Families of varieties of general type

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CHAPTER 1

Introduction

The moduli spaces of smooth projective curves of genus $g \geq 2$, and their compactifications by the moduli space of stable projective curves of genus g are, quite possibly, the most studied of all algebraic varieties.

The aim of this book is to generalize the moduli theory of curves to surfaces and to higher dimensional varieties. In the introduction we start to outline how this is done, and, more importantly, to explain why the answer for surfaces is much more complicated than for curves. On the positive side, once we get the moduli theory of surfaces right, the higher dimensional theory works the same.

Section 1.1 is a quick review of the history of moduli problems, culminating in an outline of the basic moduli theory of curves. Section 1.2 introduces canonical models, which are the basic objects of moduli theory in higher dimensions. Starting from stable curves, Section 1.3 leads up to the definition of stable varieties, their higher dimensional analogs. Then we show, by a series of examples, why flat families of stable varieties are *not* the correct higher dimensional analogs of flat families of stable curves. Finding the correct replacement has been one of the main difficulties of the whole theory.

Next we give a collection of examples showing how easy it is to end up with rather horrible moduli problems. Hypersurfaces are discussed in Section 1.4 and alternate compactification of the moduli of curves in Section 1.5. Further interesting examples are given in Section 1.6 while Section 1.7 illustrates the differences between fine and coarse moduli spaces.

In Section 1.8 we recall the most important definitions and results about singularities that occur on stable varieties.

An overview of the moduli theory of higher dimensional varieties is given in [**Kol13b**].

1.1. Short history of moduli problems

Let $\mathbf{V}$ be a “reasonable” class of objects in algebraic geometry, for instance, $\mathbf{V}$ could be all subvarieties of $\mathbb{P}^n$, all coherent sheaves on $\mathbb{P}^n$, all smooth curves or all projective varieties. The aim of the theory of moduli is to understand all “reasonable” families of objects in $\mathbf{V}$ and to construct an algebraic variety (or scheme, or algebraic space) whose points are in “natural” one-to-one correspondence with the objects in $\mathbf{V}$. If such a variety exists, we call it the *moduli space* of $\mathbf{V}$ and denote it by $M_{\mathbf{V}}$. The simplest, classical examples are given by the theory of linear systems and families of linear systems.

1.1 (Linear systems). Let X be a normal projective variety over an algebraically closed field k and L a line bundle on X. The corresponding linear system is

$$\mathcal{L}in\mathcal{S}ys(X, L) = \{\text{effective divisors } D \text{ such that } \mathcal{O}_X(D) \cong L\}.$$

The objects in $LinSys(X, L)$ are in natural one-to-one correspondence with the points of the projective space $\mathbb{P}\big(H^0(X, L)^\vee\big)$ which is classically denoted by $|L|$. Thus, for every effective divisors D such that $\mathcal{O}_X(D) \cong L$ there is a unique point $[D] \in |L|$.

Moreover, this correspondence between divisors and points is given by a universal family of divisors over $|L|$. That is, there is an effective Cartier divisor $\mathrm{Univ}_L \subset |L| \times X$ with projection $\pi : \mathrm{Univ}_L \to |L|$ such that

$$\pi^{-1}\big([D]\big) = D$$

for every effective divisor D linearly equivalent to L,

The classical literature never differentiates between the linear system as a set and the linear system as a projective space. There are, indeed, few reasons to distinguish them as long as we work over a fixed base field k. If, however, we pass to a field extension $K \supset k$, the advantages of viewing $|L|$ as a k-variety appear. For any $K \supset k$, the set of effective divisors D defined over K such that $\mathcal{O}_X(D) \cong L$ corresponds to the K-points of $|L|$. Thus the scheme theoretic version automatically gives the right answer over every field.

1.2 (Jacobians of curves). Let C be a smooth projective curve (or Riemann surface) of genus g. As discovered by Abel and Jacobi, there is a variety $\mathrm{Jac}^0(C)$ of dimension g whose points are in natural one-to-one correspondence with degree 0 line bundles on C. As before, the correspondence is given by a universal line bundle $L^{\mathrm{univ}} \to C \times \mathrm{Jac}^0(C)$, called the Poincaré bundle, That is, for any point $p \in \mathrm{Jac}^0(C)$, the restriction of L^{univ} to $C \times \{p\}$ is the degree 0 line bundle corresponding to p.

