \in V(T)$, the set of all vertices in B_u that are ever put aside will remain a small fraction of B_u, because otherwise we will obtain a contradiction to the choice of x. So eventually we will not run into this difficulty, and the construction of the $\phi(v)$'s will go through. The process of setting vertices aside is formalized in the choice of the sets Y_v and X_{uv} below. Now we return to the proof.

For $1 \le s \le r-1$, let $p_s = dt^{-Kk_s}$, and let $p_r = 1$. For $2 \le s \le r$, let $f_s = dt^2/p_{s-1} = t^{Kk_{s-1}+2}$. So $f_s \ge t^{4K+2} \ge t^{22}$. Inductively for $s = r, r-1, \dots, 2$, and each $v \in L_s$, we define $X_v \subseteq B_v$, $Y_v \subseteq B_v$, and $X_{uv} \subseteq B_u$ as follows, where u is the parent of v:

- X_v is the union of the sets X_{vw} over all children w of v;
- choose $X_{uv} \subseteq B_u$ and $Y_v \subseteq B_v \setminus X_v$, with X_{uv} maximal such that $|Y_v| \le \min(3|B_v|/4, f_s|X_{uv}|)$ and every vertex in X_{uv} has fewer than p_s neighbours in $B_v \setminus (X_v \cup Y_v)$.

This completes the inductive definition. (See figure 2.) For each $v \in L_1$, let X_v be the union of the sets X_{vw} over all children w of v.

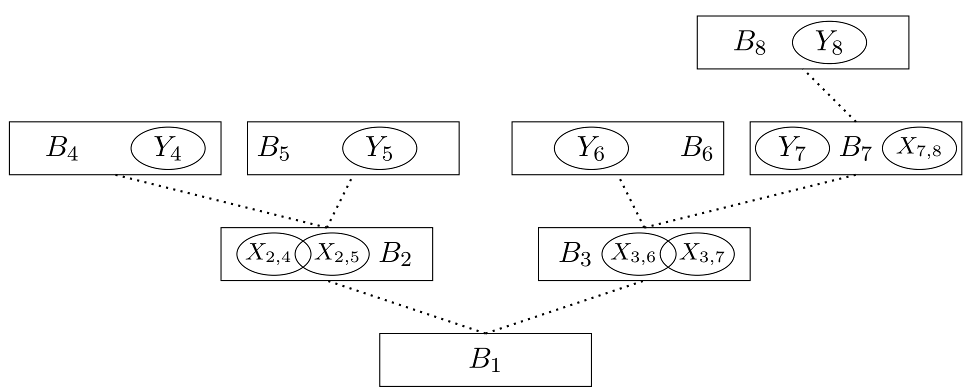

Figure 2: The intervals of figure 1 arranged as a tree, and the corresponding sets X_{uv} and Y_v.

(2) *Let $2 \le s \le r$, let $v \in L_s$ and let u be its parent. Then*

- $|X_{uv}| < nt^{-x-k_s} \le |B_u|/t^{23}$*, and* $|Y_v| < |B_v|/t$*; and*
- *there do not exist* $X \subseteq B_u \setminus X_{uv}$ *and* $Y \subseteq B_v \setminus X_v$*, such that* $X \ne \emptyset$ *and* $|Y| \le \max(|B_v|/2, f_s|X|)$ *and every vertex in* X *has fewer than* p_s *neighbours in* $B_v \setminus (X_v \cup Y)$.

We prove the first bullet by induction on $r - s$, and so we may assume that it holds with s, u, v replaced by $s+1, v, w$ for each child w of v. In particular, $|X_{vw}| \le |B_v|/t^{23}$ for each such w, and so

$|X_v| \le |B_v|/t^{22}$. Suppose that $|X_{uv}| \ge nt^{-x-k_s}$, and choose $X \subseteq X_{uv}$ with $|X| = \lceil nt^{-x-k_s} \rceil$. Since $nt^{-x-k_s} \le |B_v|/t^{23}$ (because $|B_v| \ge nt^{-1-x}$ and $k_s \ge 4(K+1) \ge 24$) and $1 \le |B_v|/t^{23}$ (because $|B_v| \ge nt^{-1-x} \ge t^{23}$ by (1)), it follows that

$$|X| \le nt^{-x-k_s} + 1 \le 2|B_v|/t^{23} \le |B_v|/t^{22}.$$

But $|X_v| \le |B_v|/t^{22}$, and $|Y_v| \le 3|B_v|/4$, and so $|B_v \setminus (X_v \cup Y_v)| \ge 2|X|$. Since each vertex in X has fewer than p_s neighbours in $B_v \setminus (X_v \cup Y_v)$, there are at most $|X|$ vertices in $B_v \setminus (X_v \cup Y_v)$ that have at least p_s neighbours in X, and so there is a subset $Y \subseteq B_v \setminus (X_v \cup Y_v)$ with $|Y| = |X|$ such that every vertex in Y has fewer than p_s neighbours in X. If $s = r$, then $p_s = 1$, and so X, Y are anticomplete, and therefore $|X| < nt^{-5K^{r-1}}$; but $x \le K^{r-1} - k_{r-1} = 5K^{r-1} - k_r$ by (1), and so $|X| < nt^{-x-k_r}$, a contradiction. If $s < r$, then since every vertex in X has fewer than $p_s = dt^{-Kk_s}$ neighbours in Y and vice versa, and $x + k_s \le K^{r-1}$ by (1), the maximality of x implies that $|X| < nt^{-x-k_s}$, a contradiction. This proves that $|X_{uv}| < nt^{-x-k_s}$.

Also,

$$|Y_v|/|B_v| \le f_s|X_{uv}|/|B_v| < t^{Kk_{s-1}+2}(nt^{-x-k_s})/(nt^{-x-1}) = t^{Kk_{s-1}-k_s+3} = t^{-1}.$$

This completes the inductive argument for the first bullet.

For the second bullet, suppose that there exist $X \subseteq B_u \setminus X_{uv}$ and $Y \subseteq B_v \setminus X_v$, such that $X \ne \emptyset$ and $|Y| \le |B_v|/2$, and $|Y| \le f_s|X|$ and every vertex in X has fewer than p_s neighbours in $B_v \setminus (X_v \cup Y)$. Then every vertex in $X_{uv} \cup X$ has fewer than p_s neighbours in $B_v \setminus (X_v \cup Y_v \cup Y)$; and $Y_v \cup Y \subseteq B_v \setminus X_v$; and $|Y_v \cup Y| \le 3|B_v|/4$ (since $|B_v| \le |B_v|/t \le |B_v|/4$ and $|Y| \le |B_v|/2$); and $|Y_v \cup Y| \le f_s|X_{uv} \cup X|$ (since $|Y_v| \le f_s|X_{uv}|$ and $|Y| \le f_s|X|$ and $X \cap X_{uv} = \emptyset$). Since $X \ne \emptyset$, this contradicts the maximality of X_{uv}, and so proves the second bullet, and hence proves (2).

A remark: the reason for the sets X_{uv} and Y_v was just to arrange the property of the second bullet of (2). We will not need the sets Y_v after this point.

Let $P_{v_0} = B_{v_0}$. For $s = 1, \dots, r-1$ we will choose a vertex $\phi(u) \in P_u$ for each $u \in L_{s-1}$, and a subset $P_v \subseteq B_v \setminus X_v$ for each $v \in L_s$, satisfying the following conditions:

- for all distinct $u, v \in V(T_{s-1})$, $\phi(u), \phi(v)$ are G-adjacent if and only if u, v are T-adjacent;
- for all $u \in V(T_{s-1})$ and $v \in L_s$, and all $y \in P_v$, $\phi(u), y$ are G-adjacent if and only if u, v are T-adjacent;
- for each $v \in L_s$, $|P_v| \ge p_s$.

