Graph Minors
XVIII. Tree-Decompositions and Well-Quasi-Ordering

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Abstract

We prove the following result. Suppose that for every graph G in a class C of graphs, and for every "highly connected component" of G, there is a decomposition of G of a certain kind centred on the component. Then C is well-quasi-ordered by minors; that is, in any infinite subset of C there are two graphs, one a minor of the other. This is another step towards Wagner's conjecture.

1 Introduction

It was shown in an earlier paper [2] that if each member G of a class C of finite graphs has a "linked tree-decomposition" into "well-behaved" pieces, then C is well-quasi-ordered by minors; that is, in every infinite subset of C there are two graphs, one a minor of the other. It was also shown, in another earlier paper [3], that for every finite graph G there is a linked tree-decomposition into pieces corresponding to the large order "tangles" in G. (A tangle of order θ in G is, more or less, a θ-connected component of G.) In the present paper we combine these results into a lemma that if, for every $G \in C$ and for every large order tangle of G, there is a decomposition of G with certain properties centred on the tangle, then C is well-quasi-ordered by minors.

This lemma is crucial in the proof of Wagner's conjecture, that the class of all finite graphs is well-quasi-ordered by minors; indeed, we shall need it twice to prove that conjecture, first to prove that the class of finite hypergraphs with edges of size 2 or 3 drawable on a fixed surface is well-quasi-ordered, and secondly to derive from this that the class of all finite graphs is well-quasi-ordered. It will also be needed in later papers, again in a hypergraph form, to prove Nash-Williams' "immersion" conjecture [1]. We shall therefore formulate it completely in terms of hypergraphs.

The paper is organized as follows. Section 2 contains basic definitions and results about tree-decompositions and tangles. Sections 3 and 4 develop the relation between the kinds of decomposition relative to a tangle that we need. In section 5 we introduce patchworks, which enable us to define minors of hypergraphs, and develop some lemmas about them. The main result is stated and proved in section 6, and section 7 contains a lemma which is often useful in applying the theorem.

Thus this work falls into two parts. Sections 2-4 are about how to convert information about the local structure of a hypergraph relative to each of its high-order tangles, into a linked tree-decomposition whose pieces (the nodes of the tree) correspond to the high-order tangles, and still have the same local structure (more or less — we may have to grow the pieces to make them fit together by adding on subhypergraphs of bounded tree-width). Here the tree-decompositions use unrooted trees; there is no reason to fix a root for the tree, and if we did so the results would appear most unnatural. The second half, sections 5-6, mostly concerns well-quasi-ordering, and in that topic we have to use *rooted* trees; we have to do complicated inductions concerning the sizes of these trees, and it is very important to fix a root of the tree. When we do so, for each piece of the tree-decomposition, there is not symmetry between its neighbouring pieces any more; one is in the direction of the root, and has to be treated differently. When we lop off the arms of the tree-decomposition growing out from a given piece, and replace these arms by new hyperedges marking where the arms used to attach (which is what we mean by the local structure at the node of the tree), it is convenient to lop off the "root arm" in a different way; instead of replacing it by a new hyperedge, we simply label the vertices where it used to attach and call them roots of the hypergraph. And also, when we lop off the "non-root" arms, we need to remember not only the set of vertices where the arm used to attach, but also which of these vertices was which; we need to remember an ordered set. So the new hyperedge replacing the arm will have to be equipped with a linear order of its vertex set. The point is that half-way through the paper, suddenly our trees become rooted trees or "arborescences", and the hypergraphs develop roots, and their hyperedges become ordered sets of vertices. This is most confusing when it happens (particular since we have to redefine all our terms for rooted trees and rooted hypergraphs, and there is not quite an exact correspondence), and we hope it will help the reader to be warned ahead of time.

2 Hypergraphs, tangles and tree-decompositions

For the purposes of this paper, a *hypergraph* G consists of a finite set $V(G)$ of *vertices*, a finite set $E(G)$ of *edges*, and an incidence relation between them. The vertices incident with an edge are the *ends* of the edge. (A hypergraph is thus a graph if every edge has one or two ends.) A hypergraph H is a *subhypergraph* of a hypergraph G (written $H \subseteq G$) if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and for every $v \in V(G)$ and $e \in E(H)$, e is incident with v in G if and only if $v \in V(H)$ and e is incident with v in H. If A, B are subhypergraphs of G we denote by $A \cup B$, $A \cap B$ the subhypergraphs with vertex sets $V(A) \cup V(B)$, $V(A) \cap V(B)$ and edge sets $E(A) \cup E(B)$, $E(A) \cap E(B)$ respectively. A *separation* of G is a pair (A, B) of subhypergraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$; its *order* is $|V(A \cap B)|$, and its *reverse* is (B, A).

A central idea in our approach is that of a *tangle* in a hypergraph, which was introduced in [3]. Intuitively, a tangle of order θ in a hypergraph G may be thought of as a "θ-connected component" of G, a highly coherent mass in G which resides almost completely on one side or the other of every separation of order $< \theta$. Formally, let G be a hypergraph and $\theta \geq 1$ an integer. A *tangle* of order θ in G is a set $\mathcal{T}$ of separations of G, each of order $< \theta$, such that

(T1) for every separation (A, B) of G of order $< \theta, \mathcal{T}$ contains one of (A, B), (B, A)

(T2) if $(A_i, B_i) \in \mathcal{T} (i = 1, 2, 3)$ then $A_1 \cup A_2 \cup A_3 \neq G$

(T3) if $(A, B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

Let us mention one lemma that we shall need later.

2.1 *Let G be a hypergraph, let $G' \subseteq G$, and let $\mathcal{T}'$ be a tangle in G' of order θ. Let $\mathcal{T}$ be the set of all separations (A, B) of G of order $< \theta$ such that $(A \cap G', B \cap G') \in \mathcal{T}'$. Then $\mathcal{T}$ is a tangle in G of order θ.*

The proof is clear.

The second concept we need is that of tree-decomposition. A *tree* is a non-null connected graph without circuits. A *tree-decomposition* of a hypergraph G is a pair (T, τ), where T is a tree and τ assigns to each $t \in V(T)$ a subhypergraph $\tau(t)$ of G, such that

- $\cup(\tau(t) : t \in V(T)) = G$
- for distinct $t_1, t_2 \in V(T), E(\tau(t_1) \cap \tau(t_2)) = \emptyset$
- if $t_1, t_2, t_3 \in V(T)$ and t_2 lies on the path between t_1 and t_3 then $\tau(t_1) \cap \tau(t_3) \subseteq \tau(t_2)$.

If T' is a subtree of T we denote $\cup(\tau(t) : t \in V(T'))$ by $\tau \times T'$. If $e \in E(T)$ and T_1, T_2 are the two components of $T \backslash e$ then $(\tau \times T_1, \tau \times T_2)$ and its reverse are the *separations made by e under* (T, τ); their common order is the *order of e in* (T, τ). The tree-decomposition (T, τ) has *width* w if $w \geq 0$ is minimum such that $|V(\tau(t))| \leq w + 1$ for each $t \in V(T)$; and the *tree-width* of a hypergraph G is the minimum width of all tree-decompositions of G. The following is proved in theorem 5.2 of [3].

2.2 *Let G be a hypergraph with no tangle of order $\geq \theta$, where $\theta \geq 1$. Then G has tree-width $\leq \frac{3}{2}\theta$.*

A *location* in G is a set $\mathcal{L}$ of separations of G such that $A_1 \subseteq B_2$ for all distinct (A_1, B_1), $(A_2, B_2) \in \mathcal{L}$. We define $M(G, \mathcal{L})$ to be $\cap(B : (A, B) \in \mathcal{L})$ if $\mathcal{L} \neq \emptyset$, and $M(G, \emptyset) = G$.

2.3 *Let $\mathcal{L} = \{(A_1, B_1), \ldots, (A_n, B_n)\}$ be a location in a hypergraph G. Then*

1. *$A_1, \ldots, A_n, M(G, \mathcal{L})$ are mutually edge-disjoint, and have union G*
2. *for $1 \le i \le n$, $B_i = M(G,\mathcal{L}) \cup \bigcup(A_j : 1 \le j \le n, j \ne i)$, and $A_i \cap M(G, \mathcal{L}) = A_i \cap B_i$*
3. *for $1 \le i < j \le n, A_i \cap A_j \subseteq M(G, \mathcal{L})$, and*
   $V(A_i \cap A_j) = V(A_i \cap B_i) \cap V(A_j \cap B_j) \cap V(M(G, \mathcal{L}))$.

Proof. For $1 \le i \le n$, $M(G, \mathcal{L}) \subseteq B_i$ and $A_j \subseteq B_i$ for $j \ne i$; since $E(A_i \cap B_i) = \emptyset$, the first assertion of 2.3.1 follows. For the second assertion of 2.3.1, we observe that any vertex or edge of G not in $M(G, \mathcal{L})$ fails to belong to some B_i, and therefore belongs to the corresponding A_i. Thus 2.3.1 holds. For 2.3.2, we have already seen that

$$M(G, \mathcal{L}) \cup \bigcup(A_j : 1 \le j \le n, j \ne i) \subseteq B_i.$$

Conversely, any vertex or edge of B_i not in $M(G, \mathcal{L})$ fails to belong to B_j for some $j \ne i$, and hence belongs to A_j. This proves the first assertion of 2.3.2, and the second will follow from the first and 2.3.3. For 2.3.3, let $1 \le i < j \le n$. By 2.3.1, $E(A_i \cap A_j) = \emptyset$; let $v \in V(A_i \cap A_j)$. For $1 \le k \le n$, if $k \ne i$ then $v \in V(A_i) \subseteq V(B_k)$, and if $k = i$ then $v \in V(A_j) \subseteq V(B_k)$. Thus $v \in V(B_k)$ for all $k\,(1 \le k \le n)$, and so $v \in V(M(G, \mathcal{L}))$. This proves the first assertion of 2.3.3. For the second, $A_i \cap A_j \subseteq A_i \cap B_i$ since $A_j \subseteq B_i$, and similarly $A_i \cap A_j \subseteq A_j \cap B_j$, and so the second assertion of 2.3.3 follows. This proves 2.3. ∎

The following is easily seen to be true (compare theorem 9.1 of [3]).

2.4 *Let (T, τ) be a tree-decomposition of a hypergraph G, let $t_0 \in V(T)$ and let $e_1, \ldots, e_k$ be the edges of T incident with t_0. For $1 \le i \le k$ let the components of $T \backslash e_i$ be T_i, T'_i, where $t_0 \in V(T'_i)$. Then*

$$(\tau \times T_i, \tau \times T'_i) : 1 \le i \le k$$

is a location.

We call this the *location of* t_0 in (T, τ). It is possible that $(\tau \times T_i, \tau \times T'_i) = (\tau \times T_j, \tau \times T'_j)$ for distinct i, j, but only if $\tau \times T'_i = G$. We say (T, τ) is *proper* if no edge of T makes a separation (A, B) with $B = G$.

2.5 *Let (T, τ) be a tree-decomposition of a hypergraph G, and let $t \in V(T)$. Let $\mathcal{L}$ be the location of t in (T, τ); then $\tau(t) = M(G, \mathcal{L})$.*

Proof. Certainly $\tau(t) \subseteq B$ for every $(A, B) \in \mathcal{L}$ and so $\tau(t) \subseteq M(G, \mathcal{L})$. For the converse inclusion, let x be a vertex or edge of G not in $\tau(t)$, and choose $t' \in V(T)$ with x in $\tau(t')$. Let e be the edge of T incident with t such that t, t' are in different components of $T \backslash e$, and let $(A, B) \in \mathcal{L}$ be the corresponding separation. Then $A \cap B \subseteq \tau(t)$, and so x is not in $A \cap B$; but x is in A, and so is not in B. Hence x is not in $M(G, \mathcal{L})$. This proves 2.5. ∎

For several purposes it would be convenient if there were at most one smallest order separation with a given property, and we can more or less arrange this by a refinement in the definition of the order of separation. A *tie-breaker* in a hypergraph G is a function λ which maps each separation (A,B) of G to some member $\lambda(A,B)$ of a linearly ordered set $(\Lambda,\leq)$ (we call $\lambda(A,B)$ the λ-*order* of (A,B)) in such a way that

- $\lambda(A,B)=\lambda(C,D)$ if and only if $(A,B)=(C,D)$ or $(A,B)=(D,C)$
- for all separations (A,B), (C,D), either $\lambda(A\cup C, B\cap D)\leq\lambda(A,B)$, or $\lambda(A\cap C, B\cup D)<\lambda(C,D)$
- if $|V(A\cap B)|<|V(C\cap D)|$ then $\lambda(A,B)<\lambda(C,D)$.

It was shown in theorem 9.2 of [3] that every hypergraph has a tie-breaker.

Let $\mathcal{T}_1,\mathcal{T}_2$ be tangles in a hypergraph G. If $(A,B)\in\mathcal{T}_1$ and $(B,A)\in\mathcal{T}_2$ we say that (A,B) *distinguishes* $\mathcal{T}_1$ *from* $\mathcal{T}_2$. If there is such an (A,B), then for a given tie-breaker λ in G there is a unique $(A,B)\in\mathcal{T}_1$ such that $(B,A)\in\mathcal{T}_2$ of minimum λ-order, called the $(\mathcal{T}_1,\mathcal{T}_2)$-*distinction*; and if (A,B) is the $(\mathcal{T}_1,\mathcal{T}_2)$-distinction then (B,A) is the $(\mathcal{T}_2,\mathcal{T}_1)$-distinction. By theorem 10.3 of [3], we have

2.6 *Let $\mathcal{T}_1,\ldots,\mathcal{T}_n$ be distinct tangles of order θ in a hypergraph G with $n\geq 1$, and let λ be a tie-breaker. Then there is a tree-decomposition (T,τ) of G where $V(T)=\{t_1,\ldots,t_n\}$, with the following properties:*

1. *if $e\in E(T)$ and T_1, T_2 are the components of $T\backslash e$ and $1\leq i\leq n$ and $t_i\in V(T_2)$ then $(\tau\times T_1,\tau\times T_2)\in\mathcal{T}_i$*
2. *for $1\leq i<j\leq n$, let e be the edge of the path of T between t_i,t_j making separations of minimum λ-order; then these separations are the $(\mathcal{T}_i,\mathcal{T}_j)$- and $(\mathcal{T}_j,\mathcal{T}_i)$-distinctions.*

We call (T,τ) a *standard* decomposition of G relative to $\mathcal{T}_1,\ldots,\mathcal{T}_n$ in which t_i *represents* $\mathcal{T}_i$ for $i=1,\ldots,n$.

A separation (A,B) of a hypergraph G is *robust* if for every separation (C,D) of A, one of the separations $(C,B\cup D)$, $(D,B\cup C)$ has order at least that of (A,B). A tree-decomposition (T,τ) of a hypergraph G is *rotund* if for every two edges $f_1,f_2\in E(T)$, making separations (A_1,B_1) and (A_2,B_2) of the same order k, where $B_2\subseteq A_1$ and $B_1\subseteq A_2$, the following holds: if there is a separation (H_1,H_2) of G with $B_1\subseteq H_1$ and $B_2\subseteq H_2$ of order $<k$, then some edge of F makes a separation of order $<k$, where F is the path of T with first and last edges f_1,f_2.

2.7 *Let $\mathcal{T}_1,\ldots,\mathcal{T}_n$ be distinct tangles of order θ in a hypergraph G with $n\geq 1$, and let λ be a tie-breaker. Let (T,τ) be as in 2.6. Then (T,τ) is proper and rotund, and every separation made by an edge of T under (T,τ) is robust.*

Proof. Let $V(T)=\{t_1,\ldots,t_n\}$ where t_i represents $\mathcal{T}_i$ $(1\leq i\leq n)$. Let $e\in E(T)$, making separations (A,B), (B,A). Then (A,B) is the $(\mathcal{T}_i,\mathcal{T}_j)$-distinction where t_i,t_j are the ends of e, and so $(A,B)\in\mathcal{T}_i$, and $V(A)\neq V(G)$ by (**T3**). Thus (T,τ) is proper. From theorem 10.2 of [3], (A,B) is robust. It remains to show that (T,τ) is rotund.

