Graph Minors
XXIII. Nash-Williams' immersion conjecture

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Abstract

We define a quasi-order of the class of all finite hypergraphs, and prove it is a well-quasi-order. This has two corollaries of interest:

- Wagner's conjecture, proved in a previous paper, states that for every infinite set of finite graphs, one of its members is a minor of another. The present result implies the same conclusion even if the vertices or edges of the graphs are labelled from a well-quasi-order and we require the minor relation to respect the labels.

- Nash-Williams' "immersion" conjecture states that in any infinite set of finite graphs, one can be "immersed" in another; roughly, embedded such that the edges of the first graph are represented by edge-disjoint paths of the second. The present result implies this, in a strengthened form where we permit vertices to be labelled from a well-quasi-order and require the immersion to respect the labels.

1 Introduction

Let G, H be graphs. (All graphs in this paper are finite.) An *immersion* of H in G is a function α with domain $V(H) \cup E(H)$, such that:

- $\alpha(v) \in V(G)$ for all $v \in V(H)$, and $\alpha(u) \neq \alpha(v)$ for all distinct $u, v \in V(H)$
- for each edge e of H, if e has distinct ends u, v then $\alpha(e)$ is a path of G with ends $\alpha(u), \alpha(v)$, and if e is a loop incident with a vertex v then $\alpha(e)$ is a circuit of G with $\alpha(v) \in V(\alpha(e))$
- for all distinct $e, f \in E(H), E(\alpha(e) \cap \alpha(f)) = \emptyset$.

(*Paths* have at least one vertex, and no "repeated" vertices. *Circuits* have at least one edge and no "repeated" vertices.) Nash-Williams [2] proposed the following conjecture, which is one of our main results.

1.1 *For every countable sequence G_i $(i = 1, 2, ...)$ of graphs, there exist $j > i \geq 1$ such that there is an immersion of G_i in G_j.*

In fact Nash-Williams also proposed another "immersion" conjecture, in [1], with another condition in the definition of immersion, that

- for all $v \in V(H)$ and $e \in E(H)$, if e is not incident with v in H then $\alpha(v) \notin V(\alpha(e))$.

Let us call this "strong immersion". Thus there are actually two immersion conjectures, but we are only proving the weaker. It seemed to us at one time that we had a proof of the stronger, but even if it was correct it was very much more complicated, and it is unlikely that we will write it down.

We prove 1.1 as a corollary of a well-quasi-ordering result about hypergraphs. A *hypergraph* G consists of a finite set $V(G)$ of *vertices*, a finite set $E(G)$ of *edges*, and an incidence relation between them. If e is an edge, $V(e)$ or $V_G(e)$ denotes the set of vertices of G incident with e, that is, the set of *ends* of e. If V is a finite set, K_V denotes the complete graph with vertex set V, that is, $E(K_V)$ is the set of all 2-element subsets of V with the natural incidence relation.

Let H, G be hypergraphs. We say a *collapse* of G to H is a function η with domain $V(H) \cup E(H)$, such that

(i) $\eta(v)$ is a non-null connected subgraph of $K_{V(G)}$ for each $v \in V(H)$, and $\eta(u), \eta(v)$ are disjoint for all distinct $u, v \in V(H)$

(ii) $\eta(e) \in E(G)$ for all $e \in E(H)$, and $\eta(e) \neq \eta(f)$ for all distinct $e, f \in E(H)$

(iii) if $v \in V(H)$ and $e \in E(H)$ and e is incident in H with v, then $\eta(e)$ is incident in G with a vertex of $\eta(v)$

(iv) for each $v \in V(H)$ and $f \in E(\eta(v))$ with ends x, y, there is an edge e of G incident with x and y.

Note that, in (iv), it is possible that $e \in \eta(E(H))$. (If we could insist that $e \notin \eta(E(H))$ this would yield Nash-Williams' *strong* immersion conjecture.)

We shall prove

1.2 *For every countable sequence $G_i\,(i = 1, 2, ...)$ of hypergraphs there exist $j > i \geq 1$ such that there is a collapse of G_j to G_i.*

If all the hypergraphs G_i are loopless graphs (that is, every edge has two ends) and there is a collapse of G_j to G_i then G_i is isomorphic to a minor of G_j in the usual sense of graph minors; and so 1.2 contains Wagner's conjecture (see [6]), at least for loopless graphs. Thus, we would expect 1.2 to be difficult. On the other hand, most of the work has already been done in previous papers.

In fact we shall prove much more than 1.2; for instance, it is permissible to label the edges of the hypergraphs from a well-quasi-order, and ask for a collapse that respects labels; and it is possible to order, independently, the elements of each edge of bounded size, and ask for the collapse to maintain the order; and indeed, if we take hypergraphs in which all edges have bounded size, we can arrange the stronger form of (iv) above, with $e \notin \eta(E(H))$. But first let us see that even 1.2 implies the immersion conjecture for loopless graphs.

Let G be a loopless graph. Its *transpose* is the hypergraph G' with $V(G') = E(G)$ and $E(G') = V(G)$, with the same incidence relation as G. We need:

1.3 *If G, H are loopless graphs, and there is a collapse of the transpose of G to the transpose of H, then there is an immersion of H in G.*

Proof. Let the transposes of G, H be G', H' respectively, and let η be a collapse of G' to H'. Choose η such that

$$\Sigma(|E(\eta(v))| : v \in V(H'))$$

is minimum.

(1) *For each $v \in V(H')$ incident with distinct $e, f \in E(H')$, $\eta(v)$ is a path with ends incident with $\eta(e), \eta(f)$ respectively, and with no internal vertex incident with $\eta(e)$ or $\eta(f)$.*

Subproof. By condition (iii) in the definition of "collapse", there exist $x, y \in V(\eta(v))$ such that x is incident in G with $\eta(e)$ and y with $\eta(f)$. By condition (i), $\eta(v)$ is connected, and so there is a path P of $\eta(v)$ with ends x, y. Define $\eta' = \eta$ except that $\eta'(v) = P$; then η' is another collapse. (To verify condition (iii) in the definition of "collapse" we use that e, f are the only edges of H' incident in H' with v.) By the choice of η, it follows that $\eta(v) = P$; and hence the choice of x, y was unique. Consequently, no vertex of $\eta(v)$ except x is incident in G' with $\eta(e)$, and similarly for y, f. This proves (1).

Let v, e, f be as in (1). Let the vertices of the path $\eta(v)$ be $v_1, v_2, \ldots, v_k$ in order, where v_1 is an end of $\eta(e)$ and v_k is an end of $\eta(f)$. For $1 \leq i < k$, there is an edge f_i of G' incident with v_i and v_{i+1}, by condition (iv) in the definition of collapse.

(2) *$f_1, \ldots, f_{k-1}$ are distinct.*

Subproof. Suppose that $f_i = f_j$ say where $1 \leq i < j \leq k-1$. Let P be the path with vertex set $v_1, \ldots, v_i, v_{j+1}, \ldots, v_k$. Since f_i is incident with v_i and v_{j+1}, if we define $\eta' = \eta$ except that $\eta'(v) = P$ we contradict the minimality of $|E(\eta(v))|$. This proves (2).

For each $v \in V(H') = E(H)$, incident with distinct $e, f \in E(H') = V(H)$, let $\alpha(v)$ be the path of G with ends $\eta(e), \eta(f)$ and with edge set $V(\eta(v))$ and vertex set $\{\eta(e), f_1, \ldots, f_{k-1}, \eta(f)\}$ where $f_1, \ldots, f_{k-1}$ are as in (2). By (2), there is indeed such a path. For each $e \in E(H') = V(H)$, define $\alpha(e) = \eta(e)$. We claim that α satisfies conditions (i)-(iii) in the definition of "immersion". Conditions (i) and (ii) are clear, since there are no loops. For (iii), if $e, f \in E(H) = V(H')$ are distinct then $E(\alpha(e) \cap \alpha(f)) = V(\eta(e)) \cap V(\eta(f)) = \emptyset$, and so (iii) holds. This proves 1.3. ∎

From 1.3 and 1.2 it follows immediately that 1.1 holds for loopless graphs. To prove 1.1 when there may be loops we need an extension of 1.2 where the edges are labelled from a well-quasi-order. A *quasi-order* Ω is a pair $(E(\Omega), \leq)$, where $E(\Omega)$ is a set and $\leq$ is a reflexive transitive relation on $E(\Omega)$. It is a *well-quasi-order* if for every countable sequence $x_i \,(i = 1, 2, \ldots)$ of elements of $E(\Omega)$ there exist $j > i \geq 1$ with $x_i \leq x_j$. We shall prove the following strengthening of 1.2:

1.4 *Let Ω be a well-quasi-order, and for $i = 1, 2, \ldots$ let G_i be a hypergraph and $\phi_i : E(G_i) \to E(\Omega)$ some function. Then there exist $j > i \geq 1$ and a collapse η of G_j to G_i such that for all $e \in E(G_i)$, $\phi_i(e) \leq \phi_j(\eta(e))$.*

The proof that 1.2 implies 1.1 can be adapted (we omit the details) to show:

1.5 *Let Ω be a well-quasi-order, and for $i = 1, 2, \ldots$ let G_i be a loopless graph and $\phi_i : V(G_i) \to E(\Omega)$ some function. Then there exist $j > i \geq 1$ and an immersion α of G_i in G_j such that for all $v \in V(G_i)$, $\phi_i(v) \leq \phi_j(\alpha(v))$.*

In particular, 1.5 implies 1.1 for graphs which may have loops. To see this, let G be a graph, and let G' be obtained from G by deleting all loops. For each $v \in V(G')$ let $\phi(v)$ be the number of loops of G incident with v. Given a countable sequence $G_i \,(i = 1, 2, \ldots)$ of graphs, we take Ω to be the non-negative integers with their natural order, and apply 1.5 to the corresponding sequences $G'_i, \phi'_i \,(i = 1, 2, \ldots)$. Thus 1.5 implies 1.1.

Here is a statement of our most general result, expressed in terms of hypergraphs.

1.6 *Let Ω be a well-quasi-order, let $k \geq 0$ be an integer, and for $i = 1, 2, \ldots$ let G_i be a hypergraph, let $\phi_i : E(G_i) \to E(\Omega)$ be some function, and let $M_i \subseteq E(G_i)$, such that $|V(e)| \leq k$ for all $e \in M_i$. For each $e \in M_i$, let $\mu_i(e)$ be some sequence $(v_1, \ldots, v_m)$ such that $v_1, \ldots, v_m$ are all distinct and $\{v_1, \ldots, v_m\} = V(e)$. Then there exist $j > i \geq 1$ and a collapse η of G_j to G_i such that for every $e \in E(G_i)$:*

- $\phi_i(e) \leq \phi_j(\eta(e))$,
- *$\eta(e) \in M_j$ if and only if $e \in M_i$, and if so then $|V(\eta(e))| = |V(e)|$,*
- *if $e \in M_i$, let $\mu_i(e) = (v_1, \ldots, v_m)$ and $\mu_j(\eta(e)) = (u_1, \ldots, u_m)$; then $u_h \in V(\eta(v_h))$ for $1 \leq h \leq m$.*

This evidently implies 1.5, taking $M_i = \emptyset$ for each i. If $M_i = E(G_i)$ for each i and each edge of each G_i has one or two ends, it yields a version of Wagner's conjecture in which labels are permitted. Since this may be of some independent interest, let us state it explicitly:

1.7 *Let Ω be a well-quasi-order. For $i \geq 1$ let G_i be a directed graph, and let $\phi_i : V(G_i) \cup E(G_i) \rightarrow E(\Omega)$ be some function. Then there exist $j > i \geq 1$ and a map η with domain $V(G_i) \cup E(G_i)$, satisfying:*

- *for each $v \in V(G_i)$, $\eta(v)$ is a connected subgraph of G_j, and there exists $w \in V(\eta(v))$ with $\phi_i(v) \leq \phi_j(w)$; and $\eta(v) \cap \eta(v')$ is null for all distinct $v, v' \in V(G_i)$*
- *for each $e \in E(G_i)$, $\eta(e)$ is an edge of G_j, a loop if and only if e is a loop, with $\phi_i(e) \leq \phi_j(\eta(e))$; and $\eta(e) \neq \eta(e')$ for all distinct $e, e' \in E(G_i)$*
- *for each $e \in E(G_i)$ with head u and tail v, $\eta(e)$ has head in $V(\eta(u))$ and tail in $V(\eta(v))$, and $\eta(e)$ is not an edge of $\eta(u)$ or $\eta(v)$ (the last is trivial unless $u = v$).*

Proof. Let Ω' be the well-quasi-order of all pairs (x, i) where $x \in \Omega$ and $i \in \{0, 1\}$, ordered by $(x, i) \leq (y, j)$ if and only if $x \leq y$ in Ω and $i = j$. For $i \geq 1$, let H_i be the hypergraph obtained from G_i by adding a new edge $e_i(v)$ incident only with v, for each $v \in V(G_i)$. Let $\psi_i(e_i(v)) = (\phi_i(v)i, 1)$. For each edge e of G_i let $\psi_i(e) = (\phi(e), 0)$. Take $M_i = E(H_i)$; then the result follows from 1.6 applied to the H_i's and ψ_i's. This proves 1.7. ∎

2 Patchworks and other terminology

To prove 1.6 we apply a result about "patchworks", proved in [6]. This needs a great mass of definition (some five pages of [6]) and it seems pointless to repeat it all here. Thus, we refer the reader to [6] (pages 326–329, 344, 346, 353) for the meaning of the following terms: subhypergraphs, separation, order of a separation, tangle, order of a tangle, march, rooted hypergraph, rooted location, θ-isolate, tie-breaker, edge-based tie-breaker defined by f and ν, patch, free patch, grouping, (partial) Ω-patchwork, free patchwork, rootless, realizable expansion, simulation, $sk(G)$, controls an H-minor.

Let λ be a tie-breaker in a hypergraph G, let $\mathcal{T}$ be a tangle in G, and let $(A, B) \in \mathcal{T}$. We say that (A, B) is λ-*linked to* $\mathcal{T}$ if there is no $(A', B') \in \mathcal{T}$ with smaller λ-order with $A \subseteq A'$ and $B' \subseteq B$.

The definition of "θ-isolate" depends implicitly on the choice of some tie-breaker, and for clarity we prefer in this paper to make the dependence explicit. Thus we shall normally speak of "θ-isolating with respect to a tie-breaker λ ". It is convenient for inductive purposes to fix (for the remainder of the paper) two disjoint countably infinite sets Γ_1, Γ_2 of "new" elements. A well-quasi-order Ω is *proper* if $E(\Omega) \cap (\Gamma_1 \cup \Gamma_2)$ is finite and there do not exist $\gamma \in E(\Omega) \cap (\Gamma_1 \cup \Gamma_2)$ and $x \in E(\Omega)$ with $x \neq \gamma$ such that $x \leq \gamma$ or $\gamma \leq x$ in Ω.

Let Ω be proper, and let $P = (G, \mu, \Delta, \phi)$ be a partial Ω-patchwork. For $Y \subseteq \Gamma_1 \cup \Gamma_2$ we denote $\{e \in E(G) : \phi(e) \in Y\}$ by $\phi^{-1}(Y)$; and for $\gamma \in \Gamma_1 \cup \Gamma_2$ we often write $\phi^{-1}(\gamma)$ for $\phi^{-1}(\{\gamma\})$. For $X \subseteq E(G)$, we denote $\{\phi(e) : e \in X \cap dom(\phi)\}$ by $\phi(X)$.

If $P = (G, \mu, \Delta, \phi)$ is a partial Ω-patchwork, a non-null set X of edges of G is a *star* of G if

- for every $e \in X$, $e \in dom(\mu)$ and $|V(e)| = 2$, and
- for some vertex v of G, v is the first term of $\mu(e)$ for all $e \in X$.

We call v the *centre* of the star; it is unique. Let Ω be proper. A partial Ω-patchwork $P = (G, \mu, \Delta, \phi)$ is *proper* if

- for each $\gamma \in \Gamma_1 \cap \phi(E(G))$, there is exactly one $e \in E(G)$ with $\phi(e) = \gamma$, and it satisfies $e \in dom(\mu)$ and $|V(e)| = 1$, and
- for each $\gamma \in \Gamma_2 \cap \phi(E(G))$, $\phi^{-1}(\gamma)$ is a star of G.

We denote by $V(\phi^{-1}(\Gamma_1))$ the set of vertices of G incident with members of $\phi^{-1}(\Gamma_1)$. We denote by $V(\phi^{-1}(\Gamma_2))$ the set of centres of the stars $\phi^{-1}(\gamma)(\gamma \in \Gamma_2 \cap \phi(E(G)))$.

Let $P = (G, \mu, \Delta, \phi)$ be a proper partial Ω-patchwork. If $u, v \in V(G)$ are distinct, we say $\{u, v\}$ is a *muscle* of P if

- there exists $e \in E(G) \setminus dom(\mu)$ with $u, v \in V(e)$, and
- there is no $e \in dom(\mu)$ with $V(e) = \{u, v\}$.