First let us assume $s = 1 < r$. Suppose that there is no $y \in B_{v_0}$ with at least p_1 neighbours in $B_v \setminus X_v$ for each child v of v_0. Consequently there is a child v of v_0 such that for at least $|B_{v_0}|/t \ge nt^{-x-2}$ vertices $y \in B_{v_0}$, y has fewer than p_1 neighbours in $B_v \setminus X_v$. Choose a set X of exactly $\lceil nt^{-x-2} \rceil$ such vertices y. Since $nt^{-x-2} \le |B_v|/t$, and $1 \le |B_v|/t^{23}$ by (1), it follows that $|X| \le nt^{-x-2} + 1 \le 3|B_v|/8$. Since $|X_v| \le |B_v|/t^{22} \le |B_v|/4$ by (2), it follows that $|B_v \setminus X_v| \ge 2|X|$, and so at least $|X|$ vertices in $B_v \setminus X_v$ have fewer than p_1 neighbours in X. Since $p_1 = dt^{-Kk_1} = dt^{-4K}$, the maximality of x and (1) imply that $|X| < nt^{-x-4}$, a contradiction. Thus there exists $y \in B_{v_0}$ with at least p_1 neighbours in $B_v \setminus X_v$ for each child v of v_0. Define $\phi(v_0) = y$, and for each $v \in L_1$ define P_v to be the set of neighbours of y in $B_v \setminus X_v$. This completes the definition when $s = 1$.

Now we assume that $2 \le s < r$, and we have chosen $\phi(u) \in P_u$ for each $v \in V(T_{s-2})$, and a subset $P_v \subseteq B_v \setminus X_v$ for each $v \in L_{s-1}$, satisfying the bullets. We must define $\phi(u) \in P_u$ for each $u \in L_{s-1}$, and $P_v \subseteq B_v \setminus X_v$ for each $v \in L_s$, satisfying the bullets. From the symmetry we may assume that $L_s \subseteq V_1(T)$.

Let C be the set of vertices in A_1 that are equal or adjacent to $\phi(v)$ for some $v \in V(T_{s-2})$. If τ is a map with domain L_{s-1}, such that $\tau(u) \in P_u$ for each $u \in L_{s-1}$, we call τ a *transversal*. A transversal τ is *valid* if for each edge uv of T with $u \in L_{s-1}$ and $v \in L_s$, there are at least p_s vertices in $B_v \setminus (C \cup X_v)$ that are adjacent to $\tau(u)$ and that have no neighbour in $\{\tau(u') : u' \in L_{s-1} \setminus \{u\}\}$. In order to complete the inductive definition, it remains to show:

(3) *There is a valid transversal.*

Suppose not. Then for every transversal τ, there exist $u \in L_{s-1}$ and a child $v \in L_s$ of u such that there are fewer than p_s vertices in $B_v \setminus (C \cup X_v)$ that are adjacent to $\tau(u)$ and that have no neighbour in $\{\tau(u') : u' \in L_{s-1} \setminus \{u\}\}$. Call (u, v) a *problem* for τ. Since there are only at most t possible problems, there exists $(u, v) \in E$ that is a problem for at least a fraction $1/t$ of all transversals. Hence there exist a choice of $\tau(u') \in P_{u'}$ for each $u' \in L_{s-1} \setminus \{u\}$, and a subset $X \subseteq P_u$ with

$$|X| \ge |P_u|/t \ge p_{s-1}/t = dt/f_s,$$

such that for all choices of $\tau(u) \in X$, (u, v) is a problem for the transversal τ. Let Y be the set of all vertices in B_v with a neighbour in

$$\{\tau(u') : u' \in L_{s-1} \setminus \{u\}\} \cup \{\phi(u') : u' \in V(T_{s-2})\}.$$

It follows that $C \subseteq Y$, and $|Y| \le dt$. Consequently

$$|Y|/|B_v| \le \varepsilon n t^{1-Kx}/(nt^{-1-x}) = \varepsilon t^{2-(K-1)x} \le \varepsilon t^2 = 1/2.$$

Moreover, $|Y| \le dt \le f_s|X|$, contrary to the second bullet of (2). This proves (3).

From (3), we may choose $\phi(u) \in P_u$ for each $u \in L_{s-1}$, such that $(\phi(u) : u \in L_{s-1})$ is a valid transversal. For each $v \in L_s$, we define P_v as follows. Let $u_0 \in L_{s-1}$ be its parent, and let P_v be the set of vertices in $B_v \setminus (C \cup X_v)$ that are adjacent to $\phi(u_0)$ and that have no neighbour in $\{\phi(u') :\ u' \in L_{s-1} \setminus \{u_0\}\}$. Since $(\phi(u) : u \in L_{s-1})$ is a valid transversal, it follows that $|P_v| \ge p_s$. This completes the inductive definition of $\phi(v)$ $(v \in V(T_{r-1}))$ and P_v $(v \in V(T))$. Choose $\phi(v) \in P_v$ for each $v \in L_r$. Then the map ϕ is an isomorphism from T to an induced ordered sub-bigraph of G. This proves 3.1. ∎

4 Parades

Now we begin the proof of 1.10, the second main result mentioned in the introduction. This proof was derived from, and still has some ingredients in common with, the proof of the main theorem of [9], but it has needed some serious modification, in order to persuade one of the two sets Z_1, Z_2 to be linear.

Let G be a bigraph (not necessarily ordered), and let I be a set of nonzero integers. We define $I^+ = \{i \in I : i > 0\}$ and $I^- = I \setminus I^+$. Let the sets B_i ($i \in I$) be nonempty, pairwise disjoint subsets of $V(G)$, such that $B_i \subseteq V_1(G)$ if $i < 0$ and $B_i \subseteq V_2(G)$ if $i > 0$. We call $\mathcal{P} = (B_i : i \in I)$ a *parade* in G. Its *length* is the pair $(|I^-|, |I^+|)$, and its *width* is the pair (w_1, w_2) where $w_1 = \min(|B_i| : i \in I^-)$ and $w_2 = \min(|B_i| : i \in I^+)$, taking $w_k = |V_k(G)|$ if the corresponding set I^- or I^+ is empty. We call the sets B_i the *blocks* of the parade. What matters is that the blocks are not too small. (We used the same word in [8] for a similar but slightly different object.)

If $I' \subseteq I$, then $(B_i : i \in I')$ is a parade, called a *sub-parade* of $\mathcal{P}$. If $B'_i \subseteq B_i$ is nonempty for each $i \in I$, then $(B'_i : i \in I)$ is a parade, called a *contraction* of $\mathcal{P}$.

In order to prove 1.10, we will prove 5.2, which is a powerful result about general parades. All the work of this section and the next is in order to prove 5.2. Then in the final section, we will deduce 1.10, by iterated applications of 5.2.

Let X, Y be disjoint nonempty subsets of $V(G)$. The *max-degree from X to Y* is defined to be the maximum over all $v \in X$ of the number of neighbours of v in Y. Let $(B_i : i \in I)$ be a parade in a bigraph G, and for all $i, j \in I$ of opposite sign, let $d_{i,j}$ be the max-degree from B_i to B_j. (For all other pairs i, j we define $d_{i,j} = 0$.) We call $d_{i,j}$ ($i, j \in I$) the *max-degree function* of the parade. The product of the numbers $d_{j,h}$ for all pairs h, j where $h \in I^-$ and $j \in I^+$ is called the *max-degree product* of $\mathcal{B}$. We just need this "product" definition for the next theorem. The idea is, we start with some parade. If we can replace two of its blocks by two slightly smaller blocks, in such a way that the max-degree product decreases significantly, do so, and continue this until we cannot do it any more. If the max-degree product is now zero, we can find the desired anticomplete pair of sets; and even if it is still nonzero, at least we have obtained what we call a "shrink-resistant" contraction of the original parade, which turns out to be a useful object, and we can replace the original parade by this new one.