Thus, let $f_1,f_2\in E(T)$, and let F be the path of T with first and last edges f_1,f_2. Let f_1,f_2 make separations (A_1,B_1), (A_2,B_2) respectively, where $B_1\subseteq A_2$ and $B_2\subseteq A_1$; and suppose that both these separations have order k. Let (H_1, H_2) be a separation of G of order $k' < k$ with $B_1 \subseteq H_1$ and $B_2 \subseteq H_2$, and let the first and last vertices of F be t_1, t_2 say. Now $(A_1, B_1) \in \mathcal{T}_1$, and so $(H_1, H_2) \notin \mathcal{T}_1$ by (**T2**), since $A_1 \cup H_1 \supseteq A_1 \cup B_1 = G$; and so $(H_2, H_1) \in \mathcal{T}_1$ by (**T1**), since $k' < k < \theta$. Similarly $(H_1, H_2) \in \mathcal{T}_2$, and so (H_2, H_1) distinguishes $\mathcal{T}_1$ from $\mathcal{T}_2$. Thus the $(\mathcal{T}_1, \mathcal{T}_2)$-distinction (A, B) has order $\leq k' < k$, and by 2.6.2 (A, B) is made by some edge of F. It follows that (T, τ) is rotund. This proves 2.7. ∎

3 Tree-width of a location

A separation (A, B) of G is *titanic* if at least one of the inequalities

$$\begin{aligned}
|V((X \cup Y) \cap Z)| &\geq |V((X \cup Y) \cap B)| \\
|V((Y \cup Z) \cap X)| &\geq |V((Y \cup Z) \cap B)| \\
|V((Z \cup X) \cap Y)| &\geq |V((Z \cup X) \cap B)|
\end{aligned}$$

holds for every choice of $X, Y, Z \subseteq A$ such that $A = X \cup Y \cup Z$ and $E(X), E(Y), E(Z)$ are mutually disjoint. We observe that whether or not (A, B) is titanic depends only on A and on $V(A \cap B)$; more precisely,

3.1 *Let (A, B) be a separation of a hypergraph G, and let (A, B') be a separation of a hypergraph G', with $A \cap B = A \cap B'$. Then (A, B) is titanic if and only if (A, B') is titanic.*

The proof is clear. From theorem 8.3 of [3], we have the following.

3.2 *Let (C, D) be a separation of a hypergraph G, and let (C', D) be a titanic separation of a hypergraph G', with $V(C \cap D) = V(C' \cap D)$. Let $\mathcal{T}$ be a tangle in G of order $\theta \geq 2$ with $(C, D) \in \mathcal{T}$. Let $\mathcal{T}'$ be the set of all separations (A', B') of G' of order $< \theta$ such that there exists $(A, B) \in \mathcal{T}$ with $E(A \cap D) = E(A' \cap D)$. Then $\mathcal{T}'$ is a tangle in G' of order θ.*

If $\mathcal{T}$ is a tangle in a hypergraph G, we say that $(A, B) \in \mathcal{T}$ is *linked* to $\mathcal{T}$ if there is no $(A', B') \in \mathcal{T}$ of smaller order with $A \subseteq A'$ and $B' \subseteq B$.

3.3 *Let $\mathcal{T}$ be a tangle of order $\geq \theta$ in a hypergraph G and let $(B, A) \in \mathcal{T}$ be linked to $\mathcal{T}$ and have order $\leq \frac{3}{4}\theta$. Then (A, B) is titanic.*

Proof. Let us suppose that (A, B) is not titanic. Hence we may choose subhypergraphs X_1, X_2, X_3 of A such that $X_1 \cup X_2 \cup X_3 = A$ and $E(X_1), E(X_2), E(X_3)$ are mutually disjoint, and

$$\begin{aligned}
|(V_1 \cup V_2) \cap V_3| &< |W_1 \cup W_2| \\
|(V_2 \cup V_3) \cap V_1| &< |W_2 \cup W_3| \\
|(V_3 \cup V_1) \cap V_2| &< |W_3 \cup W_1|
\end{aligned}$$

where $V(X_i) = V_i$ and $V(X_i \cap B) = W_i \, (i = 1, 2, 3)$. Suppose that $(X_1, X_2 \cup X_3 \cup B) \notin \mathcal{T}$. Then either $(X_2 \cup X_3 \cup B, X_1) \in \mathcal{T}$ or $(X_1, X_2 \cup X_3 \cup B)$ has order $\geq \theta$; and in either case, since (B, A) is linked to $\mathcal{T}$, we deduce that $(X_1, X_2 \cup X_3 \cup B)$ has order at least that of (B, A). Hence

$$|V_1 \cap (V_2 \cup V_3 \cup V(B))| \geq |V(A \cap B)|,$$

that is,

$$|V_1 \cap (V_2 \cup V_3)| + |W_1 \setminus (W_2 \cup W_3)| \geq |W_2 \cup W_3| + |W_1 \setminus (W_2 \cup W_3)|$$

contrary to our assumption. Hence $(X_1, X_2 \cup X_3 \cup B) \in \mathcal{T}$ and similarly $(X_2, X_3 \cup X_1 \cup B)$, $(X_3, X_1 \cup X_2 \cup B) \in \mathcal{T}$. It follows that $(X_1 \cup B, X_2 \cup X_3) \notin \mathcal{T}$ by (**T2**), since $(X_1 \cup B) \cup X_2 \cup X_3 = G$; and $(X_2 \cup X_3, X_1 \cup B) \notin \mathcal{T}$ since $(X_2 \cup X_3) \cup X_1 \cup B = G$; and so $(X_1 \cup B, X_2 \cup X_3)$ has order $\geq \theta$; that is,

$$\theta \leq |(V_1 \cup V(B)) \cap (V_2 \cup V_3)| = |(V_2 \cup V_3) \cap V_1| + |(W_2 \cup W_3) \setminus W_1|$$

$$< |W_2 \cup W_3| + |(W_2 \cup W_3) \setminus W_1| = 2|(W_2 \cup W_3) \setminus W_1| + |(W_2 \cup W_3) \cap W_1|.$$

By summing this and the two similar inequalities, we obtain

$$\begin{aligned} 3\theta &< 2|(W_2 \cup W_3) \setminus W_1| + 2|(W_3 \cup W_1) \setminus W_2| + 2|(W_1 \cup W_2) \setminus W_3| \\ &+ |(W_2 \cup W_3) \cap W_1| + |(W_3 \cup W_1) \cap W_2| + |(W_1 \cup W_2) \cap W_3| \\ &= 4|W_1 \cup W_2 \cup W_3| - |W_1 \cap W_2 \cap W_3| \\ &\leq 4|V(A \cap B)|. \end{aligned}$$

Hence the order of (A, B) is $> 3\theta/4$, a contradiction, and so our initial assumption that (A, B) is not titanic was false. This completes the proof of 3.3. ∎

Let $\mathcal{L}$ be a location in G. The *order* of $\mathcal{L}$ is the maximum order of the members of $\mathcal{L}$ (or 0 if $\mathcal{L} = \emptyset$). For each $(A, B) \in \mathcal{L}$ let $e(A, B)$ be a new element, and let H be the hypergraph with

$$V(H) = V(M(G, \mathcal{L}))$$

$$E(H) = E(M(G, \mathcal{L})) \cup \{e(A, B) : (A, B) \in \mathcal{L}\}$$

where for $e \in E(M(\mathcal{L}))$ its ends are as in G, and for $(A, B) \in \mathcal{L}$ the ends of $e(A, B)$ are the elements of $V(A \cap B)$. This is a hypergraph by 2.3.2, and we call it the *heart* of $\mathcal{L}$. We define the *tree-width* of $\mathcal{L}$ to be the tree-width of H.

3.4 *Let $\mathcal{L}$ be a location in a hypergraph G, such that each $(A, B) \in \mathcal{L}$ is titanic, and $\mathcal{L}$ has order $< \theta$, where $\theta \geq 2$. Then either there is a tangle $\mathcal{T}$ in G of order θ with $\mathcal{L} \subseteq \mathcal{T}$, or $\mathcal{L}$ has tree-width $\leq \frac{3}{2}\theta$.*

Proof. Define H as above. If there is no tangle in H of order θ, then by 2.2 the tree-width of H is at most $\frac{3}{2}\theta$, as required. So we may assume that there is a tangle $\mathcal{T}_0$ in H of order θ. Let

$$\mathcal{L} = \{(A_1, B_1), \ldots, (A_n, B_n)\},$$

and for $1 \leq i \leq n$ let C_i be the subhypergraph of H with $V(C_i) = V(A_i \cap B_i)$ and $E(C_i) = \{e(A_i, B_i)\}$. Thus,

$$H = M(G, \mathcal{L}) \cup C_1 \cup \cdots \cup C_n,$$

and

$$G = M(G, \mathcal{L}) \cup A_1 \cup \cdots \cup A_n.$$

For $0 \leq k \leq n$, let

$$H_k = M(G, \mathcal{L}) \cup A_1 \cup \cdots \cup A_k \cup C_{k+1} \cup \cdots \cup C_n.$$

Then $H_0 = H$ and $H_n = G$. For $1 \le j \le k$, let

$$B_{jk} = M(G, \mathcal{L}) \cup \bigcup(A_i : 1 \le i \le k,\, i \ne j) \cup C_{k+1} \cup \cdots \cup C_n.$$

Then (A_j, B_{jk}) is a separation of H_k, and $A_j \cap B_{jk} = A_j \cap B_j$. We claim that, for $0 \le k \le n$,

(1) *There is a tangle $\mathcal{T}_k$ in H_k of order θ such that $(A_j, B_{jk}) \in \mathcal{T}_k$ for $1 \le j \le k$.*

Subproof. We proceed by induction on k. It holds for $k = 0$, and we therefore assume that $1 \le k \le n$ and that $\mathcal{T}_{k-1}$ satisfies (1) with k replaced by $k-1$. Since (C_k, B_{kk}) has order $< \theta$ (because $\mathcal{L}$ has order $< \theta$) it follows from (**T3**) that $(C_k, B_{kk}) \in \mathcal{T}_{k-1}$. Now (A_k, B_k) is titanic, and hence so is (A_k, B_{kk}) by 3.1. Let $\mathcal{T}_k$ be the set of all separations (A', B') of H_k of order $< \theta$ such that there exists $(A, B) \in \mathcal{T}_{k-1}$ with $E(A \cap B_{kk}) = E(A' \cap B_{kk})$. By 3.2 (with $C, D, G, C', G', \mathcal{T}, \theta, \mathcal{T}'$ replaced by $C_k, B_{kk}, H_{k-1}, A_k, H_k, \mathcal{T}_{k-1}, \theta, \mathcal{T}_k$) $\mathcal{T}_k$ is a tangle in H_k of order θ. Let $1 \le j \le k$; we must verify that $(A_k, B_{jk}) \in \mathcal{T}_k$. If $j < k$, then $(A_j, B_{j,k-1}) \in \mathcal{T}_{k-1}$ from the inductive hypothesis, and so $(A_j, B_{jk}) \in \mathcal{T}_k$ from the definition of $\mathcal{T}_k$. We assume then that $j = k$. But $(C_k, B_{kk}) \in \mathcal{T}_{k-1}$ as we saw above, and $E(C_k \cap B_{kk}) = \emptyset = E(A_k \cap B_{kk})$ and so $(A_k, B_{kk}) \in \mathcal{T}_k$ from the definition of $\mathcal{T}_k$. Thus $\mathcal{T}_k$ satisfies (1); and so (1) holds, by induction on k.

From (1) with $k = n$, we deduce that $(A_j, B_j) \in \mathcal{T}_n$ for $1 \le j \le n$, since $B_j = B_{jn}$; and so $\mathcal{L} \subseteq \mathcal{T}_n$. This proves 3.4. ∎

4 Isolating locations

Let $\mathcal{T}$ be a tangle in a hypergraph G, and let λ be a tie-breaker in G. A location $\mathcal{L}$ is said to *θ-isolate* $\mathcal{T}$ if $\mathcal{L} \subseteq \mathcal{T}$ and has order $< \theta$, and for every $(C, D) \in \mathcal{L}$ and every tangle $\mathcal{T}'$ in G of order $\ge \theta$ with $(D, C) \in \mathcal{T}'$, if (A, B) is the $(\mathcal{T}, \mathcal{T}')$-distinction then $A \subseteq C$ and $D \subseteq B$. Our objective in this section is to study the global structure of a hypergraph G given, for every tangle $\mathcal{T}$ in G of high order, a location θ-isolating $\mathcal{T}$.

We shall need the following lemma (our thanks to M. Saks for its proof).

4.1 *Let T be a tree and let $\le$ be some linear order on $E(T)$. For each $t \in V(T)$, let T_t be a subtree of T such that*

- *$t \in V(T_t)$*
- *if $e \in E(T)$ has one end in $V(T_t)$ and the other end in $V(T) \setminus V(T_t)$ and f is an edge of the path of T with first vertex t and last edge e, then $e \le f$.*

Then there exists $I \subseteq V(T)$ such that the sets $V(T_t)$ $(t \in I)$ form a partition of $V(T)$.

Proof. We proceed by induction on $|V(T)|$. We may assume that $E(T) \ne \emptyset$, and may therefore choose $f \in E(T)$ minimum under $\le$. Let T^1, T^2 be the two components of $T \backslash f$, and let the ends of f be $u^1 \in V(T^1)$, $u^2 \in V(T^2)$. For each $t \in V(T^i)$, define $T^i_t = T_t \cap T^i$ $(i = 1, 2)$. These satisfy the hypotheses of 4.1, so from our inductive hypothesis, we may choose $I^i \subseteq V(T^i)$ such that the sets $V(T^i_t)$ $(t \in I^i)$ form a partition of $V(T^i)$ $(i = 1, 2)$. Now if for $i = 1, 2, T^i_t = T_t$ for every $t \in I^i$ then

$I = I^1 \cup I^2$ satisfies our requirement. We assume then that there exists $s \in I^1$ with $T_s^1 \neq T_s$. Hence $T_s \not\subseteq T^1$, and so $f \in E(T_s)$, and in particular $u^1 \in V(T_s^1)$. It follows that $T_t^1 = T_t$ for all $t \in I^1 \setminus \{s\}$, since no other $V(T_t^1)$ contains u^1. Moreover, we claim that $T^2 \subseteq T_s$. For if not, there is an edge e of T^2 with one end in $V(T_s)$ and the other in $V(T^2) \setminus V(T_s)$. Then f is in the path of T_s with first vertex s and last edge e, and so $e \leq f$. But $f < e$ from our choice of f since $e \neq f$, a contradiction. Thus $T^2 \subseteq T_s$, and so the sets $V(T_t)$ $(t \in I^1)$ partition $V(T)$. This proves 4.1. ∎

4.2 *Let $\mathcal{T}_j$ $(j \in J)$ be distinct tangles of order θ in a hypergraph G, let λ be a tie-breaker in G, and for each $j \in J$ let $\mathcal{L}_j \subseteq \mathcal{T}_j$ be a location which θ-isolates $\mathcal{T}_j$ with respect to λ. Then there exists $I \subseteq J$ such that for every $j \in J$ there is a unique $i \in I$ with $\mathcal{L}_i \subseteq \mathcal{T}_j$.*

Proof. We may assume that $J \neq \emptyset$. Let $J = \{1, \ldots, n\}$ say where $n \geq 1$. Let (T, τ) be a standard tree-decomposition relative to $\mathcal{T}_1, \ldots, \mathcal{T}_n$ in which t_i represents $\mathcal{T}_i$ for $1 \leq i \leq n$.

(1) *If $e, e' \in E(T)$ are distinct, and make separations $(A, B), (A', B')$ of G say, then*

$$(A, B), (B, A) \neq (A', B'), (B', A').$$

Subproof. Let e have ends t_i, t_j. By 2.6.2, one of $(A, B), (B, A)$ is the $(\mathcal{T}_i, \mathcal{T}_j)$-distinction, and the other is the $(\mathcal{T}_j, \mathcal{T}_i)$-distinction. Consequently, one of (A, B), (B, A) does not belong to $\mathcal{T}_i$ and the other does not belong to $\mathcal{T}_j$. But by 2.6.1, one of $(A', B'), (B', A')$ belongs to both $\mathcal{T}_i$ and $\mathcal{T}_j$. This proves (1).

For $1 \leq h \leq n$ let T_h be the restriction of T to $\{t_i : 1 \leq i \leq n,\ \mathcal{L}_h \subseteq \mathcal{T}_i\}$. For the moment, let us fix h with $1 \leq h \leq n$. Let S_h be the component of T_h containing t_h.

(2) *Let $t_i \in V(S_h)$ be adjacent in T to $t_j \in V(T) \setminus V(S_h)$, and let (A, B) be the $(\mathcal{T}_h, \mathcal{T}_j)$-distinction; then $(A, B) \in \mathcal{T}_k$ for every $t_k \in V(T_h)$, and (A, B) is the $(\mathcal{T}_i, \mathcal{T}_j)$-distinction.*

Subproof. Since $t_j \notin V(S_h)$ it follows that $t_j \notin V(T_h)$ and so $\mathcal{L}_h \not\subseteq \mathcal{T}_j$. Choose $(C, D) \in \mathcal{L}_h$ with $(C, D) \notin \mathcal{T}_j$. Then (C, D) has order $< \theta$ since $(C, D) \in \mathcal{L}_h$, and so $(D, C) \in \mathcal{T}_j$ by (**T1**). Since $\mathcal{L}_h$ θ-isolates $\mathcal{T}_h$ it follows that $A \subseteq C$ and $D \subseteq B$. For each $t_k \in V(T_h)$, $(C, D) \in \mathcal{L}_h \subseteq \mathcal{T}_k$, and so $(B, A) \notin \mathcal{T}_k$ by (**T2**) since $B \cup C = G$; and hence $(A, B) \in \mathcal{T}_k$ by (**T1**) since $\mathcal{T}_k$ has order θ. In particular, $(A, B) \in \mathcal{T}_i$, and therefore has λ-order at least that of the $(\mathcal{T}_i, \mathcal{T}_j)$-distinction (A', B'). On the other hand, (A', B') distinguishes $\mathcal{T}_h$ from $\mathcal{T}_j$ (by 2.6.1, since t_i lies on the path of T between t_h and t_j) and therefore has λ-order at least that of (A, B). Hence equality occurs, and $(A, B) = (A', B')$ by the first tie-breaker axiom. This proves (2).

