Induced subgraphs of graphs with large chromatic number. XIII. New brooms

Alex Scott[1]
Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour[2]
Princeton University, Princeton, NJ 08544, USA

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Abstract

Gyárfás [3] and Sumner [10] independently conjectured that for every tree T, the class of graphs not containing T as an induced subgraph is χ-bounded, that is, the chromatic numbers of graphs in this class are bounded above by a function of their clique numbers. This remains open for general trees T, but has been proved for some particular trees. For $k \geq 1$, let us say a *broom* of *length* k is a tree obtained from a k-edge path with ends a, b by adding some number of leaves adjacent to b, and we call a its *handle*. A tree obtained from brooms of lengths $k_1, \dots, k_n$ by identifying their handles is a $(k_1, \dots, k_n)$-*multibroom*. Kierstead and Penrice [5] proved that every $(1, \dots, 1)$-multibroom T satisfies the Gyárfás-Sumner conjecture, and Kierstead and Zhu [7] proved the same for $(2, \dots, 2)$-multibrooms.

In this paper give a common generalization; we prove that every $(1, \dots, 1, 2, \dots, 2)$-multibroom satisfies the Gyárfás-Sumner conjecture .

1 Introduction

For a graph G, let $\chi(G)$ denote the chromatic number of G, and let $\omega(G)$ denote its clique number, that is, the number of vertices in its largest clique. We say a graph G *contains* H if some induced subgraph of G is isomorphic to H, and otherwise G is H*-free*.

The Gyárfás-Sumner conjecture [3, 10] asserts that:

1.1 Conjecture: *For every forest T and every integer κ, there exists c such that $\chi(G) \le c$ for every T-free graph G with $\omega(G) \le \kappa$.*

There has been surprisingly little progress on this conjecture. It is easy to see that if the conjecture holds for every component of a forest then it holds for the forest (the first component must be present; delete it and all vertices with a neighbour in it and repeat with the next component), and so it suffices to prove the conjecture when T is a tree. Gyárfás [3] proved the conjecture when T is a path, and Scott [9] proved it when T is a subdivision of a star; and recently, with Maria Chudnovsky, we [1] proved it for trees obtained from a subdivided star by adding one more vertex with one neighbour, and for trees obtained from a star and a subdivided star by adding a path between their centres. But the results that concern us most here are theorems of Gyárfás, Szemerédi and Tuza [4], Kierstead and Penrice [5], and Kierstead and Zhu [7], which are the only other results so far on the Gyárfás-Sumner conjecture, and which we explain next.

For $k \ge 1$, let us say a *broom* of *length* k is a tree obtained from a k-edge path with ends a, b by adding some number of leaves adjacent to b, and we call a its *handle*. A tree obtained from n brooms of lengths $k_1, \dots, k_n$ respectively by identifying their handles is called a $(k_1, \dots, k_n)$-*multibroom*. Gyárfás, Szemerédi and Tuza (in the triangle-free case) and then Kierstead and Penrice (in the general case) proved that $(1, \dots, 1)$-multibrooms satisfy the Gyárfás-Sumner conjecture, and Kierstead and Zhu proved that $(2, \dots, 2)$-multibrooms satisfy it. In this paper we prove a common generalization of these results: every $(1, \dots, 1, 2, \dots, 2)$-multibroom satisfies the Gyárfás-Sumner conjecture.

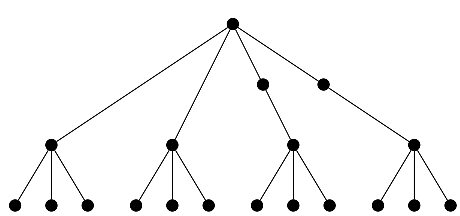

Figure 1: A $(1, 1, 2, 2)$-multibroom

Let us state this more precisely. A (k, δ)*-broom* means a broom of length k with δ leaves different from its handle (thus, it is obtained by adding δ leaves adjacent to one end of a k-edge path). For $\delta \ge 1$, let $T(\delta)$ be the tree formed from the disjoint union of δ $(1, \delta)$-brooms and δ $(2, \delta)$-brooms by identifying their handles. We will prove that

1.2 *For all $\delta \ge 0$ and all $\kappa \ge 0$ there exists c such that every $T(\delta)$-free graph with $\omega(G) \le \kappa$ has chromatic number at most c.*

The proof method is by combining ideas of [5, 7] with some new twists.

2 Inductions

There are various inductions that will give us some assistance. We can use induction on κ, and on δ (in fact with a little work we can more-or-less assume that the result holds for every tree obtained from $T(\delta)$ by deleting a leaf), and there is a third induction, core maximization, that we explain later. Next we explain these inductions in more detail.

First and easiest, by induction on κ, we may assume that there exists τ such that $\chi(G) \leq \tau$ for every $T(\delta)$-free graph with clique number less than κ. In particular, this tells us that if G is $T(\delta)$-free with clique number at most κ, then for every vertex, the subgraph induced on its neighbours has chromatic number at most τ (since this subgraph has clique number less than κ). Consequently we can use 2.1 below, taking $T = T(\delta)$.

If v is a vertex of a graph G, and $k \geq 1$, $N^k(v)$ or $N^k_G(v)$ denotes the set of vertices of G with distance exactly k from v, and $N^k[v]$ denotes the set with distance at most k from v. If G is a nonnull graph and $k \geq 1$, we define $\chi^k(G)$ to be the maximum of $\chi(N^k[v])$ taken over all vertices v of G. (For the null graph G we define $\chi^k(G) = 0$.)

The following can be obtained through by repeated application of theorem 3.2 of [1] (a similar theorem for $(2,\dots,2)$-multibrooms is proved in [7]):

2.1 *Let T be a tree formed by identifying the handles of some set of brooms (of arbitrary lengths). For all $\kappa, \tau \geq 0$ there exists c with the following property. Let G be a T-free graph, with $\omega(G) \leq \kappa$, such that for every vertex, the subgraph induced on its neighbours has chromatic number at most τ. Then $\chi^2(G) \leq c$.*

Next, let us explore induction on the size of $T(\delta)$. That will allow us to exploit "matching-covered" sets. Let $X \subseteq V(G)$. We say that X is *matching-covered* in G if for each $x \in X$ there exists $y \in V(G) \setminus X$ adjacent to x and to no other vertex in X.

We would like to be able to assume that the result holds for all trees obtained from $T(\delta)$ by deleting a leaf; but only deleting one leaf, from one of its brooms, and so the smaller tree is not equal to $T(\delta')$ for $\delta' < \delta$, and so induction on δ is not fine enough. We could change the statement of the theorem, and prove it not only for $T(\delta)$, but for any tree that is a subtree of $T(\delta)$; but that would make things notationally more complicated later. There is another way to do it that is more convenient.

Let us say that G is *(δ, κ)-good* if G is $T(\delta)$-free and $\omega(G) \leq \kappa$. An *ideal* of graphs is a class $\mathcal{C}$ of graphs such that every induced subgraph of a member of $\mathcal{C}$ also belongs to $\mathcal{C}$. If $X \subseteq V(G)$, we write $\chi(X)$ for $\chi(G[X])$ when there is no ambiguity. Let us say the *chromatic number* of a nonnull ideal of graphs is the maximum of the chromatic number of its members, if this exists, and otherwise we say the ideal has *unbounded chromatic number.*

2.2 *Let $\delta, \kappa \geq 1$. If $\mathcal{C}$ is an ideal of (δ, κ)-good graphs with unbounded chromatic number, then there exist a subideal $\mathcal{C}'$ of $\mathcal{C}$ with unbounded chromatic number, and a number c such that every matching-covered set in every member of $\mathcal{C}'$ has chromatic number at most c.*

Proof. Let R be a maximal subtree of $T(\delta)$ such that there exists c such that every R-free graph in $\mathcal{C}$ with clique number at most κ has chromatic number at most c, and choose some such number c. By hypothesis there are members of $\mathcal{C}$ with arbitrarily large chromatic number, so $R \neq T(\delta)$. Hence there is a subtree S of $T(\delta)$ with a leaf v such that $S \setminus v = R$. Let u be the neighbour of v in S. Let

$\mathcal{C}'$ be the subideal of all S-free graphs in $\mathcal{C}$. From the maximality of R, there are graphs in $\mathcal{C}'$ with arbitrarily large chromatic number.

Let $G \in \mathcal{C}'$, and let X be matching-covered in G. Suppose that there is an induced subgraph of $G[X]$ isomorphic to R, and to simplify notation we assume it equals R. Choose $y \in V(G) \setminus X$ adjacent to u and to no other vertex in X; then $G[V(R) \cup \{y\}]$ is isomorphic to S, a contradiction. Thus $G[X]$ does not contain R. Since $G[X] \in \mathcal{C}'$, the choice of c implies that $\chi(X) \leq c$. This proves 2.2. ∎

There is a third, very helpful, induction we can use, but it is more complicated. For integers $a, b \geq 1$, let us say an *(a,b)-core* in a graph G is a subset $Y \subseteq V(G)$ of cardinality ab, that admits a partition $\{A_1, \ldots, A_b\}$ such that

- $A_1, \ldots, A_b$ each have cardinality a;
- $A_1, \ldots, A_b$ are all stable sets of G; and
- for $1 \leq i < j \leq b$, every vertex in A_i is adjacent to every vertex in A_j.

(An (a,b)-core is therefore a complete multipartite induced subgraph of specified size.) This partition is unique, since $a \geq 1$, and we speak of $A_1, \ldots, A_b$ as the *parts* of Y. Thus, if there is an (a,b)-core in G then $b \leq \omega(G)$. Let $\mathbb{N}$ denote the set of nonnegative integers.

2.3 *Let $\delta, \kappa \geq 1$. If $\mathcal{C}$ is an ideal of (δ,κ)-good graphs with unbounded chromatic number, then there exist a subideal $\mathcal{C}'$ of $\mathcal{C}$ with unbounded chromatic number, and integers $\alpha \geq 1$ and $\beta \geq 2$, and a non-decreasing function $\theta : \mathbb{N} \to \mathbb{N}$, with the following properties:*

- *for all $\zeta \geq 1$, every graph in $\mathcal{C}$ with chromatic number more than $\theta(\zeta)$ admits a (ζ,β)-core; and*
- *no graph in $\mathcal{C}'$ admits an $(\alpha, \beta+1)$-core.*

Proof. For integers $\zeta \geq 1$ and $\beta \geq 2$, let us say (ζ,β) is *unavoidable* if there exists c such that every graph in $\mathcal{C}$ with chromatic number more than c admits a (ζ,β)-core. V. Rödl (see [6]) proved that for all integers $\zeta \geq 1$, $(\zeta, 2)$ is unavoidable. On the other hand, by hypothesis there are graphs in $\mathcal{C}$ with arbitrarily large chromatic number, and they do not admit $(1, \kappa+1)$-cores (because they have clique number at most κ), and so $(1, \kappa+1)$ is not unavoidable. Choose β with $2 \leq \beta \leq \kappa+1$ maximum such that for all $\zeta \geq 1$, (ζ,β) is unavoidable, and it follows that $\beta \leq \kappa$. Since (ζ,β) is unavoidable for all $\zeta \geq 1$, there is a function $\theta : \mathbb{N} \to \mathbb{N}$ such that for all $\zeta \geq 1$, every graph in $\mathcal{C}$ with chromatic number more than $\theta(\zeta)$ admits a (ζ,β)-core, and we can choose θ to be non-decreasing, so the first bullet holds.

From the maximality of β, there exists $\alpha \geq 1$ such that there are graphs in $\mathcal{C}$ with arbitrarily large chromatic number that do not admit an $(\alpha, \beta+1)$-core. Let $\mathcal{C}'$ be the ideal of graphs in $\mathcal{C}$ that do not admit an $(\alpha, \beta+1)$-core; then the second bullet holds. This proves 2.3. ∎

We combine these results in the following.

2.4 *Let $\delta \geq 1$. Suppose that for some value of $\kappa \geq 1$ there are (δ,κ)-good graphs with arbitrarily large chromatic number. Then there exist $\tau \geq 0$, $\alpha \geq 1$, $\beta, \kappa \geq 2$, a non-decreasing function $\theta : \mathbb{N} \to \mathbb{N}$, and an ideal $\mathcal{C}$ of graphs with unbounded chromatic number, such that for every $G \in \mathcal{C}$:*

- *G is $T(\delta)$-free;*
- *$\omega(G) \leq \kappa$;*
- *$\chi^2(G) \leq \tau$;*
- *every matching-covered set in G has chromatic number at most τ;*
- *for all $\zeta \geq 1$, and every induced subgraph G' of G, if $\chi(G') > \theta(\zeta)$ then G' admits a (ζ, β)-core;*
- *G does not admit an $(\alpha, \beta+1)$-core.*

Proof. Choose κ minimum such that there are (δ, κ)-good graphs with arbitrarily large chromatic number. Thus $\kappa \geq 2$. Choose τ_1 such that every $(\delta, \kappa-1)$-good graph has chromatic number at most τ_1. By 2.1 there exists τ_2 such that $\chi^2(G) \leq \tau_2$ for every (δ, κ)-good graph. By 2.2 there exist τ_3 and an ideal $\mathcal{C}_1$ of (δ, κ)-good graphs with unbounded chromatic number, such that every matching-covered set in G has chromatic number at most τ_3. By 2.3, there exist a subideal $\mathcal{C}$ of $\mathcal{C}_1$ with unbounded chromatic number, and α, β satisfying the last two bullets. Let $\tau = \max(\tau_1, \tau_2, \tau_3)$; then all six bullets are satisfied. This proves 2.4. ∎

In view of 2.4, in order to prove 1.2 it suffices to show the following:

2.5 *For all $\tau \geq 0$, and $\alpha, \delta \geq 1$, and $\beta \geq 2$, and for every non-decreasing function $\theta : \mathbb{N} \to \mathbb{N}$, there exists c such that if G satisfies*

(i) *G is $T(\delta)$-free;*

(ii) *$\chi^2(G) \leq \tau$;*

(iii) *every matching-covered set in G has chromatic number at most τ;*

(iv) *for all $\zeta \geq 1$, and every induced subgraph G' of G, if $\chi(G') > \theta(\zeta)$ then G' admits a (ζ, β)-core;*

(v) *G does not admit an $(\alpha, \beta+1)$-core*

then $\chi(G) \leq c$.

