Mat104 Fall 2002, Improper Integrals From Old Exams

For the following integrals, state whether they are convergent or divergent, and give your reasons.

(1) $\int_0^\infty \frac{dx}{x^3+2}$ converges. Break it up as $\int_0^1 \frac{dx}{x^3+2} + \int_1^\infty \frac{dx}{x^3+2}$. The first of these is proper and finite. The second behaves like the integral of $1/x^3$ on $[1, \infty)$ and thus converges.

(2) $\int_0^1 \frac{dx}{x+\sqrt{x}}$ converges. As $x \to 0$, $\sqrt{x}$ goes to 0 much more slowly than x does. (Think about the graphs.) Therefore when x is very close to 0, the denominator $x + \sqrt{x} \approx \sqrt{x}$. So this integral will behave like the integral of $1/\sqrt{x}$ on $[0, 1]$, and this integral converges.

(3) $\int_1^\infty \frac{\sqrt{1+x}}{x^3}$ converges. As x goes to ∞, the integrand behaves like $\frac{\sqrt{x}}{x^3} = \frac{1}{x^{5/2}}$.

(4) $\int_0^\infty \frac{x^2}{x^3+1}\,dx$ diverges. Break it up into two integrals $\int_0^1 \frac{x^2}{x^3+1}\,dx + \int_1^\infty \frac{x^2}{x^3+1}\,dx$. The first integral is proper and finite. The second can be compared to the integral of $1/x$ on $[1, \infty)$ which diverges.

(5) $\int_0^1 \ln x\,dx$ converges to -1. Here we can compute directly since integration by parts tells us that $\int \ln x\,dx = x\ln x - x + C$. Evaluating at the $x = 1$ endpoint gives $\ln 1 - 1 = -1$. For the other endpoint we have to take the limit as x goes to 0. For this we need L'Hôpital's rule.

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0} -x = 0.$$

So evaluating at the $x = 0$ endpoint gives 0.

(6) $\int_0^1 \frac{dx}{e^x-1}$ diverges. The only difficulty is that the denominator is 0 when $x = 0$. There are a couple of approaches we could take. The easiest is to use the Taylor series for e^x. Then we know that $e^x - 1 = x +$ higher powers of x and as x goes to zero, the higher powers of x will vanish much more rapidly. So this function behaves essentially like $1/x$ when x is close to 0. Since $\int dx/x$ diverges, this integral will also.

Alternatively, we could compute the integral, making the substitution $u = e^x$ and then use partial fractions.

(7) $\int_0^\infty \frac{dx}{x^2+2x+2}$ converges. The only difficulty is that we have an infinite endpoint. The integrand is asymptotic to $1/x^2$ as x goes to infinity. Since $\int_1^\infty dx/x^2$ converges, this integral will as well. (To compare these we should break up the integral. First integrate from 0 to 1, which gives a finite value. Then integrate further from 1 out to ∞. This gives a finite value as well by comparison to $1/x^2$.)

(8) $\int_1^\infty \frac{x^3}{\ln x + x^4}\,dx$ diverges. Again the only problem is that we have an infinite endpoint. As x goes to infinity, x^4 grows much faster than $\ln x$. Thus the integrand will be asymptotic to $x^3/x^4 = 1/x$ as x goes to infinity.

(9) $\displaystyle\int_0^\infty \frac{dx}{x^3+\sqrt{x}}$ converges. Break it up into an integral from 0 to 1 plus the integral from 1 to ∞. When x is close to 0, the integrand will behave like $1/\sqrt{x}$ since x^3 goes to 0 much more rapidly than $\sqrt{x}$ does. Since the integral of $1/\sqrt{x}$ on $[0,1]$ converges, so will $1/(x^3+\sqrt{x})$.

As x goes to infinity, $\sqrt{x}$ grows much more slowly than x^3, so $1/(x^3+\sqrt{x}) \approx 1/x^3$ when x is very large. Since the integral of $1/x^3$ on $[1,\infty)$ is converges, so will the integral of $1/(x^3+\sqrt{x})$ on $[1,\infty)$.

(10) $\displaystyle\int_0^1 \frac{dx}{1-\cos x}$ diverges. Here the easiest method is to use the Taylor series for $\cos x$. It tells us that $1-\cos x = x^2/2+$ higher powers of x. Since the higher powers of x die out more rapidly when x is close to 0, $1/(1-\cos x)$ behaves like $2/x^2$ as x goes to 0. Therefore the given integral will behave like the integral of $2/x^2$ on $[0,1]$ and this integral diverges.

(11) $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$ converges. We could use integration by parts twice to compute the integral and then take limits. On the other hand, since e^{-x} dies out more rapidly than any power of x, we can conclude that $e^{-x} < \dfrac{1}{x^2}$ once x gets big enough, say, when $x > 1$ (Check it graphically). So

$$e^{-x}\cos x < \frac{\cos x}{x^2} < \frac{1}{x^2}$$

Since the integral of $1/x^2$ on $[1,\infty)$ converges, so will the integral of $e^{-x}\cos x$ on $[1,\infty)$. Since there is no problem with our function on $[0,1]$, the given integral converges.