(3) *T_h is a tree.*

Subproof. Let $t_k \notin V(S_h)$, and let P be the path of T from t_h to t_k. Let t_i be the last vertex of P in $V(S_h)$ and t_j the next vertex of P, and define (A, B) as in (2). Then $(A, B) \notin \mathcal{T}_k$ by 2.6.1 and 2.6.2, since (A, B) is the $(\mathcal{T}_i, \mathcal{T}_j)$-distinction; and so $t_k \notin V(T_h)$ by (2). Hence $S_h = T_h$ and T_h is a tree. This proves (3).

For each $e \in E(T)$ with ends t_i, t_j say, let $\mu(e)$ be the λ-order of the $(\mathcal{T}_i, \mathcal{T}_j)$-distinction. By (1), $\mu(e) \neq \mu(e')$ for all distinct $e, e' \in E(T)$.

(4) *If $e \in E(T)$ has one end in $V(T_h)$ and the other in $V(T) \setminus V(T_h)$, and f is an edge of the path of T with first vertex t and last edge e, then $\mu(e) < \mu(f)$ unless $e = f$.*

Subproof. Let e have ends $t_i \in V(T_h)$ and $t_j \in V(T) \setminus V(T_h)$, and let (A, B) be the $(\mathcal{T}_i, \mathcal{T}_j)$-distinction. By (2) and (3), (A, B) is the $(\mathcal{T}_h, \mathcal{T}_j)$-distinction, and so its λ-order is at most the λ-order of the separation made by f, with strict inequality unless $e = f$ by (1). This proves (4).

In view of (3), (4) and 4.1, this proves 4.2. ∎

Let $\mathcal{L}, \mathcal{L}^*$ be locations in a hypergraph G, and let $\mathcal{L} = \{(C_1, D_1), \ldots, (C_k, D_k)\}$. We say that $\mathcal{L}^*$ is an *enlargement of* $\mathcal{L}$ if there exist $\mathcal{L}_1, \ldots, \mathcal{L}_k \subseteq \mathcal{L}^*$, mutually disjoint (possibly empty) and with union $\mathcal{L}^*$, such that for $1 \leq h \leq k$, every $(A, B) \in \mathcal{L}_h$ satisfies $A \subseteq C_h$ and $D_h \subseteq B$. If in addition $w \geq 0$ and $\mathcal{L}_h \cup \{(D_h, C_h)\}$ has tree-width $\leq w$ for $1 \leq h \leq k$, we say that $\mathcal{L}^*$ is an *enlargement of* $\mathcal{L}$ *by tree-width* $\leq w$.

4.3 *Let λ be a tie-breaker in a hypergraph G, let $\theta \geq 2$, and let $\mathcal{T}_1, \ldots, \mathcal{T}_n$ be distinct tangles in G, each of order θ, where $n \geq 1$. For $1 \leq i \leq n$ let $\mathcal{L}_i \subseteq \mathcal{T}_i$ be a location of order $\leq \frac{3}{4}\theta$ which θ-isolates $\mathcal{T}_i$; and suppose that for every tangle $\mathcal{T}$ in G of order θ, there is a unique i with $1 \leq i \leq n$ such that $\mathcal{L}_i \subseteq \mathcal{T}$. Let (T, τ) be a standard tree-decomposition of G relative to $\mathcal{T}_1, \ldots, \mathcal{T}_n$, where $V(T) = \{t_1, \ldots, t_n\}$ and t_i represents $\mathcal{T}_i$ for $1 \leq i \leq n$. Then for $1 \leq i \leq n$, the location of t_i in (T, τ) is an enlargement of $\mathcal{L}_i$ by tree-width $\leq \frac{9}{4}\theta$.*

Proof. Let $1 \leq i \leq n$, and let $\mathcal{L}_i = \{(C_1, D_1), \ldots, (C_k, D_k)\}$. Let $\mathcal{L}^*$ be the location of t_i in (T, τ).

(1) *If $(A, B) \in \mathcal{L}^*$ then (A, B) has order $\leq \frac{3}{4}\theta$ and there exists h with $1 \leq h \leq k$ such that $A \subseteq C_h$ and $D_h \subseteq B$.*

Subproof. Since $(A, B) \in \mathcal{L}^*$, there exists $j \neq i$ with $1 \leq j \leq n$ such that t_i, t_j are adjacent in T and (A, B) is the $(\mathcal{T}_i, \mathcal{T}_j)$-distinction. Since $\mathcal{L}_j \subseteq \mathcal{T}_j$ and $j \neq i$ it follows that $\mathcal{L} \not\subseteq \mathcal{T}_j$, and so there exists h with $1 \leq h \leq k$ such that $(C_h, D_h) \notin \mathcal{T}_j$. Since (C_h, D_h) has order $\leq \frac{3}{4}\theta < \theta$, and $\mathcal{T}_j$ has order θ, it follows that $(D_h, C_h) \in \mathcal{T}_j$. Since $\mathcal{L}_i$ θ-isolates $\mathcal{T}_i$, we deduce that $A \subseteq C_h$ and $D_h \subseteq B$. Moreover, since (C_h, D_h) distinguishes $\mathcal{T}_i$ from $\mathcal{T}_j$ and has order $\leq \frac{3}{4}\theta$, it follows that the $(\mathcal{T}_i, \mathcal{T}_j)$-distinction (A, B) also has order $\leq \frac{3}{4}\theta$. This proves (1).

(2) *Each member of $\mathcal{L}^*$ is titanic.*

Subproof. Let $(A, B) \in \mathcal{L}^*$, and choose j as above. We claim that (B, A) is linked to $\mathcal{T}_j$. For suppose that there exists $(B', A') \in \mathcal{T}_j$ of smaller order than (A, B), and with $B \subseteq B'$ and $A' \subseteq A$. Since $(A, B) \in \mathcal{T}_i$ and $A' \subseteq A$ it follows that $(A', B') \in \mathcal{T}_i$, and so (A', B') distinguishes $\mathcal{T}_i$ from $\mathcal{T}_j$; and hence has order at least that of (A, B), a contradiction. Thus there is no such (B', A'), and so (B, A) is linked to $\mathcal{T}_j$. Since (B, A) has order $\leq \frac{3}{4}\theta$ by (1), it follows that (A, B) is titanic, by 3.3. This proves (2).

By (1), there exist $\mathcal{L}_1^*, \ldots, \mathcal{L}_k^* \subseteq \mathcal{L}^*$, mutually disjoint and with union $\mathcal{L}^*$, such that for $1 \leq g \leq k$, every $(A, B) \in \mathcal{L}_g^*$ satisfies $A \subseteq C_g$ and $D_g \subseteq B$. Fix h with $1 \leq h \leq k$.

(3) *There is no tangle $\mathcal{T}$ of order θ in G with $\mathcal{L}_h^* \cup \{(D_h, C_h)\} \subseteq \mathcal{T}$.*

Subproof. Suppose that $\mathcal{T}$ is such a tangle. From the hypothesis, there exists j with $1 \leq j \leq n$ such that $\mathcal{L}_j \subseteq \mathcal{T}$. Since $(D_h, C_h) \in \mathcal{T}$ and $(C_h, D_h) \in \mathcal{L}_i$, it follows that $i \neq j$. Let (A, B) be the $(\mathcal{T}_i, \mathcal{T}_j)$-distinction. Since $\mathcal{L}_j \not\subseteq \mathcal{T}_i$, there exists $(C, D) \in \mathcal{L}_j$ such that $(C, D) \notin \mathcal{T}_i$. Therefore $(D, C) \in \mathcal{T}_i$, since $\mathcal{L}_j$ has order $< \theta$, and hence $D \subseteq A$ since (B, A) is the $(\mathcal{T}_j, \mathcal{T}_i)$-distinction and $\mathcal{L}_j$ θ-isolates $\mathcal{T}_j$. Since $\mathcal{L}_j \subseteq \mathcal{T}$ it follows that $(C, D) \in \mathcal{T}$, and hence $(B, A) \in \mathcal{T}$ since (B, A) has order $< \theta$ and $D \subseteq A$. Let $t_{j'}$ be the second vertex of the path of T from t_i to t_j, and let (A', B') be the $(\mathcal{T}_i, \mathcal{T}_{j'})$-distinction; then, since one of the edges of this path makes the separation (A, B) under $(\mathcal{T}, \tau)$ (by 2.6.2), it follows that $A \subseteq A'$ and $B' \subseteq B$. Hence $(B', A') \in \mathcal{T}$, since $(B, A) \in \mathcal{T}$. Choose g with $1 \leq g \leq k$ such that $(A', B') \in \mathcal{L}_g^*$. Then $A' \subseteq C_g$ since $\mathcal{L}_g^* \cup \{(D_g, C_g)\}$ is a location, and so $(C_g, D_g) \notin \mathcal{T}$ by (**T2**), since $(B', A') \in \mathcal{T}$ and $B' \cup C_g = G$. But (C_g, D_g) has order $< \theta$, and so $(D_g, C_g) \in \mathcal{T}$ by (**T1**). Now $(D_h, C_h) \in \mathcal{T}$ by our assumption, and so $D_g \cup D_h \neq G$ by (**T2**), and hence $g = h$ since $\mathcal{L}_i$ is a location. But $(A', B') \notin \mathcal{T}$ and

$$(A', B') \in \mathcal{L}_g^* = \mathcal{L}_h^* \subseteq \mathcal{T},$$

a contradiction. Thus there is no such $\mathcal{T}$. This proves (3).

Let $\mathcal{L}' = \{(A, B \cap C_h) : (A, B) \in \mathcal{L}_h^*\}$. Then $\mathcal{L}'$ is a location in C_h, of order $< \theta$.

(4) *There is no tangle in C_h of order θ including $\mathcal{L}'$.*

Subproof. Suppose that $\mathcal{T}'$ is such a tangle. Let $\mathcal{T}$ be the set of all separations (A, B) of G of order $< \theta$ such that $(A \cap C_h, B \cap C_h) \in \mathcal{T}'$. By 2.1, $\mathcal{T}$ is a tangle in G, of order θ. Since (D_h, C_h) has order $< \theta$ and $(D_h \cap C_h, C_h \cap C_h) \in \mathcal{T}'$ by (**T1**) and (**T3**), it follows that $(D_h, C_h) \in \mathcal{T}$. Similarly, if $(A, B) \in \mathcal{L}_h^*$, then (A, B) has order $< \theta$, and $(A \cap C_h, B \cap C_h) = (A, B \cap C_h) \in \mathcal{L}' \subseteq \mathcal{T}'$, and so $(A, B) \in \mathcal{T}$. Hence, $\mathcal{L}_h^* \cup \{(D_h, C_h)\} \subseteq \mathcal{T}$, contrary to (3). This proves (4).

Now every member of $\mathcal{L}'$ is titanic by (2) and 3.1, and so from (4) and 3.4, $\mathcal{L}'$ has tree-width $\leq \frac{3}{2}\theta$. Let $\mathcal{L} = \mathcal{L}_h^* \cup \{(D_h, C_h)\}$. The heart of $\mathcal{L}$ may be obtained from the heart of $\mathcal{L}'$ (taking the latter to be C_h if $\mathcal{L}' = \emptyset$) by adding one new edge whose set of ends is $V(C_h \cap D_h)$, and since $|V(C_h \cap D_h)| \leq \frac{3}{4}\theta$, we deduce that $\mathcal{L}$ has tree-width $\leq \frac{3}{2}\theta + \frac{3}{4}\theta = \frac{9}{4}\theta$. This proves 4.3. ∎

Now we deduce the main result of this section, by combining 2.7, 4.2 and 4.3.

4.4 *Let λ be a tie-breaker in a hypergraph G, and let $\theta \geq 1$ be an integer. For each tangle $\mathcal{T}$ in G of order $\geq \theta$ let $\mathcal{L}(\mathcal{T}) \subseteq \mathcal{T}$ be a location which θ-isolates $\mathcal{T}$, and let G have a tangle of order $\geq \frac{4}{3}\theta$. Then there is a tree-decomposition (T, τ) of G with the following properties:*

- *(T, τ) is proper and rotund*
- *for each $e \in E(T)$, the separations made by e under (T, τ) are robust*

- *for each $t \in V(T)$, let $\mathcal{L}$ be the location of t in (T,τ); then there is a tangle $\mathcal{T}$ in G of order $\geq \frac{4}{3}\theta$ with $\mathcal{L} \subseteq \mathcal{T}$, such that $\mathcal{L}$ is an enlargement of $\mathcal{L}(\mathcal{T})$ by tree-width $\leq 3\theta + 1$.*

Proof. Let θ' be the least integer with $\theta' \geq \frac{4}{3}\theta$. Then $\theta' \geq 2$. Let $\mathcal{T}_j$ $(j \in J)$ be all the tangles of order θ' in G. Then $J \neq \emptyset$, by hypothesis. For each $j \in J, \mathcal{L}(\mathcal{T}_j)$ θ'-isolates $\mathcal{T}_j$ since it θ-isolates $\mathcal{T}_j$. By 4.2, there exists $I \subseteq J$ such that for every $j \in J$ there is a unique $i \in I$ with $\mathcal{L}(\mathcal{T}_i) \subseteq \mathcal{T}_j$. Let $I = \{1, \ldots, n\}$ say. Now $n \geq 1$ since $J \neq \emptyset$. Let (T,τ) be a standard decomposition of G relative to $\mathcal{T}_1, \ldots, \mathcal{T}_n$ in which t_i represents $\mathcal{T}_i$ for $1 \leq i \leq n$. By 2.7, the first two statements of the theorem hold. Let us verify the third. Let $1 \leq i \leq n$, and let $\mathcal{L}$ be the location of t_i in (T,τ). From 4.3 (with θ replaced by θ') $\mathcal{L}$ is an enlargement of $\mathcal{L}(\mathcal{T}_i)$ by tree-width $\leq \frac{9}{4}\theta'$. Since $\theta' \leq \frac{4}{3}\theta + \frac{2}{3}$ and $\frac{9}{4}\left(\frac{4}{3}\theta + \frac{2}{3}\right) < 3\theta + 2$ we deduce that the third statement holds. This proves 4.4. ∎

5 Patchworks

Our application of 4.4 will be to prove that certain classes of "patchworks" in the sense of [2] are well-quasi-ordered by our patchwork containment relation, "simulation", and now we need to define these things. A *march* in a set V is a finite sequence of distinct elements of V; and if π is the march $v_1, \ldots, v_k$, we denote the set $\{v_1, \ldots, v_k\}$ by $\bar{\pi}$. A *rooted hypergraph* G is a pair $(G^-, \pi(G))$ where G^- is a hypergraph and $\pi(G)$ is a march in $V(G^-)$. We define $V(G) = V(G^-), E(G) = E(G^-)$. If G, H are rooted hypergraphs and $G^- \subseteq H^-$ we write $G \subseteq H$ and say that G is a *rooted subhypergraph* of H..

If V is a finite set we denote by K_V the complete graph on V, that is, the graph with vertex set V and edge set the set of all subsets of V of cardinality 2, with the natural incidence relation. A *grouping* in V is a subgraph of K_V every component of which is complete. A *pairing* in V is a grouping in V every component of which has at most two vertices. If K is a pairing in V, we say that K *pairs* X, Y if $X, Y \subseteq V$ are disjoint and

- every 2-vertex component of K has one vertex in X and the other in Y, and
- every vertex of $X \cup Y$ belongs to some 2-vertex component of K.

A *patch* Δ in V is a subset $V(\Delta)$ of V, together with a collection of groupings in V, each with vertex set $V(\Delta)$. (We shall use the same symbol Δ to denote the collection of groupings.) A patch Δ is *free* if Δ contains every grouping in V with vertex set $V(\Delta)$; and it is *robust* if for every choice of $X, Y \subseteq V(\Delta)$ with $|X| = |Y|$ and $X \cap Y = \emptyset$, there is a pairing in Δ which pairs X, Y.

A *patchwork* is a triple $P = (G, \mu, \Delta)$, where

- G is a rooted hypergraph
- μ is a function with domain $dom(\mu) \subseteq E(G)$; and for each $e \in dom(\mu)$ $\mu(e)$ is a march with $\bar{\mu}(e)$ the set of ends of e in G
- Δ is a function with domain $E(G)$, such that for each $e \in E(G)$ $\Delta(e)$ is a patch with $V(\Delta(e))$ the set of ends of e; and for each $e \in E(G) \setminus dom(\mu), \Delta(e)$ is free.

The patchwork is *robust* if each $\Delta(e)$ $(e \in E(G))$ is robust. (This is automatic for $e \notin dom(\mu)$, since free patches are robust.)

