# Polynomial bounds for chromatic number
# VI. Adding a four-vertex path

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**Abstract**

A hereditary class of graphs is $\chi$-*bounded* if there is a function $f$ such that every graph $G$ in the class has chromatic number at most $f(\omega(G))$, where $\omega(G)$ is the clique number of $G$; and the class is *polynomially* $\chi$-*bounded* if $f$ can be taken to be a polynomial. The Gyárfás-Sumner conjecture asserts that, for every forest $H$, the class of $H$-free graphs (graphs with no induced copy of $H$) is $\chi$-bounded. Let us say a forest $H$ is *good* if it satisfies the stronger property that the class of $H$-free graphs is polynomially $\chi$-bounded.

Very few forests are known to be good: for example, the goodness of the five-vertex path is open. Indeed, it is not even known that if every component of a forest $H$ is good then $H$ is good, and in particular, it was not known that the disjoint union of two four-vertex paths is good. Here we show the latter (with corresponding polynomial $\omega(G)^{16}$); and more generally, that if $H$ is good then so is the disjoint union of $H$ and a four-vertex path. We also prove an even more general result: if every component of $H_1$ is good, and $H_2$ is any path (or broom) then the class of graphs that are both $H_1$-free and $H_2$-free is polynomially $\chi$-bounded.

# 1 Introduction

A class of graphs is *hereditary* if it is closed under taking induced subgraphs; a hereditary class is $\chi$*-bounded* if there is a function $f$ such that every graph $G$ in the class has chromatic number at most $f(\omega(G))$, where $\omega(G)$ is the clique number of $G$; and the class is *polynomially* $\chi$*-bounded* if $f$ can be taken to be a polynomial. A graph is $H$*-free* if it has no induced subgraph isomorphic to $H$.

The Gyárfás-Sumner conjecture [4, 14] asserts:

**1.1 Conjecture:** *For every forest $H$, the class of $H$-free graphs is $\chi$-bounded.*

There has been a great deal of recent progress on $\chi$-bounded classes (see [9] for a survey), although the Gyárfás-Sumner conjecture remains open. In most cases, proofs of $\chi$-boundedness give fairly fast-growing functions, so it is interesting to ask: when do we get the stronger property of polynomial $\chi$-boundedness?

A provocative conjecture of Louis Esperet [3] asserted that every $\chi$-bounded hereditary class is polynomially $\chi$-bounded, but this was recently disproved by Briański, Davies and Walczak [1]. So the question now is: which hereditary classes are polynomially $\chi$-bounded? In particular, can 1.1 be strengthened to polynomial $\chi$-boundedness? Let us say a graph $H$ is *good* if the class of $H$-free graphs is polynomially $\chi$-bounded. Perhaps every forest is good, but the only trees currently known to be good are those not containing the five-vertex path $P_5$ [11]. It is not known whether $P_5$ is good (although see [12] for the best current bounds for $H = P_5$; and see [13] for the case when $H$ is a general tree of radius two).

In the case of $\chi$-boundedness, it is not hard to show that a forest $H$ satisfies the Gyárfás-Sumner conjecture if and only if all its components do. But it has *not* been shown that if every component of a forest $H$ is good then $H$ is good. Indeed, only some very restricted forests are known to be good [8, 10]. One outstanding case was when $H$ is the forest $2P_4$, the disjoint union of two copies of the four-vertex path $P_4$; and this was particularly annoying since the $P_4$-free graphs are very well-understood and rather trivial. We will prove that $2P_4$ is good, and indeed:

**1.2** *If $G$ is $2P_4$-free, then $\chi(G) \leq \omega(G)^{16}$.*

More generally, we will prove the following:

**1.3** *If $H$ is a good forest, then the disjoint union of $H$ and $P_4$ is also good.*

1.3 is a consequence of the next result, about brooms. A $(k, d)$*-broom* is a tree obtained from a $k$-vertex path with one end $v$ by adding $d$ new vertices adjacent to $v$, and a *broom* is a tree that is a $(k, d)$-broom for some $k, d$. It is known that $(3, d)$-brooms are good [6, 11], but this is not known for larger brooms (all of which contain $P_5$). We will show the following, which implies 1.3:

**1.4** *Let $H_1$ be a forest such that every component of $H_1$ is good, and let $H_2$ be either a broom, or the disjoint union of a good forest and a number of paths. Then there is a polynomial $\phi$ such that $\chi(G) \leq \phi(\omega(G))$ for every $\{H_1, H_2\}$-free graph $G$.*

($\{H_1, H_2\}$*-free* means both $H_1$-free and $H_2$-free.) To deduce 1.3 from 1.4, let $H$ be a good forest, let $H_1 = H_2$ be the disjoint union of $H$ and $P_4$, and apply 1.4.

Some notation and terminology: if $G$ is a graph and $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of $G$ induced on $X$, and we sometimes write $\chi(X)$ for $\chi(G[X])$ and $\omega(X)$ for $\omega(G[X])$. Two disjoint subsets $A, B \subseteq V(G)$ are *complete* if every vertex in $A$ is adjacent to every vertex of $B$, and *anti-complete* if there is no edge between $A, B$; and we say a vertex $v$ *is complete to* $B$ if $\{v\}$ is complete to $B$, and so on. A graph $G$ *contains* a graph $H$ if some induced subgraph of $G$ is isomorphic to $H$, and such a subgraph is a *copy* of $H$. The *cone* of a graph $H$ is obtained from $H$ by adding a new vertex adjacent to every vertex of $H$.