We could have added another constant κ and another condition that $\omega(G) \leq \kappa$, but it turns out not to be needed any more (a bound on $\omega(G)$ is implied by the second condition).

The five statements (i)–(v) of 2.5 are important for the rest of the paper, and we refer to them simply as (i)–(v). Henceforth, we fix $\tau \geq 0$, and $\alpha, \delta \geq 1$, and $\beta \geq 2$, and some non-decreasing function $\theta : \mathbb{N} \to \mathbb{N}$, for the remainder of the paper, and shall investigate the properties of a graph satisfying (i)–(v).

Let Y be a (ζ, β)-core in G. A vertex $v \in V(G) \setminus Y$ is *dense* to Y if v has at least α neighbours in each part of Y. We observe:

2.6 *Let G satisfy* (i)–(v)*, and let Y be a (ζ, β)-core in G. Then there are at most $\alpha\tau 2^{\beta\zeta}$ vertices in G that are dense to Y.*

Proof. Let $A_1, \dots, A_\beta$ be the parts of Y, and let $X_i \subseteq A_i$ with $|X_i| = \alpha$, for $1 \le i \le \beta$. The set N of vertices adjacent to all vertices in $X_1 \cup \dots \cup X_\beta$ has chromatic number at most τ by (ii), and includes no stable set of cardinality α, since G does not admit an $(\alpha, \beta+1)$-core by (v). Consequently $|N| \le \alpha\tau$. Since there are only at most $2^{\beta\zeta}$ choices for $X_1, \dots, X_\beta$, and every vertex that is dense to Y belongs to the set N corresponding to some choice of $X_1, \dots, X_\beta$, it follows that there are at most $\alpha\tau 2^{\beta\zeta}$ vertices that are dense to Y. This proves 2.6. ∎

3 Templates

We will use an extension of the template method of Kierstead-Penrice and Kierstead-Zhu, which was used in different (and not easily compatible) ways in those papers. Let $\eta \ge 1$ and $\zeta \ge \max(\eta, \alpha)$ be integers, and let G satisfy (i)–(v). If Y is a (ζ, β)-core in G, we say that a vertex $v \in V(G)$ is *η-mixed* on Y if

- v is not dense to Y; and
- v has at least η neighbours in some part of Y.

Thus every vertex in Y is η-mixed on Y. A (ζ, η)-*template* in G is a pair (Y, H), where Y is a (ζ, β)-core in G, and H is a set of vertices of G with $Y \subseteq H$ such that every vertex in H is η-mixed on Y. (Note that there may be vertices in $V(G) \setminus H$ that are η-mixed on Y.)

A (ζ, η)-*template sequence* in G is a sequence (Y_i, H_i) $(1 \le i \le n)$ of (ζ, η)-templates, such that

- for $1 \le i < j \le n$, $H_i \cap H_j = \emptyset$;
- for $1 \le i < j \le n$, there is no edge between H_i and Y_j; and
- for $1 \le i < j \le n$, no vertex in H_j is η-mixed on Y_i.

Later, we will denote $H_i \setminus Y_i$ by Z_i (see figure 2).

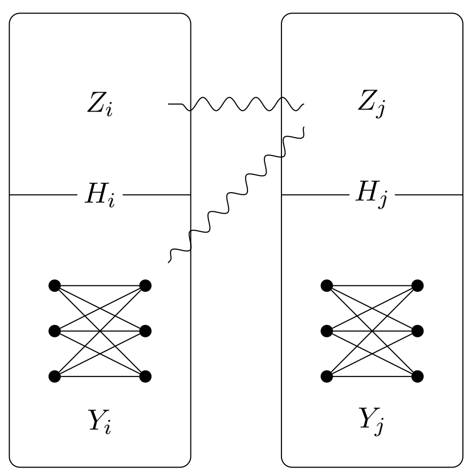

Figure 2: Two terms of a (ζ, η)-template sequence, with $j > i$. Wiggles indicate possible edges.

A (ζ,η)-*template array* $\mathcal{T}$ in G consists of a (ζ,η)-template sequence (Y_i, H_i) $(1 \le i \le n)$ together with a set $U(\mathcal{T}) \subseteq V(G)$, such that for every vertex $v \in U(\mathcal{T})$,

- v is not η-mixed on Y_i (and consequently $v \notin H_i$) for $1 \le i \le n$; and
- v has a neighbour in $H_1 \cup \cdots \cup H_n$.

We call (Y_i, H_i) $(1 \le i \le n)$ the *sequence* of $\mathcal{T}$, and define $H(\mathcal{T}) = H_1 \cup \cdots \cup H_n$ and $V(\mathcal{T}) = H(\mathcal{T}) \cup U(\mathcal{T})$.

3.1 *Let $\eta \ge 1$ and $\zeta \ge \max(\eta, \alpha)$ be integers, and let G satisfy* (i)–(v)*. Then there is a (ζ,η)-template array $\mathcal{T}$ in G such that $V(G) \setminus V(\mathcal{T})$ has chromatic number at most $\theta(\zeta)$.*

Proof. Let (Y_i, H_i) $(1 \le i \le n)$ be a (ζ,η)-template sequence with the property that for $1 \le i \le n$, H_i is the set of all vertices in G that are η-mixed on Y_i, and subject to this, with n maximal. Let $H = H_1 \cup \cdots \cup H_n$, and let U be the set of vertices in $V(G) \setminus H$ with a neighbour in H. Let $\mathcal{T}$ be the (ζ,η)-template array consisting of (Y_i, H_i) $(1 \le i \le n)$ together with $U(\mathcal{T}) = U$. Let $W = V(G) \setminus V(\mathcal{T})$, and suppose that there is a (ζ,β)-core $Y_{n+1} \subseteq W$. Let H_{n+1} be the set of vertices in G that are η-mixed on Y_{n+1}. Then (Y_{n+1}, H_{n+1}) is a (ζ,η)-template. Moreover, no vertex in H_{n+1} belongs to H, since no vertex in H has a neighbour in Y_{n+1}, and every vertex in H_{n+1} has a neighbour in Y_{n+1}. Consequently (Y_i, H_i) $(1 \le i \le n+1)$ is a (ζ,η)-template sequence, contrary to the maximality of n. Thus there is no (ζ,β)-core in W. From (iv), $\chi(W) \le \theta(\zeta)$. This proves 3.1. ∎

By setting $\phi(x) = x + \theta(\zeta)$ for $x \ge 0$, we deduce from 3.1 that:

3.2 *Let $\eta \ge 1$ and $\zeta \ge \max(\eta, \alpha)$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \to \mathbb{N}$ with the following property. For all $c \ge 0$, if G satisfies* (i)–(v) *and $\chi(G) > \phi(c)$ then there is a (ζ,η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > c$.*

We will use the following elementary fact many times in the remainder of the paper, and we leave its proof to the reader:

3.3 *Let D be a digraph with maximum outdegree at most d. Then the graph underlying D has chromatic number at most $2d+1$, and at most $d+1$ if D is acyclic (that is, has no directed cycle).*

A (ζ,η)-template array $\mathcal{T}$ with sequence (Y_i, H_i) $(1 \le i \le n)$ is *partially 1-cleaned* if

- for all distinct $i, j \in \{1, \dots, n\}$, no vertex of H_j is dense to Y_i; and
- for all distinct $i, j \in \{1, \dots, n\}$ and all $v \in U(\mathcal{T})$, if v is dense to Y_i then v has no neighbours in H_j;

and *1-cleaned* if

- for $1 \le i \le n$, no vertex in $V(\mathcal{T})$ is dense to Y_i.

3.4 *Let $\eta \ge 1$ and $\zeta \ge \max(\eta, \alpha)$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \to \mathbb{N}$ with the following property. For all $c \ge 0$, if G satisfies* (i)–(v) *and $\chi(G) > \phi(c)$ then there is a partially 1-cleaned (ζ,η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > c$.*

Proof. Let ψ satisfy 3.2 (with ϕ replaced by ψ). Let $t = \alpha\tau 2^{\beta\zeta}$, and define $\phi(x) = \psi((2t+1)x)$ for all $x \in \mathbb{N}$; we claim that ϕ satisfies 3.4. Let $c \geq 0$, and let G satisfy (i)–(v), with $\chi(G) > \phi(c)$. By 3.2, there is a (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > (2t+1)c$. Let its sequence be (Y_i, H_i) $(1 \leq i \leq n)$. Choose a partition U_i $(1 \leq i \leq n)$ of $U(\mathcal{T})$, such that

- for $1 \leq i \leq n$, every vertex of U_i has a neighbour in H_i; and
- for all distinct $i, j \in \{1, \ldots, n\}$, if $v \in U_i$ is dense to Y_i and v has a neighbour in H_j then v is dense to Y_j.

(This can be arranged by assigning each vertex $v \in U(\mathcal{T})$ to some set U_i where v has a neighbour in H_i and is not dense to Y_i if possible, and otherwise assigning v to some set U_i where v has a neighbour in H_i.)

Let D be the digraph with vertex set $\{1, \ldots, n\}$ in which for $1 \leq i, j \leq n$ with $i \neq j$, if some vertex in $H_j \cup U_j$ is dense to Y_i then j is adjacent from i. By 2.6, D has maximum outdegree at most t, and so by 3.3 the graph underlying D is $(2t+1)$-colourable. Consequently there is a partition $I_1, \ldots, I_{2t+1}$ of $\{1, \ldots, n\}$ such that for $1 \leq s \leq 2t+1$ and all distinct $i, j \in I_s$, no vertex of $H_j \cup U_j$ is dense to Y_i. Hence the subsequence of (Y_i, H_i) $(1 \leq i \leq n)$ consisting of the terms with $i \in I_s$, together with the set $\bigcup_{i \in I_s} U_i$, is a partially 1-cleaned (ζ, η)-template array, $\mathcal{T}_s$ say. But every vertex of $V(\mathcal{T})$ belongs to $V(\mathcal{T}_s)$ for some s; and so there exists $s \in \{1, \ldots, 2t+1\}$ such that

$$\chi(V(\mathcal{T}_s)) \geq \chi(V(\mathcal{T}))/(2t+1) > c.$$

This proves 3.4. ∎

3.5 *Let $\eta \geq 1$ and $\zeta \geq \max(\eta, \alpha)$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \to \mathbb{N}$ with the following property. For all $c \geq 0$, if G satisfies* (i)–(v) *and $\chi(G) > \phi(c)$ then there is a 1-cleaned (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > c$.*

Proof. Let ψ satisfy 3.4 (with ϕ replaced by ψ). Let $t = \alpha\tau 2^{\beta\zeta}$, and define $\phi(x) = \psi(x + t\tau)$ for all $x \in \mathbb{N}$; we claim that ϕ satisfies 3.5. Let $c \geq 0$, and let G satisfy (i)–(v), with $\chi(G) > \phi(c)$. By 3.4, there is a partially 1-cleaned (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > c + t\tau$. Let its sequence be (Y_i, H_i) $(1 \leq i \leq n)$; and for $1 \leq i \leq n$ let X_i be the set of all vertices in $V(\mathcal{T})$ that are dense to Y_i. Hence $X_i \subseteq U(\mathcal{T})$, and for all distinct $i, j \in \{1, \ldots, n\}$, no vertex of X_i has a neighbour in H_j. Let $X = X_1 \cup \cdots \cup X_n$. By 2.6 $|X_i| \leq t$ for each i. Hence we may partition X into t sets $W_1, \ldots, W_t$ such that $|X_i \cap W_j| \leq 1$ for all i, j with $1 \leq i \leq n$ and $1 \leq j \leq t$. For $v \in W_j$, let $v \in X_i$; then v has a neighbour in Y_i, and has no neighbours in $Y_{i'}$ for $i' \neq i$; and so each set W_j is matching-covered in G. By (iii), $\chi(W_j) \leq \tau$ for each j, and so $\chi(W_1 \cup \cdots \cup W_t) \leq t\tau$. This proves that $\chi(X) \leq t\tau$. Let $\mathcal{T}'$ be the (ζ, η)-template array in G with the same sequence as $\mathcal{T}$ and with $U(\mathcal{T}') = U(\mathcal{T} \setminus X)$; then $\mathcal{T}'$ is 1-cleaned, and

$$\chi(V(\mathcal{T}')) \geq \chi(V(\mathcal{T})) - t\tau > c.$$

This proves 3.5. ∎

4 Edges between templates

4.1 *Let $\eta \geq 1$ and $\zeta \geq \max(\eta+\delta, \alpha)$ be integers, and let G satisfy* (i)–(v). *Let $\mathcal{T}$ be a 1-cleaned (ζ, η)-template array in G, with sequence (Y_i, H_i) $(1 \leq i \leq n)$. For each $v \in V(\mathcal{T})$, there are at most 2δ values of $i \in \{1, \ldots, n\}$ such that v has a neighbour in Y_i.*

