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Directed Tree-Width

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We generalize the concept of tree-width to directed graphs and prove that every directed graph with no "haven" of large order has small tree-width. Conversely, a digraph with a large haven has large tree-width. We also show that the Hamilton cycle problem and other NP-hard problems can be solved in polynomial time when restricted to digraphs of bounded tree-width. © 2001 Academic Press

1. INTRODUCTION

All graphs and digraphs in this paper are finite and may have loops and multiple edges. A *tree-decomposition* of a graph G is a pair (T, W), where T is a tree and $W = (W_t\colon t \in V(T))$ is a family of subsets of $V(G)$, satisfying

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(W1) $\bigcup_{t \in V(T)} W_t = V(G)$, and every edge of G has both ends in some W_t; and

(W2) if t, t', $t'' \in V(T)$ and t' lies on the path from t to t'', then $W_t \cap W_{t''} \subseteq W_{t'}$.

The *width* of a tree-decomposition is $\max(|W_t| - 1 : t \in V(T))$, and the *tree-width* of G is the minimum width of a tree-decomposition of G.

Tree-width was introduced in [7], but it went unnoticed until it was rediscovered in [15] and, independently, in [2]. It has since received widespread attention, for the following reasons:

(i) It serves as a cornerstone of the Graph Minors Theory [15, 16],

(ii) it can be used to prove theorems in structural graph theory [12, 16],

(iii) it has many algorithmic applications due to the fact that many NP-hard problems can be solved in linear time when restricted to graphs of bounded tree-width [1, 2, 17], and

(iv) it has been successfully used in practical computations [5].

In Section 2 of this paper we generalize tree-width to directed graphs and point out two relationships with tree-width of undirected graphs. In Section 4 we identify a class of problems that are NP-hard in general, but can be solved in polynomial time when restricted to digraphs of bounded tree-width. In Section 5 we state what appears to be a fundamental open question concerning the tree-width of directed graphs.

The main result of Section 3 is a theorem that says that a digraph of large tree-width has a "haven" of large order and, conversely, that a digraph with a haven of large order has large tree-width. Havens of digraphs are defined in Section 3. Let us review the corresponding result for undirected graphs now.

Let $w \geqslant 0$ be an integer. A *haven of order* w in an undirected graph G is a function β which assigns to every set $Z \subseteq V(G)$ with $|Z| < w$ the vertex-set of a component of $G \backslash Z$ in such a way that if $Z' \subseteq Z \subseteq V(G)$ and $|Z| < w$, then $\beta(Z) \subseteq \beta(Z')$. (Let us remark that the definition of a haven in [18] is slightly more restrictive, but we prefer this version because it readily generalizes to directed graphs.) The following is shown in [18].

(1.1) *Let G be a graph, and let $w \geqslant 0$ be an integer. Then G has a haven of order w if and only if its tree-width is at least $w - 1$.*

Havens correspond to particularly nice winning strategies for the robber player in a certain cops-and-robbers game introduced in [18]. While not needed here, the game helps to develop intuition for the concept of the haven. In the next section we present a generalization of the game to directed graphs.

2. ARBOREAL DECOMPOSITIONS

In this section we introduce the tree-width of digraphs and present two propositions relating it to the tree-width of undirected graphs. By an *arborescence* we mean a directed graph R such that R has a vertex r_0, called the *root* of R, with the property that for every vertex $r \in V(R)$ there is a unique directed walk from r_0 to r. Thus every arborescence arises from a tree by selecting a root and directing all edges away from the root. If r, $r' \in V(R)$ we write $r' > r$ if $r' \neq r$ and there exists a directed walk in R with initial vertex r and terminal vertex r'. If $e \in E(R)$ we write $r' > e$ if either $r' = r$ or $r' > r$, where r is the head of e. We also write $e \sim r$ to mean that e is incident with r.

Let D be a digraph, and let $Z \subseteq V(D)$. The digraph obtained from D by deleting Z will be denoted by $D \backslash Z$. We say that a set $S \subseteq V(D) - Z$ is *Z-normal* if there is no directed walk in $D \backslash Z$ with first and last vertices in S that uses a vertex of $D \backslash (Z \cup S)$. It follows that every Z-normal set is the union of the vertex-sets of certain strong components of $D \backslash Z$. As one can readily check, a set S is Z-normal if and only if the vertex-sets of the strong components of $D \backslash Z$ can be numbered $S_1, S_2, ..., S_d$ in such a way that

(a) if $1 \leqslant i < j \leqslant d$, then no edge of D has its head in S_i and tail in S_j, and

(b) either $S = \emptyset$ or $S = S_i \cup S_{i+1} \cup \cdots \cup S_j$ for some integers i, j with $1 \leqslant i \leqslant j \leqslant d$.

An *arboreal decomposition* of a digraph D is a triple (R, X, W), where R is an arborescence and $X = (X_e : e \in E(R))$ and $W = (W_r : r \in V(R))$ are sets of vertices of D that satisfy

(D1) $(W_r : r \in V(R))$ is a partition of $V(D)$ into nonempty sets, and

(D2) if $e \in E(R)$, then $\bigcup \{W_r : r \in V(R), r > e\}$ is X_e-normal.

The *width* of (R, X, W) is the least integer w such that, for all $r \in V(R)$, $|W_r \cup \bigcup_{e \sim r} X_e| \leqslant w + 1$. The *tree-width* of D is the least integer w such that D has an arboreal decomposition of width w. It is easy to see that the tree-width of a subdigraph of D is at most the tree-width of D.

To aid the reader's intuition we now introduce a generalization of the cops-and-robbers game from [18]. The game is played on a directed graph D by two players, one controlling the movement of a robber and the other controlling k cops, where k is a parameter of the game. At any time a cop either stands on a vertex or is in a helicopter (that is, is temporarily removed from the game). The robber stands on a vertex of D and can at any time run at great speed to another vertex in the same strong component of $D\backslash Z$, where Z is the set of vertices occupied by the cops. In particular, the robber is not permitted to run through a cop. In other words, the robber moves along directed cop-free paths, but he is only permitted to make a move if there also exists a directed cop-free path from his intended destination back to where he is at the moment. The objective of the player controlling the movement of the cops is to land a cop via helicopter on the vertex occupied by the robber, and the robber's objective is to elude capture. (The point of the helicopters is that cops are not constrained to move along paths of the digraph—they move from vertex to vertex arbitrarily.) The robber can see the helicopter approaching its landing spot and may run to a new vertex before the helicopter actually lands.