Let us say a graph is 0-*bad* if it is good; and a graph $J$ is $\beta$-bad, where $\beta \geq 1$ is an integer, if either $J$ is the disjoint union of two $(\beta - 1)$-bad graphs, or $J$ is the cone of a $(\beta - 1)$-bad graph, or $J$ is $(\beta - 1)$-bad. In general, cones are not forests, so they are not good. Nevertheless, we will prove the following strengthening of 1.4:

**1.5** *Let $\beta \geq 0$, let $H_1$ be a $\beta$-bad graph, and let $H_2$ be either a broom, or the disjoint union of a good forest and a number of paths. Then there is a polynomial $\phi$ such that $\chi(G) \leq \phi(\omega(G))$ for every $\{H_1, H_2\}$-free graph $G$.*

This implies several results that were previously known. For instance, in [7] it is proved that:

**1.6** *Let $H_1$ be either*

- *the disjoint union of a complete graph and a good graph, or*
- *the disjoint union of some complete graphs, or*
- *the cone of the disjoint union of some complete graphs.*

*Let $H_2$ be a path. Then there is a polynomial $\phi$ such that $\chi(G) \leq \phi(\omega(G))$ for every $\{H_1, H_2\}$-free graph $G$.*

Some other results of [7, 8] are also special cases of 1.5.

## 2 Finding a disjoint union

Suppose that $H$ is the disjoint union of good forests $H_1, H_2$. Choose $c_1, c_2$ such that for $i = 1, 2$, every $H_i$-free graph $G$ satisfies $\chi(G) \leq \omega(G)^{c_i}$. Thus, if $G$ is $H$-free, we know that there do not exist disjoint, anticomplete subsets $P, Q \subseteq V(G)$ with $\chi(P) > \omega(P)^{c_1}$ and $\chi(Q) > \omega(Q)^{c_2}$; because then $G[P]$ is not $H_1$-free, and $G[Q]$ is not $H_2$-free, and the union of a copy of $H_1$ in $G[P]$ and a copy of $H_2$ in $G[Q]$ gives a copy of $H$, which is impossible.

But we do not really need $P, Q$ to be anticomplete. It is enough that $\chi(P) > \omega(P)^{c_1}$, and $\chi(Q) > |H_1|r + \omega(Q)^{c_2}$, where $r$ denotes the maximum over $v \in P$ of the chromatic number of the set of neighbours of $v$ in $Q$; because then if we choose a copy $H_1'$ of $H$ in $G[P]$, the chromatic number of the set of vertices in $Q$ with no neighbours in $V(H_1')$ is at least $\chi(Q) - |H_1|r > \omega(Q)^{c_2}$, and so this set contains a copy of $H_2$, a contradiction. In the proof to come later in the paper, this is the only way we will ever use that $G$ is $H$-free; and so we might as well prove a stronger theorem, replacing the hypothesis that $G$ is $H$-free with the weaker hypothesis that there is no suitable pair $(P, Q)$ in $G$.

Thus we will be excluding pairs of disjoint sets $P, Q$ where $\chi(P)$ is at least some power of $\omega(P)$, and for each vertex in $P$, its set of neighbours in $Q$ has chromatic number at most some $r$ that is small compared with the chromatic number of $Q$.

In our proof, it happens that when we find a suitable pair $(P, Q)$, it comes equipped with an extra vertex $v$ that is complete to $P$ and anticomplete to $Q$; so we might as well prove that there is a "suitable triple" $(v, P, Q)$. Such a thing will also allow us to handle cones.

We denote the set of nonnegative integers by $\mathbb{N}$, and say a function $\phi : \mathbb{N} \to \mathbb{N}$ is *non-decreasing* if $\phi(x) \leq \phi(x')$ for all $x, x' \in \mathbb{N}$ with $x \leq x'$.

Let $\psi : \mathbb{N} \to \mathbb{N}$ be non-decreasing, and let $q \geq 0$ be an integer. We say a $(\psi, q)$-*scattering* in a graph $G$ is a triple $(v, P, Q)$ where:

- $P, Q$ are disjoint subsets of $V(G)$, and $v \in V(G) \setminus (P \cup Q)$;
- $\{v\}$ is complete to $P$ and anticomplete to $Q$;
- $\chi(P) > \psi(\omega(P))$; and
- $\chi(Q) > qr + \psi(\omega(Q))$, where $r$ is the maximum, over $u \in P$, of the chromatic number of the set of neighbours of $u$ in $Q$.

Thus we will replace the hypothesis in 1.5 that $G$ is $H_1$-free and $H_1$ is $\beta$-bad, with the hypothesis that $G$ contains no $(\psi, q)$-scattering, for appropriate $\psi, q$. We will show:

**2.1** *Let $\psi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing polynomial and let $q \in \mathbb{N}$. Let $H_2$ be either a broom, or the disjoint union of a good forest and a number of paths. Then there is a polynomial $\phi : \mathbb{N} \to \mathbb{N}$ such that if $\chi(G) > \phi(\omega(G))$ and $G$ contains no $(\psi, q)$-scattering, then $G$ contains $H_2$.*

**Proof of 1.5, assuming 2.1.** We proceed by induction on $\beta$. Let $H_1$ be $\beta$-bad, and let $H_2$ be either a broom, or the disjoint union of a good forest and a number of paths.