A somewhat subtle point is that, unlike in (1.1), the universal line bundle L^{univ} is not unique (and need not exist if the base field is not algebraically closed). This has to do with the fact that while a divisor $D \subset X$ has no automorphisms fixing X, any line bundle $L \to C$ has automorphisms that fix C: we can multiply every fiber of L by the same nonzero constant.

1.3 (Chow varieties). Historically the next to emerge was the theory of Chow varieties, though it is a rather difficult moduli problem. It was defined by [**Cay62**] for curves in $\mathbb{P}^3$. See Section 3.1 for an outline, [**HP47**] for a classical introduction and [**Kol96**, Secs.I.3–4] for a more recent treatment.

Let k be an algebraically closed field and X a normal, projective k-variety. Fix a natural number m. An *m-cycle* on X is a finite, formal linear combination $\sum a_i Z_i$ where the Z_i are irreducible, reduced subvarieties of dimension m and $a_i \in \mathbb{Z}$. We usually assume tacitly that all the Z_i are distinct. An m-cycle is called *effective* if $a_i \geq 0$ for every i.

Let $Y \subset X$ be a closed subscheme of dimension m. Let $Y_i \subset Y$ be its m-dimensional irreducible components, $Z_i := \operatorname{red} Y_i$ and $y_i \in Y_i$ the generic point. Let a_i be the length of the Artin ring $\mathcal{O}_{y_i, Y_i}$. We define the *fundamental cycle* of Y as $[Y] := \sum a_i Z_i$. Thus the fundamental cycle ignores lower dimensional associated primes and from the m-dimensional components it keeps only the underlying reduced variety and the length at the generic points.

It turns out that there is a k-variety $\mathrm{Chow}_m(X)$, called the *Chow variety* of X whose points are in "natural" one-to-one correspondence with the set of effective m-cycles on X. (Since we did not fix the degree of the cycles, $\mathrm{Chow}_m(X)$ is not actually a variety but a countable disjoint union of projective, reduced k-schemes.)

The point of $\mathrm{Chow}_m(X)$ corresponding to a cycle $Z = \sum a_i Z_i$ is also usually denoted by $[Z]$.

As for linear systems, it is best to describe the "natural correspondence" by a universal family. The situation is, however, more complicated than before.

There is a family (or rather an effective cycle) $\mathrm{Univ}_m(X)$ on $\mathrm{Chow}_m(X) \times X$ with projection $\pi : \mathrm{Univ}_m(X) \to \mathrm{Chow}_m(X)$ such that for every effective m-cycle $Z = \sum a_i Z_i$,

(1) the support of $\pi^{-1}\big([Z]\big)$ is $\sum Z_i$, and
(2) the fundamental cycle of $u^{-1}\big([Z]\big)$ equals Z if $a_i = 1$ for every i.

If the characteristic of k is 0, then the only problem in (2) is a clash between the traditional cycle-theoretic definition of the Chow variety and the scheme-theoretic definition of the fiber. It is easy to define a cycle-theoretic notion of fiber that restores equality in (2) for every Z; see [**Kol96**, I.3]. In positive characteristic the situation is more problematic; a possible solution is described in [**Kol96**, I.4].

The example of a "perfect" moduli problem is the theory of *Hilbert schemes*, introduced in [**Gro62b**]. See [**Mum66**], [**Kol96**, I.1–2] or [**Ser06**, Sec.4.3] for detailed treatments and Section 3.1 for a summary.

1.4 (Hilbert schemes). Let k be an algebraically closed field and X a projective k-scheme. Set

$$\mathcal{H}ilb(X) = \{\text{closed subschemes of } X\}.$$

Then there is a k-scheme $\mathrm{Hilb}(X)$, called the *Hilbert scheme* of X whose points are in a "natural" one-to-one correspondence with closed subschemes of X. The point of $\mathrm{Hilb}(X)$ corresponding to a subscheme $Y \subset X$ is frequently denoted by $[Y]$. There is a universal family $\mathrm{Univ}(X) \subset \mathrm{Hilb}(X) \times X$ such that

(1) the first projection $\pi : \mathrm{Univ}(X) \to \mathrm{Hilb}(X)$ is flat, and
(2) $\pi^{-1}\big([Y]\big) = Y$ for every closed subscheme $Y \subset X$.

