ON THE CHROMATIC NUMBER OF THE UNION OF COMPARABILITY GRAPHS

WOUTER CAMES VAN BATENBURG[1], MARIA CHUDNOVSKY[2], LINDA COOK[3], JAMES DAVIES[4], SEOKBEOM KIM[6,5], AND SANG-IL OUM[5,6]

Abstract. Resolving in a strong sense a problem of Gyárfás on the union of two perfect graphs, we prove that for every pair of positive integers d and k, there is a graph G with clique number k and chromatic number k^d that is the union of d comparability graphs. We also show that the chromatic number can be replaced by the fractional chromatic number or $\frac{|V(G)|}{\alpha(G)}$.

1. Introduction

For a graph G, let us write $\omega(G)$ for the *clique number*, that is, the maximum size of a clique in G, and $\chi(G)$ for the chromatic number of G. A graph is *perfect* if $\chi(H) = \omega(H)$ for all of its induced subgraphs H. We say a graph G is the *union* of graphs $G_1, \dots, G_d$ if $V(G) = V(G_1) \cup \cdots \cup V(G_d)$ and $E(G) = E(G_1) \cup \cdots \cup E(G_d)$. By taking a product coloring, if G is the union of d perfect graphs, then $\chi(G) \leq (\omega(G))^d$. This raises the natural problem of determining the optimal χ-bounding function for the class of graphs that are unions of d perfect graphs. For $d = 2$, this is a problem of Gyárfás [14, Problem 5.3] from 1985.

A *comparability graph* is a graph on a partially ordered set $(P, \preceq)$ where two vertices are adjacent if and only if they are comparable. By Dilworth's theorem [7], comparability graphs are perfect. There are natural geometric classes of disjointness graphs that are the union of comparability graphs [10]. For instance, disjointness graphs of grounded x-monotone curves are the union of 2 comparability graphs [23], disjointness graphs of arbitrary convex sets in the plane are the union of 4 comparability graphs [18], and disjointness graphs of axis-aligned boxes in $\mathbb{R}^d$ are the union of d comparability graphs [5].

We resolve Gyárfás' problem [14, Problem 5.3] in a strong sense by showing that $(\omega(G))^d$ is the optimal χ-bounding function, even if G is the union of d comparability graphs.

Theorem 1. *For positive integers k and d, there is a graph G that is the union of d comparability graphs such that $\omega(G) = k$ and $\chi(G) = k^d$.*

Theorem 1 is tight and improves previous lower bound constructions; Dumitrescu and Tóth [8] showed that there are such graphs G with $\chi(G) \geq (\omega(G))^{d(1-o(1))}$, and Pach and Tomon [23, Theorem 14] constructed unions of two comparability graphs with chromatic number $\binom{\omega(G)+1}{2}$. For further Ramsey results on the union of comparability graphs, see [3, 8, 11, 12, 15, 24].

[1]Département d'Informatique, Université libre de Bruxelles, Belgium.
[2]Department of Mathematics, Princeton University, Princeton, USA.
[3]Mathematics Institute, Utrecht University, Utrecht, the Netherlands.
[4]Institute of Mathematics, Leipzig University, Leipzig, Germany.
[5]Discrete Mathematics Group, Institute for Basic Science (IBS), Daejeon, South Korea.
[6]Department of Mathematical Sciences, KAIST, Daejeon, South Korea.
E-mail addresses: w.p.s.camesvanbatenburg@gmail.com, mchudnov@math.princeton.edu, l.j.cook@uu.nl, jgdavies@uwaterloo.ca, seokbeom@kaist.ac.kr, sangil@ibs.re.kr.
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Theorem 1 can be strengthened for *fractional chromatic number* $\chi_f(G)$ and $\frac{|V(G)|}{\alpha(G)}$, where $\alpha(G)$ denotes the maximum size of an independent set in G. Note that $\frac{|V(G)|}{\alpha(G)} \leq \chi_f(G) \leq \chi(G)$ in general.