If $H_1$ is good, the result is true, so we assume that $H_1$ is not good, and therefore $\beta \geq 1$. Thus either $H_1$ is the disjoint union of two $(\beta-1)$-bad graphs $J_1, J_2$, or the cone of a $(\beta-1)$-bad graph $J_1$ (and in this case let $J_2$ be the null graph). From the inductive hypothesis on $\beta$, for $i = 1, 2$ there is a non-decreasing polynomial $\phi_i$ such that if $G$ is $H_2$-free and $J_i$-free then $\chi(G) \leq \phi_i(\omega(G))$, and by replacing $\phi_1, \phi_2$ by $\phi_1 + \phi_2$ we may assume that $\phi_1 = \phi_2$.

Let $q = |J_1|$. By 2.1, there is a non-decreasing polynomial $\phi$ such that if $\chi(G) > \phi(\omega(G))$ and contains no $(\phi_1, q)$-scattering, then $G$ contains $H_2$. We claim that $\phi$ satisfies 1.5.

Let $G$ be $\{H_1, H_2\}$-free, and suppose that $\chi(G) > \phi(\omega(G))$. Since $G$ is $H_2$-free, it follows from the choice of $\phi$ that $G$ contains a $(\phi_1, q)$-scattering $(w, P, Q)$ say. Let $r$ be the maximum, over $v \in P$, of the chromatic number of the set of neighbours of $v$ in $Q$. Since $\chi(P) > \phi_1(\omega(P))$, there is an induced subgraph of $G[P]$ isomorphic to $J_1$, say $J_1'$. Hence $G$ contains the cone of $J_1$, so we may assume that $H_1$ is the disjoint union of $J_1, J_2$. The set of vertices in $Q$ with a neighbour in $V(J_1')$ has chromatic number at most $r|J_1|$, and since

$$\chi(Q) > |J_1|r + \phi_2(\omega(Q)),$$

it follows that the set (say $Q'$) of vertices in $Q$ that are anticomplete to $J_1'$ has chromatic number more than $\phi_2(\omega(Q))$. From the choice of $\phi_2$, and since $G$ is $H_2$-free, it follows that $G[Q']$ is not $J_2$-free; but then, combining this copy of $J_2$ with $J_1'$, we find a copy of $H_1$ in $G$, a contradiction. This proves 1.5. ∎

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. We say a subgraph $P$ of a graph $G$ is *$\sigma$-nondominating* if there is a set $X \subseteq V(G) \setminus V(P)$, anticomplete to $V(P)$, with $\chi(X) > \sigma(\omega(X))$. Next we will show that to prove 2.1 it suffices to prove the following:

**2.2** *Let $\psi, \sigma : \mathbb{N} \to \mathbb{N}$ be non-decreasing polynomials, and let $q \geq 0$ an integer. Let $H$ be a broom, and let $J$ be a path. Then there is a non-decreasing polynomial $\phi : \mathbb{N} \to \mathbb{N}$ such that if $G$ is a graph, and $\chi(G) > \phi(\omega(G))$, and $G$ contains no $(\psi, q)$-scattering, then $G$ contains $H$ and a $\sigma$-nondominating copy of $J$.*

**Proof of 2.1, assuming 2.2.** Let $\psi, q, H_2$ be as in 2.1. If $H_2$ is a broom, then 2.1 follows immediately from 2.2 (setting $H = H_2$ and setting $J$ to be some path, for instance the one-vertex path). Thus we assume that $H_2$ is the disjoint union of a good forest $J_1$ and a forest $J_2$ that is a disjoint union of paths. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function such that every $J_1$-free graph $G$ has chromatic number at most $\sigma(\omega(G))$; and choose a path $J$ such that $J_2$ is an induced subgraph of $J$. By 2.2 (setting $H$ to be some broom, for instance with one vertex) there is a non-decreasing polynomial $\phi : \mathbb{N} \to \mathbb{N}$ such that if $\chi(G) > \phi(\omega(G))$ and $G$ contains no $(\psi, q)$-scattering, then $G$ contains a $\sigma$-nondominating copy $J'$ of $J$.

We claim that $\phi$ satisfies 2.1. Thus we must show that if $G$ is $H_2$-free and contains no $(\psi, q)$-scattering then $\chi(G) \leq \phi(\omega(G))$. Suppose not. By the choice of $f$, and since $G$ contains no $(\psi, q)$-scattering, it follows that $G$ contains a copy $J'$ of $J$, such that there is a set $X \subseteq V(G)$ with $\chi(X) > \sigma(\omega(X))$ anticomplete to $V(J_1')$. But since $\chi(X) > \sigma(\omega(X))$, it follows that $G[X]$ contains $J_1$, and since $J$ contains $J_2$, and $V(J)$ is anticomplete to $X$, it follows that $G$ contains $H_2$. This proves 2.1. ∎

We remark that there is an appealing possible strengthening of 2.2, that we could not prove:

**2.3 Conjecture:** *Let $\psi, \sigma : \mathbb{N} \to \mathbb{N}$ be non-decreasing polynomials, let $q \geq 0$ an integer, and let $H$ be a broom. Then there is a non-decreasing polynomial $\phi : \mathbb{N} \to \mathbb{N}$ such that if $G$ is a graph, and $\chi(G) > \phi(\omega(G))$, and $G$ contains no $(\psi, q)$-scattering, then $G$ a $\sigma$-nondominating copy of $H$.*

Let us say a graph $H$ is *self-isolating* if for every non-decreasing polynomial $\psi : \mathbb{N} \to \mathbb{N}$, there is a polynomial $\phi : \mathbb{N} \to \mathbb{N}$ with the following property: for every graph $G$ with $\chi(G) > \phi(\omega(G))$, there exists $A \subseteq V(G)$ with $\chi(A) > \psi(\omega(A))$, such that either

- $G[A]$ is $H$-free, or
- $G$ contains a copy $H'$ of $H$ such that $V(H')$ is disjoint from and anticomplete to $A$.

Which graphs are self-isolating? It is proved in [10] that stars are self-isolating, and we will show in [2] that complete graphs and complete bipartite graphs are self-isolating. Let us observe that 2.2 implies that:

**2.4** *Every path is self-isolating.*

**Proof.** Let $J$ be a path, and let $\psi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing polynomial. Choose $\phi$ satisfying 2.2 with $H = J$ and $\sigma = \psi$ and $q = |J|$, and let $G$ be a graph with $\chi(G) > \phi(\omega(G))$. We claim that either there is a $\psi$-nondominating copy of $J$ in $G$, or there exists $A \subseteq V(G)$ with $\chi(A) > \psi(\omega(A))$ such that $G[A]$ is $J$-free. By 2.2 we may assume that there is a $(\psi, q)$-scattering $(w, P, Q)$ in $G$. If

$G[P]$ is $J$-free, the claim holds, so we assume that there is a copy $J'$ of $J$ in $G[P]$. Thus $|J'| = q$. Let $r$ be the maximum over $v \in P$ of the chromatic number of the set of neighbours of $v$ in $Q$. The set of vertices in $Q$ with a neighbour in $V(J')$ has chromatic number at most $|J'|r = qr$; and $\chi(Q) > \psi(\omega(Q)) + qr$ from the definition of a $(\psi, q)$-scattering. Consequently $J'$ is $\psi$-nondominating, and hence $J$ is self-isolating. This proves 2.4. ∎

# 3 Constructing a horn

Let $d \geq 0$ be an integer. If $A, B \subseteq V(G)$ are disjoint, we say that $A$ is *$d$-dense* to $B$ if for every vertex $v \in A$, the set of non-neighbours of $v$ in $B$ has chromatic number at most $d$. Let us say a *$(d, z)$-horn* in a graph $G$ is a triple $(v, A, B)$ where

- $A, B$ are disjoint subsets of $V(G)$, and $v \in V(G) \setminus (A \cup B)$;
- $v$ is complete to $A$ and anticomplete to $B$; and
- there is no $Z \subseteq A \cup B$ with $\chi(Z) \leq z$ such that $A \setminus Z$ is $d$-dense to $B \setminus Z$.

We will need a $(d, z)$-horn $(v, A, B)$ where $z$ is at least some large function of the clique number of $A \cup B$, and this section produces such a horn. We show in 3.5 that if $G$ has sufficiently large chromatic number (and, for convenience, all its proper induced subgraphs have smaller chromatic number), then either $G$ contains both a $(k, s)$-broom and a $\sigma$-nondominating $k$-vertex path, or $G$ contains a $(d, z)$-horn. To complete the proof of 2.2, it therefore suffices to handle graphs $G$ that contain $(d, z)$-horns, for suitably chosen values of $d, z$, and we will do so in the next section.

We will use the following well-known version of Ramsey's theorem, proved (for instance) in [10] ($|G|$ denotes the number of vertices of $G$):

**3.1** *Let $x \geq 2$ and $y \geq 1$ be integers. For a graph $G$, if $|G| \geq x^y$, then $G$ has either a clique of cardinality $x + 1$, or a stable set of cardinality $y$.*

If $v \in V(G)$, we denote by $N(v)$ or $N_G(v)$ the set of all neighbours of $v$ in $G$. First, we need a result of Gyárfás [5] (we give the well-known proof, because it is so pretty.)

**3.2** *Let $k \geq 1$ and $x \geq 0$ be integers. Let $G$ be a connected graph such that $\chi(N(v)) \leq x$ for every vertex $v$. Let $H$ be a connected induced subgraph of $G$, and let $v \in V(G) \setminus V(H)$ with a neighbour in $V(H)$. If $\chi(H) > (k-2)x$, there is an induced $k$-vertex path of $G$ with one end $v$ and all other vertices in $V(H)$.*

**Proof.** We proceed by induction on $k$. The result is clear for $k \leq 2$, so we assume that $k \geq 3$. Let $J$ be obtained from $H$ by deleting all vertices in $N(v)$; thus $\chi(J) > (k-3)x > 0$, and so there is a component $H'$ of $J$ with chromatic number more than $(k-3)x$. Let $v' \in N(v) \cap V(H)$ with a neighbour in $V(H')$. From the inductive hypothesis applied to $v', H'$, there is an induced $(k-1)$-vertex path of $G$ with one end $v'$ and all other vertices in $V(H')$. Appending $v$ to this path proves 3.2. ∎