Proof. Suppose there exists $I \subseteq \{1, \ldots, n\}$ with $|I| = 2\delta + 1$ such that v has a neighbour in Y_i for each $i \in I$, and let i_0 be the maximum element of I. Let $I' = I \setminus \{i_0\}$. It follows that $v \notin H_i$ for all $i \in I'$. This, together with the definition of template array, implies that for $i \in I'$, v is not η-mixed on Y_i. Moreover, v is not dense to Y_i, since the template array is 1-cleaned and $v \in V(\mathcal{T})$. It follows that v has at most $\eta - 1$ neighbours in each part of Y_i. Let $i \in I'$, and let the parts of Y_i be $A_1, \ldots, A_\beta$. Since v has a neighbour in Y_i, we may assume that v has a neighbour in A_1. Since v has at most $\eta - 1$ neighbours in A_2, and $|A_2| = \zeta$, it follows that v has at least δ non-neighbours in A_2, and so there is a $(1, \delta)$-broom with handle v in $G[Y_i \cup \{v\}]$. But also, since v has at least δ non-neighbours in A_1, there is a $(2, \delta)$-broom with handle v in $G[Y_i \cup \{v\}]$. By selecting the $(1, \delta)$-broom with handle v in $G[Y_i \cup \{v\}]$ for δ values of $i \in I'$, and selecting the $(2, \delta)$-broom for the remaining δ values of $i \in I'$, and taking the union of all these brooms, we find that G contains $T(\delta)$, contrary to (i). This proves 4.1. ∎

For the remainder of the paper, let us define $\gamma = (2\delta\tau + 1)(2\delta + 1)$.

4.2 *Let $\eta \geq \delta$ and $\zeta \geq \max(\eta, \alpha) + \delta$ be integers. Let G satisfy* (i)–(v), *and let $\mathcal{T}$ be a 1-cleaned (ζ, η)-template array in G, with sequence (Y_i, H_i) $(1 \leq i \leq n)$. For each $v \in V(\mathcal{T})$, there are fewer than γ values of $i \in \{1, \ldots, n\}$ such that v has a neighbour in H_i.*

Proof. Suppose then that G, $\mathcal{T}$, and (Y_i, H_i) $(1 \leq i \leq n)$ are as in the statement of 4.2, and there are at least γ values of $i \in \{1, \ldots, n\}$ such that v has a neighbour in H_i. By 4.1, there are at most 2δ values of i such that v has a neighbour in Y_i. Consequently there exists $I_1 \subseteq \{1, \ldots, n\}$ with $|I_1| = 2\delta(2\delta + 1)\tau$ such that for each $i \in I_1$, v has a neighbour in H_i and v has no neighbour in Y_i. For $i \in I_1$, let $u_i \in H_i \setminus Y_i$ be adjacent to v.

Let D be the digraph with vertex set I_1 in which for distinct $i, j \in I_1$, i is adjacent from j in D if u_j has a neighbour in Y_i (and consequently $i < j$). From 4.1, D has maximum outdegree at most 2δ and is acyclic, and so by 3.3 the graph underlying D has chromatic number at most $2\delta + 1$. Hence there exists $I_2 \subseteq I_1$ with $|I_2| = |I_1|/(2\delta + 1) = 2\delta\tau$ such that for all distinct $i, j \in I_2$, u_j has no neighbour in Y_i. Since $\chi(\{u_i : i \in I_2\} \leq \tau$ by (ii), there exists $I_3 \subseteq I_2$ with $|I_3| = 2\delta$ such that for all $i < j$ with $i, j \in I_3$, u_i and u_j are nonadjacent. For each $i \in I_3$, since u_i is η-mixed on Y_i, and $\eta \geq \delta$, and v has no neighbour in Y_i, it follows that there is a $(1, \delta)$-broom in $G[\{v, u_i\} \cup Y_i]$ with handle v. Since u_i is η-mixed on Y_i, and has fewer than α neighbours in some part of Y_i, and $\zeta \geq \delta + \alpha$, it follows that there are two distinct parts A_1, A_2 of Y_i such that u_i has a neighbour in A_1 and has at least δ non-neighbours in A_2. Consequently there is a $(2, \delta)$-broom in $G[\{v, u_i\} \cup Y_i]$ with handle v. But then, choosing the $(1, \delta)$-broom for δ values of $i \in I_3$ and choosing the $(2, \delta)$-broom for the other δ values of $i \in I_3$, and taking their union, we find that G contains $T(\delta)$, contrary to (i). This proves 4.2. ∎

A (ζ, η)-template array in G, with sequence (Y_i, H_i) $(1 \leq i \leq n)$, is said to be *slightly 2-cleaned* if it is 1-cleaned, and in addition

- for all distinct $i, j \in \{1, \dots, n\}$, no vertex of Y_i has a neighbour in H_j.

4.3 *Let $\eta \ge \delta$ and $\zeta \ge \max(\eta, \alpha)+\delta$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \to \mathbb{N}$, with the following property. For all $c \ge 0$, if G satisfies* (i)–(v) *and $\chi(G) > \phi(c)$ then there is a slightly 2-cleaned (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > c$.*

Proof. Let $t = \zeta\beta\gamma$. Let ψ satisfy 3.5 (with ϕ replaced by ψ), and define $\phi(x) = \psi(tx)$ for $x \ge 0$. Now let $c \ge 0$, and let G satisfy (i)–(v), with $\chi(G) > \phi(c)$. By 3.5 there is a 1-cleaned (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > tc$. Let its sequence be (Y_i, H_i) $(1 \le i \le n)$. Let D be the digraph with vertex set $\{1, \dots, n\}$ in which for distinct i, j with $1 \le i, j \le n$, j is adjacent from i if some vertex of Y_i has a neighbour in H_j. (Thus if j is adjacent from i then $j > i$, from the definition of a template sequence.) Since each Y_i has cardinality $\zeta\beta$, 4.2 implies that D has maximum outdegree less than $\zeta\beta\gamma = t$, and is acyclic, so by 3.3 the graph underlying D has chromatic number at most t. Consequently there is a partition $I_1, \dots, I_t$ of $\{1, \dots, n\}$ such that for $1 \le r \le t$, if $i, j \in I_r$ are distinct then there are no edges between Y_i and H_j. For each $r \in \{1, \dots, 2s+1\}$, let $\mathcal{T}_r$ be the (ζ, η)-template array with sequence the subsequence of (Y_i, H_i) $(1 \le i \le n)$ consisting of the terms with $i \in I_r$, and with $U(\mathcal{T}_r)$ the set of vertices in $U(\mathcal{T})$ with a neighbour in $\bigcup_{i \in I_r} H_i$. Thus each $\mathcal{T}_r$ is slightly 2-cleaned; and since every vertex of $V(\mathcal{T})$ belongs to $V(\mathcal{T}_r)$ for some r, there exists $r \in \{1, \dots, t\}$ such that $\chi(V(\mathcal{T}_r)) \ge \chi(V(\mathcal{T}))/t > c$. This proves 4.3. ∎

A (ζ, η)-template array in G, with sequence (Y_i, H_i) $(1 \le i \le n)$, is said to be *moderately 2-cleaned* if it is slightly 2-cleaned, and in addition

- for all $i \in \{1, \dots, n\}$, $H_i \setminus Y_i$ is stable.

4.4 *Let $\eta \ge \delta$ and $\zeta \ge \max(\eta, \alpha)+\delta$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \to \mathbb{N}$ with the following property. For all $c \ge 0$, if G satisfies* (i)–(v) *and $\chi(G) > \phi(c)$ then there is a moderately 2-cleaned (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > c$.*

Proof. Let ψ satisfy 4.3 (with ϕ replaced by ψ). Define $\phi(x) = \psi(2\tau x)$ for $x \ge 0$; we claim that ϕ satisfies the statement of 4.4. For let $c \ge 0$, and let G satisfy (i)–(v) with $\chi(G) > \phi(c)$. By 4.3 there is a slightly $(2, d)$-cleaned (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > 2\tau c$. Let its sequence be (Y_i, H_i) $(1 \le i \le n)$. For $1 \le i \le n$, every vertex of H_i has a neighbour in Y_i, and so, if we choose two adjacent vertices in Y_i, every vertex in H_i has distance at most two from one of them. Consequently, by (ii) it follows that $\chi(H_i) \le 2\tau$. Take a partition $W_1, \dots, W_{2\tau}$ of $H_1 \cup \dots \cup H_n$ such that $H_i \cap W_r$ is stable for $1 \le i \le n$ and $1 \le r \le 2\tau$. For $1 \le r \le 2\tau$, let $\mathcal{T}_r$ be the (ζ, η)-template with sequence $(Y_i, (H_i \cap W_r) \cup Y_i)$ $(1 \le i \le n)$, where $U(\mathcal{T}_r)$ is the set of vertices in $U(\mathcal{T})$ with a neighbour in W_r. Then each $\mathcal{T}_r$ is moderately 2-cleaned. Since every vertex of $V(\mathcal{T})$ belongs to $V(\mathcal{T}_r)$ for some r, there exists $r \in \{1, \dots, 2\tau\}$ such that

$$\chi(V(\mathcal{T}_r)) \ge \chi(V(\mathcal{T}))/(2\tau) > c.$$

This proves 4.4. ∎

4.5 *Let $\eta \geq \delta$ and $\zeta \geq \max(\eta, \alpha) + \delta$ be integers. Then there exists an integer s with the following property. Let G satisfy* (i)–(v)*, and let $\mathcal{T}$ be a moderately 2-cleaned (ζ, η)-template array in G, with sequence (Y_i, H_i) $(1 \leq i \leq n)$. For $1 \leq j \leq n$, there are at most s values of $i \in \{1, \ldots, n\} \setminus \{j\}$ such that some vertex in H_j has at least δ neighbours in H_i.*

Proof. Let $s_1 = 2\delta(2(\delta+1)\gamma+1)$, and let $s = \zeta\beta s_1$. Now let $\eta, \zeta, G, \mathcal{T}$ and (Y_i, H_i) $(1 \leq i \leq n)$ be as in the statement of 4.5, and suppose that for some $j \in \{1, \ldots, n\}$ and some subset $I \subseteq \{1, \ldots, n\} \setminus \{j\}$ with $|I| > s$, and every $i \in I$, there exists $u_i \in H_j$ with at least δ neighbours in H_i. Let $W_i \subseteq H_i$ with $|W_i| = \delta$ such that every vertex in W_i is adjacent to u_i. Thus each W_i is stable and disjoint from Y_i, because the template array is moderately 2-cleaned. Since each u_i has a neighbour in Y_j, and $|Y_j| = \zeta\beta$, there exists $I_1 \subseteq I$ with $|I_1| = s_1$ and a vertex $y \in Y_j$ adjacent to every u_i $(i \in I_1)$.

Let D be the digraph with vertex set I_1 in which for distinct $i, i' \in I_1$, i' is adjacent from i if some vertex in $\{u_i\} \cup W_i$ has a neighbour in $H_{i'}$. (Thus if $u_i = u_{i'}$ then we have $i \to i'$ and $i' \to i$.) By 4.2, D has maximum outdegree at most $(\delta + 1)\gamma$, and so by 3.3 the graph underlying D has chromatic number at most $2(\delta + 1)\gamma + 1$. Hence there exists $I_2 \subseteq I_1$ with $|I_2| = |I_1|/(2(\delta + 1)\gamma + 1) = 2\delta$ such that for all distinct $i, i' \in I_2$, no vertex in $\{u_i\} \cup W_i$ has a neighbour in $H_{i'}$ (and in particular the vertices u_i $(i \in I_2)$ are all distinct). The vertices u_i $(i \in I_2)$ are pairwise nonadjacent since they all belong to $H_j \setminus Y_j$, and the template array is moderately 2-cleaned.

Now for $i \in I_2$, u_i has no neighbour in Y_i, because the template array is moderately 2-cleaned. Choose $w_i \in W_i$; then since w_i is η-mixed on Y_i, it has at least η neighbours in some part of Y_i, and so there is a stable subset of Y_i of cardinality η, all adjacent to w_i. These vertices are all nonadjacent to u_i, and so there is a $(2, \delta)$-broom in $G[\{y, u_i, w_i\} \cup Y_i]$ with handle y. There is also a $(1, \delta)$-broom in $G[\{y, u_i\} \cup W_i]$ with handle y. By choosing the $(1, \delta)$-broom for δ values of $i \in I_2$, and the $(2, \delta)$-broom for the other δ values of i, and taking their union, we find that G contains $T(\delta)$, contrary to (i). This proves 4.5. ∎

For $d \geq 0$, let us say a moderately 2-cleaned (ζ, η)-template array $\mathcal{T}$ with sequence (Y_i, H_i) $(1 \leq i \leq n)$ is *partially* $(2, d)$*-cleaned* if

- for all $i \in \{1, \ldots, n\}$, every vertex of H_i has at most d neighbours in $H(\mathcal{T}) \setminus H_i$,

and *2-cleaned* if

- for all distinct $i, j \in \{1, \ldots, n\}$, no vertex of H_i has a neighbour in H_j.