A *quasi-order* Ω is a pair $(E(\Omega), \leq)$, where $E(\Omega)$ is a class and $\leq$ is a reflexive transitive relation on $E(\Omega)$. It is a *well-quasi-order* if for every countable sequence x_i $(i = 1, 2 \ldots)$ of elements of $E(\Omega)$ there exist $j > i \geq 1$ such that $x_i \leq x_j$. If Ω_1, Ω_2 are quasi-orders with $E(\Omega_1) \cap E(\Omega_2) = \emptyset$ we denote by $\Omega_1 \cup \Omega_2$ the quasi-order Ω with $E(\Omega) = E(\Omega_1) \cup E(\Omega_2)$ in which $x \leq y$ if and only if for some $i \in \{1, 2\}, x, y \in E(\Omega_i)$ and $x \leq y$ in Ω_i. If Ω_1, Ω_2 are quasi-orders we write $\Omega_1 \subseteq \Omega_2$ if $E(\Omega_1) \subseteq E(\Omega_2)$ and for all $x, y \in E(\Omega_1), x \leq y$ in Ω_1 if and only if $x \leq y$ in Ω_2.

If Ω is a quasi-order, a *partial* Ω-*patchwork* is a quadruple (G, μ, Δ, ϕ), where (G, μ, Δ) is a patchwork and ϕ is a function from a subset $dom(\phi)$ of $E(G)$ into $E(\Omega)$. It is an Ω-*patchwork* if $dom(\phi) = E(G)$. It is *robust* if (G, μ, Δ) is robust. The underlying rooted hypergraph G of a partial Ω-patchwork $P = (G, \mu, \Delta, \phi)$ will be denoted by $||P||$.

If V is a finite set, N_V denotes the graph with vertex set V and no edges. A *realization* of a patchwork (G, μ, Δ) is a subgraph of $K_{V(G)}$ expressible in the form

$$N_{V(G)} \cup \bigcup(\delta_e : e \in E(G))$$

where $\delta_e \in \Delta(e)$ for each $e \in E(G)$. A *realization* of a partial Ω-patchwork (G, μ, Δ, ϕ) is a realization of (G, μ, Δ). If μ_1, μ_2 are marches with the same length, we denote the bijection of $\bar{\mu}_1$ onto $\bar{\mu}_2$ mapping μ_1 to μ_2 by $\mu_1 \to \mu_2$. Let $P = (G, \mu, \Delta), P' = (G', \mu', \Delta')$ be patchworks. An *expansion* of P in P' is a function η with domain $V(G) \cup E(G)$ such that

- for each $v \in V(G), \eta(v)$ is a non-empty subset of $V(G')$, and for each $e \in E(G)$, $\eta(e) \in E(G')$
- for distinct $v_1, v_2 \in V(G), \eta(v_1) \cap \eta(v_2) = \emptyset$
- for distinct $e_1, e_2 \in E(G), \eta(e_1) \neq \eta(e_2)$
- for each $e \in E(G), e \in dom(\mu)$ if and only if $\eta(e) \in dom(\mu')$
- for each $e \in E(G) \setminus dom(\mu)$, if v is an end of e in G then $\eta(v)$ contains an end of $\eta(e)$ in G'
- for each $e \in dom(\mu), \mu(e)$ and $\mu'(\eta(e))$ have the same length, k say, and for $1 \leq i \leq k, \eta(v)$ contains the i^{th} term of $\mu'(\eta(e))$ where v is the i^{th} term of $\mu(e)$
- $\pi(G)$ and $\pi(G')$ have the same length, k say, and for $1 \leq i \leq k$ $\eta(v)$ contains the i^{th} term of $\pi(G')$ where v is the i^{th} term of $\pi(G)$
- for each $e \in dom(\mu), \mu(e) \to \mu'(\eta(e))$ maps $\Delta(e)$ to $\Delta'(\eta(e))$.

If $P = (G, \mu, \Delta, \phi)$, $P' = (G', \mu', \Delta', \phi')$ are partial Ω-patchworks, an *expansion* of P in P' is an expansion η of (G, μ, Δ) in (G', μ', Δ') such that $\eta(e) \in dom(\phi')$ and $\phi(e) \leq \phi'(\eta(e))$ for each $e \in dom(\phi)$.

If G is a hypergraph and $F \subseteq E(G)$, $G \backslash F$ denotes the subhypergraph with the same vertex set and edge set $E(G) \setminus F$. If G is a rooted hypergraph, $G \backslash F$ denotes $(G^- \backslash F, \pi(G))$. If $P = (G, \mu, \Delta, \phi)$ is an Ω-patchwork and $F \subseteq E(G)$, $P \backslash F$ denotes the Ω-patchwork $(G \backslash F, \mu', \Delta', \phi')$ where μ', Δ', ϕ' are the restrictions of μ, Δ, ϕ to $dom(\mu) \cap E(G \backslash F)$, $E(G \backslash F)$, $E(G \backslash F)$ respectively. Similarly, if $P = (G, \mu, \Delta)$ is a patchwork and $F \subseteq E(G)$, $P \backslash F$ denotes the patchwork $(G \backslash F, \mu', \Delta')$, with

μ', Δ' as before. We often write $P\backslash e$ for $P\backslash\{e\}$, etc. Let η be an expansion of $P = (G, \mu, \Delta)$ in $P' = (G', \mu', \Delta')$, or of $P = (G, \mu, \Delta, \phi)$ in $P' = (G', \mu', \Delta', \phi')$. A realization H of $P'\backslash\eta(E(G))$ is said to *realize* η if for every $v \in V(G)$, $\eta(v)$ is the vertex set of some component of H; and if there is such a realization, η is said to be *realizable*. Let us say that P is *simulated* in P' if there is a realizable expansion of P in P'.

If $P = (G, \mu, \Delta)$ is a patchwork and $A \subseteq G$, we denote by $P|A$ the patchwork (A, μ', Δ'), where μ', Δ' are the restrictions of μ, Δ to $E(A) \cap dom(\mu)$, $E(A)$ respectively. If $P = (G, \mu, \Delta, \phi)$ is a partial Ω-patchwork, $P|A$ is the partial Ω-patchwork $(A, \mu', \Delta', \phi')$ where μ', Δ' are as before and ϕ' is the restriction of ϕ to $E(A) \cap dom(\phi)$.

A *separation* of a rooted hypergraph G is a pair (A, B) of rooted hypergraphs such that (A^-, B^-) is a separation of G^-, $\bar{\pi}(A) = V(A \cap B)$, and $\pi(B) = \pi(G)$. Two vertices of a graph H are *connected* in H if they belong to the same component of H. We begin with the following lemma.

5.1 *For $i = 1, 2$ let $P_i = (G_i, \mu_i, \Delta_i)$ be a patchwork, and let (G'_i, G_0) be a separation of G_i. Let $\pi(G'_1) = \pi(G'_2)$, and let $P_1|G_0 = P_2|G_0$. For $i = 1, 2$ let H'_i be a realization of $P_i|G'_i$, such that for $x, y \in \bar{\pi}(G'_1) = \bar{\pi}(G'_2)$, x and y are connected in H'_1 if and only if they are connected in H'_2. Let H_0 be a realization of $P_1|G_0 = P_2|G_0$, and let $H_i = H_0 \cup H'_i\,(i = 1, 2)$. Then for $i = 1, 2$, H_i is a realization of P_i, and for $x, y \in V(G_0)$ x and y are connected in H_1 if and only if they are connected in H_2.*

Proof. Let $x, y \in V(G_0)$ be connected in H_1 say; we shall prove that they are connected in H_2. Choose a sequence

$$x = v_0, e_1, v_1, e_2, \dots, e_t, v_t = y$$

such that $v_0, \dots, v_t \in V(H_1), e_1, \dots, e_t \in E(H_1)$ and for $1 \leq i \leq t$, e_i is incident with v_{i-1} and v_i in H_1. Let

$$I = \{i : 0 \leq i \leq t,\, v_i \in V(G_0)\}.$$

Then $0, t \in I$; let $I = \{s(1), s(2), \dots, s(r)\}$ say, in order, where $s(1) = 0$ and $s(r) = t$.

(1) *For $1 \leq j \leq r - 1$, $v_{s(j)}$ and $v_{s(j+1)}$ are connected in H_2.*

Subproof. If $e_k \in E(H_0)$ for some k with $s(j) + 1 \leq k \leq s(j + 1)$ then $v_{k-1}, v_k \in V(G_0)$ since they are both incident with e_k; hence $k - 1, k \in I$, and so from the definition of $I, k - 1 = s(j)$, $k = s(j+1)$ and $v_{s(j)}, v_{s(j+1)}$ are connected in H_2, as claimed. If $e_k \notin E(H_0)$ for $s(j)+1 \leq k \leq s(j+1)$ then $v_{s(j)}, v_{s(j+1)}$ are vertices of H'_1 and are connected in H'_1; but $v_{s(j)}, v_{s(j+1)} \in V(G_0)$ and so both belong to $\bar{\pi}(G'_1)$. Since $v_{s(j)}, v_{s(j+1)}$ are connected in H'_1 it follows from our hypothesis that they are connected in H'_2 and hence in H_2, as claimed. This proves (1).

From (1) it follows that x, y are connected in H_2. This proves 5.1. ∎

Let $P = (G, \mu, \Delta)$ be a patchwork. A grouping K is *feasible* in P if $V(K) = \bar{\pi}(G)$ and there is a realization H of P such that for distinct $x, y \in V(K)$, x and y are connected in H if and only if they are adjacent in K. A grouping is *feasible* in a partial Ω-patchwork (G, μ, Δ, ϕ) if it is feasible in (G, μ, Δ). The set of all groupings feasible in P will be denoted by $gr(P)$.

5.2 *For $i = 1,2$ let $P_i = (G_i, \mu_i, \Delta_i)$ be a patchwork, and let (G'_i, G_0) be a separation of G_i, such that $\pi(G'_1) = \pi(G'_2)$, $P_1|G_0 = P_2|G_0$, and $gr(P_1|G'_1) \subseteq gr(P_2|G'_2)$. Then for every realization H_1 of P_1 there is a realization H_2 of P_2 such that for x, $y \in V(G_0)$, x and y are connected in H_1 if and only if they are connected in H_2.*

Proof. Let H_1 be a realization of P_1; then $H_1 = H_0 \cup H'_1$, where H_0 is a realization of $P_1|G_0$ and H'_1 is a realization of $P_1|G'_1$. Let H'_2 be a realization of $P_2|G'_2$ such that for x, $y \in \bar{\pi}(G'_1)$, x and y are connected in H'_1 if and only if they are connected in H'_2. (This exists because $gr(P_1|G'_1) \subseteq gr(P_2|G'_2)$. Then $H_2 = H_0 \cup H'_2$ is a realization of P_2 satisfying the theorem, by 5.1. This proves 5.2. ∎

5.3 *For $i = 1,2$ let $P_i = (G_i, \mu_i, \Delta_i)$ be a patchwork, and let (G'_i, G_0) be a separation of G_i, such that $\pi(G'_1) = \pi(G'_2)$, $P_1|G_0 = P_2|G_0$, and $gr(P_1|G'_1) \subseteq gr(P_2|G'_2)$. Let η_1 be a realizable expansion of some patchwork $P = (G, \mu, \Delta)$ in P_1 such that $\eta_1(e) \in E(G_0)$ for every $e \in E(G)$ and $\eta_1(v) \cap V(G_0) \neq \emptyset$ for each $v \in V(G)$. Then there is a realizable expansion η_2 of P in P_2 such that $\eta_2(e) = \eta_1(e)$ for each $e \in E(G)$, and $\eta_2(v) \cap V(G_0) = \eta_1(v) \cap V(G_0)$ for each $v \in V(G)$.*

Proof. Let H_1 be a realization of $P_1 \backslash \eta_1(E(G))$ which realizes η_1. By 5.2 applied to $P_1 \backslash \eta_1(E(G))$ and $P_2 \backslash \eta_1(E(G))$, there is a realization H_2 of $P_2 \backslash \eta_1(E(G))$ such that for x, $y \in V(G_0)$, x and y are connected in H_1 if and only if they are connected in H_2. For $e \in E(G)$ let $\eta_2(e) = \eta_1(e)$. For each $v \in V(G)$ there is a component C_1 of H_1 with $V(C_1) = \eta_1(v)$, and hence a (unique) component C_2 of H_2 with

$$V(C_2) \cap V(G_0) = V(C_1) \cap V(G_0) = \eta_1(v) \cap V(G_0),$$

since $\eta_1(v) \cap V(G_0) \neq \emptyset$. Let $\eta_2(v)$ be $V(C_2)$. Then η_2 is the required expansion. This proves 5.3. ∎

If f, g are functions with domains $dom(f), dom(g)$ respectively and x is any object, the statement $f(x) \equiv g(x)$ will mean "either $x \in dom(f) \cap dom(g)$ and $f(x) = g(x)$, or $x \notin dom(f) \cup dom(g)$."

Let G be a rooted hypergraph. We say that $A \subseteq G$ is *complemented* if $\bar{\pi}(A)$ contains every vertex $v \in V(A)$ such that either $v \in \bar{\pi}(G)$ or some edge $e \in E(G) \setminus E(A)$ is incident with v. If A is complemented, we define $G \setminus A \subseteq G$ to be the rooted hypergraph with

$$\begin{aligned} V(G \setminus A) &= (V(G) \setminus V(A)) \cup \bar{\pi}(A); \\ E(G \setminus A) &= E(G) \setminus E(A); \\ \pi(G \setminus A) &= \pi(G). \end{aligned}$$

Then $(A, G \setminus A)$ is a separation of G, since $(A^-, (G \setminus A)^-)$ is a separation of G^-, $\bar{\pi}(A) = V(A) \cap V(G \setminus A)$, and $\pi(G \setminus A) = \pi(G)$. A *rooted location* $\mathcal{L}$ in a rooted hypergraph G is a set $\mathcal{L}$ of complemented rooted hypergraphs A with $A \subseteq G$ such that $A_1 \subseteq G \setminus A_2$ for all distinct $A_1, A_2 \in \mathcal{L}$. If $\mathcal{L}$ is a rooted location in G then $\{(A^-, (G \setminus A)^-) : A \in \mathcal{L}\}$ is a location in G^- which we denote by $\mathcal{L}^-$. (It is possible that $(A^-, (G \setminus A)^-) = (A'^-, (G \setminus A')^-)$ for distinct $A, A' \in \mathcal{L}$, but only if $E(A) = E(A') = \emptyset$ and $V(A) = V(A') = \bar{\pi}(A) = \bar{\pi}(A')$.) We define $M(G, \mathcal{L}) = M(G^-, \mathcal{L}^-)$.

Let $P = (G, \mu, \Delta)$ be a patchwork and let $\mathcal{L}$ be a rooted location in G. For each $A \in \mathcal{L}$ let $e(A)$ be a new element, and let G' be the rooted hypergraph with

$$\begin{aligned} V(G') &= V(M(G, \mathcal{L})) \\ E(G') &= E(M(G, \mathcal{L})) \cup \{e(A) : A \in \mathcal{L}\} \\ \pi(G') &= \pi(G) \end{aligned}$$

where for $e \in E(M(G,\mathcal{L}))$ its ends are as in G^-, and for $A \in \mathcal{L}$ the ends of $e(A)$ are the vertices in $\bar{\pi}(A)$. We define the *heart* $P|\mathcal{L}$ of $(P,\mathcal{L})$ to be the patchwork (G',μ',Δ') such that $\mu'(e(A)) = \pi(A)$ and $\Delta'(e(A)) = gr(P|A)$ for all $A \in \mathcal{L}$ and $\mu'(e) \equiv \mu(e)$ and $\Delta'(e) = \Delta(e)$ for all $e \in E(M(G,\mathcal{L}))$. (It is unique up to the choice of the new elements $e(A)$.)

5.4 *Let $P = (G,\mu,\Delta)$ be a patchwork, let $\mathcal{L}$ be a rooted location in G, and let $P' = (G',\mu',\Delta')$ be the heart of $(P,\mathcal{L})$. Then*

$$V(G) \setminus \bar{\pi}(G) = (V(G') \setminus \bar{\pi}(G')) \cup \bigcup_{A\in\mathcal{L}} (V(A) \setminus \bar{\pi}(A)),$$

and $gr(P) = gr(P')$.

Proof. For the first assertion, let $v \in V(G) \setminus \bar{\pi}(G)$. By the definition of $M(G,\mathcal{L})$, either $v \in V(M(G,\mathcal{L}))$ or there exists $A \in \mathcal{L}$ with $v \notin V(G \setminus A)$. In the first case, $v \in V(G')$, and since $\pi(G) = \pi(G')$ it follows that $v \in V(G') \setminus \bar{\pi}(G')$. In the second case $v \in V(A)$, and therefore $v \notin \bar{\pi}(A)$ since $\bar{\pi}(A) \subseteq V(G \setminus A)$. So in either case

$$v \in (V(G') \setminus \bar{\pi}(G')) \cup \bigcup_{A\in\mathcal{L}} (V(A) \setminus \bar{\pi}(A)),$$

and therefore $V(G) \setminus \bar{\pi}(G)$ is a subset of this set.

