Colouring graphs with no long holes

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Abstract

We prove a 1985 conjecture of Gyárfás that for all k, ℓ, every graph with sufficiently large chromatic number contains either a clique of cardinality more than k or an induced cycle of length more than ℓ.

1 Introduction

All graphs in this paper are finite, and without loops or parallel edges. A *hole* in a graph G is an induced subgraph which is a cycle of length at least four, and an *odd hole* means a hole of odd length. (The *length* of a path or cycle is the number of edges in it, and we sometimes call a hole of length ℓ an *ℓ-hole.*) In 1985, A. Gyárfás [2] made three famous conjectures:

1.1 Conjecture: *For every integer $k > 0$ there exists $n(k)$ such that every graph G with no clique of cardinality more than k and no odd hole has chromatic number at most $n(k)$.*

1.2 Conjecture: *For all integers $k, \ell > 0$ there exists $n(k, \ell)$ such that every graph G with no clique of cardinality more than k and no hole of length more than ℓ has chromatic number at most $n(k, \ell)$.*

1.3 Conjecture: *For all integers $k, \ell > 0$ there exists $n(k, \ell)$ such that every graph G with no clique of cardinality more than k and no odd hole of length more than ℓ has chromatic number at most $n(k, \ell)$.*

Two of us recently proved the first conjecture in [3]. The third implies the other two, and remains open, although two of us proved the third when $k = 2$ [4]. (In fact we proved much more, when $k = 2$; that for all $\ell \geq 0$, in every graph with large enough chromatic number and no triangle, there is a sequence of holes of ℓ consecutive lengths). In this paper we prove the second; thus, our main result is:

1.4 *For all integers $k, \ell > 0$ there exists $n(k, \ell)$ such that every graph G with no clique of cardinality more than k and no hole of length more than ℓ has chromatic number at most $n(k, \ell)$.*

Our proof is an extension of the method of [4]. We denote the chromatic number of a graph G by $\chi(G)$. If $X \subseteq V(G)$, the subgraph of G induced on X is denoted by $G[X]$, and we often write $\chi(X)$ for $\chi(G[X])$.

2 Multicovers

If X, Y are disjoint subsets of the vertex set of a graph G, we say

- X is *complete* to Y if every vertex in X is adjacent to every vertex in Y;
- X is *anticomplete* to Y if every vertex in X nonadjacent to every vertex in Y; and
- X *covers* Y if every vertex in Y has a neighbour in X.

(If $X = \{v\}$ we say v is complete to Y instead of $\{v\}$, and so on.) Let $x \in V(G)$, let N be some set of neighbours of x, and let $C \subseteq V(G)$ be disjoint from $N \cup \{x\}$, such that x is anticomplete to C and N covers C. In this situation we call (x, N) a *cover* of C in G. For $C, X \subseteq V(G)$, a *multicover of C* in G is a family $(N_x : x \in X)$ such that

- X is stable;

- for each $x \in X, (x, N_x)$ is a cover of C;
- for all distinct $x, x' \in X$, x' is anticomplete to N_x (and in particular all the sets $\{x\} \cup N_x$ are pairwise disjoint).

The multicover $(N_x : x \in X)$ is *stable* if each of the sets N_x $(x \in X)$ is stable. Let $(N_x : x \in X)$ be a multicover of C in G. If $X' \subseteq X$, and $N'_x \subseteq N_x$ for each $x \in X'$, we say that $(N'_x : x \in X')$ is *contained in* $(N_x : x \in X)$.

If $(N_x : x \in X)$ is a multicover of C, and F is a subgraph of G with $X \subseteq V(F)$ such that no vertex in $C \cup \bigcup_{x \in X} N_x$ belongs to or has a neighbour in $V(F) \setminus X$, we say that F is *tangent* to the multicover. We need to prove that if we are given a multicover $(N_x : x \in X)$ with $|X|$ large, of some set C with $\chi(C)$ large, then there a multicover $(N'_x : x \in X')$ of some $C' \subseteq C$, contained in $(N_x : x \in X)$, with $|X'|$ and $\chi(C')$ still large (but much smaller than before), and with a certain desirable subgraph tangent, a "tick".

Let $X \subseteq V(G)$ be stable. Let a and a_x $(x \in X)$ be distinct members of $V(G) \setminus X$, such that

- a is anticomplete to X;
- a_x is adjacent to a, x and is anticomplete to $X \setminus \{x\}$, for each $x \in X$;

We call the subgraph of G with vertex set $X \cup \{a\} \cup \{a_x : x \in X\}$ and edges x-a_x, a-a_x for each $x \in X$ a *tick* on X in G. This may not be an induced subgraph of G because the vertices a_x $(x \in X)$ may be adjacent to one another in G.

