Graph Minors
XVI. Excluding a Non-Planar Graph

Neil Robertson[1]
Ohio State University, Columbus, OH 43210

P. D. Seymour[2]
Telcordia Technologies, 445 South St., Morristown, NJ 07960

April 1986; revised September 20, 2023

[1]Research partially supported by NSF grant number DMS 8504054
[2]Current address: Princeton University, Princeton NJ 08544

Abstract

This paper contains the cornerstone theorem of the series. We study the structure of graphs with no minor isomorphic to a fixed graph L, when L is non-planar. (The case when L is planar was studied in an earlier paper.) We find that every graph with no minor isomorphic to L may be constructed by piecing together in a tree-structure graphs each of which "almost" embeds in some surface in which L cannot be embedded.

1 Introduction

Graphs in this paper are finite, undirected graphs which may have loops or multiple edges. A graph L is a *minor* of a graph G if L can be obtained from a subgraph of G by contracting edges.

In this paper we are concerned with the structure of the graphs which have no minor isomorphic to a given graph, and we begin in this section with a discussion of what kind of structure we might expect. It turns out that there are (at least) four ingredients to "structure", which we now review.

Ingredient 1: tree-structure

It is a triviality that the graphs with no minor isomorphic to a loop are the forests; this is the simplest instance of the exclusion of a minor forcing some variety of tree-structure. If we exclude K_3 instead we get almost the same answer — forests augmented by loops and multiple edges. Excluding K_4 yields the "series-parallel" graphs, by a theorem of Dirac [1] or Duffin [2], and they too have a certain tree-structure. To unify these and some other instances, we make the following definitions. Let G be a graph with a separation (H_1, H_2) (that is, H_1 and H_2 are subgraphs of G with no common edges and $H_1 \cup H_2 = G$). Let K be a complete graph with $V(K) = V(H_1) \cap V(H_2)$ and with no edges in common with H_1 and H_2. Let $G_i = H_i \cup K (i = 1, 2)$. Then we say that G is the *clique-sum* of G_1 and G_2. If G can be constructed from members of a class $\mathcal{C}$ of graphs by (repeated) clique-sums we say that G is a *tree-structure* over $\mathcal{C}$. Thus, the graphs with no K_3-minor are the tree-structures over the class of all graphs with ≤ 2 vertices; and the graphs with no K_4-minor are the tree-structures over the class of all graphs with ≤ 3 vertices. Let us say that G has *tree-width* $\leq N$ if G is a tree-structure over the class of all graphs with $\leq N+1$ vertices (and the smallest such N is of course the *tree-width* of G). The following is the main theorem of [4]. (An *L-minor* is a minor isomorphic to L.)

1.1 *For any planar graph L there is a number N such that every graph with no L-minor has tree-width $\leq N$.*

In a way this generalizes the above-mentioned result about the exclusion of K_3 and K_4. It is not a complete generalization, because 1.1 does not yield a structure which is necessary and sufficient for the exclusion of L, but merely one which is necessary. Nevertheless 1.1 has proved to be very useful. In another sense, 1.1 is best possible, because the structure given by 1.1 is necessary for the exclusion of L, and sufficient for the exclusion of some other (larger) planar graph. (A related fact; if L is non-planar there is no number N as in 1.1.) The object of this paper is to find an analogue of 1.1 for non-planar graphs L, best possible in the same sense; that is, the structure is necessary for the exclusion of L and sufficient for the exclusion of some other, larger graph L'. (In fact it can be shown that for any surface Σ in which L can be drawn, we can choose L' so that it too can be drawn in Σ.)

Ingredient 2: genus

As central to the subject as the genus of L, however, is the genus of graphs with no L-minor. If Σ is a surface and L cannot be drawn in Σ, then no graph which can be drawn in Σ has an L-minor. This provides a second type of structure associated with the exclusion of L as a minor. There are some important theorems about excluded minors which involve combinations of these two structures; for instance, K. Wagner [12] proved the following. (V_8 is obtained from an eight-vertex circuit by adding edges joining the four opposite pairs of vertices.)

1.2 *A graph G has no K_5 minor if and only if G is a tree-structure over the class of all graphs which are either planar or isomorphic to V_8.*

Ingredient 3: bounded extension

Let us consider the result of excluding K_6. The structure of the graphs with no K_6-minor has not yet been determined, but one class of such graphs consists of those with a vertex the deletion of which yields a planar graph. It can be shown that there is no class of graphs of bounded genus such that all these graphs are tree-structures over it, and that shows the necessity for our third ingredient. Let Σ be a surface, let $N \geq 0$ be an integer, and let L be a graph such that no graph obtained from L by deleting $\leq N$ vertices can be drawn in Σ. Then there will be no L minor in any graph which can be constructed by adding $\leq N$ vertices (joined arbitrarily) to a graph drawn in Σ. This then yields a third type of structure which we should anticipate. If a subgraph G' of G can be obtained from G by deleting $\leq n$ vertices of G, let us call G a $(\leq n)$-*vertex extension* of G'.

We conjectured for some time that these three were sufficient. But that was false.

Ingredient 4: vortices

Take a graph drawn in the plane, and let the vertices on the infinite region be $v_1, v_2, \ldots, v_n$ in order. Add new edges joining v_1 to v_3, v_2 to v_4, v_3 to v_5 and so on, and let the resulting graph be G. Then it can be shown that such a graph has no large clique minor; indeed, Seese and Wessel [11] found the exact bound, that G can have a K_7, but cannot have a K_8, minor. Yet such graphs G cannot be constructed from our first three ingredients, and so a fourth ingredient is needed, so-called "vortices". Roughly, a vortex is a graph with some of its vertices arranged in a circular order, so that however these special vertices are partitioned into two intervals, there are only a bounded number of disjoint paths from one interval to the other. For instance, the graph G above consists of a planar graph together with a vortex (formed by the new edges) inserted into the infinite region.

Our main result implies that these four ingredients suffice. Let us turn to the statement of the theorem. We need first to define what we mean by an "r-ring with perimeter $t_1, \ldots, t_n$". Roughly speaking, it consists a graph G and a sequence $t_1, \ldots, t_n$ of distinct vertices of G, such that G can be constructed as follows. Let $G_1, \ldots, G_n$ be mutually disjoint graphs, each with $\leq r$ vertices, and for $1 \leq i \leq n$ let $t_i \in V(G_i)$. Let us choose some i with $1 \leq i < n$, choose $u \in V(G_i)$ and $v \in V(G_{i+1})$ and identify u with v, provided that $t_1, \ldots, t_n$ remain all distinct; and repeat this process as often as we wish.

More precisely, we say G is an *r-ring with perimeter* $t_1, \ldots, t_n$ if $t_1, \ldots t_n \in V(G)$ are distinct and there is a sequence $X_1, \ldots, X_n$ of subsets of $V(G)$, such that

- $X_1 \cup \cdots \cup X_n = V(G)$, and every edge of G has both ends in some X_i
- $t_i \in X_i$ for $1 \leq i \leq n$
- $X_i \cap X_k \subseteq X_j$ for $1 \leq i \leq j \leq k \leq n$

- $|X_i| \leq r$ for $1 \leq i \leq n$.

It is easy to check that this is equivalent with the previous definition.

Now let G_0 be a graph drawn in a surface Σ, and let $\Delta_1, \ldots, \Delta_d \subseteq \Sigma$ be pairwise disjoint closed discs, each meeting the drawing only in vertices of G_0, and each containing no vertices of G_0 in its interior. For $1 \leq i \leq d$ let the vertices of G_0 in $bd(\Delta_i)$ be $t_1, \ldots, t_n$ say, in order, and choose an r-ring G_i with perimeter $t_1, \ldots, t_n$, meeting G_0 just in $t_1, \ldots, t_n$ and disjoint from every other G_j; and let G be the union of $G_0, G_1, \ldots, G_d$. Such a graph G (and any graph isomorphic to it) is called an *outgrowth by d r-rings of a graph in Σ*. Now we can state our theorem.

1.3 *Let L be a nonplanar graph, and let $\Sigma_1, \ldots, \Sigma_s$ be all the connected surfaces (up to homeomorphism) in which L cannot be drawn. Then there are numbers r, d, w such that every graph with no L-minor may be constructed by clique-sums, starting from graphs G' with the following property: G' is a $(\leq w)$-vertex extension of an outgrowth by $\leq d$ r-rings of a graph that can be drawn in one of $\Sigma_1, \ldots, \Sigma_s$.*

One can think of an r-ring as a graph with a given path-decomposition of bounded width, and in particular, its pieces come in a linear order. Since these pieces are being sewn onto a circular hole in a surface, it would perhaps be more natural if the pieces came in a circular order. One can indeed replace the third condition in the definition of an r-ring by the condition

- $X_i \cap X_k \subseteq X_h \cup X_j$ and $X_h \cap X_j \subseteq X_i \cup X_k$ for $1 \leq h \leq i \leq j \leq k \leq n$

and 1.3 remains true (because the new statement is obviously weaker than the original); and in fact the new version is equivalent to the original (we leave the equivalence to the reader). This is why we call it a "ring" rather than some more linear name. It is a question of convenience which definition of "r-ring" is used.

While 1.3 has been one of the main goals of this series of papers, it turns out to have been a red herring. There is another result (theorem 3.1) which is proved in this paper, and from which 1.3 is then derived; and in all the future applications in this series of papers, it is not 1.3 but 3.1 that will be needed. Let us explain how theorem 3.1 is used to prove 1.3.

Evidently we would like to eliminate the "tree-structure" part of 1.3 and concentrate on the internal structure of one of the "nodes" of the tree. How can we do so? An inductive argument looks plausible at first sight; if there is no low order cutset of G dividing it into two substantial pieces then G itself must be almost a "node" if the theorem is to be true, while if there is such a cutset we may express G as a clique-sum of two smaller graphs, and hope to apply our inductive hypothesis to these graphs. But there is a difficulty here; it is possible that these smaller graphs have an L-minor while G does not. Fortunately there is a way to focus in on a "node" which does not involve any decomposing, as follows. We can assume that the tree is as refined as possible in the sense that no node can be split into two smaller nodes, and so for every low order cutset of G, most of any node will lie on one side or the other of the cutset (except for nodes of bounded cardinality, which we can ignore.) Therefore if we fix some node, every small cutset has a "big" side (containing most of the node) and a "small" side — and it turns out that no three small sides have union G. Thus a node defines a "tangle", which is such an assignment of big and small sides to the low order cutsets; and conversely, it can be shown that any tangle in G of sufficiently high "order" will be associated with some node of the tree-structure. Hence a convenient way to analyze the internal structure of the nodes is to analyze the local structure of G with respect to some high order tangle, and this is the content of theorem 3.1.

We have organized the paper backwards, to try to motivate the various steps better. Thus we first prove 1.3 assuming 3.1, then prove 3.1 assuming another statement 4.1, and so on.

2 Structure relative to a tangle

A *separation* of a graph G is a pair (A, B) of subgraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$, and its *order* is $|V(A \cap B)|$. A *tangle* of *order* θ in G, where $\theta \geq 1$ is an integer, is a set $\mathcal{T}$ of separations of G, all of order $< \theta$, such that

1. For every separation (A, B) of order $< \theta$, one of $(A, B), (B, A)$ belongs to $\mathcal{T}$
2. If $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ then $A_1 \cup A_2 \cup A_3 \neq G$, and
3. If $(A, B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

We define $ord(\mathcal{T}) = \theta$.

A *design* is a pair (H, M) where H is a graph and M is a set of subsets of $V(H)$. A *location* in G is a set $\{(A_1, B_1), \ldots, (A_k B_k)\}$ of separations of G such that $A_i \subseteq B_j$ for all distinct i, j with $1 \leq i, j \leq k$. If $\{(A_1, B_1), \ldots, (A_k, B_k)\}$ is a location then

$$(G \cap B_1 \cap \cdots \cap B_k, \{V(A_i \cap B_i) : 1 \leq i \leq k\})$$

is a design, which we call the *design of the location.*

Let $\theta \geq 1$ be an integer, and let $\mathcal{D}$ be a class of designs. We say that $\mathcal{D}$ is θ-*pervasive* in a graph G if for every subgraph G' of G and every tangle $\mathcal{T}$ in G' of order $\geq \theta$ there is a location $\mathcal{L} \subseteq \mathcal{T}$ with its design in $\mathcal{D}$.

A *tree-decomposition* of a graph G is a pair (T, τ), where T is a tree and for each $t \in V(T), \tau(t)$ is a subgraph of G, such that

- $\cup(\tau(t) : t \in V(T)) = G$, and $E(\tau(t) \cap \tau(t')) = \emptyset$ for all distinct $t, t' \in V(T)$
- if $t, t', t'' \in V(T)$ and t' lies on the path of T between t and t'' then $\tau(t) \cap \tau(t'') \subseteq \tau(t')$.

If (T, τ) is a tree-decomposition of G and $t_0 \in V(T)$, and t_0 has neighbours $t_1, \ldots, t_k \in V(T)$, then

$$(\tau(t_0), \{V(\tau(t_0) \cap \tau(t_i)) : 1 \leq i \leq k\})$$

is a design, called the *design of* t_0 *in* (T, τ).

Let $(H, M), (H', M')$ be designs. We say that (H', M') is an n-*enlargement* of (H, M) if there exists $Z \subseteq V(H')$ such that

- H is a subgraph of H' and $V(H') \backslash V(H) \subseteq Z$
- every edge of H' is an edge of H
- for every $X \in M'$ with $X \neq Z, X \cap V(H) \in M$

- n is an integer and $|Z| \leq n$.

If $\mathcal{D}$ is a class of designs and $n \geq 0$ is an integer, the class of all n-enlargements of members of $\mathcal{D}$ is denoted by $\mathcal{D}^n$. The class of all designs (H, M) with $|V(H)| \leq n$ is denoted by $\mathcal{R}_n$. The following is implied by theorem 11.1 of [6].

2.1 *For any $\theta \geq 1$, let $\mathcal{D}$ be a class of designs which is θ-pervasive in a graph G. Then G has a tree-decomposition (T, τ) such that for each $t \in V(T)$, the design of t in (T, τ) belongs to $\mathcal{D}^{3\theta-2} \cup \mathcal{R}_{4\theta-3}$.*

Let (H, M) be a design. A graph H' is a *torso* of (H, M) if $V(H) = V(H')$, H is a subgraph of H', and for every $e \in E(H')\backslash E(H)$ there exists $X \in M$ including the ends of e. It is easy to see the following.

2.2 *Let $\mathcal{D}$ be a class of designs, and let $\mathcal{D}'$ be the class of torsos of members of $\mathcal{D}$. If G is a graph with a tree-decomposition (T, τ) such that $\mathcal{D}$ contains the design of t in (T, τ) for each $t \in V(T)$, then G is a tree-structure over $\mathcal{D}'$.*

Now for $n \geq 0$, if $\mathcal{D}$ is a class of designs then every torso of a member of $\mathcal{D}^n$ is a $(\leq n)$-vertex extension of a torso of a member of $\mathcal{D}$. From 2.1 and 2.2 we deduce the following.

2.3 *For any $\theta \geq 1$, let $\mathcal{D}$ be a class of designs which is θ-pervasive in a graph G. Then G may be constructed by clique-sums starting from graphs G' such that either*

- *$|V(G')| \leq 4\theta - 3$, or*
- *G' is a $(\leq 3\theta - 2)$-vertex extension of a torso of a member of $\mathcal{D}$.*

We shall use 2.3 to derive 1.3 from a theorem that a certain class of designs is θ-pervasive.

3 Surfaces, societies and segregations

In this paper, by a *surface* we mean a non-null compact connected 2-manifold without boundary. An *O-arc* in a surface Σ is a subset $F \subseteq \Sigma$ homeomorphic to a circle. *Open* and *closed discs* in Σ are defined in the natural way. For $X \subseteq \Sigma$, its closure is denoted by $\overline{X}$, and $\overline{X} \cap \overline{\Sigma \backslash X}$ is denoted by $bd(X)$. If $F \subseteq \Sigma$ is an O-arc and $X \subseteq F$ is finite then F induces two cyclic permutations on X, called the *natural orders* of X from F.