To prove the reverse inclusion, we observe that $\bar{\pi}(G) \cap V(G') \subseteq \bar{\pi}(G')$ and for each $A \in \mathcal{L}$, $\bar{\pi}(G) \cap V(A) \subseteq \bar{\pi}(A)$ since A is complemented. It follows that no vertex of

$$(V(G') \setminus \bar{\pi}(G')) \cup \bigcup_{A\in\mathcal{L}} (V(A) \setminus \bar{\pi}(A))$$

belongs to $\bar{\pi}(G)$, so this set is a subset of $V(G) \setminus \bar{\pi}(G)$. This proves the first assertion of the theorem.

For the second assertion, let $\mathcal{L} = \{A_1, \ldots, A_k\}$, and for $1 \le i \le k$ let $e(A_i) \in E(G')$ be the new element of P' corresponding to A_i. Since $\pi(G') = \pi(G)$, we must show that a grouping K with $V(K) = \bar{\pi}(G)$ is feasible in P if and only if K is feasible in P'. Thus, let K be a grouping with $V(K) = \bar{\pi}(G)$.

For $0 \le j \le k$, let G_j be the rooted hypergraph with

$$\begin{aligned} V(G_j) &= V(M(G,\mathcal{L})) \cup \bigcup (V(A_i) : j < i \le k) \\ E(G_j) &= E(M(G,\mathcal{L})) \cup \{e(A_i) : 1 \le i \le j\} \cup \bigcup (E(A_j) : j < i \le k) \\ \pi(G_j) &= \pi(G) \end{aligned}$$

where for $e \in E(M(G,\mathcal{L}))$ its ends are as in G^-, for $1 \le i \le j$ the ends of $e(A_i)$ are the vertices in $\bar{\pi}(A_i)$, and for $e \in E(A_i)$ where $j < i \le k$ its ends are as in A_i^-. For

$$e \in dom(\mu) \cap (E(M(G,\mathcal{L})) \cup E(A_{j+1}) \cup \cdots \cup E(A_k))$$

let $\mu_j(e) = \mu(e)$, and for $1 \le i \le j$ let $\mu_j(e(A_i)) = \pi(A_i)$. For

$$e \in E(M(G,\mathcal{L})) \cup E(A_{j+1}) \cup \cdots \cup E(A_k)$$

let $\Delta_j(e) = \Delta(e)$, and for $1 \le i \le j$ let $\Delta_j(e(A_i)) = gr(P|A_i)$, with $V(\Delta_j(e(A_i))) = \bar{\pi}(A_i)$. Then $P_j = (G_j, \mu_j, \Delta_j)$ is a patchwork for $0 \le j \le k$, and $P_0 = P$, and $P_k = P'$. It therefore suffices to show that for $1 \le j \le k$, K is feasible in P_{j-1} if and only if K is feasible in P_j, since $\pi(G_j) = \pi(G)$.

Let B be $G_j \backslash e(A_j)$, and let A'_j be the rooted hypergraph with $E(A'_j) = \{e(A_j)\}$, $V(A'_j) = \bar{\pi}(A_j)$ (where the ends of $e(A_j)$ are the vertices in $\bar{\pi}(A_j)$), and $\pi(A'_j) = \pi(A_j)$. Since

$$V(A'_j) = \bar{\pi}(A_j) \subseteq V(B),$$

it follows that $\bar{\pi}(A_j) = V(A_j \cap B)$ and $\pi(A'_j) = V(A'_j) \cap B$, and so (A_j, B) is a separation of G_{j-1}, and (A'_j, B) is a separation of G_j.

(1) *A grouping is feasible in $P_{j-1}|A_j$ if and only if it is feasible in $P_j|A'_j$.*

Subproof. $P_{j-1}|A_j = P|A_j$, and a grouping with vertex set $\bar{\pi}(A_j)$ is feasible in $P_j|A'_j$ if and only if it belongs to $\Delta_j(E(A_j))$; that is, it is feasible in $P|A_j = P_{j-1}|A_j$. This proves (1).

Suppose that K is feasible in one of P_{j-1}, P_j (say Q_1), and let H_1 be the corresponding realization of Q_1 such that for distinct $x, y \in V(K)$, x, y are connected in H_1 if and only if they are adjacent in K. By (1) and 5.2 there is a realization H_2 of Q_2 (where $\{P_{j-1}, P_j\} = \{Q_1, Q_2\}$) such that for distinct $x, y \in V(B)$, x, y are connected in H_1 if and only if they are connected in H_2. But $V(K) \subseteq V(B)$, and so for distinct $x, y \in V(K)$, x, y are connected in H_2 if and only if they are adjacent in K. Thus K is feasible in Q_2. This proves 5.4. ∎

Now let $P = (G, \mu, \Delta, \phi)$ be a partial Ω-patchwork, and let $\mathcal{L}$ be a rooted location in G. We call $(P, \mathcal{L})$ a *partial* Ω*-place*. If $dom(\phi) = E(G)$ we call $(P, \mathcal{L})$ an Ω*-place*. For $e \in E(M(G, \mathcal{L})) \cap dom(\phi)$ let $\phi'(e) = \phi(e)$, and let (G', μ', Δ') be the heart of $((G, \mu, \Delta), \mathcal{L})$; then $(G', \mu', \Delta', \phi')$ is a partial Ω-patchwork which we call the *heart* (again denoted by $P|\mathcal{L}$) of $(P, \mathcal{L})$.

A partial Ω-patchwork (G, μ, Δ, ϕ) has *tree-width* $\le w$, where $w \ge 0$, if there is a tree-decomposition (T, τ) of G^- of width $\le w$ such that $\bar{\pi}(G) \subseteq V(\tau(t))$ for some $t \in V(T)$. If $(P, \mathcal{L})$ is a partial Ω-place, and $P|A$ has tree-width $\le w$ for all $A \in \mathcal{L}$, we say that P is an *enlargement of $P|\mathcal{L}$ by tree-width* $\le w$.

5.5 *Let $P = (G, \mu, \Delta, \phi)$ be an Ω-patchwork, let $w \ge 0$, and let $\mathcal{L}, \mathcal{L}^*$ be rooted locations in G, such that $\mathcal{L}^{*-}$ is an enlargement of $\mathcal{L}^-$ by tree-width $\le w$. Then $P|\mathcal{L}^*$ is an enlargement of $P|\mathcal{L}$ by tree-width $\le w$.*

Proof. Let $\mathcal{L} = \{C_1, \ldots, C_k\}$ where $C_1, \ldots, C_k$ are distinct, and for $1 \le i \le k$ let $D_i = (G \setminus C_i)^-$. Then $\mathcal{L}^- = \{(C_1^-, D_1), \ldots, (C_k^-, D_k)\}$. (However, $(C_1^-, D_1), \ldots, (C_k^-, D_k)$ may not all be distinct.) Let $P|\mathcal{L}^* = (G^*, \mu^*, \Delta^*, \phi^*)$ ($= P^*$ say), using new elements $e(A)$ ($A \in \mathcal{L}^*$). Choose $\mathcal{L}_1, \ldots, \mathcal{L}_k \subseteq \mathcal{L}^*$, mutually disjoint and with union $\mathcal{L}^*$, such that for $1 \le i \le k$, every $(A, B) \in \mathcal{L}_i^-$ satisfies $A \subseteq C_i^-$ and $D_i \subseteq B$, and $\mathcal{L}_i^- \cup \{(D_i, C_i^-)\}$ is a location in G^- of tree-width $\le w$. We claim that for $1 \le i \le k$ and all $A \in \mathcal{L}_i$, A is complemented in C_i. For certainly $A^- \subseteq C_i^-$ and $D_i \subseteq (G \setminus A)^-$ since $\mathcal{L}_i^- \cup \{(D_i, C_i^-)\}$ is a location in G^-. Moreover,

$$\bar{\pi}(C_i) \cap V(A) \subseteq V(D_i) \cap V(A) \subseteq V(G \setminus A) \cap V(A) = \bar{\pi}(A);$$

and if $v \in V(A)$ is an end of some $e \in E(C_i) \setminus E(A)$, then $e \in E(G) \setminus E(A)$ and so $v \in \bar{\pi}(A)$. This proves that A is complemented in C_i. Consequently, for $1 \le i \le k$, $\mathcal{L}_i$ is a rooted location in C_i, and so $(P|C_i, \mathcal{L}_i)$ is an Ω-place; let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ be its heart (with "new" elements $e(A)$ $(A \in \mathcal{L}_i)$, some of the new elements of P^*).

(1) *P_i has tree-width $\leq w$.*

Subproof. Let H be the heart of $\mathcal{L}_i^- \cup \{(D_i, C_i^-)\}$. Then H is the hypergraph obtained from

$$M(G^-, \mathcal{L}_i^- \cup \{(D_i, C_i^-)\}) \;=\; C_i^- \cap \bigcap((G \setminus A)^- \,:\, A \in \mathcal{L}_i)$$

by adding a new edge with set of ends $V(A \cap B)$ for each $(A, B) \in \mathcal{L}_i^-$, and adding one further new edge with set of ends $V(D_i \cap C_i^-)$ (unless $(D_i, C_i^-) \in \mathcal{L}_i$). Also, G_i^- is obtained from

$$M(C_i, \mathcal{L}_i) \;=\; C_i^- \cap \bigcap((C_i \setminus A)^- \,:\, A \in \mathcal{L}_i)$$

by adding a new edge with set of ends $\bar{\pi}(A)$ for each $A \in \mathcal{L}_i$. But

$$C_i^- \cap \bigcap((G \setminus A)^- \,:\, A \in \mathcal{L}_i) \;=\; C_i^- \cap \bigcap(C_i^- \cap (G \setminus A)^- \,:\, A \in \mathcal{L}_i) \;=\; C_i^- \cap \bigcap((C_i \setminus A)^- \,:\, A \in \mathcal{L}_i);$$

and there is a surjection from $\mathcal{L}_i$ onto $\mathcal{L}_i^-$ such that if $A \in \mathcal{L}_i$ is mapped to $(A', B') \in \mathcal{L}_i^-$ then $A' = A^-$ and $V(A' \cap B') = \bar{\pi}(A)$. Consequently, a hypergraph isomorphic to G_i^- may be obtained from H by deleting an edge with set of ends $\bar{\pi}(C_i) = V(D_i \cap C_i^-)$ (unless $(D_i, C_i^-) \in \mathcal{L}_i$) and adding some new edges, each with the same ends as some edge of H. (The latter arise when distinct members of $\mathcal{L}_i$ correspond to the same member of $\mathcal{L}_i^-$). Since H has tree-width $\leq w$, there is a tree-decomposition (T, τ) of G_i^- such that $\bar{\pi}(C_i) \subseteq V(\tau(t))$ for some $t \in V(T)$; that is, P_i has tree-width $\leq w$. This proves (1).

(2) *For $1 \leq i \leq k$, G_i is a complemented rooted subhypergraph of G^*.*

Subproof. G_i^- is obtained from $C_i^- \cap \bigcap((C_i \setminus A)^- \,:\, A \in \mathcal{L}_i)$ by adding a new edge with set of ends $\bar{\pi}(A)$ for each $A \in \mathcal{L}_i$, and G^{*-} is obtained from $M(G, \mathcal{L}^*) = G^- \cap \bigcap((G \setminus A)^- \,:\, A \in \mathcal{L}^*)$ by adding a new edge with set of ends $\bar{\pi}(A)$ for each $A \in \mathcal{L}^*$. Since $C_i \setminus A \subseteq G \setminus A$ for each $A \in \mathcal{L}_i$ and $C_i \subseteq G \setminus A$ for all $A \in \mathcal{L}^* - \mathcal{L}_i$, it follows that $C_i^- \cap \bigcap((C_i \setminus A)^- \,:\, A \in \mathcal{L}_i)$ is a subhypergraph of $G^- \cap \bigcap((G \setminus A)^- \,:\, A \in \mathcal{L}^*)$, and so G_i^- is a subhypergraph of G^{*-}. Hence G_i is a rooted subhypergraph of G^*. To see that it is complemented, let $v \in V(G_i)$ be such that either $v \in \bar{\pi}(G^*)$ or some $e \in E(G^*) \setminus E(G_i)$ is incident with v; we claim that $v \in V(D_i)$. If $v \in \bar{\pi}(G^*)$, then $v \in \bar{\pi}(G)$ since $\pi(G^*) = \pi(G)$, and so $v \in V(G \setminus C_i) = V(D_i)$, as claimed. We assume then that some $e \in E(G^*) \setminus E(G_i)$ is incident with v. If $e \in E(G)$, then $e \in E(D_i)$ and so $v \in V(D_i)$ as claimed. If $e \notin E(G)$, then $e = e(A)$ for some $A \in \mathcal{L}^*$. Since $e \notin E(G_i)$ it follows that $A \notin \mathcal{L}_i$, and so $A \in \mathcal{L}_j$ for some $j \neq i$. In particular, $A^- \subseteq C_j^- \subseteq D_i$, and so $v \in V(D_i)$, as claimed. Thus in each case $v \in V(D_i)$, and so

$$v \in V(G_i) \cap V(D_i) \subseteq V(C_i^- \cap D_i) \;=\; \bar{\pi}(C_i) \;=\; \bar{\pi}(G_i).$$

Hence G_i is complemented in G^*. This proves (2).

(3) *$P_i = P^*|G_i$ for $1 \leq i \leq k$.*

Subproof. By (2), $P^*|G_i$ is well-defined, and has the same underlying rooted hypergraph as P_i, namely G_i. Let $e \in E(G_i)$; we must show that $\mu_i(e) \equiv \mu^*(e)$, $\phi_i(e) \equiv \phi^*(e)$, and $\Delta_i(e) = \Delta^*(e)$. Now

$$E(G_i) = E(M(C_i, \mathcal{L}_i)) \cup \{e(a) : A \in \mathcal{L}_i\}.$$

We recall that P_i is the heart of $(P|C_i, \mathcal{L}_i)$ and P^* is the heart of $(P, \mathcal{L}^*)$. If $e \in E(M(C_i, \mathcal{L}_i))$, then $\mu_i(e) \equiv \mu^*(e)$ (because $\mu_i(e) \equiv \mu(e)$ and $\mu^*(e) \equiv \mu(e)$), and the other two relations follow similarly. We assume then that $e = e(A)$ for some $A \in \mathcal{L}_i$. Since A belongs to both $\mathcal{L}_i$ and $\mathcal{L}^*$, it follows from the definition of "heart" that

- $\mu_i(e) = \mu^*(e)$ (for they are both equal to $\pi(A)$),
- e does not belong to $dom(\phi) \cup dom(\phi^*)$, and
- $\Delta_i(e) = \Delta^*(e)$ (for they are both equal to $gr(P|A)$).

This proves (3).

(4) $G_1, \ldots, G_k$ *are all distinct, and* $\{G_1, \ldots, G_k\}$ *is a rooted location in* G^*.

Subproof. Let $1 \leq i,\ j \leq k$ with $i \neq j$; we claim that $G_i \subseteq G^* \setminus G_j$; in other words, that $V(G_i) \cap V(G_j) \subseteq \bar{\pi}(G_j)$ and $E(G_i) \cap E(G_j) = \emptyset$. First, let $v \in V(G_i) \cap V(G_j)$. Since $V(G_i) \subseteq V(C_i)$ and $V(G_j) \subseteq V(C_j)$, it follows that $v \in V(C_i) \cap V(C_j) \subseteq \bar{\pi}(C_j)$ since $\mathcal{L}$ is a rooted location. Since $\bar{\pi}(C_j) = \bar{\pi}(G_j)$ we deduce that $V(G_i) \cap V(G_j) \subseteq \bar{\pi}(G_j)$ as required. Secondly, let $e \in E(G_i) \cap E(G_j)$. If $e \in E(G)$ then $e \in E(C_i) \cap E(C_j) = \emptyset$, which is impossible. Thus $e = e(A)$ for some $A \in \mathcal{L}^*$. Since $e \in E(G_i)$ it follows that $A \in \mathcal{L}_i$, and similarly $A \in \mathcal{L}_j$; but $\mathcal{L}_i \cap \mathcal{L}_j = \emptyset$, a contradiction. Thus $E(G_i) \cap E(G_j) = \emptyset$, as required. This proves that $G_i \subseteq G^* - G_j$. Suppose that $G_i = G_j$. Then $E(G_i) = \emptyset$, and so $\mathcal{L}_i = \emptyset$ and $G_i = C_i$; and similarly $G_j = C_j$. Consequently $C_i = C_j$, a contradiction. Thus $G_i \neq G_j$, and (4) follows.

Let $\mathcal{L}' = \{G_1, \ldots, G_k\}$.

(5) $M(G, \mathcal{L}) = M(G^*, \mathcal{L}')$.