For a graph G, we denote by $\omega(G)$ the cardinality of the largest clique of G, and if $X \subseteq V(G)$ we sometimes write $\omega(X)$ for $\omega(G[X])$. We need:

2.1 *For all $j, k, m, c, \kappa \geq 0$ there exist $m_j, c_j \geq 0$ with the following property. Let G be a graph with $\omega(G) \leq k$, such that $\chi(H) \leq \kappa$ for every induced subgraph H of G with $\omega(H) < k$. Let $(N_x : x \in X)$ be a stable multicover in G of some set C, such that $|X| \geq m_j$, $\chi(C) \geq c_j$, and $\omega(\bigcup_{x \in X} N_x) \leq j$. Then there exist $X' \subseteq X$ with $|X'| \geq m$ and $C' \subseteq C$ with $\chi(C') \geq c$ and a stable multicover $(N'_x : x \in X')$ of C' contained in $(N_x : x \in X)$, such that there is a tick in G tangent to $(N'_x : x \in X')$.*

Proof. We may assume that $k \geq 2$, for otherwise the result is vacuous. We proceed by induction on j, keeping k, m, c, κ fixed. If $j = 0$ then we may take $m_0 = c_0 = 1$ and the theorem holds vacuously; so we assume that $j > 0$ and the result holds for $j - 1$. Thus m_{j-1}, c_{j-1} exist. Let

$$
\begin{aligned}
m_j &= 2kmm_{j-1} \\
d_2 &= m_j 2^{m_j} c_{j-1} + 2^{m_j} c \\
d_1 &= d_2 + m_j \kappa \\
d_0 &= k2^{m_j} d_1 \\
c_j &= d_0 + k\kappa.
\end{aligned}
$$

We claim that m_j, c_j satisfy the theorem. Let G, $(N_x : x \in X)$, and C be as in the theorem, with $|X| \geq m_j$ and $\chi(C) \geq c_j$, such that $\omega(\bigcup_{x \in X} N_x) \leq j$. We may assume that $|X| = m_j$. Since $c_j > \kappa$, there is a clique $A \subseteq C$ with $|A| = k$. Let C_0 be the set of vertices in $C \setminus A$ with no neighbour in A;

then since every vertex in $C \setminus C_0$ has a neighbour in A, and for each $a \in A$ its set of neighbours has chromatic number at most κ (because it includes no k-clique), it follows that $\chi(C \setminus C_0) \leq k\kappa$, and so $\chi(C_0) \geq c_j - k\kappa = d_0$.

(1) *There exist $a \in A$, and $X_1 \subseteq X$ with $|X_1| \geq m_j/k$, and $C_1 \subseteq C_0$ with $\chi(C_1) \geq d_1$, such that for each $v \in C_1$ and each $x \in X_1$, there is a vertex in N_x adjacent to v and nonadjacent to a.*

For each $v \in C_0$ and each $x \in X$, v has a neighbour in N_x; and this neighbour is nonadjacent to some vertex in A, since $|A| = k = \omega(G)$. Thus there exists $a_{v,x} \in A$ such that some vertex in N_x is adjacent to v and nonadjacent to $a_{v,x}$. There are only k possible values for $a_{v,x}$ as x ranges over X, and so there exist $a_v \in A$ and $X_v \subseteq X$ with $|X_v| \geq |X|/k$, such that $a_{v,x} = a_v$ for all $x \in X_v$. There are only k possible values for a_v; so there exist $a \in A$ and $C' \subseteq C_0$ with $\chi(C') \geq \chi(C_0)/k \geq 2^{m_j} d_1$, such that $a_v = a$ for all $v \in C'$. Thus for each $v \in C'$ there exists $X_v \subseteq X$ with $|X_v| \geq |X|/k$, such that $a_{v,x} = a$ for all $x \in X_v$. There are at most 2^{m_j} possibilities for X_v; so there exists $C_1 \subseteq C'$ with $\chi(C_1) \geq d_1$, and $X_1 \subseteq X$ with $|X_1| \geq m_j/k$, such that $X_v = X_1$ for all $v \in C_1$. This proves (1).

Let a, X_1, C_1 be as in (1). For each $v \in C_1$ and each $x \in X_1$, let $n_{x,v} \in N_x$ be adjacent to v and nonadjacent to a. For each $x \in X_1$ choose $a_x \in N_x$ adjacent to a. Let C_2 be the set of all vertices in C_1 nonadjacent to each $a_x (x \in X)$; then $\chi(C_2) \geq \chi(C_1) - m_j\kappa \geq d_2$. For each $y \in X_1$, let C_y be the set of all $v \in C_2$ such that $n_{x,v}$ is adjacent to a_y, for at least m_{j-1} values of $x \in X_1 \setminus \{y\}$. Next, we show that we may assume that:

(2) *$\chi(C_y) \leq c_{j-1} 2^{m_j}$, for each $y \in X_1$.*

We will show that if (2) is false, then there is a multicover $(N'_x : x \in X')$ contained in $(N_x : x \in X)$ with $\omega(\bigcup_{x \in X'} N'_x) \leq j - 1$, to which we can apply the the inductive hypothesis on j. Suppose then that $\chi(C_y) > c_{j-1} 2^{m_j}$ for some $y \in X_1$. For each $v \in C_y$, let $X_v \subseteq X_1 \setminus \{y\}$ with $|X_v| = m_{j-1}$, such that $n_{x,v}$ is adjacent to a_y for each $x \in X_v$. There are at most 2^{m_j} choices of X_v, and so there exist $C' \subseteq C_y$ and $X' \subseteq X_1 \setminus \{y\}$ with $\chi(C') \geq \chi(C_y) 2^{-m_j} \geq c_{j-1}$ and $|X'| = m_{j-1}$, such that $X_v = X'$ for all $v \in C'$. Let N'_x be the set of neighbours of a_y in N_x, for each $x \in X'$; then $(N'_x : x \in X')$ is a multicover of C'. Moreover, since every vertex in $\bigcup_{x \in X'} N'_x$ is adjacent to a_y, it follows that $\omega(\bigcup_{x \in X'} N'_x) < j$. But then the result follows from the definition of m_{j-1}, c_{j-1}. This proves (2).

(3) *There exist $C_3 \subseteq C_2$ with $\chi(C_3) \geq c$ and $X_3 \subseteq X_1$ with $|X_3| \geq m$, such that $n_{x,v}$ is nonadjacent to a_y for all $v \in C_3$ and all distinct $x, y \in X_3$.*

Let C' be the set of all $v \in C_2$ that are not in any of the sets C_y $(y \in X_1)$, that is, such that for each $y \in X_1$, there are fewer than m_{j-1} values of $x \in X_1 \setminus \{y\}$ such that $n_{x,v}$ is adjacent to a_y. From (2), it follows that

$$\chi(C') \geq \chi(C_2) - m_j 2^{m_j} c_{j-1} \geq d_2 - m_j 2^{m_j} c_{j-1} = 2^{m_j} c.$$

Let $v \in C'$; and let G_v be the digraph with vertex set X_1 in which for distinct $x, y \in X_1$, y is adjacent from x in G_v if $n_{x,v}$ is adjacent to a_y. It follows from the definition of C_2 that every vertex of G_v has indegree at most $m_{j-1} - 1$. Consequently the undirected graph underlying G_v has degeneracy at most

$2m_{j-1}-2$, and therefore is $2m_{j-1}$-colourable. Thus there exists $X_v \subseteq X_1$ with $|X_v| \geq |X_1|/(2m_{j-1})$ such that no two members of X_v are adjacent in G_v. There are at most 2^{m_j} choices of X_v, and so there exists $C_3 \subseteq C'$ with $\chi(C_3) \geq \chi(C')2^{-m_j} \geq c$ and $X_3 \subseteq X_1$ with

$$|X_3| \geq |X_1|/(2m_{j-1}) \geq m_j/(2km_{j-1}) = m,$$

such that $X_v = X_3$ for all $v \in C_3$. This proves (3).

For each $x \in X_3$, let N'_x be the set of vertices in N_x nonadjacent to each a_y ($y \in X_3$). Thus $n_{x,v} \in N'_x$ for each $x \in X_3$ and $v \in C_3$. Hence $(N'_x : x \in X_3)$ is a multicover of C_3 contained in $(N_x : x \in X)$. Moreover, the subgraph consisting of a, the vertices a_x ($x \in X_3$) and X, together with the edges a-a_x and a_x-x for each $x \in X_3$, form a tick which is tangent to this multicover. This proves 2.1. ∎

By repeated application of 2.1 with $j = k$, we can obtain many ticks on the same subset X' of X, disjoint except for X' and with no edges joining them disjoint from X'. (Note that vertices in the same tick with degree two in that tick may be adjacent in G, but otherwise the subgraph formed by the union of the ticks is induced.) But such a "tick cluster" has a hole of length at least ℓ, if there are at least $\ell/3$ ticks and the set X has cardinality at least $\ell/3$. We deduce that:

2.2 *Let $k, \kappa, \ell \geq 0$ be integers. Then there exists m, c with the following property. Let G be a graph with no hole of length at least ℓ, with $\omega(G) \leq k$, such that $\chi(H) \leq \kappa$ for every induced subgraph H of G with $\omega(H) < k$. Then there is no stable multicover $(N_x : x \in X)$ in G of a set C, such that $|X| \geq m$ and $\chi(C) \geq c$.*

We remark that with a little more work, we can prove a version of 2.1, and of 2.3 below, which just assumes there is no odd hole of length at least ℓ, instead of assuming there is no hole of length at least ℓ. The proof is, roughly: use the argument above to get a large tick cluster, all tangent to a multicover $(N_x : x \in X)$ of some set C, with $|X|$ and $\chi(C)$ large. Use Ramsey's theorem repeatedly, to arrange that for each tick, its "knees" are stable (shrinking X to some smaller set); and then choose an odd path between two vertices $x, x' \in X$ via a vertex in N_x, a vertex in N'_x, and an $\omega(G)$-clique in C. We omit the details.