4.6 *Let $\eta \geq \delta$ and $\zeta \geq \max(\eta, \alpha) + \delta$ be integers. Then there exist $d \geq 0$ and a non-decreasing function $\phi : \mathbb{N} \to \mathbb{N}$, with the following property. For all $c \geq 0$, if G satisfies* (i)–(v) *and $\chi(G) > \phi(c)$ then there is a partially $(2, d)$-cleaned (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > c$.*

Proof. Let s be as in 4.5, and let $d = \gamma(\delta - 1)$. Let ψ satisfy 4.4 (with ϕ replaced by ψ), and define $\phi(x) = \psi((2s + 1)x)$ for $x \geq 0$. Now let $c \geq 0$, and let G satisfy (i)–(v), with $\chi(G) > \phi(c)$. By 4.4 there is a moderately 2-cleaned (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > (2s + 1)c$. Let its sequence be (Y_i, H_i) $(1 \leq i \leq n)$. Let D be the digraph with vertex set $\{1, \ldots, n\}$ in which for distinct i, j with $1 \leq i, j \leq n$, i is adjacent from j if some vertex of H_j has at least δ neighbours in H_i. By 4.5, D has maximum outdegree at most s, and so by 3.3 the graph underlying D has chromatic number at most $2s + 1$. Consequently there is a partition $I_1, \ldots, I_{2s+1}$ of $\{1, \ldots, n\}$ such that for $1 \le r \le 2s+1$, if $i, j \in I_r$ are distinct then each vertex of H_j has at most $\delta - 1$ neighbours in H_i. By 4.2 it follows that for each $j \in I_r$, each vertex of H_j has at most $\gamma(\delta-1) = d$ neighbours in $\bigcup_{i \in I_r} H_i \setminus H_j$. For each $r \in \{1, \dots, 2s+1\}$, let $\mathcal{T}_r$ be the (ζ, η)-template array with sequence the subsequence of (Y_i, H_i) $(1 \le i \le n)$ consisting of the terms with $i \in I_r$, and with $U(\mathcal{T}_r)$ the set of vertices in $U(\mathcal{T})$ with a neighbour in $\bigcup_{i \in I_r} H_i$. Thus each $\mathcal{T}_r$ is partially $(2, d)$-cleaned; and since every vertex of $V(\mathcal{T})$ belongs to $V(\mathcal{T}_r)$ for some r, there exists $r \in \{1, \dots, 2s+1\}$ such that $\chi(V(\mathcal{T}_r)) \ge \chi(V(\mathcal{T}))/(2s+1) > c$. This proves 4.6. ∎

4.7 *Let $\eta \ge \delta$ and $\zeta \ge \max(\eta, \alpha) + \delta$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \to \mathbb{N}$ with the following property. For all $c \ge 0$, if G satisfies* (i)–(v) *and $\chi(G) > \phi(c)$ then there is a 2-cleaned (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > c$.*

Proof. Let d, ψ satisfy 4.6 (with ϕ replaced by ψ). Define $\phi(x) = \psi((d+1)x)$ for $x \ge 0$; we claim that ϕ satisfies the statement of 4.7. For let $c \ge 0$, and let G satisfy (i)–(v) with $\chi(G) > \phi(c)$. By 4.6 there is a partially $(2, d)$-cleaned (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(V(\mathcal{T})) > (d+1)c$. Let its sequence be (Y_i, H_i) $(1 \le i \le n)$. Since this template array is partially $(2, d)$-cleaned, the graph $G[H(\mathcal{T})]$ has maximum degree at most d, and so is $(d+1)$-colourable. Consequently there is a partition $W_1, \dots, W_{d+1}$ of $H_1 \cup \dots \cup H_n$ into $d+1$ stable sets. For $1 \le j \le d+1$, let $\mathcal{T}_j$ be the (ζ, η)-template with sequence $(Y_i, (H_i \cap W_j) \cup Y_i)$ $(1 \le i \le n)$, where $U(\mathcal{T}_j)$ is the set of vertices in $U(\mathcal{T})$ with a neighbour in W_j. Then each $\mathcal{T}_j$ is 2-cleaned (because it is moderately 2-cleaned and there are no edges between $H_i \cap W_j$ and $H_j \cap W_j$). Since every vertex of $V(\mathcal{T})$ belongs to $V(\mathcal{T}_j)$ for some j, there exists $j \in \{1, \dots, t\}$ such that

$$\chi(V(\mathcal{T}_j)) \ge \chi(V(\mathcal{T}))/(d+1) > c.$$

This proves 4.7. ∎

5 Shadowing, and growing daisies

Let $\mathcal{T}$ be a (ζ, η)-template array in G, with sequence (Y_i, H_i) $(1 \le i \le n)$. A *shadowing* of $\mathcal{T}$ is a sequence $B_1, \dots, B_n$ of pairwise disjoint subsets of $U(\mathcal{T})$, with union $U(\mathcal{T})$, such that for $1 \le i \le n$, every vertex in B_i has a neighbour in H_i. Every template array has a shadowing, and in general it has many. Let us say a shadowing $B_1, \dots, B_n$ has *degree at most s* if for every vertex $v \in U(\mathcal{T})$, there are at most s values of $i \in \{1, \dots, n\}$ such that v has a neighbour in B_i. If $X \subseteq U(\mathcal{T})$, we say the shadowing has *degree at most s relative to X* if for every vertex $v \in U(\mathcal{T})$, there are at most s values of $i \in \{1, \dots, n\}$ such that v has a neighbour in $B_i \cap X$.

Let $B_1, \dots, B_n$ be a shadowing of $\mathcal{T}$. A *daisy* (with respect to $\mathcal{T}$ and the given shadowing) is an induced subgraph D of G isomorphic to the $(\delta+2)$-vertex star $K_{1,\delta+1}$, such that

- exactly one vertex u of D belongs to $H(\mathcal{T})$; let $u \in H_i$ say;
- u has degree one in D, and the neighbour v of u in D belongs to $U(\mathcal{T})$;
- there exists $j \ne i$ with $1 \le j \le n$ such that $V(D) \setminus \{u, v\} \subseteq B_j$; and

- there are no edges between $V(D) \setminus \{u,v\}$ and H_i.

We call u the *root*, v the *eye*, and the vertices in $V(D) \setminus \{u,v\}$ the *petals* of the daisy. We need the following, proved in [1], but we repeat the proof because it is short:

5.1 *Let $d \geq 0$ be an integer, let G be a graph with chromatic number more than d, and let $X \subseteq V(G)$ be stable, such that $\chi(G \setminus X) < \chi(G)$. Then some vertex in X has at least d neighbours in $V(G) \setminus X$.*

Proof. Let $\chi(G) = k+1$, and so $k \geq d$. Let $\phi : V(G) \setminus X \to \{1,\dots,k\}$ be a k-colouring of $G \setminus X$. For each $x \in X$, if x has at most $d-1$ neighbours in $V(G) \setminus X$ then we may choose $\phi(x) \in \{1,\dots,k\}$, different from $\phi(v)$ for each neighbour $v \in V(G) \setminus X$ of x; and this extends ϕ to a k-colouring of G, which is impossible. Thus for some $x \in X$, x has at least d neighbours in $V(G) \setminus X$. This proves 5.1. ∎

We deduce [1]

5.2 *Let $s \geq 0$ be an integer, and let $\mathcal{T}$ be a (ζ,η)-template array in G, with sequence (Y_i, H_i) $(1 \leq i \leq n)$. Let $X \subseteq U(\mathcal{T})$ with $\chi(X) > s\beta\delta\zeta\tau^2$, and let $B_1,\dots,B_n$ be a shadowing of degree at most s relative to X. Then there is a daisy in $G[H(\mathcal{T}) \cup X]$.*

Proof. For $1 \leq i \leq n$, let W_i be the set of vertices in X that have a neighbour in H_i and have none in $H_1 \cup \cdots \cup H_{i-1}$. Since every vertex in W_i has distance at most two from some vertex in Y_i, and $|Y_i| = \beta\zeta$, it follows that $\chi(W_i) \leq \beta\zeta\tau$ for each i. Consequently there is a partition $X_1,\dots,X_{\beta\zeta\tau}$ of X such that $W_i \cap X_j$ is stable for $1 \leq i \leq n$ and $1 \leq j \leq \beta\zeta\tau$. We may assume that $\chi(X_1) \geq \chi(X)/(\beta\zeta\tau) > s\delta\tau$. Choose i minimum such that $\chi(X_1 \setminus (W_1 \cup \cdots \cup W_i)) < \chi(X_1)$, and let $H = G[X_1 \setminus (W_1 \cup \cdots W_{i-1})]$. Then since $\chi(H \setminus (X_1 \cap W_i)) < \chi(H)$, and $X_1 \cap W_i$ is stable, and $\chi(H) > s\delta\tau$, 5.1 implies that some vertex $v \in W_i$ has a set P of at least $s\delta\tau$ neighbours in $W_{i+1} \cup \cdots \cup W_n$. Choose $u \in H_i$ adjacent to v. Since P is disjoint from $W_1 \cup \cdots W_i)$, there are no edges between P and H_i. By hypothesis there are at most s values of $j \in \{1,\dots,n\}$ such that v has a neighbour in $B_j \cap X$, and so there exists $j \in \{1,\dots,n\}$ such that $|P \cap B_j| \geq |P|/s \geq \delta\tau$. Since $\chi(P) \leq \tau$ (because the vertices in P are all adjacent to v), there is a stable subset $P' \subseteq P \cap B_j$ with $|P'| \geq \delta$. Now $j \neq i$ since no vertices in P have a neighbour in H_i; and so $G[\{u,v\} \cup P']$ is a daisy. This proves 5.2. ∎

Let $\mathcal{T}$ be a (ζ,η)-template array in G, with sequence (Y_i, H_i) $(1 \leq i \leq n)$; and let $B_1,\dots,B_n$ be a shadowing. A *bunch* of daisies is a set $\{D_j : j \in J\}$ of daisies where $J \subseteq \{1,\dots,n\}$ and for each $j \in J$, D_j has root u_j, eye v_j and set of petals P_j, with the following properties:

- $P_j \subseteq B_j$ for each $j \in J$;
- there exists $i \in \{1,\dots,n\} \setminus J$ such that $u_j \in H_i$ for each $j \in J$; and
- for all distinct $j, j' \in J$, $P_j \cup \{v_j\}$ is disjoint from $P_{j'} \cup \{v_{j'}\}$, and there is no edge joining these two sets.

[1] To be deleted before publication: one referee queried the range of s, whether we require $s < n$. No, we don't, it holds for all $s \geq 0$.

(Thus, the roots may not all be distinct, and the root of one daisy may be adjacent to the eye of another.)

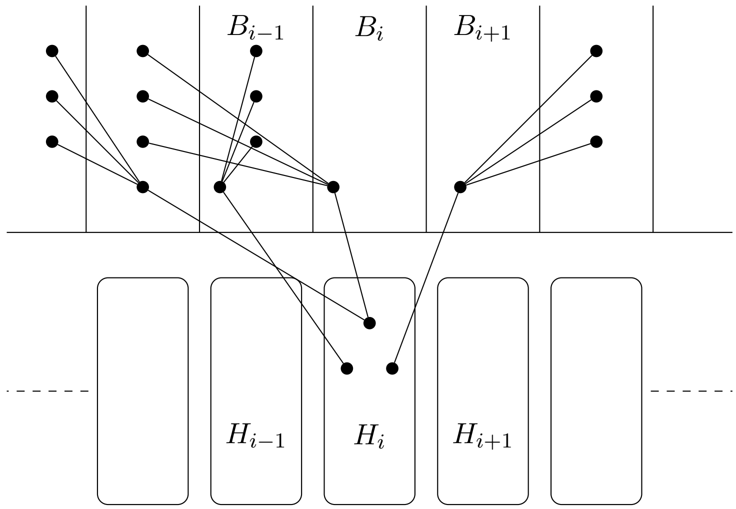

Figure 3: A bunch of daisies.

We deduce:

5.3 *Let $\mathcal{T}$ be a (ζ,η)-template array in G, with sequence (Y_i, H_i) $(1 \le i \le n)$. Let $t \ge 0$, and let $X \subseteq U(\mathcal{T})$ with*

$$\chi(X) > 2st\zeta^2\delta(2(\delta+2)s+3)\beta^2\tau^3.$$

Let $B_1, \ldots, B_n$ be a shadowing of degree at most s relative to X. Then there is a bunch $\{D_1, \ldots, D_t\}$ of daisies, such that $V(D_j) \subseteq H(\mathcal{T}) \cup X$ for $1 \le j \le t$.

Proof. Let $m_1 = t\beta\zeta\tau$, and $m = t\beta\zeta\tau(2(\delta+2)s+3)$. Let D be the digraph with vertex set $\{1, \ldots, n\}$ in which for distinct $i, j \in \{1, \ldots, n\}$, j is adjacent from i if there is a daisy with root in H_i, eye in X and set of petals in $X \cap B_j$.