There are two forms of this game. In the first, the robber is invisible, and so to capture him the cops must methodically search the whole graph. For undirected graphs this version was investigated in [3, 4, 8–11, 13]. We are concerned with a second form of the game, where the cops can see the robber at all times—the difficulty is just cornering him somewhere.

We shall see in the next section that a haven of order w in a digraph gives a winning strategy for the robber player against $w-1$ cops. Conversely, it can be shown that if a digraph D has tree-width less than w, then w cops have a winning strategy. To see this let (R, X, W) be an arboreal decomposition of D of width less than w. The winning strategy is as follows. In the first move the cops will occupy $W_{r_0} \cup \bigcup_{e \sim r_0} X_e$, where r_0 is the root of R. The robber then selects a strong component C_0 of $D\backslash(W_{r_0} \cup \bigcup_{e \sim r_0} X_e)$; it follows from (D2) that $V(C_0) \subseteq \bigcup_{r > e_1} W_r$ for some edge e_1 of R with tail r_0. Let r_1 be the head of e_1. In the next move the cops in $W_{r_0} \cup \bigcup_{e \sim r_0} X_e - X_{e_1}$ will take off, leaving only X_{e_1} occupied. By (D2) the robber remains trapped in $\bigcup_{r > e_1} W_r$. In the next move the cops land on the set $W_{r_1} \cup \bigcup_{e \sim r_1} X_e$, at which point the robber must select a strong component C_1 of $D\backslash(W_{r_1} \cup \bigcup_{e \sim r_1} X_e)$, and so on. Continuing in this way the cop moves will trace a directed path $r_0, r_1, ...$ in R, and eventually the cops will capture the robber. It is worth pointing out that the search strategy thus obtained need not be "cop-monotone" in the sense that the cops may have to revisit certain vertices. For example, consider the digraph depicted in Fig. 1, where we use the convention that an undirected edge represents two edges with the same ends, one in each direction. That digraph has tree-width three, but in order for four cops to capture the robber they must revisit a previously occupied and vacated vertex. This is in sharp contrast with the undirected case, where the existence of a tree-decomposition implies a search strategy where no vertex is revisited once it has been vacated [18].

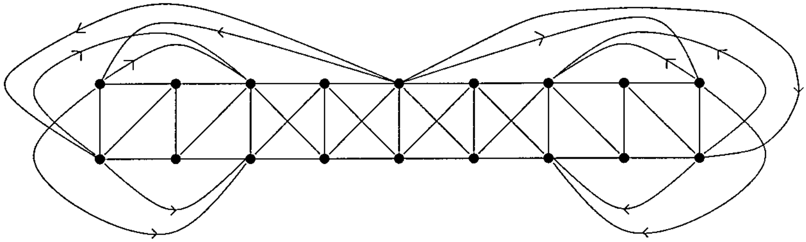

FIGURE 1

There is another notion of monotonicity. We say that a search strategy for the cops is *robber-monotone* if, for every sequence $Z_1, Z_2, \ldots$ of moves by the cops and all possible responses by the robber, the strong components of $D\backslash Z_i$ containing the robber form a nonincreasing sequence (eventually equal to the null digraph). The strategy for the cop player outlined above is indeed robber-monotone, and so it appears plausible that if k cops can capture the robber in a digraph, then they can do so robber-monotonely. We do not know if this is true, but it would be implied by the converse of (3.1), discussed prior to (3.3).

The following proposition justifies calling the concept of the tree-width of a digraph a generalization of the tree-width of graphs.

(2.1) *Let G be a graph, and let D be the directed graph obtained from G by replacing every edge with two directed edges directed in opposite directions. Then the tree-width of D is equal to the tree-width of G.*

Proof. Let (T, Z) be a tree-decomposition of G of width w, where $Z=(Z_t\colon t\in V(T))$. Let $r_0\in V(T)$ be arbitrary, and let R be the arborescence obtained from T by directing every edge away from r_0. Let $W_{r_0}=Z_{r_0}$; for an edge e of R with head r' and tail r let $X_e=Z_r\cap Z_{r'}$, and let $W_{r'}=Z_{r'}-Z_r$. Let $X=(X_e\colon e\in E(R))$ and $W=(W_r\colon r\in V(R))$. Then (R, X, W) is an arboreal decomposition of D by [15, Theorem 3.4]. Moreover, since $W_r\cup\bigcup_{e\sim r}X_e\subseteq Z_r$, we see that its width is at most w. Thus the tree-width of D is at most the tree-width of G. We postpone the proof of the other inequality until the next section. ∎

A digraph is *Eulerian* if the out-degree of every vertex is equal to its in-degree. The next proposition shows that the tree-width of an Eulerian digraph of bounded degree and that of its underlying undirected graph are within a constant factor of each other.

(2.2) *Let D be an Eulerian digraph of maximum out-degree Δ, and let G be its underlying undirected graph. Let d be the tree-width of D, and let w be the tree-width of G. Then $d\leqslant w\leqslant(2\Delta+1)(d+1)-1$.*

Proof. The fact that $d \leqslant w$ follows from (2.1), because the tree-width of D is at most the tree-width of the digraph obtained from G by replacing every edge with two edges directed in opposite directions. We postpone the proof of the other inequality until the next section. ∎

3. HAVENS IN DIGRAPHS

Let D be a digraph, and let $w \geqslant 0$ be an integer. A *haven of order w* in D is a function β assigning to every set $Z \subseteq V(D)$ with $|Z| < w$ the vertex-set of a strong component of $D \backslash Z$ in such a way that if $Z' \subseteq Z \subseteq V(D)$ with $|Z| < w$, then $\beta(Z) \subseteq \beta(Z')$. If β is a haven of order w in a digraph D, then the robber player wins against $w-1$ cops by staying in $\beta(Z)$, where Z is the set of vertices occupied by the cops. The haven axiom guarantees that however the cops change their position, the robber has a move consistent with this strategy. The following is a "directed" version of the easy half of (1.1).