Let us eliminate the "stable" hypothesis.

2.3 *Let $k, \kappa, \ell \geq 0$ be integers. Then there exists m, c with the following property. Let G be a graph with no hole of length at least ℓ, with $\omega(G) \leq k$, such that $\chi(H) \leq \kappa$ for every induced subgraph H of G with $\omega(H) < k$. Then there is no multicover $(N_x : x \in X)$ in G of a set C, such that $|X| \geq m$ and $\chi(C) \geq c$.*

Proof. Let m, c' satisfy 2.2 (with c replaced by c'). Let $c = c'\kappa^m$. We claim that m, c satisfy the theorem. Let G be a graph with no hole of length at least ℓ, with $\omega(G) \leq k$, such that $\chi(H) \leq \kappa$ for every induced subgraph H of G with $\omega(H) < k$. Suppose that $(N_x : x \in X)$ is a multicover in G of a set C, such that $|X| \geq m$ and $\chi(C) \geq c$. We may assume that $|X| = m$. For each $x \in X$, the subgraph induced on N_x is κ-colourable; choose some such colouring, with colours $1, \ldots, \kappa$, for each x. For each $v \in C$, let $f_v : X \to \{1, \ldots, \kappa\}$ such that for each $x \in X$, some neighbour of v in N_x has colour $f_v(x)$. There are only $\kappa^{|X|}$ possibilities for f_v, so there is a function $f : X \to \{1, \ldots, \kappa\}$ and a subset $C' \subseteq C$ with $\chi(C') \geq \chi(C)\kappa^{-|X|} \geq c'$, such that $f_v = f$ for all $v \in C'$. For each $x \in X$, let N'_x be the set of vertices in N_x with colour $f(x)$; then $(N'_x : x \in X)$ is a stable multicover of C', and the result follows from the choice of m, c'. This proves 2.3. ∎

3 Clique control

Let $X \subseteq V(G)$ be a clique. If $|X| = k$ we call X a *k-clique.* We denote by $N^1_G(X)$ the set of all vertices in $V(G) \setminus X$ that are complete to X; and by $N^2_G(X)$ the set of all vertices in $V(G) \setminus X$ with a neighbour in $N^1(X)$ and with no neighbour in X. When $X = \{v\}$ we write $N^i_G(v)$ for $N^i_G(X)$ $(i = 1, 2)$. Let $\mathbb{N}$ denote the set of nonnegative integers, let $\phi : \mathbb{N} \to \mathbb{N}$ be a nondecreasing function, and let $k \geq 1$ be an integer. We say a graph G is *(h, ϕ)-clique-controlled* if for every induced subgraph H of G and every integer $n \geq 0$, if $\chi(H) > \phi(n)$ then there is an h-clique X of H such that $\chi(N^2(X)) > n$. Intuitively, this means that in every induced subgraph H of large chromatic number, there is an h-clique X with $N^2_H(X)$ of large chromatic number; the function ϕ is just a way of making "large" precise. (This is different from what was called being "(ρ, ϕ)-controlled" in [4]. There, ρ was a distance, and here, h is a clique cardinality.)

We need the following. A somewhat stronger version was proved in [1], but we give a proof here to make the paper self-contained.

3.1 *Let $\ell \geq 4$, $\kappa \geq 0$ and $\tau \geq 0$ be integers, and let G be a graph with no hole of length at least ℓ, such that $\chi(N^1(v)) \leq \kappa$ and $\chi(N^2(v)) \leq \tau$ for every vertex v. Then $\chi(G) \leq 2(\ell - 3)(\kappa + \tau) + 1$.*

Proof. Let G_1 be a component of G with $\chi(G_1) = \chi(G)$, let $z_0 \in V(G_1)$, and for $i \geq 0$ let L_i be the set of vertices of G_1 with distance i from z_0. Choose k such that $\chi(L_k) \geq \chi(G_1)/2$. If $k = 0$ then the theorem holds, so we may assume that $k \geq 1$. Let C_0 be the vertex set of a component of $G[L_k]$ with maximum chromatic number. Choose $v_0 \in L_{k-1}$ with a neighbour in C_0. Let $t = \ell - 3$, and suppose that $\chi(C_0) > t\kappa + t\tau$. We claim that :