We say that P is *skeletal* if it is free, rootless, and proper, and every muscle is a subset of $V(\phi^{-1}(\Gamma_1))$. Our main result is:

2.1 *Let Ω be a proper well-quasi-order, and let $k \geq 0$ be an integer. Every set of skeletal Ω-patchworks (G, μ, Δ, ϕ) such that $|V(e)| \leq k$ for all $e \in dom(\mu)$, is well-quasi-ordered by simulation.*

Proof of 1.6, *assuming* 2.1. Let $\Omega, k, G_i, \phi_i, M_i, \mu_i$ $(i = 1, 2, \ldots)$ be as in 1.6. We may assume that $E(\Omega) \neq \emptyset$. Let Ω' be the well-quasi-order with

$$E(\Omega') = \{(x, 0) : x \in E(\Omega)\} \cup \{(x, 1) : x \in E(\Omega)\}$$

where $(x, a) \leq (y, b)$ if and only if $a = b$ and $x \leq y$ in Ω. We may assume that $E(\Omega') \cap (\Gamma_1 \cup \Gamma_2) = \emptyset$, by replacing Ω by an isomorphic quasi-order. For each $i \geq 1$, let $P'_i = (G'_i, \mu'_i, \Delta'_i, \phi'_i)$ be an Ω'-patchwork, defined as follows:

- G'_i is the rooted hypergraph with $V(G'_i) = V(G_i)$, $E(G'_i) = E(G_i) \cup L(G_i)$, and $\pi(G'_i) = 0$, with the natural incidence relation, where $L(G_i)$ is the set of all unordered pairs $\{u, v\}$ of distinct vertices of G_i such that $u, v \in V(e)$ for some $e \in E(G_i) \setminus M_i$ (we assume for notational convenience that $E(G_i) \cap L(G_i) = \emptyset$)
- $dom(\mu'_i) = M_i \cup L(G_i)$; for each $e \in M_i$, $\mu'_i(e) = \mu_i(e)$, and for each $e = \{u, v\} \in L(G_i)$, $\mu'_i(e)$ is (u, v) or (v, u)
- P'_i is free (this determines Δ'_i)
- for each $e \in E(G_i), \phi'_i(e) = (\phi_i(e), 0)$; and for $e \in L(G_i)$, $\phi'_i(e) = (x, 1)$ where $x \in E(\Omega)$ is arbitrary.

Then each P'_i is skeletal. Moreover, for each $i \geq 1$ and each $e \in dom(\mu'_i)$, $|V(e)| \leq max(k, 2)$, since $|V(e)| \leq k$ if $e \in M_i$, and $|V(e)| = 2$ otherwise. Hence by 2.1 there exist $j > i \geq 1$ such that P'_i is simulated in P'_j. Let η' be a realizable expansion of P'_i in P'_j, and let H be a realization of $P'_j \backslash \eta'(E(G'_i))$ realizing η'. For $v \in V(G_i)$, let $\eta(v)$ be the component of H with vertex set $\eta'(v)$, and for $e \in E(G_i)$ let $\eta(e) = \eta'(e)$; we claim that η is a collapse of G_j to G_i, satisfying 1.6. Certainly condition (i) in the definition of "collapse" holds. For (ii), let $e_1 \in E(G_i)$. Then $\phi'_i(e_1) = (\phi_i(e_1), 0)$. Let $\eta(e_1) = e_2$ and let $\phi'_j(e_2) = (a, b)$; then since

$$(\phi_i(e_1), 0) = \phi'_i(e) \leq \phi'_j(e_2) = (a, b)$$

it follows that $b = 0$ and $\phi_i(e_1) \le a$. Since $b = 0$ we deduce that $e_2 \in E(G_j)$, and so $a = \phi_j(e_2)$, and therefore $\phi_i(e_1) \le \phi_j(e_2)$. Hence statement (ii) of the definition of "collapse" holds, and moreover $\phi_i(e) \le \phi_j(\eta(e))$ for all $e \in E(G_i)$.

For (iii), let $e \in E(G_i)$ be incident with $v \in V(G_i)$; then since $e \in E(G'_i)$, it follows from the definition of an expansion that $\eta'(v)$ contains an end of $\eta'(e)$ in G'_j, and so (iii) holds. For (iv), let $v \in V(G_i)$ and $f \in E(\eta(v))$ with ends x, y. Since $f \in E(sk(G_j^{'-}))$, there is an edge of G'_j incident with x and y, and so (from the definition of G'_j) there is an edge of G_j incident with x and y. Hence (iv) holds.

Consequently, η is indeed a collapse of G_j to G_i. We must check the three statements of 1.6; but we have already seen the first, and the other two are immediate from the definition of "expansion". This proves 1.6. ∎

Our method of proof of 2.1 is to apply theorem 11.2 of [6], 2.2 below. The main part of the paper is devoted to constructing an appropriate set $\mathcal{C}$ such that 2.2 is satisfied.

2.2 *Let Ω be a well-quasi-order, let $\mathcal{C}$ be a well-behaved set of partial Ω-patchworks, and let $\theta \ge 1$ and $n \ge 0$. Let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2, \ldots)$ be a countable sequence of free rootless Ω-patchworks. For each $i \ge 1$, let λ_i be an edge-based tie-breaker in G_i. Suppose that for each $i \ge 1$ and each tangle $\mathcal{T}$ in G_i of order $\ge \theta$ which controls a K_n-minor of $sk(G_i^-)$, there is a rooted location $\mathcal{L}$ in G_i which θ-isolates $\mathcal{T}$ with respect to λ_i, such that $(P_i, \mathcal{L})$ has a heart in $\mathcal{F}$. Then there exist $j > i \ge 1$ such that P_i is simulated in P_j.*

3 The induction

Let Ω, Ω' be quasi-orders. We write $\Omega \subseteq \Omega'$ if $E(\Omega) \subseteq E(\Omega')$ and for $x, y \in E(\Omega)$, $x \le y$ in Ω if and only if $x \le y$ in Ω'. We say Ω is an *ideal* of Ω', and write $\Omega \le \Omega'$, if $\Omega \subseteq \Omega'$ and Ω is "closed downwards", that is, there do not exist $x \in E(\Omega') \setminus E(\Omega)$ and $y \in E(\Omega)$ with $x \le y$ in Ω'. If $\Omega \le \Omega'$ and $\Omega \ne \Omega'$ we write $\Omega < \Omega'$. We need the following well-known lemma (we omit the proof, which is elementary):

3.1 *There is no countable sequence Ω_i $(i = 1, 2, \ldots)$ such that Ω_1 is a well-quasi-order and $\Omega_{i+1} < \Omega_i$ for all $i \ge 1$.*

If Ω is a quasi-order and $X \subseteq E(\Omega)$, the unique minimal ideal of Ω containing X is called the *ideal generated by* X. A *shadow* is a finite sequence

$$(\Omega_\infty, m, \Omega_m, \Omega_{m-1}, \ldots, \Omega_1, R_2, R_1)$$

where $m \ge 2$ is an integer, R_2 is a finite subset of Γ_2, R_1 is a finite subset of Γ_1, and Ω_∞ and $\Omega_m, \ldots, \Omega_1$ are proper well-quasi-orders, each with no element in $\Gamma_1 \cup \Gamma_2$. If

$$\Sigma = (\Omega_\infty, m, \Omega_m, \Omega_{m-1}, \ldots, \Omega_1, R_2, R_1)$$

is a shadow, we define $\Omega_\infty(\Sigma) = \Omega_\infty$, $m(\Sigma) = m$, etc. We order shadows lexicographically; thus, if

$$\begin{aligned} \Sigma &= (\Omega_\infty, m, \Omega_m, \Omega_{m-1}, \ldots, \Omega_1, R_2, R_1) \\ \Sigma' &= (\Omega'_\infty, m', \Omega'_{m'}, \Omega'_{m'-1}, \ldots, \Omega'_1, R'_2, R'_1) \end{aligned}$$

are shadows, we write $\Sigma \le \Sigma'$ if either:

- $\Omega_\infty < \Omega'_\infty$, or
- $\Omega_\infty = \Omega'_\infty$ and $m < m'$, or
- $\Omega_\infty = \Omega'_\infty, m = m'$, and for some k with $1 \le k \le m$, $\Omega_k < \Omega'_k$ and $\Omega_i = \Omega'_i$ for $k < i \le m$, or
- $\Omega_\infty = \Omega'_\infty, m = m'$, $\Omega_i = \Omega'_i$ for $1 \le i \le m$, and $R_2 \subset R'_2$, or
- $\Omega_\infty = \Omega'_\infty, m = m'$, $\Omega_i = \Omega'_i$ for $1 \le i \le m$, $R_2 = R'_2$, and $R_1 \subseteq R'_1$.

The relation $\le$ is transitive on shadows, as is easily seen. If $\Sigma \le \Sigma'$ and $\Sigma \ne \Sigma'$ we write $\Sigma < \Sigma'$. It follows from 3.1 (again, we omit the elementary proof) that

3.2 *There is no countable sequence $\Sigma_i\,(i = 1, 2, \dots)$ of shadows such that $\Sigma_{i+1} < \Sigma_i$ for all $i \ge 1$.*

Now let Ω be a well-quasi-order, and let $\mathcal{S}$ be a set of partial Ω-patchworks. We define $ht(\mathcal{S})$ to be the minimum integer $k \ge 0$ such that $|V(e)| \le k$ for all $(G, \mu, \Delta, \phi) \in \mathcal{S}$ and all $e \in dom(\mu)$. (If there is no such k then $ht(\mathcal{S})$ is undefined.) If $ht(\mathcal{S})$ exists, $\mathcal{S}$ is said to be *limited*.

Let Ω be a proper well-quasi-order, and let $\mathcal{S}$ be a limited set of proper partial Ω-patchworks. We define the *shadow of* $\mathcal{S}$ to be

$$(\Omega_\infty, m, \Omega_m, \dots, \Omega_1, R_2, R_1)$$

where:

- Ω_∞ is the ideal of Ω generated by
$$\{\phi(e) : (G, \mu, \Delta, \phi) \in \mathcal{S},\ e \in dom(\phi) \setminus dom(\mu)\}$$
- $m = ht(\mathcal{S})$
- for $3 \le h \le m, \Omega_h$ is the ideal of Ω generated by
$$\{\phi(e) : (G, \mu, \Delta, \phi) \in \mathcal{S},\ e \in dom(\phi) \cap dom(\mu),\ |V(e)| = h\}$$
- Ω_2 is the ideal generated by
$$\{\phi(e) : (G, \mu, \Delta, \phi) \in \mathcal{S},\ e \in dom(\phi) \cap dom(\mu) \setminus \phi^{-1}(\Gamma_2),\ |V(e)| = 2\}$$
- Ω_1 is the ideal generated by
$$\{\phi(e) : (G, \mu, \Delta, \phi) \in \mathcal{S},\ e \in dom(\phi) \cap dom(\mu) \setminus \phi^{-1}(\Gamma_1),\ |V(e)| = 1\}$$
- $R_2 = \bigcup(\Gamma_2 \cap \phi(E(G)) : (G, \mu, \Delta, \phi) \in \mathcal{S})$
- $R_1 = \bigcup(\Gamma_1 \cap \phi(E(G)) : (G, \mu, \Delta, \phi) \in \mathcal{S})$.

A sequence $P_i\,(i = 1, 2, \dots)$ of proper partial Ω-patchworks is said to be *limited* if $\{P_i : i \ge 1\}$ is limited, and its *shadow* is the shadow of $\{P_i : i \ge 1\}$. A sequence $P_i\,(i = 1, 2, \dots)$ of Ω-patchworks is *bad* if there do not exist $j > i \ge 1$ such that P_i is simulated in P_j; and a shadow Σ is *evil* if there is some proper well-quasi-order Ω and some bad sequence of skeletal Ω-patchworks with shadow Σ. A shadow Σ is *sharp* if Σ is evil and no shadow $\Sigma' < \Sigma$ is evil. Our objective to prove that no shadow is evil (for this evidently implies 2.1); and to do so it suffices (because of 3.2) to prove that no shadow is sharp. Proving that no shadow is sharp is the objective of the remainder of the paper.

4 A sufficient condition for simulation

Let G be a hypergraph, and let $\mathcal{T}$ be a tangle in G of order θ. A subset $X \subseteq V(G)$ is *free relative to* $\mathcal{T}$ if $|X| \leq \theta$ and there is no $(A, B) \in \mathcal{T}$ of order $< |X|$ with $X \subseteq V(A)$. Theorem 12.2 of [3] asserts the following.

4.1 *Let G be a hypergraph, and let $\mathcal{T}$ be a tangle in G of order θ. For $X \subseteq V(G)$, let $r(X)$ be the least order of a separation $(A, B) \in \mathcal{T}$ with $X \subseteq V(A)$, if one exists, and otherwise $r(X) = \theta$. The free sets relative to $\mathcal{T}$ are the independent sets of a matroid on $V(G)$ with rank function r.*

Now let P be a rootless Ω-patchwork, let $\mathcal{T}$ be a tangle in G of order θ, let $\xi \in E(\Omega)$, and let $h \geq 1$ and $w \geq 0$ be integers. We say that $\mathcal{T}$ is

- (ξ, h, w)-*restricted internally* if there exists $W \subseteq V(G)$ with $|W| \leq w$ such that W is free relative to $\mathcal{T}$, and there is no edge e of G such that $e \in dom(\mu), |V(e)| = h$, $\xi \leq \phi(e)$, $V(e) \cap W = \emptyset$, and $V(e) \cup W$ is free relative to $\mathcal{T}$
- (ξ, h, w)-*restricted externally* if there are at most w edges e of G such that $e \notin dom(\mu)$, $\xi \leq \phi(e)$ and there exists $X \subseteq V(e)$ with $|X| = h$ free relative to $\mathcal{T}$.

Let $P = (G, \mu, \Delta, \phi)$ be a skeletal Ω-patchwork. For $\gamma \in \Gamma_1 \cap \phi(E(G))$ we denote the unique vertex of the unique edge e of G with $\phi(e) = \gamma$ by $\phi^{-2}(\gamma)$. For $\gamma \in \Gamma_2 \cap \phi(E(G))$, we denote the centre of the star $\phi^{-1}(\gamma)$ by $\phi^{-2}(\gamma)$. A *null edge* of P is an edge $e \in E(G)$ with $V(e) = \emptyset$.

4.2 *Let Ω be a proper well-quasi-order, and let $P_0 = (G_0, \mu_0, \Delta_0, \phi_0)$ be a skeletal Ω-patchwork with no null edge. Let $n > \frac{3}{2}|V(G_0)|\,(|E(G_0)| + 2)$ be an integer. Now let $P = (G, \mu, \Delta, \phi)$ be a skeletal Ω-patchwork with no null edge, and let $\mathcal{T}$ be a tangle in G of order $> n$, controlling a K_n-minor of $sk(G^-)$. Suppose that the following five conditions holds:*

- *$\Gamma_1 \cap \phi_0(E(G_0)) = \Gamma_1 \cap \phi(E(G))$, and all the corresponding singleton edges are pairwise disjoint, that is,*
$$|V(\phi_0^{-1}(\Gamma_1))| = |V(\phi^{-1}(\Gamma_1))| = |\Gamma_1 \cap \phi(E(G))|$$
- *$\Gamma_2 \cap \phi_0(E(G_0)) = \Gamma_2 \cap \phi(E(G))$, $V(\phi^{-1}(\Gamma_1)) \cap V(\phi^{-1}(\Gamma_2)) = \emptyset$ and for all distinct $\gamma, \gamma' \in \Gamma_2 \cap \phi(E(G))$, if $\phi^{-2}(\gamma) = \phi^{-2}(\gamma')$ then $\phi_0^{-2}(\gamma) = \phi_0^{-2}(\gamma')$*
- *$V(\phi^{-1}(\Gamma_1)) \cup V(\phi^{-1}(\Gamma_2))$ is free relative to $\mathcal{T}$, and for each $\gamma \in \Gamma_2 \cap \phi(E(G))$, there is no $(A, B) \in \mathcal{T}$ of order $\leq n$ with $\phi^{-1}(\gamma) \subseteq E(A)$*
- *for each $e \in E(G_0) \setminus dom(\mu_0), \mathcal{T}$ is not $(\phi_0(e), n, |E(G_0)|)$-restricted externally, and*
- *for each $e \in dom(\mu_0), \mathcal{T}$ is not $(\phi_0(e), |V(e)|, n)$-restricted internally.*

Then P_0 is simulated in P.

