**MAT 104 Quiz 1, due Feb 21, 2003 on simple substitutions, integration by parts and partial fractions**

1. *(10 points)* Find $\int \frac{e^{1/x}}{x^3}\,dx$.

   We make the substitution $t = 1/x$ and then use integration by parts.

   If $t = 1/x$, then $dt = (-1/x^2)dx$ so $dt = -t^2\,du$. Thus

$$\int \frac{e^{1/x}}{x^3}\,dx = -\int t^3\,e^t\,\frac{dt}{t^2} = -\int te^t\,dt = -te^t + \int e^t\,dt = -te^t + e^t + C = -\frac{e^{1/x}}{x} + e^{1/x} + C.$$

   Here we have used integration by parts with $u = t$ and $dv = e^t\,dt$ so $du = dt$ and $v = e^t$.

2. *(10 points)* Find $\int_0^{\pi/2} \frac{\cos x}{2-\cos^2 x}\,dx$. (Hint: Use the identity $\sin^2 x + \cos^2 x = 1$.)

   The denominator $2 - \cos^2 x$ is the same as $2 - (1 - \sin^2 x) = 1 + \sin^2 x$. The integral becomes

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx = \int_{\sin 0}^{\sin \pi/2} \frac{du}{1+u^2} \text{ from the subsitution } u = \sin x \text{ and } du = \cos x\,dx.$$

   So we get $\arctan(1) - \arctan 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

3. *(10 points)* Find $\int 3x^2 \arctan x^3\,dx$.

   Note: You may be used to calling the inverse tangent function $\tan^{-1}$ instead of arctan. Both notations are standard. So $\arctan x^3$ means exactly the same as $\tan^{-1}(x^3)$.

   If $t = x^3$ then $dt = 3x^2\,dx$ and we get $\int \arctan(t)\,dt$. Now use integration by parts with $u = \arctan t$ and $dv = dt$. Thus $du = dt/(1+t^2)$ and $v = t$. So

$$\int \arctan t\,dt = t\arctan t - \int \frac{t\,dt}{1+t^2} = t\arctan t - \frac{1}{2}\int \frac{2t\,dt}{1+t^2} = t\arctan t - \frac{1}{2}\ln(1+t^2) + C.$$

   Going back to the original variable

$$\int 3x^2 \arctan x^3\,dx = x^3 \arctan x^3 - \ln\sqrt{1+x^6} + C.$$

   **Note:** You can also do this problem using integration by parts directly, without first making a $u$-substitution. In that case, you let $u = \arctan(x^3)$ and $dv = 3x^2\,dx$. In order to do this problem correctly, you must remember the CHAIN RULE – which tells you that $du = \frac{1}{1+(x^3)^2} \cdot 3x^2\,dx$.

4. *(10 points)* Find $\int_1^2 \frac{\ln x}{(x-3)^2}\,dx$.

Start with integration by parts. Let $u = \ln x$ and $dv = (x-3)^{-2}dx$. Then $du = dx/x$ and $v = -(x-3)^{-1}$. So

$$\int_1^2 \frac{\ln x}{(x-3)^2}\,dx = -\frac{\ln x}{x-3}\bigg|_1^2 + \int_1^2 \frac{dx}{x(x-3)}$$

To compute this integral, we find the partial fraction decomposition of the integrand.

$$\frac{1}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3} = \frac{A(x-3)+Bx}{x(x-3)} \qquad \Rightarrow \qquad A(x-3)+Bx = 1.$$

Setting $x = 3$ we find that $3B = 1$ so $B = 1/3$. Setting $x = 0$ we find that $-3A = 1$ so $A = -1/3$. So

$$\begin{aligned}
\int_1^2 \frac{\ln x}{(x-3)^2}\,dx &= \left(-\frac{\ln x}{x-3} + \frac{1}{3}\ln|x-3| - \frac{1}{3}\ln|x|\right)\bigg|_1^2 \\
&= -\frac{\ln 2}{2-3} + \frac{\ln 1}{1-3} + \frac{1}{3}\ln|2-3| - \frac{1}{3}\ln 2 - \frac{1}{3}\ln|1-3| - \frac{1}{3}\ln 1 \\
&= \ln 2 - \frac{\ln 2}{3} - \frac{\ln 2}{3} = \frac{\ln 2}{3} \quad (\text{since } \ln 1 = 0.)
\end{aligned}$$

5. *(10 points)* Find $\displaystyle\int \frac{x^3}{x^2+2x+5}\,dx$.

The rational function we want to integrate here is not proper. So ther first thing we must do is long division. This tells us that

$$\frac{x^3}{x^2+2x+5} = x - 2 - \frac{x-10}{x^2+2x+5}$$

So our main work is to integrate the new, proper, fraction:

$$\begin{aligned}
\int \frac{x-10}{x^2+2x+5}\,dx &= \int \frac{x-10}{x^2+2x+1+4}\,dx \\
&= \int \frac{x-10}{(x+1)^2+4}\,dx \\
&= \int \frac{u-11}{u^2+4}\,du \quad \text{from } u = x+1, du = dx, u-11 = x-10. \\
&= \frac{1}{2}\int \frac{2u}{u^2+4}\,du - 11\int \frac{du}{u^2+4} \\
&= \frac{1}{2}(u^2+4) - \frac{11}{2}\arctan\left(\frac{u}{2}\right) + C. \\
&= \ln\sqrt{x^2+2x+5} - \frac{11}{2}\arctan\left(\frac{x+1}{2}\right) + C.
\end{aligned}$$

Putting it all together we have

$$\int \frac{x^3}{x^2+2x+5}\,dx = \frac{x^2}{2} - 2x - \ln\sqrt{x^2+2x+5} + \frac{11}{2}\arctan\left(\frac{x+1}{2}\right) + C.$$