(1) *For all i with $0 \leq i < t$, there is an induced path v_0-v_1-$\cdots$-v_i where $v_1, \ldots, v_i \in C$, and a subset C_i of C such that $G[C_i]$ is connected, $\chi(C_i) > (t - i)\kappa + t\tau$, v_i has a neighbour in C_i, and $v_0, \ldots, v_{i-1}$ have no neighbours in C_i.*

For this is true when $i = 0$; suppose it is true for some value of $i < t$, and we prove it is also true for $i + 1$. Let N be the set of neighbours of v_i in C_i. Thus

$$\chi(C_i \setminus N) \geq \chi(C_i) - \kappa > (t - i - 1)\kappa + t\tau \geq 0,$$

and so $C_i \setminus N \neq \emptyset$; let C_{i+1} be the vertex set of a component of $G[C_i \setminus N]$ with maximum chromatic number. Thus $\chi(C_{i+1}) > (t - i - 1)\kappa - (i + 1)\kappa$. Choose $v_{i+1} \in N$ with a neighbour in C_{i+1}. This completes the inductive definition of $v_1, \ldots, v_i$ and C_i, and so proves (1).

In particular, such a path v_0-$\cdots$-v_t and subset C_t exist. Since $\chi(C_t) > t\tau$, there is a vertex $v \in C_t$ in none of the sets $N^2_G(v_i)$ $(0 \leq i \leq t - 1)$, and therefore with distance at least three from all of $v_0, \ldots, v_{t-1}$, since $t \geq 1$. Choose $u \in L_{k-1}$ adjacent to v; then u has distance at least two from all of $v_0, \ldots, v_{t-1}$. Let P be an induced path of $G[C_t \cup \{u, v_t\}]$ between u, v_t; thus P has length at least one. Let Q be an induced path of G between u, v_0 with all internal vertices in $L_0 \cup \cdots \cup L_{k-2}$; then Q has length at least two. The union of P, Q and v_0-v_1-$\cdots$-v_t is a hole of length at least $t + 3 = \ell$, which is impossible.

This proves that $\chi(C_0) \leq t\kappa + t\tau$. Consequently $\chi(L_k) \leq t(\kappa + \tau)$, and so $\chi(G) \leq 2t(\kappa + \tau)$. This proves 3.1. ∎

From 3.1 we deduce:

3.2 *Let $\ell \geq 4$, and let $k \geq 1$ and $\kappa \geq 0$ be such that $\chi(H) \leq \kappa$ for every graph H with no hole of length at least ℓ and $\omega(H) < k$. For $x \geq 0$ let $\phi_1(x) = 2(\ell - 3)(\kappa + x) + 1$. Then every graph G with no hole of length at least ℓ and with $\omega(G) \leq k$ is $(1, \phi_1)$-clique-controlled.*

Proof. Let G be a graph with no hole of length at least ℓ and with $\omega(G) \leq k$. Let $n \geq 0$, and let H be an induced subgraph of G with $\chi(H) > \phi(n)$. Consequently $V(H) \neq \emptyset$; choose $v \in V(H)$ with $\chi(N^2_H(v))$ maximum, $\chi(N^2_H(v)) = \tau$ say. Since H has no hole of length at least ℓ, and $\chi(N_H(u)) \leq \kappa$ and $\chi(N^2_H(u)) \leq \tau$ for every vertex u of H, 3.1 implies that $\chi(H) \leq 2(\ell - 3)(\kappa + \tau) + 1$, and so $\phi_1(n) < \chi(H) \leq \phi_1(\chi(N^2_H(v)))$. Consequently $\chi(N^2_H(v)) > n$. This proves 3.2. ∎

We are going to prove, by induction on h, that:

3.3 *Let $\ell \geq 4$, and let $k \geq 1$ and $\kappa \geq 0$ be such that $\chi(H) \leq \kappa$ for every graph H with no hole of length at least ℓ and $\omega(H) < k$. For all $h \geq 1$ there is a nondecreasing function $\phi_h : \mathbb{N} \to \mathbb{N}$ such that every graph G with no hole of length at least ℓ and with $\omega(G) \leq k$ is (h, ϕ_h)-clique-controlled.*

Suppose 3.3 is true. Let G be a graph with no hole of length at least ℓ and with $\omega(G) \leq k$; then G is $(k+1, \phi_{k+1})$-clique-controlled, by 3.3 with $h = k+1$, and since G has no $(k+1)$-clique, it follows that $\chi(G) \leq \phi_{k+1}(0)$; and this proves 1.4.

Thus it suffices to prove 3.3. In order to do so, in view of 3.2, it suffices to prove that if ϕ_h exists for a given value of $h \geq 1$, then ϕ_{h+1} also exists. To prove the latter, we need to prove that for every integer $\tau \geq 0$, there exists $c(\tau)$ such that if G has no hole of length at least ℓ and $\omega(G) \leq k$, and $\chi(N^2_G(X)) \leq \tau$ for every $(h+1)$-clique X in G, then $\chi(G) \leq c(\tau)$. (If we can prove this, we define $\phi_{h+1}(n) = \max_{0 \leq \tau \leq n} c(\tau)$ for every $n \geq 0$, and then ϕ_{h+1} satisfies 3.3 as required.)