Proof. For $\gamma \in \Gamma_2$, let $N(\gamma)$ denote the set of vertices v of G such that $V(e) = \{\phi^{-2}(\gamma), v\}$ for some $e \in \phi^{-1}(\gamma)$. Let $D_0 = V(\phi^{-1}(\Gamma_1)) \cup V(\phi^{-1}(\Gamma_2))$. By hypothesis, D_0 is free relative to $\mathcal{T}$. For each $\gamma \in \Gamma_2 \cap \phi(E(G))$ choose $D(\gamma) \subseteq N(\gamma)$ such that

- for each γ, $|D(\gamma)| \le |\phi_0^{-1}(\gamma)|$, and
- D_0 and the sets $D(\gamma)$ are all pairwise disjoint and their union, U say, is free relative to $\mathcal{T}$.

(This is possible by taking each $D(\gamma) = \emptyset$.) Choose such $D(\gamma)$, U with U maximal.

(1) *For each* $\gamma \in \Gamma_2 \cap \phi(E(G))$, $|D(\gamma)| = |\phi_0^{-1}(\gamma)|$.

Subproof. The sets $\phi_0^{-1}(\gamma)$ ($\gamma \in \Gamma_2 \cap \phi(E(G))$) are pairwise disjoint subsets of $E(G_0)$, and so the sum of their cardinalities is at most $|E(G_0)|$. Consequently

$$|U| \le |D_0| + |E(G_0)| \le |V(G_0)| + |E(G_0)| < n.$$

Suppose that $\gamma \in \Gamma_2 \cap \phi_2(E(G))$ and $|D(\gamma)| < |\phi_0^{-1}(\gamma)|$. But by hypothesis, there is no separation $(A, B) \in \mathcal{T}$ of order $< n$ with $\phi^{-1}(\gamma) \subseteq E(A)$, and hence $N(\gamma) \cup \{\phi^{-2}(\gamma)\}$ has rank $\ge n$ in the matroid defined by the free subsets of $V(G)$ relative to $\mathcal{T}$ by 4.1. Hence there exists $v \in N(\gamma) \setminus U$ such that $U \cup \{v\}$ is free relative to $\mathcal{T}$; but then adding v to $D(\gamma)$ contradicts the maximality of U. This proves (1).

Consequently, there is a subset $F \subseteq E(G_0)$ and an injection $\eta : F \to E(G)$, and for each $f \in F \setminus dom(\mu_0)$ a subset X_f of $V(G)$, with the following properties:

- $\phi_0^{-1}(\Gamma_1) \cup \phi_0^{-1}(\Gamma_2) \subseteq F$
- $\phi_0(f) \le \phi(\eta(f))$ for all $f \in F$
- $\eta(f) \in dom(\mu)$ and $|V(\eta(f))| = |V(f)|$ for all $f \in F \cap dom(\mu_0)$
- $\eta(f) \in E(G) \setminus dom(\mu)$ and $X_f \subseteq V(\eta(f))$ and $|X_f| = |V(f)|$, for all $f \in F \setminus dom(\mu_0)$
- for all distinct $f, f' \in F$
  - if $f, f' \notin dom(\mu_0)$ then $X_f \cap X_{f'} = \emptyset$;
  - if $f \in dom(\mu_0)$ and $f' \notin dom(\mu_0)$ then $V(\eta(f)) \cap X_{f'} = \emptyset$; and
  - if $f, f' \in dom(\mu_0)$, then either $V(\eta(f)) \cap V(\eta(f')) = \emptyset$ or all the following hold:
    - $\phi_0(f), \phi_0(f') \in \Gamma_2$, and therefore the sequences $\mu_0(f), \mu_0(f'), \mu(\eta(f)), \mu(\eta(f'))$ all have length 2
    - $\mu_0(f), \mu_0(f')$ have the same first term
    - $\mu(\eta(f)), \mu(\eta(f'))$ have the same first term v say, and $V(\eta(f)) \cap V(\eta(f')) = \{v\}$.
- the union of all the sets $V(\eta(f))$ ($f \in F \cap dom(\mu_0)$) and X_f ($f \in F \setminus dom(\mu_0)$) is free relative to $\mathcal{T}$.

(To see this, set $F = \phi_0^{-1}(\Gamma_1) \cup \phi_0^{-1}(\Gamma_2)$ and use (1).) Choose such F, η and sets X_f with F maximal, and let W be the union of all the sets $V(\eta(f))$ ($f \in F \cap dom(\mu_0)$) and X_f ($f \in F \setminus dom(\mu_0)$). Since $|X_f| = |V(f)| \le |V(G_0)|$ for $f \in F \setminus dom(\mu_0)$ and $|V(\eta(f))| = |V(f)| \le |V(G_0)|$ for $f \in F \cap dom(\mu_0)$, it follows that

$$|W| \le |E(G_0)||V(G_0)|,$$

and $|W| \leq (|E(G_0)| - 1)|V(G_0)|$ if $F \neq E(G_0)$.

(2) *$dom(\mu_0) \subseteq F$.*

Subproof. Suppose that $dom(\mu_0) \not\subseteq F$, and let $f \in dom(\mu_0) \setminus F$. By the final condition of the theorem, $\mathcal{T}$ is not $(\phi_0(f), |V(f)|, n)$-restricted internally. But $|W| \leq n$ and W is free relative to $\mathcal{T}$, and so there is an edge e of G such that $e \in dom(\mu), |V(e)| = |V(f)|$, $\phi_0(f) \leq \phi(e)$, $V(e) \cap W = \emptyset$, and $V(e) \cup W$ is free relative to $\mathcal{T}$. Suppose that $e = \eta(f')$ for some $f' \in F$. Since $V(e) \cap W = \emptyset$, and $V(\eta(f')) \subseteq W$, it follows that $V(e) = \emptyset$, contrary to hypothesis. Thus $e \notin \eta(F)$, and so we may define $\eta(f) = e$ and add f to F, contradicting the maximality of F. This proves (2).

(3) *$F = E(G_0)$.*

Subproof. Suppose not; then by (2) there exists $f \in E(G_0)$ with $f \notin dom(\mu_0) \cup F$. By the fourth condition of the theorem, $\mathcal{T}$ is not $(\phi_0(f), n, |E(G_0)|)$-restricted externally. Hence there are at least $|E(G_0)| + 1$ edges e of G such that $e \notin dom(\mu)$, $\phi_0(f) \leq \phi(e)$, and there exists $X \subseteq V(e)$ with $|X| = n$ such that X is free relative to $\mathcal{T}$. Choose some one of these edges, e say, such that $e \notin F$ (this is possible since $|F| \leq |E(G_0)|$), and let X be as above. Now the subsets of $V(G)$ free relative to $\mathcal{T}$ are the independent sets of a matroid, by 4.1; and so since W and X are both free, and $|X| = n \geq |W| + |V(f)|$, there exists $X_f \subseteq X \setminus W$ with $|X_f| = |V(f)|$ such that $W \cup X_f$ is free relative to $\mathcal{T}$. But then setting $\eta(f) = e$ and adding f to F contradicts the maximality of F. This proves (3).

For each $v \in V(G_0)$ and each $e \in E(G_0)$ incident with v, let $\beta(v, e) \in V(G)$, such that

- if $e \in dom(\mu_0)$ and v is the ith vertex of $\mu_0(e)$ then $\beta(v, e)$ is the ith vertex of $\mu(\eta(e))$
- if $e \notin dom(\mu_0)$ then $\beta(v, e) \in X_e$

in such a way that every vertex in W is $\beta(v, e)$ for some choice of v, e. (Note that if $(v_1, e_1) \neq (v_2, e_2)$, then $\beta(v_1, e_1) = \beta(v_2, e_2)$ only if $v_1 = v_2 \in V(\phi_0^{-1}(\Gamma_2))$.)

For each $v \in V(G_0)$ choose $w(v) \in V(G) \setminus W$, such that all the vertices $w(v)\,(v \in V(G_0))$ are distinct, and

$$W' = W \cup \{w(v) : v \in V(G_0)\}$$

is free relative to $\mathcal{T}$. (This is possible by 4.1, since $\mathcal{T}$ has order $> n \geq |W| + |V(G_0)|$.) For each $v \in V(G_0)$ let

$$\alpha(v) = \{w(v)\} \cup \{\beta(v, e) : e \in E(G_0) \text{ is incident with } v\}.$$

Thus, the sets $\alpha(v)\,(v \in V(G_0))$ form a partition of W' into non-empty sets. Since W' is free relative to $\mathcal{T}$, and $\mathcal{T}$ controls a K_n-minor of $sk(G^-)$, and $n \geq \frac{3}{2}|W'|$, by theorem 5.4 of [4] it follows that there are disjoint connected subgraphs $\eta(v)\,(v \in V(G_0))$ of $sk(G^-)$ such that

$$W' \cap V(\eta(v)) = \alpha(v)\,(v \in V(G_0)).$$

(4) *For each $v \in V(G_0)$, and every edge xy of $\eta(v)$, there is an edge $e \in E(G) \setminus \eta(E(G_0))$ with $x, y \in V(e)$.*

Subproof. Certainly there is an edge $e \in E(G)$ with $x, y \in V(e)$, since $xy \in E(sk(G^-))$. Suppose that $x, y \in V(\phi^{-1}(\Gamma_1))$. Since η maps $\phi_0^{-1}(\Gamma_1)$ bijectively onto $\phi^{-1}(\Gamma_1)$, there are edges a, b of $\phi_0^{-1}(\Gamma_1)$ such that $V(\eta(a)) = \{x\}$ and $V(\eta(b)) = \{y\}$. Since $xy \in E(\eta(v))$ it follows that $V(\eta(v))$ meets both $V(\eta(a))$ and $V(\eta(b))$, and so, since $V(\eta(a)), V(\eta(b)) \subseteq W'$, we deduce that $\alpha(v)$ meets both $V(\eta(a))$ and $V(\eta(b))$. Hence, from the definition of $\alpha(v)$, it follows that a and b are both incident in G_0 with v, contradicting that $|V(\phi_0^{-1}(\Gamma_1))| = |\phi_0^{-1}(\Gamma_1)|$. Consequently, not both x, y belong to $V(\phi^{-1}(\Gamma_1))$. Since P is skeletal it follows that $\{x, y\}$ is not a muscle of P, and so e may be chosen to belong to $dom(\mu)$. Suppose that $e = \eta(f)$ for some $f \in E(G_0)$. Then $f \in dom(\mu_0)$, and $|V(e)| = |V(f)| = k$ say. Now $x, y \in V(\eta(f))$, and $V(\eta(f)) \subseteq W$ since $f \in dom(\mu_0)$, and since $W' \cap V(\eta(v)) = \alpha(v)$, it follows that $x, y \in \alpha(v)$. But $\beta(v, f)$ is the only member of $\alpha(v) \cap V(\eta(f))$ (since $f \in dom(\mu_0)$) and so $x = y$, a contradiction. Hence $e \notin \eta(E(G_0))$. This proves (4).

Since P is free, it follows from (4) that η is a realizable expansion of P_0 in P, and so P_0 is simulated in P. This proves 4.2. ∎

5 Freeing the roots

4.2 gives us five conditions which together are sufficient to force P_0 to be simulated in P. When P_0 and P are two terms of a bad sequence, one of the five conditions must therefore fail; and in each case there turns out to be a suitable well-behaved set satisfying 2.2. We consider the five possibilities separately. In this section we handle the failures of the first three conditions.

Let Ω be a proper well-quasi-order, and let $P = (G, \mu, \Delta, \phi)$ be a rootless proper partial Ω-patchwork. Let $v \in V(G)$, and let $\gamma \in \Gamma_1 \cap E(\Omega) \setminus \phi(E(G))$. Let $P' = (G', \mu', \Delta', \phi')$ be the partial Ω-patchwork defined as follows:

- $V(G') = V(G), E(G') = E(G) \cup \{f\}$ where f is a new element, $\pi(G') = 0$, G^- is a subhypergraph of G'^-, and the only end of f in G' is v
- $dom(\mu') = dom(\mu) \cup \{f\}$; for $e \in dom(\mu), \mu'(e) = \mu(e)$, and $\mu'(f) = (v)$
- for $e \in E(G), \Delta'(e) = \Delta(e)$, and $\Delta(f)$ is free
- $dom(\phi') = dom(\phi) \cup \{f\}$; for $e \in dom(\phi), \phi'(e) = \phi(e)$, and $\phi'(f) = \gamma$.

We say that P' is obtained from P by *tieing v to γ*.

5.1 *Let Ω be a proper well-quasi-order, and let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2)$ be proper Ω-patchworks. Let*

$$\gamma \in \Gamma \cap E(\Omega) \setminus (\phi_1(E(G_1)) \cup \phi_2(E(G_2))).$$

Let $v_i \in V(G_i)$ $(i = 1, 2)$, and for $i = 1, 2$ let P'_i be obtained from P_i by tieing v_i to γ. If η' is a realizable expansion of P'_1 in P'_2, then $\eta'(f_1) = f_2$, where f_1, f_2 are the new elements, and the restriction η of η' to G_1 is a realizable expansion of P_1 in P_2 satisfying $v_2 \in \eta(v_1)$.

The proof of 5.1 is easy and we omit it.

If Ω is a proper well-quasi-order and $\Gamma' \subseteq \Gamma_1 \cup \Gamma_2$ is finite with $\Gamma' \cap E(\Omega) = \emptyset$, we denote by $\Omega \cup \Gamma'$ the well-quasi-order Ω' with $E(\Omega') = E(\Omega) \cup \Gamma'$, in which $x \leq y$ for distinct $x, y \in E(\Omega')$ if and only if $x, y \in E(\Omega)$ and $x \leq y$ in Ω. We see that $\Omega' = \Omega \cup \Gamma'$ is also proper, and $\Omega \leq \Omega'$.

Now let Ω, Ω' be proper well-quasi-orders with $\Omega \leq \Omega'$. Let $P' = (G, \mu, \Delta, \phi')$ be an Ω'-completion of a proper partial Ω-patchwork $P = (G, \mu, \Delta, \phi)$. We say P' is a *strict* Ω'-completion of P if it is proper and $\phi'(e) \notin E(\Omega)$ for all $E(G') \setminus dom(\phi)$. The next lemma says roughly that to check if a set is well-behaved, it is enough to examine strict completions.

5.2 *Let Ω be a proper well-quasi-order, and let $\mathcal{C}$ be a set of proper partial Ω-patchworks. Suppose that for every proper well-quasi-order Ω' with $\Omega \leq \Omega'$ and $E(\Omega') \cap (\Gamma_1 \cup \Gamma_2) \subseteq E(\Omega)$, there is no bad sequence of strict Ω'-completions of members of $\mathcal{C}$. Then $\mathcal{C}$ is well-behaved.*

Proof. We must show that if Ω' is a well-quasi-order with $\Omega \subseteq \Omega'$, and $P'_i = (G_i, \mu_i, \Delta_i, \phi'_i)$ is an Ω'-completion of $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ for $i = 1, 2, \ldots$, then P'_i $(i = 1, 2, \ldots)$ is not a bad sequence.

For each $x \in E(\Omega')$ let x^* be a new element (not in $\Gamma_1 \cup \Gamma_2$), and let Ω'' be the well-quasi-order with $E(\Omega'') = E(\Omega) \cup \{x^* : x \in E(\Omega')\}$ in which

- for $x, y \in E(\Omega), x \leq y$ in Ω'' if and only $x \leq y$ in Ω
- for $x, y \in E(\Omega')$, $x^* \leq y^*$ in Ω'' if and only if $x \leq y$ in Ω'
- for $x \in E(\Omega)$ and $y \in E(\Omega')$, $x \not\leq y^*$ in Ω'' and $y^* \not\leq x$ in Ω''.

Thus Ω'' is proper, $\Omega \leq \Omega''$, and $(\Gamma_1 \cup \Gamma_2) \cap E(\Omega'') \subseteq E(\Omega)$. For each $i \geq 1$, let $\phi''_i(e) = \phi_i(e)$ for all $e \in dom(\phi_i)$, and $\phi''_i(e) = (\phi'_i(e))^*$ for all $e \in E(G_i) \setminus dom(\phi_i)$. Then each $P''_i = (G_i, \mu_i, \Delta_i, \phi''_i)$ is a strict Ω''-completion of P_i, and so, by hypothesis, there exist $j > i \geq 1$ such that P''_i is simulated in P''_j. But then it follows easily that P'_i is simulated in P'_j. This proves 5.2. ∎

5.3 *Let Ω be a proper well-quasi-order, and let Σ be a sharp shadow. Let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2, \ldots)$ be a bad sequence of skeletal Ω-patchworks, with shadow $\Sigma' \leq \Sigma$. Then $\Sigma' = \Sigma$, and there exist $\Gamma_1^* \subseteq \Gamma_1$ and $\Gamma_2^* \subseteq \Gamma_2$ such that for all except finitely many values of i,*

$$\begin{aligned} \Gamma_1 \cap \phi_i(E(G_i)) &= \Gamma_1^* \\ \Gamma_2 \cap \phi_i(E(G_i)) &= \Gamma_2^* \\ |V(\phi_i^{-1}(\Gamma_1))| &= |\phi_i^{-1}(\Gamma_1)|, \end{aligned}$$

and

$$V(\phi_i^{-1}(\Gamma_1)) \cap V(\phi_i^{-1}(\Gamma_2)) = \emptyset.$$

Proof. Certainly $\Sigma' = \Sigma$ since Σ is sharp. For each $i \geq 1$, let $\Gamma_j(i) = \Gamma_j \cap \phi_i(E(G_i))$ $(j = 1, 2)$. Let R_2, R_1 be the last two terms of Σ; then $R_j = \bigcup(\Gamma_j(i) : i \geq 1)$ $(j = 1, 2)$.

