Induced subgraph density. I. A loglog step towards Erdős-Hajnal

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Abstract

In 1977, Erdős and Hajnal made the conjecture that, for every graph H, there exists $c > 0$ such that every H-free graph G has a clique or stable set of size at least $|G|^c$; and they proved that this is true with $|G|^c$ replaced by $2^{c\sqrt{\log |G|}}$. Until now, there has been no improvement on this result (for general H).

We prove a strengthening: that for every graph H, there exists $c > 0$ such that every H-free graph G with $|G| \geq 2$ has a clique or stable set of size at least

$$2^{c\sqrt{\log |G| \log\log |G|}}.$$

Indeed, we prove the corresponding strengthening of a theorem of Fox and Sudakov, which in turn was a common strengthening of theorems of Rödl, Nikiforov, and the theorem of Erdős and Hajnal mentioned above.

1 Introduction

A graph G *contains* a graph H if H is isomorphic to an induced subgraph of G, and G is H*-free* otherwise. $|G|$ denotes the number of vertices of the graph G; and we write $\kappa(G)$ for the largest t such that G has a clique or stable set of cardinality t. For most n-vertex graphs G, $\kappa(G) = O(\log n)$, but this changes dramatically if we forbid some induced subgraph. In 1977, Erdős and Hajnal [3, 4] proposed the following well-known conjecture:

1.1 Conjecture: *For every graph H there exists $c > 0$ such that $\kappa(G) \geq |G|^c$ for every H-free graph G.*

This has attracted a great deal of attention over the years, but despite this, it is only known to be true for a few graphs H. Until recently, it was only known for the graphs with at most five vertices, except the five-vertex path and its complement, and graphs that can be made from these by vertex-substitution. In two recent papers [6, 8], three of us have got further; we have shown it for the five-vertex path, and for infinitely many other graphs that are cannot be built from smaller graphs by vertex-substitution. But it remains the case that the graphs known to satisfy 1.1 are rare and highly restricted.

On the other hand, it is known that excluding any fixed induced subgraph will guarantee that $\kappa(G)$ is much bigger than $\log |G|$. Erdős and Hajnal themselves proved the following:

1.2 *For every graph H there exists $c > 0$ such that $\kappa(G) \geq 2^{c\sqrt{\log |G|}}$ for every non-null H-free graph G.*

(All logarithms in this paper are to base 2.) Indeed they proved something slightly stronger, that for all H and all $c > 0$ the same conclusion holds, provided that $|G|$ is sufficiently large. Rather surprisingly, until now there has been no improvement on this, for a general graph H. Our result is such an improvement:

1.3 *For every graph H there exists $c > 0$ such that $\kappa(G) \geq 2^{c\sqrt{\log |G| \log\log |G|}}$ for every H-free graph G with $|G| \geq 2$.*

A *cograph* means a P_4-free graph, where P_4 denotes the path with four vertices; and we denote by $\mu(G)$ the largest t such that some t-vertex induced subgraph of G is a cograph. Cliques and stable sets induce cographs, and every cograph J has a clique or stable set of size at least $|J|^{1/2}$; so 1.1, 1.2, and 1.3 are equivalent to the same statements with $\kappa(G)$ replaced by $\mu(G)$, and in that form they are often easier to work with.

If $X \subseteq V(G)$, $G[X]$ denotes the induced subgraph with vertex set X. A *pure pair* in G is a pair of disjoint subsets A, B of $V(G)$ such that either there are no edges between A, B, or all edges between A, B are present. If we are trying to prove that $\mu(G) \geq f(|G|)$ for all H-free graphs G, where f is some function, it is enough to know that all H-free graphs G with $|G| > 1$ have pure pairs A, B with $|A|, |B|$ appropriately large in terms of $|G|$. Because then we could deduce by induction on $|G|$ that $G[A]$ contains a cograph C with $|C| \geq f(|A|)$, and similarly $G[B]$ contains a large cograph D, and so $V(C) \cup V(D)$ induces a cograph in G, and therefore $\mu(G) \geq f(|A|) + f(|B|)$; and if $|A|, |B|$ are large enough, then $f(|A|) + f(|B|) \geq f(|G|)$ and the inductive step is complete.

In fact, for this purpose, we do not really need the pair A, B to be pure. Suppose that either every vertex in B has at most $|A|/(2\mu(G))$ neighbours in A, or every vertex in B has at most $|A|/(2\mu(G))$ non-neighbours in A. Choose $D \subseteq B$ as before; then, since $|D| \le \mu(G)$, there exists $A' \subseteq A$ with $|A'| \ge |A|/2$ such that A', D is a pure pair, and we apply the inductive hypothesis to $G[A']$, and reach the same conclusion as before.

Do such pairs A, B necessarily exist with A, B large? If $A, B \subseteq V(G)$ are disjoint, we say B is *x-sparse to A in G* if every vertex in B has at most $x|A|$ neighbours in A. Erdős and Hajnal [4] proved:

1.4 *For every graph H, there exists $c > 0$ such that for every H-free graph G with $|G| \ge 2$, and all $x \in (0, 1/2)$, there exist disjoint $A, B \subseteq V(G)$ with $|A|, |B| \ge x^c|G|$, such that B is x-sparse to A in one of $G, \overline{G}$.*

Then they used 1.4 (with an appropriate choice of x) and the inductive argument sketched above, to prove 1.2. But perhaps 1.4 can be strengthened. There is a pretty conjecture of Conlon, Fox and Sudakov [2] that would strengthen it:

1.5 Conjecture: *For every graph H there exists $c_1, c_2 > 0$ such that for every H-free graph G with $|G| \ge 2$, and all $x \in (0, 1/2)$, there exist disjoint $A, B \subseteq V(G)$ with $|A| \ge x^{c_1}|G|$ and $|B| \ge c_2|G|$, such that B is x-sparse to A in one of $G, \overline{G}$.*

If this is true, then, as Conlon, Fox and Sudakov observed, the inductive argument would yield exactly our result 1.3 (by choosing $x = 1/(2\mu(G))$).

Our argument takes a different approach and leaves this conjecture open, however. Rather than look for a sparse pair of large subsets, we look for a large number of smaller subsets, with a sparseness condition between each pair. We will explain this in more detail in the next section.

Over the years, there have been several theorems discovered that are related to the Erdös-Hajnal conjecture 1.1, and our proof method allows us to strengthen some of them. First, there is a fundamental theorem of Rödl [10]:

1.6 *For every graph H and all $x > 0$, there exists $\delta > 0$ with the following property. For every H-free graph G, there exists $S \subseteq V(G)$ with $|S| \ge \delta|G|$ such that one of $G[S], \overline{G}[S]$ has at most $x\binom{|S|}{2}$ edges.*

How large can we take δ as a function of x? Rödl's original proof gave a tower-type bound, because it used the regularity lemma, but Fox and Sudakov [5] made a significant improvement, proving a version of 1.6 that implies 1.2:

1.7 *There exists $c > 0$ such that for every graph H and all $x \in (0, 1/2)$, setting $\delta = 2^{-c|H|(\log \frac{1}{x})^2}$ satisfies 1.6.*

(To deduce 1.2, just set $x = 2^{-\sqrt{\frac{\log |G|}{2c|H|}}}$, and apply Turán's theorem. The proof is similar to the proof that 1.10 implies 1.3, which we give later.)