Theorem 2. *For positive integers k and d, there is a graph G that is the union of d comparability graphs such that $\omega(G) = k$ and $\frac{|V(G)|}{\alpha(G)} = \chi_f(G) = \chi(G) = k^d$.*

2. The construction

For a positive integer k, we let $[k] = \{1, \ldots, k\}$. The *girth* of a graph is the length of its shortest cycle. Our construction uses graphs with arbitrarily large girth and chromatic number. Erdős [9] first showed that such graphs exist using the probabilistic method and Lovász [20] gave the first explicit construction. For further explicit constructions, see [1, 4, 6, 16, 17, 21, 22].

Theorem 3 (Erdős [9])**.** *For every pair of positive integers g and k, there exists a graph G of girth more than g and with $\chi(G) = k$.*

The idea of how to partition into comparability graphs is roughly based on Pach and Tomon's [23, Theorem 14] aforementioned construction of the union of 2 comparability graphs with chromatic number $\binom{\omega(G)+1}{2}$, which was probabilistic. Instead of a probabilistic approach, we also use a trick of starting with a graph with large girth and fixed chromatic number. A somewhat similar trick to how we use these graphs was also pointed out by Nešetřil for a different purpose (see [13]). Our construction is explicit and it can also be modified so that for any positive integer h, G contains no hole of length at most h.

Proof of Theorem 1. We may assume $k \geq 2$. By Theorem 3, there is a graph G_0 with girth at least $3k-2$ and chromatic number equal to k^d. We let $\phi : V(G_0) \to [k]^d$ be a fixed k^d-coloring of G_0. For each $1 \leq r \leq d$, let $(V(G_0), \preceq_r)$ be the partial order such that $u \preceq_r v$ if G_0 has a path $p_1 \cdots p_s$ for $s \geq 1$ with $p_1 = u$, $p_s = v$ such that for every $1 < i \leq s$, the first $r-1$ coordinates of $\phi(p_{i-1})$ and $\phi(p_i)$ are equal and the r-th coordinate of $\phi(p_i)$ is strictly greater than the r-th coordinate of $\phi(p_{i-1})$. Note that such a path $p_1 \cdots p_s$ forms a chain of $(V(G_0), \preceq_r)$, and that $s \leq k$. Let H_r be the comparability graph of $(V(G_0), \preceq_r)$. Then, H_r has clique number at most k since $(V(G_0), \preceq_r)$ clearly has no chain of length $k+1$. Since comparability graphs are perfect, we have that $\chi(H_r) = \omega(H_r) \leq k$. Let $G = H_1 \cup \cdots \cup H_d$. Then by considering a product coloring, we have that $\chi(G) \leq \omega(H_1) \cdots \omega(H_d) \leq k^d$. For every edge uv of G_0, we have that $u \prec_r v$ or $v \prec_r u$ for some unique $1 \leq r \leq d$, and in particular that uv is an edge of H_r. Therefore, G_0 is a subgraph of G, and so $\chi(G) \geq \chi(G_0) = k^d$. Hence, $\chi(G) = k^d$. Since $\omega(H_1), \ldots, \omega(H_d) \leq k$ and $k^d = \chi(G) \leq \omega(H_1) \cdots \omega(H_d)$, it follows that $\omega(H_1) = \cdots = \omega(H_d) = k$, and therefore that $\omega(G) \geq k$.

It remains to show that $\omega(G) \leq k$. Suppose for sake of contradiction that $\omega(G) > k$. Then G contains a triangle on vertices u, v, w such that the edges uv, uw, vw are not all contained in a single H_r. In particular, this implies without loss of generality that there exist $1 \leq a, b, c \leq d$ with a, b, c not all equal such that $u \prec_a v$, $v \prec_b w$, and either $u \prec_c w$ or $w \prec_c u$. Therefore, G_0 has paths $x_1 \cdots x_{s_1}$, $y_1 \cdots y_{s_2}$, $z_1 \cdots z_{s_3}$ such that $x_1 = u = z_{s_3}$, $x_{s_1} = v = y_1$, $y_{s_2} = w = z_1$, $x_1 \prec_a \cdots \prec_a x_{s_1}$, $y_1 \prec_b \cdots \prec_b y_{s_2}$, and either $z_1 \prec_c \cdots \prec_c z_{s_3}$ or $z_{s_3} \prec_c \cdots \prec_c z_1$. The ends of every edge of G_0 are comparable in exactly one of the partial orders $(V(G_0), \preceq_1), \ldots, (V(G_0), \preceq_d)$. Since a, b, c are not all equal, it therefore follows that one of the paths $x_1 \cdots x_{s_1}$, $y_1 \cdots y_{s_2}$, $z_1 \cdots z_{s_3}$ is edge-disjoint from the other two. As $s_1, s_2, s_3 \leq k$, it therefore follows that the union of these three paths contains a cycle of length at most $3k-3$. Since this cycle is contained in G_0, this contradicts our choice of G_0 having girth at least $3k-2$. Hence $\omega(G) = k$, as desired. $\square$