(1) $\Gamma_1(i) = R_1$ *and* $\Gamma_2(i) = R_2$ *for all* $i \geq 1$ *except finitely many.*

Subproof. Suppose not; then there are $\Gamma'_1 \subseteq R_1$ and $\Gamma'_2 \subseteq R_2$ with either $\Gamma'_1 \neq R_1$ or $\Gamma'_2 \neq R_2$, such that for infinitely many $i \geq 1$, $\Gamma_1(i) = \Gamma'_1$ and $\Gamma_2(i) = \Gamma'_2$. But then the corresponding subsequence of $P_i (i = 1, 2, \ldots)$ is bad and has shadow $< \Sigma$, a contradiction. This proves (1).

(2) $|V(\phi_i^{-1}(\Gamma_1))| = |\phi_i^{-1}(\Gamma_1)|$ *and* $V(\phi_i^{-1}(\Gamma_1)) \cap V(\phi_i^{-1}(\Gamma_2)) = \emptyset$ *for all* $i \geq 1$ *except finitely many.*

Subproof. Suppose not; then there exist distinct $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_1 \cup \Gamma_2$ such that $\phi_i^{-2}(\gamma_1) = \phi_i^{-2}(\gamma_2)$ for infinitely many values of $i \geq 1$. By restricting to a subsequence, we may therefore assume that $\phi_i^{-2}(\gamma_1) = \phi_i^{-2}(\gamma_2)$ for all $i \geq 1$. For all $i \geq 1$, let $f_i \in E(G_i)$ with $\phi_i(f_i) = \gamma_1$, and let P'_i be obtained from P_i by deleting f_i. Then the sequence $P'_i\,(i = 1, 2, \ldots)$ has shadow $< \Sigma$, and each P'_i is skeletal, so there exist $j > i \geq 1$ such that P'_i is simulated in P'_j. Let η be a realizable expansion of P'_i in P'_j, and let v_i, v_j be the (unique) vertices of f_i, f_j in P_i, P_j respectively. Now let $g_i \in E(G_i)$ with $\phi_i(g_i) = \gamma_2$, and let $g_j = \eta(g_i)$. Then $g_j \in E(G_j)$ with $\phi_j(g_j) = \gamma_2$. Now v_i is the first term of $\mu_i(g_i)$ ($\mu_i(g_i)$ has length one or two, depending whether $\gamma_2 \in \Gamma_1$ or $\gamma_2 \in \Gamma_2$), and v_j is the first term of $\mu_j(g_j)$, and since $g_j = \eta(g_i)$ it follows that $v_j \in \eta(v_i)$. Consequently by defining $\eta(f_i) = f_j$ we obtain a realizable expansion of P_i in P_j, a contradiction. This proves (2).

From (1) and (2), this proves 5.3. ∎

5.4 *Let Ω be a proper well-quasi-order, and let Σ be a sharp shadow. Let $\mathcal{C}$ be a set of rootless proper partial Ω-patchworks with shadow $\Sigma' \leq \Sigma$ such that for each $P = (G, \mu, \Delta, \phi) \in \mathcal{C}$ there exists $g \in E(G)$ satisfying:*

- *$dom(\phi) = E(G) \setminus \{g\}$, and $g \in dom(\mu)$, and $|V(g)| < |R_1(\Sigma)|$*
- *$\Delta(e)$ is free for every edge $e \in E(G) \setminus \{g\}$*
- *$\phi^{-1}(\Gamma_1) = \emptyset$*
- *$V(g)$ includes every muscle of P.*

Then $\mathcal{C}$ is well-behaved.

Proof. Since $\phi^{-1}(\Gamma_1) = \emptyset$ for each $(G, \mu, \Delta, \phi) \in \mathcal{C}$, we may assume that $E(\Omega) \cap \Gamma_1 = \emptyset$. Let Ω' be a well-quasi-order with $\Omega \leq \Omega'$ and $\Gamma_1 \cap E(\Omega') = \emptyset$, and for $i \geq 1$ let $P'_i = (G_i, \mu_i, \Delta_i, \phi'_i)$ be a strict Ω'-completion of $P_i = (G_i, \mu_i, \Delta_i, \phi_i) \in \mathcal{C}$. By 5.2, it suffices to show that $P'_i\,(i = 1, 2, \ldots)$ is not a bad sequence.

For each $i \geq 1$, let $E(G_i) \setminus dom(\phi_i) = \{g\}$. Now $|V(g_i)| < |R_1(\Sigma)|$, and hence there are only finitely many possibilities for $|V(g_i)|$; and by restricting to a subsequence, we may therefore assume that $|V(g_i)| = s$ for all $i \geq 1$.

Since there are only finitely many possibilities for $\Delta_i(g_i)$, we may assume they are all "equal"; more precisely, that for all $i \neq j$, the bijection from $V(g_i)$ to $V(g_j)$ that maps $\mu_i(g_i)$ onto $\mu_j(g_j)$ maps $\Delta_i(g_i)$ onto $\Delta_j(g_j)$. We may also assume that $\phi'_1(g_1) \leq \phi'_2(g_2) \leq \ldots$, by restricting to a subsequence.

Let $\gamma_1, \ldots, \gamma_s \in R_1(\Sigma)$ be distinct, and let $\Omega'' = \Omega' \cup \{\gamma_1, \ldots, \gamma_s\}$. For $i \geq 1$, let $P''_i = (G''_i, \mu''_i, \Delta''_i, \phi''_i)$ be the Ω''-patchwork obtained from P'_i by tieing the jth term of $\mu(g_i)$ to γ_j for $1 \leq j \leq s$, and then deleting g_i. Then P''_i is a skeletal Ω''-patchwork, and $P''_i\,(i = 1, 2, \ldots)$ has shadow $\Sigma'' \subseteq \Sigma'$ say; and since $|R_1(\Sigma'')| = s < |R_1(\Sigma)|$ and $R_1(\Sigma'') \subseteq R_1(\Sigma)$, it follows that $\Sigma'' \neq \Sigma$. Hence Σ'' is not evil, and so $P''_i\,(i = 1, 2, \ldots)$ is not bad, and consequently (by 5.1) $P'_i\,(i = 1, 2, \ldots)$ is not bad. This proves 5.4. ∎

5.5 *Let Ω be a proper well-quasi-order, and let $\mathcal{S}$ be a set of skeletal Ω-patchworks with shadow Σ. Let Σ be sharp. There is a well-behaved set $\mathcal{C}$ of proper partial Ω-patchworks, with the following property. Let $P = (G, \mu, \Delta, \phi) \in \mathcal{S}$ with $|V(\phi^{-1}(\Gamma_1))| = |R_1(\Sigma)|$, let $\mathcal{T}$ be a tangle in G of order $> |R_1(\Sigma)|$, let λ be a tie-breaker in G, and let there be a separation $(A, B) \in \mathcal{T}$ of order $< |R_1(\Sigma)|$ with $\phi^{-1}(\Gamma_1) \subseteq E(A)$. Then there is a rooted location $\mathcal{L}$ in G which $|R_1(\Sigma)|$-isolates $\mathcal{T}$ with respect to λ, such that $(P, \mathcal{L})$ has a heart in $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be the set of all hearts of $(P, \mathcal{L})$, for all $P = (G, \mu, \Delta, \phi) \in \mathcal{S}$ with $|V(\phi^{-1}(\Gamma_1))| = |\phi^{-1}(\Gamma_1)| = |R_1(\Sigma)|$ and rooted locations $\mathcal{L}$ in G with $|\mathcal{L}| = 1$, $\mathcal{L} = \{A\}$ say, where $\phi^{-1}(\Gamma_1) \subseteq E(A)$ and $|\bar{\pi}(A)| < |R_1(\Sigma)|$. By 5.4, $\mathcal{C}$ is well-behaved. Now let $P = (G, \mu, \Delta, \phi) \in \mathcal{S}$, $\mathcal{T}, \lambda, (A, B)$ be as in the theorem; and choose (A, B) with A minimal. Let A' be a rooted subhypergraph of G with $A'^{-} = A$ and $\bar{\pi}(A') = V(A \cap B)$, and let $\mathcal{L} = \{A'\}$. Then since (A, B) is λ-linked to $\mathcal{T}$, it follows that $\mathcal{L}$ $|R_1(\Sigma)|$-isolates $\mathcal{T}$ with respect to λ (by theorem 7.1 of [5]), and $(P, \mathcal{L})$ has heart in $\mathcal{C}$. This proves 5.5. ∎

5.6 *Let Ω be a proper well-quasi-order, and let Σ be a sharp shadow. Let $n \geq 1$ be an integer, and let $\gamma \in R_2(\Sigma)$. Let $\mathcal{C}$ be a set of rootless proper partial Ω-patchworks with shadow $\Sigma' \leq \Sigma$ such that for each $P = (G, \mu, \Delta, \phi) \in \mathcal{C}$ there exists $g \in E(G)$ satisfying:*

- *$dom(\phi) = E(G) \setminus \{g\}$, and $g \in dom(\mu)$, and $|V(g)| \leq n$*
- *$\Delta(e)$ is free for every edge $e \in E(G) \setminus \{g\}$*
- *$\gamma \notin \phi(E(G) \setminus \{g\})$*
- *$V(g) \cup V(\phi^{-1}(\Gamma_1))$ includes every muscle of P.*

Then $\mathcal{C}$ is well-behaved.

Proof. Let Ω' be a proper well-quasi-order with $\Omega \leq \Omega'$ and $E(\Omega') \cap (\Gamma_1 \cup \Gamma_2) \subseteq E(\Omega)$. For $i \geq 1$ let $P'_i = (G_i, \mu_i, \Delta_i, \phi'_i)$ be a strict Ω'-completion of $P_i = (G_i, \mu_i, \Delta_i, \phi_i) \in \mathcal{C}$. By 5.2, it suffices to show that P'_i $(i = 1, 2, \ldots)$ is not a bad sequence.

For each $i \geq 1$ let $E(G_i) \setminus dom(\phi_i) = \{g_i\}$. As in the proof of 5.4, we may assume that $|V(g_i)| = s \leq n$ for all $i \geq 1$; and that for all $j > i \geq 1$ the bijection taking $\mu_i(g_i)$ to $\mu_j(g_j)$ maps $\Delta_i(g_i)$ to $\Delta_j(g_j)$; and that $\phi'_1(g_1) \leq \phi'_2(g_2) \leq \ldots$.

Choose distinct $\gamma_1, \ldots, \gamma_s \in \Gamma_1 \setminus R_1(\Sigma)$. Let Ω'' and P''_i $(i = 1, 2, \ldots)$ be as in 5.4. Then each P''_i is skeletal, and the sequence P''_i $(i = 1, 2, \ldots)$ has shadow $\Sigma'' \leq \Sigma'$ say, with $\gamma \notin R_2(\Sigma'')$. Consequently $\Sigma'' \neq \Sigma$, and so Σ'' is not evil. The conclusion follows as in 5.4. This proves 5.6. ∎

5.7 *Let Ω be a proper well-quasi-order, and let $\mathcal{S}$ be a set of skeletal Ω-patchworks with shadow Σ. Let Σ be sharp, let $n \geq 1$ be an integer, and let $\gamma \in R_2(\Sigma)$. There is a well-behaved set $\mathcal{C}$ of proper partial Ω-patchworks with the following property. Let $P = (G, \mu, \Delta, \phi) \in \mathcal{S}$, let $\mathcal{T}$ be a tangle in G of order $> n$, let λ be a tie-breaker in G, and let there be a separation $(A, B) \in \mathcal{T}$ of order $\leq n$ with $\phi^{-1}(\gamma) \subseteq E(A)$. Then there is a rooted location $\mathcal{L}$ in G which $(n+1)$-isolates $\mathcal{T}$ with respect to λ, such that $(P, \mathcal{L})$ has a heart in $\mathcal{C}$.*

The proof is like that of 5.5, using 5.6 in place of 5.4, and we omit it.

6 External restriction

In this section we handle the failure of the fourth condition of 4.2. If Ω is a proper well-quasi-order, an Ω-patchwork $P = (G, \mu, \Delta, \phi)$ is *near-skeletal* if there exists $g \in dom(\mu)$ such that

- P is free, rootless and proper, and
- $\phi^{-1}(\Gamma_1) \cup V(g)$ includes every muscle of P.

6.1 *Let Σ be a sharp shadow. Let Ω be a proper well-quasi-order, and let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2, \ldots)$ be a sequence of skeletal or near-skeletal Ω-patchworks with shadow Σ', where $\Omega_\infty(\Sigma') \leq \Omega_\infty(\Sigma)$. Let $t \geq 0$ be an integer, and let $\xi \in E(\Omega_\infty(\Sigma))$. Suppose that for each $i \geq 1$ there are at most t edges $e \in E(G_i) \setminus dom(\mu_i)$ such that $\xi \leq \phi_i(e)$. Then there exist $j > i \geq 1$ such that P_i is simulated in P_j.*

Proof. We may assume (by replacing $P_1, P_2, \ldots$ by an infinite subsequence and reducing t) that for each $i \geq 1$ there are exactly t edges $e \in E(G_i) \setminus dom(\mu_i)$ with $\xi \leq \phi_i(e)$. Let these edges be $e_{i1}, \ldots, e_{it}$. Again, since $\Omega_\infty(\Sigma)$ is a well-quasi-order we may assume that for $1 \leq h \leq t$,

$$\phi_1(e_{1h}) \leq \phi_2(e_{2h}) \leq \ldots$$

by restricting to an infinite subsequence. Also, we may assume that either each P_i is skeletal, or each is near-skeletal. First, we assume they are all skeletal.

We may assume that $E(\Omega)$ contains no t-tuple (by replacing Ω by an isomorphic well-quasi-order.) Let Ω' be the well-quasi-order with

$$E(\Omega') = E(\Omega) \cup \{(x_1, \ldots, x_t) : x_i = 0 \,\text{or}\, 1 \,\text{for}\, 1 \leq i \leq t\}$$

where for distinct $a, b \in E(\Omega')$, $a \leq b$ in Ω' if and only if $a, b \in E(\Omega)$ and $a \leq b$ in Ω. For each $i \geq 1$ let $P'_i = (G'_i, \mu'_i, \Delta'_i, \phi'_i)$ be an Ω'-patchwork defined as follows. Roughly, G'_i is obtained from G_i by removing $e_{i1}, \ldots, e_{it}$ and adding $|V(G_i)|$ new edges each with only one end, one at each vertex. More precisely, let $V(G'_i) = V(G_i)$, and

$$E(G'_i) = (E(G_i) \setminus \{e_{i1}, \ldots, e_{it}\}) \cup \{e_i(v) : v \in V(G_i)\}$$

where the elements $e_i(v)$ are new and all distinct; and for $v \in V(G'_i)$ and $e \in E(G'_i)$, e is incident with v in G'_i if and only if either $e = e_i(v)$, or $e \in E(G_i)$ and e is incident with v in G_i. Let $\pi(G'_i) = 0$. Let

$$dom(\mu'_i) = dom(\mu_i) \cup \{e_i(v) : v \in V(G_i)\}$$

for $e \in dom(\mu_i)$ let $\mu'_i(e) = \mu_i(e)$, and let $\mu'_i(e_i(v)) = (v)$. Let P'_i be free (this determines Δ'_i). For $e \in E(G'_i)$, let $\phi'_i(e) = \phi_i(e)$ if $e \in E(G_i)$. For $v \in V(G_i)$ let $\phi'_i(e_i(v)) = (x_1, \ldots, x_t)$ where for $1 \leq h \leq t$, $x_h = 1$ if $v \in V(e_{ih})$ and otherwise $x_h = 0$. Thus P'_i is a skeletal Ω'-patchwork.

Let the sequence P'_i $(i = 1, 2, \ldots)$ have shadow Σ''. Then $\Omega_\infty(\Sigma'') \leq \Omega_\infty(\Sigma)$, and since $\xi \notin E(\Omega_\infty(\Sigma''))$ it follows that $\Omega_\infty(\Sigma'') < \Omega_\infty(\Sigma)$. Hence $\Sigma'' < \Sigma$. Since Σ is sharp, it follows that Σ'' is not evil, and so P'_i $(i = 1, 2, \ldots)$ is not a bad sequence. Hence there exist $j > i \geq 1$ such that P'_i is simulated in P'_j.