Consequently, it remains to prove the following:

3.4 *Let $\ell, k, \kappa, \tau \geq 0$, let $h \geq 1$, and let $\phi : \mathbb{N} \to \mathbb{N}$ be nondecreasing. Then there exists $c \geq 0$ with the following property. Let G be a graph such that*

- *G has no hole of length at least ℓ;*
- *$\omega(G) \leq k$;*
- *$\chi(H) \leq \kappa$ for every induced subgraph H of G with $\omega(H) < k$;*
- *G is (h, ϕ)-clique-controlled; and*
- *$\chi(N^2_G(X)) \leq \tau$ for every clique X in G with $|X| = h + 1$.*

Then $\chi(G) \leq c$.

This is the goal of the remainder of the paper.

4 Cables

Let G be a graph and let $t \geq 0$ and $h \geq 1$ be integers. A *t-cable* of *order h* in G consists of:

- t h-cliques $X_1, \dots, X_t$, pairwise disjoint and anticomplete;
- for $1 \leq i \leq t$, a subset N_i of $N^1(X_i)$, such that the sets $N_1, \dots, N_t$ are pairwise disjoint;
- for $1 \leq i \leq t$, disjoint subsets $Z_{i,i+1}, \dots, Z_{i,t}, Y_{i,t}$ of N_i; and
- a subset $C \subseteq V(G)$ disjoint from $X_1 \cup \dots \cup X_t \cup N_1 \cup \dots \cup N_t$

satisfying the following conditions:

- for $1 \leq i \leq t$, $Y_{i,t}$ covers C, and C is anticomplete to $Z_{i,j}$ for $i+1 \leq j \leq t$, and C is anticomplete to X_i;
- for $i < j \leq t$, X_i is anticomplete to N_j;
- for all $i < j \leq t$, every vertex in $Z_{i,j}$ has a non-neighbour in X_j;
- for $i < j < k \leq t$, $Z_{i,j}$ is anticomplete to $X_k \cup N_k$;
- for all $i < j \leq t$, either
  - some vertex in X_j has no neighbours in $Y_{i,t}$, and $Z_{i,j} = \emptyset$, or
  - X_j is complete to $Y_{i,t}$, and $Z_{i,j}$ covers N_j.

We call C the *base* of the t-cable, and say $\chi(C)$ is the *chromatic number* of the t-cable. Given a t-cable in this notation, let $I \subseteq \{1, \dots, t\}$; then the cliques X_i $(i \in I)$, the sets N_i $(i \in I)$, the sets $Z_{i,j}$ $(i, j \in I)$, the sets Y_i $(i \in I)$ and C (after appropriate renumbering) define an $|I|$-cable. We call this a *subcable.*

Thus there are two types of pair (i, j) with $i < j \leq t$, and later we will apply Ramsey's theorem on these pairs to get a large subcable where all the pairs have the same type. Consequently, two special kinds of t-cables are of interest:

- t-cables of *type 1*, where for all $i < j \leq t$, some vertex in X_j has no neighbours in $Y_{i,t}$, and $Z_{i,j} = \emptyset$; and
- t-cables of *type 2*, where for all $i < j \leq t$, X_j is complete to $Y_{i,t}$, and $Z_{i,j}$ covers N_j.

From 2.3 we deduce:

4.1 *For all $k, \kappa, \ell \geq 0$ and $h \geq 1$, there exist $t, c \geq 0$ with the following property. Let G be a graph with no hole of length at least ℓ, with $\omega(G) \leq k$, such that $\chi(H) \leq \kappa$ for every induced subgraph H of G with $\omega(H) < k$. Then G admits no t-cable of type 1 and order h with chromatic number more than c.*

Proof. Choose m, c to satisfy 2.3. By Ramsey's theorem there exists t such that for every partition of the edges of K_t into h sets, there is an m-clique of K_t for which all edges joining its vertices are in the same set. We claim that t, c satisfy the theorem.

For let G be as in the theorem, and suppose that G admits a t-cable of type 1 and order h with chromatic number more than c. In the usual notation for t-cables, fix an ordering of the members of X_i for each i; thus we may speak of the rth member of X_i for $1 \leq r \leq h$. For each pair (i, j) with $i < j \leq t$, let $f(i, j) = r$ where the rth member of X_j has no neighbours in $Y_{i,t}$. From the choice of t, there exist $I \subseteq \{1, \dots, t\}$ with $|I| = m$ and $r \in \{1, \dots, h\}$ such that $f(i, j) = r$ for all $i, j \in I$ with $i < j$. For each $j \in I$, let x_j be the rth member of X_j. Then the sets (x_j, N_j) $(j \in I)$ form a multicover of C, which is impossible by 2.3. This proves 4.1. ∎

We need an analogue for cables of type 2, but it needs an extra hypothesis. On the other hand, we only need to assume that there is no hole of length exactly ℓ.