(3.1) *Let D be a digraph, and let w be an integer. If D has a haven of order w, then its tree-width is at least $w-1$.*

Proof. Suppose for a contradiction that D has a haven β of order w and an arboreal decomposition (R, X, W) of width at most $w-2$. Let us choose a vertex $r \in V(R)$ such that (letting X_r denote the union of X_e over all edges e of R incident r)

(i) $\beta(X_r \cup W_r) \subseteq \bigcup \{W_t : t \in V(R), t > r\}$, and

(ii) subject to (i), the distance of r from the root of R is maximum.

Such a choice is possible, because $|X_r \cup W_r| \leqslant w-1$ and the root of R satisfies (i). Since $\beta(X_r \cup W_r)$ is the vertex-set of a strong component of $D \backslash (X_r \cup W_r)$ we deduce from (D2) that there exists an edge $e \in E(R)$ with tail r such that $\beta(X_r \cup W_r) \subseteq W^e$, where $W^e = \bigcup \{W_r : r \in V(R), r > e\}$.

Let r' be the head of e. By the haven axiom $\beta(X_r \cup W_r) \subseteq \beta(X_e)$, and so $\beta(X_e) \cap W^e \neq \emptyset$. Thus $\beta(X_e) \subseteq W^e$ by (D2). But $\beta(X_{r'} \cup W_{r'}) \subseteq \beta(X_e)$ by the haven axiom, and hence $\beta(X_{r'} \cup W_{r'}) \subseteq \bigcup \{W_t : t \in V(R), t > r'\}$, contrary to the choice of r. Thus our assumption that D has both a haven of order w and an arboreal decomposition of width at most $w-2$ was incorrect, and the result follows. ∎

Now we are ready to complete the proofs of (2.1) and (2.2).

Proof of (2.1). We have already shown that the tree-width of D is at most the tree-width of G. To show the converse, let w be the tree-width of G. By (1.1) G has a haven β of order $w+1$. Then β is a haven in D, and hence D has tree-width at least w by (3.1), as desired. ∎

To complete the proof of (2.2) we need the following easy lemma.

(3.2) *Let G be a graph, let w be an integer, let β be a haven in G of order w, and let V, $V' \subseteq V(G)$ be such that $|V \cup V'| < w$. Then* $\beta(V) \cap \beta(V') \neq \emptyset$.

Proof. By the haven axiom $\beta(V \cup V') \subseteq \beta(V) \cap \beta(V')$, and hence $\beta(V) \cap \beta(V') \neq \emptyset$, as desired. ∎

If D is a digraph and $S \subseteq V(D)$ we define $\partial^+(S)$ to be the set of all vertices $v \in S$ such that v is the tail of an edge of D with head in $V(D) - S$, and we define $\partial^-(S)$ to be the set of all vertices $v \in S$ such that v is the head of an edge of D with tail in $V(D) - S$.

Proof of (2.2). We have already shown that $d \leqslant w$. To prove that $w \leqslant (2\Delta+1)(d+1)-1$, suppose for a contradiction that G has tree-width at least $(2\Delta+1)(d+1)$. Then G has a haven β of order $(2\Delta+1)(d+1)+1$ by (1.1). Our aim is to construct a haven γ in D of order $d+2$. To this end we first prove that if $S \subseteq V(D) - Z$ is Z-normal for some set $Z \subseteq V(D)$ with $|Z| \leqslant d+1$, then both $\partial^+(S)$ and $\partial^-(S)$ have size at most $\Delta(d+1)$. To see this let $S = S_i \cup S_{i+1} \cup \cdots \cup S_j$, where $S_1, S_2, ..., S_t$ are the vertex-sets of the strong components of $D \backslash Z$, numbered so that for $1 \leqslant i' < j' \leqslant t$ no edge of D has head in $D_{i'}$ and tail in $D_{j'}$. Then the size of $\partial^+(S)$ is at most the number of edges of D with tail in S and head not in S. The heads of such edges belong to $Z \cup Q$, where $Q = S_{j+1} \cup S_{j+2} \cup \cdots \cup S_t$. Since D is Eulerian, the number of edges with tail not in Q and head in Q is equal to the number of edges directed the opposite way. But every edge with tail in Q and head not in Q has head in Z, and so we see that the size of $\partial^+(S)$ is at most the number of edges with head in Z, which, in turn, is at most $\Delta(d+1)$, as desired. The bound on the size of $\partial^-(S)$ is analogous.

We claim that for every set $Z \subseteq V(D)$ with $|Z| \leqslant d+1$ there exists a strong component C of $D \backslash Z$ such that $\beta(Z \cup \partial^+(V(C)) \cup \partial^-(V(C))) \subseteq V(C)$. To prove this claim let Z be as stated and let $S_1, S_2, ..., S_t$ be the vertex-sets of all the strong components of $D \backslash Z$ numbered so that if $1 \leqslant i < j \leqslant t$, then no edge of D has head in S_i and tail in S_j. Let us choose the maximum integer $i \in \{1, 2, ..., t\}$ such that

$$\beta(Z \cup \partial^+(S_1 \cup S_2 \cup \cdots \cup S_{i-1})) \subseteq S_i \cup S_{i+1} \cup \cdots \cup S_t.$$

Such a choice is possible, because $i=1$ satisfies the inclusion. We shall prove that the strong component with vertex-set S_i satisfies the claim. By (3.2)

$$\beta(Z \cup \partial^-(S_i \cup S_{i+1} \cup \cdots \cup S_t)) \subseteq S_i \cup S_{i+1} \cup \cdots \cup S_t,$$

and so either

$$\beta(Z \cup \partial^-(S_i \cup S_{i+1} \cup \cdots \cup S_t) \cup \partial^+(S_1 \cup S_2 \cup \cdots \cup S_i)) \subseteq S_i$$

or

$$\beta(Z \cup \partial^-(S_i \cup S_{i+1} \cup \cdots \cup S_t) \cup \partial^+(S_1 \cup S_2 \cup \cdots \cup S_i)) \subseteq S_{i+1} \cup S_{i+2} \cup \cdots \cup S_t.$$

The latter case cannot occur, because then (3.2) would give a contradiction to the choice of i. Thus the former case holds, and (3.2) implies

$$\beta(Z \cup \partial^-(S_i) \cup \partial^+(S_i)) \subseteq S_i,$$

as desired. This proves the claim, and hence we can pick a component C of $D \backslash Z$ satisfying the claim and define $\gamma(Z) = V(C)$.