Let η' be a realizable expansion of P'_i in P'_j. Define η by

$$\begin{aligned}\eta(v) &= \eta'(v)\,(v \in V(G_i))\\ \eta(e) &= \eta'(e)\,(e \in E(G_i) \setminus \{e_{i1}, \dots, e_{it}\})\\ \eta(e_{ih}) &= e_{jh}\,(1 \le h \le t)\,.\end{aligned}$$

We claim that η is an expansion of P_i in P_j. This is mostly clear; let us check that for $e \in E(G_i) \setminus dom(\mu_i)$, if v is an end of e in G_i then $\eta(v)$ contains an end of $\eta(e)$ in G_j. If $e \neq e_{i1}, \dots, e_{it}$ this is true since η' is an expansion. If $e = e_{ih}$ say, then since v is incident with e_{ih} in G_i it follows that $\phi'_i(e_i(v)) = (x_1, \dots, x_t)$ say where $x_h = 1$. Let $f = \eta'(e_i(v))$. Since $(x_1, \dots, x_t) \le \phi_j(f)$ it follows that $f = e_j(w)$ for some $w \in V(G_j)$, and $\phi_j(e_j(w)) = (x_1, \dots, x_t)$ since $\phi_i(e_i(v)) \le \phi_j(e_j(w))$; and so w is incident with e_{jh}. Now since $e_i(v)$ is incident with v in G_i, it follows $\eta'(e_i(v))$ is incident with some vertex in $\eta'(v)$, that is, $w \in \eta'(v)$. This proves that e_{jh} is incident with a vertex in $\eta'(v)$, and so $\eta(v)$ contains an end of $\eta(e_{ih})$, as required.

Hence η is an expansion of P_i in P_j. Let H be a realization of $P'_j \backslash \eta'(E(G_i))$ that realizes η'. Then H is also a realization of $P_j \backslash \eta(E(G_i))$, realizing η. Hence P_i is simulated in P_j.

Thus, if each P_i is skeletal then the result holds. Now we suppose that each P_i is near-skeletal, and $g_i \in dom(\mu_i)$ is such that every muscle $\{x, y\}$ is a subset of $V(\phi_i^{-1}(\Gamma)) \cup V(g_i)$. Since $g_i \in dom(\mu_i)$ and the sequence has a shadow Σ', it follows that the sequence is limited. By restricting to an infinite subsequence we may therefore assume that $|V(g_i)| = s$ for all $i \ge 1$. Let $\gamma_1, \dots, \gamma_s \in \Gamma_1 \setminus \phi(E(G))$, and let $\Omega' = \Omega \cup \{\gamma_1, \dots, \gamma_s\}$. For each $i \ge 1$, let $P'_i = (G'_i, \mu'_i, \Delta'_i, \phi'_i)$ be the Ω'-patchwork obtained from P_i by tieing v_h to γ for $1 \le h \le s$, where $\mu_i(g_i) = (v_1, \dots v_s)$.

Then P'_i is a skeletal Ω'-patchwork, and the sequence $P'_i\,(i = 1, 2, \dots)$ has shadow Σ'' say, where $\Omega_\infty(\Sigma'') \le \Omega_\infty(\Sigma') \le \Omega_\infty(\Sigma)$. Hence, by the first assertion of the theorem, there exist $j > i \ge 1$ such that P'_i is simulated in P'_j. But then by 5.1, P_i is simulated in P_j. This proves 6.1. ∎

6.2 *Let Ω be a proper well-quasi-order, and let $\mathcal{S}$ be a set of skeletal Ω-patchworks with shadow Σ, where Σ is sharp. Let $\xi \in E(\Omega_\infty(\Sigma))$, and let $h \ge 1$ and $t \ge 0$ be integers. Then there is a well-behaved set $\mathcal{C}$ of proper partial Ω-patchworks with the following property. Let $P = (G, \mu, \Delta, \phi) \in \mathcal{S}$, let $\mathcal{T}$ be a tangle in G of order $\ge h$ that is (ξ, h, t)-restricted externally, and let λ be an edge-based tie-breaker in G. There is a rooted location $\mathcal{L}$ in G which h-isolates $\mathcal{T}$ with respect to λ, such that $(P, \mathcal{L})$ has a heart in $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be the set of all proper partial Ω-patchworks $P' = (G', \mu', \Delta', \phi')$ with the following properties:

(i) for some $P = (G, \mu, \Delta, \phi) \in \mathcal{S}$ and some rooted location $\mathcal{L}$ in G, P' is a heart of $(P, \mathcal{L})$

(ii) P' is skeletal or near-skeletal

(iii) $|\bar{\mu}'(e)| < h$ for all $e \in E(G') \setminus dom(\phi')$ (since P' is a heart, every such edge belongs to $dom(\mu')$)

(iv) there are at most t edges $e \in E(G') \setminus dom(\mu')$ such that $\xi \le \phi'(e)$.

Since $E(G') \setminus dom(\mu') \subseteq dom(\phi')$, it follows that for every proper well-quasi-order Ω'' with $\Omega \leq \Omega''$ and every set $\mathcal{S}''$ of strict Ω''-completions of members of $\mathcal{C}$, the shadow Σ'' of $\mathcal{S}''$ exists and satisfies $\Omega_\infty(\Sigma'') \leq \Omega_\infty(\Sigma)$. Hence, from 6.1, there is no bad sequence of members of $\mathcal{S}''$. Thus $\mathcal{C}$ is well-behaved, by 5.2.

Now let $P = (G, \mu, \Delta, \phi) \in \mathcal{S}, \mathcal{T}, \lambda$ be as in the theorem. We must find $\mathcal{L}$ as in the theorem. Let

$$F = \{f \in E(G) \setminus dom(\mu) : \xi \leq \phi(f)\}.$$

Let F_0 be the set of $f \in F$ such that there exists $X \subseteq V(f)$ with $|X| = h$, free relative to $\mathcal{T}$. Since $\mathcal{T}$ is (ξ, h, t)-restricted externally, $|F_0| \leq t$.

For each $f \in F \setminus F_0$, there exists $(A(f), B(f)) \in \mathcal{T}$ of order $< h$ with $f \in E(A(f))$. Choose $(A(f), B(f))$ with minimum λ-order. Evidently it has order $\leq |V(f)|$.

(1) *For all distinct $f_1, f_2 \in F \setminus F_0$, either*

- $A(f_1) \subseteq A(f_2)$ *and* $B(f_2) \subseteq B(f_1)$, *or*
- $A(f_2) \subseteq A(f_1)$ *and* $B(f_1) \subseteq B(f_2)$, *or*
- $A(f_1) \subseteq B(f_2)$ *and* $A(f_2) \subseteq B(f_1)$.

Subproof. $(A(f_1) \cup A(f_2), B(f_1) \cap B(f_2))$ has λ-order at least that $(A(f_2), B(f_2))$, since otherwise it belongs to $\mathcal{T}$ and $f_2 \in E(A(f_1) \cup A(f_2))$, contrary to the choice of $(A(f_2), B(f_2))$. Since λ is a tie-breaker, it follows that $(A(f_1) \cap A(f_2), B(f_1) \cup B(f_2))$ has λ-order at most that of $(A(f_1), B(f_1))$. Consequently, if $f_1 \in E(A(f_2))$ then $A(f_1) \cap A(f_2) = A(f_1)$ and $B(f_1) \cup B(f_2) = B(f_1)$, that is, $A(f_1) \subseteq A(f_2)$ and $B(f_2) \subseteq B(f_1)$, as required. We may therefore assume that $f_1 \notin E(A(f_2))$, and similarly $f_2 \notin E(A(f_1))$.

Now $(A(f_1) \cap B(f_2), B(f_1) \cup A(f_2))$ has λ-order at least that of $(A(f_1), B(f_1))$ since $f_1 \in E(A(f_1) \cap B(f_2))$, and similarly $(A(f_1) \cup B(f_2), B(f_1) \cap A(f_2))$ has λ-order at least that of $(A(f_2), B(f_2))$. We therefore have equality throughout; and so $A(f_1) \cap B(f_2) = A(f_1)$ and $B(f_1) \cup A(f_2) = B(f_1)$, that is, $A(f_1) \subseteq B(f_2)$ and $B(f_1) \subseteq A(f_2)$, as required. This proves (1).

Choose a maximal subset $F_1 \subseteq F \setminus F_0$ such that

- the separations $(A(f), B(f))$ $(f \in F_1)$ are all distinct
- for each $f \in F$ there is no $f' \in F$ with $(A(f'), B(f')) \neq (A(f), B(f))$ and $A(f) \subseteq A(f')$ and $B(f') \subseteq B(f)$.

It follows from (1) and the maximality of F_1 that

(2) *For all distinct $f, f' \in F_1, A(f) \subseteq B(f')$; and for every $f \in F \setminus F_0$ there exists $f_1 \in F_1$ with $A(f) \subseteq A(f_1)$ and $B(f_1) \subseteq B(f)$.*

For each $f \in F_1$ let $C(f)$ be a rooted hypergraph with $C(f)^- = A(f)$ and $\bar{\pi}(C(f)) = V(A(f) \cap B(f))$. By (2), $\mathcal{L} = \{C(f) : f \in F_1\}$ is a rooted location.

(3) *$\mathcal{L}$ h-isolates $\mathcal{T}$ with respect to λ.*

Subproof. Certainly $\mathcal{L}^- \subseteq \mathcal{T}$ and has order $< h$. Moreover, each member of $\mathcal{L}^-$ is λ-linked to $\mathcal{T}$, from the choice of $(A(f), B(f))$, so by theorem 7.1 of [5], $\mathcal{L}$ h-isolates $\mathcal{T}$ with respect to λ. This proves (3).

Let $P' = (G', \mu', \Delta', \phi')$ be a heart of $(P, \mathcal{L})$. We must show that $P' \in \mathcal{C}$; and to do so, we check conditions (i)-(iv) in the definition of $\mathcal{C}$. Now (i) is obvious, and (iv) holds since $|F_0| \leq t$ and every $f \in F \setminus F_0$ belongs to $E(C(f_1))$ for some $f_1 \in F_1$, by (2). Also (iii) holds, since if $e \in E(G') \setminus dom(\phi')$ then $\mu'(e) = \pi(C(f))$ for some $f \in F_1$, and $\pi(C(f))$ has length $|V(A(f) \cap B(f))| < h$. It remains to check (ii). Now P' is proper, since P is, and rootless; we must check that it is free, and verify the condition about muscles.

(4) *P' is free.*

Subproof. Let $f \in F_1$; we must show that every grouping with vertex set $\bar{\pi}(C(f))$ is feasible in $P|C(f)$. Let $k = |\bar{\pi}(C(f))|$. From the choice of $(A(f), B(f))$, there are k disjoint paths of $sk(C(f^-))$ from $\bar{\pi}(C(f))$ to $V(f)$. But $\Delta(f)$ is free, and so the claim follows. This proves (4).

(5) *P' is skeletal or near-skeletal.*

Subproof. We recall that λ is edge-based; let λ be defined by g, ν say. If $g \in E(A(f))$ for some (necessarily unique) edge $f \in F_1$, let $f^* = f$, and otherwise let f^* be undefined. We shall show that for every muscle $\{u, v\}$ of P', if f^* is defined then $\{u, v\} \subseteq \phi'^{-1}(\Gamma_1) \cup \bar{\pi}(C(f^*))$ (and hence P' is near-skeletal), and if f^* is not defined then $\{u, v\} \subseteq \phi'^{-1}(\Gamma_1)$ (and hence P' is skeletal). Thus, let $\{u, v\}$ be a muscle of P'. Suppose first that it is a muscle of P. Then $\{u, v\} \subseteq V(\phi^{-1}(\Gamma_1))$ since P is skeletal. Let $e_1, e_2 \in \phi^{-1}(\Gamma_1)$ such that e_1 is incident with u and e_2 with v in G. If $e_1 \in E(G')$ then $u \in V(\phi^{-1}(\Gamma_1))$. If not, then $e_1 \in E(A(f))$ for some $f \in F_1$. Since $f \neq e_1$ (because $f \notin dom(\mu)$) it follows that $f \in E(A')$ where $A' = A(f) \backslash e$ and $B' = B(f) + e$ (with the natural notation). Also, $(A', B') \in \mathcal{T}$, and so from the definition of $(A(f), B(f))$ it follows that (A', B') has λ-order at least that of $(A(f), B(f))$. But their orders are the same, and so $g \in E(A(f))$; and hence f^* is defined and $f = f^*$, and $u \in \bar{\pi}(C(f^*))$. Thus if $\{u, v\}$ is a muscle of P then either $\{u, v\} \subseteq \phi'^{-1}(\Gamma_1)$, or f^* is defined and $\{u, v\} \subseteq \phi^{-1}(\Gamma_1) \cup \bar{\pi}(C(f^*))$, as required.

Now suppose that $\{u, v\}$ is not a muscle of P. Then there exists $e \in dom(\mu)$ with $V(e) = \{u, v\}$. Since $\{u, v\}$ is a muscle of P', $e \notin E(G')$, and so $e \in E(A(f))$ for some $f \in F_1$. Now $f \neq e$ since $e \in dom(\mu)$, and so as before f^* is defined and $f^* = f$; and hence $\{u, v\} \subseteq \bar{\pi}(C(f^*))$. This proves (5).

Consequently, $P' \in \mathcal{C}$. This proves 6.2. ∎

7 A hypergraph lemma

Finally we need an analogue of 6.2 for internal restriction. This is more difficult, however, and needs some preparation.

Let $\mathcal{T}$ be a tangle in a hypergraph G. Let $(A, B) \in \mathcal{T}$, and let $v \in V(A \cap B)$. A separation $(A', B') \in \mathcal{T}$ is called the *$\mathcal{T}$-successor* of (A, B) *via v* if

(i) $v \notin V(B')$, and $A \subseteq A'$ and $B' \subseteq B$

(ii) subject to (i), (A', B') has minimum order

(iii) subject to (i) and (ii), B' is minimal.

7.1 *With $\mathcal{T}, A, B, v$ as above, there is at most one $\mathcal{T}$-successor of (A, B) via v.*

Proof. Suppose that (A', B') and (A'', B'') are both $\mathcal{T}$-successors of (A, B) via v. By (ii) they both have the same order, k say. Let $(C, D) = (A' \cap A'', B' \cup B'')$; then (C, D) satisfies condition (i) above, and $(C, D) \in \mathcal{T}$, and so it has order $\geq k$. Hence $(A' \cup A'', B' \cap B'')$ has order $\leq k$. From (iii), $B' \subseteq B' \cap B''$ and $B'' \subseteq B' \cap B''$, that is, $B' = B''$. Consequently $(A' \cap A'', B')$ satisfies (i), and so has order $\geq k$, and so $A' = A''$. Thus $(A', B') = (A'', B'')$. This proves 7.1. ∎

7.2 *Let $\mathcal{T}$ be a tangle in a hypergraph G, and let $W \subseteq V(G)$ be free relative to $\mathcal{T}$, with $|W| \leq w$. Let $h \geq 1$ be an integer, and let $\mathcal{T}$ have order $\geq (w+h)^{h+1} + h$. Then there exists $W' \subseteq V(G)$ with $W \subseteq W'$ and $|W'| \leq (w+h)^{h+1}$ such that for every $(C, D) \in \mathcal{T}$ of order $< |W| + h$ with $W \subseteq V(C)$, there exists $(A', B') \in \mathcal{T}$ with $W' \subseteq V(A' \cap B')$, $|V(A' \cap B') \setminus W'| < h$, $C \subseteq A'$ and $E(B') \subseteq E(D)$.*

Proof. Let (A_0, B_0) be the separation of G with $B_0 = G$ and $V(A_0) = W$ and $E(A_0) = \emptyset$. Then $(A_0, B_0) \in \mathcal{T}$. Let $\mathcal{A}_0 = \{(A_0, B_0)\}$. For $1 \leq i \leq h+1$, let $\mathcal{A}_i$ be the set of all $\mathcal{T}$-successors (A', B') of members (A, B) of $\mathcal{A}_{i-1}$ via some vertex in $V(A \cap B)$, such that (A', B') has order $< |W| + h$. By 7.1, $|\mathcal{A}_i| \leq (|W| + h)|\mathcal{A}_{i-1}|$. Let

$$W' = \bigcup_{0 \leq i \leq h} \bigcup (V(A \cap B) : (A, B) \in \mathcal{A}_i).$$

Thus $W \subseteq W'$ and

$$|W'| \leq w + w(w+h) + w(w+h)^2 + \ldots + w(w+h)^h \leq (w+h)^{h+1}.$$

We claim that W' satisfies the theorem. For let $(C, D) \in \mathcal{T}$ of order $< |W| + h$ with $W \subseteq V(C)$. We may assume that

(1) *There is no $(C', D') \in \mathcal{T}$ of order $< |W| + h$ with $(C', D') \neq (C, D)$, $C \subseteq C'$ and $D' \subseteq D$.*

Subproof. If there is such a (C', D'), then we can replace (C, D) by (C', D'), and continue until (1) holds. This proves (1).