A *society* is a pair (A, Ω), where A is a graph and Ω is a cyclic permutation of a subset (denoted by $\overline{\Omega}$) of $V(A)$. A *segregation* of G is a set $\mathcal{S}$ of societies such that

- $A \subseteq G$ for every $(A, \Omega) \in \mathcal{S}$, and $\cup(A : (A, \Omega) \in \mathcal{S}) = G$
- $V(A \cap A') \subseteq \overline{\Omega} \cap \overline{\Omega'}$ and $E(A \cap A') = \emptyset$ for all distinct $(A, \Omega), (A', \Omega') \in \mathcal{S}$.

We write $V(\mathcal{S}) = \cup(\overline{\Omega} : (A, \Omega) \in \mathcal{S})$.

Let Σ be a surface, and $\mathcal{S} = \{(A_1, \Omega_1), \ldots, (A_k, \Omega_k)\}$ a segregation of G. It is convenient always to assume (as we may) that $S \cap V(\mathcal{S}) = \emptyset$. An *arrangement* of $\mathcal{S}$ in Σ is a function α with domain $\mathcal{S} \cup V(\mathcal{S})$, such that (writing $\alpha(A, \Omega)$ for $\alpha((A, \Omega))$):

- For $1 \leq i \leq k, \alpha(A_i, \Omega_i)$ is a closed disc $\Delta_i \subseteq \Sigma$, and $\alpha(x) \in bd(\Delta_i)$ for each $x \in \overline{\Omega}_i$
- For $1 \leq i < j \leq k$, if $x \in \Delta_i \cap \Delta_j$ then $x = \alpha(v)$ for some $v \in \overline{\Omega}_i \cap \overline{\Omega}_j$
- For all distinct $x, y \in V(\mathcal{S}), \alpha(x) \neq \alpha(y)$
- For $1 \leq i \leq k, \Omega_i$ is mapped by α to a natural order of $\alpha(\overline{\Omega}_i)$ from $bd(\Delta_i)$.

An arrangement is *proper* if $\Delta_i \cap \Delta_j = \emptyset$ for all $1 \leq i < j \leq k$ such that $|\overline{\Omega_i}|, |\overline{\Omega_j}| > 3$. A society (G, Ω) is a *ρ-vortex*, where $\rho \geq 0$ is an integer, if for all distinct $u, v \in \overline{\Omega}$ there do not exist $\rho + 1$ mutually vertex-disjoint paths of G between $I \cup \{u\}$ and $J \cup \{v\}$, where I denotes the set of vertices in $\overline{\Omega}$ after u and before v and J is the set after v and before u, in the natural sense (so $(I \cup \{u\}, J \cup \{v\})$ is a partition of $\overline{\Omega}$). A segregation $\mathcal{S}$ is of *type* (ρ, κ), where $\rho, \kappa \geq 0$ are integers, if $|\overline{\Omega}| > 3$ for at most κ members (A, Ω) of $\mathcal{S}$, and each such member is a ρ-vortex.

Let L be a graph, and let $\mathcal{T}^*$ be a tangle in a graph G. We say that $\mathcal{T}^*$ *controls an L-minor of* G if there is a function α, with domain $V(L) \cup E(L)$, such that

- for each $v \in V(L), \alpha(v)$ is a non-null connected subgraph of G, and for all distinct $u, v \in V(L), \alpha(u)$ and $\alpha(v)$ are vertex-disjoint
- for each $e \in E(L)$ with distinct ends $u, v, \alpha(e) \in E(G)$ with one end in $V(\alpha(u))$ and the other in $V(\alpha(v))$
- for every loop $e \in E(L)$ with end $v, \alpha(e) \in E(G) \backslash E(\alpha(v))$ with both end in $V(\alpha(v))$
- for all distinct $e, f \in E(L), \alpha(e) \neq \alpha(f)$.
- there is no $(A, B) \in \mathcal{T}^*$ of order $< |V(L)|$ and $v \in V(L)$ such that $V(\alpha(v)) \subseteq V(A)$.

If $Z \subseteq V(G)$, we denote the graph obtained by deleting Z by $G \backslash Z$. If $\mathcal{T}$ is a tangle in G of order θ and $Z \subseteq V(G)$ with $|Z| < \theta$, we denote by $\mathcal{T} \backslash Z$ the set of all separations (A', B') of $G \backslash Z$ of order $< \theta - |Z|$ such that there exists $(A, B) \in \mathcal{T}$ with $Z \subseteq V(A \cap B), A \backslash Z = A'$ and $B \backslash Z = B'$. It is shown in theorem 8.5 of [6] that $\mathcal{T} \backslash Z$ is a tangle in $G \backslash Z$ of order $\theta - |Z|$.

If $\mathcal{T}$ is a tangle in G, a segregation $\mathcal{S}$ of G is said to be *$\mathcal{T}$-central* if for all $(A, \Omega) \in \mathcal{S}$ there is no $(A', B') \in \mathcal{T}$ with $B' \subseteq A$. Now we can state our main result.

3.1 *For any graph L, there are integers $\kappa, \rho, \zeta \geq 0$ and $\theta \geq 1$ with the following property. Let $\mathcal{T}$ be a tangle of order $\geq \theta$ in a graph G, controlling no L-minor of G. Then there exists $Z \subseteq V(G)$ with $|Z| \leq \zeta$, and a $\mathcal{T} \backslash Z$-central segregation of $G \backslash Z$ of type (ρ, κ) which has a proper arrangement in some surface in which L cannot be drawn.*

In the remainder of this section we shall show that 3.1 implies 1.3, by means of 2.3; and then the rest of the paper is devoted to proving 3.1. To deduce 1.3 from 3.1 we need the following lemma.

3.2 *Let (G, Ω) be a ρ-vortex, and let the vertices in $\overline{\Omega}$ be $t_1, \ldots, t_n$ in order, where $n \geq 1$. Then there are separations $(A_1, B_1), \ldots, (A_n, B_n)$ of G, such that:*

1. *$t_1, \ldots, t_n \in V(B_i)$ for $1 \leq i \leq n$,*
2. *$t_i \in V(A_i)$ for $1 \leq i \leq n$,*

3. $\mathcal{L} = \{(A_1, B_1), \ldots, (A_n, B_n)\}$ *is a location in* G*,*

4. (A_i, B_i) *has order* $\le 2\rho + 1$ *for* $1 \le i \le n$*, and*

5. every torso of the design of $\mathcal{L}$ *is a* $(2\rho+1)$*-ring with perimeter* $t_1, \ldots, t_n$*.*

Proof. By theorem 8.1 of [5], there is a sequence $A_1, \ldots, A_n$ of subgraphs of G, such that:

(a) $A_1 \cup \cdots \cup A_n = G$.

(b) $E(A_i \cap A_j) = \emptyset$ for $1 \le i < j \le n$.

(c) $t_i \in V(A_i)$ for $1 \le i \le n$

(d) $A_i \cap A_k \subseteq A_j$ for $1 \le i \le j \le k \le n$.

(e) $|V(A_i \cap A_j)| \le \rho$ for $1 \le i < j \le n$.

Let A_0 and A_{n+1} be the null graph. For $1 \le i \le n$, let B_i be the unique subgraph of G such that (A_i, B_i) is a separation of G and

$$V(A_i \cap B_i) = V(A_{i-1} \cap A_i) \cup V(A_i \cap A_{i+1}) \cup \{t_i\}.$$

(This exists, because it is easy to see that for $i \neq j$ and $1 \le i, j \le n$,

$$V(A_i \cap A_j) \subseteq V(A_{i-1} \cap A_i) \cup V(A_i \cap A_{i+1}).)$$

We claim that $(A_1, B_1), \ldots, (A_n, B_n)$ satisfies the theorem.

(1) $A_i \subseteq B_j$ *for* $1 \le i, j \le n$ *with* $i \neq j$*.*

Subproof. To show this it suffices to show that $A_i \cap A_j \subseteq B_j$, since (A_j, B_j) is a separation. But $E(A_i \cap A_j) = \emptyset$ by (b), and

$$V(A_i \cap A_j) \subseteq V(A_{i-1} \cap A_i) \cup V(A_i \cap A_{i+1}) \subseteq V(B_j),$$

and so $A_i \cap A_j \subseteq B_j$. This proves (1).

(2) $t_i \in V(B_j)$ *for* $1 \le i, j \le n$*.*

Subproof. If $i = j$ this is immediate. If $i \neq j$ then again the claim holds, since $t_i \in V(A_i) \subseteq V(B_j)$ by (1). This proves (2).

From (1) and (2) we see that statements 1-4 of 3.2 hold. It remains to show statement 5. For $1 \le i \le n$ let $X_i = V(A_i \cap B_i)$. Let H be the subgraph of G with vertex set $X_1 \cup \cdots \cup X_n$ and with no edges; then it is easy to see that $H = B_1 \cap \cdots \cap B_n$. Let $M = \{X_i : 1 \le i \le n\}$. Then $\mathcal{L}$ has design (H, M).

(3) *$X_i \cap X_k \subseteq X_j$ for $1 \le i \le j \le k \le n$.*

Subproof. Let $v \in X_i \cap X_k$. Then $v \in V(A_i \cap A_k)$, and hence $v \in V(A_j)$ by (d). But we may assume that $j \neq i$, and so $V(A_i) \subseteq V(B_j)$ by (1); and hence $v \in V(B_j)$, and consequently $v \in X_j$. This proves (3).

Since $|X_i| \le 2\rho + 1$ for $1 \le i \le n$, it follows from (3) that every torso of (H, M) is a $(2\rho+1)$-ring with perimeter $t_1, \dots, t_n$. This proves statement 5 of 3.2, and hence completes the proof of 3.2. ∎

Proof of 1.3, assuming 3.1

Let $\theta, \kappa, \rho, \zeta$ be as in 3.1. We may assume (by replacing θ by a larger number if necessary) that $\theta \ge 2\rho + 4 + \zeta$. Let $r = 2\rho + 1, d = \kappa$ and $w = \zeta + 4\theta - 3$. We claim that r, d, w satisfy 1.3. For let $\mathcal{D}$ be the class of all designs (G', M) such that every torso of (G', M) is isomorphic to an outgrowth by $\le d$ r-rings of a graph in one of $\Sigma_1, \dots, \Sigma_s$, where $\Sigma_1, \dots, \Sigma_s$ are the surfaces in which L cannot be drawn (up to homeomorphism).

(1) *For any graph G' with no L-minor, $\mathcal{D}^\zeta$ is θ-pervasive in G'.*

Subproof. Let G be a subgraph of G', and let $\mathcal{T}$ be a tangle in G of order $\ge \theta$. Certainly $\mathcal{T}$ controls no L-minor of G, and so, by 3.1, there exists $Z \subseteq V(G)$ with $|Z| \le \zeta$, and a segregation $\mathcal{S}$ of $G \backslash Z$ of type (ρ, κ), such that

- there is a proper arrangement α of $\mathcal{S}$ in one of $\Sigma_1, \dots, \Sigma_s$, and
- for each $(A, \Omega) \in \mathcal{S}$ there is no $(A', B') \in \mathcal{T}$ with $Z \subseteq V(A' \cap B')$ and $B' \backslash Z \subseteq A$.

Let $(A_1, \Omega_1), \dots, (A_k, \Omega_k)$ be the members (A, Ω) of $\mathcal{S}$ with $|\overline{\Omega}| > 3$. For $1 \le j \le k$, let the vertices of $\overline{\Omega_j}$ be $t^j_{n_1}, \dots, t^j_{n_j}$, in order. Since (A_j, Ω_j) is a ρ-vortex, by 3.2 there are separations $(A^j_1, B^j_1), \dots, (A^j_{n_j}, B^j_{n_j})$ of A_j as in 3.2. For each $(A, \Omega) \in \mathcal{S}$, let $C(A, \Omega)$ be the unique subgraph B of $G \backslash Z$ such that (A, B) is a separation of $G \backslash Z$ and $V(A \cap B) = \overline{\Omega}$. For $1 \le j \le k$ and $1 \le i \le n_j$, let $C^j_i = B \cup C(A_j, \Omega_j)$. Let

$$\mathcal{L} = \{(A^j_i, C^j_i) : 1 \le j \le k, 1 \le i \le n_j\} \cup \{(A, C(A, \Omega)) : (A, \Omega) \in \mathcal{S}, |\overline{\Omega}| \le 3\}.$$

Then since $\mathcal{S}$ is a segregation of G, and $(A^j_1, B^j_1), \dots, (A^j_{n_j}, B^j_{n_j})$ satisfy 3.2, it is straightforward to verify (we omit the details) that $\mathcal{L}$ is a location in $G \backslash Z$ and every torso of the design of $\mathcal{L}$ is an outgrowth by $\le d$ r-rings of a graph in one of $\Sigma_1, \dots, \Sigma_s$, that is, $\mathcal{D}$ contains the design of $\mathcal{L}$. (We use here that for $(A, \Omega) \in \mathcal{S}$ with $|\overline{\Omega}| \le 3$, the edges for the torso of $\mathcal{L}$ with ends in $\overline{\Omega}$ may all be drawn within $\alpha(A, \Omega)$.)

For each $(A, B) \in \mathcal{L}$, let (A^+, B^+) be the separation of G such that $Z \subseteq V(A^+ \cap B^+), A^+ \backslash Z = A$, $B^+ \backslash Z = B$ and $E(B^+)$ contains every edge of $E(G) \backslash E(A)$ with both ends in $V(B^+)$. Let $\mathcal{L}^+ = \{(A^+, B^+) : (A, B) \in \mathcal{L}\}$. Then $\mathcal{L}^+$ is a location in G, and its design belongs to $\mathcal{D}^\zeta$. To complete the proof of (1) we must show that $\mathcal{L}^+ \subseteq \mathcal{T}$. Let $(A, B) \in \mathcal{L}$, and suppose that $(A^+, B^+) \notin \mathcal{T}$. Now (A, B) has order $\le \max(3, 2\rho + 1)$ from the definition of $\mathcal{L}$, and so (A^+, B^+) has order at most $\max(3, 2\rho + 1) + \zeta < \theta$. Consequently, $(B^+, A^+) \in \mathcal{T}$. Let $(A', B') = (B^+, A^+)$. Then $Z \subseteq V(A' \cap B')$, and $B' \backslash Z \subseteq A$. But there exists $(A^*, \Omega^*) \in \mathcal{S}$ with $A \subseteq A^*$ from the construction of

$\mathcal{L}$, and so $B'\backslash Z \subseteq A^*$. This contradicts the second condition in the definition of $\mathcal{S}$. Hence $\mathcal{L}^+ \subseteq \mathcal{T}$. This completes the proof of (1).

From (1) and 2.3, we deduce that every graph G with no L-minor may be constructed by clique-sums, starting from graphs G' such that either

- $|V(G')| \leq 4\theta - 3 \leq w$, or
- G' is a $(\leq 3\theta - 2)$-vertex extension of a torso of a member of $\mathcal{D}^\zeta$.

But since L is nonplanar, it follows that there is a surface in which L cannot be drawn, and so the null graph is a torso of a member of $\mathcal{D}^\zeta$. Consequently any graph G' satisfying either of the conditions above is a $(\leq w)$-vertex extension of a torso of a member of $\mathcal{D}$, and hence is a $(\leq w)$-vertex extension of an outgrowth by $\leq d$ r-rings of a graph in one of $\Sigma_1, \ldots, \Sigma_s$. This proves 1.3. ∎

4 Induction on the surface

The remainder of the paper is devoted to proving 3.1. To get much further in this paper the reader will need to be familiar with [8, 9], and so there seems little point in repeating the large number of definitions that we shall need. The reader should therefore see [8] for the meaning of the following terms and notation: drawing, $U(H)$,region, $A(H)$, radial drawing, H-path, respectful tangle, metric of a tangle, free, λ-zone, clearing, rigid, dial, regional distance, battlefield. Also, the reader should see [9] for the terms Σ-span, λ-compression, rearranging within λ of $z, (\lambda, \mu)$-level. (As these terms turn up in the text, we shall remind the reader again where to look for the definition.)