Nikiforov [9] gave a different strengthening of 1.6:

1.8 *For every graph H and all $x > 0$, there exists $\delta > 0$ such that if G is a graph containing fewer than $(\delta|G|)^{|H|}$ induced copies of H, then there exists $S \subseteq V(G)$ with $|S| \ge \delta|G|$ such that one of $G[S], \overline{G}[S]$ has at most $x\binom{|S|}{2}$ edges.*

Fox and Sudakov [5] were able to incorporate the analogous strengthening of 1.8 into 1.7:

1.9 *There exists $c > 0$ such that for every graph H and all $x \in (0, 1/2)$, setting $\delta = 2^{-c|H|(\log \frac{1}{x})^2}$ satisfies 1.8.*

Our main result is:

1.10 *For every graph H there exists c such that, if $x \in (0, 1/2)$ and*

$$\delta = 2^{-c(\log \frac{1}{x})^2 / \log\log \frac{1}{x}},$$

and G is a graph containing fewer than $(\delta|G|)^{|H|}$ induced copies of H, then there exists $S \subseteq V(G)$ with $|S| \geq \delta|G|$ such that one of $G[S], \overline{G}[S]$ has at most $x\binom{|S|}{2}$ edges.

The proof of 1.10 is by induction on $|H|$. Let us mention that it is essential for inductive purposes that we only assume the weaker hypothesis that G contains few copies of H, rather than that G is H-free.

1.10 strengthens the result 1.9 of Fox and Sudakov, and improves the best known quantitative bounds in Nikiforov's theorem 1.8 and Rödl's theorem 1.6. It also implies 1.3, as we will show later.

In the final section we discuss analogous results for tournaments and ordered graphs.

2 Blockades, and a sketch of the proof

If G, H are graphs, a *copy* of H in G is an isomorphism from H to an induced subgraph of G, and we denote by $\text{ind}_H(G)$ the number of copies of H in G. A *blockade* in G is a sequence $\mathcal{B} = (B_1, \ldots, B_k)$ of pairwise disjoint subsets of $V(G)$, and we call $B_1, \ldots, B_k$ its *blocks*. (In some earlier papers, the blocks of a blockade must be nonempty, but here it is convenient to allow empty blocks.) The *length* of the blockade $\mathcal{B} = (B_1, \ldots, B_k)$ is k, and its *width* is the minimum of the cardinalities of its blocks. For $\varepsilon > 0$, the blockade $\mathcal{B} = (B_1, \ldots, B_k)$ is *x-sparse* in G if for all i with $1 \leq i \leq k$, $B_{i+1} \cup \cdots \cup B_k$ is ε-sparse to B_i in G; and *ε-restricted* if for all i with $1 \leq i \leq k$, $B_{i+1} \cup \cdots \cup B_k$ is ε-sparse to B_i in one of $G, \overline{G}$.

Let us give a sketch of the proof of 1.3. The key step is the following:

2.1 *For all H, there exist $k_1, k_2 > 0$ such that for every non-null graph G and every x with $0 < x \leq \frac{1}{8|H|}$, if $\text{ind}_H(G) < x^{k_1}|G|^{|H|}$, then there is an x-restricted blockade in G of length at least $2\log(1/x)$, and width at least $\lfloor x^{k_2}|G| \rfloor$.*

First let us see that 2.1 implies 1.3. Choose $c > 0$ sufficiently small; we will show that c satisfies 1.3 by induction on $|G|$. Let $x := 1/(2\mu(G))$. It is easy to arrange that $x \leq \frac{1}{8|H|}$, and so we can apply 2.1 to the H-free graph G, and obtain a blockade $(B_1, \ldots, B_k)$. Then choose subsets $D_i \subseteq B_i$ for $i = k, k-1, \ldots, 1$ in turn, such that $D_i \cup \cdots \cup D_k$ induces a cograph, as follows. Having chosen $D_{i+1}, \ldots, D_k$, since $D_{i+1} \cup \cdots \cup D_k$ induces a cograph, it has cardinality at most $\mu(G)$, and so (assuming every vertex of $B_{i+1} \cup \cdots \cup B_k$ has at most $x|B_i|$ neighbours in B_i; the other case is similar), at least half the vertices in B_i have no neighbour in $D_{i+1} \cup \cdots \cup D_k$. By induction, we may choose D_i from this half, inducing a cograph and of cardinality at least $(|B_i|/2)^c$. Then $D_i \cup \cdots \cup D_k$ induces a cograph, completing the inductive definition. Consequently $\mu(G) \geq \sum (|B_i|/2)^c$, and the result follows after some calculation, which we omit. (We will not actually prove 1.3 this way; our proof goes via the stronger theorem 1.10, which can also be derived from 2.1 with more work.)

The main issue is how to prove 2.1. It is a consequence of the following:

2.2 *Let H be a graph, let $g \in V(H)$, and let $H' := H \setminus \{g\}$. Let $b, c > 0$, and let $a := b + (1+c)|H|$. Let G be a graph, let A, B be disjoint subsets of $V(G)$, and let $0 < x \leq 1/2$. Suppose that every vertex in A has at least $x|B|$ non-neighbours in B. Then either:*

- *there exists $B' \subseteq B$ with $|B'| \geq x|B|$ such that $\text{ind}_{H'}(G[B']) < x^b|B'|^{|H'|}$; or*
- *$\text{ind}_H(G) \geq x^a|A| \cdot |B|^{|H|-1}$; or*
- *there exists $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq x^a|A|$ and $|B'| \geq x^a|B|$ such that the number of edges between A', B' is at most $2x^c|A'| \cdot |B'|$.*

We will sketch a proof of 2.2, and then sketch how we use it to prove 2.1.

The idea of the proof of 2.2 is as follows. Let g have degree d in H, let $H_d := H$, and for $i = d-1, d-2, \ldots, 0$ let H_i be obtained from H_{i+1} by deleting one of the $i+1$ edges of H_{i+1} incident with g. We are interested in copies of H_i in G, where g is mapped into A and the other vertices of H_i are mapped into B. (Let us call such copies "special".) Each $v \in A$ has at least $x|B|$ non-neighbours in B, and we may assume that this set of non-neighbours (B' say) induces a subgraph that contains at least $x^b|B'|^{|H|-1} \geq x^{b+|H|}|B|^{|H|-1}$ copies of $H \setminus \{g\}$, because otherwise the first bullet holds. Since this is true for each v, there are at least $x^{b+|H|}|A| \cdot |B|^{|H|-1}$ special copies of H_0. On the other hand, we may assume that there are fewer than $x^a|A| \cdot |B|^{|H|-1}$ special copies of $H_d = H$, because otherwise the second bullet holds. So for some t, the number of special copies of H_t is at least $x^{b+|H|+ct}|A| \cdot |B|^{|H|-1}$, and the number of special copies of H_{t+1} is less than $x^{b+|H|+c(t+1)}|A| \cdot |B|^{|H|-1}$. Let us focus on this value of t.

Now H_t was obtained from H_{t+1} by deleting some edge incident with g, say gh. Fix a copy ϕ of $H \setminus \{g, h\}$ in $G[B]$; let U be the set of $u \in B$ such that mapping h to u extends ϕ to a copy of $H \setminus \{g\}$, and let V be the set of $v \in A$ such that mapping g to v extends ϕ to a copy of $H \setminus \{h\}$. The number of special copies of H_t is the sum, over all ϕ, of the number of nonedges between U, V, and the number of special copies of H_{t+1} is the sum over all ϕ of the number of edges between U, V. Since the second sum is at most x^c times the first, there is a choice of ϕ such that the number of edges between U, V is at most $x^c|U| \cdot |V|$ (and by allowing a factor of two here and averaging, we can arrange that also $|U| \cdot |V| \geq x^a|A| \cdot |B|$). But then U, V satisfy the third bullet, and this will prove 2.2.

Next, let us explain how to use 2.2 to prove 2.1. Let $g \in V(H)$ and $F = H \setminus \{g\}$. We will assume inductively that 2.1 holds for F, with k_1, k_2 replaced by k_1', k_2' say. Choose k_1, k_2 sufficiently large, and let G be a graph with $\text{ind}_H(G) < x^{k_1}|G|^{|H|}$. We must show that there is an x-restricted blockade in G of length at least $\log(1/x)$, and width at least $\lfloor x^{k_2}|G| \rfloor$.