To prove Theorem 2, we use the following strengthening of Theorem 3, due to Lin and Zhu [19]. Although their result explicitly states that $\chi_f(G) = \chi(G) = h$, the construction in their proof also yields $\frac{|V(G)|}{\alpha(G)} = h$. For completeness, we include an alternative proof in the appendix.

Theorem 4. *For every pair of positive integers g and h, there exists a graph $H = H_{g,h}$ such that the girth of H is more than g and $\frac{|V(H)|}{\alpha(H)} = \chi_f(H) = \chi(H) = h$.*

Proof of Theorem 2. Let $G_0 = H_{3k-2,k^d}$ be the graph given by Theorem 4. Since adding edges does not decrease the inverse independence ratio, we obtain a graph G that is the union of d comparability graphs such that $\omega(G) = k$ and $\frac{|V(G)|}{\alpha(G)} = k^d$ by the proof of Theorem 1 starting with G_0. □

We leave as an open problem to find a constructive proof for Theorem 4. This would immediately yield a construction for Theorem 2 by substituting the construction from Theorem 4 into the proof of Theorem 2.

Tool and computational resource disclosure. The authors used ChatGPT 5.5 Pro to find a proof of Pach and Tomon [23, Theorem 14] without using the probabilistic method, which allowed the authors to simplify the proof in the present paper.

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Appendix A. Graphs of large girth and prescribed Hall ratio

We present a direct proof of Theorem 4. Let us denote by $\mathrm{girth}(G)$ the girth of G. We will prove the following lemma.

Lemma 5. *Let $h \geq 3$ and $g \geq 1$ be fixed integers. Then there exist an integer $m \geq 1$ and an h-partite graph G with parts $V_1, \ldots, V_h$, all of the same size m, such that the girth of G is more than g and $\alpha(G) = m$.*

Assuming Lemma 5, we can prove Theorem 4.

Proof of Theorem 4 assuming Lemma 5. Let $H_{g,1} = K_1$ and let $H_{g,2}$ be an even cycle of length greater than g. For $h \geq 3$, by Lemma 5, we can choose $H = H_{g,h}$ to be an h-partite graph such that all its parts have size m, its girth is greater than g, and $\alpha(H) = m$. Then we have $h = |V(H)|/\alpha(H) \leq \chi_f(H) \leq \chi(H) \leq h$. This proves Theorem 4. □

Proof of Lemma 5. Fix an integer Λ so that $e^{\Lambda} > 2e^{2h}$ and $2^{\Lambda} > 4e^2h$. For a sufficiently large integer m to be specified later, let $V_i = \{(i,x) : x \in [m]\}$ for each $i \in [h]$. We will construct a random subgraph G of the complete multipartite graph with parts $V_1, V_2, \ldots, V_h$, as follows. For every ordered pair (i,j) such that $i, j \in [h]$ and $i \neq j$, every $\lambda \in [\Lambda]$, and every $x \in [m]$, choose $f_{ij}^{\lambda}(x) \in [m]$ independently and uniformly, add a directed edge from (i,x) to $(j, f_{ij}^{\lambda}(x))$, and label the directed edge with λ. In other words, we draw Λ random labelled directed edges from (i,x) to V_j, where parallel edges are allowed. This yields a directed multigraph Γ. After this, construct G from Γ by putting an undirected edge between two vertices whenever at least one labelled directed edge joins them, ignoring multiple edges. This yields a simple h-partite graph G on mh vertices.