If $\mathcal{T}^*$ is a tangle in a graph G, we shall usually abbreviate "Σ-span in G with respect to $\mathcal{T}^*$" by "Σ-span" when there is no danger of ambiguity. (For "Σ-span", see section 1 of [9].)

If H is a minor of G, and so $E(H) \subseteq E(G)$, and $\mathcal{T}'$ is a tangle in H of order ≥ 2, let $\mathcal{T}$ be the set of all separations (A, B) of G of order $< ord(\mathcal{T}')$ such that there exists $(A', B') \in \mathcal{T}'$ with $E(H) \cap E(A) = E(A')$. By theorem 6.1 of [6], $\mathcal{T}$ is a tangle in G, called the tangle in G *induced* by $\mathcal{T}'$.

We shall show that 3.1 is implied by the following.

4.1 *Let Σ be a surface and let $p \geq 0, \phi \geq 1$. Then there exist $\kappa, \rho, \zeta \geq 0$ and $\theta \geq 1$ such that if $\mathcal{T}^*$ is a tangle in a graph G, and there is a Σ-span of order $\geq \theta$, then either*

- *there is a Σ'-span of order $\geq \phi$, for some surface Σ' obtained from Σ by adding a handle or crosscap, or*
- *there exists $Z \subseteq V(G)$ with $|Z| \leq \zeta$, and a $\mathcal{T}^*\backslash Z$-central segregation of $G\backslash Z$ of type (ρ, κ) with a proper arrangement in Σ, or*
- *$\mathcal{T}^*$ controls a K_p minor of G.*

We remark that in the first outcome of 4.1 there are three possibilities for Σ' (up to homeomorphism) if Σ is orientable, because a handle can be added in two ways, preserving or destroying orientability. If Σ is non-orientable, there are only two possibilities.

To show that 4.1 implies 3.1, we need two lemmas. The $m \times n$ *grid* has vertex set

$$\{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\},$$

and $(i,j),(i',j')$ are adjacent if $|i-i'|+|j-j'| = 1$. For $1 \leq j \leq n$, the set of edges joining (i,j) and $(i+1,j)$ for $1 \leq i \leq m-1$ is called a *row* of the grid. Our first lemma is as follows.

4.2 *For any integer $\theta \geq 1$ there exists $\theta^* \geq 1$ such that if $\mathcal{T}^*$ is a tangle of order $\geq \theta^*$ in a graph G, there is a Σ-span of order $\geq \theta$, where Σ is the sphere.*

Proof. We may assume that θ is even, by increasing θ if necessary. Let N_0 be the $(2\theta+1) \times \theta$ grid, and let its rows be $P_1, \ldots, P_\theta$. Let N_1 be obtained from N_0 by deleting all edges with ends (i,j) and $(i, j+1)$, where $1 \leq i \leq 2\theta+1, 1 \leq j \leq \theta - 1$ and $i+j$ is odd. We see that there is a rigid drawing (see sections 2 and 4 of [8]) in a sphere, isomorphic to N_1.

Let $\mathcal{T}_1$ be the set of all separations (A,B) of N_1 of order $< \theta$ such that $E(B)$ includes one of $P_1, \ldots, P_\theta$.

(1) *$\mathcal{T}_1$ is a tangle in N_1 of order θ.*

Subproof. Let us delete the vertices $(2\theta+1, 1), \ldots, (2\theta+1, \theta)$ from N_1, and contract the edges joining (i,j) and $(i+1,j)$ for all i,j with $1 \leq i \leq 2\theta$ and $1 \leq j \leq \theta$ such that i is odd. We thereby obtain a graph N_2 isomorphic to the $\theta \times \theta$ grid. Let $\mathcal{T}_2$ be the set of all separations (A', B') of N_2 of order $< \theta$ such that $E(B')$ includes $P_j \cap E(N_2)$ for some j. By theorem 7.3 of [6], $\mathcal{T}_2$ is a tangle of order θ in N_2. But $\mathcal{T}_1$ is the tangle in N_1 induced by $\mathcal{T}_2$. This proves (1).

By theorem 6.1 of [10], there exists $\theta^* \geq 1$ such that

(2) *For every graph G and tangle $\mathcal{T}^*$ in G of order $\geq \theta^*$, there is a $(2\theta+1) \times (2\theta+1)$ grid minor of G such that for all $(A,B) \in \mathcal{T}^*$ of order $\leq 2\theta, E(A)$ includes no row of the grid.*

We claim that θ^* satisfies the theorem. For let $\mathcal{T}^*$ be a tangle of order $\geq \theta^*$ in a graph G. By (2), G has a $(2\theta+1) \times (2\theta+1)$ grid minor as in (2). Since this grid has a subgraph isomorphic to N_1 (and to simplify notation we may assume that it is N_1) it follows that N_1 is a minor of G, and for every $(A,B) \in \mathcal{T}^*$ of order $\leq 2\theta, E(A)$ includes none of $P_1, \ldots, P_\theta$. Let $\mathcal{T}_3$ be the tangle in G induced by $\mathcal{T}_1$. If $(A,B) \in \mathcal{T}_3$ then $E(B) \cap E(N_1)$ (and hence $E(B)$) includes one of $P_1, \ldots, P_\theta$, and so $(B,A) \notin \mathcal{T}^*$. Consequently $(A,B) \in \mathcal{T}^*$, and so $\mathcal{T}_3 \subseteq \mathcal{T}^*$.

Now N_1 has maximum degree ≤ 3, and so there is a subgraph N_4 of G isomorphic to a subdivision of N_1 (and hence N_1 is a minor of N_4). We may assume that N_4 is a rigid drawing in a sphere Σ. Let $\mathcal{T}_4$ be the tangle in N_4 induced by $\mathcal{T}_1$. Then $\mathcal{T}_3$ is the tangle in G induced by $\mathcal{T}_4$, and since $\mathcal{T}_3 \subseteq \mathcal{T}*$ it follows that $N_4, \mathcal{T}_4$ is a Σ-span of order θ. This proves 4.2. ∎

Our second lemma is the following, theorem 4.3 of [7].

4.3 *If L is a graph and Σ is a surface such that L can be drawn in Σ, there is an integer $\theta \geq 1$ with the following property. If $\mathcal{T}^*$ is a tangle in a graph G controlling no L minor, then there is no Σ-span of order $\geq \theta$.*

Proof of 3.1, assuming 4.1.

Up to homeomorphism, let $\Sigma_1, \dots, \Sigma_n$ be all the surfaces on which L cannot be drawn together with the sphere and all the surfaces which can be obtained from a surface on which L cannot be drawn by adding a handle or crosscap. (There are only finitely many up to homeomorphism.) Let us number $\Sigma_1, \dots, \Sigma_n$ so that for $1 \le i, j \le n$, if Σ_j can be obtained from Σ_i by adding a handle or crosscap then $j > i$.

Now we define $\kappa_i, \rho_i, \zeta_i$ for $1 \le i \le n$ as follows. Choose $p \ge 0$ so that K_p has a minor isomorphic to L. Inductively, suppose that $1 \le i \le n$, and $\kappa_j, \rho_j, \zeta_j, \theta_j$ are defined for all j with $i < j \le n$. If L can be drawn in Σ_i, choose θ_i so that 4.3 is satisfied (with Σ, θ replaced by Σ_i, θ_i), and let $\kappa_i, \rho_i, \zeta_i = 0$. If L cannot be drawn in Σ_i, then $i < n$; choose $\theta_i, \kappa_i, \rho_i, \zeta_i$ so that 4.1 is satisfied (with $\phi, \kappa, \rho, \zeta, \theta$ replaced by $\max(\theta_{i+1}, \dots, \theta_n), \kappa_i, \rho_i, \zeta_i, \theta_i$). This completes the inductive definition.

Choose $\theta^* \ge 1$ so that 4.2 is satisfied (with θ replaced by θ_1). Let $\kappa = \max(\kappa_1, \dots, \kappa_n), \rho = \max(\rho_1, \dots, \rho_n), \zeta = \max(\zeta_1, \dots, \zeta_n)$ and $\theta = \theta^*$. We claim that $\kappa, \rho, \zeta, \theta$ satisfy 3.1.

For let $\mathcal{T}^*$ be a tangle of order $\ge \theta = \theta^*$ in a graph G, controlling no L-minor. By 4.2, there is a Σ_1-span of order $\ge \theta_1$ (because Σ_1 is the sphere). Choose i with $1 \le i \le n$ maximum so that there is a Σ_i-span of order $\ge \theta_i$. If L can be drawn in Σ_i then θ_i satisfies 4.3 and yet $\mathcal{T}^*$ controls no L-minor, a contradiction. Thus, L cannot be drawn in Σ_i, and so one of the outcomes of 4.1 holds (with $\phi, \kappa, \rho, \zeta, \theta$ replaced by $\max(\theta_{i+1}, \dots, \theta_n), \kappa_i, \rho_i, \zeta_i, \theta_i$). Since L cannot be drawn in Σ_i it follows that every surface which can be obtained from Σ_i by adding a handle or crosscap occurs in the list $\Sigma_{i+1}, \dots, \Sigma_n$ (up to homeomorphism), and so the first outcome of 4.1 does not hold from the maximality of i. Moreover the third outcome of 4.1 does not hold, since $\mathcal{T}^*$ controls no L-minor and hence controls no K_p-minor, by theorem 4.2 of [7]. Thus the second outcome of 4.1 holds, and so 3.1 holds as required. ∎

5 Induction on horns

Thus, it remains to prove 4.1. Let Σ be a surface, let $\mathcal{T}^*$ be a tangle in a graph G, and let $H, \eta, \mathcal{T}$ be a Σ-span in G with respect to $\mathcal{T}^*$. Let $v \in V(G) \backslash V(\eta(H))$. A *horn* at v of *breadth* $\ge \theta$ (over $H, \eta, \mathcal{T}$) is a set $\{P_1, \dots, P_\sigma\}$ of paths of G, such that

- $P_1, \dots, P_\sigma$ each have one end v and are otherwise mutually disjoint,
- for $1 \le i \le \sigma$, P_i has precisely one vertex in $V(\eta(H))$, its end different from v; let this end be $\eta(u_i)$, where $u_i \in V(H)$,
- for $1 \le i < j \le \sigma, d(u_i, u_j) \ge \theta$ where d is the metric of $\mathcal{T}$ (see section 3 of [8]).

Let Σ be a surface. An *animal* (in G, with respect to $\mathcal{T}^*$) is a quintuple $(H, \eta, \mathcal{T}, X, Y)$, such that $H, \eta, \mathcal{T}$ is a Σ-span, $X \subseteq V(G) \setminus V(\eta(H))$ and $Y \subseteq V(H)$. If $|X| = \chi$ and $|Y| = \delta$, we call this an animal with χ *horns* and δ *hairs*. The animal has *strength* $\ge (\theta, \sigma)$, where $\theta \ge 1$ and $\sigma \ge 0$ are integers, if

- $\mathcal{T}$ has order $\ge \theta$,
- for each $v \in X$ there is a horn at v of cardinality $\ge \sigma$ and breadth $\ge \theta$,

- for each $u \in Y$ there is an $\eta(H)$-path (see section 3 of [8]) P with ends $\eta(u), \eta(v)$ say, such that $X \cap V(P) = \emptyset$ and $d(u,v) \geq \theta$, where d is the metric of $\mathcal{T}$,

- $d(u_1, u_2) \geq \theta$ for all distinct $u_1, u_2 \in Y$.

As usual, we shall omit reference to η if η is the identity, and instead speak of the animal $(H, \mathcal{T}, X, Y)$. The animal $(H, \eta, \mathcal{T}, X, Y)$ is *hairless* if $Y = \emptyset$, and in that case we shall speak of $(H, \eta, \mathcal{T}, X)$ or $(H, \mathcal{T}, X)$. In this section we shall only be concerned with hairless animals.

We shall show that the following implies 4.1.

5.1 *Let Σ be a surface and let $p, \tau, \chi \geq 0$ and $\phi, \psi \geq 1$. Then there exist $\sigma, \kappa, \rho, \zeta \geq 0$ and $\theta \geq 1$ such that if $\mathcal{T}^*$ is a tangle in a graph G, and there is a hairless animal with χ horns, of strength $\geq (\theta, \sigma)$, then either:*

1. *there is a Σ'-span of order $\geq \phi$, for some surface Σ' obtained from Σ by adding a handle or a crosscap, or*

2. *there is a hairless animal with $\chi + 1$ horns, of strength $\geq (\psi, \tau)$, or*

3. *there exists $Z \subseteq V(G)$ with $|Z| \leq \zeta$, and a $\mathcal{T}^* \backslash Z$-central segregation of $G \backslash Z$ of type (ρ, κ) with a proper arrangement in Σ, or*

4. *$\mathcal{T}^*$ controls a K_p-minor of G.*

To prove that 5.1 implies 4.1 we need several lemmas. The first is theorem 4.4 of [7].

5.2 *Let Σ be a surface and let $p \geq 0$. Then there exists $\theta \geq 1$ such that if $\mathcal{T}^*$ is a tangle in a graph G, and $H, \eta, \mathcal{T}$ is a Σ-span of order $\geq \theta$ with metric d, and*

- *$s_1, t_1, \ldots, s_\mu, t_\mu$ are distinct vertices of H, where $\mu = \frac{1}{2}p(p-1)$, such that $d(u,v) \geq \theta$ for all distinct $u, v \in \{s_1, t_1, \ldots, s_\mu, t_\mu\}$, and*

- *$Q_1, \ldots, Q_\mu$ are mutually disjoint $\eta(H)$-paths, such that Q_i has ends $\eta(s_i), \eta(t_i) (1 \leq i \leq \mu)$,*

then $\mathcal{T}^*$ controls a K_p-minor of G.

We also need

5.3 *Let G be a graph and let $Y \subseteq V(G)$, such that no component of G contains exactly one vertex in Y. If $\mu, \tau \geq 0$ are integers and $|Y| > \tau(\mu - 1)$, then either:*

- *There are μ disjoint paths of G each with both ends distinct and in Y and with no internal vertex in Y, or*

- *There exist $v \in V(G)$ and τ paths $P_1, \ldots, P_\tau$ of G, with ends $v, u_i (1 \leq i \leq \tau)$ respectively, mutually disjoint except for v, and such that for $1 \leq i \leq \tau$, $u_i \in Y$ and $u_i \neq v$, and no internal vertex of P_i is in Y.*

Proof. Let T be a minimal subgraph of G with the properties that $Y \subseteq V(T)$ and any two vertices in Y which belong to the same component of G also belong to the same component of Y. We deduce that T is a forest, every component of T contains at least two vertices in Y, and every vertex T with degree 1 belongs to Y.

Now let $\mathcal{P}$ be the set of all paths P of T with distinct ends both in Y, and with no internal vertex in Y. Since $\mathcal{P}$ is a set of subtrees of a forest, it follows by an elementary theorem that either there are μ members of $\mathcal{P}$, mutually vertex-disjoint, or there exists $X \subseteq V(T)$ with $|X| < \mu$ meeting all members of $\mathcal{P}$. In the first case we are finished, and so we assume that the second case holds. For each $y \in Y, X$ meets the component T' of T containing y, since $|V(T') \cap Y| \geq 2$ and X meets every member of $\mathcal{P}$. Hence there is a minimal path P_y of T from y to some member v_y of X. Since $|Y| > \tau(\mu - 1) \geq \tau|X|$, there exists $x \in X$ and $Y' \subseteq Y$ such that $|Y'| \geq \tau + 1$ and $v_y = x$ for all $y \in Y'$. For each $y \in Y$, no vertex of P_y belongs to X except v_y, and so for distinct $y, y' \in Y', P_y$ and $P_{y'}$ intersect in precisely x (since X meets every path between y and y'). Since there are at least τ members of Y' different from x, it follows that the second outcome of the theorem holds. This proves 5.3. ∎

5.4 *Let G be a graph, and let $v \in V(G)$. For $1 \leq i \leq n$ let P_i be a path of G with distinct ends v, u_i, such that $P_1, \dots, P_n$ are mutually disjoint except for v. Let $Q_1, \dots, Q_m$ be mutually disjoint paths of G, not passing through v, such that Q_i has ends s_i, t_i for $1 \leq i \leq m$. Let*

$$Z = \{u_1, \dots, u_n, s_1, t_1, \dots, s_m, t_m\}.$$

If $n \geq 5m + 2$, there are $m + 1$ mutually disjoint paths of G, each with distinct ends both in Z.