Let $\phi, \mu > 0$ be real numbers. We say that $\mathcal{B}$ is *(ϕ, μ)-shrink-resistant* if for all $h \in I^-$ and $j \in I^+$, and for all $X \subseteq B_h$ and $Y \subseteq B_j$ with $|X| \geq \mu|B_h|$ and $|Y| \geq \mu|B_j|$, the max-degree from Y to X is more than $d_{j,h}|V_1(G)|^{-\phi}$. We begin with:

4.1 *Let $\mathcal{B} = (B_i : i \in I)$ be a parade in a bigraph G, and let $\phi, \mu > 0$ be real numbers with $\mu \leq 1$. Let $\beta = \mu^{1+|I|^2/\phi}$. Then either*

- *there exist $h \in I^-$ and $j \in I^+$, and $X \subseteq B_h$ and $Y \subseteq B_j$ with $\frac{|X|}{|B_h|}, \frac{|Y|}{|B_j|} \geq \beta$, such that X, Y are anticomplete; or*
- *there is a (ϕ, μ)-shrink-resistant contraction $(B'_i : i \in I)$ of $\mathcal{B}$, such that $|B'_i| \geq \beta|B_i|$ for each $i \in I$.*

Proof. Let $S = \lfloor |I|^2/\phi \rfloor$. Choose an integer s with $0 \leq s \leq S + 1$ and with s maximum such that there is a contraction $\mathcal{B}' = (B'_i : i \in I)$ of $\mathcal{B}$ with

- $|B'_i| \geq \mu^s |B_i|$ for each $i \in I^-$; and
- max-degree product at most $|V_1(G)|^{|I|^2 - \phi s}$.

(This is possible since we may take $s = 0$ and $\mathcal{B}' = \mathcal{B}$.) Let $d_{h,j}$ ($h, j \in I$) be the max-degree function of $\mathcal{B}'$.

(1) *We may assume that $d_{j,h} \geq 1$ for all $h \in I^-$ and $j \in I^+$, and so $s \leq S$.*

If $d_{j,h} < 1$, then $d_{j,h} = 0$, since it is an integer. Thus B'_h, B'_j are anticomplete. Since $s \leq S+1$ and hence $\mu^s \geq \mu^{S+1} \geq \beta$, it follows that $|B'_h|/|B_h|, |B'_j|/|B_j| \geq \beta$, and the first outcome of the theorem holds. Thus we may assume that $d_{j,h} \geq 1$, for all $h \in I^-$ and $j \in I^+$. Hence the max-degree product of $\mathcal{B}'$ is at least one, and since it is at most $|V_1(G)|^{|I|^2-\phi s}$, it follows that $|I|^2 - \phi s \geq 0$. Hence $s \leq S$. This proves (1).

(2) *$(B'_i : i \in I)$ is (ϕ, μ)-shrink-resistant.*

Let $h \in I^-$ and $j \in I^+$, and let $C_h \subseteq B'_h$ and $C_j \subseteq B'_j$, with $|C_h| \geq \mu|B'_h|$ and $|C_j| \geq \mu|B'_j|$. For all $i \in I$ with $i \neq h, j$ let $C_i = B'_i$, and let d be the max-degree from C_j to C_h. From the maximality of s, and since $s \leq S$, it follows that the max-degree product of $(C_i : i \in I)$ is more than $|V_1(G)|^{|I|^2-\phi(s+1)}$. Since the first is at most $d/d_{j,h}$ times the max-degree product of $(B'_i : i \in I)$, which is at most $|V_1(G)|^{|I|^2-\phi s}$, it follows that $d/d_{j,h} > |V_1(G)|^{-\phi}$. This proves (2).

Since $|B'_i| \geq \mu^S|B_i| \geq \beta|B_i|$ for each $i \in I$, the second outcome of the theorem holds. This proves 4.1. ∎

Shrink-resistance tells us that for each $h \in I^-$ and $j \in I^+$, most vertices in B_j have about the same number of neighbours in B_h. The next result will arrange that in addition, this "same number of neighbours" does not depend on the choice of h, j, and also gives us bounds on the analogous quantities with h, j exchanged.

Let $(B_i : i \in I)$ be a parade in a bigraph G, and let $\phi, \mu > 0$ be real numbers. We say that a real number $\tau > 0$ is a *(ϕ, μ)-band* for $(B_i : i \in I)$ if for all $h \in I^-$ and $j \in I^+$:

- the max-degree from B_j to B_h is at most $\tau|B_h|$; and
- for all $X \subseteq B_h$ and $Y \subseteq B_j$ with $|X| \geq \mu|B_h|$ and $|Y| \geq \mu|B_j|$, the max-degree from Y to X is more than $\tau|V_1(G)|^{-\phi}|B_h|$.

4.2 *Let $k \geq 1$ be an integer, and let $\phi, \mu > 0$ be real numbers. Then there exists an integer $K \geq k$ with the following property. Let G be a bigraph, and let $(B_i : i \in I)$ be a (ϕ, μ)-shrink-resistant parade in G, of length at least (K, K). Then there exists $J \subseteq I$ with $|J^-| = |J^+| = k$ such that $(B_i : i \in J)$ has a $(2\phi, \mu)$-band.*

Proof. Let $K \geq k$ be an integer such that for every complete bipartite graph with bipartition (H, J) where $|H|, |J| \geq K$, and every partition of its edge set into $\lfloor 1/\phi + 1 \rfloor$ classes, there exist $H' \subseteq H$ and $J' \subseteq J$ with $|H'|, |J'| = k$ such that all edges between H', J' belong to the same class. (The existence of such a number K follows from Ramsey's theorem by an easy exercise.)

Now let $(B_i : i \in I)$ be a (ϕ, μ)-shrink-resistant parade in G, with max-degree function $d_{i,j}$ $(i, j \in I)$.

For all $h \in I^-$ and $j \in I^+$, there is an integer s such that

$$|V_1(G)|^{-(s+1)\phi} < \frac{d_{j,h}}{|B_h|} \leq |V_1(G)|^{-s\phi}.$$

We call s the *type* of the pair (h, j). Since $|V_1(G)|^{-(s+1)\phi} < d_{j,h}/|B_h| \leq 1$, it follows that $-(s+1)\phi < 0$, and so $s \geq 0$; and since $1/|V_1(G)| \leq d_{j,h}/|B_h| \leq |V_1(G)|^{-s\phi}$ (because $d_{j,h} > 0$ from the definition of (ϕ, μ)-shrink-resistant), it follows that $1 \leq |V_1(G)|^{1-s\phi}$, and so $s \leq 1/\phi$. Hence s is one of the integers $0, 1, \ldots, \lfloor 1/\phi \rfloor$. From the choice of K, there exists $J \subseteq I$ with $|J^-| = |J^+| = k$, such that every pair (h, j) with $h \in J^-$ and $j \in J^+$ has the same type, s say. Let $\tau = |V_1(G)|^{-s\phi}$; then for all $h \in J^-$ and $j \in J^+$,

$$\tau|V_1(G)|^{-\phi} < d_{j,h}/|B_h| \leq \tau.$$

We claim that τ is a $(2\phi, \mu)$-band for $(B_i : i \in J)$. To show this, it remains to show that for all $h \in J^-$ and $j \in J^+$, and for all $X \subseteq B_h$ and $Y \subseteq B_j$ with $|X| \geq \mu|B_h|$ and $|Y| \geq \mu|B_j|$, the max-degree from Y to X is more than $\tau|V_1(G)|^{-2\phi}|B_h|$. But $\mathcal{B}$ is (ϕ, μ)-shrink-resistant, and so the max-degree from Y to X is more than $d_{j,h}|V_1(G)|^{-\phi}$; and since $d_{j,h} \geq \tau|V_1(G)|^{-\phi}|B_h|$, the claim follows. This proves 4.2. ∎