4.2 *Let $\tau \geq 0$, $\ell \geq 5$ and $h \geq 1$, and let G be a graph with no ℓ-hole, such that $\chi(N^2(X)) \leq \tau$ for every $(h+1)$-clique X of G. Then G admits no $(\ell - 3)$-cable of type 2 and order h with chromatic number more than $(\ell - 3)\tau$.*

Proof. Let $t = \ell - 3$, let G be as in the theorem, and suppose that G admits a t-cable of type 2 and order h with chromatic number more than $t\tau$. In the usual notation, choose $z_t \in Y_{t,t}$, and choose $z_{t-1} \in Z_{t-1,t}$ adjacent to z_t. Since $z_{t-1} \in Z_{t-1,t}$, it has a non-neighbour $x_t \in X_t$. Neither of x_t, z_t has a neighbour in $Z_{i,i+1}$ for $1 \leq i \leq t-2$. Now z_{t-1} has a neighbour $z_{t-2} \in Z_{t-2,t-1}$; and similarly for $i = t-3, \dots, 1$ let $z_i \in Z_{i,i+1}$ be a neighbour of z_{i+1}. It follows that

$$z_1\text{-}z_2\text{-}\cdots\text{-}z_{t-1}\text{-}z_t\text{-}x_t$$

is an induced path.

For $1 \leq i \leq t$, let C_i be the set of vertices $v \in C$ such that some vertex in $Y_{1,t}$ is adjacent to both v, z_i. Since X_i is complete to $Y_{1,t}$, it follows that $C_i \subseteq N_G^2(X_i \cup \{z_i\})$; and since $X_i \cup \{z_i\}$ is an $(h+1)$-clique, it follows from the hypothesis that $\chi(C_i) \leq \tau$. Thus the union $C_1 \cup \cdots \cup C_t$ has chromatic number at most $t\tau$; and since $\chi(C) > t\tau$, there exists $u \in C$ not in any of the sets $C_i\,(1 \leq i \leq t)$. Choose $v \in Y_{1,t}$ adjacent to u; then v is not adjacent to any of $z_1, \dots, z_t$, by definition of $C_1, \dots, C_t$. Choose $x_1 \in X_1$; then

$$v\text{-}x_1\text{-}z_1\text{-}z_2\text{-}\cdots\text{-}z_{t-1}\text{-}z_t\text{-}x_t\text{-}v$$

is a hole of length $t + 3 = \ell$, a contradiction. This proves 4.2. ∎

From 4.1, 4.2 and Ramsey's theorem, we deduce that:

4.3 *For all $k, \kappa, \tau, \ell \geq 0$ and $h \geq 1$, there exist $t, c \geq 0$ with the following property. Let G be a graph such that:*

- *G has no hole of length at least ℓ;*
- *$\omega(G) \leq k$;*

- $\chi(H) \leq \kappa$ *for every induced subgraph* H *of* G *with* $\omega(H) < k$*; and*
- $\chi(N^2(X)) \leq \tau$ *for every* $(h+1)$*-clique* X *of* G.

Then G *admits no* t*-cable of order* h *with chromatic number more than* c.

On the other hand, we have the following:

4.4 *Let* $t, c, \tau, \kappa \geq 0$ *and* $h > 0$*, and let* $\phi : \mathbb{N} \to \mathbb{N}$ *be nondecreasing. Then there exists* c' *with the following property. Let* G *be a graph such that*

- $\chi(N^1(v)) \leq \kappa$ *for every* $v \in V(G)$*;*
- G *is* (h, ϕ)*-clique-controlled; and*
- $\chi(N^2(X)) \leq \tau$ *for every* $(h+1)$*-clique* X *of* G.

If G *admits no* t*-cable of order* h *with chromatic number more than* c*, then* $\chi(G) \leq c'$.

Proof. Let $\sigma_t = c$, and for $s = t-1, \ldots, 0$ let

$$\sigma_s = \max(2^s \phi((h+1)^s \sigma_{s+1}), \tau + h\kappa).$$

Let $c' = \sigma_0$. We claim that c' satisfies the theorem.

For let G be a graph satisfying the hypotheses of the theorem, and suppose that $\chi(G) > c'$. Consequently G admits a 0-cable with chromatic number more than σ_0. We claim that for $s = 1, \ldots, t$, G admits an s-cable of order h with chromatic number more than σ_s. For suppose the result holds for some $s < t$; we prove it also holds for $s + 1$.