Choose an induced subgraph J of H maximal such that G contains a large "approximate blowup" of J; that is, $|J|$ disjoint subsets A_j ($j \in V(J)$) of $V(G)$, each of size about $x^k|G|$ (where k is an appropriate constant depending on $|J|$), and such that for all distinct $i, j \in V(J)$, if $ij \notin E(J)$ then A_i, A_j are x-sparse to each other in G, and the same in the complement if $ij \in E(J)$. It cannot be that $J = H$ since otherwise there would be $x^{k_1}|G|^{|H|}$ copies of H in G, contrary to the hypothesis; let $h \in V(H) \setminus V(J)$, and let $J' := H[V(J) \cup \{h\}]$. There is no large approximate blowup of J'

in G (even allowing its sets to be a little smaller than the A_j's, and the sparsity between them to be relaxed a little), and we will exploit this. Let W be the set of vertices of G in none of the sets A_j ($j \in V(J)$). Thus W contains almost all vertices of G.

Let us assume, first, that there exists $X_0 \subseteq W$ with $|X_0| \geq |W|/2$, such that for each $j \in V(J)$, if $hj \in E(H)$ then every vertex in X_0 has at least $x|A_j|$ neighbours in A_j, and if $hj \notin E(H)$ then every vertex in X_0 has at least $x|A_j|$ non-neighbours in A_j. Then we can obtain a blockade with the properties we want, as follows. Let $j \in V(J)$, and suppose that h, j are nonadjacent in H (the other case is the same in the complement). Each vertex in X_0 has at least $x|A_j|$ non-neighbours in A_j, and so we can apply 2.2 with X_0, A_j in place of A, B. If the first outcome of 2.2 holds, then since $F = H \setminus \{g\}$ satisfies 2.1, there is an x-restricted blockade in $G[A_j]$ of length at least $\log(1/x)$, and width at least $\lfloor x^{k_2'}|A_j|\rfloor$; and that blockade has the properties we want, if we arrange the constants properly. The second outcome of 2.2 cannot hold, since it would contradict the hypothesis of 2.1. So we assume that the third outcome holds; and we can therefore choose $X_1 \subseteq X_0$ and $C_j \subseteq A_j$ such that $|X_1| \geq \text{poly}(x)|X_0|$, and $|C_j| \geq \text{poly}(x)|A_j|$, such that there are at most $\text{poly}(x)|X_1| \cdot |A_j'|$ edges between X_1, C_j. Note what has happened: we started with every vertex in X_0 just having a few (at least $x|A_j|$) non-neighbours in A_j, and now there are almost no edges between X_1 and C_j. We can assume that $|C_j|$ has size about equal to its lower bound $\text{poly}(x)|A_j|$; and by removing some "outlier" vertices from X_1, we can assume that in addition, X_1 is $\text{poly}(x)$-sparse to C_j. (The advantage of this is that, we will choose successively smaller subsets of X_1, and they will all be $\text{poly}(x)$-sparse to C_j.)

Now choose some other vertex $j' \in V(J)$ different from j, and apply 2.2 to the pair $X_1, A_{j'}$; we may assume this gives us $X_2 \subseteq X_1$ with $|X_2| \geq \text{poly}(x)|X_1|$, and $C_{j'} \subseteq A_{j'}$ with $|C_{j'}| \geq \text{poly}(x)|A_{j'}|$, such that X_2 is $\text{poly}(x)$-sparse to $C_{j'}$ in G if $ij' \notin E(H)$, and the same in the complement if $ij' \in E(H)$. Continue in this way until we have processed each vertex of J. This gives us $X_{|J|} \subseteq W$ with $|X_{|J|}| \geq \text{poly}(x)|W|$, and $C_j \subseteq A_j$ with $|C_j| \geq \text{poly}(x)|A_j|$ for each $j \in V(J)$, such that $X_{|J|}$ is $\text{poly}(x)$-sparse to C_j in $\overline{G}$ or in G (depending whether $hj \in E(H)$ or not) for each $j \in V(J)$. We can assume that each C_j is $\text{poly}(x)$-sparse to $X_{|J|}$ (again, by removing a few outliers). Since the sets A_j ($j \in V(J)$) are $\text{poly}(x)$-sparse to each other in G or $\overline{G}$ (where $\text{poly}(x)$ is some polynomial of large degree), as in the definition of an approximate blowup, and since $|C_j| \geq \text{poly}(x)|A_j|$ for each j, it follows that the sets C_j ($j \in V(J)$) are still $\text{poly}(x)$-sparse to each other in G or $\overline{G}$ (where $\text{poly}(x)$ is now some polynomial of somewhat smaller degree). This gives an approximate blowup of $J' := H[V(J) \cup \{h\}]$, contradicting the choice of J. So this cannot happen; and therefore, at some step, the first outcome of 2.2 held, and so we obtained the blockade we want.

Consequently, we may assume that there is no such X_0; and so, for some $j \in V(J)$, there is a subset $A \subseteq W$ with $|A| \geq |W|/(2|H|)$ that is x-sparse to A_j in G or in $\overline{G}$. Now repeat the proof, working completely within A. After $2\log(1/x)$ iterations of the argument, we will produce the x-restricted blockade that we want; and until that stage, the various subsets we must deal with are still large enough that the argument is valid. This completes the sketch of the proof of 1.3.

3 The proof of 2.2

In this section we prove 2.2, which we restate:

3.1 *Let H be a graph, and let $g \in V(H)$. Let $b, c > 0$, and define $a := b + (1+c)|H|$. Let G be a graph, let A, B be disjoint subsets of $V(G)$, and let $0 < x \le 1/2$, such that A is $(1-x)$-sparse to B. Then either:*

- *there exists $B' \subseteq B$ with $|B'| \ge x|B|$ such that $\text{ind}_{H\setminus\{g\}}(G[B']) < x^b|B'|^{|H|-1}$; or*
- $\text{ind}_H(G) \ge x^a|A| \cdot |B|^{|H|-1}$*; or*
- *there exists $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \ge x^a|A|$ and $|B'| \ge x^a|B|$ such that the number of edges between A', B' is at most $2x^c|A'| \cdot |B'|$.*

Proof. Let g have degree d, let $H_d := H$, and inductively for $t = d-1, \ldots, 0$, let H_t be obtained from H_{t+1} by deleting one of the $t+1$ edges of H_{t+1} incident with g. Let $k := |H|$, and $s := |A| \cdot |B|^{k-1}$.

Let $v \in A$. By hypothesis, the set B' (say) of non-neighbours of v in B has cardinality at least $x|B|$. If $G[B']$ contains fewer than $x^b|B'|^{k-1}$ copies of $H \setminus \{g\} = H_0 \setminus \{g\}$, then the first bullet holds, so we assume not. Consequently there are at least $x^b|B'|^{k-1} \ge x^{k-1+b}|B|^{k-1}$ copies ϕ of H_0 in G such that $\phi(g) = v$ and $\phi(j) \in B$ for each $j \in V(H) \setminus \{g\}$. It follows by summing over all $v \in A$ that there are at least $x^{k-1+b}s$ copies ϕ of H_0 such that $\phi(g) \in A$ and $\phi(j) \in B$ for each $j \in V(H) \setminus \{g\}$.

For $0 \le t \le d$, let τ_t be the number of copies ϕ of H_t in G such that $\phi(g) \in A$ and $\phi(j) \in B$ for each $j \in V(H) \setminus \{g\}$. We have just seen that $\tau_0 \ge x^{k-1+b}s$. We may assume that $\tau_d < sx^a \le x^{k-1+b+cd}s$, because otherwise the second bullet holds. Consequently, for some t with $1 \le t \le d$,

- $\tau_{t-1} \ge x^{k-1+b+c(t-1)}s \ge 2x^a s$, and
- $\tau_t < x^{k-1+b+ct}s$, and therefore $\tau_t < x^c\tau_{t-1}$.