By combining 4.1 and 4.2, we deduce:

4.3 *Let $k \geq 1$ be an integer, and let $\phi, \mu > 0$ be real numbers with $\mu \leq 1$. Then there exists an integer $K \geq k$ with the following property. Let $\mathcal{B} = (B_i : i \in I)$ be a parade of length at least (K, K) in a bigraph G. Let $\beta = \mu^{1+2K^2/\phi}$. Then either*

- *there exist $h \in I^-$ and $j \in I^+$, and $X \subseteq B_h$ and $Y \subseteq B_j$ with $\frac{|X|}{|B_h|}, \frac{|Y|}{|B_j|} \geq \beta$, such that X, Y are anticomplete; or*
- *there exist $J \subseteq I$ with $|J^-| = |J^+| = k$, and a subset $B'_i \subseteq B_i$ with $|B'_i| \geq \beta|B_i|$ for each $i \in J$, such that $(B'_i : i \in J)$ has a (ϕ, μ)-band.*

Proof. Let K satisfy 4.2 with ϕ replaced by $\phi/2$. Let G be a bigraph, and let $\mathcal{B} = (B_i : i \in I)$ be a parade in G, of length at least (K, K). By 4.1, either

- there exist $h \in I^-$ and $j \in I^+$, and $X \subseteq B_i$ and $Y \subseteq B_j$ with $|X|/|B_h|, |Y|/|B_j| \geq \beta$, such that X, Y are anticomplete; or
- there is a $(\phi/2, \mu)$-shrink-resistant contraction $\mathcal{B}' = (B'_i : i \in I)$ of $\mathcal{B}$, such that $|B'_i| \geq \beta|B_i|$ for each $i \in I$.

In the first case the first outcome of the theorem holds. In the second case, by 4.2 applied to $\mathcal{B}'$, the second outcome of the theorem holds. This proves 4.3. ∎

5 Covering with leaves

Again, this section concerns graphs rather than ordered graphs. If G is a graph and $A, B \subseteq V(G)$ are disjoint, we say A *covers* B if every vertex of B has a neighbour in A. A parade with a (ϕ, μ)-band is easier to work with than a general parade, and in particular the next result holds for such a parade.

5.1 *Let $k \geq 1$ be an integer, and let $\tau, \phi, \mu > 0$ be real numbers with $\mu \leq 1/(8k)$ and $\tau \leq 1/(8k^4)$. Let G be a bigraph and let $\mathcal{A} = (A_i : i \in I)$ be a parade in G, with $|I^+|, |I^-| \leq k$, such that τ is a (ϕ, μ)-band for $\mathcal{A}$. Then for each $h \in I^-$ there exist $C_h \subseteq B_h \subseteq A_h$, and for each $h \in I^-$ and $j \in I^+$ there exists $D_{h,j} \subseteq A_j$, with the following properties:*

- $|B_h| \ge |A_h|/2$ *and* $|C_h| \ge (64k^4)^{-k}|V_1(G)|^{-k\phi}|A_h|/16$, *for each* $h \in I^-$; *and*
- $D_{h,j}$ *is anticomplete to* B_i *for all* $i \in I^- \setminus \{h\}$, *and is anticomplete to* $B_h \setminus C_h$, *and covers* C_h, *for each* $h \in I^-$ *and* $j \in I^+$.

(See figure 3.)

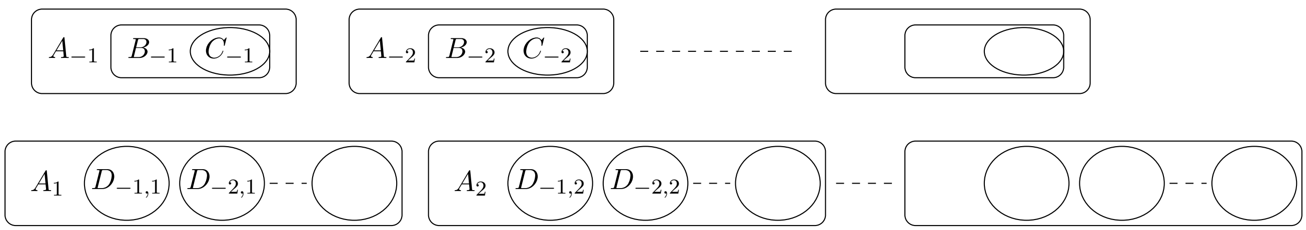

Figure 3: $D_{h,j}$ is anticomplete to B_i for all $i \in I^- \setminus \{h\}$, and is anticomplete to $B_h \setminus C_h$, and covers C_h, for each $h \in I^-$ and $j \in I^+$.

Proof. For each $h \in I^-$ and $j \in I^+$, every vertex in A_j has at most $\tau|A_h|$ neighbours in A_h, and so there are at most $\tau|A_h| \cdot |A_j|$ edges between A_h, A_j. Hence at most $|A_h|/(2k)$ vertices in A_h have at least $2k\tau|A_j|$ neighbours in A_j. For each $h \in I^-$, let P_h be the set of vertices $v \in A_h$ such that for each $j \in I^+$, v has fewer than $2k\tau|A_j|$ neighbours in A_j. It follows that $|P_h| \ge |A_h|/2$ for all $h \in I^-$.

Choose $H \subseteq I^-$ maximal such that for each $h \in H$ there exists $Q_h \subseteq P_h$ with

$$|Q_h| \ge (64k^4)^{-k}|V_1(G)|^{-k\phi}|A_h|/8,$$

and for all $h \in H$ and $j \in I^+$ there exists $D_{h,j} \subseteq A_j$, satisfying:

- $|D_{h,j}| \le 1/(8k^4\tau)$;
- $D_{h,j}$ covers Q_h; and
- every vertex in $D_{h,j}$ has at most $4k^2\tau|Q_i|$ neighbours in Q_i for all $i \in H \setminus \{h\}$.

(This is possible since setting $H = \emptyset$ satisfies the bullets.) We will show that $H = I^-$.

Let $j \in I^+$. For each $h \in H$, each vertex in Q_h has at most $2k\tau|A_j|$ neighbours in A_j, and so there are at most $2k\tau|Q_h| \cdot |A_j|$ edges between Q_h and A_j; and hence at most $|A_j|/(2k)$ vertices in A_j have at least $4k^2\tau|Q_h|$ neighbours in Q_h. Let S_j be the set of vertices $v \in A_j$ such that for each $h \in H$, v has fewer than $4k^2\tau|Q_h|$ neighbours in Q_h. It follows that $|S_j| \ge |A_j|/2$.

(1) *Suppose that* $g \in I^- \setminus H$. *Then there exists* $T_g \subseteq P_g$ *with* $|T_g| \ge |A_g|/4$ *such that, for all* $h \in H$ *and* $j \in H^+$, T_g *is anticomplete to* $D_{h,j}$. *Moreover, for each* $j \in I^+$ *there exists* $Y_j \subseteq T_g$ *with* $|T_g \setminus Y_j| < \mu|A_g|$, *and* $X_j \subseteq S_j$ *with* $|X_j| \le 2|V_1(G)|^{\phi}/\tau$, *such that* X_j *covers* Y_j.