Proof. Let H be the graph formed by the union of $P_1, \dots, P_n$ and $Q_1, \dots, Q_m$; we see that no vertex of H has degree > 4 except possibly v. Let J be the union of all components of $H \backslash v$ which contain at least two vertices in Z, and let $Y = Z \cap V(J)$. Let us apply 5.3 to J, Y, setting $\mu = m + 1$ and $\tau = 5$. Certainly the second statement of 5.3 does not hold, because no vertex of J has degree ≥ 5. If the first statement holds then the theorem is satisfied, and so we assume not. By 5.3, it follows that $|Y| \leq 5m$. Now certainly each Q_j is a path of J because it has at least two vertices in Z, and so $P_i \backslash v \subseteq J$ for each i such that P_i meets some Q_j. Consequently,

$$|Z \backslash Y| = |\{i : 1 \leq i \leq n, P_i \text{ meets no } Q_j\}|.$$

But $|Z \backslash Y| = |Z| - |Y| \geq n - 5m \geq 2$, and so there exist distinct i, i' with $1 \leq i, i' \leq n$ such that $P_i \cup P_{i'}$ meets no Q_j. But then $\{Q_1, \dots, Q_m, P_i \cup P_{i'}\}$ is a set of $m + 1$ paths satisfying the theorem. This proves 5.4. ∎

We deduce

5.5 *For any surface Σ and integer $p \geq 0$, there exists $\theta \geq 1$ such that, if $\mathcal{T}^*$ is a tangle in a graph G and there is a hairless animal with $\frac{1}{2}p(p-1)$ horns and with strength $\geq (\theta, 4p(p-1))$, then $\mathcal{T}^*$ controls a K_p-minor of G.*

Proof. Let $\mu = \frac{1}{2}p(p-1)$, and choose $\theta_1 \geq 1$ so that 5.2 is satisfied (with θ replaced by θ_1). Let $\theta = 2\theta_1$: we claim that 5.5 is satisfied. For let $\mathcal{T}^*$ be a tangle in G, and let $(H, \mathcal{T}X)$ be a hairless animal of strength $\geq (\theta, 8\mu)$ and with $|X| = \mu$.

(1) *For $0 \leq m \leq \mu$ there are m H-paths $Q_1, \dots, Q_m$, mutually vertex-disjoint, such that*

- $|V(Q_1 \cup \cdots \cup Q_m) \cap X| \le m$, *and*
- $d(u,v) \ge \theta_1$ *for all distinct* $u, v \in \{s_1, t_1, \ldots, s_m, t_m\}$, *where* Q_i *has ends* $s_i, t_i (1 \le i \le m)$.

Subproof. We proceed by induction on m. Certainly (1) holds if $m = 0$; we suppose that it holds for some $m < \mu$, and shall prove that it also holds for $m+1$. Let $Q_1, \ldots, Q_m, s_1, t_1, \ldots, s_m, t_m$ be as in (1).

Since $|(V(Q_1) \cup \cdots \cup V(Q_m)) \cap X| \le m$, and $|X| = \mu > m$, there exists $x \in X$ with $x \notin V(Q_1), \ldots, V(Q_m)$. Let $\{P_1, \ldots, P_{8\mu}\}$ be a horn at x with breadth $\ge \theta$, and let P_i have ends $x, u_i (1 \le i \le 8\mu)$. Since $|X| = \mu$, at most $\mu - 1$ of $P_1, \ldots, P_{8\mu}$ have a vertex in X different from x, and so we may assume that $P_1, \ldots, P_{7\mu}$ have no vertex in $X \backslash \{x\}$. Let $W = \{s_1, t_1, \ldots, s_m, t_m\}$. Now for each $w \in W$ there is at most one $i (1 \le i \le 7\mu)$ such that $d(u_i, w) < \frac{1}{2}\theta = \theta_1$, since $d(u_i, u_j) \ge \theta$ for all distinct i, j. Consequently, we may assume that for $1 \le i \le 5\mu, d(u_i, w) \ge \frac{1}{2}\theta$ for all $w \in W$. Let $n = 5\mu$ and let $G' = P_1 \cup \cdots \cup P_n \cup Q_1 \cup \cdots \cup Q_m$. Now $n = 5\mu \ge 5m + 2$, and so by 5.4 (with G replaced by G') there are $m+1$ mutually disjoint paths $R_1, \ldots, R_{m+1}$ of G', each with distinct ends both in $\{u_1, \ldots, u_n\} \cup W$, and we may assume that they have no internal vertex in this set. Each of these paths R_i is therefore an H-path since $V(G' \cap H) = \{u_1, \ldots, u_n\} \cup W$. But $d(u,v) \ge \theta_1$ for all distinct $u, v \in \{u_1, \ldots, u_n\} \cup W$. Moreover, $|V(G') \cap X| \le m + 1$, and so

$$|(V(R_1) \cup \cdots \cup V(R_{m+1})) \cap X| \le m + 1.$$

This completes the inductive argument, and hence proves (1).

Consequently, (1) holds with $m = \mu$. Let $Q_1, \ldots, Q_\mu$ be the corresponding paths, and let Q_i have ends $s_i, t_i (1 \le i \le \mu)$. Since 5.2 is satisfied (replacing θ by θ_1) and $d(u,v) \ge \theta_1$ for all distinct $u, v \in \{s_1, t_1, \ldots, s_\mu, t_\mu\}$, it follows from 5.2 that $\mathcal{T}^*$ controls a K_p-minor. This proves 5.5. ∎

Proof of 4.1, assuming 5.1

Let Σ be a surface and let $p \ge 0, \phi \ge 1$. Let $\mu = \frac{1}{2}p(p-1)$, and choose θ_μ so that 5.5 holds (with θ replaced by θ_μ). Let $\sigma_\mu = 8\mu$ and $\kappa_\mu = \rho_\mu = \zeta_\mu = 0$. We define $\sigma_i, \theta_i, \kappa_i, \rho_i, \zeta_i$ for $0 \le i \le \mu$ inductively as follows. Suppose that $0 \le i < \mu$ and $\sigma_{i+1}, \theta_{i+1}$ have already been defined. Choose $\sigma_i, \theta_i, \kappa_i, \rho_i, \zeta_i$ so that 5.1 is satisfied (with $\tau, \chi, \psi, \sigma, \kappa, \rho, \zeta, \theta$ replaced by $\sigma_{i+1}, i, \theta_{i+1}, \sigma_i, \kappa_i, \rho_i, \zeta_i, \theta_i$). This completes the inductive definition.

Let $\kappa = \max(\kappa_0, \ldots, \kappa_\mu), \rho = \max(\rho_0, \ldots, \rho_\mu), \zeta = \max(\zeta_0, \ldots, \zeta_\mu)$, and let $\theta = \theta_0$. We claim that $\kappa, \rho, \zeta, \theta$ satisfy 4.1. For let $\mathcal{T}*$ be a tangle in G with a Σ-span of order $\ge \theta = \theta_0$, and hence with a hairless animal with 0 horns, of strength $\ge (\theta_0, \sigma_0)$. Choose i with $0 \le i \le \mu$ maximum so that there is a hairless animal with i horns, of strength $\ge (\theta_i, \sigma_i)$. If $i = \mu$ then since θ_μ satisfies 5.5, it follows that the third outcome of 4.1 holds. We assume then that $i < \mu$. From the maximality of i, there is no hairless animal with $i+1$ horns, of strength $\ge (\theta_{i+1}, \sigma_{i+1})$. By 5.1, we deduce that one of outcomes 1,3,4 of 5.1 holds, and so 4.1 holds, as required. ∎

6 Induction on hairs

Next, we show that 5.1 is implied by the following.

6.1 *Let Σ be a surface, and let $\rho,\tau,\chi,\delta \geq 0$ and $\phi,\psi \geq 1$. Then there exist $\sigma,\kappa,\rho,\zeta \geq 0$ and $\theta \geq 1$ such that if $\mathcal{T}^*$ is a tangle in a graph G, and there is an animal with χ horns and δ hairs, of strength $\geq (\theta,\sigma)$, then either*

1. *there is a Σ'-span of order $\geq \phi$, for some surface Σ' obtained from Σ by adding a handle or a crosscap, or*
2. *there is an animal with χ horns and $\delta+1$ hairs, of strength $\geq (\psi,\tau)$, or*
3. *there exists $Z \subseteq V(G)$ with $|Z| \leq \zeta$, and a $\mathcal{T}^*\backslash Z$-central segregation of $G\backslash Z$ of type (ρ,κ) with a proper arrangement in Σ, or*
4. *$\mathcal{T}^*$ controls a K_p-minor of G.*

To show that 6.1 implies 5.1 we need the following lemmas. (For the meaning of "λ-zone", see section 3 of [8], and for "λ-compression" see section 1 of [9].)

6.2 *Let Σ be a surface and let $\sigma \geq 0$ be an integer. Let $\mathcal{T}^*$ be a tangle in a graph G, and let $H,\mathcal{T}$ be a Σ-span of order θ. Let $\lambda \geq 2$ be an integer with $\theta \geq 4\lambda+3$, let $\Lambda \subseteq \Sigma$ be a λ-zone such that $H \cap (\Sigma\backslash\Lambda)$ is rigid, and let $\mathcal{T}'$ be the tangle in $H' = H \cap (\Sigma\backslash\Lambda)$ of order $\theta-4\lambda-2$ which is a $(4\lambda+2)$-compression of $\mathcal{T}$. Let $v \in V(G)\backslash V(H)$, such that there is a horn at v over $H,\mathcal{T}$ with cardinality $\sigma+1$ and breadth $\geq \theta$. Then there is a horn at v over $H',\mathcal{T}'$ with cardinality σ and breadth $\geq \theta-4\lambda-2$.*

Proof. First, we remark that $\mathcal{T}'$ exists, by theorem 7.10 of [7]. Let $\{P_1,\ldots,P_{\sigma+1}\}$ be a horn at v over $H,\mathcal{T}$, of breadth $\geq \theta$. For $1 \leq i \leq \sigma+1$ let P_i have ends v,u_i. Now if $u_i,u_j \in \Lambda$ where $i \neq j$, then $d(u_i,u_j) \leq 2\lambda$ since Λ is a λ-zone, and yet $d(u_i,u_j) \geq \theta > 2\lambda$ by hypothesis, a contradiction. Thus, Λ contains at most one of $u_1,\ldots,u_{\sigma+1}$, and hence we may assume that $u_1,\ldots,u_\sigma \notin \Lambda$. Since $\mathcal{T}'$ is the $(4\lambda+2)$-compression of $\mathcal{T}$, it follows that for $1 \leq i < j \leq \sigma$,

$$d'(u_i,u_j) \geq d(u_i,u_j) - 4\lambda - 2 \geq \theta - 4\lambda - 2.$$

Thus $\{P_1,\ldots,P_\sigma\}$ is a horn at v over $H',\mathcal{T}'$ of breadth $\geq \theta-4\lambda-2$. This proves 6.2. ∎

6.3 *Let Σ be a surface, and let $p,\tau,\chi \geq 0$ and $\psi \geq 1$ be integers. Then there exists $\delta \geq 0$ and $\theta \geq 1$ such that if $\mathcal{T}^*$ is a tangle in a graph G, and there is an animal with χ horns and δ hairs of strength $\geq (\theta,\tau+1)$, then either*

- *there is a hairless animal with $\chi+1$ horns of strength $\geq (\psi,\tau)$, or*
- *$\mathcal{T}^*$ controls a K_p-minor of G.*

Proof. We may assume that $p \geq 2$, for otherwise the result holds trivially. Let $\mu = \frac{1}{2}p(p-1)$, and choose θ_1 so that 5.2 holds (with θ replaced by θ_1). We may assume, by increasing θ_1, that $\theta_1 \geq \max(\psi,3)$. Let $\theta = 5\theta_1 + 34$, $\tau' = \max(\tau,2)$, and $\delta = 2\mu^2\tau'^2$.

We claim that δ,θ satisfy the theorem. For let $\mathcal{T}^*$ be a tangle in a graph G, and let $(H,\mathcal{T},X,Y)$ be an animal of strength $\geq (\theta,\tau+1)$, with $|X| = \chi$ and $|Y| = \delta$. For each $y \in Y$ let P_y be an H-path with ends $y,v(y)$, such that $X \cap V(P_y) = \emptyset$ and $d(y,v(y)) \geq \theta$ where d is the metric of $\mathcal{T}$. For each

$y \in Y$ let $Q_y = P_y \backslash \{v(y)\}$, and let $G_0 = \cup(Q_y : y \in Y)$. Let G_1 be the union of all components of G_0 which contain at least two vertices in Y, and let $Y_1 = Y \cap V(G_1)$. We may assume that $\mathcal{T}^*$ controls no K_p-minor of G. Since $d(y, y') \geq \theta \geq \theta_1$ for all distinct $y, y' \in Y_1$, it follows from 5.2 that:

(1) *There do not exist μ disjoint paths of G_1, each with distinct ends, both in Y_1, and with no internal vertex in Y_1.*

We may assume that:

(2) *There do not exist $v \in V(G_1)$ and τ' paths of G_1 each from v to $Y_1 \backslash \{v\}$, mutually disjoint except for v and with no internal vertex in Y_1.*

Subproof. Otherwise, $v \notin V(H)$ since v has degree $\geq \tau' \geq 2$ in G_1 and every vertex of G_1 in $V(H)$ has degree 1 in G_1. Since $d(y, y') \geq \theta \geq \psi$ for all distinct y, y' it follows that there is a horn at v over $H, \mathcal{T}$ with cardinality τ and breadth $\geq \psi$. But $v \notin X$ since $v \in V(G_1)$ and $X \cap V(G_1) = \emptyset$ (since $X \cap V(P_y) = \emptyset$ for each $y \in Y$) and so $(H, \mathcal{T}, X \cup \{x\})$ is an animal of strength $\geq (\psi, \tau)$ with $\chi + 1$ horns. Thus we may assume (2).

From (1), (2) and 5.3 applied to G_1, Y_1, we deduce that $|Y_1| \leq (\mu - 1)\tau'$, and so

$$|Y \backslash Y_1| \geq \delta - (\mu - 1)\tau' \geq (2\mu - 1)\mu\tau'.$$

We see that Q_{y_1} is disjoint from Q_{y_2} for all distinct $y_1, y_2 \in Y \backslash Y_1$.

Choose $Y_2 \subseteq Y \backslash Y_1$ maximal such that the set $W = \{y, v(y) : y \in Y_2\}$ has cardinality $2|Y_2|$ and $d(u, v) \geq \theta_1$ for any two distinct $u, v \in W$. By 5.2, we may assume that $|Y_2| \leq \mu - 1$. From the maximality of Y_2 we deduce that for any $y \in Y_2$ there exists $w \in W$ such that either $d(y, w) < \theta_1$ or $d(v(y), w) < \theta_1$. Consequently there exist $w \in W$ and a set $Y_3 \subseteq Y \backslash Y_1$ with

$$|Y_3| \geq |Y \backslash Y_1|/(2|Y_2|) > \mu\tau'$$

such that for all $y \in Y_3$, either $d(y, w) < \theta_1$ or $d(v(y), w) < \theta_1$. Now $d(y, w) < \theta_1$ for at most one value of $y \in Y$ since $2\theta_1 < \theta$ and $d(y, y') \geq \theta$ for all distinct $y, y' \in Y$. Consequently there exists $Y_4 \subseteq Y_3$ with $|Y_4| \geq \mu\tau'$, such that $d(v(y), w) < \theta_1$ for all $y \in Y_4$.