(12) $\displaystyle\int_0^\infty \frac{e^{-x^2}}{x^2}\,dx$ diverges. We have to split it up and think about what happens as we approach 0 and what happens as we approach infinity separately. To think about what is happening at the 0 endpoint, we notice that the numerator goes to 1. So $e^{-x^2}/x^2 \sim 1/x^2$ as x goes to zero. Since the integral of $1/x^2$ on $[0,1]$ diverges, so will the integral of e^{-x^2}/x^2. (Remark: The integral of this function on $[1,\infty)$ will converge – again because the exponential dies out very very rapidly.)

(13) $\displaystyle\int_0^\infty \frac{x^2+10}{3x^5+6x+8}\,dx$ converges. The only problem is that we have an infinite endpoint. SInce the integrand is asymptotic to $1/x^3$ the integral will converge.

(14) $\displaystyle\int_0^\infty \frac{x^4+3x+1}{x^5+2x^2+3}\,dx$ diverges. The only problem is that we have an infinite endpoint. The integrand is asymptotic to $1/x$ so the integral diverges.

(15) $\displaystyle\int_0^1 \frac{e^x}{x}\,dx$ diverges. The only issue is what happens at 0. Since the numerator approaches 1 this function will behave like $1/x$ as x goes to zero.

(16) $\displaystyle\int_0^1 \frac{\sin x}{\sqrt{x}}\,dx$ converges. Again the only issue is what happens as we approach 0. Since $\sin x \approx x$ when x is close to zero, we see that the integrand behaves like $x/\sqrt{x} = \sqrt{x}$.

(17) $\displaystyle\int_0^1 \frac{dx}{x^2+\sqrt{x}}$ converges. As x goes to zero, $\sqrt{x}$ dominates. (the other term dies out much faster) So this integral behaves like $1/\sqrt{x}$ near zero.

(18) $\int_0^1 (1-x)^{-2/3}\,dx$ converges. Compute directly.

(19) $\int_2^\infty \frac{x^2+4x+4}{(\sqrt{x}-1)^3\cdot\sqrt{x^3-1}}\,dx$ diverges. The only issue is that we have an infinite endpoint. As x goes to infinity, the highest powers of x will dominate. So the integrand will behave like $x^2/(x^{3/2}\cdot x^{3/2}) = x^2/x^3 = 1/x$.

(20) $\int_0^{\pi/2} \tan x\,dx$ diverges. Compute directly, using the substitution $y = \cos x$.

(21) $\int_0^\infty \frac{e^x-1}{e^{2x}+1}\,dx$ converges. The only issue is the infinite endpoint. When x is large the integrand will behave like $e^x/e^{2x} = 1/e^x$. Compute directly or use the fact that e^x grows faster than any power of x so $1/e^x$ dies out faster than any power of x.

(22) $\int_2^\infty \frac{\sin x}{x^2-1}\,dx$ converges. Compare to $1/x^2$.

(23) $\int_1^\infty \frac{\sin\sqrt{x}}{x+x^4}\,dx$ converges. Compare to $1/(x+x^4)$ and then to $1/x^4$.

(24) $\int_0^1 \frac{\sin\sqrt{x}}{x+x^4}\,dx$ converges. When x is small the numerator will be well-approximated by $\sqrt{x}$ and the denominator will be well-approximated by x. So the integrand behaves like $1/\sqrt{x}$ when x goes to zero.

(25) $\int_0^2 \frac{dx}{|x-1|}$ diverges. This is the same as integrating $1/(1-x)$ which behaves like integrating $1/x$.

(26) $\int_1^\infty \frac{dx}{x^{0.99}}$ diverges. Compute directly or use the p-test.

(27) $\int_0^\infty \frac{dx}{x^4+x^{2/3}}\,dx$ converges. Near 0 the integrand behaves like $1/x^{2/3}$ which gives a convergent integral on $[0,1]$. When x is large the integrand behaves like $1/x^4$ which gives a convergent integral on $[1,\infty)$.

(28) $\int_0^\infty x^3\,e^{-x}\,dx$ converges. Compute directly (a pain) or use the fact that the exponential dies out faster than any power of x, say faster than x^{-5}. This allows you to compare the integral to that of $1/x^2$ which gives convergence.

(29) $\int_1^\infty \frac{\ln x}{1+x^2}\,dx$ converges. Since $\ln x$ grows more slowly than any power of x we can say that $\ln x/(1+x^2) < \sqrt{x}/(1+x^2)$ when x is large enough. Since $\sqrt{x}/(1+x^2) \sim 1/x^{3/2}$ we get convergence at the infinite endpoint, the only possible problem.