For each $h \in H$ and $j \in I^+$, the set $D_{h,j}$ has cardinality at most $1/(8k^4\tau)$, and since each of its vertices has at most $\tau|A_g|$ neighbours in A_g, it follows that at most $|A_g|/(8k^4)$ vertices in A_g have a neighbour in $D_{h,j}$. Consequently at most $|A_g|/(8k^2)$ vertices in A_g have a neighbour in some $D_{h,j}$;

and since $|P_g| \geq |A_g|/2$, there is a subset $T_g \subseteq P_g$ with $|T_g| \geq |A_g|/4$ such that, for all $h \in H$ and $j \in J^+$, T_g is anticomplete to $D_{h,j}$. This proves the first assertion.

For the second, let $j \in I^+$, and choose $X_j \subseteq S_j$ maximal (possibly empty) such that

- $|X_j| \leq 2|V_1(G)|^{\phi}/\tau$; and
- $|Y_j| \geq \tau|V_1(G)|^{-\phi}|X_j| \cdot |A_g|$, where Y_j is the set of vertices in T_g that have a neighbour in X_j.

Suppose that $|T_g \setminus Y_j| \geq \mu|A_g|$. Since $|S_j| \geq \mu|A_j|$ and τ is a (ϕ, μ)-band for $(A_i : i \in I)$, it follows that the max-degree from S_j to $T_g \setminus Y_j$ is more than $\tau|V_1(G)|^{-\phi}|A_g|$. Choose $v \in S_j$ with more than $\tau|V_1(G)|^{-\phi}|A_g|$ neighbours in $T_g \setminus Y_j$. Since v has a neighbour in $T_g \setminus Y_j$, it follows that $v \notin X_j$, and from the maximality of X_j, adding v to X_j contradicts one of the two bullets in the definition of X_j. The second bullet is satisfied, and so the first is violated; and hence $|X_j| + 1 > 2|V_1(G)|^{\phi}/\tau$. Since $2|V_1(G)|^{\phi}/\tau \geq 1$, it follows that $X_j \neq \emptyset$, and so $2|X_j| \geq |X_j| + 1 > 2|V_1(G)|^{\phi}/\tau$, and therefore $|X_j| > |V_1(G)|^{\phi}/\tau$. So

$$|Y_j| > \tau|V_1(G)|^{-\phi}(|V_1(G)|^{\phi}/\tau)|A_g| = |A_g|,$$

a contradiction. This proves that $|T_g \setminus Y_j| < \mu|A_g|$, and so proves (1).

(2) $H = I^-$.

Suppose that $H \neq I^-$, and choose $g \in I^- \setminus H$. For each $j \in I^+$, let T_j, X_j, Y_j be as in (1). Let Y be the intersection of the sets Y_j ($j \in I^+$). Since each Y_j satisfies $|T_g \setminus Y_j| < \mu|A_g|$, it follows that $|Y| \geq |T_g| - k\mu|A_g| \geq |A_g|/8$ (since $k\mu \leq 1/8$ and $|T_g| \geq |A_g|/4$). Let $j \in I^+$. Since $\lfloor 1/(8k^4\tau)\rfloor \geq 1/(16k^4\tau)$ (because $8k^4\tau \leq 1$), there is a partition of X_j into at most $\lceil 16k^4\tau|X_j|\rceil$ sets each of cardinality at most $1/(8k^4\tau)$. But $|X_j| \leq 2|V_1(G)|^{\phi}/\tau$, and so

$$\lceil 16k^4\tau|X_j|\rceil \leq \lceil 32k^4|V_1(G)|^{\phi}\rceil \leq 64k^4|V_1(G)|^{\phi}$$

since $32k^4|V_1(G)|^{\phi} \geq 1$. Thus X_j admits a partition $\mathcal{R}_j$ into at most $64k^4|V_1(G)|^{\phi}$ sets each of cardinality at most $1/(8k^4\tau)$. For each $v \in Y$, there exists $u \in X_j$ adjacent to Y; choose some such u, choose $R \in \mathcal{R}_j$ containing u, and say R is the *j-type* of v. Each vertex of Y has a j-type, for each $j \in I^+$; and since there are only at most $64k^4|V_1(G)|^{\phi}$ j-types for each j, and $|I^+| \leq k$, it follows that there exists $Q_g \subseteq Y$ with

$$|Q_g| \geq (64k^4)^{-k}|V_1(G)|^{-k\phi}|Y| \geq (64k^4)^{-k}|V_1(G)|^{-k\phi}|A_g|/8,$$

such that for all $j \in I^+$, all members of Q_g have the same j-type, say $D_{g,j} \in \mathcal{R}_j$, and each $D_{g,j}$ covers Q_g. Since

- $|D_{g,j}| \leq 1/(8k^4\tau)$ for each $j \in I^+$;
- for every $j \in I^+$ and $h \in H$, every vertex in $D_{h,j}$ has no neighbours in Q_g, since $Q_g \subseteq T_g$; and
- for every $j \in I^+$ and $h \in H$, every vertex of $D_{g,j}$ has at most $4k^2\tau|Q_h|$ neighbours in Q_h, because $D_{g,j} \subseteq S_j$,

this contradicts the maximality of H, and so proves (2).

Now let $i \in I^-$. Since there are only at most k^2 sets $D_{h,j}$, and each has cardinality at most $1/(8k^4\tau)$, and every vertex in a set $D_{h,j}$ for $h \neq i$ has at most $4k^2\tau|Q_i|$ neighbours in Q_i, it follows that at most $|Q_i|/2$ vertices in Q_i have a neighbour in some set $D_{h,j}$ with $h \neq i$; and so there exists $C_i \subseteq Q_i$ with $|C_i| \geq |Q_i|/2$ such that C_i is anticomplete to $D_{h,j}$ for all $h \in I^- \setminus \{i\}$ and $j \in I^+$. Hence $|C_i| \geq (64k^4)^{-k}|V_1(G)|^{-k\phi}|A_h|/16$ for each $i \in I^-$. Moreover, since the union of all the sets $D_{h,j}$ has cardinality at most $1/(8\tau k^2)$, and each vertex of this union has at most $\tau|A_i|$ neighbours in A_i, it follows that at most $|A_i|/(8k^2)$ vertices in A_i have a neighbour that belongs to some $D_{h,j}$; and consequently there exists $B_i \subseteq A_i$ with $C_i \subseteq B_i$, and with $|B_i| \geq |A_i|/2$, such that $B_i \setminus C_i$ is anticomplete to all the sets $D_{h,j}$. This proves 5.1. ∎

In 5.1 there is no removal of blocks, but the hypothesis that there is a (ϕ, μ)-band is awkward. We can eliminate it at the cost of having to remove some blocks, by means of 4.3. The next result is a combination of 4.3 and 5.1, specifying values for ϕ and μ. Note that that the "anticomplete pair" outcome of this theorem provides two anticomplete sets both of linear size (relative to the original blocks that contain them). This is the theorem that will provide the anticomplete pair of 1.10, but it will be applied to a parade where the blocks in one part of the bipartition already have sublinear size, and consequently the anticomplete pair found in 1.10 will have one set of sublinear size.