(2) *For all i with $0 \leq i \leq h+1$, every member of $\mathcal{A}_i$ has order $\geq |W| + i - 1$.*

Subproof. Every member of $\mathcal{A}_0$ has order $|W|$, and since W is free relative to $\mathcal{T}$, all $\mathcal{T}$-successors of (A_0, B_0) have order $\geq |W|$. Thus (2) holds for $i = 0, 1$. It follows in general by induction, since if $(A, B) \in \mathcal{A}_{i-1}$ where $i \geq 2$ then all its $\mathcal{T}$-successors have larger order. This proves (2).

Now $A_0 \subseteq C$ and $D \subseteq B_0$; choose i with $0 \leq i \leq h$ maximum such that $A \subseteq C$ and $D \subseteq B$ for some $(A, B) \in \mathcal{A}_i$. We claim that

(3) $V(A\cap B)\subseteq V(C\cap D)$.

Subproof. Suppose that $v\in V(A\cap B)\setminus V(C\cap D)$. Then $v\notin V(D)$, since $v\in V(A)\subseteq V(C)$; and so there is a $\mathcal{T}$-successor (A',B') of (A,B) via v, with order $\leq |V(C\cap D)|<|W|+h$. Hence $\mathcal{A}_{i+1}$ has a member of order $<|W|+h$; and so by (2), $i\leq h-1$. From the maximality of i, it is not the case that $A'\subseteq C$ and $D\subseteq B'$. Now $(A'\cap C, D\cup B')\in\mathcal{T}$ and $v\notin V(D\cup B')$, and so $(A'\cap C, D\cup B')$ has order at least that of (A',B'), since (A',B') is a $\mathcal{T}$-successor of (A,B) via v. Hence $(A'\cup C, D\cap B')$ has order at most that of (C,D), and so $(A'\cup C, D\cap B')\in\mathcal{T}$. From (1) it follows that $C=A'\cup C$ and $D=D\cap B'$, that is, $A'\subseteq C$ and $D\subseteq B'$, a contradiction. This proves (3).

Now since W is free relative to $\mathcal{T}$, it follows that (A,B) has order $\geq |W|$; and so $|V(C\cap D)\setminus W'|<h$, by (3), since $|V(C\cap D)|<|W|+h$ and $V(A\cap B)\subseteq W'$. Let $A',B'\subseteq G$ with $E(A')=E(C), E(B')=E(D)$, $V(A')=V(C)\cup W'$, and $V(B')=V(D)\cup W'$; then $(A',B')\in\mathcal{T}$, since (A',B') has order $<|W'|+h$ and $\mathcal{T}$ has order $\geq (w+h)^{h+1}+h$. Then (A',B') satisfies the theorem. This proves 7.2. ∎

Let G be a hypergraph. A *location* in G is a set $\{(A_1,B_1),\ldots,(A_k,B_k)\}$ of separations of G such that $A_i\subseteq B_j$ for all distinct i,j with $1\leq i,j\leq k$. Let G be a hypergraph, let $\mathcal{T}$ be a tangle in G, and let $J\subseteq E(G)$. A *(J,h)-separator* $(X,\mathcal{L})$ consists of a subset $X\subseteq V(G)$ and a location $\mathcal{L}\subseteq\mathcal{T}$ such that

- $X\subseteq V(A\cap B)$ for all $(A,B)\in\mathcal{L}$
- $|V(A\cap B)\setminus X|<h$ for all $(A,B)\in\mathcal{L}$
- for each $e\in J$ there exists $(A,B)\in\mathcal{L}$ with $e\in E(A)$.

7.3 *Let $G,\mathcal{T},J$ be as above, and let $(X_0,\mathcal{L}_0)$ be a (J,h)-separator. Let λ be an edge-based tie-breaker in G, defined by f,ν. There is a (J,h)-separator $(X,\mathcal{L})$ such that*

(i) $|X|\leq|X_0|$

(ii) *$\mathcal{L}$ is λ-linked to $\mathcal{T}$*

(iii) *for each $(A,B)\in\mathcal{L}$ there exists $e\in J\cap E(A)$ such that there are $|V(A\cap B)\setminus X|$ disjoint paths of $sk(A)\backslash X$ between $V(A\cap B)\setminus X$ and $V(e)$, and*

(iv) *there is at most one $(A,B)\in\mathcal{T}$ such that $V(e)\subseteq V(A\cap B)$ for some $e\in E(A)\setminus J$.*

Proof. We recall that λ is an edge-based tie-breaker, defined by f,ν. For a separation (A,B), its λ-order is a triple, $(\lambda_1(A,B),\lambda_2(A,B),\ \lambda_3(A,B))$ say. Thus, $\lambda_1(A,B)=|V(A\cap B)|$. Let $\lambda_0(A,B)=(|E(G)|+1)^{\lambda_1(A,B)}$. For a set $\mathcal{L}$ of separations, $\lambda_i(\mathcal{L})$ denotes $\Sigma(\lambda_i(A,B):(A,B)\in\mathcal{L})$.

If $(X_1,\mathcal{L}_1),(X_2,\mathcal{L}_2)$ are (J,h)-separators, we say the first is *better* than the second if $|X_1|\leq|X_2|$ and either

- $\lambda_0(\mathcal{L}_1)<\lambda_0(\mathcal{L}_2)$, or

- $\lambda_0(\mathcal{L}_1) = \lambda_0(\mathcal{L}_2)$ and $\lambda_2(\mathcal{L}_1) < \lambda_2(\mathcal{L}_2)$.

The relation "better than" is transitive and irreflexive, and so since there is a (J,h)-separator $(X_0, \mathcal{L}_0)$, there is a (J,h)-separator $(X, \mathcal{L})$ with $|X| \le |X_0|$ such that no (J,h)-separator is better. We shall show that $(X, \mathcal{L})$ satisfies statements (i), (ii), (iii) of the theorem. Certainly it satisfies statement (i).

To check (ii), let $(A_0, B_0) \in \mathcal{L}$, and suppose that there exists $(A_0', B_0') \in \mathcal{T}$ with $A_0 \subseteq A_0'$ and $B_0' \subseteq B_0$, and with λ-order less than that of (A_0, B_0). Then $X \subseteq V(A_0')$ since $X \subseteq V(A_0)$. Choose (A_0', B_0') with minimum order k say; then there are k disjoint paths P_v ($v \in V(A_0' \cap B_0')$) of $sk(G)$ from $V(A_0' \cap B_0')$ to $V(A_0 \cap B_0)$, where each P_v has initial vertex v and has no other vertex in $V(B_0')$. Let

$$X' = \{v \in V(A_0' \cap B_0') : P_v \text{ has last vertex in } X\}.$$

Let $\mathcal{L} = \{(A_i, B_i) : 0 \le i \le n\}$ say. For $1 \le i \le n$, let A_i^+ be the subhypergraph of G with $E(A_i^+) = E(A_i)$ and $V(A_i^+) = V(A_i) \cup X'$. Then (A_i^+, B_i) is a separation of G. Let $A_i' = A_i^+ \cap B_0'$, $B_i' = B_i \cup A_0'$. Then (A_i', B_i') is also a separation of G, and $X' \subseteq V(A_i' \cap B_i')$.

(1) *For $1 \le i \le n$, $|V(A_i' \cap B_i') \setminus X'| \le |V(A_i \cap B_i) \setminus X|$, and so $|V(A_i' \cap B_i')| \le |V(A_i \cap B_i)|$.*

Subproof. Let $Y = V(A_i \cap B_i) \setminus V(A_0')$. Then $Y \subseteq V(A_i \cap B_i) \setminus X$ and $Y \subseteq V(A_i' \cap B_i') \setminus X'$, and so it suffices for the first claim to show that

$$|V(A_i' \cap B_i') \setminus (X' \cup Y)| \le |V(A_i \cap B_i) \setminus (X \cup Y)|.$$

Let $v \in V(A_i' \cap B_i') \setminus (X' \cup Y)$. Since

$$V(A_i' \cap B_i') \subseteq V(A_i \cap B_i) \cup X' \cup V(A_0') \subseteq Y \cup X' \cup V(A_0'),$$

it follows that $v \in V(A_0')$, and so $v \in V(A_0' \cap B_0')$. We claim that some vertex of P_v is in $V(A_i \cap B_i) \setminus (X \cup Y)$. For since $v \notin X'$ it follows that $v \in V(A_i)$, and since the other end of P_v is in $V(A_0 \cap B_0) \subseteq V(B_i)$, there is a vertex u of P_v in $V(A_i \cap B_i)$, and $V(P_v) \cap X = \emptyset$. But $u \in V(A_0')$ since $V(P_v) \subseteq V(A_0')$, and so $u \notin Y$, and hence

$$u \in V(A_i \cap B_i) \setminus (X \cup Y).$$

Since $u \in V(P_v)$ and the paths P_v are pairwise disjoint, it follows that

$$|V(A_i' \cap B_i') \setminus (X' \cup Y)| \le |V(A_i \cap B_i) \setminus (X \cup Y)|.$$

This proves the first claim. Since

$$|V(A_i' \cap B_i') \cap X'| = |X'| \le |X| = |V(A_i \cap B_i) \cap X|$$

the second claim follows. This proves (1).

Let $\mathcal{L}' = \{(A_i', B_i') : 0 \le i \le n\}$.

(2) *$(X', \mathcal{L}')$ is a (J,h)-separator.*

Subproof. We have seen that $X' \subseteq V(A'_i \cap B'_i)$ $(0 \le i \le n)$ and

$$|V(A'_i \cap B'_i) \setminus X'| \le |V(A_i \cap B_i) \setminus X| \le h$$

from (1); and for $e \in J$, if $e \in E(A_i)$ then

$$e \in E(A_i) \subseteq E(A'_i) \cup E(A'_0),$$

so it suffices to show that $\mathcal{L}'$ is a location and $\mathcal{L}' \subseteq \mathcal{T}$. For $1 \le i \le n$, certainly $A'_0 \subseteq B'_i$ and $A'_i \subseteq B'_0$, so let $1 \le i < j \le n$. Then $A_i \subseteq B_j$ and $A_j \subseteq B_i$, and so

$$A'_i = A^+_i \cap B'_0 \subseteq B_j \cup A'_0 = B'_j$$

and similarly $A'_j \subseteq B'_i$. Thus, $\mathcal{L}'$ is a location.

To show that $\mathcal{L}' \subseteq \mathcal{T}$, certainly $(A'_0, B'_0) \in \mathcal{T}$. Let $1 \le i \le n$ and suppose that $(A'_i, B'_i) \notin \mathcal{T}$. Since $(A_i, B_i) \in \mathcal{T}$ and by (1) (A'_i, B'_i) has order at most that of (A_i, B_i), it follows that $(B'_i, A'_i) \in \mathcal{T}$. But $(A'_0, B'_0), (A_i, B_i) \in \mathcal{T}$, and $B'_i \cup A'_0 \cup A_i = G$ since $A'_i \subseteq A_i \cup A'_0$, contrary to the second tangle axiom. Thus $\mathcal{L}' \subseteq \mathcal{T}$. This proves (2).

(3) $|X'| = |X|$, *and* $|V(A'_i \cap B'_i)| = |V(A_i \cap B_i)|$ *for* $0 \le i \le n$.

Subproof. From (2) and the choice of $(X, \mathcal{L})$ it follows that $(X', \mathcal{L}')$ is not better than $(X, \mathcal{L})$. But certainly $|X'| \le |X|$, from the definition of X'; and so $\lambda_0(\mathcal{L}') \ge \lambda_0(\mathcal{L})$. From (1) it follows that $|V(A'_i \cap B'_i)| = |V(A_i \cap B_i)|$ for $0 \le i \le n$. Consequently each vertex of $V(A_0 \cap B_0)$ is an end of some P_v, and so $|X'| = |X|$. This proves (3).

(4) $f \in E(A'_0)$.

Subproof. From (3), $\lambda_1(A'_0, B'_0) = \lambda_1(A_0, B_0)$. Suppose that $f \in E(B'_0)$. Since $A_0 \subseteq A'_0$ and $B'_0 \subseteq B_0$, it follows that $f \in B_0$, and hence $\lambda_2(A'_0, B'_0) \ge \lambda_2(A_0, B_0)$. Since (A'_0, B'_0) has smaller λ-order than (A_0, B_0) it follows that $Z(B'_0) = Z(B_0)$ and hence $B'_0 = B_0$. Since $A_0 \subseteq A'_0$ and $(A_0, B_0), (A'_0, B'_0)$ have the same order, it follows that $A_0 = A'_0$, a contradiction. This proves (4).

(5) *If* $i \ge 0$ *then* $\lambda_2(A'_i, B'_i) \le \lambda_2(A_i, B_i)$.

Subproof. If $i = 0$ this follows from (4), since $B'_0 \subseteq B_0$. We assume $i \ge 1$, and then there are two cases. First, let $f \notin E(A_i)$. Then since $f \notin E(A'_i)$, we have

$$\begin{aligned} \lambda_2(A'_i, B'_i) &= \Sigma(\nu(x) : x \in Z(G) \setminus Z(B'_i)) \\ \lambda_2(A_i, B_i) &= \Sigma(\nu(x) : x \in Z(G) \setminus Z(B_i)). \end{aligned}$$

Since $B_i \subseteq B'_i$ and hence $Z(B_i) \subseteq Z(B'_i)$, it follows that

$$\lambda_2(A'_i, B'_i) \le \lambda_2(A_i, B_i)$$

as required.

Now let $f \in E(A_i)$, and suppose that the desired inequality is false. Then

$$\Sigma(\nu(x) : x \in Z(G) \setminus Z(B'_i)) > \Sigma(\nu(x) : x \in Z(G) \setminus Z(A_i)) .$$

Since (A'_0, B'_0) has λ-order less than that of (A_0, B_0), and $f \in E(A'_0 \cap A_i) \subseteq E(A'_0 \cap B_0)$, it follows that

$$\Sigma(\nu(x) : x \in Z(G) \setminus Z(B_0)) \geq \Sigma(\nu(x) : x \in Z(G) \setminus Z(A'_0)) .$$

In other words,

$$\Sigma(\nu(x) : x \in Z(B'_i)) < \Sigma(\nu(x) : x \in Z(A_i))$$

and

$$\Sigma(\nu(x) : x \in Z(B_0)) \leq \Sigma(\nu(x) : x \in Z(A'_0)) .$$

But $Z(A'_0) \subseteq Z(B'_i)$ and $Z(A_i) \subseteq Z(B_0)$, which is impossible. This proves (5).

From (5) it follows that $\lambda_2(\mathcal{L}') \leq \lambda_2(\mathcal{L})$; and so we have equality, from (3) and the choice of $(X, \mathcal{L})$. Hence, in particular, $\lambda_2(A'_0, B'_0) = \lambda_2(A_0, B_0)$, and from (3) we deduce that $B_0 = B'_0$. Since $A_0 \subseteq A'_0$ and the two separations have the same order, it follows that $A_0 = A'_0$, contradicting that (A'_0, B'_0) has smaller λ-order than (A_0, B_0). We have proved therefore that there is no such (A_0, B_0), and so $\mathcal{L}$ is λ-linked to $\mathcal{T}$. Hence statement (ii) of the theorem holds.

Now we verify (iii). Let $(A_0, B_0) \in \mathcal{L}$, and let $|V(A_0 \cap B_0) \setminus X| = t$. Let $J_0 = J \cap E(A_0)$, and suppose that for each $e \in J_0$ there is a separation (A_e, B_e) of A_0 of order $< t + |X|$ with $e \in E(A_e)$, $V(A_0 \cap B_0) \subseteq V(B_e)$ and $X \subseteq V(A_e \cap B_e)$. For each e, choose (A_e, B_e) of minimum order, and subject to that with A_e minimal.

(6) *If $e, e' \in J_0$ are distinct then either $A_e \subseteq A_{e'}$ and $B_{e'} \subseteq B_e$, or $A_{e'} \subseteq A_e$ and $B_e \subseteq B_{e'}$, or $A_e \subseteq B_{e'}$ and $A_{e'} \subseteq B_e$.*

Subproof. Now $(A_e \cup A_{e'}, B_e \cap B_{e'})$ has order at least that of $(A_{e'}, B_{e'})$ from the choice of $(A_{e'}, B_{e'})$; and so $(A_e \cap A_{e'}, B_e \cup B_{e'})$ has order at most that of (A_e, B_e). If $e \in E(A_{e'})$, it follows from the minimality of A_e that $A_e = A_e \cap A_{e'}$, that is, $A_e \subseteq A_{e'}$; and so since $(A_e \cap A_{e'}, B_e \cup B_{e'})$ and (A_e, B_e) have the same order, it follows that $B_{e'} \subseteq B_e$, as required. We may assume then that $e \notin E(A_{e'})$, and similarly $e' \notin E(A_e)$. Then $e \in E(A_e \cap B_{e'})$, and $e' \in E(A_{e'} \cap B_e)$. Consequently, $(A_e \cap B_{e'}, B_e \cup A_{e'})$ has order at least that of (A_e, B_e), and $(A_{e'} \cap B_e, B_{e'} \cup A_e)$ has order at least that of $(A_{e'}, B_{e'})$. We therefore have equality throughout, and so $A_e \cap B_{e'} = A_e$ by the minimality of A_e, that is, $A_e \subseteq B_{e'}$; and similarly $A_{e'} \subseteq B_e$, as required. This proves (6).