Next we show that γ is a haven in D of order $d+2$. To this end let $Z' \subseteq Z \subseteq V(D)$ with $|Z| \leqslant d+1$. Suppose for a contradiction that $\gamma(Z) \nsubseteq \gamma(Z')$; then $\gamma(Z) \cap \gamma(Z') = \emptyset$. By reversing the directions of all the edges of D if necessary we may assume that $D \backslash Z'$ has no directed path from a vertex in $\gamma(Z)$ to a vertex in $\gamma(Z')$. Then the vertex-sets of the strong components of $D \backslash Z'$ may be numbered $S_1, S_2, \ldots, S_t$ in such a way that for $1 \leqslant i < j \leqslant t$ no edge of D has head in S_i and tail in S_j and such that $k < l$, where $\gamma(Z') = S_k$ and $\gamma(Z) \subseteq S_l$. Let $S' = S_1 \cup S_2 \cup \cdots \cup S_k$. Then S' is the union of vertex-sets of some strong components of $D \backslash Z'$ including $\gamma(Z')$ but not $\gamma(Z)$ such that $\partial^+(\gamma(Z')) \subseteq \partial^+(S')$ and every edge of $D \backslash Z'$ with one end in S' and the other in $V(D) - Z' - S'$ has tail in S'. By definition of γ,

$$\beta(Z' \cup \partial^-(\gamma(Z')) \cup \partial^+(\gamma(Z'))) \subseteq S'.$$

By (3.2), $\beta(Z' \cup \partial^+(S')) \subseteq S'$. Similarly, let S be the union of the vertex-sets of some strong components of $D \backslash Z$ including $S' - Z$ but not $\gamma(Z)$ such that $\partial^-(\gamma(Z)) \subseteq \partial^-(V(D) - Z - S)$. From (3.2) we deduce that $\beta(Z \cup \partial^-(V(D) - Z - S)) \subseteq S$, and by applying (3.2) once again we conclude that $\beta(Z \cup \partial^-(\gamma(Z)) \cup \partial^+(\gamma(Z)))$ is disjoint from $\gamma(Z)$, contrary to the definition of γ.

Thus γ is a haven in D of order $d+2$, contrary to (3.1), because D has tree-width d. ∎

It appears possible that the converse of (3.1) holds, in other words, that (1.1) holds also for directed graphs. However, at the time of writing we are unable to decide that. We are only able to prove the following weaker result.

(3.3) *Let D be a directed graph, and let $w > 0$ be an integer. Then either D has tree-width at most $3w - 2$ or it has a haven of order w.*

Proof. Let us select an arboreal decomposition (R, X, W) of D such that

(1) $|W_r \cup \bigcup_{e \sim r} X_e| \leqslant 3w - 1$ for all $r \in V(R)$ of out-degree at least one,

(2) $|X_e| \leqslant 2w - 1$ for every $e \in E(R)$, and

(3) subject to (1) and (2), the number of vertices $v \in V(D)$ such that $v \in W_r$ and $|W_r| \geqslant w + 1$ for some $r \in V(R)$ is minimum.

Such a choice is possible, because the arboreal decomposition (R^0, X^0, W^0) of D with X^0 null, $V(R^0) = \{r\}$, $W^0 = (W_r)$, and $W_r = V(D)$ satisfies (1) and (2). If the inequality in (1) holds for all vertices $r \in V(R)$, then D has tree-width at most $3w - 2$, and the theorem holds. We may therefore assume that there exists a vertex $r_0 \in V(D)$ of out-degree zero such that $|W_{r_0} \cup \bigcup_{e \sim r_0} X_e| \geqslant 3w$. If $|V(R)| > 1$, then there exists a unique edge $e_0 \in E(R)$ with head r_0, and we put $Y = X_{e_0}$; otherwise e_0 is undefined and we put $Y = \emptyset$. In either case $|Y| \leqslant 2w - 1$, $|W_{r_0}| \geqslant w + 1$, and W_{r_0} is Y-normal.

We claim that if for every set $Z \subseteq V(D)$ with $|Z| < w$ there exists a strong component C of $D \backslash Z$ with vertex-set $\beta(Z)$ such that $|\beta(Z) \cap Y| \geqslant w$, then the theorem holds. To prove this claim we shall show that β is a haven in D of order w. Indeed, let $Z' \subseteq Z \subseteq V(D)$ with $|Z| \leqslant w - 1$. Since $|Y| \leqslant 2w - 1$ we deduce that $\beta(Z) \cap \beta(Z') \cap Y \neq \emptyset$; in particular, $\beta(Z) \cap \beta(Z') \neq \emptyset$. But $\beta(Z)$ is the vertex-set of a strongly connected subdigraph of D and is disjoint from Z', and hence $\beta(Z) \subseteq \beta(Z')$, as desired. This proves the claim.

Thus we may assume that there exists a set $Z' \subseteq V(D)$ with $|Z'| < w$ such that every strong component C of $D \backslash Z'$ satisfies $|V(C) \cap Y| < w$. By adding one element of W_{r_0} to such a set Z' we produce a set $Z \subseteq V(D)$ such that

(4) $|Z| \leqslant w$, $Z \cap W_{r_0} \neq \emptyset$, *and every strong component C of $D \backslash Z$ satisfies* $|V(C) \cap Y| < w$.