5.2 *Let $k \geq 1$ be an integer, and let $c > 0$ be a real number. Then there exists an integer $K > 0$ with the following property. Let $\beta = (8k)^{-1-2K^2k/c}$, and let $\mathcal{A} = (A_i : i \in I)$ be a parade of length at least (K, K) in a bigraph G. Then either*

- *there exist $h \in I^-$ and $j \in I^+$, and $X \subseteq A_h$ and $Y \subseteq A_j$ with $\frac{|X|}{|A_h|}, \frac{|Y|}{|A_j|} \geq \beta$, such that X, Y are anticomplete; or*
- *there exist $h \in I^-$ and $j \in I^+$ such that some $v \in A_j$ has at least $\frac{\beta}{8k^4}|A_h|$ neighbours in A_h; or*
- *there exist $J \subseteq I$ with $|J^-| = |J^+| = k$, and for each $h \in J^-$ there exist $C_h \subseteq B_h \subseteq A_h$ with $|B_h| \geq \beta|A_h|/2$ and $|C_h| \geq (64k^4)^{-k}\beta|V_1(G)|^{-c}|A_h|/16$; and for each $h \in J^-$ and $j \in J^+$ there exists $D_{h,j} \subseteq A_j$, such that $D_{h,j}$ is anticomplete to B_i for all $i \in J^- \setminus \{h\}$, and is anticomplete to $B_h \setminus C_h$, and covers C_h.*

Proof. By 4.3, taking $\mu = 1/(8k)$ and $\phi = c/k$, we may assume that there exist $J \subseteq I$ with $|J^-| = |J^+| = k$, and a subset $F_i \subseteq A_i$ with $|F_i| \geq \beta|A_i|$ for each $i \in J$, such that $(F_i : i \in J)$ has a (ϕ, μ)-band τ. We may assume that for all $j \in J^+$ and $v \in A_j$ and $h \in I^-$, v has fewer than $(\beta/(8k^4))|A_h|$ neighbours in A_h, and hence has fewer than $|F_h|/(8k^4)$ neighbours in F_h. Consequently we may assume that $\tau \leq 1/(8k^4)$. By 5.1 applied to $\mathcal{F} = (F_i : i \in J)$, for each $h \in J^-$ there exists $B_h \subseteq F_h$ with $|B_h| \geq |F_h|/2 \geq \beta|A_h|/2$, and there exists $C_h \subseteq B_h$ with

$$|C_h| \geq (64k^4)^{-k}|V_1(G)|^{-k\phi}|F_h|/16 \geq (64k^4)^{-k}\beta|V_1(G)|^{-k\phi}|A_h|/16;$$

and for each $h \in J^-$ and $j \in J^+$ there exists $D_{h,j} \subseteq F_j$ covering C_h, such that $D_{h,j}$ is anticomplete to B_i for all $i \in I^- \setminus \{h\}$, and is anticomplete to $B_h \setminus C_h$, and covers C_h. This proves 5.2. ∎

This result 5.2 is not symmetric under exchanging the two parts of the bipartition, and so there is a similar result, that we call the "switched version of 5.2", with the two parts exchanged.

6 The proof of 1.10

Let us sketch the ideas of the remainder of the proof of 1.10. We are given an ordered bigraph G, an ordered tree bigraph T, and a real number c with $0 < c < 1$. We choose a large number K (independent of G), and partition $V_1(G)$ into K intervals, all of size about $|V_1(G)|/K$, and the same for $V_2(G)$. This makes a parade. We need to prove that, if the max-degrees between blocks are at most ε times the size of the corresponding block, where ε is some very small constant, then either the desired anticomplete pair exists, or a "rainbow" copy of T exists in G, that is, one where each vertex belongs to a different block of the parade. (The advantage of using intervals is that, to check that a rainbow copy of T as an unordered bigraph is also a copy of T as an ordered bigraph, we only have to know which vertex of T is mapped into which block of the parade, and this is easy to control.)

Let us apply 5.2 to this parade. We can assume that neither of the first two outcomes of 5.2 applies, because they lead immediately to the first two outcomes of 1.10. Thus, in the notation of 5.2, for each $h \in J^-$ we have subsets $C_h \subseteq B_h \subseteq A_h$ with $|B_h| \geq \beta|A_h|/2$ and $|C_h| \geq (64k^4)^{-k}\beta|V_1(G)|^{-c}|A_h|/16$; and for each $h \in J^-$ and $j \in J^+$ we have $D_{h,j} \subseteq A_j$, such that $D_{h,j}$ is anticomplete to B_i for all $i \in J^- \setminus \{h\}$, and is anticomplete to $B_h \setminus C_h$, and covers C_h.

How can we use this? It is good for the trivial case when T has radius one and its root is in $V_1(T)$ with degree d. Choose any $h \in J^-$, and any $v \in C_h$, and choose any distinct $j_1, \dots, j_d$ in J^+. Then for $1 \leq i \leq d$ there is a vertex in D_{h,j_i} adjacent to v, and these vertices together with v make a copy of T. (If the root of T belongs to $V_2(T)$, we apply the switched version of 5.2 instead.) It also works for the much less trivial case when T is a "star forest", a forest in which every component is a tree of radius one with root in $V_1(T)$. Let us see this more carefully. We can get a copy of each component of the forest T as before (using disjoint sets of blocks for distinct components of T), but there are two new issues: we need to make sure there are no edges of G between these copies of components; and we need to map the vertices of T to vertices of G in the right order. The first issue is fine, because the only vertices of $V_2(G)$ we use belong to sets $D_{h,j}$, and $D_{h,j}$ is anticomplete to B_i (and hence to C_i) for all $i \in J^- \setminus \{h\}$. And the second issue is fine, because we can choose any block we want to contain any vertex we want, and so we can arrange the orders as we want, since the blocks are intervals.

If T is a tree of radius two, we need to apply 5.2 a second time. We have a parade formed by the sets C_h ($h \in J^-$) and A_j ($j \in J^+$); let us apply the switched version of 5.2 to this parade. It has three possible outcomes. The first is good for us; although, note that the sets C_h are no longer of linear cardinality, so if we obtain this outcome, one of its two sets will be sublinear (the other will still be linear). The second outcome is good for us, because it says (since we are using the switched version of 5.2) that some vertex in some C_j is adjacent to a large fraction of some A_h. Thus we may assume the third outcome applies, and so (after removing many of the blocks, and changing the meaning of J correspondingly) for each $j \in J^+$ there exists $C_j \subseteq B_j \subseteq A_j$ with $|B_j| \geq \beta|A_j|/2$ and $|C_j| \geq (64k^4)^{-k}\beta|V_2(G)|^{-c}|A_j|/16$; and for each $h \in J^-$ and $j \in J^+$ there exists $D_{j,h} \subseteq C_h$, such that $D_{j,h}$ is anticomplete to B_i for all $i \in J^+ \setminus \{j\}$, and is anticomplete to $B_j \setminus C_j$, and covers C_j. Now, to get our tree T of radius two, and with root $w \in V_2(T)$ say,

- if w is the nth vertex in the order of $V_2(T)$, let j be the nth member of J^+ and map w to some vertex $\phi(w) \in C_j$;
- if w has d neighbours $v_1, \dots, v_d$ in T, choose distinct $h_1, \dots, h_d \in J^-$, and for $1 \leq i \leq d$ choose

$\phi(v_i)$ in D_{j,h_i} adjacent to $\phi(w)$; and

- now choose the remainder of T in the same way that we handled the star forest in the previous paragraph, being careful to map the vertices in $V_2(T)$ to vertices in $V_2(G)$ before or after $\phi(w)$ appropriately. (Note that we choose $\phi(v_i)$ in $D_{j,h_i} \subseteq C_{h_i}$, and so the method used for the star forest still works.)

If T has radius three, we iterate 5.2 a third time, applying it to the parade formed by the sets B_h ($h \in J^-$) and C_h ($h \in J^+$) . Note that we use the sets B_h ($h \in J^-$), not the original sets A_h ($h \in J^-$), because we want to avoid neighbours of the vertices in the sets $D_{h,j}$; but the sets B_h are still of linear size, so it still works. And so on; the number of times we iterate 5.2 is the radius of T. This finishes the sketch of the proof.