Let $\mathcal{L}_0$ be the set of all members (A, B) of $\{(A_e, B_e) : e \in J_0\}$ with A_e maximal and B_e minimal. By (6), $\mathcal{L}_0$ is a location, and hence so is

$$\mathcal{L}' = \{(A, B \cup B_0) : (A, B) \in \mathcal{L}_0\} \cup (\mathcal{L} \setminus \{(A_0, B_0)\}).$$

It follows that $(X, \mathcal{L}')$ is a (J, h)-separator. For each $(A, B) \in \mathcal{L}_0$, $(A, B \cup B_0)$ has the same order as (A, B), and hence has order $< |X| + t$; and since $|\mathcal{L}_0| \leq |E(G)|$, it follows that

$$\lambda_0(\{(A, B \cup B_0) : (A, B) \in \mathcal{L}_0\}) < \lambda_0(A_0, B_0) .$$

Hence $\lambda_0(\mathcal{L}') < \lambda_0(\mathcal{L})$, contrary to the choice of $(X, \mathcal{L})$. It follows that for some $e \in J_0$ there is no such (A_e, B_e), and hence statement (iii) of the theorem holds.

Now let us choose $(X, \mathcal{L})$ satisfying (i)-(iii) of the theorem, with $\bigcup(A : (A, B) \in \mathcal{L})$ minimal. We claim that $(X, \mathcal{L})$ also satisfies (iv). For let $(A, B) \in \mathcal{L}$, and let $e \in E(A) \setminus J$ with $V(e) \subseteq V(A \cap B)$. Let $A' = A \backslash e, B' = B + e$, and $\mathcal{L}' = (\mathcal{L} \setminus \{(A, B)\}) \cup \{(A', B')\}$. Then $(X, \mathcal{L})$ is a (J, h)-separator, since $e \notin J$, and it satisfies (i) and (iii) of the theorem. From the minimality of $\bigcup(A : (A, B) \in \mathcal{L})$, it does not satisfy (ii), and so (A', B') is not λ-linked to $\mathcal{T}$. Hence there exists $(C, D) \in \mathcal{T}$ with $A' \subseteq C$ and $D \subseteq B'$, with λ-order less than that of (A', B').

Suppose that (C, D) has λ-order less than that of (A, B). Since (A, B) is λ-linked to $\mathcal{T}$, it is not the case that $A \subseteq C$ and $D \subseteq B$, and so $e \in E(D)$, and $A \cap C = A'$ and $B \cup D = B'$. Now since λ is a tie-breaker, either $(A \cup C, B \cap D)$ has λ-order less than that of (A, B) (which is impossible since (A, B) is λ-linked to $\mathcal{T}$) or $(A \cap C, B \cup D)$ has λ-order at most that of (C, D) (which is impossible since $(A \cap C, B \cup D) = (A', B')$). Thus (C, D) has λ-order at least that of (A, B), and so (A, B) has λ-order less than that of (A', B'). Since they have the same order and $Z(A') \subset Z(A)$, it follows that $f^* \in E(A)$, where λ is defined by f^*, ν say, and therefore A is unique. Hence (iv) holds. This proves 7.3. ∎

By combining 7.2 and 7.3, we obtain:

7.4 *Let $\mathcal{T}$ be a tangle in a hypergraph G, and let $W \subseteq V(G)$, free relative to $\mathcal{T}$. Let $w \geq 0, h \geq 1$ be integers, such that $|W| \leq w$ and $\mathcal{T}$ has order $\geq (w+h)^{h+1} + h$. Let λ be an edge-based tie-breaker in G, and let $J \subseteq E(G)$ such that for each $e \in J$, $|V(e)| = h$ and either $V(e) \cap W \neq \emptyset$ or $W \cup V(e)$ is not free relative to $\mathcal{T}$. Then there is a (J, h)-separator $(X, \mathcal{L})$ such that $|X| \leq (w+h)^{h+1}$, $\mathcal{L}$ is λ-linked to $\mathcal{T}$, and for each $(A, B) \in \mathcal{L}$ there exists $e \in J \cap E(A)$ such that there are $|V(A \cap B) \setminus X|$ disjoint paths of $sk(A) \backslash X$ between $V(A \cap B) \setminus X$ and $V(e)$; and there is at most one $(A, B) \in \mathcal{L}$ such that $V(e) \subseteq V(A \cap B)$ for some $e \in E(A) \setminus J$.*

Proof. Let W' be as in 7.2. For each $e \in J$, there is by hypothesis a separation $(A, B) \in \mathcal{T}$ of order $< |W| + h$ with $e \in E(A)$. Hence, since W' satisfies 7.2, there is a separation $(A(e), B(e)) \in \mathcal{T}$ with $W' \subseteq V(A(e) \cap B(e))$, $|V(A(e) \cap B(e)) \setminus W'| < h$, and $e \in E(A(e))$. Choose such a separation $(A(e), B(e))$ with minimum λ-order.

(1) *For all distinct $e_1, e_2 \in J$, either*

- *$A(e_1) \subseteq A(e_2)$ and $B(e_2) \subseteq B(e_1)$, or*
- *$A(e_2) \subseteq A(e_1)$ and $B(e_1) \subseteq B(e_2)$, or*
- *$A(e_1) \subseteq B(e_2)$ and $A(e_2) \subseteq B(e_1)$.*

Subproof. This is the same as the proof of (1) in 6.2, and we omit it.

As in 6.2, by (1) there exists $J' \subseteq J$ such that

- $(A(e_1), B(e_1)) \neq (A(e_2), B(e_2))$ and $A(e_1) \subseteq B(e_2)$ for all distinct $e_1, e_2 \in J'$, and
- for all $e \in J$ there exists $e' \in J'$ such that $A(e) \subseteq A(e')$ and $B(e') \subseteq B(e)$.

Hence $(W', \{(A(e), B(e)) : e \in J'\})$ is a (J, h)-separator. By 7.3, there is a (J, h)-separator satisfying the theorem. This proves 7.4. ∎

8 Internal restriction

With the aid of 7.4 we can prove an analogue of 6.2 for internal restriction, to handle the failure of the fifth condition of 4.2. We begin with the following.

8.1 *Let Ω be a proper well-quasi-order, and let Σ be a sharp shadow. Let $h \geq 1$, $w \geq 0$ be integers, and let $\xi \in E(\Omega_h(\Sigma))$. Let $\mathcal{C}$ be a set of rootless proper partial Ω-patchworks with shadow $\leq \Sigma$, such that for each $P = (G, \mu, \Delta, \phi) \in \mathcal{C}$ there exists k with $0 \leq k \leq w$ and distinct vertices $w_1, \dots, w_k$ of G with the following properties:*

- (i) *for every edge $e \in E(G) \setminus dom(\phi)$, $|V(e)| < h + k$, $e \in dom(\mu)$, $w_1, \dots w_k$ are the first k terms of $\mu(e)$, and $\Delta(e)$ contains every grouping with vertex set $V(e)$ in which $w_1, \dots, w_k$ all have degree 0*
- (ii) *$\Delta(e)$ is free for all $e \in dom(\phi)$*
- (iii) *there is no $e \in dom(\mu) \cap dom(\phi)$ with $|V(e)| = h$ and $\xi \leq \phi(e)$*
- (iv) *if $dom(\phi) \neq E(G)$ then there exists $g \in dom(\mu) \setminus dom(\phi)$ such that either*
  - (a) *$V(\phi^{-1}(\Gamma_1)) \cup V(g)$ includes every muscle of P, or*
  - (b) *$h = 2$, and for each $e \in E(G) \setminus dom(\phi)$ there is a subgraph F_e of $K_{V(e)}$ with $V(F_e) = V(e)$ satisfying*
    1. *for every muscle $\{u, v\}$ not included in $V(\phi^{-1}(\Gamma_1)) \cup V(g)$ there exists $e \in E(G) \setminus dom(\phi)$ such that u, v are adjacent in F_e*
    2. *for each $e \in E(G) \setminus dom(\phi)$, $\Delta(e)$ contains every grouping δ with $V(\delta) = V(e)$, $|E(\delta)| = 1$ and $E(\delta) \subseteq E(F_e)$*
    3. *for each $e \in E(G) \setminus dom(\phi)$, every edge of F_e has an end (and hence exactly one end, since $|V(e) \setminus \{w_1, \dots, w_k\}| < h = 2$) in $V(e) \setminus V(g)$.*

Then $\mathcal{C}$ is well-behaved.

Proof. Let Ω' be a proper well-quasi-order with $\Omega \leq \Omega'$ and $E(\Omega') \cap (\Gamma_1 \cup \Gamma_2) \subseteq E(\Omega)$; and for each $i \geq 1$ let $P'_i = (G_i, \mu_i, \Delta_i, \phi'_i)$ be a strict Ω'-completion of $P_i = (G_i, \mu_i, \Delta_i, \phi_i) \in \mathcal{S}$. By 5.2, it suffices to show that P'_i $(i = 1, 2, \dots)$ is not a bad sequence.

By restricting to a subsequence, we may assume that either $dom(\phi_i) = E(G_i)$ for all $i \geq 1$, or $dom(\phi_i) \neq E(G_i)$ for all $i \geq 1$. In the first case, each P_i is a skeletal Ω-patchwork, and $P'_i = P_i$. Let P_i $(i = 1, 2, \dots)$ have shadow $\Sigma' \leq \Sigma$; then $\Sigma' \neq \Sigma$, since $\xi \notin \Omega_h(\Sigma')$ by condition (iii). Thus Σ' is not evil, and so P_i $(i = 1, 2, \dots)$ is not bad, as required.

We may therefore assume that $dom(\phi_i) \neq E(G_i)$ for all $i \geq 1$. By condition (iv), for each $i \geq 1$ there exists $g_i \in dom(\mu_i) \setminus dom(\phi_i)$ as in (iv). Since there are only finitely many possibilities for $|\bar{\pi}(g_i)|$, we may assume they are all equal, to some integer s say, and also, that the number called k in the statement of the theorem is the same for all P_i. Thus, for each $i \geq 1$ there are distinct $w_{i1}, \dots, w_{ik} \in V(G)$ such that $w_{i1}, \dots, w_{ik}$ are the first k terms of $\mu_i(e)$ for each $e \in E(G_i) \setminus dom(\phi_i)$.

Let $\mathcal{C}$ have shadow Σ'. Let Ω^* be an isomorphic copy of Ω' with $E(\Omega^*) \cap E(\Omega') = \emptyset$, and $E(\Omega^*) \cap (\Gamma_1 \cup \Gamma_2) = \emptyset$, and let $E(\Omega^*) = \{x^* : x \in E(\Omega')\}$, such that for $x, y \in E(\Omega')$, $x \leq y$ in Ω if and only if $x^* \leq y^*$ in Ω^*. Let $\gamma_1, \dots \gamma_s \in \Gamma_1 \setminus R_1(\Sigma')$ be distinct.

We assume first that $h \neq 2$. Let Ω'' be the well-quasi-order with $E(\Omega'') = E(\Omega') \cup E(\Omega^*) \cup \{\gamma_1, \ldots, \gamma_s\}$, where for distinct $x, y \in E(\Omega'')$, $x \leq y$ in Ω'' if and only if either $x, y \in E(\Omega')$ and $x \leq y$ in Ω', or $x, y \in E(\Omega^*)$ and $x \leq y$ in Ω^*. For each $i \geq 1$, let $\mu_i(g_i) = (w_{i1}, \ldots, w_{is})$ say, and let $P_i'' = (G_i'', \mu_i'', \Delta_i'', \phi_i'')$ be the Ω''-patchwork defined as follows:

- $V(G_i'') = V(G_i)$, $E(G_i'') = E(G_i) \cup \{e_{i1}, \ldots, e_{is}\}$, where $e_{i1}, \ldots, e_{is}$ are new elements, $\pi(G_i'') = 0$, and for $v \in V(G_i'')$ and $e \in E(G_i'')$, e is incident with v in G_i'' if and only if either
  - $e \in dom(\phi_i)$ and e is incident with v in G_i, or
  - $e \in E(G_i) \setminus dom(\phi_i)$ and e is incident with v in G_i and $v \neq w_{i1}, \ldots, w_{ik}$, or
  - $e \in \{e_{i1}, \ldots, e_{is}\}$, $e = e_{ij}$ say, and $v = w_{ij}$
- for $e \in dom(\phi_i)$, $\mu_i''(e) = \mu_i(e)$; for $e \in E(G_i) \setminus dom(\phi_i)$, $\mu_i''(e)$ is obtained from $\mu_i(e)$ by removing the first k terms; and for $1 \leq j \leq s$, $\mu_i''(e_{ij}) = (w_{ij})$
- P_i'' is free (this determines Δ_i'')
- for $e \in dom(\phi_i)$, $\phi_i''(e) = \phi_i'(e)$; for $e \in E(G_i) \setminus dom(\phi_i)$, $\phi_i''(e) = (\phi_i'(e))^*$; and for $1 \leq j \leq s$, $\phi_i''(e_{ij}) = \gamma_j$.

Then P_i'' is a skeletal Ω''-patchwork (since $h \neq 2$). Let the sequence P_i'' $(i = 1, 2, \ldots)$ have shadow Σ'' say. Since $\mathcal{C}$ has shadow $\Sigma' \leq \Sigma$, and $\xi \in E(\Omega_h(\Sigma))$, and for all $i \geq 1$ there is no edge $e \in dom(\mu_i'') \cap dom(\phi_i'')$ with $|V(e)| = h$ and $\xi \leq \phi_i''(e)$, and every edge $e \in E(G_i'') \setminus dom(\phi_i)$ satisfies $e \in dom(\mu_i'')$ and $|V(e)| < h$, it follows that $\Sigma'' < \Sigma$, and so Σ'' is not evil. Hence there exist $j > i \geq 1$ such that P_i'' is simulated in P_j''. But then it follows easily that P_i' is simulated in P_j', as required.

Now we assume that $h = 2$. Let $\gamma_1^2, \ldots, \gamma_k^2 \in \Gamma_2 \setminus R_2(\Sigma')$ be distinct, and let Ω'' be the well-quasi-order with

$$E(\Omega'') = E(\Omega') \cup E(\Omega^*) \cup \{\gamma_1^1, \ldots, \gamma_s^1, \gamma_1^2, \ldots, \gamma_k^2\}$$

where for distinct $x, y \in E(\Omega'')$, $x \leq y$ in Ω'' if and only if either $x, y \in E(\Omega')$ and $x \leq y$ in Ω', or $x, y \in E(\Omega^*)$ and $x \leq y$ in Ω^*. For each $i \geq 1$, and each $e \in E(G_i) \setminus dom(\phi_i)$, let $F_{e,i}$ be the subgraph called F_e in statement (iv) of the theorem. Let F_i be the union of all the subgraphs $F_{e,i}$ $(e \in E(G_i) \setminus dom(\phi_i))$; thus, $F_i \subseteq K_{V(G_i)}$. We may assume without loss of generality that $E(F_i) \cap E(G_i) = \emptyset$ (by replacing G_i by an isomorphic hypergraph). For $\mu_i(g_i)$ be $(w_{i1}, \ldots, w_{is})$ say. Let $P_i'' = (G_i'', \mu_i'', \Delta_i'', \phi_i'')$ be the Ω''-patchwork defined as follows:

- $V(G_i'') = V(G_i)$, $E(G_i'') = E(G_i) \cup \{e_{i1}, \ldots e_{is}\} \cup E(F_i)$, where $e_{i1}, \ldots, e_{is}$ are new elements, $\pi(G_i'') = \emptyset$, and for $v \in V(G_i'')$ and $e \in E(G_i'')$, e is incident with v in G_i'' if and only if either
  - $e \in dom(\phi_i)$ and e is incident with v in G_i, or
  - $e \in E(G_i) \setminus dom(\phi_i)$ and e is incident with v in G_i and $v \neq w_{i1}, \ldots, w_{ik}$, or
  - $e \in \{e_{i1}, \ldots, e_{is}\}$, $e = e_{ij}$ say, and $v = w_{ij}$, or
  - $e \in E(F_i)$, and v is incident with e in F_i.
- for $e \in dom(\phi_i)$, $\mu_i''(e) = \mu_i(e)$; for $e \in E(G_i) \setminus dom(\phi_i)$, $\mu_i''(e)$ is obtained from $\mu_i(e)$ by removing the first k terms; for $1 \leq j \leq s$, $\mu_i''(e_{ij}) = (w_{ij})$; and for $e \in E(F_i)$, $\mu_i''(e) = (u, v)$ where u, v are the ends of e in F_i, and $u \in \{w_{i1}, \ldots, w_{ik}\}$ and $v \notin \{w_{i1}, \ldots, w_{ik}\}$

- P_i'' is free (this determines Δ_i'')
- for $e \in dom(\phi_i)$, $\phi_i''(e) = \phi_i'(e)$; for $e \in E(G_i) \setminus dom(\phi_i)$, $\phi_i''(e) = (\phi'(e))^*$; for $1 \le j \le s$, $\phi_i''(e_{ij}) = \gamma_j^1$; and for $1 \le j \le k$ and $e \in F_i$ incident with w_{ij}, $\phi_i''(e) = \gamma_j^2$.