By theorem 9.2 of [8], there is a $(\theta_1 + 2)$-zone Λ_1 around $\{w\}$ such that $x \subseteq \Lambda_1$ for every $x \in A(H)$ ($A(H)$ is the set of atoms of H — see section 2 of [8]) with $d(\{w\}, x) < \theta_1$, since $2 \leq \theta_1 \leq \theta - 3$. Consequently $v(y) \in \Lambda_1$ for all $y \in Y_4$. By theorem 9.3 of [8] there is a $(\theta_1 + 8)$-zone Λ around $\{w\}$ such that $H \cap (\Sigma \backslash \Lambda)$ is rigid and $x \subseteq \Lambda$ for every $x \in A(H)$ with $d(z, x) < \theta_1 + 3$, since $2 \leq \theta_1 + 3 \leq \theta - 6$. Consequently, $\overline{\Lambda}_1 \subseteq \Lambda$, and so every two vertices in Y_4 are joined by a path P of H with $U(P) \subseteq \Lambda$. (For $U(P)$ see section 2 of [8].) Let H_2 be a connected subgraph of H with $Y_4 \subseteq V(H_2)$ and $U(H_2) \subseteq \Lambda$.

Let $H' = H \cap (\Sigma \backslash \Lambda)$, and let $\mathcal{T}'$ be the $(4(\theta_1 + 8) + 2)$-compression of $\mathcal{T}$ in H' (this exists, by theorem 7.10 of [7]). Then $H', \mathcal{T}'$ is a Σ-span of order $\theta - (4\theta_1 + 34) = \theta - 1$.

(3) $Y_4 \subseteq V(H')$.

Subproof. Let $y \in Y_4$. Then $d(w, v(y)) < \theta_1$, but $d(y, v(y)) = \theta_1$ and so $d(w, y) > \theta - \theta_1$. Since $\theta - \theta_1 \geq \theta_1 + 8$ and Λ is a $(\theta_1 + 8)$-zone, it follows that $y \notin \Lambda$, and so $y \in V(H')$. This proves (3).

Let d' be the metric of $\mathcal{T}'$. Since $\mathcal{T}'$ is a $(4\theta_1 + 34)$-compression of $\mathcal{T}$, and $d(y_1, y_2) = \theta$, it follows that $d'(y_1, y_2) = \theta_1$ for all distinct $y_1, y_2 \in Y_4$. Let G_2 be the union of H_2 and all $P_y (y \in Y_4)$. From 5.2, since $H', \mathcal{T}'$ is a Σ-span of order θ_1, it follows that there do not exist μ mutually disjoint paths of G_2 each with distinct ends both in Y_4 and with no internal vertex in Y_4.

Now G_2 is connected and $Y_4 \subseteq V(G_2)$ and $|Y_4| \geq \mu\tau' \geq 2$. From 5.3 (applied to G_2, Y_4), we deduce there exists $v \in V(G_2)$ and τ' paths of G_2 from v to $Y_4 \backslash \{v\}$, mutually disjoint except for v, and each with no internal vertex in Y_4. Since each $y \in Y_4$ has degree 1 in G_2, and $\tau' > 1$, it follows that $v \notin Y_4$ and so $v \in V(G) \backslash V(H')$, since $V(G_2 \cap H') = Y_4$. Consequently there is a horn at v over $H', \mathcal{T}'$, with breadth $\geq \theta_1$ and cardinality τ. But $v \notin X$ since $X \cap V(G_2) = \emptyset$; and from 6.2, for each $x \in X$ there is a horn at x over $H', \mathcal{T}'$ with breadth $\geq \theta_1$ and cardinality τ, since $\theta \geq 4\theta_1 + 35$. Hence $(H', \mathcal{T}', X \cup \{v\})$ is a hairless animal with $\chi + 1$ horns, of strength $\geq (\theta_1, \tau) \geq (\psi, \tau)$, and so the first outcome of 6.3 holds. This proves 6.1. ▮

Proof of 5.1, assuming 6.1

Let Σ be a surface and let $p, \tau, \chi \geq 0$ and $\phi, \psi \geq 1$ be integers. Choose δ and θ_δ so that 6.3 is satisfied (with θ replaced by θ_δ). Let $\kappa_\delta = \rho_\delta = \zeta_\delta = 0$ and $\sigma_\delta = \tau + 1$. For $0 \leq i \leq \delta$ we define $\theta_i, \sigma_i, \kappa_i, \rho_i, \zeta_i$ inductively, as follows. Suppose that $0 \leq i < \delta$ and $\sigma_{i+1}, \theta_{i+1}$ have already been defined. Choose $\theta_i, \sigma_i, \kappa_i, \rho_i, \zeta_i$ so that 6.1 is satisfied (with $\tau, \delta, \psi, \sigma, \kappa, \rho, \zeta, \theta$ replaced by $\sigma_{i+1}, i, \theta_{i+1}, \sigma_i, \kappa_i, \rho_i, \zeta_i, \theta_i$). This completes the inductive definition.

Let $\kappa = \max(\kappa_0, \ldots, \kappa_\delta), \rho = \max(\rho_0, \ldots, \rho_\delta), \zeta = \max(\zeta_0, \ldots, \zeta_\delta), \sigma = \sigma_0$ and $\theta = \theta_0$. We claim that $\sigma, \kappa, \rho, \zeta, \theta$ satisfy 5.1. For let $\mathcal{T}^*$ be a tangle in a graph G, with a hairless animal with χ horns of strength $\geq (\theta, \sigma) = (\theta_0, \sigma_0)$. Choose i with $\theta \leq i \leq \delta$ maximum such that there is an animal with χ horns and i hairs of strength $\geq (\theta_i, \sigma_i)$. If $i = \delta$ then by 6.3, either

- there is a hairless animal with $\chi + 1$ horns, of strength $\geq (\psi, \tau)$, or
- $\mathcal{T}^*$ controls a K_p-minor of G

and in either case 5.1 holds. We assume then that $i < \delta$. From the maximality of i, there is no animal with χ horns and $i + 1$ hairs and strength $\geq (\theta_{i+1}, \sigma_{i+1})$. By 6.1, either

- there is a Σ'-span of order $\geq \phi$, for some surface Σ' obtained from Σ by adding a handle or a crosscap, or
- there exists $Z \subseteq V(G)$ with $|Z| \leq \zeta_i \leq \zeta$, and a $\mathcal{T}^* \backslash Z$-central segregation of $G \backslash Z$ of type (ρ_i, κ_i) (and hence of type (ρ, κ)) with a proper arrangement in Σ, or
- $\mathcal{T}^*$ controls a K_p-minor of G.

In each case 5.1 holds, as required. ▮

7 The Giant Steps application

The previous paper of this series contains a result designed to be used at this point. We apply it to prove that 6.1 is implied by the following. (For "(λ, μ)-level", see section 8 of [9].)

7.1 *Let Σ be a surface, and let $\tau, \chi, \delta, \lambda \geq 0, \theta' \geq 1$ and $\phi, \psi \geq 3$. Then there exist $\theta \geq 1$ and $\sigma \geq 0$ such that if $\mathcal{T}^*$ is a tangle in a graph G, and there is an animal with χ horns and δ hairs, of strength $\geq (\theta, \sigma)$, then either*

- *there is a Σ'-span of order $\geq \phi$, for some surface Σ' obtained from Σ by adding a handle, or*
- *there is an animal with χ horns and $\delta + 1$ hairs, of strength $\geq (\psi, \tau)$, or*
- *there exists $Z \subseteq V(G)$ with $|Z| \leq \chi + \frac{1}{2}\delta^2\phi^2$ such that some Σ-span of order $\geq \theta'$ in $G \backslash Z$ with respect to $\mathcal{T}^* \backslash Z$ is $(\lambda, 2\psi)$-level.*

To encourage the reader, let us point out that we are making progress. 7.1 no longer involves segregations of type (ρ, κ) nor does it involve controlling minors, which were the two fundamental ingredients of 3.1. Also, crosscaps have gone from the first outcome.

To show that 7.1 implies 6.1 we use the following, which is almost theorem 8.4 of [9].

7.2 *Let Σ be a surface and let $p, \phi, \mu \geq 0$. Then there exist $\kappa, \lambda, \rho \geq 0$ and $\theta \geq 1$ such that if $\mathcal{T}^*$ is a tangle in a graph G, and some Σ-span of order $\geq \theta$ is (λ, μ)-level, then either:*

1. *there is a Σ'-span of order ϕ, for some surface Σ' obtained from Σ by adding a crosscap, or*
2. *there is a $\mathcal{T}^*$-central segregation of G of type (ρ, κ) with a proper arrangement in Σ, or*
3. *$\mathcal{T}^*$ controls a K_p minor of G.*

However, let us point out a discrepancy. We were not farsighted enough in [9], and omitted to include the term "proper" in the statement of the theorem, although the *proof* in that paper does yield a proper arrangement. (All the arrangements in [9] first come into being via the proof of theorem 7.7 of that paper, so it is enough to check that they are proper at that stage; and they are, as we can see from theorem 7.6 of [9], or from statement (i) of 7.5 in [9].) Alternately, we could pay for the mistake by starting with a non-proper arrangement, and showing how to convert it to a proper one; that is straightforward, but not very short, and it seems unnecessary to inflict it on the reader.

We also need two other lemmas.

7.3 *Let $\mathcal{T}^*$ be a tangle in a graph G, let $Z \subseteq V(G)$ with $|Z| < ord(\mathcal{T}^*)$, and let (A, B) be a separation of G of order $< ord(\mathcal{T}^*) - |Z|$. Then $(A, B) \in \mathcal{T}^*$ if and only if*

$$(A \backslash (Z \cap V(A)), B \backslash (Z \cap V(B))) \in \mathcal{T}^* \backslash Z.$$

Proof. Choose a separation (A^*, B^*) of G such that $Z \subseteq V(A^*) \cap V(B^*)$, and $A^* \backslash Z = A$, $B^* \backslash Z = B$. Then (A^*, B^*) has order at most $|Z|$ more than that of (A, B), and hence less than $ord(\mathcal{T}^*)$. By theorem 2.9 of [6], $(A, B) \in \mathcal{T}^*$ if and only if $(A^*, B^*) \in \mathcal{T}^*$. But $(A^*, B^*) \in \mathcal{T}^*$ if and only if $(A^* \backslash Z, B^* \backslash Z) \in \mathcal{T}^* \backslash Z$. This proves 7.3. ∎

7.4 *Let $\mathcal{T}^*$ be a tangle in G, and let $Z \subseteq V(G)$ with $|Z| < ord(\mathcal{T}^*)$. If $H, \eta, \mathcal{T}$ is a Σ-span in $G \backslash Z$ with respect to $\mathcal{T}^* \backslash Z$ then $H, \eta, \mathcal{T}$ is a Σ-span in G with respect to $\mathcal{T}^*$.*

Proof. For simplicity we assume that η is the identity. Let $(A, B) \in \mathcal{T}^*$ have order $< ord(\mathcal{T})$; it suffices to show that $(A \cap H, B \cap H) \in \mathcal{T}$. Now

$$|V(A \cap B)| < ord(\mathcal{T}) \le ord(\mathcal{T}^* \backslash Z) = ord(\mathcal{T}^*) - |Z|$$

and so by 7.3, $(A', B') \in \mathcal{T}^* \backslash Z$, where $A' = A \backslash (Z \cap V(A)), B' = B \backslash (Z \cap V(B))$. Since $H, \eta, \mathcal{T}$ is a Σ-span in $G \backslash Z$ with respect to $\mathcal{T}^* \backslash Z$ and (A', B') has order $< ord(\mathcal{T})$, it follows that $(A' \cap H, B' \cap H) \in \mathcal{T}$. But $A \cap H = A' \cap H$ since $Z \cap V(H) = \emptyset$, and similarly $B \cap H = B' \cap H$. Hence $(A \cap H, B \cap H) \in \mathcal{T}$. This proves 7.4. ∎

Proof of 6.1, assuming 7.1.

Let Σ be a surface, and let $p, \tau\chi, \delta \ge 0$ and $\phi, \psi \ge 1$. By increasing ϕ, ψ we may assume that $\phi, \psi \ge 3$. Choose $\kappa, \lambda, \rho \ge 0$ and $\theta' \ge 1$ so that 7.2 is satisfied (with θ, μ replaced by $\theta', 2\psi$). Choose θ, σ as in 7.1 and let $\zeta = \lfloor \frac{1}{2}\delta^2\phi^2 \rfloor + \chi$. We claim that 6.1 holds. For let $\mathcal{T}^*$ be a tangle in G, with an animal with χ horns and δ hairs, of strength $\ge (\theta, \sigma)$. Let us apply 7.1. If 7.1.1 or 7.1.2 holds then 6.1.1 or 6.1.2 holds, and so we may assume that 7.1.3 holds, that is,

(1) *There exists $Z \subseteq V(G)$ with $|Z| \le \zeta$ such that some Σ-span of order $\ge \theta'$ in $G \backslash Z$ with respect to $\mathcal{T}^* \backslash Z$ is $(\lambda, 2\psi)$-level.*

Let us apply 7.2 (with $\theta, \mathcal{T}^*, G, \mu$ replaced by $\theta', \mathcal{T}^* \backslash Z, G \backslash Z, 2\psi$). By (1), the hypotheses of 7.2 are satisfied. We deduce that one of the outcomes of 7.2 holds. But if 7.2.1 holds, that is, there is a Σ'-span of order $\ge \phi$ in $G \backslash Z$ with respect to $\mathcal{T}^* \backslash Z$, then 6.1.1 holds, by 7.4. If 7.2.2 holds then 6.1.3 holds, while if 7.2.3 holds then 6.1.4 holds, because it is easy to see that $\mathcal{T}^*$ controls every minor controlled by $\mathcal{T}^* \backslash Z$. The result follows. ∎

8 Non-level Σ-spans

Let $\mathcal{T}$ be a respectful tangle (see section 3 of [8]) of order θ in a rigid drawing H in a surface Σ, and let $Y \subseteq V(H)$. If $\gamma \ge 0$, a *γ-envelope around* Y is a family $(\Lambda_y : y \in Y)$ such that:

- for each $y \in Y, \Lambda_y$ is a γ-zone around y
- for all distinct $y, y' \in Y, \overline{\Lambda}_y \cap \overline{\Lambda}_{y'} = \emptyset$
- for every subset $Y' \subseteq Y$, the drawing $H \cap (\Sigma \backslash \cup (\Lambda_y : y \in Y'))$ is rigid.

(Actually, the third statement here is implied by the same statement for all singleton subsets Y', but it is convenient to present it this way.)

In this section we show that 7.1 is implied by the following. (8.1 will be proved in the next section.)

