MAT 104 Spring 2003 Quiz 2 (Sections 7.6, 8.1, 8.3, 8.4)

1. *(10 points)* Find $\displaystyle\int \frac{dx}{(1-x^2)^{3/2}}$

   Make the substitution $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$ and $1 - x^2 = \cos^2\theta$.

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta}{\cos^3\theta}\, d\theta = \int \sec^2\theta\, d\theta = \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C.$$

2. *(10 points)* Find $\displaystyle\int (\sin 5x \cos 3x + \cos 5x \sin 3x)\, dx$

   Here we can recognize the addition formula for the sine function:

$$\sin(8x) = \sin(3x + 5x) = \sin 3x \cos 5x + \cos 3x \sin 5x.$$

   So we have simply

$$\int \sin(8x)\, dx = -\frac{\cos 8x}{8} + C.$$

   Alternatively, if you did not recognize the addition formula for sine, you can use the formula from the text:

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

   Applying this with $\alpha = 5x$ and $\beta = 3x$ we find

$$\sin 5x \cos 3x = \frac{1}{2}[\sin 8x + \sin 2x]$$

   Taking $\alpha = 3x$ and $\beta = 5x$ we get

$$\sin 3x \cos 5x = \frac{1}{2}[\sin 8x + \sin(-2x)] = \frac{1}{2}[\sin 8x - \sin 2x].$$

   So

$$\sin 5x \cos 3x + \sin 3x \cos 5x = \frac{1}{2}[2 \sin 8x] = \sin 8x$$

   and now we can integrate to get $-\cos(8x)/8 + C$.

3. *(10 points)* Consider the "triangular region" in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$. Compute the volume of the solid S obtained by revolving this region about the x-axis.

   Sketching the region we find that the top boundary is the curve $y = \cos x$ and the bottom boundary is $y = \sin x$. The curves cross when $x = \pi/4$. So we can fill this region with vertical slices whose length is $\cos x - \sin x$ as x runs from 0 to $\pi/4$. When we rotate these slices about the x axis, we get washers whose outer radius is $\cos x$ and whose inner radius is $\sin x$. Thus

$$\text{Volume } = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x)\, dx = \pi \int_0^{\pi/4} (1 - 2\sin^2 x)\, dx$$

   Replacing $\sin^2 x$ by $\dfrac{1-\cos 2x}{2}$ we get

$$\text{Volume } = \pi \int_0^{\pi/4} [1 - (1 - \cos 2x)]\, dx = \pi \int_0^{\pi/4} \cos 2x\, dx = \pi \cdot \left.\frac{\sin 2x}{2}\right|_0^{\pi/4} = \frac{\pi}{2}.$$

4. *(10 points)* The region R is bounded by the curves $y = \dfrac{1}{\sqrt[4]{3x^2+1}}$, $x = 1$, the y-axis, and the x-axis.

(a) Compute the volume of the solid S_1 obtained by revolving R around x-axis.
Slice vertically and revolve around the x-axis to get disks of radius $1/\sqrt[4]{3x^2+1}$. So cross-sectional area is $1/\sqrt{3x^2+1}$. Integrate to get volume:

$$V = \pi \int_0^1 \frac{dx}{\sqrt{3x^2+1}} = \frac{\pi}{\sqrt{3}} \int_0^{\sqrt{3}} \frac{du}{\sqrt{u^2+1}} = \frac{\pi}{\sqrt{3}} \int_0^{\pi/3} \frac{\sec^2\theta}{\sec\theta}\, d\theta.$$

(First substitute $u = \sqrt{3}x$, then $u = \tan\theta$.) So we have

$$V = \frac{\pi}{\sqrt{3}} \int_0^{\pi/3} \sec\theta\, d\theta = \frac{\pi}{\sqrt{3}} \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/3} = \frac{\pi}{\sqrt{3}} \cdot \ln|2+\sqrt{3}|.$$

(b) Compute the volume of the solid S_2 obtained by revolving R around y-axis.

Again we slice vertically and rotate to get shells. The height of a typical shell will be $1/\sqrt[4]{3x^2+1}$. The radius through which it rotates will be x. So

$$V = 2\pi \int_0^1 \frac{x}{\sqrt[4]{3x^2+1}}\, dx = \frac{2\pi}{6} \int_1^4 \frac{du}{\sqrt[4]{u}} = \frac{\pi}{3} \cdot \frac{4}{3} u^{3/4}\Big|_1^4 = \frac{4\pi}{9}(2\sqrt{2}-1)$$

5. *(10 points)* Find $\displaystyle\int \frac{dx}{(\sqrt[3]{x}+1)\sqrt{x}}$

Here, to get rid of the cube root and the square root, we make the substitution $u = \sqrt[6]{x}$. Then $u^2 = \sqrt[3]{x}$ and $u^3 = \sqrt{x}$. Since $du = (1/6)x^{-5/6}\, dx$ we have $6u^5\, du = dx$. Thus

$$\begin{aligned}
\int \frac{dx}{(\sqrt[3]{x}+1)\sqrt{x}} &= \int \frac{6u^5}{u^3(u^2+1)}\, du = 6\int \frac{u^2}{u^2+1}\, du \\
&= 6\int \left(\frac{u^2+1}{u^2+1} - \frac{1}{u^2+1}\right) du = 6\int \left(1 - \frac{1}{u^2+1}\right) du \\
&= 6[u - \arctan u] + C \\
&= 6\sqrt[6]{x} - 6\arctan(\sqrt[6]{x}) + C.
\end{aligned}$$