Let B be a strong component of $C \backslash Y$ for some strong component C of $D \backslash Z$. The digraph B is strongly connected and its vertex-set is disjoint from Y, and hence either $V(B) \subseteq W_{r_0}$ or $V(B) \cap W_{r_0} = \emptyset$. Let $B_1, B_2, ..., B_d$ be all the digraphs such that $V(B_i) \subseteq W_{r_0}$ and there exists a strong component C_i of $D \backslash Z$ (depending on B_i) such that B_i is a strong component of $C_i \backslash Y$. Then for $i = 1, 2, ..., d$ the strong component C_i is uniquely determined. It follows that

(5) $Z \cap W_{r_0}$, $V(B_1)$, $V(B_2)$, ..., $V(B_d)$ *is a partition of* W_{r_0} *into nonempty sets.*

For $i=1, 2, ..., d$ let $r_i \notin V(R)$ be a new vertex, and let R' be the arborescence obtained from R by adding $r_1, r_2, ..., r_d$ and a directed edge e_i with tail r and head r_i for all $i=1, 2, ..., d$. Let $X'_e = X_e$ for $e \in E(R)$, let $W'_r = W_r$ for $r \in V(R) - \{r_0\}$, let $W'_{r_0} = Z \cap W_{r_0}$, and for $i=1, 2, ..., d$ let $X'_{e_i} = Z \cup (V(C_i) \cap Y)$ and $W'_{r_i} = V(B_i)$. Finally, let $X' = (X'_e : e \in E(R'))$ and $W' = (W'_r : r \in V(R'))$.

We claim that (R', X', W') is an arboreal decomposition of D. Condition (D1) follows immediately from (5). Condition (D2) clearly holds for edges whose tails are not r_0, because all the sets referenced in (D2) remained unchanged. Thus it remains to verify (D2) for edges with tail r_0. To that end let $i \in \{1, 2, ..., d\}$. Then $\bigcup \{W'_r : r \in V(R'), r > e_i\} = W'_{r_i}$ and $W'_{r_i} \cap X'_{e_i} = \emptyset$, because $W'_{r_i} = V(B_i)$ and B_i is a strong component of $C_i \backslash Y$, and hence has no vertex in $Z \cup Y$. To complete the verification of (D2) we shall show that B_i is a strong component of $D \backslash X'_{e_i}$. Indeed, B_i is strongly connected and has no vertex in X'_{e_i}, and hence B_i is a subdigraph of a strong component H of $D \backslash X'_{e_i}$. But $V(H) \cap Z = \emptyset$, and yet $V(H) \cap V(C_i) \neq \emptyset$ (because $V(B_i) \subseteq V(H)$), and hence H is a subdigraph of C_i. But H has no vertex in $V(C_i) \cap Y$, and so H is a subdigraph of a strong component of $C_i \backslash Y$. Thus $H = B_i$, as desired. This proves that B_i is a strong component of $D \backslash X'_{e_i}$, and thus completes the proof of the fact that (R', X', W') is an arboreal decomposition of D.

We claim that (R', X', W') satisfies (1) and (2). Condition (1) is clear for all $r \in V(R') - \{r_0\}$, and for $r = r_0$ we have

$$\left| W'_r \cup \bigcup_{e \sim r} X'_e \right| \leqslant |Y \cup Z| \leqslant |Y| + |Z| \leqslant 2w - 1 + w = 3w - 1.$$

Condition (2) is clear for all $e \in E(R)$, and for $i = 1, 2, ..., d$ we have $|X'_{e_i}| \leqslant |Z| + |V(C_i) \cap Y| \leqslant w + w - 1 = 2w - 1$ by (4), as desired. Since $Z \cap W_{r_0} \neq \emptyset$ we see that the existence of (R', X', W') contradicts (3), as desired. ∎

4. ALGORITHMS

In this section we give a generic algorithm to solve many NP-hard problems in polynomial time, provided the input digraph has bounded tree-width.

Let D be a digraph, let $S \subseteq V(D)$, and let w be an integer. Let us recall that Z-normal sets were defined at the beginning of Section 2. We say that S is *w-protected in D* if it is Z-normal for some $Z \subseteq V(D)$ with $|Z| \leqslant w$. In particular, the empty set is w-protected for all $w \geqslant 0$. In the course of the algorithm we shall compute certain information about w-protected subsets of $V(D)$. It turns out that this information needs to satisfy very little in order for the algorithm to perform correctly, and so it seems worthwhile to isolate the conditions we need. We shall refer to the information we are interested in as an *itinerary* for a set $A \subseteq V(D)$, and we require that for every integer w there be a real number α and two algorithms such that the following two axioms hold. (In fact, the first axiom does not depend on w.)

(4.1) AXIOM 1. *Let D be a digraph, and let $A, B \subseteq V(D)$ be disjoint sets such that no edge of D has head in A and tail in B. Then an itinerary for $A \cup B$ can be computed from itineraries of A and B in time $O((|A|+|B|)^{\alpha})$.*

(4.2) AXIOM 2. *Let D be a digraph, and let $A, B \subseteq V(D)$ be disjoint sets such that A is w-protected, and $|B| \leqslant w$. Then an itinerary for $A \cup B$ can be computed from itineraries of A and B in time $O((|A|+1)^{\alpha})$.*

We remark that the constants hidden in the $O(...)$ notation depend on w, but not on D, A, or B. Before we present the main algorithm we need the following lemma.

(4.3) *Let D be a digraph, let (R, X, W) be an arboreal decomposition of D of width at most $w-1$, let $X=(X_e\colon e \in E(R))$, let $W=(W_r\colon r \in V(R))$, and let $r_0 \in V(R)$. Then the set $S=\bigcup \{W_r\colon r \in V(R), r > r_0\}$ is w-protected in D.*

Proof. Let $Z = W_{r_0} \cup \bigcup_{e \sim r_0} X_e$. Then $|Z| \leqslant w$, because (R, X, W) has width at most $w-1$. We claim that S is Z-normal. Indeed, let $e_1, e_2, ..., e_d$ be the edges of R with tail r_0. Then $S = \bigcup W_r$, the union taken over all $r \in V(R)$ such that $r > e_i$ for some $i = 1, 2, ..., d$. We must show that there is no directed walk in $D \backslash Z$ from a vertex of S to a vertex of S using a vertex $x \in V(D) - Z - S$. To this end suppose for a contradiction that such a walk, say P, exists. Since x exists, r_0 is not the root of R. Let e_0 be the unique edge of R with head r_0. Then $S \subseteq \bigcup \{W_r\colon r \in V(R), r > e_0\}$, but the latter is X_{e_0}-normal by (D2), contrary to the existence of P. This completes the proof of the claim that S is Z-normal. By the claim S is w-protected, as desired. ∎

(4.4) ALGORITHM. For every fixed integer w there exists an algorithm satisfying the following specifications.

