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Extending partial 3-colourings in a planar graph

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Abstract

Let D be a disc, and let X be a finite subset of points on the boundary of D. An essential part of the proof of the four colour theorem is the fact that many sets of 4-colourings of X do not arise from the proper 4-colourings of any graph drawn in D. In contrast to this, we show that every set of 3-colourings of X arises from the proper 3-colourings of some graph drawn in D.

1. Introduction

Let X be a finite subset of the boundary of a disc D. Call a set Q of k-colourings of X *k-feasible* if there exists a drawing G in D with $X \subseteq V(G)$ such that the k-colourings of X which can be extended to k-colourings of G are precisely those in Q. We are interested in the following question: what sets of colourings are k-feasible? Kempe chain arguments show that for $k \geqslant 4$ only certain heavily restricted sets of k-colourings are k-feasible, and this is an important technique in the proof of the four colour theorem. For $k = 1, 2$ it is easy to see that only certain structured sets of k-colourings are k-feasible. In contrast, we shall show that any set of 3-colourings is 3-feasible.

One may ask the question: given a set of k-colourings of X which is k-feasible, how large is the smallest graph which admits precisely this set of k-colourings? For 3-colouring, our proof yields a bound of $O(9^{|X|})$ on the size of this graph. For $k = 4$ and 5 we do not know of any bound, but for $k = 6$, we will prove a quadratic bound

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in Section 3. When $k \geqslant 7$ there is a simple linear bound resulting from Euler's formula.

It will be convenient for us to work with vertex colouring in terms of partitions. We will consider a *k-colouring* of a set to be a partition of its elements into at most k nonempty sets. A *k-colouring* of a graph G is a k-colouring of $V(G)$ such that each member of the partition is a stable set. For any set X, we define $C(X)$ to be the set of all 3-colourings of X. If $\tau \in C(X)$ and $x \in T \in \tau$, we define $\tau(x) = T$. If G is a graph and $X \subseteq V(G)$, we define

$$\Phi_G(X) = \{\tau \in C(X) \mid \tau \text{ can be extended to a } 3-\text{colouring of } G\}.$$

2. 3-Feasible colourings

Our main result is the following theorem.

Theorem 2.1. *Let D be a disc, let X be a finite subset of the boundary of D, and let $Q \subseteq C(X)$ be a set of 3-colourings of X. Then there exists a drawing G in D with $X \subseteq V(G)$, such that $\Phi_G(X) = Q$.*

The proof of this theorem will require three lemmas. The first two lemmas will be used to construct a (possibly nonplanar) graph G_0 with $X \subseteq V(G_0)$ and with the property that $\Phi_{G_0}(X) = Q$. The third lemma will define a particular planar graph *Cloverleaf* which provides a planar simulation of a crossing. We will then draw G_0 in a disc (with crossings) with X on the boundary as required, and then use *Cloverleaf* as a gadget to remove the crossings. The resulting graph G will satisfy the theorem.

Lemma 2.2. *For every finite set X, and every 3-colouring τ of X, there exists a graph G with $X \subseteq V(G)$ such that $\Phi_G(X) = C(X) \backslash \{\tau\}$.*

Proof. We proceed by induction on $|X|$. If $|X| < 3$ or if there do not exist $x_1, x_2 \in X$ with $\tau(x_1) = \tau(x_2)$, then one of the graphs $K_4, K_2, K_{1,3}, K_4 - e$ has the required properties. Hence we may assume that $|X| \geqslant 3$ and that there exist distinct $x_1, x_2 \in X$ such that $\tau(x_1) = \tau(x_2) = T$. Let z be a new vertex (not in X), let $X' = (X \backslash \{x_1, x_2\}) \cup \{z\}$, let $T' = (T \backslash \{x_1, x_2\}) \cup \{z\}$ and $\tau' = (\tau \backslash \{T\}) \cup \{T'\}$. Inductively, we may choose a graph G' with the property that $X' \subseteq V(G'')$ and $\Phi_{G'}(X') = C(X') \backslash \{\tau'\}$. Let F be the graph of Fig. 1, and let G be the graph obtained from the disjoint union of G' and F by identifying the vertex z of G' and the vertex z of F. Let $\sigma \in C(X)$ be given. We claim that σ is extendable to G if and only if $\sigma \neq \tau$.

Case 1: $\sigma(x_1) \neq \sigma(x_2)$. In this case $\sigma \neq T$. Since only one colouring of X' does not extend to G', and $|X'| \geqslant 2$, we may always choose a colour for z such that the resulting colouring of X' will extend to G'. Since this colouring of z can also be completed to a proper 3-colouring of F, we have found a proper 3-colouring of G, and we conclude that $\sigma \in \Phi_G(X)$.