8.1 *Let Σ be a surface and let $\delta \ge 0$ and $\phi, \psi \ge 3$. Then there exists $\gamma \ge 0$ and $\theta > (4\gamma + 2)\delta$ such that if $\mathcal{T}^*$ is a tangle in a graph G and $H, \mathcal{T}$ is a Σ-span of order $\ge \theta$, and $Y \subseteq V(H)$ satisfies $|Y| = \delta$ and $d(y, y') \ge \theta$ for all distinct $y, y' \in Y$ (where d is the metric of $\mathcal{T}$), then either:*

1. *there is a Σ'-span of order $\ge \phi$ for some surface Σ' obtained from Σ by adding a handle, or*

2. *there is an H-path in G with ends s,t, such that $d(s,t) \geq \psi$ and $d(s,y) \geq \psi$ for all $y \in Y$, or*

3. *there is a γ-envelope $(\Lambda_y : y \in Y)$ around Y, and there exists $Z \subseteq V(G) \backslash V(H')$ with $|Z| \leq \frac{1}{2}\delta^2\phi^2$, such that Z meets every H'-path in G with ends s,t, satisfying $d'(s,t) \geq 2\psi$, where $H' = H \cap (\Sigma \backslash \cup (\Lambda_y : y \in Y))$ and d' is the metric of the $(4\gamma+2)\delta$-compression of $\mathcal{T}$ in H'.*

To show that 8.1 implies 7.1, we need two lemmas. The first is a kind of converse of 7.4. If $\mathcal{T}$ is a tangle in G and $1 \leq \theta \leq ord(\mathcal{T})$, the set of members of $\mathcal{T}$ of order $< \theta$ is called the *truncation of $\mathcal{T}$ to order θ.*

8.2 *Let $\mathcal{T}^*$ be a tangle in a graph G, and let $H, \mathcal{T}$ be a Σ-span. Let $Z \subseteq V(G) \backslash V(H)$ with $|Z| < ord(\mathcal{T})$, let $1 \leq \theta' \leq ord(\mathcal{T}) - |Z|$, and let $\mathcal{T}'$ be the truncation of $\mathcal{T}$ to order θ'. Then $H, \mathcal{T}'$ is a Σ-span in $G \backslash Z$ with respect to $\mathcal{T}^* \backslash Z$.*

Proof. Since H is a subgraph of $G \backslash Z$ and

$$ord(\mathcal{T}') = \theta' \leq ord(\mathcal{T}) - |Z| \leq ord(\mathcal{T}^*) - |Z| = ord(\mathcal{T}^* \backslash Z),$$

it suffices to show that $(A \cap H, B \cap H) \in \mathcal{T}'$ for all $(A,B) \in \mathcal{T}^* \backslash Z$ of order $< \theta'$. Thus, let $(A,B) \in \mathcal{T}^* \backslash Z$ have order $< \theta'$. By definition of $\mathcal{T}^* \backslash Z$, there exists $(A^*, B^*) \in \mathcal{T}^*$ with $Z \subseteq V(A^* \cap B^*)$, such that $A^* \backslash Z = A$ and $B^* \backslash Z = B$. Then (A^*, B^*) has order

$$|Z| + |V(A \cap B)| < |Z| + \theta' = ord(\mathcal{T})$$

and so $(A^* \cap H, B^* \cap H) \in \mathcal{T}$, since $H, \mathcal{T}$ is a Σ-span with respect to $\mathcal{T}$. Now $(A^* \cap H, B^* \cap H) = (A \cap H, B \cap H)$ since $Z \cap V(H) = \emptyset$, and so $(A \cap H, B \cap H) \in \mathcal{T}$; and consequently $(A \cap H, B \cap H) \in \mathcal{T}'$, since it has order $\leq |V(A \cap B)| < ord(\mathcal{T}')$. This proves 8.2. ∎

Our second lemma is the following.

8.3 *Let Σ be a surface, and let $\tau, \chi, \delta, \lambda, \zeta \geq 0$ and $\psi \geq 3$. Then there exists $\theta > \zeta$ such that if $\mathcal{T}^*$ is a tangle in a graph G, and $(H, \mathcal{T}, X, Y)$ is an animal with χ horns and δ hairs, of strength $\geq (\theta, \zeta + \tau)$, and $Z \subseteq V(G) \backslash V(H)$ with $X \subseteq Z$ and $|Z| \leq \zeta$, then either*

- *there is an animal with χ horns and $\delta + 1$ hairs of strength $\geq (\psi, \tau)$, or*
- *there is an H-path in $G \backslash Z$ with ends s_1, s_2 and there are distinct $y_1, y_2 \in Y$ with $d(s_1, y_1)$, $d(s_2, y_2) < \psi$, where d is the metric of $\mathcal{T}$, or*
- *$H, \mathcal{T}_0$ is a $(\lambda, 2\psi)$-level Σ-span of order $\theta - |Z|$ in $G \backslash Z$ with respect to $\mathcal{T}^* \backslash Z$, where $\mathcal{T}_0$ is the truncation of $\mathcal{T}$ to order $\theta - |Z|$.*

Proof. Let $\theta = 4\psi + 10\lambda + 3\zeta + 11$. We claim that θ satisfies 8.3. For let $\mathcal{T}^*, G, H, \mathcal{T}, X, Y, Z$ be as in the hypothesis of 8.3, and let d be the metric of $\mathcal{T}$. We may assume that

(1) *$ord(\mathcal{T}) = \theta$.*

Subproof. Let $\mathcal{T}'$ be the truncation of $\mathcal{T}$ to order θ. Then $(H, \mathcal{T}', X, Y)$ is another animal satisfying the same hypotheses, and since $\theta \geq \psi$, it is easy to see that if the theorem is true for

$(H, \mathcal{T}', X, Y)$ then it is true for $(H, \mathcal{T}, X, Y)$. Thus it suffices to prove the theorem for $(H, \mathcal{T}', X, Y)$ and (1) follows.

Also, we may assume that

(2) *If there is an H-path in $G \backslash Z$ with ends s_1, s_2 then $d(s_1, s_2) < 2\psi$.*

Subproof. Suppose that $d(s_1, s_2) \geq 2\psi$. If $d(s_1, y) \geq \psi$ for all $y \in Y$, then $(H, \mathcal{T}, X, Y \cup \{s_1\})$ is an animal with χ horns and $\delta + 1$ hairs of strength $\geq (\psi, \tau)$, since $X \subseteq Z$, and so the first outcome of 8.3 holds. We may assume then that $d(s_1, y_1) < \psi$ for some $y_1 \in Y$, and similarly $d(s_2, y_2) < \psi$ for some $y_2 \in Y$. But $y_1 \neq y_2$ since

$$2\psi \leq d(s_1, s_2) \leq d(s_1, y_1) + d(y_1, y_2) + d(y_2, s_2) < 2\psi + d(y_1, y_2)$$

and so the second outcome of 8.3 holds. Thus we may assume (2).

Let $\mathcal{T}_0$ be the truncation of $\mathcal{T}$ to order $\theta - |Z|$. By 8.2, $H, \mathcal{T}_0$ is a Σ-span in $G \backslash Z$ with respect to $\mathcal{T}^* \backslash Z$, and we may assume that it is not $(\lambda, 2\psi)$-level, for otherwise the third outcome of 8.3 holds. But by (2), there is no H-path in $G \backslash Z$ with ends s_1, s_2 such that $d_0(s_1, s_2) \geq 2\psi$, for $d_0(s_1, s_2) \leq d(s_1, s_2)$ (where d_0 is the metric of $\mathcal{T}_0$). Yet $\theta - |Z| \geq 4\lambda + 2\psi + 2$, and so from the definition of $(\lambda, 2\psi)$-level we deduce the following. (For "rearranging", see section 1 of [9].)

(3) *There is a Σ-span $H', \eta', \mathcal{T}''$ in $G \backslash Z$ with respect to $\mathcal{T}^* \backslash Z$, of order $\theta - |Z| - 4\lambda - 2$, obtained from $H, \mathcal{T}_0$ by rearranging within λ of some $z \in A(H)$, and there is an $\eta'(H')$-path in $G \backslash Z$ with ends $\eta'(s), \eta'(t)$, such that $d(z, s), d(z, t) \leq \lambda$ and $d''(s, t) \geq 2\psi$, where d'' is the metric of $\mathcal{T}''$.*

Let λ' be an integer with

$$\lambda + \frac{1}{4}|Z| \leq \lambda' < \lambda + \frac{1}{4}|Z| + 1.$$

Now $\mathcal{T}''$ has order $\theta - |Z| - 4\lambda - 2$; let $\mathcal{T}'$ be the truncation of $\mathcal{T}''$ to order $\theta - 4\lambda' - 2$.

(4) *$H', \eta', \mathcal{T}'$ is a Σ-span in G with respect to $\mathcal{T}^*$, obtained from $H, \mathcal{T}$ by rearranging within λ' of z, and $d'(s, t) \geq 2\psi$ where d' is the metric of $\mathcal{T}'$.*

Subproof. Now $H', \eta', \mathcal{T}'$ is a Σ-span in $G \backslash Z$ with respect to $\mathcal{T}^* \backslash Z$,and hence in G with respect to $\mathcal{T}^*$ by 7.4. Let $a \in A(H)$ with $d(a, z) > \lambda'$. Then

$$d_0(a, z) = \min(d(a, z), \theta - |Z|) > \lambda' \geq \lambda$$

and so $z \in A(H')$ (because $H', \eta', \mathcal{T}''$ is obtained from $H, \mathcal{T}_0$ by rearranging within λ of z). Similarly if x is a vertex or edge of H with $d(x, z) > \lambda'$, then $d_0(x, z) > \lambda$ and so $\eta'(x) = x$. Since $\mathcal{T}'$ is the $(4\lambda' + 2)$-compression of $\mathcal{T}$ in H', the first claim follows. For the second, we observe that

$$d'(s, t) = \min(d''(s, t), \theta - 4\lambda' - 2) \geq 2\psi.$$

This proves (4).

(5) *For each vertex* $w \in V(\eta'(H')))\backslash V(H)$*, there is a path* Q *of* $\eta'(H')$ *from* w *to a vertex* $x \in V(H)$*, such that no vertex of* Q *belongs to* H *except* x*, and* $d(z,x) \leq \lambda' + 2$.

Subproof. Let Q be a minimal path of $\eta'(H')$ from w to $V(H)$, and let its ends be w, x. Then $x \in V(H)$, and no vertex of Q belongs to H except x, by the minimality of Q. Suppose that $d(z,x) \geq \lambda' + 3$. Then $x \in V(H')$ and $\eta'(x) = x$ by (4). Since $x \neq w$, there is an edge e of Q incident with x; let $e = \eta'(f)$ where $f \in E(H')$. Since $\eta'(f)$ is incident with $\eta'(x) = x$, it follows that f is incident with x. Let $a \in A(H)$, with $x \in \overline{a}$ and $a \cap f \neq \emptyset$. Since $x \in \overline{a}$, it follows that $d(a,x) \leq 2$, and so $d(z,a) \geq \lambda' + 1$, since $d(z,x) \geq \lambda' + 3$. By (4), we deduce that $a \in A(H')$ and so $a = f$, since $a \cap f \neq \emptyset$ and $f \in E(H')$. Consequently $f \in E(H)$ and $d(z,f) \geq \lambda' + 1$. By (4) again, $e = \eta'(f) = f$, and so $e \in E(H)$, contrary to the minimality of Q. This proves (5).

Let $v \in X$, and let $\{P_1, \ldots, P_{\zeta+\tau}\}$ be a horn at v over $H, \mathcal{T}$ of breadth $\geq \theta$, and let P_i have ends $v, u_i (1 \leq i \leq \zeta + \tau)$.

(6) $P_1, \ldots, P_{\zeta+\tau}$ *may be numbered so that* $d(z,u_i) \geq \lambda' + 2 + 2\psi$ *and* $V(P_i) \cap Z = \{v\}$ *for* $1 \leq i \leq \tau$.

Subproof. Since $d(u_i, u_j) \geq \theta$ for $1 \leq i < j \leq \zeta + \tau$, and since $\theta \geq 2(\lambda' + 2 + 2\psi)$, it follows that $d(z, u_i) < \lambda' + 2 + 2\psi$ for at most one value of i. Moreover, since $|Z| \leq \zeta$, there are at most $\zeta - 1$ values of i such that $Z \cap V(P_i) \neq \{v\}$. This proves (6).

(7) *For* $1 \leq i \leq \tau$*, no internal vertex of* P_i *belongs to* $V(\eta'(H'))$.

Subproof. Suppose that w is an internal vertex of P_i, where $1 \leq i \leq \tau$ and $w \in V(H')$. Let R be the subpath of P_i between w and u_i, and let Q be a path in $\eta'(H')$ between w and some $x \in V(H)$ with $d(z,x) \leq \lambda' + 2$, such that no vertex of Q is in H except x (this exists by (5).) Then by (6),

$$\lambda' + 2 + 2\psi \leq d(z,u_i) \leq d(z,x) + d(x,u_i) \leq \lambda' + 2 + d(x,u_i)$$

and so $d(x,u_i) \geq 2\psi$. In particular, $x \neq u_i$, and so $Q \cup R$ is an H-path. But $Z \cap V(Q \cup R) = \emptyset$ since $V(Q) \subseteq V(\eta'(H')) \subseteq V(G \backslash Z)$ and $V(R) \subseteq V(P_i) \backslash \{v\} \subseteq V(G) \backslash Z$ by (6). This contradicts (2), and hence proves (7).

(8) *For* $1 \leq i \leq \tau, u_i \in V(H')$ *and* $u_i = \eta'(u_i) \in V(\eta'(H'))$*; and for* $1 \leq i < j \leq \tau, d'(u_i, u_j) \geq \psi$.

Subproof. By (6), $d(z, u_i) > \lambda'$ and so by (4), $u_i \in V(H')$ and $u_i = \eta'(u_i) \in V(\eta'(H'))$. For the second claim, let $1 \leq i < j \leq \tau$. Then $d(u_i, u_j) \geq \theta$ since $\{P_1, \ldots, P_{\zeta+\tau}\}$ has breadth $\geq \theta$, and so $d'(u_i, u_j) \geq \theta - 4\lambda' - 2$ since $\mathcal{T}'$ is a $(4\lambda' + 2)$-compression of $\mathcal{T}$. Since $\theta - 4\lambda' - 2 \geq \psi$, this proves (8).

(9) *For each* $v \in X$ *there is a horn at* v *over* $H', \eta', \mathcal{T}'$ *of cardinality* τ *and breadth* $\geq \psi$.

Subproof. This follows from (7) and (8), because $v \notin V(\eta'(H')$ since $v \in X \subseteq Z$ and $V(\eta'(H')) \subseteq V(G \backslash Z)$. This proves (9).

Let $Y_1 = \{y \in Y : d(z,y) \geq \psi + 5\lambda + 2\}$.

(10) $|Y_1| \geq \delta - 1$, *and* $Y_1 \subseteq V(H')$, *and* $\eta'(y) = y$ *for all* $y \in Y_1$.

Subproof. Suppose that $y, y' \in Y \backslash Y_1$ are distinct. Then $d(z,y), d(z,y') \leq \psi + 5\lambda' + 1$, and so $d(y,y') \leq 2(\psi + 5\lambda' + 1) < \theta$, a contradiction. Hence $|Y'| \geq |Y| - 1 = \delta - 1$. Now if $y \in Y_1$, then $d(z,y) \geq \psi + 5\lambda' + 2 \geq \lambda'$, and so $y \in V(H')$ and $\eta'(y) = y$ by (4). This proves (10).

Let $Y' = Y \cup \{s,t\}$ (we recall that s and t are defined in (3)). Then $Y' \subseteq V(H')$, by (10).

(11) $d'(y_1, y_2) \geq \psi$ *for all distinct* $y_1, y_2 \in Y'$.

Subproof. If $\{y_1 y_2\} = \{s,t\}$ this is true by (3). If $y_1, y_2 \in Y_1$, then

$$d'(y_1,y_2) \geq d(y_1,y_2) - 4\lambda' - 2 \geq \theta - 4\lambda' - 2 \geq \psi$$

since $\mathcal{T}'$ is a $(4\lambda' + 2)$-compression of $\mathcal{T}$. Finally, if $y_1 \in Y_1$ and $y_2 \in \{s,t\}$, then

$$d'(y_1,y_2) \geq d'(y_1,z) - d'(y_2,z) \geq d(y_1,z) - 4\lambda' - 2 - d(y_2,z) \geq (\psi + 5\lambda' + 2) - 4\lambda' - 2 - \lambda \geq \psi$$

since $d(y_2, z) \leq \lambda$ by (3). This proves (11).