Input: A digraph D on n vertices and an arboreal decomposition (R, X, W) of D of width at most $w-1$.

Output: An itinerary for $V(D)$.

Running time: $O(n^{\alpha+1})$, assuming Axioms 1 and 2 are satisfied.

Description. Let $X=(X_e\colon e \in E(R))$ and $W=(W_r\colon r \in V(R))$. Let $r_0 \in V(R)$ be a vertex of out-degree zero. By (4.2) applied to $A=\varnothing$ and $B=W_{r_0}$ we can compute an itinerary for W_{r_0} in constant time. Now let $r_0 \in V(R)$, let $e_1, ..., e_d$ be the edges of R with tail r_0, let e_0 be the edge of R with head r_0 (if r_0 is the root of R, then e_0 is undefined, but we define X_{e_0} to be the empty set), for $i = 1, 2, ..., d$ let $S_i = \bigcup \{W_r : r \in V(R), r > e_i\}$, let $S_0 = W_{r_0} \cup S_1 \cup S_2 \cup \cdots \cup S_d$, and suppose that we have already computed itineraries for $S_1, S_2, ..., S_d$. Our objective is to compute an itinerary for S_0. Let $Z = X_{e_0} \cup X_{e_1} \cup \cdots \cup X_{e_d}$. Since for all $i = 1, 2, ..., d$ the set S_i is Z-normal, we may assume that the edges $e_1, e_2, ..., e_d$ are numbered in such a way that for i, j with $1 \leqslant i < j \leqslant d$ no edge of D has its head in S_i and tail in S_j. By using (4.1) repeatedly we can compute the itineraries of $S_1 \cup S_2$, $S_1 \cup S_2 \cup S_3, ...,$ $S_1 \cup S_2 \cup \cdots \cup S_d$ in total time $O(dn^{\alpha})$. By (4.3) the set $S' = S_1 \cup S_2 \cup \cdots \cup S_d$ is w-protected in D, and so by (4.2) we may compute an itinerary for $S_0 = S' \cup W_{r_0}$ in time $O(n^{\alpha})$. When r_0 is the root of R, then $S_0 = V(G)$, and hence an itinerary for $V(G)$ can be computed in time $O(n^{\alpha+1})$, because $|V(R)| \leqslant n$ by (D1). ∎

For the purpose of applications we now make a more specific choice of itineraries. If D is a digraph and $S \subseteq V(D)$, then $D \mid S$ denotes the digraph $D \backslash (V(D) - S)$. A *linkage* in a digraph D is a subdigraph L of D such that every weak component of L (that is, a component of the underlying undirected graph) is a directed path. Let $\sigma = (s_1, t_1, s_2, t_2, ..., s_k, t_k)$ be a sequence of $2k$ (not necessarily distinct) vertices of D. We say that a linkage L is a σ-linkage if the weak components of L can be numbered $P_1, P_2, ..., P_k$ in such a way that, for $i = 1, 2, ..., k$, the digraph P_i is a directed path with first vertex s_i and last vertex t_i. Let D be a digraph, let $S \subseteq V(D)$, let L be a linkage in D, and let $k, w \geqslant 0$ be integers. We say that L is (k, w)-*limited* in S if $V(L) \subseteq S$, and for every w-protected set $S' \subseteq S$ the digraph $L \mid S'$ has at most $k + w$ weak components. The following is an easy but important fact.

(4.5) *Let $k, w \geqslant 0$ be integers, let D be a digraph, let $S \subseteq V(D)$, and let L be a linkage in D with k weak components and $V(L) \subseteq S$. Then L is (k, w)-limited in S.*

Proof. Let $P_1, P_2, ..., P_k$ be the weak components of L, and let $S' \subseteq S$ be w-protected. Thus there exists a set $Z \subseteq V(D)$ such that $|Z| \leqslant w$ and S' is Z-normal. Then for $i = 1, 2, ..., k$, if $P_i \mid S'$ has j weak components, then $|V(P_i) \cap Z| \geqslant j - 1$ by the Z-normality of S'. Thus $L \mid S'$ has at most $k + w$ weak components, as desired. ∎

Let D be a digraph, let $k, w \geqslant 0$ be integers, and let $S \subseteq V(D)$. Let l be an integer with $1 \leqslant l \leqslant |S|$, and let σ be a sequence of $2j$ (not necessarily distinct) vertices of S, where $0 \leqslant j \leqslant k + w$. A mapping f which assigns to every pair (l, σ) as above a value 0 or 1 is called a (k, w)-*itinerary* for S in D provided that

(i) if $f(l, \sigma)=0$, then there exists no σ-linkage L in $D \mid S$ with $|V(L)|=l$ such that L is (k, w)-limited in S, and

(ii) if $f(l, \sigma)=1$, then there exists a σ-linkage L in $D \mid S$ with $|V(L)|=l$.

We now show that this concept satisfies Axioms (4.1) and (4.2).

(4.6) *Let D be a digraph, let k, $w \geqslant 0$ be integers, and let A, $B \subseteq V(D)$ be disjoint sets such that no edge of D has head in A and tail in B. Then a (k, w)-itinerary for $A \cup B$ can be computed from (k, w)-itineraries for A and B in time $O((|A|+|B|)^{4(k+w)+2})$.*