(30) $\int_1^\infty \frac{dx}{x^2\ln x}$ diverges. This integral has problems at both endpoints. This means we have to split the integration, say integrating first from 1 to 2 and then integrating again from 2

to ∞. To understand what is happening at $x = 1$ we could make the substitution $x = u+1$

$$\int_1^2 \frac{dx}{x^2 \ln x} = \int_0^2 \frac{du}{(u-1)^2 \ln(1+u)}$$

and then use a known Taylor series to understand this integral. Since $\ln(1+u) = u - u^2/2 + u^3/3 - \ldots$ we see that the denominator $(1-2u+u^2)(u - u^2/2 + u^3/3 - \ldots)$ is of the form $u +$ higher powers of u. So the integrand behaves like $1/u$ as u goes to zero, and therefore this integral (as well as the original integral) diverges.

While we're here let me say that the other integral, as x runs from 2 to ∞ converges. To see this, observe that $x^2 \ln x > x^2$ and thus $1/(x^2 \ln x) < 1/x^2$. By comparison $\displaystyle\int_2^\infty \frac{dx}{x^2 \ln x}$ converges.

(31) $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}}$ converges. The only problem is the $x = 0$ endpoint. When x is small, $\sin x \approx x$, so this integral behaves like that of $1/\sqrt{x}$ and converges.

(32) $\displaystyle\int_0^\infty e^x(1+e^{-2x})\,dx$ diverges. Multiplying out the integrand we get $\displaystyle\int_0^\infty e^x\,dx + \int_0^\infty e^{-x}\,dx$ The second integral here is finite, and the first is infinite since e^x goes to infinity as x does.

(33) $\displaystyle\int_0^1 \sqrt{x} \ln x\,dx$ converges. Compute directly using integration by parts and take the limit using L'Hôpital's Rule.

(34) $\displaystyle\int_2^\infty \frac{dx}{x^3-1}$ converges. The only problem is the infinite endpoint. The integrand is asymptotic to $1/x^3$ as x goes to infinity.

(35) $\displaystyle\int_0^{\pi/2} \frac{1+\cos x}{x}\,dx$ diverges. The only problem here is that denominator vanishes at $x = 0$. Since the numerator approaches 2 as x goes to 0, the integrand behaves like $2/x$ when x goes to 0.

(36) $\displaystyle\int_1^\infty \frac{\ln x \cdot \cos x}{x^2+1}\,dx$ converges. Here the only problem is the infinite endpoint.

$$\frac{\ln x \cdot \cos x}{x^2+1} \le \frac{\ln x}{x^2+1} < \frac{\sqrt{x}}{x^2+1} \sim \frac{1}{x^{3/2}}$$

since $\ln x$ grows more slowly than any power of x.

(37) $\displaystyle\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ converges. When x is close to zero, then $\sqrt{x}$ dominates. That is $\dfrac{1}{(1+x)\sqrt{x}} \sim \dfrac{1}{\sqrt{x}}$ as $x \to 0$. When x is very large, then $x\sqrt{x} = x^{3/2}$ dominates – $\dfrac{1}{(1+x)\sqrt{x}} \sim \dfrac{1}{x^{3/2}}$.

(38) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{1+x^4}}$ converges. $\dfrac{dx}{\sqrt{1+x^4}} \sim \dfrac{1}{x^2}$ as $x \to \infty$.

(39) $\displaystyle\int_0^\infty \frac{dx}{\sqrt[3]{x}+x^2}$ converges. When x is close to 0, $\sqrt[3]{x}$ dominates. When x is very large, x^2 dominates.

Other problems involving improper integrals

(1) Find the arc length of the curve given by $x = e^{-t}\cos t$ and $y = e^{-t}\sin t$ for $0 \le t < \infty$.

$$\frac{dx}{dt} = -e^{-t}\sin t - e^{-t}\cos t$$

$$\frac{dy}{dt} = e^{-t}\cos t - e^{-t}\sin t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \cdots = 2e^{-2t}$$

So the arc length is given by the improper integral

$$\int_0^\infty \sqrt{2}e^{-t}\,dt = \sqrt{2}.$$

(2) Find $\int_0^\infty te^{-t}\,dt$ or show that it diverges. Use integration by parts to show that $\int te^{-t}\,dt = -te^{-t} - e^{-t}$ and then $\int_0^\infty te^{-t}\,dt = 1$.

(3) Evaluate $\int_1^{\sqrt{e}} \frac{\arcsin(\ln x)}{x}\,dx$. Make the substitution $w = \ln x$ and the integral becomes

$$\int_1^{\sqrt{e}} \frac{\arcsin(\ln x)}{x}\,dx = \int_0^{1/2} \arcsin(w)\,dw$$

Using integration by parts with $u = \arcsin w$ and $dv = dw$ we find that

$$\int \arcsin w\,dw = w\arcsin w + \sqrt{1-w^2}$$

and the definite integral works out to be $\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$.

(4) Evaluate $\int_1^\infty \frac{dx}{x^2+1}$. Here we get $\lim_{t\to\infty} \arctan t - \arctan 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.