The beauty of the Hilbert scheme is that it describes not just subschemes but all flat families of subschemes as well. To see what this means, note that for any morphism $g : T \to \mathrm{Hilb}(X)$, by pull-back we obtain a flat family of subschemes of X parametrized by T

$$T \times_{g,\mathrm{Hilb}(X)} \mathrm{Univ}(X) \subset T \times X.$$

It turns out that every family is obtained this way:

(3) For every T and for every closed subscheme $Z_T \subset T \times X$ that is flat and proper over T, there is a unique $g : T \to \mathrm{Hilb}(X)$ such that
$$Z_T = T \times_{g,\mathrm{Hilb}(X)} \mathrm{Univ}(X).$$

This takes us to the next, functorial approach to moduli problems.

1.5 (Hilbert functor and Hilbert scheme). Let $X \to S$ be a morphism of schemes. Define the *Hilbert functor* of X/S as a functor that associates to a scheme $T \to S$ the set

$$\mathcal{H}ilb_{X/S}(T) = \big\{\text{subschemes } Z \subset T \times_S X \text{ that are flat and proper over } T\big\}.$$

The basic existence theorem of Hilbert schemes then says that, if $X \to S$ is quasi-projective, there is a scheme $\mathrm{Hilb}_{X/S}$ such that for any S scheme T,

$$\mathcal{H}ilb_{X/S}(T) = \mathrm{Mor}_S\big(T, \mathrm{Hilb}_{X/S}\big).$$

Moreover, there is a universal family $\pi : \mathrm{Univ}_{X/S} \to \mathrm{Hilb}_{X/S}$ such that the above isomorphism is given by pulling back the universal family.

We can summarize these results as follows

PRINCIPLE 1.6. *$\pi : \mathrm{Univ}_{X/S} \to \mathrm{Hilb}_{X/S}$ contains all the information about proper, flat families of subschemes of X/S and does it in the most succinct way.*

This example leads us to a general definition:

DEFINITION 1.7 (Fine moduli spaces). Let $\mathbf{V}$ be a "reasonable" class of projective varieties (or schemes, or sheaves, or ...). In practice "reasonable" may mean several restrictions, but for the definition we only need the following weak assumption:

(1) Let $K \supset k$ be a field extension. Then a k-variety X_k is in $\mathbf{V}$ iff $X_K := X_k \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K$ is in $\mathbf{V}$.

Following (1.5), define the corresponding moduli functor as

$$\mathcal{V}\!\mathit{arieties}_{\mathbf{V}}(T) := \left\{ \begin{array}{c} \text{Flat families } X \to T \text{ such that} \\ \text{every fiber is in } \mathbf{V}, \\ \text{modulo isomorphisms over } T. \end{array} \right\} \tag{1.7.2}$$

We say that a scheme $\mathrm{Moduli}_{\mathbf{V}}$, or, more precisely, a flat morphism

$$u : \mathrm{Univ}_{\mathbf{V}} \to \mathrm{Moduli}_{\mathbf{V}}$$

is a *fine moduli space* for the functor $\mathcal{V}\!\mathit{arieties}_{\mathbf{V}}$ if the following holds:

(3) For every scheme T, pulling back gives an equality

$$\mathcal{V}\!\mathit{arieties}_{\mathbf{V}}(T) = \mathrm{Mor}\big(T, \mathrm{Moduli}_{\mathbf{V}}\big).$$

Applying the definition to $T = \mathrm{Spec}\, K$, where K is a field, we see that every fiber of $u : \mathrm{Univ}_{\mathbf{V}} \to \mathrm{Moduli}_{\mathbf{V}}$ is in $\mathbf{V}$ and the K-points of the fine moduli space $\mathrm{Moduli}_{\mathbf{V}}$ are in one-to-one correspondence with the K-isomorphism classes of objects in $\mathbf{V}$.

We consider the existence of a fine moduli space as the ideal possibility. Unfortunately, it is rarely achieved.