Proof. Let D, A, B, k, w be as stated, let f_1 be an itinerary for A in D, and let f_2 be an itinerary for B in D. Let l be an integer with $1 \leqslant l \leqslant |A \cup B|$, and let $\sigma=(s_1, t_1, s_2, t_2, ..., s_j, t_j)$ be a sequence of $2j$ vertices of $A \cup B$, where $0 \leqslant j \leqslant k+w$. If $s_i \in B$ and $t_i \in A$ for some $i=1, 2, ..., j$, then there is no σ-linkage in $D \mid (A \cup B)$ and we set $f(l, \sigma)=0$. Otherwise we proceed as follows. Let $i=1, 2, ..., j$. If s_i, $t_i \in A$ we define $s_i^1=s_i$, $t_i^1=t_i$, and declare s_i^2, t_i^2 undefined. If s_i, $t_i \in B$ we define $s_i^2=s_i$, $t_i^2=t_i$, and declare s_i^1, t_i^1 undefined. If $s_i \in A$ and $t_i \in B$ we define $s_i^1=s_i$, $t_i^2=t_i$, and select $t_i^1 \in A$, $s_i^2 \in B$ arbitrarily in such a way that some edge of D has tail t_i^1 and head s_i^2. Finally, we select an integer l_1 with $1 \leqslant l_1 \leqslant |A|$ arbitrarily. Let σ_1 be the subsequence of $(s_1^1, t_1^1, s_2^1, t_2^1, ..., s_j^1, t_j^1)$ consisting of those entries that are defined, and let σ_2 be defined analogously. If for some choice of s_1^1, t_1^1, ..., s_j^2, t_j^2 and l as above we find $f(l_1, \sigma_1)=f(l-l_1, \sigma_2)=1$, then we set $f(l, \sigma)=1$. Otherwise we set $f(l, \sigma)=0$. If $f(l, \sigma)=1$, then condition (ii) in the definition of a (k, w)-itinerary is clearly satisfied. Let us then assume that $f(l, \sigma)=0$; we must show that condition (i) holds. Suppose for a contradiction that there exists a σ-linkage L in $G \mid (A \cup B)$ with $|V(L)|=l$ such that L is (k, w)-limited in $A \cup B$. Let $L_1=L \mid A$ and $L_2=L \mid B$. Then L_1 is a σ_1-linkage in $G \mid A$ for some sequence σ_1 considered above, and similarly L_2 is a σ_2-linkage in $G \mid B$. Furthermore, L_1 is (k, w)-limited in A, and L_2 is (k, w)-limited in B, and hence, letting $l_1=|V(L_1)|$, we see that $f_1(l_1, \sigma_1)=f(l-l_1, \sigma_2)=1$, a contradiction. Thus f is a (k, w)-itinerary for $A \cup B$, as desired. The bound on running time follows immediately. ∎

(4.7) *Let D be a digraph, let k, $w \geqslant 0$ be integers, and let A, $B \subseteq V(D)$ be disjoint sets such that A is w-protected and $|B| \leqslant w$. Then a (k, w)-itinerary for $A \cup B$ in D can be computed from a (k, w)-itinerary for A in D in time $O((|A|+1)^{4(k+w)+1})$.*

Proof. Let D, A, k, w be as stated, and let f' be an itinerary for A in D. Let l be an integer with $1 \leqslant l \leqslant |A \cup B|$, and let $\sigma=(s_1, t_1, s_2, t_2, ..., s_j, t_j)$ be a sequence of vertices of $A \cup B$, where $j \leqslant k+w$. Let Σ be the set of all sequences $\sigma'=(s'_1, t'_1, s'_2, t'_2, ..., s'_{j'}, t'_{j'})$ of vertices of A of length at most $2(k+w)$ such that $\{s_1, s_2, ..., s_j\} \cap A \subseteq \{s'_1, s'_2, ..., s'_{j'}\}$ and $\{t_1, t_2, ..., t_j\} \cap A \subseteq \{t'_1, t'_2, ..., t'_{j'}\}$. For $\sigma' \in \Sigma$ as above let H be the digraph with vertex-set $\{s'_1, t'_1, s'_2, t'_2, ..., s'_{j'}, t'_{j'}\}$ and j' edges, where the ith edge has tail s'_i and head t'_i. Also, for $\sigma' \in \Sigma$ as above let $\mathscr{L}_{\sigma'}$ be the set of all linkages Q in $A \cup B$ such that every weak component of Q has its first and last vertex in $B \cup \{s'_1, t'_1, s'_2, t'_2, ..., s'_{j'}, t'_{j'}\}$ and all other vertices in B and $H \cup Q$ is a σ-linkage. Thus the size of $\mathscr{L}_{\sigma'}$ is bounded by a function of k and w and does not depend on $|A|$.

We set $f(l, \sigma) = 1$ if there exists a sequence $\sigma' \in \Sigma$ and a linkage $Q \in \mathscr{L}_{\sigma'}$ such that $f'(l', \sigma') = 1$, where $l' = l - |B \cap V(Q)|$. We claim that f thus defined satisfies conditions (i) and (ii) in the definition of a (k, w)-itinerary. Condition (ii) is clearly satisfied by the construction. To prove that condition (i) holds let us assume that there exists a σ-linkage L in $G \mid (A \cup B)$ with $|V(L)| = l$ such that L is (k, w)-limited in $A \cup B$. Then $L \mid A$ is a σ'-linkage in $G \mid A$ for some $\sigma' \in \Sigma$ (because L is (k, w)-limited in $A \cup B$), and it follows from the construction that there exists a linkage $Q \in \mathscr{L}_{\sigma'}$ (in fact, Q is a subdigraph of L). Furthermore, $L \mid A$ is (k, w)-limited in A, and hence $f'(l', \sigma') = 1$, where $l' = |V(L|A)|$. We deduce that $f(l, \sigma) = 1$, as desired. ∎

From (4.4), (4.5), (4.6) and (4.7) we deduce

(4.8) *For all fixed integers k, $w \geqslant 0$ there exists a polynomial-time algorithm to solve the following problem.*

Instance: *A directed graph D on n vertices, an arboreal decomposition (R, X, W) of D of width at most $w-1$, a sequence σ of $2k$ vertices of D, and a set $M \subseteq \{1, 2, ..., n\}$.*

Question: *Does there exist a σ-linkage L in D with $|V(L)| \in M$?*

Proof. By (4.6) and (4.7) the (k, w)-itinerary satisfies Axioms 1 and 2. By (4.4) a (k, w)-itinerary f for $V(D)$ can be computed in polynomial time. If $f(l, \sigma) = 1$ for some $l \in M$, then we answer "yes"; otherwise we answer "no." The "yes" answer is clearly correct, and the "no" answer is correct by (4.5). ∎

This includes the linkage problem for a fixed number of terminals, the Hamilton path and Hamilton cycle problems, the Hamilton path problem with prescribed ends, the even cycle problem through a specified vertex, etc. Since acyclic digraphs have tree-width zero, (4.8) generalizes the algorithm of Fortune *et al.* [6].