We deduce:

**3.3** *Let $\sigma : \mathbb{N} \to \mathbb{N}$ be non-decreasing, let $k, x \geq 1$ be integers, and let $G$ be a graph. If $\chi(N(v)) \leq x$ for every $v \in V(G)$, and $\chi(G) > kx + \sigma(\omega(G))$, then there is a $\sigma$-nondominating $k$-vertex induced path $P$ in $G$.*

**Proof.** We may assume that $G$ is connected; choose $v \in V(G)$. Since $\chi(G \setminus v) > kx - 1 \geq (k-2)x$, 3.2 (applied to $v$ and to a component of $G \setminus v$ of maximum chromatic number) implies that $G$ contains a $k$-vertex induced path $P$. The set of vertices of $G$ with a neighbour in $V(P)$ has chromatic number at most $kx$, and the result follows. This proves 3.3. ∎

The next result is also essentially due to Gyárfás (mentioned in [5]):

**3.4** *Let $H$ be a $(k, s)$-broom, and suppose that $G$ is $H$-free, and $\chi(N(v)) \leq x$ for every $v \in V(G)$. Then*

$$\chi(G) \leq \max(\omega(G)^{2s}, (2s+1)(x+1) + (k-2)x).$$

**Proof.** Suppose that $\chi(G) > \max(\omega(G)^{2s}, (2s+1)(t+1) + (k-2)x)$. We may assume that $G$ is connected. If every vertex of $G$ has degree less than $\omega(G)^{2s}$ then $\chi(G) \leq \omega(G)^{2s}$, a contradiction, so some vertex $v$ has at least $\omega(G)^{2s}$ neighbours. By 3.1 applied to $G[N(v)]$, there is a stable set $S$ of neighbours of $v$, with $|S| = 2s$. Let $M$ be the set of all vertices of $G$ that do not belong to $S \cup \{v\}$ and have a neighbour in $S \cup \{v\}$. Thus $\chi(M) \leq (2s+1)x$. Let $H$ be a component of $G \setminus (M \cup S \cup \{v\})$ of maximum chromatic number; then $\chi(H) \geq \chi(G) - (2s+1)(x+1) > (k-2)x$. Choose $u \in M \cup S \cup \{v\}$ with a neighbour in $V(H)$. Since no vertex of $S \cup \{v\}$ has a neighbour in $V(H)$, from the definition of $M$, it follows that $u \in M$. By 3.2 applied to $u, H$, there is an induced $k$-vertex path $P$ of $G$ with one end $u$ and all other vertices in $V(H)$. Thus $u$ is the only vertex of $P$ with a neighbour in $S \cup \{v\}$. If $u$ is adjacent to at least $s$ vertices in $S$, then the subgraph induced on $V(P)$ and some $s$ of these neighbours is a $(k, s)$-broom, a contradiction. Thus there exists $S' \subseteq S$ with $|S'| = s$, such that all vertices in $S'$ are nonadjacent to $u$. If $u$ is adjacent to $v$, the subgraph induced on $V(P) \cup S \cup \{v\}$ is a $(k+1, s)$-broom, a contradiction. Thus $u$ is adjacent to some $w \in S \setminus S'$, and nonadjacent to $v$. But then the subgraph induced on $V(P) \cup S' \cup \{v, w\}$ is a $(k+2, s)$-broom, a contradiction. This proves 3.4. ∎

**3.5** *Let $\sigma : \mathbb{N} \to \mathbb{N}$ be non-decreasing. Let $k, s, d, z \geq 0$ and $c \geq 2s$ be integers. Let $G$ be a graph such that*

$$\begin{aligned}
&\chi(G) > \omega(G)^c;\\
&\chi(G') \leq \omega(G')^c \text{ for every induced subgraph } G' \text{ of } G \text{ with } G' \neq G;\\
&\omega(G)^c \geq (\omega(G)-1)^c + z + d\omega(G) + 2;\\
&\omega(G)^c \geq (2s+1)(z+1) + (k-2)z; \text{ and}\\
&\omega(G)^c \geq kz + \sigma(\omega(G)).
\end{aligned}$$

*Then either*

- *$G$ contains a $(d, z)$-horn; or*

- *$G$ contains a $(k,s)$-broom, and a $\sigma$-nondominating $k$-vertex path.*

**Proof.** Suppose that $\chi(N(v)) \le z$ for every vertex $v \in V(G)$. By 3.4, and since

$$\chi(G) > \omega(G)^c \ge \max(\omega(G)^{2s}, (2s+1)(z+1) + (k-2)z)$$

(because $c \ge 2s$), it follows that $G$ contains a $(k,s)$-broom. By 3.3, since $\chi(G) - kz > \sigma(\omega(G))$, there is a $\sigma$-nondominating $k$-vertex induced path $P$ in $G$, and so the second bullet holds.