1.8 (Remarks on flatness). The definition (1.7) is very natural within our usual framework of algebraic geometry, but it hides a very strong supposition:

Assumption 1.8.1. If $\mathbf{V}$ is a "reasonable" class then any flat family whose fibers are in $\mathbf{V}$ is a "reasonable" family.

In Grothendieck's foundations of algebraic geometry flatness is one of the cornerstones and there are many "reasonable" classes for which flat families are indeed the "reasonable" families. Nonetheless, (1.8.1) should not be viewed as self evident.

Even when the base of the family is a smooth curve, (1.8.1) needs arguing, but the assumption is especially surprising when applied to families over non-reduced schemes T. Consider, for instance, the case when T is the spectrum of an Artinian k-algebra. Then T has only one closed point $t \in T$. A flat family $p : X \to T$ has only one fiber X_t, and our only restriction is that X_t be in our class $\mathbf{V}$. Thus (1.8.1) declares that we care only about X_t. Once X_t is in $\mathbf{V}$, every flat deformation of X_t over T is automatically "reasonable."

A crucial conceptual point in the moduli theory of higher dimensional varieties is the realization that in (1.7) flatness of the map $X \to T$ is not enough: allowing all flat families whose fibers are in a "reasonable" class leads to the wrong moduli problem. Problems arise even for families of surfaces over smooth curves.

The difficulty of working out the correct concept has been one of the main stumbling blocks of the general theory.

Next we see what happens with the simplest case, for smooth curves of fixed genus.

1.9 (Moduli functor and moduli space of smooth curves). Following (1.7) we define the moduli functor of smooth curves of genus g as

$$\mathcal{C}urves_g(T) := \left\{ \begin{array}{c} \text{Smooth, proper families } S \to T, \\ \text{every fiber is a curve of genus } g, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

It turns out that there is no fine moduli space for curves of genus g. Every curve C with nontrivial automorphisms causes problems; there can not be any point $[C]$ corresponding to it in a fine moduli space. Actually, problems arise already when $\mathbf{V}$ consist of a single curve! See Section 1.7 for such examples.

It has been, however, understood for a long time that there is some kind of an object, denoted by M_g, and called the *coarse moduli space* (or simply *moduli space*) of curves of genus g that comes close to being a fine moduli space:

(1) For any algebraically closed field k, the k-points of M_g are in a "natural" one-to-one correspondence with isomorphism classes of smooth curves of genus g defined over k. Let us denote the correspondence by $C \mapsto [C] \in M_g$.
(2) For any family of smooth genus g curves $h : S \to T$ there is a "moduli map" $m_{h,T} : T \to M_g$ such that for every geometric point $p \in T$, the image $m_{h,T}(p)$ is the point corresponding to the fiber $[h^{-1}(p)]$.

For elliptic curves we get $M_1 = \mathbb{A}^1$ and the moduli map is given by the j-invariant, as was known to Euler and Lagrange. They also knew that there is no universal family over M_1. The theory of Abelian integrals due to Abel, Jacobi and Riemann does essentially the same for all curves, though in this case a clear moduli theoretic interpretation seems to have been done only later [**?**]. For smooth plane curves, and more generally for smooth hypersurfaces in any dimension, the invariant theory of Hilbert produces coarse moduli spaces. Still, a precise definition and proof of existence of M_g appeared only in [**Tei44**] in the analytic case and in [**Mum65**] in the algebraic case. See [**AJP16**] for a historical account.

1.10 (Coarse moduli spaces). As in (1.7), let $\mathbf{V}$ be a "reasonable" class. When there is no fine moduli space, we still can ask for a scheme that best approximates its properties.

We look for schemes M for which there is a natural transformation of functors

$$T_M : \mathcal{V}arieties_g(*) \longrightarrow \operatorname{Mor}(*, M).$$

Such schemes certainly exist, for instance, if we work over a field k then we can take $M = \operatorname{Spec} k$. All schemes M for which T_M exists form an inverse system which is closed under fiber products. Thus, as long as we are not unlucky, there is a universal (or largest) scheme with this property. Though it is not usually done, it should be called the *categorical moduli space.*

This object can be rather useless in general. For instance, fix n, d and let $\mathbf{H}_{n,d}$ be the class of all hypersurfaces of degree d in $\mathbb{P}_k^{n+1}$ up to isomorphisms. It is easy to see (cf. (1.52)) that a categorical moduli space exists and it is $\operatorname{Spec} k$.