In the above problems we were assuming that an arboreal decomposition of the input digraph is given as part of the input instance. What if we were only told that the input digraph D has bounded tree-width, but were given no arboreal decomposition of D? It is easy to convert the proof of

(3.3) into a polynomial-time algorithm that for a fixed integer w either finds an arboreal decomposition of the input digraph D of width at most $3w-2$ (and such that the underlying arborescence has at most $|V(D)|$ vertices) or constructs a haven in D of order w, in which case D has tree-width at least $w-1$ by (3.1). Thus if D has tree-width at most $w-1$, this algorithm will allow us to construct an arboreal decomposition of D of width at most $3w+1$, and we can then apply (4.8) (with w replaced by $3w+2$).

5. A CONJECTURE

The result of [16] states that an undirected graph has large tree-width if and only if it has a large grid minor. The purpose of this section is to formulate an analogous conjecture for directed graphs. To do so we need to define minors of digraphs and directed grids.

Let D be a digraph. We say that an edge $e \in E(D)$ with head v and tail u is *contractible* if either e is the only edge of D with head v, or it is the only edge of D with tail u, or both. We say that a digraph D is a *minor* of a directed graph D' if D can be obtained from a subdigraph of D' by repeatedly contracting contractible edges. It is easy to see that if a digraph D is a minor of a digraph D', then the tree-width of D is at most the tree-width of D'.

For $k=1, 2, \ldots$ we define a digraph J_k as the union of k directed circuits $C_1, C_2, \ldots, C_k$, and $2k$ directed paths $P_1, P_2, \ldots, P_k, Q_1, Q_2, \ldots, Q_k$, where, for $i=1, 2, \ldots, k$, C_i has vertex-set $\{u_{i,1}, u_{i,2}, \ldots, u_{i,k}, v_{i,1}, v_{i,2}, \ldots, v_{i,k}\}$ (in order), P_i has vertex-set $\{u_{1,i}, u_{2,i}, \ldots, u_{k,i}\}$ (in order), and Q_i has vertex-set $\{v_{k,i}, v_{k-1,i}, \ldots, v_{1,i}\}$ (in order). Thus J_k has a planar drawing, where

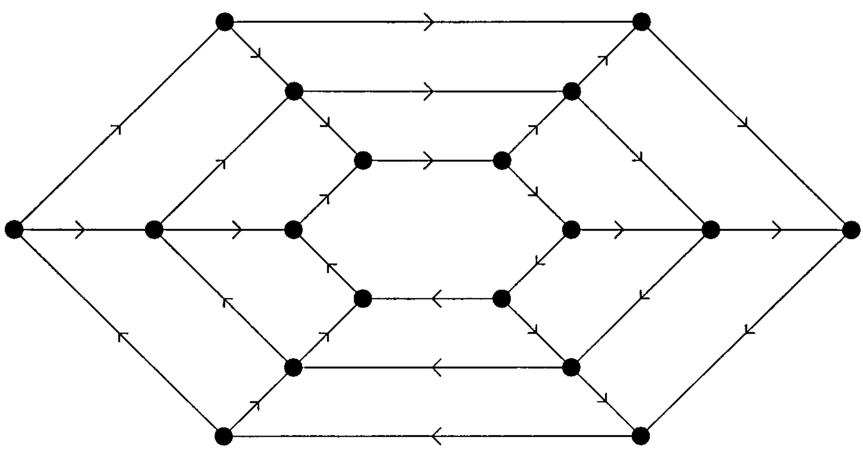

FIGURE 2

the circuits $C_1, C_2, ..., C_k$ are concentric (in the order listed); $P_1, P_2, ..., P_k$ are disjoint paths linking C_1 to C_k; and $Q_1, Q_2, ..., Q_k$ are disjoint paths linking C_k to C_1. See Fig. 2.

The conjecture is as follows. It was formulated using havens during a conversation of the last three authors with Noga Alon and Bruce Reed at a conference held in Annecy, France, in 1995 and appeared in equivalent form in [14].

(5.1) CONJECTURE. *For every integer k there exists an integer N such that every digraph with tree-width N or more has a minor isomorphic to J_k.*

Conversely, if $X \subseteq V(J_k)$ and $|X| < k$, then there exist indices i, p, q such that $C_i \cup P_p \cup Q_q$ is disjoint from X. Let $\beta(X)$ be the strong component of $J_k \backslash X$ that includes C_i. It follows that β is well-defined and that it is a haven in J_k of order k, and hence J_k has tree-width at least $k-1$ by (3.1). We have convinced ourselves that (5.1) holds for planar digraphs, but the general case is open.

Finally, we mention two attempts at defining tree-width for a directed graph that did not work out. Given a digraph D, one can consider pairs (T, Z), where T is a tree and $Z = (Z_t: t \in V(T))$ satisfies

(i) $\bigcup (Z_t: t \in V(T)) = V(D)$,

(ii) if t' lies on the path in T between t and t'', then $Z_{t'} \subseteq Z_t \cap Z_{t''}$, and

(iii) if $e \in E(T)$, then the components of $T \backslash e$ can be numbered T_1 and T_2 in such a way that no edge of D has head in $\bigcup_{t \in V(T_1)} Z_t - \bigcup_{t \in V(T_2)} Z_t$ and tail in $\bigcup_{t \in V(T_2)} Z_t - \bigcup_{t \in V(T_1)} Z_t$.

Another possibility is to consider triples (T, Z, π), where T is a tree, $Z = (Z_t: t \in V(T))$ satisfies (i) and (ii) above, and $\pi = (\pi_t: t \in V(T))$ satisfies

(iii′) for every $t_0 \in V(T)$, π_{t_0} is a linear ordering of the components of $T \backslash t_0$ such that if T_1 is strictly before T_2 in π_{t_0}, then no edge of D has head in $\bigcup_{t \in V(T_1)} Z_t - Z_{t_0}$ and tail in $\bigcup_{t \in V(T_2)} Z_t - Z_{t_0}$.

For either of these two concepts one can define *width* as $\min\{|Z_t| - 1: t \in V(T)\}$, leading to two possible variations of tree-width. However, neither seems satisfactory, because they are not closed under directed unions. (A digraph D is a *directed union* of digraphs D_1 and D_2 if D_1 and D_2 are induced subdigraphs of D, $V(D_1) \cup V(D_2) = V(D)$, and no edge of D has head in $V(D_1)$ and tail in $V(D_2)$.)

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