We claim that P_i'' is skeletal. For it is free, rootless and proper; let $\{u, v\}$ be a muscle of P_i'', and suppose that $\{u, v\} \not\subseteq V(\phi_i''^{-1}(\Gamma_1))$. Thus, $\{u, v\} \not\subseteq V(\phi_i^{-1}(\Gamma_1)) \cup V(g)$, and so by statement (iv) of the theorem, there exists $e \in E(G_i) \setminus dom(\phi_i)$ such that u, v are adjacent in $F_{i,e}$. But then there is an edge of F_i incident with u and v, contradicting that $\{u, v\}$ is a muscle of P_i''. This proves that P_i'' is skeletal.

Let the sequence P_i'' $(i = 1, 2, \ldots)$ have shadow Σ''. We claim that $\Sigma'' < \Sigma$. For $\Sigma' \le \Sigma$, and $\Omega_\infty(\Sigma'') \le \Omega_\infty(\Sigma')$ and $m(\Sigma'') \le m(\Sigma')$, so we may assume that $m(\Sigma'') = m(\Sigma')$. For $2 \le j \le m(\Sigma')$, $\Omega_j(\Sigma'') \le \Omega_j(\Sigma)$, since the only edges of P_i'' which are "new" have either ≤ 1 end in G_i'' or have two ends and belong to $\phi_i''^{-1}(\Gamma_2)$. Since $\xi \notin E(\Omega_2(\Sigma''))$ and $\xi \in E(\Omega_2(\Sigma))$ it follows that $\Sigma'' < \Sigma$, as claimed.

Thus Σ'' is not evil, and so there exist $j > i \ge 1$ and a realizable expansion η of P_i'' in P_j''. For each $e \in E(G_j'') \setminus \eta(E(G_i''))$ let $\delta_e'' \in \Delta_j''(e)$, such that for each $v \in V(G_i''), \eta(v)$ is the vertex set of a connected component of

$$H = N(G_j'') \cup \bigcup(\delta_e'' : e \in E(G_j'') \setminus \eta(E(G_i''))) \, .$$

For each $e \in E(G_j) \setminus dom(\phi_j)$ we define $\delta_e \in \Delta_j(e)$ as follows. $V(\delta_e)$ is the set of vertices incident with e in G_j. If there is no $f \in E(F_{j,e}) \setminus \eta(E(G_i''))$ with $E(\delta_f'') \neq \emptyset$, we take $E(\delta_e) = \emptyset$. If there is some such edge, it is necessarily unique, for otherwise two distinct members of $V(\phi_j^{-1}(\{\gamma_1^1, \ldots, \gamma_s^1\}))$ would belong to the same component of H, contradicting that η is an expansion of P_i'' in P_j''. Thus, if f is such an edge we define $E(\delta_e) = \{f\}$ (we recall that $f \in E(K_{V(e)})$). From statement (iv) of the theorem, in both cases $\delta_e \in \Delta_j(e)$.

We claim that if $e \in E(G_j) \setminus dom(\phi_j)$ and $E(\delta_e) \neq \emptyset$, then $e \notin \eta(E(G_i''))$. For suppose that $e = \eta(e_0)$ say, where $e_0 \in E(G_i'')$. Since $\phi_i''(e_0) \le \phi_j''(e) \in E(\Omega^*)$, it follows that $\phi_i''(e_0) \in E(\Omega^*)$ and so $e_0 \in E(G_i) \setminus dom(\phi_i)$. Consequently, none of $w_{i1}, \ldots, w_{ik}$ are incident with e_0 in G_i''; and hence (since $w_{jt} \in \eta(w_{it})$ for $1 \le t \le k$) it follows that none of $w_{j1}, \ldots, w_{jk}$ are in the same component of H as any end of e in G_j''. Hence there is no $f \in E(F_{j,e}) \setminus \eta(E(G_i''))$ with $E(\delta_f'') \neq \emptyset$, and so $E(\delta_e) = \emptyset$, as claimed.

It follows that, if we define $\delta_e = \delta_e''$ for $e \in E(G_j) \cap E(G_j'') \setminus \eta(E(G_i''))$, then

$$H = N(G_j) \cup \bigcup(\delta_e : e \in E(G_j) \setminus \eta(E(G_i''))) \, ,$$

for the edges of G_j'' with only one end that are missing in G_j contribute nothing to H. Hence, the restriction of η to $V(G_i) \cup E(G_i)$ provides a realizable expansion of P_i' in P_j'. This proves 8.1. ∎

Our analogue of 5.3 is the following.

8.2 *Let Ω be a proper well-quasi-order, and let $\mathcal{S}$ be a set of skeletal Ω-patchworks with shadow Σ. Let Σ be sharp. Let $w \ge 0$, $h \ge 1$ be integers and let $\xi \in E(\Omega_h(\Sigma))$. Then there is a well-behaved set $\mathcal{C}$ of proper partial Ω-patchworks with the following property. Let $P = (G, \mu, \Delta, \phi) \in \mathcal{S}$, and let $\mathcal{T}$ be a tangle in G of order $\ge (w+h)^{h+1} + h$ that is (ξ, h, w)-restricted internally, and let λ be an edge-based tie-breaker in G. There is a rooted location $\mathcal{L}$ in G which $((w+h)^{h+1} + h)$-isolates $\mathcal{T}$ with respect to λ such that $(P, \mathcal{L})$ has a heart in $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be the set of all rootless proper partial Ω-patchworks $P = (G, \mu, \Delta, \phi)$ satisfying conditions (i)-(iv) of 8.1, such that P is a heart of $(P', \mathcal{L}')$ for some $P' = (G', \mu', \Delta', \phi') \in \mathcal{S}$ and some rooted location $\mathcal{L}$ in G. Since $\mathcal{S}$ has shadow Σ, it follows that $\mathcal{C}$ has shadow $\leq \Sigma$, and so by 8.1, $\mathcal{C}$ is well-behaved.

We claim that $\mathcal{C}$ satisfies the theorem. Let $P = (G, \mu, \Delta, \phi) \in \mathcal{S}$ and let $\mathcal{T}$ be a tangle in G of order $\geq (w+h)^{h+1} + h$ that is (ξ, h, w)-restricted internally, and let λ be an edge-based tie-breaker in G. Let

$$J = \{e \in dom(\mu) : |V(e)| = h \,\text{and}\, \xi \leq \phi(e)\}.$$

Since $\mathcal{T}$ is (ξ, h, w)-restricted internally, there exists $W \subseteq V(G)$ with $|W| \leq w$, free relative to $\mathcal{T}$, such that for every $e \in J$, either $V(e) \cap W \neq \emptyset$ or $W \cup V(e)$ is not free relative to $\mathcal{T}$. By 7.4, there is a (J, h)-separator $(X, \mathcal{L})$ such that $|X| \leq (w+h)^{h+1}$, $\mathcal{L}$ is λ-linked to $\mathcal{T}$, and for each $(A, B) \in \mathcal{L}$ there exists $e \in J \cap E(A)$ such that there are $|V(A \cap B) \setminus X|$ disjoint paths of $sk(A) \backslash X$ between $V(A \cap B) \setminus X$ and $V(e)$, and there is at most one $(A, B) \in \mathcal{L}$ such that $V(e) \subseteq V(A \cap B)$ for some $e \in E(A) \setminus J$.

Let $X = \{w_1, \ldots, w_k\}$, say. Let $\mathcal{L}'$ be a rooted location in G such that $\mathcal{L}'^{-} = \mathcal{L}$, and for each $A \in \mathcal{L}'$ the first k terms of $\pi(A)$ are $w_1, \ldots, w_k$. Then $\mathcal{L}'$ $((w+h)^{h+1} + h)$-isolates $\mathcal{T}$ with respect to λ, by theorem 7.1 of [5].

Let the heart of $(P, \mathcal{L}')$ be $P' = (G', \mu', \Delta', \phi')$. We wish to show that $P' \in \mathcal{C}$, and therefore it suffices to check conditions (i)-(iv) of 8.1. For condition (i), let $A \in \mathcal{L}'$; then there exists $e \in J \cap E(A)$ and $|\bar{\pi}(A) \setminus X|$ disjoint paths of $sk(A^{-}) \backslash X$ between $\bar{\pi}(A) \setminus X$ and $V(e)$, and since $\Delta(e)$ is free it follows that every grouping with vertex set $\bar{\pi}(A)$ in which $w_1, \ldots, w_k$ all have degree 0 is feasible in $P|A$. Thus condition (i) of 8.1 is satisfied. Condition (ii) holds since P is free, and condition (iii) since each $e \in J$ belongs to some member of $\mathcal{L}'$.

It remains to check condition (iv). We may therefore assume that $dom(\phi') \neq E(G')$. From the choice of $X, \mathcal{L}$ there is at most one $(A, B) \in \mathcal{L}$ such that $V(e) \subseteq V(A \cap B)$ for some $e \in E(A) \setminus J$; and hence, since $dom(\phi') \neq E(G')$, we may choose $g \in E(G') \backslash dom(\phi') \subseteq dom(\mu')$ such that $V(e) \subseteq V'(g)$ for every edge e of G not in J that satisfies $V(e) \subseteq V(G')$ and $e \notin E(G')$. (To disambiguate $V(e)$ we shall henceforth in this proof use $V(e)$ to denote the set of ends of e in G, and $V'(e)$ to denote its ends in G'.)

If $V(\phi'^{-1}(\Gamma_1)) \cup V'(g)$ includes every muscle of P' then 8.1(iv) holds and we are done. Thus, let $\{u, v\}$ be a muscle of P' not included in $V(\phi'^{-1}(\Gamma_1)) \cup V'(g)$. We may assume that $v \notin V(\phi'^{-1}(\Gamma_1)) \cup V'(g)$. Suppose first that $v \in V(\phi^{-1}(\Gamma_1))$, and let $e \in E(G)$ with $V(e) = \{v\}$ and $\phi(e) \in \Gamma_1$. Since $v \notin V(\phi'^{-1}(\Gamma_1))$ it follows that $e \notin E(G')$, and so either $e \in J$ or $v \in V'(g)$. The first is impossible since $\phi(e) \in \Gamma_1$, and the second since $v \notin V'(g)$. Thus, $v \notin V(\phi^{-1}(\Gamma_1))$. Hence $\{u, v\}$ is not a muscle of P, and so there exists $e_0 \in dom(\mu)$ with $V(e_0) = \{u, v\}$. Since $e_0 \notin E(G')$, and $V(e_0) \subseteq V(G')$ and $V(e_0) \not\subseteq V'(g)$, it follows from the choice of g that $e_0 \in J$. Hence $h = 2$.

For each $e \in E(G') \setminus dom(\phi')$, let A be the corresponding member of $\mathcal{L}'$, and let F_e be the subgraph of $K_{V'(e)}$ in which distinct u, v are adjacent if $\{u, v\} \not\subseteq V'(g)$ and $V(e_0) = \{u, v\}$ for some $e_0 \in E(A) \cap dom(\mu)$. It follows that 8.1(iv)(b) (1)-(3) hold. Hence $P' \in \mathcal{C}$. This proves 8.2. ∎

9 Completing the proof

Let us combine the results of sections 5, 6 and 8 with those of sections 2 and 4 to prove the following, which implies 2.1 as we saw in section 3.

9.1 *There is no sharp shadow.*

Proof. Suppose that Σ is a sharp shadow. Let Ω be a proper well-quasi-order such that there is a bad sequence P_i $(i = 0, 1, 2, \ldots)$ of skeletal Ω-patchworks with shadow Σ. Let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i \geq 0)$.

(1) *We may assume that $V(e) \neq \emptyset$ for all $i \geq 0$ and all $e \in E(G_i)$.*

Subproof. Let (A_i, B_i) be a separation of G_i, where $\bar{\pi}(A_i) = \emptyset = \bar{\pi}(B_i)$, every edge of B_i has ≥ 1 end, and every edge of A_i has no ends. Since the Ω-patchworks $P_i|A_i$ are well-quasi-ordered by simulation, we may assume that $P_i|A_i$ is simulated in $P_j|A_j$ for all $j > i \geq 0$. But then $P_i|B_i$ $(i = 0, 1, \ldots)$ is a bad sequence with shadow $\leq \Sigma$, in which $V(e) \neq \emptyset$ for every edge. This proves (1).

By restricting to a subsequence, we may assume from 5.3 that there exist $\Gamma_1^* \subseteq \Gamma_1$ and $\Gamma_2^* \subseteq \Gamma_2$ such that for all $i \geq 1$,

$$\begin{aligned}
\Gamma_1 \cap \phi_i(E(G_i)) &= \Gamma_1^* \\
\Gamma_2 \cap \phi_i(E(G_i)) &= \Gamma_2^* \\
|V(\phi_i^{-1}(\Gamma_1))| &= |\phi_i^{-1}(\Gamma_1)|
\end{aligned}$$

and

$$V(\phi_i^{-1}(\Gamma_1)) \cap V(\phi_i^{-1}(\Gamma_2)) = \emptyset.$$

Let $\Gamma_2^* = \{\gamma_1, \ldots, \gamma_k\}$ say. For $i \geq 1, \phi_i^{-2}(\gamma_1), \ldots, \phi_i^{-2}(\gamma_k)$ need not all be distinct; but by restricting to a subsequence, we can assume that

(2) *For all $i \geq 1$, and all a, b with $1 \leq a < b \leq k$, $\phi_i^{-2}(\gamma_a) = \phi_i^{-2}(\gamma_b)$ if and only if $\phi_0^{-2}(\gamma_a) = \phi_0^{-2}(\gamma_b)$.*

Let $\mathcal{S} = \{P_i : i \geq 0\}$. Let $\mathcal{C}_1$ be such that setting $\mathcal{C} = \mathcal{C}_1$ satisfies 5.5. Let n be an integer with $n \geq \frac{3}{2}|V(G_0)|\ (|E(G_0)| + 2)$. For each $\gamma \in \Gamma_2^*$, let $\mathcal{C}_2(\gamma)$ be such that setting $\mathcal{C} = \mathcal{C}_2(\gamma)$ satisfies 5.7. Let $\mathcal{C}_2 = \bigcup(\mathcal{C}_2(\gamma) : \gamma \in \Gamma_2^*)$.

For each $e \in E(G_0) = dom(\mu_0)$, let $\mathcal{C}_3(e)$ be such that setting $h = n$, $t = |E(G_0)|$, $\xi = \phi_0(e)$ and $\mathcal{C} = \mathcal{C}_3(e)$ satisfies 6.2. Let $\mathcal{C}_3 = \bigcup(\mathcal{C}_3(e) : e \in E(G_0) \setminus dom(\mu_0))$.

For each $e \in dom(\mu_0)$, let $\mathcal{C}_4(e)$ be such that setting $w = n$, $h = |V(e)|$, $\xi = \phi_0(e)$ and $\mathcal{C} = \mathcal{C}_4(e)$ satisfies 8.2. Let $\mathcal{C}_4 = \bigcup(\mathcal{C}_4(e) : e \in dom(\mu_0))$.

Since all these sets are well-behaved, their union $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ is also well-behaved.

Let $\theta = (2n)^{|V(G_0)|+1}$. Let $i \geq 1$, let λ be an edge-based tie-breaker in G_i, and let $\mathcal{T}$ be a tangle in G_i of order $\geq \theta$ controlling a K_n-minor of $sk(G_i^-)$. We claim that there is a rooted location $\mathcal{L}$ in G_i that θ-isolates $\mathcal{T}$ with respect to λ, such that $(P_i, \mathcal{L})$ has heart in $\mathcal{C}$. For P_0 is not simulated in P_i, and so by (1) and 4.2, one of the five conditions of 4.2 is false. In each case (by (2), 5.5, 5.7, 6.2 and 8.2) it follows that the required $\mathcal{L}$ exists.

By 2.2, there exist $j > i \geq 1$ such that P_i is simulated in P_j, a contradiction. This proves 9.1. ∎

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