(1) *There exists $i \in \{1, \ldots, n\}$ with outdegree in D at least m.*

Suppose not; then by 3.3 the graph underlying D is $2m$-colourable. Consequently there exists $I \subseteq \{1, \ldots, n\}$ with

$$\chi(X') \ge \chi(\bigcup_{1 \le i \le n} B_i \cap X)/(2m) > s\delta\beta\zeta\tau^2$$

where $X' = \bigcup_{i \in I} B_i \cap X$, such that for all distinct $i, j \in I$, there is no daisy with root in H_i, eye in X and set of petals in $X \cap B_j$. In particular, applying 5.2 to the (ζ,η)-template array $\mathcal{T}'$ with sequence (Y_i, H_i) $(i \in I)$ and $U(\mathcal{T}') = X'$, it follows that $\chi(X') \le s\beta\delta\zeta\tau^2$, a contradiction. This proves (1).

From (1), there exist $i \in \{1, \ldots, n\}$ and $J \subseteq \{1, \ldots, n\} \setminus \{i\}$, with $|J| = m$, such that for each $j \in J$ there is a daisy D_j with root $u_j \in H_i$, eye $v_j \in X$ and set of petals $P_j \subseteq B_j \cap X$. Now let D' be the digraph with vertex set J in which for all distinct $j, j' \in J$, j is adjacent from j' if some vertex in D_j belongs to or has a neighbour in $P_{j'}$. Then D' has maximum outdegree at most $(\delta+2)s+1$, and so by 3.3 the graph underlying D' is $2(\delta+2)s+3$-colourable. Hence there exists $J_1 \subseteq J$ with $|J_1| = m/(2(\delta+2)s+3) = m_1$, such that for all distinct j, j', no vertex in D_j has a neighbour in $P_{j'}$. In particular, the vertices v_j ($j \in J_1$) are all distinct. By an argument used in the proof of 5.2, $\{v_j : j \in J_1\}$ has chromatic number at most $\beta\zeta\tau$, and so there exists $J_2 \subseteq J_1$ with $|J_2| = m_1/(\beta\zeta\tau) = t$ such that the vertices v_j ($j \in j_2$) are pairwise nonadjacent. But then $\{D_j : j \in J_2\}$ is a bunch of daisies of cardinality t. This proves 5.3. ▮

6 Privatization

Let A, B be disjoint subsets of $V(G)$; we say A *covers* B if every vertex in B has a neighbour in A. We claim:

6.1 *Let $A, B \subseteq V(G)$ be disjoint, and let A cover B. Let $d \geq 0$ be an integer. Then there exist $A' \subseteq A$ and $B' \subseteq B$ (possibly $A' = \emptyset$ and $B' = B$) such that*

- *A' covers $B \setminus B'$;*
- *B' is the union of d matching-covered sets;*
- *every vertex in B' has at most one neighbour in A'; and*
- *every vertex in A' has exactly d neighbours in B'.*

Proof. We proceed by induction on d. The result is trivial for $d = 0$, because we can set $A' = A$ and $B' = \emptyset$; so we assume that $d > 0$ and the result holds for $d - 1$. Hence there exist $A' \subseteq A$ and $B'' \subseteq B$ such that

- A' covers $B \setminus B''$;
- B'' is the union of $d-1$ matching-covered sets;
- every vertex in B'' has at most one neighbour in A'; and
- every vertex in A' has exactly $d-1$ neighbours in B''.

Choose A' minimal with this property. Consequently for each $u \in A'$ there exists $v_u \in B \setminus B''$ such that u is the unique neighbour of v_u in A'. (There may be more than one choice for v_u, but if so select one and call it v_u.) Let $X = \{v_u : u \in A'\}$ and let $B' = B'' \cup X$. Then X is matching-covered, and every vertex in A' has a unique neighbour in X and exactly $d-1$ in B'', and so exactly d in B'. Consequently B' satisfies the statement of 6.1. This proves 6.1. ▮

Let $\mathcal{T}$ be a 2-cleaned (ζ, η)-template array in G, with sequence (Y_i, H_i) $(1 \leq i \leq n)$. We recall that $H(\mathcal{T})$ denotes $\bigcup_{1\leq i\leq n} H_i$; and let $Y(\mathcal{T})$ denote $\bigcup_{1\leq i\leq n} Y_i$ and $Z(\mathcal{T}) = H(\mathcal{T}) \setminus Y(\mathcal{T})$. A *privatization* for $\mathcal{T}$ is a subset $\Pi \subseteq U(\mathcal{T})$ such that

- Π is the union of $\delta\tau$ matching-covered sets;

- every vertex in Π has exactly one neighbour in $Z(\mathcal{T})$ and none in $Y(\mathcal{T})$; and
- every vertex in $Z(\mathcal{T})$ has exactly $\delta\tau$ neighbours in Π.

We deduce:

6.2 *Let G satisfy* (i)–(v)*, and let $\mathcal{T}$ be a 2-cleaned (ζ,η)-template array in G, with sequence (Y_i,H_i) $(1 \leq i \leq n)$. Then there is a 2-cleaned (ζ,η)-template array $\mathcal{T}'$, with sequence (Y_i,H_i') $(1 \leq i \leq n)$ and a privatization Π for $\mathcal{T}'$, such that*

- *$H_i' \subseteq H_i$ for $1 \leq i \leq n$;*
- *$U(\mathcal{T}') \subseteq U(\mathcal{T})$; and*
- *$\chi(U(\mathcal{T}') \setminus \Pi) \geq \chi(U(\mathcal{T})) - \delta\tau^2$.*

Proof. We will obtain the desired $\mathcal{T}'$ by removing some elements of each H_i and also removing some elements of $U(\mathcal{T})$. We cannot remove from H_i any element of Y_i, but the only role of the elements of $H_i \setminus Y_i$ is to provide neighbours for the vertices in $U(\mathcal{T})$; so we can happily remove some of them if we also remove from $U(\mathcal{T})$ the vertices which no longer have neighbours in any of the (shrunken) sets H_i.

Let B be the set of vertices in $U(\mathcal{T})$ with no neighbour in $Y(\mathcal{T})$. By 6.1, since $Z(\mathcal{T})$ covers B, there exist $A' \subseteq Z(\mathcal{T})$ and $B' \subseteq B$ such that

- A' covers $B \setminus B'$;
- B' is the union of $\delta\tau$ matching-covered sets;
- every vertex in B' has at most one neighbour in A'; and
- every vertex in A' has exactly $\delta\tau$ neighbours in B'.

Let Π be the set of vertices in B' that have a neighbour in A'; for $1 \leq i \leq n$ let $H_i' = (H_i \cap A') \cup Y_i$; and let $\mathcal{T}'$ be the template array with sequence (Y_i,H_i') $(1 \leq i \leq n)$ and $U(\mathcal{T}') = (U(\mathcal{T}) \setminus B') \cup \Pi$. Since $\chi(B') \leq \delta\tau^2$ by (iii), this proves 6.2. ∎

The advantage of privatization is the following lemma, used when we have a shadowing of bounded degree.

6.3 *Let $\eta \geq 1$ and $\zeta \geq \max(\eta,\alpha)$, and let $q,s \geq 0$. Let G satisfy* (i)–(v)*, and let $\mathcal{T}$ be a 2-cleaned (ζ,η)-template array in G, with sequence (Y_i,H_i) $(1 \leq i \leq n)$, that admits a privatization Π. Let $B_1,\dots,B_n$ be a shadowing of degree at most s relative to $U(\mathcal{T}) \setminus \Pi$. Let $1 \leq i \leq n$, and let $\{D_j : j \in J\}$ be a bunch of daisies, each with root in H_i and with $V(D_j) \cap \Pi = \emptyset$, with*

$$|J| = 2q\zeta\beta\tau((q+s)(\delta^2+1) + 2\delta + \delta\tau) + 2q\zeta\beta$$

Then there exist $u \in H_i \cup B_i$ and $J' \subseteq J$ with $|J'| = q$, such that for each $j \in J'$, u is adjacent to the eye of D_j and nonadjacent to the petals of D_j.

Proof. For each $j \in J$, let u_j, v_j, P_j be the root, eye, and set of petals of D_j respectively. Let D be the digraph with vertex set J in which for all distinct $j, j' \in J$, j' is adjacent from j if u_j is adjacent to $v_{j'}$. If some vertex of D has outdegree at least q we are done, so we assume not. Hence by 3.3 the graph underlying D has chromatic number at most $2q$, and so there exists $J_1 \subseteq J$ with $|J_1| = |J|/(2q)$ such that u_j is nonadjacent to $v_{j'}$ for all distinct $j, j' \in J_1$. In particular, the vertices u_j $(j \in J_1)$ are all distinct.

Since $|Y_i| = \zeta\beta$, it follows that $u_j \notin Y_i$ for at least $|J_1| - \zeta\beta$ values of $j \in J_1$. Now each such u_j has a neighbour in Y_i, and so there exist $J_2 \subseteq J_1$ with

$$|J_2| = (|J_1| - \zeta\beta)/|Y_i| = ((q+s)(\delta^2+1) + 2\delta + \delta\tau)\tau$$

and a vertex $y \in Y_i$, such that $u_j \notin Y_i$ and y is adjacent to u_j for each $j \in J_2$. Since $\{u_j : j \in J_2\}$ is τ-colourable, there exists $J_3 \subseteq J_2$ with

$$|J_3| = |J_2|/\tau = (q+s)(\delta^2+1) + 2\delta + \delta\tau$$

such that the vertices u_j $(j \in J_3)$ are pairwise nonadjacent. Consequently for all distinct $j, j' \in J_3$ the daisies $D_j, D_{j'}$ are vertex-disjoint and no edge joins them.

For each $j \in J_3$, $u_j \in Z(\mathcal{T})$ and so u_j has $\delta\tau$ neighbours in Π. We claim that all these neighbours belong to B_i. Let $x \in \Pi$ be adjacent to u_j, and let $x \in B_k$ say. Then x has a neighbour in H_k, from the definition of B_k; but x has a unique neighbour in $H(\mathcal{T})$, since $x \in \Pi$, and this neighbour is u_j, and so $u_j \in H_k$. Since $u_j \in H_i$ it follows that $k = i$. Thus u_j has $\delta\tau$ neighbours in Π, and they all belong to B_i.

The set of vertices in $\Pi \cap B_i$ with distance two from y has chromatic number at most τ; fix some partition of this set into τ stable sets. Consequently, for each $j \in J_3$, there are δ neighbours of u_j that belong to the same stable set of the partition. Since $|J_3| \geq \delta\tau$, there exist $J_4 \subseteq J_3$ with $|J_4| = \delta$ and a stable subset Π' of $\Pi \cap B_i$, such that for each $j \in J_4$, u_j has at least δ neighbours in Π'. For each $j \in J_4$, choose $\Pi_j \subseteq \Pi' \cap B_i$ with $|\Pi_j| = \delta$, such that every vertex in Π_j is adjacent to u_j. Thus $G[\{y, u_j\} \cup \Pi_j]$ is a $(1,\delta)$-broom with handle y, for each $j \in J_4$; and there are no edges between these brooms not incident with y.

Let $\{y\} \cup \bigcup_{j \in J_4} \Pi_j = Q$ say. Since the shadowing has degree at most s relative to $U(\mathcal{T}) \setminus \Pi$, each vertex in $Q \setminus \{y\}$ has a neighbour in P_j for at most s values of $j \in J_3 \setminus J_4$, and y has no neighbours in P_j for $j \in J_3 \setminus J_4$. Thus we may assume that each vertex $v \in Q$ has a neighbour in $P_j \cup \{v_j\}$ for at most $q+s$ values of $j \in J_3 \setminus J_4$, since if there are q values of j such that v is adjacent to v_j and has no neighbour in P_j, we are done. Since $|Q| = \delta^2 + 1$, there are at most $(q+s)(\delta^2+1)$ values of $j \in J_3 \setminus J_4$ such that some vertex in Q has a neighbour in P_j. Since $|J_3| - |J_4| - (q+s)(\delta^2+1) \geq \delta$, there exists $J_5 \subseteq J_3 \setminus J_4$ with $|J_5| = \delta$ such that for each $j \in J_5$, no vertex in Q has a neighbour in $P_j \cup \{v_j\}$. It follows that $G[\{y, u_j, v_j\} \cup P_j]$ is a $(2,\delta)$-broom with handle y for each $j \in J_5$. By taking the union of the $(1,\delta)$-brooms $G[\{y, u_j\} \cup \Pi_j]$ for each $j \in J_4$ and the $(2,\delta)$-brooms $G\{y, u_j, v_j\} \cup P_j]$ for each $j \in J_5$, we find that G contains $T(\delta)$, a contradiction. This proves 6.3 ∎

7 Edges between $H(\mathcal{T})$ and $U(\mathcal{T})$.

Our next goal is to bound the number of neighbours each vertex of $U(\mathcal{T})$ has in $H(\mathcal{T})$.