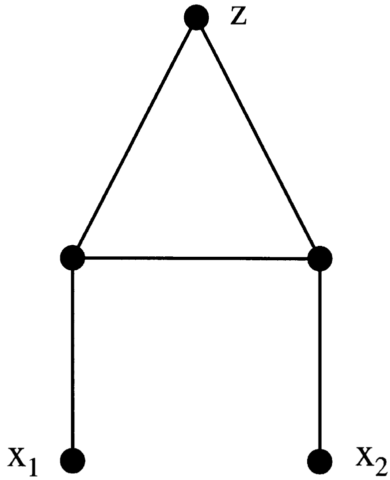

Fig. 1. A basic five vertex graph.

Case 2: $\sigma(x_1) = \sigma(x_2)$. Let $S = \sigma(x_1)$ and $S' = (S \backslash \{x_1, x_2\}) \cup \{z\}$. When x_1 and x_2 are given the same colour, σ can only be completed to a proper 3-colouring of F so that z has the same colour as x_1, x_2. Thus, σ cannot be extended to a proper 3-colouring of G if and only if the colouring $\sigma' \in C(X')$ given by $\sigma' = (\sigma \backslash \{S\}) \cup \{S'\}$ cannot be extended to a proper 3-colouring of G'. This is true if and only if $\sigma' = \tau'$, which is true if and only if $\sigma = \tau$. Thus, we have that $\Phi_G(X) = C(X) \backslash \{\tau\}$ as desired. □

Lemma 2.3. *For every finite set* X*, and* $Q \subseteq C(X)$*, there exists a graph* G *with* $X \subseteq V(G)$ *such that* $\Phi_G(X) = Q$.

Proof. If $Q = C(X)$ then the graph $(X, \emptyset)$ satisfies the lemma, so we may assume that $C(X) \backslash Q \neq \emptyset$. For each $\tau \in C(X) \backslash Q$, we may choose a graph G_τ such that $X \subseteq V(G_\tau)$ and $\Phi_{G_\tau}(X) = C(X) \backslash \{\tau\}$ by Lemma 1. Now, we construct G by taking the disjoint union of the G_τ graphs and then identifying all of the copies of each vertex in X. Now, a colouring $\sigma \in C(X)$ is not extendable to all of G if and only if σ is not extendable to G_τ for some $\tau \in C(X) \backslash Q$, which holds if and only if $\sigma \in C(X) \backslash Q$. Thus, we have $\Phi_G(X) = Q$ as desired. □

Lemma 2.4. *Let Cloverleaf and* $W = \{w_1, w_2, w_3, w_4\}$ *be defined by Fig.* 2. *Then*

$$\Phi_{\mathrm{Cloverleaf}}(W) = \{\tau \in C(W) \mid \tau(w_1) = \tau(w_3), \tau(w_2) = \tau(w_4)\}.$$

Proof. *Cloverleaf* is made up of four triangular pieces by identifying their outermost vertices. Each triangular piece only accepts 3-colourings for which the outermost vertices all have the same colour, or all have distinct colours. The proof follows easily from this. □

For the remainder of the paper, it will be helpful to consider graphs with two kinds of edges, ordinary edges and special edges. We redefine a colouring τ of such a graph to be a colouring of the vertex set so that for any adjacent vertices x, y, we have $\tau(x) \neq \tau(y)$ if xy is an ordinary edge, and $\tau(x) = \tau(y)$ if xy is a special edge. The colourings of G are in one to one correspondence with the colourings of the graph obtained from G by contracting all of its special edges.

Proof of Theorem 2.1. Let D be a disc, and let X be a finite subset of the boundary of D. It will be helpful for us to consider graphs which are drawn in D with crossings. Let a *scribble* G be a drawing of a graph in D such that $X \subseteq V(G)$, and with the additional properties that any two edges of G have at most one point in common, either an endpoint or a crossing, no three edges have a common crossing point, and the interior of every edge is disjoint from the vertex set. Now, let Q be a set of 3-colourings of X. By Lemma 2.3 (and since every graph is isomorphic to some scribble) we may choose a scribble G_0 in D such that $\Phi_{G_0}(X) = Q$.

We construct a new scribble G_1 from G_0 as follows: If e is an edge of G_0 which crosses k other edges, we subdivide it k times, forming a path P of length $k+1$ consisting of k special edges and one ordinary edge. This can be done in such a way that each special edge of P crosses exactly one other edge, and the ordinary edge of P does not cross another edge. Let G_1 be the scribble formed by repeating this process on each edge of G_0. Since G_0 is precisely the graph obtained by contracting the special edges of G_1, we have that $\Phi_{G_1}(X) = Q$. Furthermore, G_1 also has the properties that no ordinary edge crosses another edge, and each special edge crosses exactly one other edge.