Let Φ be the set of all copies ϕ of H_{t-1} in G such that $\phi(g) \in A$ and $\phi(j) \in B$ for each $j \in V(H)\setminus\{g\}$. There is one edge of H_t that is not an edge of H_{t-1}, say gh. Let J be the graph obtained from H by deleting both g and h, and let Ψ be the set of all copies of J in $G[B]$. Each member of Φ is an extension of some member of Ψ. For each $\psi \in \Psi$, let $n(\psi)$ be the number of $\phi \in \Phi$ that are extensions of ψ. Thus the sum of $n(\psi)$ over all $\psi \in \Psi$ equals τ_{t-1}. Let Ψ' be the set of all $\psi \in \Psi$ such that $n(\psi) \ge \tau_{t-1}/(2|B|^{k-2})$, and $\Psi'' = \Psi \setminus \Psi'$. Thus

$$\sum_{\phi \in \Psi''} n(\psi) \le |B|^{k-2}\left(\frac{\tau_{t-1}}{2|B|^{k-2}}\right) = \tau_{t-1}/2$$

since $|\Psi''| \le |B|^{k-2}$ and $n(\psi) \le \tau_{t-1}/(2|B|^{k-2})$ for each $\phi \in \Psi''$. Since

$$\sum_{\psi \in \Psi''} n(\psi) + \sum_{\psi \in \Psi'} n(\psi) = \tau_{t-1},$$

it follows that

$$\sum_{\psi \in \Psi'} n(\psi) \ge \tau_{t-1}/2.$$

For each $\psi \in \Psi'$, let $U(\psi)$ be the set of all $u \in B$ such that mapping i to u extends ψ to a copy of $H \setminus \{g\}$, and let $V(\psi)$ be the set of all $v \in A$ such that mapping g to v extends ψ to a copy of $H \setminus \{h\}$. Thus $n(\psi)$ is the number of pairs (u, v) with $u \in U(\psi)$ and $v \in V(\psi)$ such that u, v are nonadjacent. Let $p(\psi)$ be the number of edges between $U(\psi)$ and $V(\psi)$. Thus τ_t is at least the sum of $p(\psi)$ over all $\psi \in \Psi'$. Since $\tau_t < x^c\tau_{t-1}$, it follows that

$$\sum_{\psi\in\Psi'} p(\psi) \le \tau_t < x^c\tau_{t-1} \le 2x^c \sum_{\psi\in\Psi'} n(\psi);$$

and consequently there exists $\psi \in \Psi'$ such that $p(\psi) \le 2x^c n(\psi)$.

Since $\psi \in \Psi'$, it follows that

$$|U(\psi)| \cdot |V(\psi)| \ge n(\psi) \ge \frac{\tau_{t-1}}{2|B|^{k-2}} \ge \frac{2sx^a}{2|B|^{k-2}} = x^a|A| \cdot |B|;$$

and since $|U(\psi)| \le |A|$ and $|V(\psi)| \le |B|$, it follows that $|U(\psi)| \ge x^a|A|$ and similarly $|V(\psi)| \ge x^a|B|$. Since

$$p(\psi) \le 2x^c n(\psi) \le 2x^c|U(\psi)| \cdot |V(\psi)|,$$

there are at most $2x^c|U(\psi)| \cdot |V(\psi)|$ edges between $U(\psi)$ and $V(\psi)$. Hence the third bullet holds, setting $A' = U(\psi)$ and $B' = V(\psi)$. This proves 3.1. ∎

4 The proof of 2.1

Now we turn to the proof of 2.1. We will need:

4.1 *Let A, B be disjoint subsets of $V(G)$, such that there are at most $c|A| \cdot |B|$ edges between A and B. Then there exists $A' \subseteq A$ with $|A'| \ge |A|/2$ such that A' is $2c$-sparse to B.*

Proof. There are at most $c|A| \cdot |B|$ edges between A and B, and so at most $|A|/2$ vertices in A have more than $2c|B|$ neighbours in B. This proves 4.1. ∎

Let J be a graph, and $t > 0$ an integer, and $0 \le q \le 1$ a real number. Let G be a graph, and let A_j ($j \in V(J)$) be pairwise disjoint subsets of $V(G)$. We say that the family $(A_j : j \in V(J))$ is a *(t, q)-blowup of J* if

- each set A_j ($j \in V(J)$) has cardinality t;
- for all distinct $i, j \in V(J)$, if $ij \notin E(J)$ then A_i, A_j are q-sparse to each other in G, and if $ij \in E(J)$ then A_i, A_j are q-sparse to each other in $\overline{G}$.

We observe:

4.2 *Let J be a graph, and $t > 0$ an integer. If there is a $(t, 1/|J|)$-blowup of J in G, then* $\text{ind}_J(G) \ge (t/|J|)^{|J|}$.

Proof. Let $(A_j : j \in V(J))$ be a $(t, 1/|J|)$-blowup of J in G. If I is an induced subgraph of J, a copy ϕ of I is *good* if $\phi(i) \in A_i$ for each $i \in I$.

(1) *Let I be an induced subgraph of J, and suppose that ϕ is a good copy of I. Then there are at least $(t/|J|)^{|J|-|I|}$ good copies of J that extend ϕ.*

The proof is by induction on $|V(J)| - |V(I)|$. If this is zero then the claim is true, so we may assume that there exists $j \in V(J) \setminus V(I)$. Let I' be the induced subgraph of J with vertex set $V(I) \cup \{j\}$. For each $i \in V(I)$, let us say that $v \in A_j$ is *i-conforming* if either $ij \in E(J)$ and $\phi(i), v$ are adjacent in G, or $ij \notin E(J)$ and $\phi(i), v$ are nonadjacent in G. From the definition of a (t, q)-blowup, for each $i \in V(I)$ there are at most $t/|J|$ vertices in A_j that are not i-conforming; and so there are at least $t - t|I|/|J| \geq t/|J|$ vertices $v \in A_j$ such that v is i-conforming for each $i \in V(I)$. For each such v, let ϕ' be the extension of ϕ obtained by mapping j to v; then ϕ' is a good copy of I'. From the inductive hypothesis, there are at least $(t/|J|)^{|J|-|I|-1}$ good copies of J that extend ϕ'; and since there are at least $t/|J|$ choices of v and hence of ϕ', the claim follows. This proves (1).

But then the theorem follows from (1) by setting I to be the null graph. This proves 4.2. ∎

The bulk of the proof of 2.1 consists of the following lemma:

4.3 *For all graphs H, all $g \in V(H)$, and all $\alpha > 0$, there exist $\beta, \gamma > 0$ such that for every graph G with $|G| \geq 2$ and all x with $0 < x \leq 1/(8|H|)$, either:*

- *there exists $A \subseteq V(G)$ with $|A| \geq x^\beta |G|$ such that $\text{ind}_{H\setminus\{g\}}(G[A]) < x^\alpha |A|^{|H|-1}$; or*
- *$\text{ind}_H(G) \geq x^\gamma |G|^{|H|}$; or*
- *there are disjoint subsets $A, B \subseteq V(G)$ with $|A| \geq x^\beta |G|$ and $|B| \geq |G|/(2|H|)$, such that B is x-sparse to A in one of $G, \overline{G}$.*

Proof. We may assume that $|H| \geq 2$, because otherwise the theorem holds taking $\gamma = 1$. It suffices to prove the result assuming that $1/x$ is an integer. Indeed, suppose that H, g, α are given, and setting $\beta = \beta'$ and $\gamma = \gamma'$ satisfies the theorem for all G and x with $1/x$ an integer. Then setting $\beta = 2\beta'$ and $\gamma = 2\gamma'$ satisfies the theorem for all G and x. To see this, let $0 < x \leq 1/(8|H|)$, and let $x' = 1/(\lceil 1/x \rceil)$. Then $1/x'$ is an integer, and $1/x' = \lceil 1/x \rceil \leq \frac{8|H|+1}{8|H|x}$, and so

$$x^2 \leq \frac{x}{8|H|} \leq \frac{8|H|x}{8|H|+1} \leq x' \leq x.$$

Consequently $(x')^{\beta'} \geq (x^2)^{\beta'} = x^\beta$ and similarly $(x')^{\gamma'} \geq x^\gamma$ and hence, whichever bullet of the theorem holds for x', β', γ', the same bullet holds for $x, 2\beta', 2\gamma'$. So to prove the theorem, we just need to exhibit values of β, γ that work when $1/x$ is an integer.