In the usual notation, let C be the base of the s-cable. For each $v \in C$ and $1 \leq i \leq s$, let $C_{i,v}$ be the set of vertices $u \in C \setminus \{v\}$ nonadjacent to v, such that some vertex in $Y_{i,s}$ is adjacent to both u, v. Let $f_{i,v} = 1$ if $\chi(C_{i,v}) > \tau + h\kappa$, and $f_{i,v} = 0$ otherwise. There are only 2^s possibilities for the sequence $f_{1,v}, \ldots, f_{t,v}$, so there is a subset $C_1 \subseteq C$ with $\chi(C_1) \geq 2^{-s}\chi(C) > 2^{-s}\sigma_s$ and a 0,1-sequence $f_1, \ldots, f_s$ such that $f_{i,v} = f_i$ for $1 \leq i \leq t$ and all $v \in C_1$. For $0 \leq i \leq s$ let $d_i = (h+1)^{s-i}\sigma_{s+1}$. Let $H = G[C_1]$; then since $2^{-s}\sigma_s = \phi(d_0)$, there is an h-clique X_{s+1} of H such that $\chi(D_0) > d_0$, where $D_0 = N^2_H(X_{s+1})$. Let $N_{s+1} = Y_{s+1,s+1} = N^1_H(X_{s+1})$.

For $1 \leq i \leq s$, we define $Y_{i,s+1}, Z_{i,s+1} \subseteq Y_{i,s}$ and $D_i \subseteq D_{i-1}$ as follows. Assume that we have defined D_{i-1}, and $\chi(D_{i-1}) > d_{i-1}$. Let W be the set of vertices in $Y_{i,s}$ that are complete to X_{s+1}, and for each $x \in X_{s+1}$, let U_x be the set of vertices in D_{i-1} with a neighbour in $Y_{i,s}$ that is nonadjacent to x. If $\chi(U_x) > d_i$ for some $x \in X_{h+1}$, let $D_i = U_x$, let $Y_{i,s+1}$ be the set of all vertices in $Y_{i,s}$ that are nonadjacent to x, and let $Z_{i,s+1} = \emptyset$.

Thus we assume that $\chi(U_x) \leq d_i$ for each $x \in X_{h+1}$; and so $\bigcup_{x \in X_{s+1}} U_x$ has chromatic number at most hd_i. Let $D_i = D_{i-1} \setminus \bigcup_{x \in X_{s+1}} U_x$; then $\chi(D_i) > d_{i-1} - hd_i = d_i$. For each vertex in D_i, all its neighbours in $Y_{i,s}$ belong to W. In particular, let $x \in X_{s+1}$; then $C_{i,x}$ (defined earlier) has chromatic number more than

$$d_{i-1} - hd_i = d_i \geq \sigma_{s+1} \geq \tau + h\kappa,$$

and so $f_{i,x} = 1$. Since $x \in C_1$, it follows that $f_i = 1$, and so $\chi(C_{i,v}) > \tau + h\kappa$ for each $v \in C_1$.

Now let $v \in N_{s+1}$. If $u \in C$, and u has no neighbour in $X_{s+1} \cup \{v\}$, and some vertex in W is adjacent to both u, v, then $u \in N_G^2(X_{s+1} \cup \{v\}$; and so the set of all such u has chromatic number at most κ. On the other hand, the set of $u \in C$ with a neighbour in X_{s+1} and are nonadjacent to v has chromatic number at most $h\kappa$, since for each $x \in X_{s+1}$ its set of neighbours has chromatic number at most κ. Consequently the set of vertices in C that are nonadjacent to v and adjacent to a neighbour of v in W has chromatic number at most $\tau + h\kappa$. Since $\chi(C_{i,v}) > \tau + h\kappa$, it follows that there exists $u \in C_{i,v}$ such that no neighbour of v in W is adjacent to u. From the definition of $C_{i,v}$, it follows that v has a neighbour in $Y_{i,s} \setminus W$.

Since this is true for every vertex $v \in N_{s+1}$, we may define $Y_{i,s+1} = W$ and $Z_{i,s+1} = Y_{i,s} \setminus W$. This completes the definition of $Y_{i,s+1}, Z_{i,s+1}$ and D_i.

Thus $\chi(D_s) > d_s$, and so $X_1, \ldots, X_{s+1}$, the sets $N_1, \ldots, N_{s+1}$, the sets $Z_{i,j}$ for $1 \le i < j \le s+1$, the sets $Y_{i,s+1}$ for $1 \le i \le s+1$, and D_s, define an $(s+1)$-cable of order h with chromatic number more than d_s.

This proves that G admits a t-cable of order h with chromatic number more than $\sigma_t = c$, a contradiction, and so proves 4.4. ∎

3.4 follows immediately from 4.3 and 4.4. This proves 3.4, and hence completes the proof of 1.4.

References

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