Now, we construct a new scribble G from G_1 as follows: If x_1x_2 and y_1y_2 are special edges that cross, then x_1, x_2, y_1, y_2 are all distinct, and we may choose a disc $D' \subseteq D$ such that D' contains all of x_1x_2, y_1y_2 with x_1, x_2, y_1, y_2 on the boundary of D', and such that no other edges of G_1 intersect D' except at the points x_1, x_2, y_1, y_2. Let G_1' denote the scribble $G_1 \backslash \{x_1x_2, y_1y_2\}$. Since x_1x_2 and y_1y_2 were crossing edges, we may assume that x_1, y_1, x_2, y_2 occur on the boundary of D' in this clockwise order, and we may modify G_1' by placing *Cloverleaf* in D' and identifying the points x_1, y_1, x_2, y_2 of G_1' with w_1, w_2, w_3, w_4 of Fig. 2, respectively. Call this new scribble G_1''. Now, the proper 3-colourings of G_1 are precisely those proper 3-colourings τ of G_1' in which $\tau(x_1) = \tau(x_2)$ and $\tau(y_1) = \tau(y_2)$, but these are precisely the 3-colourings of w_1, w_2, w_3, w_4 which can be extended to *Cloverleaf*. Thus we find that $\Phi_{G_1''}(X) = \Phi_{G_1}(X) = Q$. Let G be the graph obtained by repeating this process for each pair of crossing edges in G_1. Then, $\Phi_G(X) = Q$, and G has no special edges or crossings, so G is an ordinary graph drawn in D with all of the required properties, and we are done. □

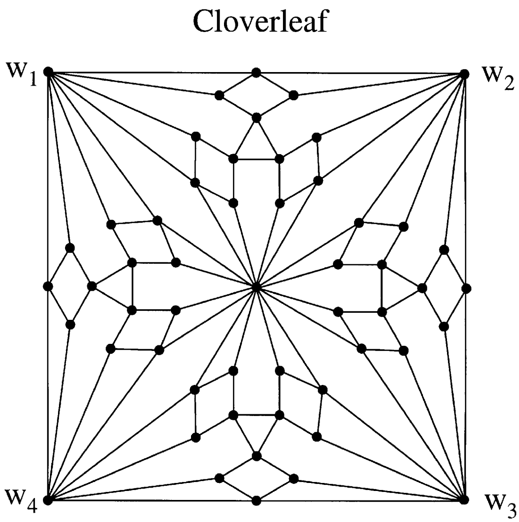

Fig. 2. *Cloverleaf* A square-shaped planar graph.

3. Bounding the graph size

If a set of k-colourings is k-feasible, one may ask how large a graph realizing it needs to be. From the proof of Theorem 1 in the previous section, $O(9^{|X|})$ is a bound when $k=3$. We do not know of a bound for the cases $k=4$ and 5, but when $k \leqslant 6$ we have a bound again. Indeed, in general we may assume that no vertex in $V(G)\backslash X$ has degree $<k$. If $k>6$, it follows from Euler's formula that $|V(G)| \leqslant O(|X|)$. In the remainder of this section, we will prove a bound of $O(|X|^2)$ for the case $k=6$.

Theorem 3.1. *Let G be a simple planar graph with the infinite region bounded by a cycle C, and such that the degree of every vertex in $V(G)\backslash V(C)$ is at least* 6. *Then* $|V(G)| \leqslant |V(C)|^2/12 + |V(C)|/2 + 1$.

Although this theorem does not directly concern graph colouring, we are including it in part because of its own interest. We note that the theorem is tight for a regular hexagonal piece of the triangular lattice.

A *quilt* is a simple planar drawing G with a cycle C bounding the infinite region, such that every finite region is bounded by a triangle, and such that the degree of any vertex in $V(G)\backslash V(C)$ is at least 6. If $P \subseteq C$ is a path with distinct terminal vertices of degree 3 and all internal vertices of degree 4, we will call P a *convenient* path (of the quilt).