By increasing α if necessary, we may assume that α is an integer and $\alpha \geq |H|(|H|+1)$. Define $r_{|H|} = 0$, and inductively for $i = |H|-1, \ldots, 1$ define

$$r_i := \alpha + 2|H| + 1 + (|H|+1)r_{i+1}.$$

Let $\beta := r_1 + 3$ and $\gamma := 2r_1 + \beta|H|$. We claim that β, γ satisfy the theorem (when $1/x$ is an integer).

Thus, let G be a graph with $|G| \geq 2$, and let $x > 0$ such that $0 < x \leq 1/(8|H|)$, where $1/x$ is an integer. If $x^\beta |G| \leq 1$, the third bullet is true taking $|A| = 1$ (unless $|G| - 1 < |G|/|H|$, which is impossible since $|G|, |H| \geq 2$), so we may assume that $x^\beta |G| > 1$. We assume the first two bullets of the theorem are false, and we will show that the third holds.

Let $t := \lfloor x^{\beta-1}|G|\rfloor$; thus $t \geq x^{\beta}|G|$, because $x^{\beta-1}|G| \geq 1$ and $x \leq 1/2$. Let $t_i := x^{-r_i}t$ and $q_i := x^{r_i}/|H|$ for $1 \leq i \leq |H|$. Thus $t_1, \ldots, t_{|H|}$ are integers.

Since $\gamma/|H| \geq \beta + 1$, it follows that

$$t \geq x^{\beta}|G| \geq (1/x)x^{\gamma/|H|}|G| \geq |H|x^{\gamma/|H|}|G|,$$

and consequently $(t/|H|)^{|H|} \geq x^{\gamma}|G|^{|H|}$. Hence by 4.2, there is no $(t, 1/|H|)$-blowup (that is, no $(t_{|H|}, q_{|H|})$-blowup) of H in G. Let J be a maximal induced subgraph of H such that there is a $(t_{|J|}, q_{|J|})$-blowup $(A_j : j \in V(J))$ of J in G, and let $k := |J|$.

Thus $J \neq H$; let $h \in V(H) \setminus V(J)$, and $L := \bigcup_{j \in V(J)} A_j$. For each $j \in V(J)$, let M_j be the set of vertices $v \in V(G) \setminus L$ such that

- if $hj \in E(H)$, then v has at most $x|A_j|$ neighbours in A_j;
- if $hj \notin E(H)$, then v has at most $x|A_j|$ non-neighbours in A_j.

For each $j \in V(J)$, since $t_k \geq t \geq x^{\beta}|G|$, we may assume that $|M_j| < |G|/(2|H|)$, since otherwise the third bullet of the theorem holds. Since

$$|L| = kt_k \leq kx^{\beta-1-r_k}|G| \leq x^2|H| \cdot |G| \leq |G|/(2|H|),$$

it follows that the union of L and the sets M_j $(j \in V(J))$ has cardinality at most $|G|/2$. Let Z be the set of vertices of G that do not belong to L or to any of the sets M_j $(j \in V(J))$. Thus $|Z| \geq |G|/2$; and for each $j \in V(J)$, if $hj \notin E(H)$ then Z is $(1-x)$-sparse to A_j, and if $hj \in E(H)$ then Z is $(1-x)$-sparse to A_j in $\overline{G}$. Let $s := x^{r_k - r_{k+1}}$. Thus $t_{k+1} = t_k$ and $q_{k+1} = q_k/s$.

(1) *Let $j \in V(J)$, and let $Y \subseteq Z$ with $|Y| \geq |Z|s^{k-1}$. Then there exist $C \subseteq A_j$ with $|C| = 2t_{k+1}$, and $X \subseteq Y$ with $|X| \geq s|Y|$, such that X is $\frac{1}{2}q_{k+1}$-sparse to C in G if $hj \notin E(H)$, and X is $\frac{1}{2}q_{k+1}$-sparse to C in $\overline{G}$ if $hj \in E(H)$.*

By taking complements if necessary, we may assume that $hj \notin E(H)$, and so Y is $(1-x)$-sparse to A_j. We will apply 2.2 with b, c, A, B replaced by $\alpha, r_{k+1}+1, Y, A_j$; note that the expression $b + (1+c)|H|$ of 2.2 becomes $\alpha + (r_{k+1}+2)|H| = r_k - r_{k+1} - 1$. By 2.2, we deduce that either:

- there exists $A' \subseteq A_j$ with $|A'| \geq x|A_j|$ such that $\text{ind}_{H \setminus \{g\}} H(G[A']) < x^{\alpha}|A'|^{|H|-1}$; or
- $\text{ind}_H(G) \geq x^{r_k - r_{k+1} - 1}|Y| \cdot |A_j|^{|H|-1}$; or
- there exist $A' \subseteq A_j$ and $D \subseteq Y$ with $|A'| \geq x^{r_k - r_{k+1}-1}|A_j|$ and $|D| \geq x^{r_k - r_{k+1}-1}|Y|$ such that the number of edges between A', D is at most $2x^{r_{k+1}+1}|A'| \cdot |D|$.

If the first bullet above holds, then the first bullet of the theorem holds, since $|A'| \geq x|A_j| = xt_{|J|} = x^{1-r_{|J|}}t \geq t \geq x^{\beta}|G|$ (because $|J| < |H|$ and so $r_{|J|} \geq 1$), a contradiction. If the second holds, then the second bullet of the theorem holds, also a contradiction, since

$$x^{r_k}|Y| \cdot |A_j|^{|H|-1} \geq x^{r_k + r_1 + \beta|H|}|G|^{|H|} \geq x^{2r_1 + \beta|H|}|G|^{|H|} = x^{\gamma}|G|^{|H|}$$

(because

$$|A_j|^{|H|-1} \geq t^{|H|-1} \geq x^{\beta(|H|-1)}|G|^{|H|-1} \geq x^{\beta|H|}|G|^{|H|-1}.$$

and $|Y| \geq |Z|s^{k-1} \geq x^{(k-1)r_k}|G| \geq x^{r_1}|G|$.)

Thus the third bullet above holds. Let A', D be the corresponding subsets. Since

$$|A'| \geq x^{r_k - r_{k+1} - 1} t_k = x^{r_k - r_{k+1} - 1} x^{-r_k} t = t_{k+1}/x \geq 2t_{k+1},$$

it follows by averaging that we may choose $C \subseteq A'$ with $|C| = 2t_{k+1}$ such that the number of edges between C, D is at most $2x^{r_{k+1}+1}|C| \cdot |D|$. By 4.1, there exists $X \subseteq D$ with

$$|X| \geq \frac{|D|}{2} \geq \frac{1}{2} x^{r_k - r_{k+1} - 1}|Y| \geq s|Y|$$

such that X is $4x^{r_{k+1}+1}$-sparse to C, and hence $\frac{1}{2}q_{k+1}$-sparse to C, since $4x^{r_{k+1}+1} \leq \frac{1}{2|H|}x^{r_{k+1}}$ (because $x \leq 1/(8|H|)$). This proves (1).