(12) *For all* $y \in Y'$ *there is an* $\eta'(H')$*-path in* $G \backslash X$ *with ends* $\eta'(y), \eta'(x)$ *say, where* $d'(x,y) \geq \psi$.

Subproof. If $y = s$ or t this is true by (3), since $d'(s,t) \geq 2\psi \geq \psi$ by (4). We assume then that $y \in Y_1$. Since $Y_1 \subseteq Y$ and $(H, \mathcal{T}, X, Y)$ is an animal over $H, \mathcal{T}$ of strength $\geq (\theta, \zeta + \tau)$, it follows that there is an H-path Q in $G \backslash X$ with ends y, x say, where $d(x,y) \geq \theta$. Suppose first that some vertex of Q different from y belongs to $\eta'(H')$. Let w be the first such vertex, and let Q' be the subpath of Q between y and w. Let $w = \eta'(y')$ where $y' \in V(H')$. We claim that $d'(y,y') \geq \psi$. For let $a \in A(H)$ with $y' \in a$. If $d(a,z) > \lambda'$ then by (4), $a \in A(H')$ and so $a = \{y'\}$, and $y' \in V(H)$; and by (4) again, $w = \eta'(y') = y'$. Thus $w \in V(H)$, and so $w = x$, since Q is an H-path. But $d(x,y) \geq \theta$, and so $d'(x,y) \geq \theta - 4\lambda' - 2$, since $\mathcal{T}'$ is a $(4\lambda + 2)$-compression of $\mathcal{T}$; that is, $d'(y',y) \geq \theta - 4\lambda - 2 \geq \psi$, as claimed.

We may therefore assume that $d(a,z) \leq \lambda'$, and so

$$d(a,y) \geq d(z,y) - d(a,z) \geq (\psi + 5\lambda' + 2) - \lambda' = \psi + 4\lambda' + 2.$$

Since $y' \in a$ and $\mathcal{T}'$ is a $(4\lambda' + 2)$-compression of $\mathcal{T}$, it follows that $d(y,y') \geq d(a,y) - 4\lambda' - 2 \geq \psi$, and so again our claim is true. Thus we have shown that $d'(y,y') \geq \psi$, and consequently $y \neq y'$ and y, y' are not adjacent in H' (since $\psi \geq 3$). We deduce that Q' is an $\eta'(H')$-path with ends y, y' (for no vertex of Q' belongs to $\eta'(H')$ except y, w, and $E(Q') \cap E(\eta'(H') = \emptyset$ even if $|E(Q')| = 1$, since y, y' are not adjacent in H'). Hence (12) holds in this case.

We may therefore assume that y is the only vertex of Q in $\eta'(H')$. Since $x \in V(H)$, there is a minimal path R of H between x and $V(\eta'(H'))$; let its ends be x, w where $w = \eta'(x')$ and $x' \in V(H')$. Suppose that $d'(y,x') < \psi$. Let $a \in A(H)$ with $x' \in a$; then $d(y,a) < \psi + 4\lambda' + 2$ since $\mathcal{T}'$ is a $(4\lambda'+2)$-compression of $\mathcal{T}$. Since $d(z,y) \geq \psi + 5\lambda' + 2$ (because $y \in Y_1$) it follows that $d(z,a) > \lambda'$. By (4), $a \in A(H')$ and so $a = \{x'\}$, and $x' \in V(H)$, and $d(z,x') > \lambda'$; and by (4) again $w = \eta'(x') = x'$. Let e be the edge of R incident with w. Then $e \in E(H)$, and $d(z,e) \geq d(z,x') > \lambda'$. By (4),

$e \in E(H')$ and $\eta'(e) = e$, and so both ends of e belong to $V(\eta'(H'))$, contrary to the minimality of R. We deduce that $d'(y, x') \geq \psi$. Hence $y \neq x'$ and so $Q \cup R$ is an $\eta'(H')$-path, satisfying (12). This proves (12).

From (9), (10), (11) and (12) we see that $(H', \eta', \mathcal{T}', X, Y')$ is an animal with χ horns and $\geq \delta + 1$ hairs, and so the first outcome of 8.3 holds. This proves 8.3. ∎

Proof of 7.1, assuming 8.1

Let Σ be a surface, and let $\tau, \chi, \delta, \lambda \geq 0, \theta' \geq 1$, and $\phi, \psi \geq 3$. Choose $\gamma \geq 0$ and $\theta_1 > (4\gamma + 2)\delta$ so that 8.1 holds (with θ replaced by θ_1). Let $\zeta = \lfloor \frac{1}{2}\delta^2\phi^2 \rfloor + \chi$, and choose $\theta_2 > \zeta$ so that 8.3 holds (with θ replaced by θ_2). We may assume that $\theta_2 \geq \max(\theta' + \zeta, 4\psi + 2)$. Let $\sigma = \delta + \tau + \zeta$ and $\theta = \chi + \theta_1 + 3\gamma + (4\gamma + 2)\delta + \theta_2$.

We claim that σ, θ satisfy 7.1. For let $\mathcal{T}^*$ be a tangle in G, and let $(H, \mathcal{T}, X, Y)$ be an animal with χ horns and δ hairs, of strength $\geq (\theta, \sigma)$. Let $\mathcal{T}'$ be the truncation of $\mathcal{T}$ to order $\theta - \chi$. By 8.2, $H, \mathcal{T}'$ is a Σ-span in $G \backslash X$ with respect to $\mathcal{T}^* \backslash X$. Let d' be the metric of $\mathcal{T}'$. From 7.4 we deduce

(1) *If there is a Σ'-span of order $\geq \phi$ in $G \backslash X$, for some surface Σ' obtained from Σ by adding a handle, then 7.1.1 holds.*

Next, we observe

(2) *If there is an H-path in $G \backslash X$ with ends s, t such that $d'(s, t) \geq \psi$ and $d'(s, y) \geq \psi$ for all $y \in Y$, then 7.1.2 holds.*

Subproof. Then $d(s, t) \geq d'(s, t)$, and $d(s, y) \geq d'(s, y)$ for all $y \in Y$, and so $(H, \mathcal{T}, X, Y \cup \{s\})$ is an animal (in G, with respect to $\mathcal{T}^*$) with χ horns and $\delta + 1$ hairs, of strength $\geq (\psi, \sigma) \geq (\psi, \tau)$. This proves (2).

Now $\mathcal{T}'$ has order $\theta - \chi \geq \theta_1$, and for all distinct $y_1, y_2 \in Y$,

$$d'(y_1, y_2) = \min(d(y_1, y_2), \theta - \chi) \geq \min(\theta, \theta - \chi) = \theta - \chi \geq \theta_1$$

and so 8.1 may be applied (with $\mathcal{T}^*, G, \mathcal{T}, \theta$ replaced by $\mathcal{T}^* \backslash X, G \backslash X, \mathcal{T}', \theta_1$). From (1) and (2) we may therefore assume that 8.1.3 holds, that is,

(3) *There is a γ-envelope $(\Lambda_y : y \in Y)$ around Y, and there exists $Z \subseteq V(G \backslash X) \backslash V(H'')$ with $|Z| \leq \frac{1}{2}\delta^2\phi^2$, such that Z meets every H-path in $G \backslash X$ with ends s, t satisfying $d''(s, t) \geq 2\psi$, where $H'' = H \cap (\Sigma \backslash (\Lambda_y : y \in Y))$, and d'' is the metric of the $(4\gamma + 2)\delta$-compression $\mathcal{T}''$ of $\mathcal{T}'$ in H''.*

Now $(H, \mathcal{T}', X, Y)$ is an animal of strength $\geq (\theta - \chi, \sigma)$, and $H'', \mathcal{T}''$ is obtained from $H, \mathcal{T}'$ by repeating δ times the operation of clearing a γ-zone (see section 3 of [8]). From δ applications of 6.2 we deduce

(4) *For each $v \in X$ there is a horn at v over $H'', \mathcal{T}''$ of cardinality $\sigma - \delta = \tau + \zeta$ and breadth $\geq \theta - \chi - (4\gamma + 2)\delta \geq \theta_2$.*

For each $y \in Y$, let P_y be a minimal path of H between y and $V(H \cap bd(\Lambda_y))$, with ends $y, p(y)$ say.

(5) *$d''(p(y_1), p(y_2)) \geq \theta_2$ for all distinct $y_1, y_2 \in Y$.*

Subproof. Now

$$\theta - \chi \leq d'(y_1, y_2) \leq d'(y_1, p(y_1)) + d'(p(y_1), p(y_2)) + d'(p(y_2), y_2) \leq 2\gamma + d'(p(y_1), p(y_2))$$

since for $i = 1, 2, \Lambda_{y_i}$ is a γ-zone around y_i and $p(y_i) \in \overline{\Lambda}_{y_i}$. But

$$d'(p(y_1), p(y_2)) \leq d''(p(y_1), p(y_2)) + (4\gamma + 2)\delta$$

since $\mathcal{T}''$ is a $(4\gamma + 2)\delta$-compression of $\mathcal{T}'$. Consequently,

$$d''(p(y_1), p(y_2)) \geq \theta - \chi - 2\gamma - (4\gamma + 2)\delta \geq \theta_2.$$

This proves (5).

(6) *For each $y \in Y$ there is a H''-path in $G\backslash X$ with ends $p(y), q(y)$ say, such that $d''(p(y), q(y)) \geq \theta_2$.*

Subproof. Let Q_y be an H-path in $G\backslash X$ with ends $y, r(y)$ say, such that $d'(y, r(y)) \geq \theta - \chi$. Let R_y be a minimal path of H between $r(y)$ and $V(H')$, with ends $r(y), q(y)$ say. Then for some $y' \in Y, \overline{\Lambda}_{y'}$ contains both $r(y)$ and $q(y)$, and so

$$d'(y', r(y)), d'(y', q(y)) \leq \gamma.$$

Hence $d'(q(y), r(y)) \leq 2\gamma$. But $d'(y, p(y)) \leq \gamma$ since $p(y) \in \overline{\Lambda}_y$, and so

$$\theta - \chi \leq d'(y, r(y)) \leq d'(y, p(y)) + d'(p(y), q(y)) + d'(q(y), r(y)) \leq 3\gamma + d'(p(y), q(y)).$$

Thus $d'(p(y), q(y)) \geq \theta - \chi - 3\gamma \geq \theta_2$, and so $P_y \cup Q_y \cup R_y$ is an H''-path satisfying (6). This proves (6).

Let $Y'' = \{p(y) : y \in Y\}$. From (4), (5) and (6) it follows that $(H'', \mathcal{T}'', X, Y'')$ is an animal in G with respect to $\mathcal{T}^*$ with χ horns and δ hairs, of strength $\geq (\theta_2, \zeta + \tau)$. By 8.3 (with $\theta, H, \mathcal{T}, Y, Z$ replaced by $\theta_2, H'', \mathcal{T}'', Y'', Z \cup X$) we deduce that one of the outcomes of 8.3 holds.

If the first outcome of 8.3 holds, then 7.1.3 is true. Suppose that the second outcome of 8.3 holds, that is, there is an H''-path P in $G\backslash(X \cup Z)$ with ends s_1, s_2 and there are distinct $y_1, y_2 \in Y''$ with $d''(s_1, y_1), d''(s_2, y_2) < \psi$. Then P is an H-path in $G\backslash X$, not meeting Z, and so $d''(s_1, s_2) < 2\psi$ by (3). Hence

$$d''(y_1, y_2) \leq d''(y_1, s_1) + d''(s_1, s_2) + d'(s_2, y_2) < 4\psi \leq \theta_2$$

contrary to (5). Thus this case does not occur. We may therefore assume that the third outcome of 8.3 holds, that is, there is a $(\lambda, 2\psi)$-level Σ-span $H'', \mathcal{T}_0$ of order $\theta_2 - |X \cup Z|$ in $G\backslash(X \cup Z)$ with respect to $\mathcal{T}^*\backslash(X \cup Z)$. But then 7.1.3 holds, since $\theta_2 - |X \cup Z| \geq \theta_2 - \zeta \geq \theta'$. In all cases, therefore, 7.1 holds, as required. ∎

9 The main proof

Now we complete the series of reductions by proving 8.1.

Proof of 8.1

Let Σ be a surface, and let $\delta \geq 0$ and $\phi, \psi \geq 3$ be integers. Let $\kappa = \lfloor \frac{1}{2}\phi^2\delta^2 \rfloor + 1$, let β be even with $\kappa(\psi + 6) \leq \beta \leq \kappa(\psi + 6) + 1$, let $\gamma = 3\beta + 3$, and let $\theta = 12(\delta + 3)(\beta + 1)$. We claim that γ, θ satisfy 8.1. For let $\mathcal{T}^*$ be a tangle in a graph G, let $H, \mathcal{T}$ be a Σ-span of order $\geq \theta$ with metric d, and let $Y \subseteq V(H)$ with $|Y| = \delta$, such that $d(y, y') \geq \theta$ for all distinct $y, y' \in Y$. We may assume that 8.1.2 is false, that is,

(1) *There is no H-path with ends s, t such that $d(s,t) \geq \psi$ and $d(s, y) \geq \psi$ for all $y \in Y$.*

Let $Y = \{y_1, \ldots, y_\delta\}$. For $1 \leq i \leq \delta$ there is a battlefield (Λ_i, C_i, X_i) around y_i of size β (see section 9 of [8]), by theorem 9.5 of [8], since $\beta \geq 4$ is even and $\theta \geq 16\beta + 17$. We may assume that $|X_i| = \beta$ for each i.

For $1 \leq i < j \leq \delta$, if $\sigma \in \overline{\Lambda}_i \cap \overline{\Lambda}_j$ then $d(y_1, \sigma), d(y_j, \sigma) \leq 3\beta + 3$, and so

$$\theta \leq d(y_i, y_j) \leq d(y_i, \sigma_i) + d(y_j, \sigma_j) \leq 6\beta + 6,$$

a contradiction. Thus the discs $\overline{\Lambda}_1, \ldots, \overline{\Lambda}_\delta$ are mutually disjoint.

For $i = 0, 1, \ldots, \delta$, let $H_i = H \cap (\Sigma \backslash (\Lambda_1 \cup \cdots \cup \Lambda_i))$.

(2) *For $0 \leq i \leq \delta, H_i$ is a rigid drawing in Σ.*

Subproof. Certainly $H_0 = H$ is rigid, and so we may assume that $i \geq 1$. Let $F \subseteq \Sigma$ be an O-arc with $F \cap U(H_i) \subseteq V(H_i)$ and $|F \cap V(H_i)| \leq 2$. Since the circuits $H \cap bd(\Lambda_1), \ldots, H \cap bd(\Lambda_i)$ are mutually disjoint and $|F \cap V(H_i)| \leq 2$, and $i \geq 1$, it follows that for some $j (1 \leq j \leq i)$, the sets $F \cap \Lambda_1, \ldots, F \cap \Lambda_i$ are all empty except possibly $F \cap \Lambda_j$. Consequently, if H' denotes $H \cap (\Sigma \backslash \Lambda_j)$, then $F \cap U(H') \subseteq V(H')$ and $|F \cap V(H')| \leq 2$. But H' is rigid, since (Λ_j, C_j, X_j) is a battlefield, and so there is a dial Δ for F, H' (see section 4 of [8]). Now Δ includes none of $bd(\Lambda_1), \ldots, bd(\Lambda_i)$ since it includes no circuit of H', and F meets none of $\Lambda_1, \ldots, \Lambda_i$ except possibly Λ_j, and it follows that Δ is disjoint from all of $\Lambda_1, \ldots, \Lambda_i$ except possibly Λ_j. Consequently, Δ is a dial for F, H_i. This proves (2).