Lemma 3.2. *If G is a quilt with no vertices of degree* 2, *then G has* $\leqslant 6$ *convenient paths.*

Proof. Let C be the cycle bounding the infinite region, and let $|V(G)| = n$ and $|V(C)| = m$. Construct a new graph G' by adding a new vertex u in the infinite region of G, and adding edges joining u to each vertex of $V(C)$. Now, G' is a planar triangulation with $n+1$ vertices, so we have

$$\begin{aligned} 6(n+1) - 12 &= \sum_{v \in V(G')} deg_{G'}(v) \\ &= \sum_{v \in V(C)} (deg_G(v) + 1) + m + \sum_{v \in V(G) \setminus V(C)} deg_G(v) \\ &\geqslant \sum_{v \in V(C)} deg_G(v) + 6(n-m) + 2m. \end{aligned}$$

Rearranging, we find that $\sum_{v \in V(C)} deg_G(v) \leqslant 4m - 6$. Thus, there are at least 6 more vertices of degree 3 than vertices of degree $\geqslant 5$ in C. It follows that G has at least 6 convenient paths. $\square$

Proof of Theorem 3.1. It suffices to prove the theorem for quilts, so we will let G to be a quilt and C the cycle of G bounding the infinite region. Let $\mathscr{C}_G$ be the set of all convenient paths in G, and let

$$\mu(G) = \begin{cases} 1 & \text{if } G \text{ has a vertex of degree 2,} \\ \min_{P \in \mathscr{C}_G} |E(P)| & \text{otherwise,} \end{cases}$$

$$\Psi(G) = |V(C)|^2/12 + |V(C)|/2 + 1,$$

$$\Psi_0(G) = \mu(G) + |V(C)|^2/12 + |V(C)|/3 + 1.$$

Claim 1. *If* $|V(C)| < 6$, *then* $|V(G)| \leqslant \Psi(G)| \leqslant \Psi_0(G)$.

If $|V(C)| < 6$, then G must have a vertex of degree 2 by Lemma 3.2. Deleting this vertex and repeating the argument proves that $V(G) = V(C)$. But for all k, we have $k \leqslant k^2/12 + k/2 + 1$. Thus, $|V(G)| \leqslant \Psi(G) \leqslant \Psi_0(G)$.

Claim 2. *If* $|V(G)| \geqslant 6$, *then* $|V(G)| \leqslant \Psi_0(G) \leqslant \Psi(G)$.

We prove the claim by induction on $|V(G)|$. If G has a vertex of degree 2, then $\Psi_0(G) \leqslant \Psi(G)$. Also, if G has no vertices of degree 2, then by the lemma and the fact that the convenient paths of G are edge-disjoint it follows that $\Psi_0(G) \leqslant \Psi(G)$. Thus, to prove the claim, it will suffice to show that $|V(G)| \leqslant \Psi_0(G)$. Let $m = |V(C)|$.

Suppose that C has a chord edge e, and let C_1, C_2 be the two cycles such that $E(C_1 \cup C_2) = E(C) \cup \{e\}$ and $E(C_1 \cap C_2) = \{e\}$. Let G_1, G_2 be the quilts bounded by the cycles C_1 and C_2, respectively. Let $k = |V(C_1)|$; then $(k-3)(m-k-1) \geqslant 0$.

Thus, by induction

$$\begin{aligned}|V(G)| &= |V(G_1)| + |V(G_2)| - 2 \leqslant \Psi(G_1) + \Psi(G_2) - 2\\ &\leqslant k^2/12 + k/2 + 1 + (m-k+2)^2/12 + (m-k+2)/2 + 1 - 2\\ &= m^2/12 - mk/6 + 5m/6 + k^2/6 - k/3 + \tfrac{4}{3}\\ &= m^2/12 + m/3 + 11/6 - (k-3)(m-k-1)/6 \leqslant \Psi_0(G).\end{aligned}$$

Thus, we may assume that C does not have a chord, so in particular G has minimum degree 3. Let P be the shortest convenient path in C.

Case 1: $|E(P)| = 1$. Let u, v be the endvertices of P. Since C does not have any chords, $G' = G\backslash\{u, v\}$ is a quilt. Let C' be the cycle bounding the infinite region of G'. Then $|V(C')| = m - 1$, so by induction we have:

$$\begin{aligned}|V(G)| &= |V(G')| + 2 \leqslant \Psi(G') + 2\\ &= (m-1)^2/12 + (m-1)/2 + 1\\ &= m^2/12 + m/3 + 7/12 \leqslant \Psi_0(G).\end{aligned}$$

Case 2: $|E(P)| \geqslant 2$. Let v be an endvertex of P, and let $G' = G\backslash v$. Then G is a quilt with boundary C' and $|V(C')| = |V(C)| = m$. Since G and G' have no vertices of degree two, it follows from our construction that $\mu(G') = \mu(G) - 1$. Thus, by induction we have that $|V(G)| = |V(G')| + 1 \leqslant \Psi_0(G') + 1 = \Psi(G)$ as desired. $\square$