Starting with $Y = Z$, and applying (1) recursively to each $j \in V(J)$, we obtain a subset $X \subseteq Z$ with $|X| \geq |Z|s^k$, and a subset $C_j \subseteq A_j$ with $|C_j| = 2t_{k+1}$ for each $j \in V(J)$, such that for each $j \in V(J)$, X is $\frac{1}{2}q_{k+1}$-sparse to C_j in G if $hj \notin E(H)$, and X is $\frac{1}{2}q_{k+1}$-sparse to C_j in $\overline{G}$ if $hj \in E(H)$. Since (from the choice of β)

$$|X| \geq s^k|Z| \geq \frac{1}{2}x^{kr_k - kr_{k+1}}|G| \geq x^{\beta - 1 - r_{k+1}}|G| \geq tx^{-r_{k+1}} = t_{k+1}$$

we may choose $D_h \subseteq X$ with $|D_h| = t_{k+1}$. By 4.1, for each $j \in V(J)$ there exists $D_j \subseteq C_j$ with $|D_j| = t_{k+1}$ such that D_j, D_h are q_{k+1}-sparse to each other in G if $hj \notin E(H)$, and D_j, D_h are q_{k+1}-sparse to each other in $\overline{G}$ if $hj \in E(H)$. Let $i, j \in V(J)$ be distinct. We assume that $ij \notin E(H)$ (the other case is the same in the complement). Since $(A_j : j \in V(J))$ is a (t_k, q_k)-blowup of J, it follows that A_i is q_k-sparse to A_j. Since $|D_j| = t_{k+1} = x^{-r_{k+1}}t = s|A_j|$, this implies that A_i (and hence D_i) is (q_k/s)-sparse to D_j, that is, q_{k+1}-sparse to D_j. Consequently $(D_j : j \in V(J'))$ is a (t_{k+1}, q_{k+1})-blowup of J', where J' is the induced subgraph of H with vertex set $V(J) \cup \{h\}$, contrary to the maximality of J. This proves 4.3. ∎

Now we use 4.3 to prove 2.1, which we restate, slightly strengthened:

4.4 *For all H, there exist $k_1, k_2 > 0$ such that for every non-null graph G and every x with $0 < x \leq \frac{1}{8|H|}$, if $\text{ind}_H(G) < x^{k_1}|G|^{|H|}$, there is an x-restricted blockade in G with length at least $2\log(1/x)$ and width at least $\lfloor x^{k_2}|G| \rfloor$. Consequently, for all such G, x, there is a blockade in G with length at least $\log(1/x)$ and width at least $\lfloor x^{k_2}|G| \rfloor$ that is x-sparse in one of $G, \overline{G}$.*

Proof. The first statement is trivially true when $|H| \leq 1$, and we proceed by induction on $|H|$. So we may assume that $|H| \geq 2$, and $g \in V(H)$, and the theorem holds for $H \setminus \{g\}$; let k_1', k_2' be the corresponding constants (using $H \setminus \{g\}$ instead of H). Let $d := \log(2|H|)$. Choose β, γ satisfying 4.3, taking $\alpha = k_1'$. Let $k_1 := \gamma + 2d|H|$ and $k_2 := k_2' + \beta + 2d$. We will show that k_1, k_2 satisfy the theorem.

Thus, let G be a graph and x with $0 < x \leq \frac{1}{8|H|}$ such that $\text{ind}_H(G) < x^{k_1}|G|^{|H|}$. Choose an x-restricted blockade $\mathcal{B} = (B_1, \ldots, B_k)$ in G with k maximum such that $B_1, \ldots, B_{k-1}$ have cardinality at least $x^{k_2}|G|$ and $|B_k| \geq (2|H|)^{1-k}|G|$.

(1) *We may assume that* $|B_k| > x^{2d}|G|$.

We may assume that $k-1 < 2\log(1/x)$ and so

$$|B_k| \geq (2|H|)^{1-k}|G| = 2^{(1-k)d}|G| > 2^{-2d\log(1/x)}|G| = x^{2d}|G|.$$

This proves (1).

If $x^{k_2}|G| < 1$, the result holds trivially (because $\lfloor x^{k_2}|G|\rfloor = 0$ and blockades may contain empty blocks), so we may assume that $|G| \geq x^{-k_2}$. By (1), $|B_k| \geq x^{2d}|G| \geq x^{2d-k_2} > 1$ and so $|B_k| \geq 2$. Let us apply 4.3 to $G[B_k]$, taking $\alpha = k_1'$. We deduce that either:

- there exists $A \subseteq B_k$ with $|A| \geq x^\beta|B_k|$ such that $\text{ind}_{H\setminus\{g\}}(G[A]) < x^\alpha|A|^{|H|-1}$; or
- $\text{ind}_H(G[B_k]) \geq x^\gamma|B_k|^{|H|}$; or
- there are disjoint subsets $A, B \subseteq B_k$ with $|A| \geq x^\beta|B_k|$ and $|B| \geq |B_k|/(2|H|)$, such that B is x-sparse to A in one of $G, \overline{G}$.

The second is impossible, since by (1), $x^\gamma|B_k|^{|H|} \geq x^\gamma x^{2d|H|}|G|^{|H|} = x^{k_1}|G|^{|H|}$. Also, the third is impossible, from the maximality of k, because $x^\beta|B_k| \geq x^\beta x^{2d}|G| \geq x^{k_2}|G|$. Thus the first holds. Let A be the corresponding subset. Since $|A| \geq x^\beta|B_k| \geq x^{\beta+d}|G|$, the inductive hypothesis gives an x-restricted blockade in $G[A]$ with length at least $2\log(1/x)$ and width at least $\lfloor x^{k_2'}|A|\rfloor \geq \lfloor x^{k_2'}x^{\beta+d}|G|\rfloor = \lfloor x^{k_2}|G|\rfloor$. This proves the first statement of the theorem.

For the second statement, let G, x be as before, and let $(B_1, \ldots, B_k)$ be an x-restricted blockade in G with length at least $2\log(1/x)$ and width at least $\lfloor x^{k_2}|G|\rfloor$. Let I be the set of $i \in \{1, \ldots, k\}$ such that $B_{i+1} \cup \cdots \cup B_k$ is x-sparse to B_i in G, and $J = \{1, \ldots, k\} \setminus I$; then for all $i \in J$, $B_{i+1} \cup \cdots \cup B_k$ is x-sparse to B_i in $\overline{G}$. So the blockade $(B_i : i \in I)$ is x-sparse in G, and $(B_i : i \in J)$ is x-sparse in $\overline{G}$, and one of them has length at least $k/2 \geq \log(1/x)$. This proves 4.4. ∎

5 Deriving the main theorem

It remains to show that 2.1 implies 1.10, and that 1.10 implies 1.3, and we do so in this section. 2.1 says that graphs that do not contain many copies of H admit blockades with certain properties, but the length of this blockade is critical. 2.1 gives blockades of length $\log(1/x)$, but one might hope that for some graphs H, we could obtain a version of 2.1 that gave longer blockades; and then there would be corresponding improvements in 1.10 and 1.3. With that in mind, we have written the argument of this section in greater generality than is needed for this paper. (We will use this generality in [6, 7], for instance.) Let us say the *edge-density* of a nonnull graph J is the number of edges of J divided by $\binom{|J|}{2}$.

A function $\ell\colon (0, \frac{1}{2}) \to \mathbb{R}^+$ is *subreciprocal* if it is non-increasing and $1 < \ell(x) \leq 1/x$ for all $x \in (0, \frac{1}{2})$. (To prove 1.10 we will only need the subreciprocal function $\ell(x) := \log(1/x)$.) If ℓ is a subreciprocal function, a graph H is *ℓ-divisive* if there exist $c \in (0, \frac{1}{2})$ and $d > 1$ such that for every $x \in (0, c)$ and every graph G with $\text{ind}_H(G) \leq x^d|G|^{|H|}$, there is a blockade $(B_1, \ldots, B_k)$ in G with length at least $\ell(x)$ and width at least $\lfloor x^d|G|\rfloor$, that is x-sparse in one of $G, \overline{G}$.