To get something more like a fine moduli space, we require that it give a one-to-one parametrization, at least set theoretically. Thus we say that a scheme $\mathrm{Moduli}_{\mathbf{V}}$ is a *coarse moduli space* for $\mathbf{V}$ if the following hold.

(1) There is a natural transformation of functors

$$\mathrm{ModMap} : \mathcal{V}\!\mathit{arieties}_{\mathbf{V}}(*) \longrightarrow \mathrm{Mor}(*, \mathrm{Moduli}_{\mathbf{V}}),$$

(2) $\mathrm{Moduli}_{\mathbf{V}}$ is universal satisfying (1), and

(3) for any algebraically closed field $K \supset k$,

$$\mathrm{ModMap} : \mathcal{V}\!\mathit{arieties}_{\mathbf{V}}(\operatorname{Spec} K) \xrightarrow{\cong} \mathrm{Mor}(\operatorname{Spec} K, \mathrm{Moduli}_{\mathbf{V}}) = \mathrm{Moduli}_{\mathbf{V}}(K)$$

is an isomorphism (of sets).

1.11 (Moduli functors versus moduli spaces). While much of the early work on moduli, especially since [**Mum65**], put the emphasis on the construction of fine or coarse moduli spaces, recently the emphasis shifted towards the study of the families of varieties, that is towards moduli functors and moduli stacks. The main task is to understand what kind of objects form “nice” families. Once a good concept of “nice families” is established, the existence of a coarse moduli space should be nearly automatic. The coarse moduli space is not the fundamental object any longer, rather it is only a convenient way to keep track of certain information that is only latent in the moduli functor or moduli stack.

1.12 (Compactifying M_g). While the basic theory of algebraic geometry is local, that is, it concerns affine varieties, most really interesting and important objects in algebraic geometry and its applications are global, that is, projective or at least proper.

The moduli spaces M_g are not compact, in fact the moduli functor of smooth curves discussed so far has a definitely local flavor. Most naturally occurring smooth families of curves live over affine schemes, and it is not obvious how to write down any family of smooth curves over a projective base. For many reasons it is useful to find geometrically meaningful compactifications of M_g. The answer to this situation is to allow not just smooth curves but also singular curves in our families.

Concentrating on 1-parameter families, the main question is the following:

(1.12.1) *Let B be a smooth curve, $B^0 \subset B$ an open subset and $\pi^0 : S^0 \to B^0$ a smooth family of genus g curves. Find a “natural” extension*

$$\begin{array}{rcl} S^0 & \subset & S \\ \pi^0 \downarrow & & \downarrow \pi \\ B^0 & \subset & B, \end{array}$$

where $\pi : S \to B$ is a flat family of (possibly singular) curves.

We would like the extension to be unique and behave well with respect to pulling back families over curves and for families over higher dimensional bases.

The answer, proposed in [**DM69**] has been so successful that it is hard to imagine a time when it was not the “obvious” solution. Let us first review the definition of [**DM69**]. In Section 1.4 we see, by examples, why this concept has not been so obvious.

DEFINITION 1.13 (Stable curve). A *stable curve* over an algebraically closed field k is a proper, connected k-curve C such that the following hold:

(Local property) The only singularities of C are ordinary nodes.

(Global property) The canonical (or dualizing) sheaf ω_C is ample.

A *stable curve* over a scheme T is a flat, proper morphism $\pi : S \to T$ such that every geometric fiber of π is a stable curve. (The arithmetic genus of the fibers is a locally constant function on T, but we usually also tacitly assume that it is constant.)

The moduli functor of stable curves of genus g is

$$\overline{\mathcal{C}urves}_g(T) := \left\{ \begin{array}{c} \text{Stable curves of genus } g \text{ over } T, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

THEOREM 1.14. [**DM69**] *For every $g \geq 2$, the moduli functor of stable curves of genus g has a coarse moduli space $\bar{M}_g$. Moreover, $\bar{M}_g$ is projective, normal, has only quotient singularities and contains M_g as an open dense subset.*

$\bar{M}_g$ has a rich and intriguing intrinsic geometry which is related to major questions in many branches of mathematics and theoretical physics; see [**FM13**] for a collection of surveys.