7.1 *Let $\eta \geq \delta$ and $\zeta \geq \max(\eta, \alpha) + \delta$ be integers; then there exists $\ell \geq 0$ with the following property. Let G satisfy* (i)–(v)*, and let $\mathcal{T}$ be a 2-cleaned (ζ, η)-template array in G, with sequence (Y_i, H_i) $(1 \leq i \leq n)$, that admits a privatization. Let X be the set of vertices in $U(\mathcal{T})$ that have at least $(\beta+1)\gamma\delta$ neighbours in $H(\mathcal{T})$. Then $\chi(X) \leq \ell$.*

Proof. Let

$$\begin{aligned} s &= 2(2\gamma+1)\delta\tau + \gamma, \\ q &= 2\delta(2\gamma(\delta+1)+1) + \gamma, \\ m &= 2q\zeta\beta\tau((q+s)(\delta^2+1) + 2\delta + \delta\tau) + 2q\zeta\beta, \text{ and} \\ \ell &= 2sm\zeta^2\delta(2(\delta+2)s+3)\beta^2\tau^3. \end{aligned}$$

We claim that ℓ satisfies the statement of 7.1. Let $\mathcal{T}$, (Y_i, H_i) $(1 \leq i \leq n)$ and X be as in the theorem. By 4.2, for each $v \in V(\mathcal{T})$, and hence for each $v \in X$, there are fewer than γ values of $i \in \{1, \dots, n\}$ such that v has a neighbour in H_i. For each $v \in X$, since v has at least $(\beta+1)\gamma\delta$ neighbours in $H(\mathcal{T})$, it follows that there exists $i \in \{1, \dots, n\}$ such that v has at least $(\beta+1)\delta$ neighbours in H_i. Choose a shadowing $B_1, \dots, B_n$ such that for $1 \leq i \leq n$, every vertex in $B_i \cap X$ has at least $(\beta+1)\delta$ neighbours in H_i.

(1) *The shadowing $B_1, \dots, B_n$ has degree less than s relative to X.*

Suppose not, and choose $y \in U(\mathcal{T})$ and $J \subseteq \{1, \dots, n\}$ with $|J| = s$ such that for each $j \in J$ there exists $u_j \in X \cap B_j$ adjacent to y. Since y has a neighbour in H_j for at most γ values of j by 4.2, there exists $J_1 \subseteq J$ with $|J_1| = |J| - \gamma$ such that y has no neighbour in H_j for each $j \in J_1$. Since the subgraph induced on $\{u_j : j \in J_1\}$ is τ-colourable, there exists $J_2 \subseteq J_1$ with $|J_2| = |J_1|/\tau = 2(2\gamma+1)\delta$ such that the vertices u_j $(j \in J_2)$ are pairwise nonadjacent. Let D be the digraph with vertex set J_2 in which for all distinct $j, j' \in J_2$, j' is adjacent from j if u_j has a neighbour in $H_{j'}$. Since D has maximum outdegree at most γ, by 3.3 the graph underlying D is $(2\gamma+1)$-colourable, and so there exists $J_3 \subseteq J_2$ with $|J_3| = |J_2|/(2\gamma+1) = 2\delta$ such that for all distinct $j, j' \in J_3$, u_j has no neighbour in $H_{j'}$. For each $j \in J_3$, since $u_j \in X$, u_j has at least $\delta(\beta+1)$ neighbours in H_j; and since H_j is $(\beta+1)$-colourable, there is a stable set P_j of δ such neighbours. Thus $G[\{y, u_j\} \cup P_j]$ is a $(1, \delta)$-broom with handle y. Let v_j be a neighbour of u_j in H_j, choosing $v_j \in Y_j$ if possible. If $v_j \in Y_j$, let A be a part of Y_j not containing v_j; then since $u_j \in U(\mathcal{T})$, u_j has at most $\eta - 1$ neighbours in A, and since $\zeta \geq \eta - 1 + \delta$, there is a set $Q_j \subseteq A$ of δ vertices all nonadjacent to u_j. If $v_j \notin Y_j$, then u_j has no neighbour in Y_j, and since v_j is η-mixed on Y_j and $\eta \geq \delta$, it follows that v_j has a set Q_j of δ neighbours in some part of Y_j. In either case $G[\{y, u_j, v_j\} \cup Q_j]$ is a $(2, \delta)$-broom with handle y. By choosing the $(1, \delta)$-broom for δ values of $j \in J_3$, and the $(2, \delta)$-broom for the other δ values of $j \in J_3$, and taking their union, we find that G contains $T(\delta)$, a contradiction. This proves (1).

(2) *There is no bunch of daisies $\{D_j : j \in J\}$ with $|J| = m$ such that $V(D_j) \subseteq X \cup H(\mathcal{T})$ for each $j \in J$.*

Suppose such a bunch $\{D_j : j \in J\}$ exists. Let $i \in \{1, \dots, n\}$ such that the root of D_j belongs to H_i for each $j \in J$. Let Π be a privatization. (Thus $\Pi \cap X = \emptyset$, since $(\beta+1)\gamma\delta \geq 2$.) By 6.3

applied to the (ζ, η)-template array $\mathcal{T}'$ with sequence (Y_j, H_j) $(1 \le j \le n)$ and $U(\mathcal{T}') = X \cup \Pi$, there exist $y \in H_i \cup B_i$ and $J_1 \subseteq J$ with $|J_1| = q$, such that for each $j \in J_1$, y is adjacent to the eye (u_j say) of D_j and has no neighbour in the set of petals (P_j say) of D_j. Thus $G[\{y, u_j\} \cup P_j]$ is a $(1, \delta)$-broom with handle y, for each $j \in J_1$. Since there are at most γ values of $j \in J_1$ such that y has a neighbour in H_j, there exists $J_2 \subseteq J_1$ with $|J_2| = |J_1| - \gamma$ such that y has no neighbour in H_j for $j \in J_2$.

Let D be the digraph with vertex set J_2 in which for all distinct $j, j' \in J_2$, j' is adjacent from j if some vertex in $\{u_j\} \cup P_j$ has a neighbour in $H_{j'}$. Since D has maximum outdegree at most $\gamma(\delta + 1)$, by 3.3 the graph underlying D is $(2\gamma(\delta + 1) + 1)$-colourable, and so there exists $J_3 \subseteq J_2$ with $|J_3| = |J_2|/(2\gamma(\delta + 1) + 1) = 2\delta$ such that for all distinct $j, j' \in J_3$, no vertex in $\{u_j\} \cup P_j$ has a neighbour in $H_{j'}$.

For each $j \in J_3$, choose a neighbour v_j of u_j, such that

- if u_j has a neighbour in Y_j then $v_j \in Y_j$;
- if u_j has no neighbour in Y_j and has a neighbour in H_j then $v_j \in H_j$;
- if u_j has no neighbour in H_j then $v_j \in P_j$.

We claim that in each case, y is nonadjacent to v_j, and there is a stable set $Q_j \subseteq H_j$ of neighbours of v_j, all nonadjacent to u_j, with $|Q_j| = \delta$. To see this, if $v_j \in H_j$ the proof is as in the proof of (1), so we assume that $v_j \in P_j$. This implies that u_j has no neighbours in H_j, and y is nonadjacent to v_j. Since $P_j \subseteq X$, v_j has at least $\delta(\beta + 1)$ neighbours in H_j, and since H_j is $(\beta + 1)$-colourable, the claim follows. In particular, $G[\{y, u_j, v_j\} \cup Q_j]$ is a $(2, \delta)$-broom for each $j \in J_3$. By choosing the $(1, \delta)$-broom for δ values of $j \in J_3$, and the $(2, \delta)$-broom for the other δ values of $j \in J_3$, and taking their union, we find that G contains $T(\delta)$, a contradiction. This proves (2).

From (1), (2) and 5.3, it follows that $\chi(X) \le \ell$. This proves 7.1. ∎

The bound $(\beta + 1)\gamma\delta$ will be very useful in the remainder of the proof, and for convenience let us define $\varepsilon = (\beta + 1)\gamma\delta$, for the remainder of the paper. Let us say a 2-cleaned (ζ, η)-template array $\mathcal{T}$ in G is *3-cleaned* if every vertex in $U(\mathcal{T})$ has fewer than ε neighbours in $H(\mathcal{T})$. We deduce:

7.2 *Let $\eta \ge \delta$ and $\zeta \ge \max(\eta, \alpha) + \delta$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \to \mathbb{N}$ with the following property. For all $c \ge 0$, if G satisfies* (i)–(v) *and $\chi(G) > \phi(c)$ then there is a 3-cleaned (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(U(\mathcal{T})) > c$.*

Proof. Let ψ satisfy 4.7 (with ϕ replaced by ψ), and let ℓ be as in 7.1. For all $c \ge 0$ define $\phi(c) = \psi(c + \delta\tau^2 + \ell)$; we claim this satisfies the statement of 7.2. For let $c \ge 0$, and let G satisfy (i)–(v) with $\chi(G) > \phi(c)$. By 4.7, there is a 2-cleaned (ζ, η)-template array $\mathcal{T}$ in G such that $\chi(U(\mathcal{T})) > c + \delta\tau^2 + \ell$. By 6.2, there is a 2-cleaned (ζ, η)-template array $\mathcal{T}_1$ in G that admits a privatization, such that $\chi(U(\mathcal{T}_1)) > c + \ell$. Let X be the set of vertices in $U(\mathcal{T}_1)$ that have at least ε neighbours in $H(\mathcal{T}_1)$. By 7.1, $\chi(X) \le \ell$. Let $\mathcal{T}'$ be the (ζ, η)-template array with the same sequence as $\mathcal{T}_1$ and with $U(\mathcal{T}') = U(\mathcal{T}_1) \setminus X$. It follows that $\mathcal{T}'$ is 3-cleaned, and $\chi(U(\mathcal{T}')) \ge \chi(U(\mathcal{T}_1)) - \ell > c$. This proves 7.2. ∎

8 Edges within a shadowing

Now we have come to the final stage of the proof: we investigate the edges between different sets of a shadowing. First we prove that there is some template array such that every shadowing has bounded degree; and privatize it; and then we will show that for a privatized template array, if every shadowing has bounded degree then the graph has bounded chromatic number.

So far, our technique in this paper has been to start with a template array, and make nicer and nicer ones at the cost of reducing the chromatic number of $U(\mathcal{T})$. This has more-or-less reached its limit, with 7.2, so now we need to do something different. To prove the next result, we will start with a 3-cleaned template array $\mathcal{T}$, and apply 7.2 to $G[U(\mathcal{T})]$ to get a second one, with vertex set a subset of $U(\mathcal{T})$; and repeat, generating a nested sequence of template arrays.

8.1 *Let $\eta \geq \alpha+2(\delta+1)^3(\varepsilon+1)^2$ and $\zeta \geq \eta+\delta$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \to \mathbb{N}$, and an integer s, with the following property. Let G satisfy* (i)–(v)*, with $\chi(G) > \phi(c)$. Then there is a 3-cleaned (ζ,η)-template array $\mathcal{T}$ in G, with sequence (Y_i, H_i) $(1 \leq i \leq n)$, such that $\chi(U(\mathcal{T})) > c$, and such that for each vertex $v \in U(\mathcal{T})$, there are fewer than s values of $i \in \{1,\ldots,n\}$ such that some neighbour of v in $U(\mathcal{T})$ has a neighbour in H_i.*

Proof.

Let

$$\begin{aligned}
s_3 &= (\delta(\delta+1)+1)\varepsilon+\delta\\
s_2 &= ((2\delta\varepsilon+1)\delta\tau+\varepsilon)s_3\\
s_1 &= (2\varepsilon+1)\tau s_2\\
s &= s_1+\varepsilon\\
t_4 &= 2\delta\\
t_3 &= 2\delta t_4\\
t_2 &= (\alpha\tau 2^{\beta\zeta}+1)t_3\\
t_1 &= t_2^{s_2}, \text{ and}\\
t &= 1+2^{s_2}\delta\tau+t_1.
\end{aligned}$$

Let ψ satisfy 7.2 (with ϕ replaced by ψ). Define $\phi^0(x)=x$ for $x \geq 0$, and inductively for $i \geq 1$, let $\phi^i(x)=\psi(\phi^{i-1}(x))$ for $x \geq 0$. Let $\phi=\phi^t$; we claim that ϕ, s satisfy the statement of 8.1.

Let G satisfy (i)–(v), with $\chi(G) > \phi(c)$. Define $U^0 = V(G)$; thus $\chi(U^0) > \phi^t(c)$. For $1 \leq j \leq t$ define $\mathcal{T}^j$ and U^j inductively as follows. Let j satisfy $1 \leq j \leq t$, and suppose we have defined U^{j-1}, and $\chi(U^{j-1}) > \phi^{t-j+1}(c)$. By 7.2 applied to $G[U^{j-1}]$, there is a 3-cleaned (ζ,η)-template array $\mathcal{T}^j$ in $G[U^{j-1}]$ such that $\chi(U(\mathcal{T}^j)) > \phi^{t-j}(c)$. Let $U^j = U(\mathcal{T}^j)$. This completes the inductive definition. Let $\mathcal{T}$ be the (ζ,η)-template array with the same sequence as $\mathcal{T}^1$, and with $U(\mathcal{T}) = U^t$; we will show that $\mathcal{T}$ satisfies the statement of 8.1. Certainly it is 3-cleaned, and $\chi(U(\mathcal{T})) > c$.