Erdős and Hajnal [4] proved that every graph is ℓ-divisive where $\ell(x) = 2$ for $x \in (0, 1/2)$; and 2.1 implies the following:

5.1 *Every graph is ℓ-divisive where $\ell(x) := \log(1/x)$ for $0 < x < 1/2$.*

The next theorem implies our main result 1.10, by defining ℓ as in 5.1. The proof is an adaptation of an argument of Fox and Sudakov [5].

5.2 *Let H be an ℓ-divisive graph for some subreciprocal function ℓ. Then there exists $C > 0$ such that, if $\varepsilon \in (0, \frac{1}{2})$ and*

$$\delta = 2^{-C \log^2(1/\varepsilon)/\log(\ell(\varepsilon))},$$

then for every graph G with $\text{ind}_H(G) \leq (\delta|G|)^{|H|}$, there exists $S \subseteq V(G)$ with $|S| \geq \delta|G|$ such that one of $G[S], \overline{G}[S]$ has edge-density at most ε.

Proof. Let $c \in (0, \frac{1}{2})$ and $d > 1$, as in the definition of ℓ-divisive. Let $z := \ell(c)^{-1/2} \in (0, 1)$, and let $b > 2$ be such that $2^{2-b} = 1 - z$. We will first show that setting $C = 20bd$ satisfies the theorem when $\varepsilon \in (0, c)$, and then at the end of the proof, give a value of C that works in general.

Thus, let $\varepsilon \in (0, c)$, and choose δ such that

$$\log(1/\delta) = \frac{20bd \log^2(1/\varepsilon)}{\log(\ell(\varepsilon))}.$$

Let $x := \frac{1-z}{2}\varepsilon = 2^{1-b}\varepsilon$, and let $p := \ell(x)z$; then $p^2 \geq \ell(x) > 1$ since $\ell(x) \geq \ell(\varepsilon) \geq \ell(c) = z^{-2}$.

Let t be the least integer such that $p^t \geq \varepsilon^{-2}$; then since $\ell(\varepsilon) \leq \min(\ell(x), 1/\varepsilon) \leq \min(p^2, 1/\varepsilon)$, we obtain

$$1 \leq t = \left\lceil \frac{2\log(1/\varepsilon)}{\log p} \right\rceil \leq \left\lceil \frac{4\log(1/\varepsilon)}{\log(\ell(\varepsilon))} \right\rceil \leq \frac{5\log(1/\varepsilon)}{\log(\ell(\varepsilon))}.$$

Let $\eta := \frac{1}{4}x^d$; then $x = 2^{1-b}\varepsilon > \varepsilon^b$ since $\varepsilon < \frac{1}{2}$, and so

$$x^d\eta^t = 4^{-t}x^{d+dt} > \varepsilon^{2t}\varepsilon^{bd(t+1)} > \varepsilon^{4bdt} = 2^{-4bdt\log(1/\varepsilon)} \geq \delta$$

since $t \leq \frac{5\log(1/\varepsilon)}{\log(\ell(\varepsilon))}$.

Now, let $h := |H|$, and let G be such that $\text{ind}_H(G) \leq (\delta|G|)^h$. We will show that there exists S as in the theorem. If $\delta|G| \leq 1$ then we are done, so we may assume $\delta|G| > 1$, and hence $|G| > \delta^{-1} > \eta^{-t}$. For all $\varepsilon_1, \varepsilon_2 \geq \varepsilon$ and every integer s with $0 \leq s \leq t$, let $\beta_s(\varepsilon_1, \varepsilon_2)$ be the maximum $\beta > 0$ such that for every induced subgraph F of G with $|F| \geq \eta^s|G|$, there exists $T \subseteq V(F)$ such that $|T| \geq \beta|F|$ and $F[T]$ has edge-density either at most ε_1 or at least $1 - \varepsilon_2$. Since $\delta \leq x^d\eta^t$, it suffices to show that $\beta_0(\varepsilon, \varepsilon) \geq x^d\eta^t$. We claim the following.

(1) *For every integer s with $1 \leq s \leq t$, and for all $\varepsilon_1, \varepsilon_2 \geq \varepsilon$, we have*

$$\beta_{s-1}(\varepsilon_1, \varepsilon_2) \geq \eta \cdot \min(\beta_s(p\varepsilon_1, \varepsilon_2), \beta_s(\varepsilon_1, p\varepsilon_2)).$$

Put $\gamma_1 := \beta_s(p\varepsilon_1, \varepsilon_2)$ and $\gamma_2 := \beta_s(\varepsilon_1, p\varepsilon_2)$, and let $\gamma = \min(\gamma_1, \gamma_2)$. Let F be an induced subgraph of G with $|F| \geq \eta^{s-1}|G|$; we will prove that there exists $T \subseteq V(F)$ such that $|T| \geq \eta\gamma|F|$ and $F[T]$ has edge-density either at most ε_1 or at least $1 - \varepsilon_2$. Since $|F| \geq \eta^{s-1}|G|$, it follows that $|F| \geq \eta^{s-1-t} \geq \eta^{-1}$, since $|G| \geq \eta^{-t}$. Since

$$\text{ind}_H(F) \leq \text{ind}_H(G) \leq (\delta|G|)^h \leq (\eta^{-(s-1)}\delta|F|)^h = (\eta^{t-(s-1)}x^d|F|)^h \leq (x^d|F|)^h \leq x^d|F|^h,$$

there is a blockade $(B_1, \dots, B_k)$ in F, x-sparse in one of $F, \overline{F}$, of length $k \ge \ell(x)$ and width at least

$$\lfloor x^d |F| \rfloor = \lfloor 4\eta |F| \rfloor \ge 2\eta |F|$$

(since $|F| \ge \eta^{-1}$). By the symmetry, we may assume that $(B_1, \dots, B_k)$ is x-sparse in F. Let $m := \lceil \eta\gamma_1 |F| \rceil$.

Inductively for $i = k, k-1, \dots, 1$, we choose $C_i \subseteq B_i$ with $|C_i| = m$, as follows. Assume $C_k, C_{k-1}, \dots, C_{i+1}$ have been defined, and let D_i be their union. Thus D_i is x-sparse to B_i, and so by 4.1 there exists $B_i' \subseteq B_i$ with $|B_i'| \ge \frac{1}{2}|B_i|$ such that B_i' is $2x$-sparse to D_i; and in particular

$$|B_i'| \ge \frac{1}{2}|B_i| \ge \eta |F| \ge \eta^s |G|.$$

Thus, by the definition of β_s, there exists $C_i \subseteq B_i'$ with $|C_i| \ge \gamma_1 |B_i'| \ge \eta\gamma_1 |F|$ such that $F[C_i]$ has edge-density either at most $p\varepsilon_1$ or at least $1 - \varepsilon_2$. If its edge-density is at least $1 - \varepsilon_2$ then we may set $T = C_i$ and be done; so we may assume that $F[C_i]$ has edge-density at most $p\varepsilon_1$. By averaging, we may assume $|C_i| = \lceil \eta\gamma_1 |F| \rceil = m$. This completes the inductive definition of $C_k, C_{k-1}, \dots, C_1$.

For $1 \le i \le k$, C_i is $2x$-sparse to $D_i = C_k \cup C_{k-1} \cup \dots \cup C_{i+1}$, and so there are at most $2xm^2\binom{k}{2}$ edges between $C_1, \dots, C_k$. Therefore, setting $T := \bigcup_{i=1}^k C_i$, we have

$$|T| = km \ge m \ge \eta\gamma_1 |F| \ge \eta\gamma |F|;$$

and since $k \ge \ell(x) \ge p/z$ and $2x = (1-z)\varepsilon \le (1-z)\varepsilon_1$, the number of edges of $G[T]$ is at most

$$kp\varepsilon_1 \binom{m}{2} + 2xm^2 \binom{k}{2} \le z\varepsilon_1 k^2 \binom{m}{2} + (1-z)\varepsilon_1 m^2 \binom{k}{2} \le \varepsilon_1 \binom{km}{2} = \varepsilon_1 \binom{|T|}{2}.$$

So T is a subset of $V(F)$ with the desired property. This proves (1).