Subproof. If $k = 0$ then $\mathcal{L} = \emptyset$ and $\mathcal{L}' = \emptyset$; and $\mathcal{L}^* = \emptyset$, since $\mathcal{L}^{*-}$ is an enlargement of $\mathcal{L}^-$. Hence $G^* = G$, and $M(G, \mathcal{L}) = G^- = G^{*-} = M(G^*, \mathcal{L}')$ as claimed. We may assume then that $k \neq 0$. Hence $M(G, \mathcal{L}) = D_1 \cap \cdots \cap D_k$ and $M(G^*, \mathcal{L}') = \cap((G^* \setminus G_i)^- : 1 \leq i \leq k)$. If $A \in \mathcal{L}^*$, then $e(A) \notin E(M(G^*, \mathcal{L}'))$, because $e(A) \in E(G_i)$ and hence $e(A) \notin E(G^* \setminus G_i)$ for some $i\,(1 \leq i \leq k)$, namely, the value of i such that $A \in \mathcal{L}_i$. Since $M(G^*, \mathcal{L}')$ is a subhypergraph of G^{*-} and $e(A) \notin E(M(G^*, \mathcal{L}'))$ for each $A \in \mathcal{L}^*$, it follows that $M(G^*, \mathcal{L}')$ is a subhypergraph of G^-. But also $M(G, \mathcal{L})$ is a subhypergraph of G^-, and therefore to show that $M(G, \mathcal{L}) = M(G^*, \mathcal{L}')$ it suffices to show that $M(G, \mathcal{L})$ and $M(G^*, \mathcal{L}')$ have the same vertex- and edge-sets. Let $v \in V(G)$. Then from 5.4 applied to $(P|C_i, \mathcal{L}_i)$, we have:

$$\begin{aligned}
v \in V(M(G,\mathcal{L})) \;&\Leftrightarrow\; v \notin V(C_i) \setminus \bar{\pi}(C_i) \;\text{ for }\; 1 \le i \le k \\
&\Leftrightarrow\; v \notin V(G_i) \setminus \bar{\pi}(G_i) \;\text{ and }\; v \notin V(A) \setminus \bar{\pi}(A) \;\text{ for } 1 \le i \le k \\
&\;\; \text{and for all } \; A \in \mathcal{L}_i \\
&\Leftrightarrow\; v \notin V(A) \setminus \bar{\pi} \;\text{ for all }\; A \in \mathcal{L}^* \;\text{ and }\; v \notin V(G_i) \setminus \bar{\pi}(G_i) \;\text{ for }\; 1 \le i \le k \\
&\Leftrightarrow\; v \in V(G^*) \;\text{ and }\; v \notin V(G_i) \setminus \bar{\pi}(G_i) \;\text{ for }\; 1 \le i \le k \\
&\Leftrightarrow\; v \in V(M(G^*, \mathcal{L}')).
\end{aligned}$$

Thus $V(M(G,\mathcal{L})) = V(M(G^*,\mathcal{L}'))$, and a similar (somewhat easier) proof shows that $E(M(G,\mathcal{L})) = E(M(G^*,\mathcal{L}'))$. This proves (5).

Let $P' = (G', \mu', \Delta', \phi')$ be the heart of the partial Ω-place $(P^*, \mathcal{L}')$.

(6) *P' is the heart of $(P, \mathcal{L})$.*

Subproof. Since $\pi(G_h) = \pi(C_h)$ for $1 \le h \le k$, it follows from (5) that $(P,\mathcal{L})$ has heart $(G', \mu'', \Delta'', \phi'')$ for some μ'', Δ'', ϕ''. We claim that $\mu' = \mu'', \phi' = \phi''$, and $\Delta' = \Delta''$. Let the edges of G' which are not edges of G be $e_1, \ldots, e_k$, numbered in the natural way. (Here we use the fact that $G_1, \ldots, G_k$ are distinct, from (4).) Let $e \in E(G')$, and assume first that $e \ne e_1, \ldots, e_k$. Then $\mu'(e) \equiv \mu^*(e)$ since P' is the heart of $(P^*, \mathcal{L}')$; $\mu^*(e) \equiv \mu(e)$ since P^* is the heart of $(P, \mathcal{L}^*)$; and $\mu''(e) \equiv \mu(e)$ since $(G', \mu'', \Delta'', \phi'')$ is the heart of $(P, \mathcal{L})$. Consequently $\mu'(e) \equiv \mu''(e)$; and similarly $\phi'(e) = \phi''(e)$ and $\Delta'(e) = \Delta''(e)$ as required. Now we assume that $e = e_i$ for some i with $1 \le i \le k$. Then

$$\mu'(e_i) = \pi(G_i) = \pi(C_i) = \mu''(e_i);$$

and $e_i \notin dom(\phi') \cup dom(\phi'')$. Moreover, $\Delta'(e_i) = gr(P^*|G_i)$, and and $\Delta''(e_i) = gr(P|C_i)$. But $P^*|G_i = P_i$ by (3), and P_i is the heart of $(P|C_i, \mathcal{L}_i)$, and so $\Delta'(e_i) = \Delta''(e_i)$ from 5.4. This proves (6).

Since P^* is by (1) and (3) an enlargement of P' by tree-width $\le w$, it follows from (6) that the heart of $(P, \mathcal{L}^*)$ is an enlargement of the heart of $(P, \mathcal{L})$ by tree-width $\le w$. This proves 5.5. ∎

5.6 *Let P_1, P_2 be partial Ω-patchworks, and let (A_1, B_1), (A_2, B_2) be separations of $||P_1||, ||P_2||$ respectively. Let η' be a realizable expansion of $P_1|A_1$ in $P_2|A_2$ (whence $|\bar{\pi}(A_1)| = |\bar{\pi}(A_2)| = k$, say) and let η'' be a realizable expansion of $P_1|B_1$ in $P_2|B_2$ such that for $1 \le i \le k$, $\eta''(v)$ contains the i^{th} term of $\pi(A_2)$, where v is the i^{th} term of $\pi(A_1)$. Define η by:*

$$\eta(v) = \begin{cases} \eta'(v) & : \; v \in V(A_1) \setminus V(B_1) \\ \eta''(v) & : \; v \in V(B_1) \setminus V(A_1) \\ \eta'(v) \cup \eta''(v) & : \; v \in V(A_1 \cap B_1) \end{cases}$$

$$\eta(e) = \begin{cases} \eta'(e) & : \; e \in E(A_1) \\ \eta''(e) & : \; e \in E(B_1). \end{cases}$$

Then η is a realizable expansion of P_1 in P_2.

Proof. Let $*$ be a new element and let Ω' be the well-quasi-order with $\Omega \subseteq \Omega'$ and $E(\Omega') = E(\Omega) \cup \{*\}$, in which $* < x$ for all $x \in E(\Omega)$. For $i = 1, 2$, let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$; and for all $e \in E(G_i)$, define $\phi'_i(e) = \phi_i(e)$ if $e \in dom(\phi_i)$, and otherwise $\phi'_i(e) = *$. Let $P'_i = (G_i, \mu_i, \Delta_i, \phi'_i)$; then P'_i is an Ω'-patchwork. Since η' is a realizable expansion of $P_1|A_1$ in $P_2|A_2$, it follows that it is also a realizable expansion of $P'_1|A_1$ in $P'_2|A_2$. (Here we use that $* \leq x$ for all $x \in E(\Omega)$.) Similarly, η'' is a realizable expansion of $P'_1|B_1$ in $P'_2|B_2$.

By theorem 8.1 of [2] applied to these two Ω'-patchworks, we deduce that η is a realizable expansion of P'_1 in P'_2. For each $e \in dom(\phi_1)$, it follows that $\phi_2(\eta(e)) \neq *$, and therefore $\eta(e) \in dom(\phi_2)$; and consequently η is a realizable expansion of P_1 in P_2. This proves 5.6. ∎

If $P = (G, \mu, \Delta, \phi)$ is an Ω-patchwork, we write $V(P) = V(G), E(P) = E(G)$.

5.7 *For $i = 1, 2$ let $(P_i, \mathcal{L}_i)$ be an Ω-place with heart Q_i, using new elements $e_i(A)$ $(A \in \mathcal{L}_i)$. Suppose that η is a realizable expansion of Q_1 in Q_2 such that*

- *if $e \in E(Q_1)$ and $\eta(e) = e_2(A_2)$ for some $A_2 \in \mathcal{L}_2$ then $e = e_1(A_1)$ for some $A_1 \in \mathcal{L}_1$,*
- *for each $A_1 \in \mathcal{L}_1$ there exists $A_2 \in \mathcal{L}_2$ such that $\eta(e_1(A_1)) = e_2(A_2)$ and $P_1|A_1$ is simulated in $P_2|A_2$.*

Then P_1 is simulated in P_2.

Proof. We proceed by induction on $|\mathcal{L}_2|$. If $\mathcal{L}_2 = \emptyset$ then by (ii), $\mathcal{L}_1 = \emptyset$, and so $Q_1 = P_1$ and $Q_2 = P_2$, and η is a realizable expansion of P_1 in P_2, as required. We assume then that $\mathcal{L}_2 \neq \emptyset$. Choose $A_2 \in \mathcal{L}_2$. There are two cases, depending on whether or not $e_2(A_2) = \eta(e)$ for some $e \in E(Q_1)$.

First, we assume that $e_2(A_2) \neq \eta(e)$ for all $e \in E(Q_1)$. Let $\mathcal{L}'_2 = \mathcal{L}_2 \setminus \{A_2\}$, and let Q'_2 be the heart of the Ω-place $(P_2, \mathcal{L}'_2)$, using new elements $e_2(A)(A \in \mathcal{L}'_2)$. Let $Q_2 = (G, \mu, \Delta, \phi)$, and $Q'_2 = (G', \mu', \Delta', \phi')$; then $e_2(A_2) \in E(G)$, and $(A_2, G \backslash e_2(A_2))$ is a separation of G'. Let K be the rooted subhypergraph of G formed by $e_2(A_2)$ and its ends, with $\pi(K) = \pi(A_2)$; then $(K, G \backslash e_2(A_2))$ is a separation of G. Now $\pi(K) = \pi(A_2)$ and $Q_2|(G \backslash e_2(A_2)) = Q'_2|(G \backslash e_2(A_2))$, and every grouping feasible in $Q_2|K$ is also feasible in $P_2|A_2 = Q'_2|A_2$ (by definition of $\Delta(e_2(A_2))$). Moreover, η is a realizable expansion of Q_1 in Q_2, and $\eta(e) \in E(G \backslash e_2(A_2))$ for all $e \in E(Q_1)$, and $\eta(v) \cap V(G \backslash e_2(A_2)) \neq \emptyset$ for each $v \in V(Q_1)$ (because $V(G \backslash e_2(A_2)) = V(G)$). From 5.3 with $P_1, G_1, \mu_1, \Delta_1, P_2, G_2, \mu_2, \Delta_2, G'_1, G'_2, G_0, \eta_1$ replaced by (G, μ, Δ), G, μ, Δ, (G', μ', Δ'), $G', \mu', \Delta', K, A_2$, $G \backslash e_2(A_2)$, η respectively, and with P replaced by the patchwork formed by the first three components of the quadruple Q_1, we deduce that there is a realizable expansion η_0 of Q_1 in Q'_2 such that $\eta_0(e) = \eta(e)$ for all $e \in E(Q_1)$. In particular, if $e \in E(Q_1)$ and $\eta_0(e) = e_2(A'_2)$ for some $A'_2 \in \mathcal{L}'_2$ then $e = e_1(A_1)$ for some $A_1 \in \mathcal{L}_1$; and for each $A_1 \in \mathcal{L}_1$, $\eta_0(e_1(A_1)) = e_2(A'_2)$ and $P_1|A_1$ is simulated in $P_2|A'_2$ for some $A'_2 \in \mathcal{L}'_2$. From the inductive hypothesis we deduce that P_1 is simulated in P_2, as required.

In the second case, we assume that $e_2(A_2) = \eta(e_1(A_1))$ for some $A_1 \in \mathcal{L}_1$. For $i = 1, 2$, let $\mathcal{L}'_i = \mathcal{L}_i \setminus \{A_i\}$, let $Q'_i = (G'_i, \mu'_i, \Delta'_i, \phi'_i)$ be the heart of $(P_i, \mathcal{L}'_i)$ using new elements $e_i(A)$ $(A \in \mathcal{L}'{}_i)$, and let $B_i = G'_i \setminus A_i$. We claim that for $i = 1, 2$, $Q_i \backslash e_i(A_i) = Q'_i|B_i$. For let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ say. Then

$$M(G_i, \mathcal{L}_i) = M(G_i, \mathcal{L}'_i) \cap (G_i \setminus A_i)^- = M(G_i, \mathcal{L}'_i) \cap (G'_i \setminus A_i)^-$$

since $M(G_i, \mathcal{L}'_i) \subseteq G'^-_i$ and $M(G_i, \mathcal{L}'_i) \subseteq G^-_i$. Hence

$$Q_i \backslash \{e_i(A) : A \in \mathcal{L}_i\} = (Q'_i|B_i) \backslash \{e_i(A) : A \in \mathcal{L}'_i\},$$

and so $Q_i \backslash e_i(A_i) = Q'_i|B_i$, as claimed.

Since $e_2(A_2) = \eta(e_1(A_1))$, the second hypothesis of the theorem implies that there is a realizable expansion η' of $P_1|A_1$ in $P_2|A_2$, and hence of $Q'_1|A_1$ in $Q'_2|A_2$, since $P_i|A_i = Q'_i|A_i$ $(i = 1, 2)$. Let η'' be the restriction of η to $V(Q'_1) \cup E(Q'_1)$; then η'' is a realizable expansion of $Q_1 \backslash e_1(A_1)$ in $Q_2 \backslash e_2(A_2)$; that is, of $Q'_1|B_1$ in $Q'_2|B_2$. Let $|\bar{\pi}(A_1)| = |\bar{\pi}(A_2)| = k$ say. For $1 \leq i \leq k$, let v be the i^{th} term of $\pi(A_1)$; we claim that $\eta''(v)$ contains the i^{th} term of $\pi(A_2)$. For η is a realizable expansion of Q_1 in Q_2, and since $\eta(e_1(A_1)) = e_2(A_2)$ and v is the i^{th} end of $e_1(A_1)$, it follows that $\eta(v)$ contains the i^{th} end of $e_2(A_2)$; that is, the i^{th} term of $\pi(A_2)$. From 5.6 with P_1, P_2 replaced by Q'_1, Q'_2 respectively, there is a realizable expansion η_0 of Q'_1 in Q'_2 such that $\eta_0(e) = \eta(e)$ for all $e \in E(Q_1) \setminus \{e_1(A_1)\}$. In particular, if $e \in E(Q'_1)$ and $\eta_0(e) = e_2(A'_2)$ for some $A'_2 \in \mathcal{L}'_2$ then $e = e_1(A'_1)$ for some $A'_1 \in \mathcal{L}'_1$; and for each $A'_1 \in \mathcal{L}'_1$, $\eta_0(e_1(A'_1)) = e_2(A'_2)$ and $P_1|A'_1$ is simulated in $P_2|A'_2$ for some $A'_2 \in \mathcal{L}'_2$. From the inductive hypothesis applied to $\mathcal{L}'_1$ and $\mathcal{L}'_2$, we deduce that P_1 is simulated in P_2. This proves 5.7. ∎

6 Well-behavedness

Let $P = (G, \mu, \Delta, \phi)$ be a partial Ω-patchwork, and let Ω' be a quasi-order with $\Omega \subseteq \Omega'$. By an Ω'-*completion* of P we mean an Ω'-patchwork (G, μ, Δ, ϕ') such that $\phi'(e) = \phi(e)$ for each $e \in dom(\phi)$. If Ω is a well-quasi-order, a class $\mathcal{C}$ of partial Ω-patchworks is *well-behaved* if for every well-quasi-order Ω' with $\Omega \subseteq \Omega'$ and every countable sequence P'_i $(i = 1, 2, \ldots)$ of Ω'-completions of members of $\mathcal{C}$ there exist $j > i \geq 1$ such that P'_i is simulated in P'_j. (We remark that whether $\mathcal{C}$ is well-behaved depends *prima facie* not only on $\mathcal{C}$, but also on Ω; we leave this dependence implicit. In fact, it is an easy exercise to show that there is no dependence on Ω.)

A partial Ω-patchwork $P = (G, \mu, \Delta, \phi)$ is *rootless* if $\bar{\pi}(G) = \emptyset$. Let $P = (G, \mu, \Delta, \phi)$ be a rootless partial Ω-patchwork, let $e \in dom(\mu) \backslash dom(\phi)$, and let $P' = (G', \mu', \Delta', \phi')$ be the partial Ω-patchwork with $G'^- = G^- \backslash e, \pi(G') = \mu(e)$, and $P' = P|G'$. We call P' a *rooting* of P.