We remark that for all $j < j' \leq t$, every vertex in $H(\mathcal{T}^{j'})$ has fewer than ε neighbours in $H(\mathcal{T}^j)$ (since $H(\mathcal{T}^{j'}) \subseteq U(\mathcal{T}^j)$ and $\mathcal{T}^j$ is 3-cleaned), and for all j every vertex in U^t has fewer than ε neighbours in $H(\mathcal{T}^j)$ (for the same reason); but when $j' > j$ we know nothing about the number of neighbours a vertex in $H(\mathcal{T}^j)$ has in $H(\mathcal{T}^{j'})$.

For $1 \le j \le t$, let the sequence of $\mathcal{T}^j$ be (Y_i^j, H_i^j) $(1 \le i \le n_j)$. We assume for a contradiction that there exist $y \in U^t$ and a subset $I \subseteq \{1, \dots, n_1\}$ with $|I| = s$, such that for each $i \in I$, there exists $u_i \in U^t$ adjacent to y and with a neighbour in H_i^1. Since there are at most ε values of i such that y has a neighbour in H_i^1, there exists $I_1 \subseteq I$ with $|I_1| = |I| - \varepsilon = s_1$ such that for each $i \in I_1$, y has no neighbour in H_i^1.

(1) *There exist $I_2 \subseteq I_1$ with $|I_2| = s_2$, and for each $i \in I_2$, a $(2, \delta)$-broom B_i of $G[\{y, u_i\} \cup H_i^1]$ with handle y, such that every edge joining two of these brooms is incident with y.*

Let D be the digraph with vertex set I_1 in which for all distinct $i, i' \in I_1$, i' is adjacent from i if u_i has a neighbour in $H_{i'}^1$. Since D has maximum outdegree at most ε, by 3.3 the graph underlying D has chromatic number at most $2\varepsilon + 1$. Hence there exists $I_1' \subseteq I_1$ with $|I_1'| = s_1/(2\varepsilon + 1)$, such that for all distinct $i, i' \in I_1'$, u_i has no neighbour in $H_{i'}^1$. Since the set $\{u_i : i \in I_1'\}$ has chromatic number at most τ (because each u_i is adjacent to y), there exists $I_2 \subseteq I_1'$ with $|I_2| = |I_1'|/\tau = s_2$ such that the vertices u_i $(i \in I_2)$ are pairwise nonadjacent. For each $i \in I_2$, choose $w_i \in H_i^1$ adjacent to u_i, with $w_i \in Y_i^1$ if possible. If $w_i \in Y_i^1$, let A be a part of Y_i^1 not containing w_i; then since u_i has fewer than η neighbours in A, and w_i is adjacent to every vertex in A, there exists $R_i \subseteq A$ of cardinality δ all adjacent to w_i and nonadjacent to u_i. If $w_i \notin Y_i^1$, then u_i has no neighbour in Y_i^1; since $w_i \in H_i^1$, w_i has at least $\eta \ge \delta$ neighbours in some part of Y_i^1; and so there exists a stable set $R_i \subseteq Y_i^1$ of cardinality δ, all adjacent to w_i and not to u_i. In either case, $G[\{y, u_i, w_i\} \cup R_i]$ is a $(2, \delta)$-broom with handle y. This proves (1).

Let J_1 be the set of all $j \in \{2, \dots, t\}$ such that some vertex in $H(\mathcal{T}^j)$ is adjacent to at least s_3 of the vertices u_i $(i \in I_2)$.

(2) *If $j \in \{2, \dots, t\} \setminus J_1$, there is a subset $X^j \subseteq H(\mathcal{T}^j)$ and a subset $I^j \subseteq I_2$, such that*

- *$|X^j| = |I^j| = (2\delta\varepsilon + 1)\delta\tau$;*
- *y has no neighbour in X^j;*
- *every vertex in X^j has a unique neighbour in $\{u_i : i \in I^j\}$; and*
- *every vertex in $\{u_i : i \in I^j\}$ has a unique neighbour in X^j.*

Since for all $i \in I_2$, u_i has a neighbour in $H(\mathcal{T}^j)$, there exists $X \subseteq H(\mathcal{T}^j)$, minimal such that each u_i $(i \in I_2)$ has a neighbour in X. Since every vertex in X is adjacent to fewer than s_3 vertices in $\{u_i : i \in I_2\}$, it follows that $|X| \ge s_2/s_3 = (2\delta\varepsilon + 1)\delta\tau + \varepsilon$. From the minimality of X, for each $x \in X$ there exists $i(x) \in I_2$ such that x is the unique neighbour of $u_{i(x)}$ in X. (There may be more than one choice for $i(x)$; if so, choose one and call it $i(x)$.) Since y has at most ε neighbours in $H(\mathcal{T}^j)$, there exists $X^j \subseteq X$ with $|X^j| = (2\delta\varepsilon + 1)\delta\tau$ such that y has no neighbours in X^j. Let $I^j = \{i(x) : x \in X^j\}$; then this proves (2).

(3) *$|\{2, \dots, t\} \setminus J_1| \le 2^{s_2}\delta\tau$.*

Suppose not. For each $j \in \{2, \dots, t\} \setminus J_1$, there are at most 2^{s_2} possibilities for the set I^j, and so there exists $J' \subseteq \{2, \dots, t\} \setminus J_1$ with $|J'| = \delta\tau$, and a subset $I_3 \subseteq I_2$ (necessarily with $|I_3| = (2\delta\varepsilon + 1)\delta\tau$), such that $I^j = I_3$ for each $j \in J'$. Let $X = \bigcup_{j \in J'} X^j$; then $\chi(X) \le \tau$. Take a partition $Z_1, \dots, Z_\tau$ of X into stable sets. For each $i \in I_3$, since u_i has $\delta\tau$ neighbours in X, there exists $r \in \{1, \dots, \tau\}$ such that u_i has at least δ neighbours in Z_r. Since there are only τ possibilities for r, there exist $I_4 \subseteq I_3$ with $|I_4| = |I_3|/\tau = (2\delta\varepsilon + 1)\delta$, and $r \in \{1, \dots, \tau\}$, such that for each $i \in I_4$, u_i has at least δ neighbours in Z_r; let N_i be a set of δ such neighbours. Since for each $j \in J'$, every vertex in X_j has a unique neighbour in $\{u_i : i \in I_2\}$, the sets N_i $(i \in I_4)$ are pairwise disjoint. Moreover, the union of these sets is stable. Let D be the digraph with vertex set I_4, in which for all distinct $i, i' \in I_4$, i' is adjacent from i if some vertex in N_i has a neighbour in $V(B_{i'})$ (as defined in (1)). Then D has maximum outdegree $\delta\varepsilon$, since $|N_i| = \delta$ and each vertex in N_i has at most ε neighbours in H^1 (and none in $V(B_{i'}) \setminus H^1$). By 3.3 the graph underlying D has chromatic number at most $2\delta\varepsilon + 1$, and so there exists $I_5 \subseteq I_4$ with

$$|I_5| = |I_4|/(2\delta\varepsilon + 1) = \delta,$$

such that for all distinct $i, i' \in I_5$, no vertex in N_i has a neighbour in $V(B_{i'})$. It follows that G contains $T(\delta)$, a contradiction. This proves (3).

From (3), it follows that $|J_1| \ge t - 1 - 2^{s_2}\delta\tau = t_1$. For each $j \in J_1$, choose $v^j \in H(\mathcal{T}^j)$ adjacent to at least s_3 of the vertices u_i $(i \in I_2)$. Consequently there exists $J_2 \subseteq J_1$ with $|J_2| = t_2 \le |J_1|2^{-s_2}$ and a subset $I_3 \subseteq I_2$ with $|I_3| = s_3$, such that every vertex v^j $(j \in J_2)$ is adjacent to every vertex u_i $(i \in I_3)$. For each $j \in J_2$, there exists $i \in \{1, \dots, n_j\}$ such that $v^j \in H_i^j$; let us write $H^j = H_i^j$ and $Y^j = Y_i^j$, since we will have no need for the other terms of the sequence of $\mathcal{T}^j$.

(4) *There exists $J_3 \subseteq J_2$ with $|J_3| = |J_2|/(\alpha\tau 2^{\beta\zeta} + 1) = t_3$ such that for all distinct $j, j' \in J_3$, v^j is not dense to $Y^{j'}$.*

Let D be the digraph with vertex set J_2 in which for all distinct $j, j' \in J_2$, j' is adjacent from j if v^j is dense to $Y^{j'}$. Since v^j is not dense to $Y^{j'}$ if $j > j'$, it follows that D is acyclic, and has maximum indegree at most $\alpha\tau 2^{\beta\zeta}$, by 2.6, and so by 3.3 the graph underlying D has chromatic number at most $\alpha\tau 2^{\beta\zeta} + 1$. This proves (4).

(5) *For each $j_0 \in J_3$ there are fewer than δ values of $j \in J_3 \setminus \{j_0\}$ such that v^{j_0} has at least $\delta(\delta + 1)\varepsilon + s_3\varepsilon$ neighbours in some part of Y^j.*

For suppose that there exists $J_4 \subseteq J_3 \setminus \{j_0\}$ with $|J_4| = \delta$ such that for each $j \in J_4$, v^{j_0} has at least $\delta(\delta + 1)\varepsilon + s_3\varepsilon$ neighbours in some part of Y^j. For each $j \in J_4$, since there is a part of Y^j in which v^{j_0} has fewer than α neighbours (because v^{j_0} is not dense to Y^j), it follows that there are distinct parts A, A' of Y^j, such that v^{j_0} has at least $\delta(\delta + 1)\varepsilon + s_3\varepsilon$ neighbours in A and at least $\zeta - \alpha + 1 \ge \delta(\delta + 1)\varepsilon + s_3\varepsilon$ non-neighbours in A'. Since at most $s_3\varepsilon$ vertices of Y^j have a neighbour in $\{u_i : i \in I_3\}$, there is a subset $P^j \subseteq A$ with cardinality $\delta(\delta + 1)\varepsilon$, such that all vertices in P^j are adjacent to v^{j_0} and have no neighbours in $\{u_i : i \in I_3\}$; and there is a subset $Q^j \subseteq A'$ with cardinality $\delta(\delta + 1)\varepsilon$, such that all vertices in Q^j are nonadjacent to v^{j_0} and have no neighbours in $\{u_i : i \in I_3\}$.

For each $j \in J_4$, we choose a $(1, \delta)$-broom C^j of $G[Y^j \cup \{v^{j_0}\}]$ with handle v^{j_0}, inductively as follows. Let $j \in J_4$, and assume that $C^{j'}$ is defined for all $j' \in J_4$ with $j' > j$. Let S be the union of all the sets $V(C^{j'}) \setminus \{v^{j_0}\}$ for $j' \in J_4$ with $j' > j$. Then $|S| \le (\delta - 1)(\delta + 1)$, and since each vertex in S has at most ε neighbours in H^j, it follows that at most $(\delta-1)(\delta+1)\varepsilon$ vertices in H^j have a neighbour in S. Since

$$|P^j|, |Q^j| = \delta(\delta+1)\varepsilon > (\delta-1)(\delta+1)\varepsilon + \delta$$

there exist a vertex in P^j, and a set of δ vertices in Q^j, each with no neighbours in S. Hence there is a $(1,\delta)$-broom C^j of $G[Y^j \cup \{v^{j_0}\}]$ containing no neighbours of S different from v^{j_0}. This completes the inductive definition of C^j for $j \in J_4$.

Now let S be the union of the sets $V(C^j)$ for $j \in J_4$. Then $|S| = \delta(\delta+1)+1$, and so there are at most $(\delta(\delta+1)+1)\varepsilon$ vertices in $H(\mathcal{T}^1)$ with neighbours in S. Since $s_3 \geq (\delta(\delta+1)+1)\varepsilon + \delta$, there are at least δ values of $i \in I_3$ such that the edge $u_i v^{j_0}$ is the only edge between $V(B_i)$ and $V(C^j)$ for each $j \in J_4$; and so G contains $T(\delta)$ (with handle v^{j_0}), a contradiction. This proves (5).

(6) *There exists $J_4 \subseteq J_3$ with $|J_4| = t_4$ such that for all distinct $j, j' \in J_4$, v^j has fewer than $\delta(\delta+1)\varepsilon + s_3\varepsilon$ neighbours in each part of $Y^{j'}$.*

Let D be the digraph with vertex set J_3 in which for all distinct $j, j' \in J_3$, j' is adjacent from j if v^j has at least $\delta(\delta+1)\varepsilon + s_3\varepsilon$ neighbours in some part of Y^j. By (5), D has maximum outdegree at most $\delta-1$, and so by 3.3 the graph underlying D has chromatic number at most 2δ, and the claim follows. This proves (6).