There are two steps to go: to formulate and prove what is obtained by iterating 5.2; and then to use it to extract T. The next result is the first of these two steps. Let $\mathcal{A} = (A_i : i \in I)$ be a parade, and let $r \geq 0$ be an integer. Suppose there are subsets satisfying the following:

- For each $h \in I^-$ and for each odd $q \in \{0, \dots, r\}$ let $\emptyset \neq C_h^q \subseteq B_h^q \subseteq A_h$, with
$$B_h^1 \supseteq B_h^3 \supseteq B_h^5 \supseteq \cdots .$$
- For each $j \in I^+$ and for each even $q \in \{0, \dots, r\}$, let $\emptyset \neq C_j^q \subseteq B_j^q \subseteq A_j$ with $C_j^0 = B_j^0 = A_j$ and with
$$B_j^0 \supseteq B_j^2 \supseteq B_j^4 \supseteq \cdots .$$
- For each $h \in I^-$ and $j \in I^+$, and for each odd $q \in \{1, \dots, r\}$, let $D_{h,j}^q \subseteq C_j^{q-1}$, such that $D_{h,j}^q$ is anticomplete to B_i^q for all $i \in I^- \setminus \{h\}$, and $D_{h,j}^q$ is anticomplete to $B_h^q \setminus C_h^q$, and $D_{h,j}^q$ covers C_h^q.
- For each $h \in I^-$ and $j \in I^+$, and for each even $q \in \{1, \dots, r\}$, let $D_{j,h}^q \subseteq C_h^{q-1}$, such that $D_{j,h}^q$ is anticomplete to B_i^q for all $i \in I^+ \setminus \{j\}$, and $D_{j,h}^q$ is anticomplete to $B_j^q \setminus C_j^q$, and $D_{j,h}^q$ covers C_j^q.

We call such an array of subsets an *r-fold cover* in $\mathcal{A}$.

6.1 *Let $0 < c \leq 1$. For all integers $r \geq 0$ and $k \geq 1$, there exist an integer $K > 0$, and a real number $\gamma > 0$ with the following property. Let G be a bigraph and let $\mathcal{A} = (A_i : i \in I)$ be a parade in G with length at least (K, K). Then either:*

- *there exist $h \in I^-$ and $j \in I^+$, and $X \subseteq A_h$ and $Y \subseteq A_j$, either with $|X| \geq \gamma|A_h|$ and $|Y| \geq \gamma|V_2(G)|^{-c}|A_j|$, or with $|X| \geq \gamma|V_1(G)|^{-c}|A_h|$ and $|Y| \geq \gamma|A_j|$, such that X, Y are anticomplete; or*
- *there exist $h, j \in I$ with opposite sign, and $v \in A_h$, such that v has at least $\gamma|A_j|$ neighbours in A_j; or*
- *there is a subparade of $\mathcal{A}$ of length (k, k) that admits an r-fold cover, such that (in the notation of the definition) for each $q \in \{0, \dots, r\}$, if q is odd then $|B_h^q| \geq \gamma|A_h|$ and $|C_h^q| \geq \gamma|V_1(G)|^{-c}|A_h|$ for each $h \in I^-$, and if q is even then $|B_j^q| \geq \gamma|A_j|$ and $|C_j^q| \geq \gamma|V_2(G)|^{-c}|A_j|$ for each $j \in I^+$.*

Proof. We prove this by induction on r; the result is trivial when $r = 0$ (take $K = k$ and $\gamma = 1$), and follows from 5.2 when $r = 1$, so we may assume that it holds (for all k) with r replaced by $r - 1$. Choose k' such that setting $K = k'$ satisfies 5.2 (and therefore also satisfies the switched version of 5.2). From the inductive hypothesis, there exist an integer $K > 0$ and $\gamma' > 0$ such that the assertion of 6.1 holds with r, k, γ replaced by $r-1, k', \gamma'$ respectively. Let $\beta = (8k)^{-1-2k'^2k/c}$, and $\gamma = (64k^4)^{-k}\beta\gamma'/16$. We claim that the theorem is satisfied. To see this, let G be a bigraph and let $\mathcal{A} = (A_i : i \in I)$ be a parade in G with length at least (K, K). Let us apply the inductive hypothesis to $\mathcal{A}$ with r, k, γ replaced by $r-1, k', \gamma'$ respectively. Since $\gamma' \geq \gamma$, we may assume that the third outcome holds: and so there exists $L \subseteq I$ with $|L^-|, |L^+| = k'$, that admits an $(r-1)$-fold cover, as in the third outcome of the theorem with r, k, γ replaced by $r-1, k', \gamma'$ respectively. We use notation for this cover as in the definition of an r-fold cover.

We assume that r is odd (the even case is similar, using the switched version of 5.2 in place of 5.2 itself). Let $A'_i = B_i^{r-2}$ for $i \in L^-$, and $A'_i = C_i^{r-1}$ for $i \in L^+$. Then $(A'_i : i \in L)$ is a parade of length at least (k', k'), and so from 5.2 and the choice of k', either

- there exist $h \in L^-$ and $j \in L^+$, and $X \subseteq A'_h$ and $Y \subseteq A'_j$ with $\frac{|X|}{|A'_h|}, \frac{|Y|}{|A'_j|} \geq \beta$, such that X, Y are anticomplete; or
- there exist $h \in L^-$ and $j \in L^+$ such that some $v \in A_j$ has at least $\frac{\beta}{8k^4}|A'_h|$ neighbours in A'_h; or
- there exists $J \subseteq L$ with $|J^-| = |J^+| = k$, and for each $h \in J^-$ there exist $C_h^r \subseteq B_h^r \subseteq B_h^{r-2}$ with $|B_h^r| \geq \beta|B_h^{r-2}|/2$ and $|C_h^r| \geq (64k^4)^{-k}\beta|V_1(G)|^{-c}|B_h^{r-2}|/16$; and for each $h \in J^-$ and $j \in J^+$ there exists $D_{h,j}^r \subseteq C_j^{r-1}$ covering C_h^r, such that $D_{h,j}^r$ is anticomplete to B_i^r for all $i \in J^- \setminus \{h\}$, and is anticomplete to $B_h^r \setminus C_h^r$, and covers C_h^r.

In the first case, the first outcome of the theorem holds, since

$$|X| \geq \beta|A'_h| = \beta|B_h^{r-2}| \geq \beta\gamma'|A_h| \geq \gamma|A_h|$$

and

$$|Y| \geq \beta|A'_j| = \beta|C_j^r| \geq \beta\gamma'|V_2(G)|^{-c}|A_j| \geq \gamma|V_2(G)|^{-c}|A_j|.$$

In the second case, the second outcome of the theorem holds, since

$$\frac{\beta}{8k^4}|A'_h| = \frac{\beta}{8k^4}|B_h^{r-2}| \geq \frac{\beta}{8k^4}\gamma'|A_h| \geq \gamma|A_h|.$$

In the third case, the third outcome of the theorem holds, since

$$|B_h^r| \geq \beta|B_h^{r-2}|/2 \geq \beta\gamma'|A_h|/2 \geq \gamma|A_h|,$$

and

$$|C_h^r| \geq (64k^4)^{-k}\beta|V_1(G)|^{-c}|B_h^{r-2}|/16 \geq (64k^4)^{-k}\beta|V_1(G)|^{-c}\gamma'|A_h|/16 = \gamma|V_1(G)|^{-c}|A_h|$$

for each $h \in J^-$. This proves 6.1. ∎

When we apply 6.1, all we will need from its third bullet is that an r-fold cover exists. The information about cardinalities is only used for inductive purposes in the proof of 6.1. Now we can prove 1.10, which we restate:

6.2 *Let T be an ordered tree bigraph. For all $c > 0$ there exists $\varepsilon > 0$ with the following property. Let G be an ordered bigraph not containing T, such that every vertex in $V_1(G)$ has degree less than $\varepsilon|V_2(G)|$, and every vertex in $V_2(G)$ has degree less than $\varepsilon|V_1(G)|$. Then there are subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, either with $|Z_1| \geq \varepsilon|V_1(G)|$ and $|Z_2| \geq \varepsilon|V_2(G)|^{1-c}$, or with $|Z_1| \geq \varepsilon|V_1(G)|^{1-c}$ and $|Z_2| \geq \varepsilon|V_2(G)|$, such that Z_1, Z_2 are anticomplete.*

Proof. We may assume that $|V(T)| \geq 2$; let T have radius r, and choose $w \in V(T)$ with T-distance at most r from every other vertex of T. We define the *parent* of $v \in V(T) \setminus \{w\}$ to be the neighbour of v in the path to w. By exchanging $V_1(T)$ and $V_2(T)$ if necessary, we may assume that every vertex of T with T-distance r from w belongs to $V_2(T)$, that is, $w \in V_1(T)$ if and only if r is odd. Choose an integer $k \geq 1$ such that $|V_1(T)|, |V_2(T)| \leq k$. Choose K, γ as in 6.1, and let $\varepsilon = \gamma/(2K)$. We claim that ε satisfies the theorem.

Let G be an ordered bigraph that does not contain T, such that every vertex in $V_1(G)$ has degree less than $\varepsilon|V_2(G)|$, and every vertex in $V_2(G)$ has degree less than $\varepsilon|V_1(G)|$. If G has no edges then Z_1, Z_2 exist as required, so we may assume that G has an edge; and so $\varepsilon|V_i(G)| > 1$ for $i = 1, 2$. Let $p = \lceil |V_1(G)|/(2K) \rceil$; then $p \leq |V_1(G)|/K$, since $|V_1(G)| > 1/\varepsilon \geq K$. Let the vertices of $V_1(G)$ be $u_1, \ldots, u_{n_1}$, ordered according to the linear order of $V_1(G)$ imposed by G. Let

$$A_i = \{u_{(K+i)p+1}, \ldots, u_{(K+i+1)p}\}$$

for $-K \leq i \leq -1$. Similarly, let $q = \lceil |V_2(G)|/(2K) \rceil$, and $V_2(G) = \{v_1, \ldots, v_{n_2}\}$ in order, and for $1 \leq i \leq K$ let

$$A_i = \{v_{(i-1)q+1}, \ldots, v_{iq}\}.$$

Let $I = \{-K, \ldots, -1, 1, \ldots, K\}$. Then $\mathcal{A} = (A_i : i \in I)$ is a parade in G, of length (K, K) and width (p, q), and all its blocks are intervals of the linear order, in the natural sense.

Let us apply 6.1 to $(A_i : i \in I)$, and deduce that one of the three outcomes of 6.1 holds. Suppose that the first outcome holds, that is, there exist $h \in I^-$ and $j \in I^+$, and $Z_1 \subseteq A_h$ and $Z_2 \subseteq A_j$, either with $|Z_1| \geq \gamma|A_h|$ and $|Z_2| \geq \gamma|V_2(G)|^{-c}|A_j|$, or with $|Z_1| \geq \gamma|V_1(G)|^{-c}|A_h|$ and $|Z_2| \geq \gamma|A_j|$, such that Z_1, Z_2 are anticomplete. Since

$$\gamma|A_h| = \gamma p \geq \gamma|V_1(G)|/(2K) = \varepsilon|V_1(G)|$$

and

$$\gamma|A_j| \geq \gamma q \geq \gamma|V_2(G)|/(2K) = \varepsilon|V_2(G)|,$$

in this case the theorem holds.

Now suppose that the second outcome holds, that is, there exist $h, j \in I$ with opposite sign, and $v \in A_h$, such that v has at least $\gamma|A_j|$ neighbours in A_j. From the symmetry we may assume that $h \in I^-$. Since

$$\gamma|A_j| = \gamma q \geq \gamma|V_2(G)|/(2K) = \varepsilon|V_2(G)|$$

this is impossible.

Thus we may assume that the third outcome holds, and so there exists $J \subseteq I$ with $|J^-| = |J^+| = k$ such that the sub-parade $(A_i : i \in J)$ of $\mathcal{A}$ admits an r-fold cover. We use notation for this cover as in the definition of r-fold cover. Choose a map θ from $V(T)$ to J with the following properties:

- for each $v \in V(T)$, $\theta(v) \in J^-$ if and only if $v \in V_1(T)$; and
- for $i = 1, 2$, for all distinct $u, v \in V_i(T)$, if u is earlier than v in the ordering of $V_i(T)$ imposed by T, then $\theta(u) < \theta(v)$.

This is possible since $|V_1(T)|, |V_2(T)| \le k$. Choose $\phi(w) \in C^r_{\theta(w)}$. (Note that $w \in V_1(T)$ if and only if r is odd, and $w \in V_1(T)$ if and only if $\theta(w) < 0$, so $C^r_{\theta(w)}$ exists, from the definition of an r-fold cover.) Inductively for $q = r-1, \ldots, 0$, if $v \in V(T)$ and the T-distance from w to v is $r-q$, let u be the parent of v, and choose

$$\phi(v) \in D^{q+1}_{\theta(u),\theta(v)} \subseteq C^q_{\theta(v)}$$

adjacent to $\phi(u)$. (This is possible since $\phi(u) \in C^{q+1}_{\theta(u)}$ and $D^{q+1}_{\theta(u),\theta(v)}$ covers $C^{q+1}_{\theta(u)}$.) Thus ϕ is an isomorphism from the unordered graph of T to an unordered subtree $\phi(T)$ of G. We need to check that ϕ respects the orders, and that this subtree is induced.

First, we observe that $w \in V_1(T)$ if and only if $\theta(w) \in J^-$, that is, if and only if $\phi(w) \in V_1(G)$; and so, since T is connected, $v \in V_1(T)$ if and only if $\phi(v) \in V_1(G)$ for each vertex v of T. Now let $i \in \{1, 2\}$, and let $u, v \in V_i(T)$ be distinct, where u is earlier than v in the ordering of $V_i(T)$ imposed by T. Consequently $\theta(u) < \theta(v)$, and so $\phi(u)$ is earlier than $\phi(v)$ in the ordering of $V_i(G)$ imposed by G. Thus ϕ is an isomorphism from the ordered bigraph T to an ordered sub-bigraph of G.

To check that $\phi(T)$ is induced, let $v, v' \in V(T)$ be distinct, such that $\phi(v), \phi(v')$ are adjacent in G. We must show that v, v' are adjacent in T. Since G is bipartite, v, v' do not have the same T-distance from w, and so we may assume that the T-distances from w to v, v' are $r-q, r-q'$ respectively, where $q < q'$. Let u be the parent of v in T. Thus $\phi(v) \in D^{q+1}_{\theta(u),\theta(v)}$ and

$$\phi(v') \in C^{q'}_{\theta(v')} \subseteq B^{q'}_{\theta(v')} \subseteq B^{q+1}_{\theta(v')}.$$

If $\theta(v') \neq \theta(u)$, the definition of an r-fold cover implies that $D^{q+1}_{\theta(u),\theta(v)}$ is anticomplete to $B^{q+1}_{\theta(v')}$, a contradiction. Thus $\theta(v') = \theta(u)$, and so $v' = u$, as required. This proves 6.2. ∎

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