We first show that, with probability bounded below by a positive constant depending only on h, Λ, and g, there is no cycle of length at most g in G. For $3 \leq \ell \leq g$, consider a potential labelled oriented cycle of length ℓ, by which we mean ℓ distinct vertices $(i_1, x_1), \ldots, (i_\ell, x_\ell)$ with $i_t \neq i_{t+1}$ cyclically, together with one of the two directions and a label $\lambda_t \in [\Lambda]$ between each consecutive pair of vertices. A *bad event* is that all these labelled directed edges occur in Γ. Observe that every actual cycle of G of length at most g originates from at least one such bad event.

Let B be a fixed potential labelled oriented cycle of length ℓ in Γ. Since B involves ℓ distinct random variables, it holds that $\mathbb{P}(B) \leq m^{-\ell}$. In addition, the number of potential labelled oriented cycles of length ℓ is at most a constant multiple of m^ℓ. Indeed, to specify such a potential labelled cycle, we have at most h^ℓ choices for the ℓ parts, at most m^ℓ choices for the vertices $x_1, \ldots, x_\ell$, and 2Λ choices for the direction and the label for each consecutive pair. This shows that there are at most $h^\ell m^\ell (2\Lambda)^\ell = O_{h,\Lambda,\ell}(m^\ell)$ potential labelled oriented cycles of length ℓ in G.

We use the standard asymmetric Lovász local lemma (see, for example, [2, Chapter 5]). Let $\mathcal{B}$ be a family of bad events with a dependency graph. If there are numbers $0 < x_B < 1$ such that $\mathbb{P}(B) \leq x_B \prod_{B' \sim B}(1 - x_{B'})$ for every $B \in \mathcal{B}$, then $\mathbb{P}(\bigcap_{B \in \mathcal{B}} \overline{B}) \geq \prod_B (1 - x_B)$.

For a bad event B of length ℓ, set $x_B = 2m^{-\ell}$. Two bad events are dependent only if they use a common random variable $f_{ij}^{\lambda}(x)$. A fixed bad event shares a random variable with only $O_{h,\Lambda,g}(m^{r-1})$ bad events of length r, and therefore $\sum_{B' \sim B} x_{B'} = \sum_{r=3}^{g} O_{h,\Lambda,g}(m^{r-1}) \cdot 2m^{-r} = O_{h,\Lambda,g}(m^{-1})$. Thus, for all sufficiently large m, $\prod_{B' \sim B}(1 - x_{B'}) \geq 1 - \sum_{B' \sim B} x_{B'} \geq 1/2$. Hence $\mathbb{P}(B) \leq m^{-\ell} \leq x_B \prod_{B' \sim B}(1 - x_{B'})$, so the conditions of the local lemma are satisfied. Since $\sum_B x_B = O_{h,\Lambda,g}(1)$ and $\max_B x_B = o(1)$, the local lemma gives $\mathbb{P}(\bigcap_B \overline{B}) \geq \prod_B (1 - x_B) \geq \prod_B e^{-2x_B} > 0$, and hence $\mathbb{P}(\mathrm{girth}(G) > g) \geq c > 0$, for all sufficiently large m. Here, c is some constant independent of m.

It remains to show that $\mathbb{P}(\alpha(G) > m) \to 0$ as $m \to \infty$. Let $A_i \subseteq V_i$, put $a_i = |A_i|$, and set $A = A_1 \cup \cdots \cup A_h$ and $a = a_1 + \cdots + a_h$. Suppose $a = m + 1$. For fixed $A_1, \ldots, A_h$, independence of A requires every directed edge from A_i to V_j to avoid A_j, for all $i \neq j$, and hence $\mathbb{P}(A \text{ is independent}) = \prod_{i \neq j}(1 - a_j/m)^{\Lambda a_i} \leq \prod_{i \neq j} \exp(-\frac{a_j}{m} \cdot \Lambda a_i) = \exp(-\frac{\Lambda}{m} \sum_{i \neq j} a_i a_j)$.

First, assume $a_i \leq m/2$ for every i. We have $\sum_i a_i^2 \leq \frac{m}{2}\sum_i a_i = \frac{ma}{2}$, and hence $\sum_{i\neq j} a_i a_j = a^2 - \sum_i a_i^2 \geq a^2 - \frac{ma}{2} > \frac{m^2}{2}$. Therefore, $\mathbb{P}(A \text{ is independent}) \leq \exp(-\frac{\Lambda m}{2})$. Since there are at most 2^{hm} choices for A, the probability of having an independent set A with $|A| = m+1$ and $|A \cap V_i| \leq m/2$ for all i is at most $2^{hm}\exp(-\frac{\Lambda m}{2}) = o(1)$, by our choice of Λ.