By applying (1) for $s = 1, 2, \dots, t$, we obtain

$$\beta_0(\varepsilon, \varepsilon) \ge \eta^t \cdot \min_{0 \le i \le t} \beta_t(p^i \varepsilon, p^{t-i}\varepsilon).$$

Since $p^t \varepsilon^2 \ge 1$ by the choice of t, and so $\max\left(p^i\varepsilon, p^{t-i}\varepsilon\right) \ge 1$ if $0 \le i \le t$, we deduce that $\beta_t(p^i\varepsilon, p^{t-i}\varepsilon) = 1$ for all i with $0 \le i \le t$; and hence $\beta_0(\varepsilon, \varepsilon) \ge \eta^t \ge \delta$, as claimed. Thus, setting $C = 20bd$ satisfies the theorem for all $\varepsilon \in (0, c)$.

Let $a > 1$ be such that $2^{-a} = c$; we will prove that setting $C = 20a^2bd$ works for all $\varepsilon \in (0, \frac{1}{2})$. Thus, let $\varepsilon \in (0, \frac{1}{2})$, let

$$\log(1/\delta) = \frac{20a^2bd \log^2(1/\varepsilon)}{\log(\ell(\varepsilon))},$$

and let G be a graph with $\text{ind}_H(G) \le (\delta|G|)^{|H|}$. Let $\varepsilon' := \varepsilon^a$, and let

$$\log(1/\delta') = \frac{20bd \log^2(1/\varepsilon')}{\log(\ell(\varepsilon'))};$$

then $\varepsilon' \le \varepsilon$, and so $\delta' \ge \delta$ since ℓ is non-increasing. Consequently $\text{ind}_H(G) \le (\delta'|G|)^{|H|}$. But $\varepsilon' < 2^{-a} = c$, so by what we already proved, there exists $S \subseteq V(G)$ with $|S| \ge \delta'|G| \ge \delta|G|$ such that one of $G[S], \overline{G}[S]$ has edge-density at most $\varepsilon' \le \varepsilon$. This proves 5.2. ∎

Finally, let us deduce 1.3, which we restate:

5.3 *For every graph H there exists $c > 0$ such that $\mu(G) \geq 2^{c\sqrt{\log |G| \log\log |G|}}$ for every H-free graph G with $|G| \geq 2$.*

Proof. Let c be as in 1.10, and let $d = 1/(8c)^{1/2}$. Choose n_0 such that for all $n \geq n_0$,

$$\frac{1}{4}\log\log n + \frac{1}{2}\log\log\log n \geq \log(1/d), \text{ and}$$
$$n^{1/2} \geq 2^{d\sqrt{\log n \log\log n}} \geq 4.$$

Choose $c' \leq d/2$ with $c' > 0$, such that $n_0^{c'} \leq 2$. We will show that $\mu(G) \geq 2^{c'\sqrt{\log |G| \log\log |G|}}$ for every H-free graph G with $|G| \geq 2$.

Thus, let G be H-free. If $|G| \leq n_0$, then $|G|^{c'} \leq n_0^{c'} \leq 2$, and so $\mu(G) \geq |G|^{c'}$. Hence we may assume that $|G| > n_0$. Let $\varepsilon = 2^{-d\sqrt{\log |G| \log\log |G|}}$, and let $\delta = 2^{-c(\log \frac{1}{\varepsilon})^2/\log\log\frac{1}{\varepsilon}}$; then

$$\log\delta = -\frac{c(\log\frac{1}{\varepsilon})^2}{\log\log\frac{1}{\varepsilon}} = -\frac{cd^2 \log|G| \log\log|G|}{\log\log\frac{1}{\varepsilon}}.$$

Since

$$\log\log\frac{1}{\varepsilon} = \frac{1}{2}\log\log|G| + \frac{1}{2}\log\log\log|G| - \log(1/d) \geq \frac{1}{4}\log\log|G|$$

(because $\frac{1}{4}\log\log|G| + \frac{1}{2}\log\log\log|G| \geq \log(1/d)$), it follows that

$$\log\delta \geq -\frac{cd^2\log|G|\log\log|G|}{\frac{1}{4}\log\log|G|} = -4cd^2\log|G| = -\frac{1}{2}\log|G|,$$

and so $\delta \geq |G|^{-1/2}$. By 1.10 and the choice of c, there exists $S \subseteq V(G)$ with $|S| \geq |G|^{1/2}$ such that one of $G[S], \overline{G}[S]$ has edge-density at most ε. Since $|G| \geq n_0$ it follows that $|S| > 1/\varepsilon$. By Turán's theorem, $G[S]$ has a clique or stable set of size at least

$$\frac{|S|}{1+\varepsilon|S|} \geq \frac{1}{2\varepsilon} = \frac{1}{2}2^{d\sqrt{\log|G|\log\log|G|}} \geq 2^{(d/2)\sqrt{\log|G|\log\log|G|}} \geq 2^{c'\sqrt{\log|G|\log\log|G|}}.$$

This proves 1.3. ∎

6 Ordered graphs

An influential paper of Alon, Pach and Solymosi [1] showed that the Erdős-Hajnal conjecture admits equivalent formulations for ordered graphs and for tournaments. In this section we observe that our result 1.3 also extends to ordered graphs and tournaments. An *ordered graph* is a pair $(G, <)$, where G is a graph and $<$ is a linear order of its vertex set. If $(G, <)$ and $(H, <')$ are ordered graphs, we say $(G, <)$ is *$(H, <')$-free* if no induced subgraph of G (made into an ordered graph with the order inherited from $<$ in the natural way) is isomorphic to $(H, <')$. Alon, Pach and Solymosi [1] showed that the Erdős-Hajnal conjecture 1.1 is equivalent to the following analogous conjecture for ordered graphs:

6.1 Conjecture: *For every ordered graph $(H, <')$ there exists $\tau > 0$ such that $\kappa(G) \geq |G|^{\tau}$ for every $(H, <')$-free ordered graph $(G, <)$.*

Our theorem 1.3 translates to:

6.2 *For every ordered graph $(H, <')$ there exists $c > 0$ such that $\kappa(G) \geq 2^{c\sqrt{\log |G| \log\log |G|}}$ for every $(H, <')$-free ordered graph $(G, <)$ with $|G| \geq 2$.*

To prove this, we use a theorem of Rödl and Winkler [11], that says:

6.3 *For every ordered graph $(H, <')$, there exists a graph P such that, for every linear ordering of $V(P)$, the resulting ordered graph is not $(H, <')$-free.*

Proof of 6.2. Let $(H, <')$ be an ordered graph, and choose P as in 6.3. Choose c satisfying 1.3 with H replaced by P. Now let $(G, <)$ be an $(H, <')$-free ordered graph. It follows from the property of P that G is P-free, and so by 1.3, $\kappa(G) \geq 2^{c\sqrt{\log |G| \log\log |G|}}$. This proves 6.2. ∎

These results also have analogues for tournaments. If G is a tournament, define $\kappa(G)$ to be the size of the largest transitive subset of $V(G)$. Our result becomes:

6.4 *For every tournament H there exists $c > 0$ such that $\kappa(G) \geq 2^{c\sqrt{\log |G| \log\log |G|}}$ for every H-free tournament G with $|G| \geq 2$.*

Proof. Fix a linear order $<'$ of $V(H)$, and let H' be the graph with vertex set $V(H)$, in which uv is an edge if u is earlier than v in the linear order $<'$ and v is adjacent from u in H. Thus $(H', <')$ is an ordered graph. Choose c as in 6.2 (with H replaced by H'). Now let G be an H-free tournament. Derive an ordered graph $(G', <)$ from G similarly. Since G is H-free, we deduce that $(G', <)$ is $(H', <')$-free, and the result follows from 6.2, since every clique or stable set of G' is a transitive set of G. This proves 6.4. ∎

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