Let $\mathcal{T}_0 = \mathcal{T}$, and for $i = 1, \ldots, \delta$ let $\mathcal{T}_i$ be the tangle obtained from $\mathcal{T}_{i-1}$ by clearing the $(3\beta+3)$-zone Λ_i. (We observe that Λ_i is indeed a $(3\beta + 3)$-zone with respect to $\mathcal{T}_{i-1}$ since it is for $\mathcal{T}_0$, and thus this clearing operation is possible since $\mathcal{T}_{i-1}$ has order $\geq \theta - (12\beta + 14)(i - 1) > 12\beta + 14$. We see that $\mathcal{T}_i$ has order $ord(\mathcal{T}) - (12\beta + 14)i$ and is the $(12\beta + 14)i$-compression of $\mathcal{T}$ in H_i.

For "free", see section 3 of [8].

(3) *For $1 \leq i \leq \delta$, X_1 is free with respect to $\mathcal{T}_i$.*

Subproof. This is true for $i = 1$, since (Λ_1, C_1, X_1) is a battlefield. We proceed by induction on i, and suppose that $1 < i \leq \delta$, and that X_1 is free with respect to $\mathcal{T}_{i-1}$. Let d_{i-1} be the metric of

$\mathcal{T}_{i-1}$. Then for each $v \in X_1$,

$$d_{i-1}(y_i, v) \geq d_{i-1}(y_i, \Lambda_1) - 1 \geq d(y_i, y_1) - (12\beta+14)(i-1) - 1 \geq \theta - (12\beta+14)(\delta-1) - 1 > 2\beta + 5(3\beta+3) + 2$$

and so X_1 is free with respect to $\mathcal{T}_i$, by theorem 4.7 of [9] (with $\mathcal{T}, H, z, \lambda, X, H'$ replaced by $\mathcal{T}_{i-1}, H_{i-1}, y_i, 3\beta+3, X_1, H_i$). This proves (3).

By theorem 4.1 of [9] there is at most one $(12\beta+14)i$-compression of $\mathcal{T}$ in H_i, and so $\mathcal{T}_i$ is unique. In particular, $\mathcal{T}_i$ does not depend on the order in which $\Lambda_1, \ldots, \Lambda_i$ are cleared. From this observation and (2),(3), we deduce that for $1 \leq i \leq j \leq \delta, X_i$ is free with respect to $\mathcal{T}_j$, and $(\Lambda_i : 1 \leq i \leq \delta)$ is a γ-envelope around Y. Let $H' = H_\delta, \mathcal{T}' = \mathcal{T}_\delta$.

(4) *For $1 \leq i < j \leq \delta$, if there are ϕ^2 mutually disjoint H'-paths in G with one end in X_i and the other in X_j, then 8.1.1 holds.*

Subproof. Let $P_1, \ldots, P_{\phi^2}$ be such a set of paths, and let P_k have ends $a_k \in X_i, b_k \in X_j$ where $a_1, \ldots, a_{\phi^2}$ are in order in $bd(\Lambda_i)$. By a theorem of Erdös and Szekeres [3] there exist $1 \leq i_1 < i_2 < \cdots < i_\phi \leq \phi^2$ such that $b_{i_1}, b_{i_2}, \ldots, b_{i_\phi}$ are in order in $bd(\Lambda_2)$ (under one of the orientations of $bd(\Lambda_2)$). The claim follows from theorem 3.5 of [9], since $\phi \geq 3$. This proves (4).

In view of (4), we may assume that there exists Z_0 such that

(5) *$Z_0 \subseteq V(G), Z_0 \cap V(H') \subseteq X_1 \cup \cdots \cup X_\delta, |Z_0| < \kappa$, and Z_0 meets every H'-path with ends in distinct X_i, X_j.*

Subproof. Let $1 \leq i < j \leq \delta$. By (4) and Menger's theorem, we may assume that there exists $Z_{ij} \subseteq V(G)$ with $Z_{ij} \cap V(H') \subseteq X_i \cup X_j$ and $|Z_{ij}| \leq \frac{1}{2}\phi^2$, such that Z_{ij} meets every H'-path with ends in X_i and X_j. Set $Z_0 = \cup(Z_{ij} : 1 \leq i < j \leq \delta)$; then Z_0 satisfies (5).

We suppose, for a contradiction, that setting $Z = Z_0 \cap (V(G) \setminus V(H'))$ does not satisfy 8.1.3. Consequently, if d' is the metric of $\mathcal{T}'$, it follows that there is an H'-path P_0 in G with ends x_1, s_2, such that:

(6) *$Z_0 \cap V(P_0) \subseteq \{s_1, s_2\} \cap (X_1 \cup \cdots \cup X_\delta)$ and $d'(s_1, s_2) \geq 2\psi$.*

For $i = 1, 2$ let t_i be the second vertex of P_0 in $V(H)$ as P_0 is traversed from s_i (s_i is the first).

(7) *$d(s_1 t_1) < \psi$ and $d(s_2, t_2) < \psi$.*

Subproof. If s_1, t_1 are adjacent in P_0, joined by an edge of H, then $d(s_1, t_1) \leq 2 < \psi$, so we assume not. Then the subpath of P_0 between s_1 and t_1 is an H-path. But for $1 \leq i \leq \delta$, since $s_1 \notin \Lambda_i$ it follows from the definition of a battlefield that $d(s_1, y_i) \geq \beta \geq \psi$. Consequently, $d(s_1, t_1) < \psi$ by (1). This proves (7).

We may assume that

(8) *$t_1 \in \Lambda_1$ and $t_2 \in \Lambda_2$.*

Subproof. Now $t_1 \neq s_2$ since $d(s_1, s_2) \geq d'(s_1, s_2) \geq 2\psi$ and $d(s_1, t_1) < \psi$. Consequently, $t_1 \in \Lambda_1 \cup \cdots \cup \Lambda_\delta$ since P_0 is an H'-path. Similarly, $t_2 \in \Lambda_1 \cup \cdots \cup \Lambda_\delta$. If for some $i, \{t_1, t_2\} \subseteq \Lambda_i$, then by (7), $d'(s_1, \Lambda_i) \leq d(s_1, t_1) < \psi$ and similarly $d'(s_2, \Lambda_i) < \psi$, and so $d'(s_1, s_2) < 2\psi$ contrary to (6). Thus t_1, t_2 belong to distinct members of $\Lambda_1, \ldots, \Lambda_\delta$, and we may assume from the symmetry that $t_1 \in \Lambda_1$ and $t_2 \in \Lambda_2$. This proves (8).

We may assume that

(9) *There is no path Q of H from X_1 to $V(P_0)$ with $V(Q) \cap V(H') \subseteq X_1$ and with $Z_0 \cap V(Q) = \emptyset$.*

Subproof. Suppose that for $i = 1, 2, Q_i$ is a path of H from X_i to $V(P_0)$ with $V(Q_i) \cap V(H') \subseteq X_i$ and with $Z_0 \cap V(Q_i) = \emptyset$. The union of Q_1, Q_2 and P_0 includes a path P of G from X_1 to X_2 with $V(P) \cap V(H') \subseteq X_1 \cup X_2$ and with $Z_0 \cap V(P) = \emptyset$. But this contradicts (5). Thus either Q_1 or Q_2 does not exist, and without loss of generality we may assume the former. This proves (9).

For $3 \leq i \leq \beta - 3$ there is an $(i+2)$-zone M_i around y_1 including every $x \in A(H)$ with $d(y_1, x) < i$, since $\theta > \beta$.

(10) *There are κ mutually disjoint paths of H between $V(H) \cap bd(M_\kappa)$ and $V(H) \cap bd(\Lambda_1)$.*

Subproof. Suppose not. By a form of Menger's theorem for planar graphs, there is a circuit C of a radial drawing (see section 2 of [8]) K of H, of length $< 2\kappa$ with $U(C) \subseteq \overline{\Lambda}_1$, bounding an open disc in $\overline{\Lambda}_1$ including M_κ. By theorem 7.5 of [8], $ins(C) \subseteq \overline{\Lambda}_1$ since

$$|E(C)| < 2\kappa < 2(\theta - (3\beta + 3)).$$

Let r be a region of H with $r \cap U(C) \neq \emptyset$ (see section 2 of [8]]). Then $d(r, y_1) \leq \frac{1}{2}|E(C)| < \kappa$ since $r \cap U(C) \neq \emptyset$ and $y_1 \in ins(C)$; and so $r \subseteq M_\kappa$ by the choice of M_κ. But then r is a subset of the open disc bounded by $U(C)$, and so $r \cap U(C) = \emptyset$, a contradiction. This proves (10).

(11) *There is a path P_1 of H between $V(H) \cap bd(M_\kappa)$ and X_1, with only one vertex in $V(H')$ and such that $Z_0 \cap V(P_1) = \emptyset$.*

Subproof. We recall that (Λ_1, C_1, X_1) is a battlefield of size β, and hence $\overline{M}_\kappa$ is included in the open disc in Λ_1 bounded by $U(C_1)$. Moreover, the regional distance between $U(C_1)$ and $bd(\Lambda_1)$ (see section 5 of [8]) is at least $1 + \frac{1}{2}\beta$, and so from (10) and theorem 10.5 of [5], there are κ disjoint paths of H between $V(H) \cap bd(M_\kappa)$ and X_1, each with only one vertex in $V(H')$. Since $|Z_0| < \kappa$, one of these paths is disjoint from Z_0. This proves (11).

(12) *For $3 \leq i < j \leq \beta - 3$ if $j - i \geq 3$ then $\overline{M}_i \subseteq M_j$.*

Subproof. Now $d(y_1, x) \leq i + 2$ for every $x \in A(H)$ with $x \subseteq \overline{M}_i$, and since $i + 2 < j$ it follows that $x \subseteq M_j$. This proves (12).

Let i be such that $3 \leq i \leq \beta - \psi - 5$. Then M_i and $M_{i+\psi+2}$ are both defined, and $\overline{M}_i \subseteq M_{i+\psi+2}$ by

(12). Consequently $\Sigma_i = \overline{M}_{i+\psi+2} \backslash M_i$ is homeomorphic to a closed cylinder. Let J_i be the subdrawing of H formed by the vertices and edges of H in Σ_i. Now the cylinders $\Sigma_{\kappa+r(\psi+5)} (r = 0, \ldots, \kappa-1)$ all exist since $\kappa + (\kappa-1)(\psi+5) \leq \beta - \psi - 5$, and by (12) they are mutually disjoint. Consequently the graphs $J_{\kappa+r(\psi+5)} (r = 0, \ldots, \kappa-1)$ are mutually disjoint, and since $|Z_0| < \kappa$, there exists i with $\kappa \leq i \leq \beta - \psi - 5$ such that $Z_0 \cap V(J_i) = \emptyset$.

(13) *$\overline{M}_\kappa \subseteq M_{i+\psi+2}$, and $\overline{M}_{i+\psi+2} \subseteq \Lambda_1$.*

Subproof. Now $i \geq \kappa$ and $\psi \geq 1$, so the first inclusion follows from (12). For the second, let $x \in A(H)$ with $x \subseteq \overline{M}_{i+\psi+2}$. Then $d(y_1, x) \leq i + \psi + 4$ by definition of $M_{i+\psi+2}$; and so $x \subseteq \Lambda_1$ since Λ_1, C_1, X_1 is a battlefield of size β around y_1 and $i + \psi + 4 < \beta$. This proves (13).

We recall that P_0 was defined before (6).

(14) *$P_0 \cap J_i$ is null.*

Subproof. Now H is connected, and both O-arcs in $bd(\Sigma_i)$ correspond to circuits of J_i, and so J_i is connected. Let P_1 be as in (11); then P_1 has one end in $bd(M_\kappa)$ and the other in $bd(\Lambda_1)$, and so from (13), P_1 meets $H \cap bd(M_{i+\psi+2}) \subseteq J_i$. Suppose that P_0 also meets J_i. Now $U(J_i) \subseteq \Lambda_1$ by (13), and so there is a path of $P_1 \cup J_i$ from X_1 to $V(P_0)$ with only one vertex in H'. But $V(P_1 \cup J_i) \cap Z_0 = \emptyset$, contrary to (9). This proves (14).

Let v be the first vertex of P_0 (as P_0 is traversed from s_1) such that $v \in V(H), v \neq s_1$ and $v \notin \Lambda_1 \backslash M_i$. (Such a vertex exists, because $s_2 \in V(H), s_2 \neq s_1$ and $s_2 \notin \Lambda_1 \backslash M_i$.) Let u be the last vertex of P_0 in $V(H)$ before v. (This exists, because $s_1 \in V(H)$ is before v.) Let P be the subpath of P_1 between u and v.

(15) *$i + \psi + 2 \leq d(u, y_1) \leq 3\beta + 3 + \psi$.*

Subproof. Now $u \notin M_i$ and hence $u \notin M_{i+\psi+2}$ by (14). The first inequality follows from the choice of $M_{i+\psi+2}$. For the second, it follows from (7) and (8) if $u = s_1$, because

$$d(s_1, y_1) \geq d(s_1, t_1) + d(t_1, y_1) < \psi + 3\beta + 3.$$

If $u \neq s_1$, then $u \in \Lambda_1$ by the choice of v and so $d(v, y_1) \leq 3\beta + 3$. This proves (15).

From (15), $d(u, y_1) \geq \psi$; and for $2 \leq j \leq \delta$,

$$d(u, y_j) \geq d(y_1, y_j) - d(u, y_1) \geq \theta - (3\beta + 3 + \psi) \geq \psi$$

by (15). Since P is an H-path with ends u, v, it follows from (1) that $d(u, v) < \psi$. Now if $v \in M_i$ then $d(y_1, v) \leq i + 2$, and so

$$d(u, y_1) \leq d(u, v) + d(y_1, v) < \psi + i + 2,$$

contrary to (15). Thus $v \notin M_i$, and so $v \notin \Lambda_1$ from the definition of v. But $v \neq s_2$ by (1), and so $v \in \Lambda_j$ for some j with $2 \leq j \leq \delta$, since P_0 is an H'-path. Consequently $d(v, y_j) \leq 3\beta + 3$. Hence

$$\theta \leq d(y_1, y_j) \leq d(y_1, u) + d(u, v) + d(v, y_j) < (3\beta + 3 + \psi) + \psi + (3\beta + 3) < \theta,$$

a contradiction. Thus, our assumption that 8.1.3 is not satisfied was false. This proves 8.3. ∎

References

[1] G. Dirac, "A property of 4-chromatic graphs and remarks on critical graphs" *J. London Math. Soc.* 27 (1952), 85-92.

[2] R. J. Duffin, "Topology of series-parallel networks", *J. Math. Anal. Appl.* 10 (1965), 303-318.

[3] P. Erdös and G. Szekeres, "A combinatorial problem in geometry", *Composito. Math.* 2 (1935), 463-470.

[4] N. Robertson and P. D. Seymour, "Graph minors. V. Excluding a planar graph", *J. Combinatorial Theory, Ser. B*, 41 (1986), 92-114.

[5] N. Robertson and P. D. Seymour, "Graph minors. IX. Disjoint crossed paths", *J. Combinatorial Theory, Ser. B*, 49 (1990), 40-77.

[6] N. Robertson and P. D. Seymour, "Graph minors. X. Obstructions to tree-decomposition", *J. Combinatorial Theory, Ser. B*, 52 (1991), 153-190.

[7] N. Robertson and P. D. Seymour, "Graph minors. XII. Distance on a surface", *J. Combinatorial Theory, Ser. B*, 64 (1995), 240-272.

[8] N. Robertson and P. D. Seymour, "Graph minors. XIV. Extending an embedding", *J. Combinatorial Theory, Ser. B*, 65 (1995), 23-50.

[9] N. Robertson and P. D. Seymour, "Graph minors. XV. Giant steps", *J. Combinatorial Theory, Ser. B*, 68 (1996), 112-148.

[10] N. Robertson , P. D. Seymour and R. Thomas, "Quickly excluding a planar graph", *J. Combinatorial Theory, Ser. B*, 62 (1994), 323-348.

[11] D. G. Seese and W. Wessel, "Grids and their minors", *J. Combinatorial Theory, Ser. B*, 47 (1989), 349-360.

[12] K.A. Wagner, "Über eine Eigenshaft der ebenen Komplexe", *Math. Ann.* 114 (1937), 570-590.