6.1 *Let Ω be a well-quasi-order, and let $\mathcal{C}$ be a well-behaved class of partial Ω-patchworks. Let $\mathcal{C}'$ be the class of all rootings of rootless members of $\mathcal{C}$. Then $\mathcal{C}'$ is well-behaved.*

Proof. Let Ω' be a well-quasi-order with $\Omega \subseteq \Omega'$, and let Q'_i $(i = 1, 2, \ldots)$ be a countable sequence of Ω'-completions of members of $\mathcal{C}'$. Let $*$ be a new element and let Ω'' be the well-quasi-order with $\Omega' \subseteq \Omega''$ and $E(\Omega'') = E(\Omega') \cup \{*\}$, in which if $x \leq *$ or $* \leq x$ then $x = *$. For each i, let $Q'_i = (G'_i, \mu'_i, \Delta'_i, \psi'_i)$ be an Ω'-completion of $P'_i = (G'_i, \mu'_i, \Delta'_i, \phi'_i) \in \mathcal{C}'$ and choose $P_i = (G_i, \mu_i, \Delta_i, \phi_i) \in \mathcal{C}$ and $e_i \in dom(\mu_i) \setminus dom(\phi_i)$ such that $\bar{\pi}(G_i) = \emptyset$, $\pi(G'_i) = \mu_i(e_i)$, $G^-_i \backslash e_i = G'^-_i$ and $P'_i = P_i|G'_i$. Let $Q_i = (G_i, \mu_i, \Delta_i, \psi_i)$ be the Ω''-completion of P_i where

$$\begin{aligned} \psi_i(e) &= \psi'_i(e) \ (e \in E(G'_i)) \\ \psi_i(e_i) &= *. \end{aligned}$$

Since $\mathcal{C}$ is well-behaved, there exist $j > i \geq 1$ such that Q_i is simulated in Q_j; let η be a realizable expansion of Q_i in Q_j. Then $\eta(e_i) = e_j$ since e_j is the only edge e of G_j with $\psi_j(e) = *$; and hence there is a realizable expansion of $Q_i|G'_i$ in $Q_j|G'_j$; that is, of Q'_i in Q'_j. This proves 6.1. ∎

The following is a consequence of theorem 9.1 of [2].

6.2 *If Ω is a well-quasi-order and $w \geq 0$, the class of all robust partial Ω-patchworks of tree-width $\leq w$ is well-behaved.*

Let $\mathcal{C}$ be a class of partial Ω-patchworks, and let $(P, \mathcal{L})$ be a partial Ω-place. If $P|A \in \mathcal{C}$ for all $A \in \mathcal{L}$ we say that P is an *enlargement of $P|\mathcal{L}$ by $\mathcal{C}$.*

6.3 *Let Ω be a well-quasi-order, and let $\mathcal{C}_1$, $\mathcal{C}_2$ be well-behaved classes of partial Ω-patchworks. Then the class of all enlargements of members of $\mathcal{C}_1$ by $\mathcal{C}_2$ is well-behaved.*

Proof. Let $\mathcal{C}$ be the class of all enlargements of members of $\mathcal{C}_1$ by $\mathcal{C}_2$. Let Ω' be a well-quasi-order with $\Omega \subseteq \Omega'$. Let Ω'' be the class of all Ω'-completions of members of $\mathcal{C}_2$, ordered by simulation; then Ω'' is a well-quasi-order, since $\mathcal{C}_2$ is well-behaved. By replacing Ω, Ω' by isomorphic well-quasi-orders we may assume that $E(\Omega') \cap E(\Omega'') = \emptyset$. Let $\Omega^* = \Omega' \cup \Omega''$.

Let P_1 be an Ω'-completion of a member of $\mathcal{C}$. We construct an Ω^*-patchwork $enc(P_1)$ as follows. (Throughout, for $i = 1, 2, 3, 4, P_i = (G_i, \mu_i, \Delta_i, \phi_i)$.) Choose $P_2 \in \mathcal{C}$ so that P_1 is an Ω'-completion of P_2. Choose $P_3 \in \mathcal{C}_1$ so that P_2 is an enlargement of P_3 by $\mathcal{C}_2$ and let $\mathcal{L}$ be the corresponding rooted location in G_2, so that $(P_2, \mathcal{L})$ has heart P_3. Let the new elements of P_3 be $\{e(A) : A \in \mathcal{L}\}$. Since $G_2 = G_1$, $\mathcal{L}$ is also a rooted location in G_1, and so $(P_1, \mathcal{L})$ is an Ω'-place; let its heart be Q (using the same new elements as for P_3). Let $(G_4, \mu_4, \Delta_4) = (G_3, \mu_3, \Delta_3)$ and define $\phi_4 : E(G_4) \to E(\Omega^*)$ by

$$\phi_4(e) = \begin{cases} \phi_3(e) & \text{if } e \in dom(\phi_3) \\ \phi_1(e) & \text{if } e \in E(G_3) \setminus dom(\phi_3) \text{ and } e \neq e(A) \text{ for all } A \in \mathcal{L} \\ P_1|A & \text{if } e = e(A) \text{ for some } A \in \mathcal{L}. \end{cases}$$

Let $P_4 = (G_4, \mu_4, \Delta_4, \phi_4)$. Thus, P_4 is an Ω^*-completion of both P_3 and Q. We define $enc(P_1) = P_4$.

Now let P_1' be another Ω'-completion of a member of $\mathcal{C}$, and suppose that $enc(P_1)$ is simulated in $enc(P_1')$. We claim that P_1 is simulated in P_1'. For let $P_2, P_3, \mathcal{L}, Q, P_4$ and $(G_i, \mu_i, \Delta_i, \phi_i)(i = 1, \ldots, 4)$ be as above for P_1, and define $P_2', P_3', \mathcal{L}', Q', P_4'$ and $(G_i', \mu_i', \Delta_i', \phi_i')$ $(i = 1, \ldots, 4)$ similarly for P_1'. Let η be a realizable expansion of $enc(P_1) = P_4$ in $enc(P_1') = P_4'$. Then η is a realizable expansion of Q in Q' (since P_4, P_4' are Ω^*-completions of Q, Q' respectively). Moreover, for each $e \in E(Q)$, $\phi_4(e) \leq \phi_4'(\eta(e))$, and so $\phi_4(e) \in E(\Omega'')$ if and only if $\phi_4'(\eta(e)) \in E(\Omega'')$; that is, e is one of the new elements of Q if and only if $\eta(e)$ is one of the new elements of Q'. Moreover, if $e = e(A)$ say for some $A \in \mathcal{L}$, and $\eta(e) = e'(A')$ say for some $A' \in \mathcal{L}'$, then $P_1|A$ is simulated in $P_1'|A'$ since $\phi_4(e) \leq \phi_4'(\eta(e))$. Consequently the hypotheses of 5.7 are satisfied (with $P_1, \mathcal{L}_1, Q_1, P_2, \mathcal{L}_2, Q_2, \Omega$ replaced by $P_1, \mathcal{L}, Q, P_1', \mathcal{L}', Q', \Omega'$ respectively) and so by 5.7, P_1 is simulated in P_1'.

Let $P_1, P_2, \ldots$ be a countable sequence of Ω'-completions of members of $\mathcal{C}$. Since Ω^* is a well-quasi-order and $\mathcal{C}_1$ is well-behaved, there exist $j > i \geq 1$ such that $enc(P_i)$ is simulated in $enc(P_j)$. It follows that P_i is simulated in P_j. Hence $\mathcal{C}$ is well-behaved. This proves 6.3. ∎

An *arborescence* is a directed graph T, whose underlying graph is a tree (denoted by T^-), such that every vertex is the head of at most one edge. It follows that there is a unique vertex of T that is the head of no edge of T, and we call it the *root* of T and denote it by $o(T)$. If T is an arborescence and $t \in V(T)$, T^t denotes the maximal subarborescence with root t. If $f \in E(T)$, then T^f, T_f denote the two components of $T \backslash f$, where the head of f belongs to T^f (and hence $o(T)$ belongs to T_f).

Let $P = (G, \mu, \Delta, \phi)$ be a partial Ω-patchwork. A *rooted decomposition* of P is a pair (T, τ), where

- T is an arborescence, and for each $t \in V(T)$, $\tau(t) \subseteq G$ is a rooted hypergraph
- (T^-, τ^-) is a tree-decomposition of G^-, where $\tau^-(t) = \tau(t)^-$ for each $t \in V(T)$
- $\pi(\tau(o(T))) = \pi(G)$
- for every subarborescence S of T, let $\tau \times S$ denote the rooted hypergraph H with $H^- = \tau^- \times S^-$ and $\pi(H) = \pi(\tau(o(S)))$; then for every edge $f \in E(T)$ with head t,
$$\bar{\pi}(\tau(t)) = V(\tau \times T^f) \cap V(\tau \times T_f)$$
- for every directed path F of T with first edge f_1 and last edge f_2 such that f_1, f_2 make separations under (T^-, τ^-) of the same order and no edge of F makes a separation of smaller order, $P|\tau \times T^{f_2}$ is simulated in $P|\tau \times T^{f_1}$.

If (T, τ) is a rooted decomposition of a partial Ω-patchwork (G, μ, Δ, ϕ) and $f \in E(T)$, we define $\tau \times (T, f)$ to be the rooted hypergraph $((\tau \times T_f)^-, \pi(\tau \times T^f))$. (This makes sense because of the fourth condition above.)

We need the following, an immediate consequence of a result of [2].

6.4 *Let $P = (G, \mu, \Delta, \phi)$ be a rootless, robust Ω-patchwork, and let (S, σ) be a rotund tree-decomposition of G^-. Let T be an arborescence with $T^- = S$. Then there is a rooted decomposition (T, τ) of P such that $\tau(t)^- = \sigma(t)$ for each $t \in V(T)$.*

Proof. For each $t \in V(T)$ let $\sigma^+(t)$ be a rooted hypergraph chosen so that $(\sigma^+(t))^- = \sigma(t)$ and

- if $t = o(T)$ then $\bar{\pi}(\sigma^+(T)) = \emptyset$
- if t is the head of an edge $f \in E(T)$ then $\bar{\pi}(\sigma^+(t)) = V(\sigma \times T^f) \cap V(\sigma \times T_f)$.

Since (S, σ) is rotund (in the sense defined in section 2 above), it follows that (T, σ^+) is a "rotund tree-decomposition" in the sense of [2] (which is different from the sense in the present paper). Let $\mathcal{R}$ be the set of all rooted hypergraphs H with $H^- \subseteq G^-$; and let us say that $H_1 \in \mathcal{R}$ is *simulated* in $H_2 \in \mathcal{R}$ if $P|H_1$ is simulated in $P|H_2$. Then, as in section 9 of [2], Axioms 1-3 of [2] are satisfied, and so we can apply theorem 4.1 of [2] (with $T, \tau, \mathcal{R}, F$ replaced by $T, \sigma^+, \mathcal{R}, E(T)$). We deduce that there is a rooted decomposition (T, τ) of P such that $\tau(t)^- = \sigma(t)$ for each $t \in V(T)$. This proves 6.4. ∎

We need another lemma about rooted decompositions.

6.5 *Let (T, τ) be a rooted decomposition of a rootless Ω-patchwork $P = (G, \mu, \Delta, \phi)$.*

1. *If $f \in E(T)$, then $\tau \times T^f$ is complemented in G, and $G \setminus \tau \times T^f = \tau \times T_f$.*
2. *If $f \in E(T)$ then $\tau \times (T, f)$ is complemented in G and $G \setminus \tau \times (T, f)$ is the rooted hypergraph H with $H^- = (\tau \times T^f)^-$ and $\bar{\pi}(H) = \emptyset$.*
3. *If $f_0 \in E(T)$ has head t and $f_1, \dots, f_n$ are the edges of T with tail t, then*
$$\mathcal{L} = \{\tau \times T^{f_1}, \dots, \tau \times T^{f_n}\}$$
is a rooted location in $\tau \times T^t$ and $\mathcal{L}^* = \mathcal{L} \cup \{\tau \times (T, f_0)\}$ is a rooted location in G and $M(\tau \times T^t, \mathcal{L}) = M(G, \mathcal{L}^*)$.

Proof. For 6.5.1, we observe that $\bar{\pi}(\tau \times T^f) = V(\tau \times T^f) \cap V(\tau \times T_f)$, from the fourth condition in the definition of a rooted decomposition; and also, that $((\tau \times T^f)^-, (\tau \times T_f)^-)$ is one of the separations made by f under the tree-decomposition (T^-, τ^-). From these two facts it follows that $\tau \times T^f$ is complemented in G, and consequently $\tau \times T_f = G \setminus \tau \times T^f$ since $\tau \times T_f$ and G are both rootless.

For 6.5.2, let $G_0 = \tau \times (T, f)$. Since $((\tau \times T_f)^-, (\tau \times T^f)^-)$ is one of the separations made by f under (T^-, τ^-), and since $G_0^- = (\tau \times T_f)^-$ and

$$\bar{\pi}(G_0) = V(\tau \times T_f) \cap V(\tau \times T^f)$$

it follows that G_0 is complemented; and since G and H are rootless and $H^- = (\tau \times T^f)^-$, we deduce that $H = G \setminus G_0$.

For 6.5.3, we observe first that

(1) *If $f \in E(T)$ has tail t then $\tau \times T^f$ is complemented in $\tau \times T^t$, and*

$$(\tau \times T^t \setminus \tau \times T^f)^- = (\tau \times T_f)^- \cap (\tau \times T^t)^-.$$

Subproof. We have

$$\bar{\pi}(\tau \times T^t) = \bar{\pi}(\tau(t)) \subseteq V(\tau \times T_f),$$

and by 6.5.1, $G \setminus \tau \times T^f = \tau \times T_f$. Since $\tau \times T^f \subseteq \tau \times T^t$, it follows that $\tau \times T^f$ is complemented in $\tau \times T^t$, and the equation of (1) holds. This proves (1).

(2) *If distinct $f_1, f_2 \in E(T)$ have a common tail t, then $\tau \times T^{f_1} \subseteq \tau \times T^t \setminus \tau \times T^{f_2}$.*

Subproof. $\tau \times T^{f_1}$ and $\tau \times T^{f_2}$ have no edges in common, and any vertex in them both lies in $V(\tau(t))$, from the third condition in the definition of a tree-decomposition. Hence

$$V(\tau \times T^{f_1}) \cap V(\tau \times T^{f_2}) \subseteq V(\tau(t)) \cap V(\tau \times T^{f_2}) \subseteq \bar{\pi}(\tau \times T^{f_2}),$$

and so $\tau \times T^{f_1} \subseteq \tau \times T^t \setminus \tau \times T^{f_2}$. This proves (2).

(3) *If $f_0, f_1 \in E(T)$ and the head of f_0 equals the tail of f_1, let $G_0 = \tau \times (T, f)$; then $\tau \times T^{f_1} \subseteq G \setminus G_0$ and $G_0 \subseteq G \setminus \tau \times T^{f_1}$.*

The proof of (3) is very similar to that of (2) and we leave it to the reader.

Now we complete the proof of 6.5.3. By (1) and (2), $\mathcal{L}$ is a rooted location in $\tau \times T^t$. By 6.5.1 and 6.5.2, all members of $\mathcal{L}^*$ are complemented in G. If $i, j \in \{1, \dots, n\}$ and $i \neq j$ then

$$\tau \times T^{f_i} \subseteq \tau \times T_{f_j} = G \setminus \tau \times T^{f_j}$$

by 6.5.1. If $i \in \{1, \dots, n\}$ then $\tau \times (T, f_0) \subseteq G \setminus \tau \times T^{f_i}$ by (3), and

$$(\tau \times T^{f_i})^- \subseteq (\tau \times T^{f_0})^- = (G \setminus \tau \times T^{f_0})^-$$

by 6.5.2. Hence $\mathcal{L}^*$ is a rooted location in G.

To prove that $M(\tau \times T^t, \mathcal{L}) = M(G, \mathcal{L}^*)$, we observe first that by 6.5.1, $G \setminus \tau \times T^{f_i} = \tau \times T_{f_i}$ for $1 \leq i \leq n$, and by 6.5.2, $(G \setminus \tau \times (T, f))^- = (\tau \times T^t)^-$, and so

$$M(G, \mathcal{L}^*) = (\tau \times T^t)^- \cap \bigcap_{i=1}^{n} (\tau \times T_{f_i})^- .$$

By (1), this is equal to $M(\tau \times T^t, \mathcal{L})$. This proves 6.5.3, and hence completes the proof of 6.5. ▮

Let $P = (G, \mu, \Delta, \phi)$ be a rootless Ω-patchwork, and let (T, τ) be a tree-decomposition of G^-. If $(P, \mathcal{L})$ is an Ω-place such that $\mathcal{L}^-$ is the location of t_0 in (T, τ) for some $t_0 \in V(T)$, we call the heart of $(P, \mathcal{L})$ a *piece of P (at t_0, under (T, τ))*. For each $t_0 \in V(T)$, there is at least one piece of P at t_0, and in general there are many, because of the arbitrary choices of the marches $\pi(A) (A \in \mathcal{L})$.

6.6 *Let Ω be a well-quasi-order, and let $\mathcal{C}$ be a well-behaved class of rootless partial Ω-patchworks. Let $\mathcal{C}'$ be the class of all rootless, robust Ω-patchworks P such that there is a rotund, proper tree-decomposition of $||P||$ under which all pieces of P belong to $\mathcal{C}$. Then $\mathcal{C}'$ is well-quasi-ordered by simulation.*

Proof. Let $P = (G, \mu, \Delta, \phi) \in \mathcal{C}'$. From the definition of $\mathcal{C}'$ there exist an arborescence T and a rotund, proper tree-decomposition (T^-, σ) of G^- such that all pieces of P under (T^-, σ) belong to $\mathcal{C}$. By 6.4 we may choose a rooted decomposition (T, τ) of P such that $\tau(t)^- = \sigma(t)$ for each $t \in V(T)$ and consequently $\sigma = \tau^-$. Let $\mathcal{C}^*$ be the union of $\mathcal{C}$ and the class of all rootings of members of $\mathcal{C}$.