Now, assume instead that $a_i > m/2$ for some i. By symmetry it suffices to handle the case $a_1 > m/2$. If A is independent, every random edge directed from a selected vertex outside V_1 into V_1 must land in the complement $V_1 \setminus A_1$, which has size $m - a_1$. Hence, $\mathbb{P}(A \text{ is independent}) \leq (\frac{m-a_1}{m})^{\Lambda(a-a_1)}$. The case $a_1 = m$ has probability 0, so we can ignore it from now on. Fix $\frac{m}{2} < a_1 < m$. There are $\binom{m}{a_1}$ choices for A_1. Summing over all $\binom{(h-1)m}{a-a_1}$ possible choices of the remaining $a - a_1$ vertices outside V_1 gives an upper bound

$$\binom{m}{a_1}\binom{(h-1)m}{a-a_1}\left(\frac{m-a_1}{m}\right)^{\Lambda(a-a_1)} \tag{1}$$

on the probability that an independent set A with $|A \cap V_1| = a_1$ and $|A| = m+1$ exists. Now, we deduce that

$$\begin{aligned}
\binom{m}{a_1}\binom{(h-1)m}{a-a_1}\left(\frac{m-a_1}{m}\right)^{\Lambda(a-a_1)} &\leq \left(\frac{em}{m-a_1}\right)^{m-a_1}\left(\frac{e(h-1)m}{a-a_1}\right)^{a-a_1}\left(\frac{m-a_1}{m}\right)^{\Lambda(a-a_1)} \\
&\leq \left(\frac{em}{m-a_1}\right)^{m-a_1}\left(\frac{ehm}{m-a_1}\right)^{a-a_1}\left(\frac{m-a_1}{m}\right)^{\Lambda(a-a_1)} \\
&= \left(\frac{em}{m-a_1}\right)^{m-a_1}\left(\frac{ehm}{m-a_1}\right)^{m-a_1+1}\left(\frac{m-a_1}{m}\right)^{\Lambda(m-a_1+1)} \\
&= eh(e^2h)^{m-a_1}\left(\frac{m-a_1}{m}\right)^{(\Lambda-2)(m-a_1)}\left(\frac{m-a_1}{m}\right)^{\Lambda-1} \\
&\leq eh(e^2h)^{m-a_1}\left(\frac{1}{2}\right)^{(\Lambda-2)(m-a_1)}\left(\frac{m-a_1}{m}\right)^{\Lambda-1}.
\end{aligned}$$

Summing over all $m/2 < a_1 < m$ gives that the probability of having an independent set A of size $m+1$ with $|A \cap V_1| > m/2$ is at most

$$ehm^{1-\Lambda}\sum_{m/2<a_1<m}(m-a_1)^{\Lambda-1}(e^2h2^{2-\Lambda})^{m-a_1}. \tag{2}$$

Let $\theta = e^2h2^{2-\Lambda}$. Then $\theta < 1$ by the choice of Λ, so the sum over a_1 in (2) is bounded from above by the convergent series $\sum_{k\geq 1} k^{\Lambda-1}\theta^k$. Thus, the quantity in (2) is $O_{h,\Lambda}(m^{1-\Lambda})$, and hence tends to 0 as $m \to \infty$. Multiplying by h to account for the h possible choices of an index i for which $a_i > m/2$, it follows that the probability of having an independent set of size $m+1$ with $a_i > m/2$ for some i is $o(1)$. Thus, with probability tending to 1, there is no independent set of size greater than m. Since each part V_i is independent, we have $\alpha(G) \geq m$, and therefore $\mathbb{P}(\alpha(G) = m) \to 1$ as $m \to \infty$.

Combining the estimates, for sufficiently large m, we have $\mathbb{P}(\mathrm{girth}(G) > g \text{ and } \alpha(G) = m) \geq c - o(1) > 0$. Thus, for every sufficiently large m, there is a desired h-partite graph G. □