Thus we assume that $\chi(N(v)) > z$ for some vertex $v$. Let $A$ be the set of neighbours of $v$, and $B = V(G) \setminus (A \cup \{v\})$. We claim that $(v, A, B)$ is a $(d,z)$-horn. Suppose not; then there exists $Z \subseteq A \cup B$ with $\chi(Z) \le z$, such that $A \setminus Z$ is $d$-dense to $B \setminus Z$. Let $P \subseteq A \setminus Z$ be a clique with cardinality $p = \omega(A \setminus Z)$. Then $p \ge 1$, since $\chi(Z) \le z < \chi(A)$; and $p < \omega(G)$ since otherwise adding $v$ would give a clique of cardinality $\omega(G)+1$. For each $u \in P$, the set of vertices in $B \setminus Z$ nonadjacent to $u$ has chromatic number at most $d$, since $A \setminus Z$ is $d$-dense to $B \setminus Z$; and so the set of vertices in $B$ with a non-neighbour in $P$ has chromatic number at most $pd \le d\omega(G)$. The set of vertices in $B$ complete to $P$ has clique number at most $\omega(G) - p$ and so has chromatic number at most $(\omega - p)^c$. Hence $\chi(B \setminus Z) \le pd + (\omega(G) - p)^c$, and so

$$\chi(G) \le \chi(Z) + \chi(A \setminus Z) + \chi(B \setminus Z) + 1 \le z + p^c + d\omega(G) + (\omega(G) - p)^c + 1.$$

Since $1 \le p \le \omega(G) - 1$, $p^c + (\omega(G) - p)^c \le (\omega(G) - 1)^c + 1$, and so

$$\omega(G)^c < \chi(G) \le z + d\omega(G) + (\omega(G) - 1)^c + 2,$$

a contradiction. This proves 3.5. ∎

# 4 Making taller horns

In this section we prove 2.2, and hence complete the proofs of 2.1, 1.5, 1.4, and therefore 1.3. Because of 3.5, we may assume that $G$ contains a $(d,z)$-horn, for some suitable values of $d, z$; and now we will show that, provided that $G$ does not contain the proscribed scattering, we can use this horn to make a "$k$-tall" $(d', z')$-horn, which is a horn with a $k$-vertex path appended to its distinguished vertex. From such a horn, it is easy to obtain a $(k,s)$-broom and a $\sigma$-nondominating $k$-vertex path, to satisfy 2.2. The main step is therefore to convert an $\ell$-tall horn to an $(\ell+1)$-tall horn, and for that we need the next result.

If $d, z, \omega \ge 0$ are integers, a graph $G$ is *$(d,z,\omega)$-unsplittable* if there is no partition $(A, B, Z)$ of $V(G)$ such that $\chi(Z) \le z$, and $\chi(A), \chi(B) > d\omega$, and $A$ is $d$-dense to $B$. We begin with:

**4.1** *If $d, z \ge 0$ are integers, every graph $G$ admits a partition $(D_0, D_1, \dots, D_k)$ of its vertex set with $k \le \omega(G)$ such that $\chi(D_0) \le z\omega(G)$ and $G[D_i]$ is $(d, z, \omega(G))$-unsplittable for $1 \le i \le k$.*

**Proof.** We may assume that $G$ is not $(d, z, \omega(G))$-unsplittable, and so it admits a partition $(D_0, D_1, D_2)$ such that $\chi(D_0) \le z$, $\chi(D_1), \chi(D_2) > d\omega(G)$, and $D_1$ is $d$-dense to $D_2$. Hence we may choose $k \ge 2$ maximum such that there is a sequence $D_0, D_1, \dots, D_k$ of pairwise disjoint subsets of $V(G)$ with union $V(G)$, and with the following properties:

- $\chi(D_0) \le (k-1)z$
- $D_i$ is $d$-dense to $D_j$ for $1 \le i < j \le k$; and
- $\chi(D_i) > d\omega(G)$ for $1 \le i \le k$.

We claim:

(1) *$k \le \omega(G)$.*

Suppose that $k > \omega(G)$, and define $d_i \in D_i$ for $1 \le i \le \omega(G)+1$ inductively as follows. Let $1 \le i \le \omega(G)+1$, and suppose that $d_1, \dots, d_{i-1}$ have been defined, all pairwise adjacent. The set of vertices in $D_i$ that have a non-neighbour among $d_1, \dots, d_{i-1}$ has chromatic number at most

$$(i-1)d \le d\omega(G) < \chi(D_i),$$

and so some vertex $d_i \in D_i$ is adjacent to all of $d_1, \dots, d_{i-1}$. This completes the inductive definition. But then $\{d_1, \dots, d_{\omega(G)+1}\}$ is a clique of $G$, contradicting the definition of $\omega(G)$. This proves (1).

(2) *For $1 \le i \le k$, $G[D_i]$ is $(d, z, \omega(G))$-unsplittable.*

Suppose that $(A, B, Z)$ is a partition of $D_i$ such that $\chi(Z) \le z$, and $\chi(A), \chi(B) > d\omega(G)$, and $A$ is $d$-dense to $B$. Then the sequence

$$(D_0 \cup Z, D_1, \dots, D_{i-1}, A, B, D_{i+1}, \dots, D_k)$$

contradicts the maximality of $k$. This proves (2).