Fix $i \in I_3$; and partition J_4 into two sets J_4', J_4'', both of cardinality δ. For each $j \in J_4'$, we define a $(2,\delta)$-broom C^j of $G[Y^j \cup \{v^j, u_i\}]$ with handle u_i, and for each $j \in J_4''$, we define a $(1,\delta)$-broom C^j of $G[Y^j \cup \{v^j, u_i\}]$ with handle u_i, inductively as follows. Let $j \in J_4$, and assume that $C^{j'}$ is defined for all $j' \in J_4$ with $j' > j$. Let S be the union of $\{u_i\}$ and all the sets $V(C^{j'}) \setminus \{v^{j'}\}$ for $j' \in J_4$ with $j' > j$. Then $|S| \leq (\delta+1)(t_4-1)$. There are distinct parts A, A' of Y^j such that v^j has at least η neighbours in A and at most $\alpha - 1$ neighbours in A'. But at most $\varepsilon(\delta+1)(t_4-1)$ vertices of A have a neighbour in S, and at most $(t_4-1)(\delta(\delta+1)\varepsilon + s_3\varepsilon)$ vertices in A have a neighbour in $\{v^{j'} : j' \in J_4 \setminus \{j\}\}$, by (6) and the definition of J_4, and the same for A'. Since

$$\eta, \zeta - \alpha + 1 \geq (\delta+1)(t_4-1)\varepsilon + (t_4-1)(\delta(\delta+1)\varepsilon + s_3\varepsilon) + \delta,$$

there exist a set of δ neighbours of v^j in A, and a set of δ non-neighbours of v^j in A', all with no neighbours in $S \cup \{v^{j'} : j' \in J_4 \setminus \{j\}\}$. Consequently the desired broom C^j can be chosen as specified. This completes the inductive definition. But by taking the union of all the C^j $(j \in J_4)$, we see that G contains $T(\delta)$, a contradiction. This proves 8.1. ∎

8.2 *Let $\zeta, \eta, s \geq 0$. Let $\mathcal{T}$ be a 2-cleaned (ζ,η)-template array in G, with sequence (Y_i, H_i) $(1 \leq i \leq n)$, that admits a privatization Π. Suppose that for each vertex $v \in U(\mathcal{T})$, there are fewer than s values of $i \in \{1, \dots, n\}$ such that some neighbour of v in $U(\mathcal{T})$ has a neighbour in H_i. Let $q = 2\delta + s$ and*

$$r = 4(\delta+1)s\left(2q\zeta^2\beta^2\tau^2((q+s)(\delta^2+1) + 2\delta + \delta\tau) + 2q\zeta^2\beta^2\tau\right).$$

Then

$$\chi(U(\mathcal{T}) \setminus \Pi) \leq 3rs\beta\delta\zeta\tau^2.$$

Proof. It follows that every shadowing has degree less than s. For $1 \le i \le n$, let B_i be the set of vertices in $U(\mathcal{T})$ with a neighbour in H_i and with no neighbour in $H_1 \cup \dots \cup H_{i-1}$. It follows that $(B_1, \dots, B_n)$ is a shadowing, and hence has degree less than s.

For distinct $u, v \in U(\mathcal{T})$, we say that v is *later* than u if $u \in B_i$ and $v \in B_j$ where $j > i$. For $i, a, b, c \in \{1, \dots, n\}$, we say that i is *strong* to (a, b, c) if

- $i < \min(a, b, c)$ and $a < c$;
- there exist $u \in B_i \setminus \Pi$ and $v \in B_a \setminus \Pi$, adjacent;
- there exist δ vertices in $B_b \setminus \Pi$, pairwise nonadjacent, and all adjacent to u (and possibly also adjacent to v); and
- there exist δ vertices in $B_c \setminus \Pi$, pairwise nonadjacent, and all adjacent to v and not to u.

(1) *For $1 \le i \le n$, there do not exist r triples $(a_1, b_1, c_1), \dots, (a_r, b_r, c_r)$, such that i is strong to them all, and a_j, b_j, c_j are different from $a_{j'}, b_{j'}, c_{j'}$ for $1 \le j < j' \le r$.*

Suppose that r such triples exist. For $1 \le j \le r$, choose $u_j \in B_i \setminus \Pi$ and $v_j \in B_{a_j} \setminus \Pi$ adjacent to u_j, and a stable set P_j of δ vertices in $B_{b_j} \setminus \Pi$, all adjacent to u_j, and a stable set Q_j of δ vertices in $B_{c_j} \setminus \Pi$, all adjacent to v_j and not to u_j. Let D be the digraph with vertex set $\{1, \dots, r\}$ in which for all distinct $j, j' \in \{1, \dots, r\}$, j' is adjacent from j if some vertex $u \in \{u_j, v_j\} \cup P_j \cup Q_j$ is adjacent in G to a vertex $v \in \{v_{j'}\} \cup P_{j'} \cup Q_{j'}$ and u is earlier than v. Then D has maximum outdegree less than $2(\delta+1)s$, and so by 3.3 there exists $J \subseteq \{1, \dots, r\}$ with

$$|J| \ge r/(4(\delta+1)s) = 2q\zeta^2\beta^2\tau^2((q+s)(\delta^2+1) + 2\delta + \delta\tau) + 2q\zeta^2\beta^2\tau$$

such that for all distinct $j, j' \in J$, there are no edges between $\{u_j, v_j\} \cup P_j \cup Q_j$ and $\{v_{j'}\} \cup P_{j'} \cup Q_{j'}$. In particular, the vertices u_j $(j \in J)$ are all distinct. Since the set $\{u_j : j \in J\}$ has chromatic number at most $\zeta\beta\tau$ (because each u_j has distance at most two from some vertex of Y_i, and $|Y_i| = \zeta\beta$), there exists $J_1 \subseteq J$ with $|J_1| = |J|/(\zeta\beta\tau)$ such that the vertices u_j $(j \in J_1)$ are pairwise nonadjacent. For $j \in J_1$, there are no edges between H_i and P_j, from the definition of the shadowing. For each $j \in J_1$ choose $w_j \in H_j$ adjacent to u_j; then $G[\{w_j, u_j\} \cup P_j]$ is a daisy D_{b_j} say, and $\{D_{b_j} : j \in J_1\}$ is a bunch of daisies. But

$$|J|/(\zeta\beta\tau) \ge 2q\zeta\beta\tau((q+s)(\delta^2+1) + 2\delta + \delta\tau) + 2q\zeta\beta$$

and so by 6.3, applied to $\{D_{b_j} : j \in J_1\}$, we deduce that there exist $w \in H_i \cup B_i$ and $J_2 \subseteq J_1$ with $|J_2| = q$, such that for each $j \in J_2$, w is adjacent to the eye u_j of D_{b_j} and nonadjacent to the petals P_j of D_{b_j}. Moreover there are no edges between $\{u_j, v_j\} \cup P_j \cup Q_j$ and $\{u_{j'}, v_{j'}\} \cup P_{j'} \cup Q_{j'}$ for all distinct $j, j' \in J_2$. Now w has neighbours in at most s of the sets $\{v_j\} \cup P_j \cup Q_j$ $(j \in J_2)$, and so there exists $J_3 \subseteq J_2$ with $|J_3| = 2\delta$ such that w has no neighbours in $\{v_j\} \cup P_j \cup Q_j$ for $j \in J_3$. Hence for $j \in J_3$, $G[\{w, u_j\} \cup P_j]$ is a $(1, \delta)$-broom with handle w, and $G[\{w, u_j, v_j\} \cup Q_j]$ is a $(2, \delta)$-broom with handle w; and taking the first for δ choices of $j \in J_3$, and the second for the remaining δ choices of j, and taking their union, we find that G contains $T(\delta)$, a contradiction. This proves (1).

By (1), for each $i \in \{1, \dots, n\}$ there is a subset $J_i \subseteq \{i+1, \dots, n\}$ with $|J_i| < 3r$, such that for all a, b, c, if i is strong to (a, b, c) then one of a, b, c is in J_i. Let D be the digraph with vertex set

$\{1, \dots, n\}$ in which for $i < j$, j is adjacent from i if $j \in J_i$. Thus D has maximum outdegree less than $3r$, and since D is acyclic, by 3.3 the graph underlying D has chromatic number at most $3r$. It follows that there is a subset $I \subseteq \{1, \dots, n\}$ such that

$$3r\chi(\bigcup_{i \in I} B_i \setminus \Pi) \geq \chi(\bigcup_{1 \leq i \leq n} B_i \setminus \Pi),$$

with the property that for all $i, a, b, c \in I$, i is not strong to (a, b, c). Let $W = \bigcup_{i \in I} B_i \setminus \Pi$.

(2) $\chi(G[W]) \leq s^2 \delta\zeta\beta\tau^2$.

Let D be the digraph obtained from $G[W]$ by deleting all edges uv such that $\{u, v\}$ is a subset of some B_i ($i \in I$), and directing every remaining edge such that the head of every edge is later than its tail. Let X be the set of all $v \in W$, where $v \in B_i$ say, such that for some $j > i$, there are $s\delta$ neighbours of v in B_j, pairwise nonadjacent. Suppose that $D[X]$ has a directed path with $s + 1$ vertices, say $v_0\text{-}\cdots\text{-}v_s$ in order. Let $v_i \in B_{j_i}$ where $j_i \in I$; then $j_0 < \cdots < j_s$. For $0 \leq i \leq s$, since $v_i \in X$, there exist $b_i \in I$ with $b_i > j_i$ and a stable subset P_i of B_{b_i} with $|P_i| = s\delta$, all adjacent to v_i. For $1 \leq i \leq s$, let Q_i be the set of vertices in P_s that are adjacent to v_i and not to v_{i-1}. Since j_{i-1} is not strong to (j_i, b_{i-1}, b_s), it follows that $|Q_i| < \delta$ for $1 \leq i \leq s$, and so there is a vertex $v \in P_s$ that belongs to none of the sets $Q_1, \dots, Q_s$. Consequently v is adjacent to all of $v_0, v_1, \dots, v_s$, and hence has a neighbour in B_j for $s + 1$ different values of j, contrary to the hypothesis. This proves that $D[X]$ has no directed path with $s + 1$ vertices, and hence the graph underlying $D[X]$ has chromatic number at most s, from the Gallai-Roy theorem [2, 8].

For each $v \in W \setminus X$, there are fewer than s values of $j \in I$ such that $B_j \setminus \Pi$ contains some vertex adjacent from v in D, by hypothesis. Moreover, since $v \notin X$, if $v \in B_i \setminus \Pi$ and some vertex in $B_j \setminus \Pi$ is adjacent from v in D, then $j > i$, and there do not exist $s\delta$ vertices in $B_j \setminus \Pi$, pairwise nonadjacent in G and all adjacent to v in G. Since the set of neighbours of v is τ-colourable, it follows that fewer than $s\delta\tau$ vertices in $B_j \setminus \Pi$ are adjacent in D from v; and so $D[W \setminus X]$ has maximum outdegree less than $(s-1)s\delta\tau$. Thus by 3.3 the graph underlying $D[W \setminus X]$ has chromatic number at most $(s-1)s\delta\tau$ (since D is acyclic).

Consequently the graph underlying D has chromatic number at most $s + (s-1)s\delta\tau \leq s^2\delta\tau$. Since for each $i \in I$, $G[B_i \setminus \Pi]$ has chromatic number at most $\beta\zeta\tau$, it follows that $\chi(G[W]) \leq s^2\beta\delta\zeta\tau^2$. This proves (2).

From the choice of I, $3r\chi(W) \geq \chi(\bigcup_{1 \leq i \leq n} B_i \setminus \Pi)$, and so from (2),

$$\chi(\bigcup_{1 \leq i \leq n} B_i \setminus \Pi) \leq 3rs\beta\delta\zeta\tau^2.$$

This proves 8.2. ∎

We deduce, finally:

8.3 *There exists c such that if G satisfies* (i)–(v) *then $\chi(G) \leq c$.*

Proof. Let $\eta = \alpha + 2(\delta + 1)^3(\varepsilon + 1)^2$ and $\zeta = \eta + \delta$. Let ϕ and s satisfy 8.1. Let r be as in 8.2. Let $c_2 = 3rs\beta\delta\zeta\tau^2$, let $c_1 = c_2 + \delta\tau^2$, and let $c = \phi(c_1)$.

Let G satisfy (i)–(v), and suppose that $\chi(G) > c$. By 8.1, there is a 3-cleaned (ζ, η)-template array $\mathcal{T}$ in G, with sequence (Y_i, H_i) $(1 \le i \le n)$, such that $\chi(U(\mathcal{T})) > c_1$, and such that for each vertex $v \in U(\mathcal{T})$, there are fewer than s values of $i \in \{1, \dots, n\}$ such that some neighbour of v in $U(\mathcal{T})$ has a neighbour in H_i. By 6.2, there is a 2-cleaned (in fact 3-cleaned) (ζ, η)-template array $\mathcal{T}'$, with sequence (Y_i, H_i') $(1 \le i \le n)$ and a privatization Π for $\mathcal{T}'$ such that

- $H_i' \subseteq H_i$ for $1 \le i \le n$;
- $U(\mathcal{T}') \subseteq U(\mathcal{T})$; and
- $\chi(U(\mathcal{T}') \setminus \Pi) \ge \chi(U(\mathcal{T})) - \delta\tau^2 > c_2$.

It follows that for each vertex $v \in U(\mathcal{T}')$, there are fewer than s values of $i \in \{1, \dots, n\}$ such that some neighbour of v in $U(\mathcal{T}')$ has a neighbour in H_i'. By 8.2 applied to $\mathcal{T}'$,

$$\chi(U(\mathcal{T}') \setminus \Pi) \le 3rs\beta\delta\zeta\tau^2,$$

a contradiction. This proves 8.3. ∎

Consequently, this completes the proof of 2.5, and hence of 1.2.

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