(1) *Let $t \in V(T)$ and let $N(t)$ be the set of all $y \in V(T)$ such that there is an edge of T with head y and tail t. Then $P|\tau \times T^t$ is an enlargement of a member of $\mathcal{C}^*$ by the set of Ω-patchworks $\mathcal{C}_t = \{P|\tau \times T^y : y \in N(t)\}$.*

Subproof. Let $B = P|\tau \times T^t = (G^*, \mu^*, \Delta^*, \phi^*)$. Let $N(t) = \{t_1, \ldots, t_n\}$, and let F be the path of T between t and $o(T)$. For $1 \leq i \leq n$, let $P_i = P|\tau \times T^{t_i}$, and let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$. Since (T^-, τ^-) is proper, $G_1, \ldots, G_n$ are distinct; and $\mathcal{L} = \{G_1, \ldots, G_n\}$ is a rooted location in G^* by 6.5.3. Thus $(B, \mathcal{L})$ is an Ω-place. If $t = o(T)$, then by 6.5.1 $\mathcal{L}^-$ is the location of t in (T^-, τ^-), and so $B|\mathcal{L}$ is a piece of P under (T^-, τ^-), and consequently belongs to $\mathcal{C}$. But for each $A \in \mathcal{L}$,

$$P|A = P_i = P|\tau \times T^{t_i} \in \mathcal{C}_t$$

for some i, and so B is an enlargement of a member of $\mathcal{C} \subseteq \mathcal{C}^*$ by $\mathcal{C}_t$. We may assume then that $t \neq o(T)$. Let t_0 be the neighbour of t in $V(F)$ and let $f_0 \in E(T)$ have ends t, t_0. Let $G_0 = \tau \times (T, f_0)$. Then $G_0, G_1, \ldots, G_n$ are all distinct since (T^-, τ^-) is proper; and $\mathcal{L}^* = \{G_0, G_1, \ldots, G_n\}$ is a rooted location in G, by 6.5.3; and $\mathcal{L}^{*-}$ is the location of t in (T^-, τ^-), by 6.5.1 and 6.5.2. Consequently, $P|\mathcal{L}^*)$ is a piece of P under (T^-, τ^-), and hence belongs to $\mathcal{C}$. Now $B|\mathcal{L}$ is a rooting of $P|\mathcal{L}^*$, because $\mathcal{L} = \mathcal{L}^* \setminus \{G_0\}$, and $M(G^*, \mathcal{L}) = M(G, \mathcal{L}^*)$ by 6.5.3, and $\pi(B) = \pi(G_0)$. Consequently, $B|\mathcal{L} \in \mathcal{C}^*$, and since

$$B|G_i = (P|\tau \times T^t)|\tau \times T^{t_i} = P|\tau \times T^{t_i}$$

for $1 \leq i \leq n$, we deduce that B is an enlargement of a member of $\mathcal{C}^*$ by $\mathcal{C}_t$. This proves (1).

Now let $P_1, P_2, \ldots$ be a countable sequence of members of $\mathcal{C}'$. For all $i \geq 1$, let (T_i, τ_i) be the corresponding rooted decomposition of P_i; that is, such that (T_i^-, τ_i^-) is a rotund, proper tree-decomposition of $||P_i||^-$ such that all pieces of P_i under this decomposition belong to $\mathcal{C}$. We may assume that $T_1, T_2, \ldots$ are mutually disjoint; let their union be M. For $X \subseteq V(M)$, let $N(X)$ be the set of all $y \in V(M)$ such that for some $x \in X$, there is an edge xy of M with head y. Let $B(X)$ be the set of all $P_i|\tau_i \times T_i^x$, for $i \geq 1$ and $x \in X \cap V(T_i)$.

(2) *If $X \subseteq V(M)$ and $B(N(X))$ is well-quasi-ordered by simulation, then so is $B(X)$.*

Subproof. By (1), each member of $B(X)$ is an enlargement of a member of $\mathcal{C}^*$ by $B(N(X))$. Since $\mathcal{C}^*$ is well-behaved by 6.1 (for the union of two well-behaved classes is well-behaved) and $B(N(X))$ is well-behaved by hypothesis, the claim follows from 6.3. This proves (2).

We may assume that for $1 \leq i < j$, $(V(P_i) \cup E(P_i)) \cap (V(P_j) \cup E(P_j)) = \emptyset$. Let $\mathcal{R}$ be the set of all rooted hypergraphs G such that $G \subseteq ||P_i||$ for some $i > 0$; then $\mathcal{R}$ satisfies axioms 1 and 2 of [2] (as is explained at the start of section 9 of [2]). Let $i > 0$ and let $s \in V(T_i)$. Let S be the subtree of T_i induced on $\{s\} \cup N(s)$; that is, the star formed by s and its outneighbours. Define $\sigma(s) = \tau_i(s)$, and for each $t \in N(s)$ define $\sigma(t) = \tau_i \times T_i^t$. Let $\mathcal{S}$ be the set of all such pairs (S, σ) (for all $i > 0$ and all $s \in V(T_i)$). We see that $\mathcal{S}$ is a set of "star-decompositions", in the sense of section 3 of [2]. We claim that $\mathcal{S}$ is "good", in the sense of that paper. We have to check that:

- $\sigma \times S \in \mathcal{R}$ for each $(S, \sigma) \in \mathcal{S}$; this is clear.
- There exists $k \geq 0$ such that $|\bar{\pi}(t)| \leq k$ for every $(S, \sigma) \in \mathcal{S}$ and every $t \in V(S)$. To see this, observe that since $\mathcal{C}$ is well-behaved, there exists $k \geq 0$ such that $|\bar{\pi}(G)| \leq k$ for every $(G, \mu, \Delta, \phi) \in \mathcal{C}$; and since all pieces of each P_i under (T_i^-, τ_i) belong to $\mathcal{C}$, it follows that $|\bar{\pi}(t)| \leq k$ for all $i > 0$ and all $t \in V(T_i)$, and so the claim follows.
- The third condition to be verified is just (2) above, in different language.

Hence we may apply theorem 3.3 of [2]. We deduce that there exist $j > i \geq 1$ such that P_i is simulated in P_j. Hence $\mathcal{C}'$ is well-quasi-ordered by simulation, as required. ∎

Now we can prove our main result, the following.

6.7 *Let Ω be a well-quasi-order, let $\mathcal{C}$ be a well-behaved class of rootless partial Ω-patchworks, and let $\theta \geq 1$ be an integer. Let $\mathcal{D}$ be a class of rootless, robust Ω-patchworks and suppose that for each $P \in \mathcal{D}$ there is a tie-breaker λ in $||P||^-$ such that for every tangle $\mathcal{T}$ in G^- of order $\geq \theta$, there is an Ω-place $(P, \mathcal{L})$ with heart in $\mathcal{C}$ such that $\mathcal{L}^-$ θ-isolates $\mathcal{T}$. Then $\mathcal{D}$ is well-quasi-ordered by simulation.*

Proof. Let $\mathcal{C}'$ be the class of all robust partial Ω-patchworks of tree-width $\leq 3\theta + 1$. By 6.2, $\mathcal{C}'$ is well-behaved. Let $\mathcal{C}^*$ be the class of all enlargements of members of $\mathcal{C}$ by $\mathcal{C}'$. By 6.3 it follows that

(1) *$\mathcal{C}^*$ is well-behaved.*

Now let $P = (G, \mu, \Delta, \phi) \in \mathcal{D}$ be such that G^- has a tangle of order $\geq \frac{4}{3}\theta$. Let λ be a tie-breaker in G^-, as in the theorem. By 4.4 we deduce that

(2) *There is a tree-decomposition* (T,τ) *of* G^- *such that*

(a) (T,τ) *is proper and rotund*

(b) *for each* $e \in E(T)$*, the separations made by* e *under* (T,τ) *are robust, and*

(c) *for each* $t \in V(T)$*, if* $\mathcal{L}_t$ *is the location of* t *in* (T,τ)*, then there is an* Ω*-place* $(P,\mathcal{L})$ *with heart in* $\mathcal{C}$*, such that* $\mathcal{L}_t$ *is an enlargement of* $\mathcal{L}^-$ *by tree-width* $\leq 3\theta+1$*.*

(3) *Let* (T,τ) *be as in* (2) *and let* $t \in V(T)$*. Then every piece of* P *at* t *under* (T,τ) *is in* $\mathcal{C}^*$*.*

Subproof. Let $\mathcal{L}_t$, $\mathcal{L}$ be as in (2)(c), and let Q be a piece of P at t. Then $Q = P|\mathcal{L}^*$ for some rooted location $\mathcal{L}^*$ in G with $\mathcal{L}^{*-} = \mathcal{L}_t$. By (2)(c), $\mathcal{L}^{*-}$ is an enlargement of $\mathcal{L}^-$ by tree-width $\leq 3\theta+1$. By 5.5, $Q = P|\mathcal{L}^*$ is an enlargement of $P|\mathcal{L}$ by tree-width $\leq 3\theta+1$; and since $P|\mathcal{L} \in \mathcal{C}$ by hypothesis, it follows that Q is an enlargement of a member of $\mathcal{C}$ by the class of all partial Ω-patchworks of tree-width $\leq 3\theta+1$. However, the latter differs from $\mathcal{C}'$, because the members of $\mathcal{C}'$ are robust. To show that Q $(= (G_0,\mu_0,\Delta_0,\phi_0)$ say) is an enlargement of a member of $\mathcal{C}$ by $\mathcal{C}'$, we must show that $\Delta_0(e)$ is a robust patch for every $e \in E(Q)$ which is not an edge of $P|\mathcal{L}$. Actually, we shall prove more, that $\Delta_0(e)$ is robust for every $e \in E(Q)$. Let $e \in E(Q)$. If $e \in E(P)$ then $\Delta_0(e) = \Delta(e)$ and hence is robust since P is robust, as required. We assume then that $e \notin E(P)$. Since $Q = P|\mathcal{L}^*$, it follows that G_0 is obtained from $M(G,\mathcal{L}^*)$ by adding a new edge $e(A)$ for each $A \in \mathcal{L}^*$, where $e(A)$ has set of ends $\bar{\pi}(A)$; and $\Delta_0(e(A))$ is the set of all groupings feasible in $P|A$. Since $e \notin E(P)$ and hence $e \notin E(M(G,\mathcal{L}^*))$, it follows that $e = e(A)$ for some $A \in \mathcal{L}^*$. Let $B = G \setminus A$; then $(A^-,B^-) \in \mathcal{L}^{*-} = \mathcal{L}_t$. Hence (A^-,B^-) is robust by (2)(b). Let $X_1, X_2 \subseteq \bar{\pi}(A) = V(\Delta_0(e))$ with $|X_1| = |X_2|$ say, and $X_1 \cap X_2 = \emptyset$. Define $k = |\bar{\pi}(A)| - |X_1|$. Let (H_1,H_2) be any separation of A^- such that $\bar{\pi}(A) \setminus X_i \subseteq V(H_i)$ for $i = 1,2$. Since (A^-,B^-) is robust, there exists $i \in \{1,2\}$ such that $|V(H_i) \cap V(H_j \cup B^-)| \geq |V(A^- \cap B^-)|$, where $j = 3-i$. Subtracting $|V(H_i \cap B) \setminus V(H_j)|$ from both sides gives

$$|V(H_1 \cap H_2)| \geq |V(B^- \cap H_j)| \geq |\bar{\pi}(A) \setminus X_j| = k,$$

since $\bar{\pi}(A) \setminus X_j \subseteq V(B^- \cap H_j)$. From theorem 6.1 of [2], applied to $P|A$, there is a realization of $P|A$ such that k of its components have nonempty intersection with both $\bar{\pi}(A) \setminus X_1$ and $\bar{\pi}(A) \setminus X_2$. Therefore there is a pairing with vertex set $\bar{\pi}(A)$, feasible in $P|A$, which pairs X_1, X_2. Since $\Delta_0(e)$ is the set of all groupings feasible in $P|A$, it follows that $\Delta_0(e)$ is robust, as required. This proves (3).

Let $\mathcal{D}'$ be the class of all members $(G,\mu,\Delta,\phi) \in \mathcal{D}$ such that G has a tangle of order $\geq \frac{4}{3}\theta$. We have shown then that

(4) *For all* $P = (G,\mu,\Delta,\phi) \in \mathcal{D}'$*, there is a rotund, proper tree-decomposition* (T,τ) *of* G^- *such that all pieces of* P *under* (T,τ) *belong to* $\mathcal{C}^*$*.*

By (1), (4) and 6.6, $\mathcal{D}'$ is well-quasi-ordered by simulation. If $P = (G,\mu,\Delta,\phi) \in \mathcal{D} \setminus \mathcal{D}'$ then G^- has tree-width $\leq 2\theta$ by 2.2, and hence so does P since P is rootless. By 6.2, $\mathcal{D} \setminus \mathcal{D}'$ is well-quasi-ordered by simulation, and hence so is $\mathcal{D} = \mathcal{D}' \cup (\mathcal{D} \setminus \mathcal{D}')$. This proves 6.7. ∎

7 More on isolation

Here is a useful way to prove that locations θ-isolate tangles. Let λ be a tie-breaker in a hypergraph G, let $\mathcal{T}$ be a tangle in G, and let $(A, B) \in \mathcal{T}$. We say that (A, B) is *λ-linked to $\mathcal{T}$* if there is no $(A', B') \in \mathcal{T}$ with smaller λ-order with $A \subseteq A'$ and $B' \subseteq B$.

7.1 *Let $\mathcal{T}$ be a tangle of order $\geq \theta \geq 1$ in a hypergraph G with a tie-breaker λ, and let $\mathcal{L} \subseteq \mathcal{T}$ be a location of order $< \theta$, every member of which is λ-linked to $\mathcal{T}$. Then $\mathcal{L}$ θ-isolates $\mathcal{T}$.*

Proof. Let $\mathcal{T}'$ be a tangle of order $\geq \theta$, and let $(D, C) \in \mathcal{T}'$ for some $(C, D) \in \mathcal{L}$. Let (A, B) be the $(\mathcal{T}, \mathcal{T}')$-distinction.

(1) *$(A \cup C, B \cap D)$ has λ-order at least that of (C, D) .*

Subproof. Suppose not. Since $C \subseteq A \cup C$ and $B \cap D \subseteq D$, and (C, D) is λ-linked to $\mathcal{T}$, it follows that $(A \cup C, B \cap D) \notin \mathcal{T}$. But its order is at most that of (C, D) (by the third tie-breaker axiom) and hence less than the order of $\mathcal{T}$, and so $(B \cap D, A \cup C) \in \mathcal{T}$. Yet $(A, B), (C, D) \in \mathcal{T}$, contrary to (**T2**), since $(B \cap D) \cup A \cup C = G$. This proves (1).

(2) *$(A \cap C, B \cup D)$ has λ-order at least that of (A, B).*

Subproof. Suppose not. As before, the order of $(A \cap C, B \cup D)$ is at most that of (A, B), and hence less than the orders of $\mathcal{T}$ and $\mathcal{T}'$. Since $(A, B) \in \mathcal{T}$ and $A \cap C \subseteq A$ it follows that $(A \cap C, B \cup D) \in \mathcal{T}$. Since $(B, A), (D, C) \in \mathcal{T}'$ and $B \cup D \cup (A \cap C) = G$ it follows that $(A \cap C, B \cup D) \notin \mathcal{T}'$ from (**T2**), and so $(B \cup D, A \cap C) \in \mathcal{T}'$. Thus $(A \cap C, B \cup D)$ distinguishes $\mathcal{T}$ from $\mathcal{T}'$, and yet its λ-order is less than that of the $(\mathcal{T}, \mathcal{T}')$-distinction, a contradiction. This proves (2).

From (1), (2) and the second tie-breaker axiom, we deduce that $(A \cup C, B \cap D)$ has the same λ-order as (C, D), and hence $(A \cup C, B \cap D) = (C, D)$ or (D, C), from the first tie-breaker axiom. Since $(B \cap D, A \cup C)$ has the same order as (C, D) and hence belongs to $\mathcal{T}'$ (because $(B, A) \in \mathcal{T}'$) and $(C, D) \notin \mathcal{T}'$, it follows that $(B \cap D, A \cup C) \neq (C, D)$. Hence $(A \cup C, B \cap D) = (C, D)$, and so $A \subseteq C$ and $D \subseteq B$. Thus $\mathcal{L}$ θ-isolates $\mathcal{T}$. This proves 7.1. ∎

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Symbols

Greek: η, θ, λ, μ, π, τ, ϕ, Δ, Ω

Script: $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{L}$, $\mathcal{T}$

Math: $\cup$, $\cap$, $\setminus$, $\bigcup$, $\bigcap$ (cup, cap, union, intersection), $\sum$ (summation), $\lceil\,\rceil$, $\lfloor\,\rfloor$ (rounding), $\emptyset$ (null set), $*$, A^-, $P|A$, $G\backslash F$, G/F, $\tau \times T$, $\bar{\pi}$, τ^+.