From (1), (2), this proves 4.1. ∎

Let $(v, A, B)$ be a $(d, z)$-horn in a graph $G$, and let $k \ge 1$ be an integer. We say that $(v, A, B)$ is *$k$-tall* if there is an induced path $R$ in $G$ with $k$ vertices, with one end $v$, such that $V(R) \setminus \{v\}$ is disjoint from and anticomplete to $A \cup B$. Thus every $(d, z)$-horn is 1-tall. We use 4.1 to prove a result which is the heart of the paper:

**4.2** *Let $G$ be a graph, let $d, z, d', z', q \ge 0$ be integers, and let $\psi : \mathbb{N} \to \mathbb{N}$ be non-decreasing, satisfying:*

$$\begin{aligned} z &\ge \left(2\psi(\omega(G)) + (1+q)z' + qd'\omega(G)\right)\omega(G) \\ d &\ge \left(z' + d'\omega(G)\right)\omega(G). \end{aligned}$$

*Let $(v, A, B)$ be an $\ell$-tall $(d, z)$-horn in a graph $G$, for some $\ell \ge 1$. Then either*

- *there exist $P \subseteq A$ and $Q \subseteq B$ such that $(v, P, Q)$ is a $(\psi, q)$-scattering; or*
- *there exist $v' \in A$ and disjoint subsets $A', B'$ of $B$ such that $(v', A', B')$ is an $(\ell+1)$-tall $(d', z')$-horn.*

**Proof.** Let $p = \psi(\omega(G))$. By 4.1, $B$ admits a partition $(D_0, D_1, \ldots, D_k)$ with $k \le \omega(G)$ such that $\chi(D_0) \le z'\omega(G)$ and $G[D_i]$ is $(d', z', \omega(G))$-unsplittable for $1 \le i \le k$. For $1 \le i \le k$, if $\chi(D_i) \le q(z' + d'\omega(G)) + p$ let $P_i = \emptyset$, and if $\chi(D_i) > q(z' + d'\omega(G)) + p$ let $P_i$ be the set of vertices $a \in A$ such that $\chi(U) \le z' + d'\omega(G)$, where $U$ is the set of neighbours of $a$ in $D_i$. Let $P = P_1 \cup \cdots \cup P_k$.

Suppose that $\chi(P_i) > p$, for some $i \in \{1, \ldots, k\}$. Consequently $P_i \ne \emptyset$, and so

$$\chi(D_i) > q(z' + d'\omega(G)) + p \ge q(z' + d'\omega(G)) + \psi(\omega(D_i));$$

and for each $a \in P_i$, $\chi(U) \le z' + d'\omega(G)$, where $U$ is the set of neighbours of $a$ in $D_i$. It follows that $(v, P_i, D_i)$ is a $(\psi, q)$-scattering and the first bullet of the theorem holds. Thus we may assume that $\chi(P_i) \le p$ for $1 \le i \le k$, and consequently $\chi(P) \le p\omega(G)$.

Let $Z$ be the union of $P, D_0$, and all the sets $D_i$ with $1 \le i \le k$ such that

$$\chi(D_i) \le q(z' + d'\omega(G)) + p.$$

Consequently

$$\chi(Z) \le 2p\omega(G) + z'\omega(G) + q(z' + d'\omega(G))\omega(G) \le z.$$

Since $(v, A, B)$ is a $(d, z)$-horn, it follows that $A \setminus Z$ is not $d$-dense to $B \setminus Z$; and so there exists $v' \in A \setminus P$ such that the set of vertices in $B \setminus Z$ that are nonadjacent to $v'$ has chromatic number more than $d$. Since $B \setminus Z$ is the union of the sets $D_i$ with $\chi(D_i) > q(z' + d'\omega(G)) + p$, there exists $i \in \{1, \ldots, k\}$ with $\chi(D_i) \ge q(z' + d'\omega(G)) + p$ such that the set $B'$ of vertices in $D_i$ nonadjacent to $v'$ has chromatic number more than $d/\omega(G)$. Since $v' \notin P$, the set $A'$ of neighbours of $v'$ in $D_i$ has chromatic number more than $d'\omega(G) + z'$.

Let $Z' \subseteq D_i$ with $\chi(Z') \le z'$. Thus $\chi(A' \setminus Z') \ge \chi(A') - \chi(Z') > d'\omega(G)$; and $\chi(B' \setminus Z') > d/\omega(G) - z' \ge d'\omega(G)$. Since $G[D_i]$ is $(d', z', \omega(G))$-unsplittable, it follows that $A' \setminus Z'$ is not $d'$-dense to $B' \setminus Z'$. This proves that $(v', A', B')$ is a $(d', z')$-horn.

Since $(v, A, B)$ is $\ell$-tall, there is an $\ell$-vertex induced path $R$ of $G$ with one end $v$, such that $V(R) \setminus \{v\}$ is disjoint from and anticomplete to $A \cup B$. Then $R' = G[V(R) \cup \{v'\}]$ is an $(\ell + 1)$-vertex path, and since $V(R)$ is anticomplete to $B$ and hence to $A' \cup B'$, it follows that $(v', A', B')$ is $(\ell + 1)$-tall, and so the second bullet of the theorem holds. This proves 4.2. ∎

Now we prove 2.2, which we restate:

**4.3** *Let $k, s \ge 1$ and $q \ge 0$ be integers, and let $\psi, \sigma : \mathbb{N} \to \mathbb{N}$ be non-decreasing polynomials. Then there exists an integer $c \ge 0$ such that if $G$ is a graph with $\chi(G) > \omega(G)^c$, and $G$ contains no $(\psi, q)$-scattering, then $G$ contains a $(k, s)$-broom and a $\sigma$-nondominating $k$-vertex path.*

**Proof.** Let $\zeta_k : \mathbb{N} \to \mathbb{N}$ be the polynomial defined by $\zeta_k(x) = \sigma(x) + x^s$, and let $\delta_k(x) = 0$. For $i = k - 1, \ldots, 1$, define polynomials $\zeta_i, \delta_i : \mathbb{N} \to \mathbb{N}$ by

$$\zeta_i(x) = 2x\psi(x) + (1 + q)x\zeta_{i+1}(x) + qx^2\delta_{i+1}(x)$$
$$\delta_i(x) = x\zeta_{i+1}(x) + x^2\delta_{i+1}(x).$$

Choose an integer $c \ge 2s$ such that

$$x^c \ge (x - 1)^c + \zeta_1(x) + x\delta_1(x) + 2$$
$$x^c \ge (2s + 1)(\zeta_1(x) + 1) + (k - 2)\zeta_1(x), \text{ and}$$
$$x^c \ge k\zeta_1(x) + \sigma(x)$$

for all integers $x \geq 2$. We claim that $c$ satisfies 4.3. To see this, let $G$ be a graph with $\chi(G) > \omega(G)^c$, and suppose that $G$ contains no $(\psi, q)$-scattering. We must show that $G$ contains a $(k,s)$-broom and a $\sigma$-nondominating $k$-vertex path. We show this by induction on $|G|$. If there is an induced subgraph $G'$ of $G$ with $G' \neq G$ and $\chi(G') > \omega(G')^c$, then $G'$ contains no $(\psi, q)$-scattering, and from the inductive hypothesis, $G'$ contains a $(k,s)$-broom and a $\sigma$-nondominating $k$-vertex path, and hence so does $G$, as required. We may assume then that there is no such $G'$. Since $\chi(G) > \omega(G)^c$, it follows that $\omega(G) \geq 2$, and so the five displayed inequalities of 3.5 hold with $z, d$ replaced by $\zeta_1(\omega(G)), \delta_1(\omega(G))$ respectively. From 3.5, we may assume that $G$ contains a $(\delta_1(\omega(G)), \zeta_1(\omega(G)))$-horn, which is therefore 1-tall.

From 4.2, it follows that for $i = 2, \ldots, k$, $G$ contains an $i$-tall $(\delta_i(\omega(G)), \zeta_i(\omega(G)))$-horn, and so contains a $k$-tall $(0, z)$-horn $(v, A, B)$ say, where $z = \zeta_k(\omega(G))$. Since this horn is $k$-tall, there is a $k$-vertex induced path $R$ of $G$ with one end $v$, such that $V(R) \setminus \{v\}$ is disjoint from and anticomplete to $A \cup B$. From the definition of a $(0, z)$-horn, $\chi(A), \chi(B) > z$. Since $\chi(A) > z \geq \omega(A)^s$, 3.1 implies that there is a stable set $S \subseteq A$ with $|S| = s$, and so $G[V(R) \cup S]$ is a $(k,s)$-broom. Since $\chi(B) > z > \sigma(\omega(B))$, and $V(R)$ is anticomplete to $B$, $R$ is $\sigma$-nondominating. This proves 4.3. ∎

Finally, we will go through the calculations of the proof of 4.3, to prove 1.2, which we restate:

**4.4** *If $G$ is $2P_4$-free, then $\chi(G) \leq \omega(G)^{16}$.*

**Proof.** Let $\alpha$ be the polynomial where $\alpha(x) = x$ for all $x$. If $G$ is $2P_4$-free, then $G$ contains no $(\alpha, 4)$-scattering, and contains no $\alpha$-nondominating 4-vertex path; so we will follow the proof of 4.3, taking $k = q = 4$, $s = 1$, and $\psi = \sigma = \alpha$.

Thus, from the definitions, we have that for all $x \geq 0$:

$$\begin{aligned}
\zeta_4(x) &= 2x \\
\delta_4(x) &= 0 \\
\zeta_3(x) &= 12x^2 \\
\delta_3(x) &= 2x^2 \\
\zeta_2(x) &= 2x^2 + 60x^3 + 8x^4 \\
\delta_2(x) &= 12x^3 + 2x^4 \\
\zeta_1(x) &= 2x^2 + 10x^3 + 300x^4 + 88x^5 + 8x^6 \\
\delta_1(x) &= 2x^3 + 60x^4 + 20x^5 + 2x^6.
\end{aligned}$$

Then we must choose an integer $c \geq 2$ such that

$$\begin{aligned}
x^c &\geq (x-1)^c + 2 + 2x^2 + 10x^3 + 302x^4 + 148x^5 + 28x^6 + 2x^7 \\
x^c &\geq 3 + 10x^2 + 50x^3 + 1500x^4 + 440x^5 + 40x^6, \text{ and} \\
x^c &\geq x + 8x^2 + 40x^3 + 1200x^4 + 352x^5 + 32x^6.
\end{aligned}$$

for all integers $x \geq 2$. Thus we may take $c = 16$. This proves 4.4. ∎

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