1.15 (Moduli for varieties of general type).

The aim of this book is to use the moduli of stable curves as guideline, and develop a moduli theory for varieties of general type. (For the non-general type case, see (1.23).)

In some sense, this is a hopeless task since higher dimensional varieties are much more complicated than curves. For instance, even for smooth surfaces with ample canonical class, the moduli spaces can have arbitrarily complicated singularities and scheme structures [**Vak06**]. Thus we approach the question in four stages:

(1) Develop the correct higher dimensional analog of smooth, projective curves of genus ≥ 2.
(2) Following the example of stable curves, define the notion of "stable" varieties in higher dimensions.
(3) Show that the functor of "stable" varieties with suitably fixed numerical invariants gives a well behaved moduli functor/stack and has a projective coarse moduli space.
(4) Show that, in many important cases, these moduli spaces are interesting and useful objects.

Let us now see in some detail how these goals are accomplished.

1.16 (Higher dimensional analogs of smooth curves of genus ≥ 2). It has been understood since the beginnings of the theory of surfaces that, for surfaces of Kodaira dimension ≥ 0, the correct moduli theory should be birational, not biregular. That is, the points of the moduli space should correspond not to *isomorphism* classes of surfaces but to *birational* equivalence classes of surfaces. There are two ways to deal with this problem.

First, one can work with smooth families but consider two families equivalent of there is a rational map between them that induces a birational equivalence on every fiber. This seems rather complicated technically.

The second, much more useful method relies on the observation that every birational equivalence class of surfaces of Kodaira dimension ≥ 0 contains a unique *minimal model*, that is, a smooth projective surface S^m whose canonical class is nef. Therefore, one can work with families of minimal models, modulo isomorphisms. With the works of [**Mum65, Art74**] it became clear that, for surfaces of general type, it is even better to work with the *canonical model*, which is a mildly singular projective surface S^c whose canonical class is ample. The resulting class of singularities has been since established in all dimensions; they are called *canonical singularities* (1.35). See Section 1.2 for details.

Principle 1.16.1. In moduli theory, the main objects of study are projective varieties with ample canonical class and with canonical singularities.

The correct definition of the higher dimensional analogs of stable curves was much less clear. An approach through geometric invariant theory was investigated [**Mum77**], but never fully developed. In essence, the GIT approach starts with a particular method of construction of moduli spaces and then tries to see for which class of varieties does it work. The examples of [**WX14**] suggest that geometric invariant theory is unlikely to give a good compactification for the moduli of surfaces.

A different framework was proposed in [**KSB88**]; see also [**Ale96**]. Instead of building on geometric invariant theory, it focuses on 1-parameter families and uses Mori's program as its basic tool. Before we give the definition, it is very helpful to go through a key step of the proof of (1.14) that establishes separatedness and properness of $\bar{M}_g$. Keeping in mind the valuative criteria of separatedness and properness (1.21.1–2.), we expect the difficulties to be essentially 1-dimensional. This is the topic of the next theorem.

THEOREM 1.17 (Stable reduction for curves). *Let B be a smooth curve, $B^0 \subset B$ an open subset and $\pi^0 : S^0 \to B^0$ a flat family of genus g stable curves. Then there is a finite surjection $p : A \to B$ such that there is a unique extension*

$$\begin{array}{ccccc} S^0 \times_B A & =: & T^0 & \subset & T \\ \downarrow & & \pi^0_A \downarrow & & \downarrow \pi_A \\ B^0 \times_B A & =: & A^0 & \subset & A, \end{array}$$

where $\pi_A : T \to A$ is a flat family of genus g stable curves.

1.18 (Outline of proof of (1.17)). Let us present the process in a way that generalizes to higher dimensions.

Main case 1.18.1. The generic fiber of $\pi^0 : S^0 \to B^0$ is smooth.

Step 1.1. Take any (possibly singular) projective surface $S_1 \supset S^0$ such that π^0 extends to a morphism $\pi_1 : S_1 \to B$.

Step 1.2. Resolve the singularities of S_1 to obtain a smooth surface $\pi_2 : S_2 \to B$ such that the reduced fibers of π_2 have only nodes as singularities.

Step 1.3. Run the relative minimal model program. That is, repeatedly contract all smooth rational curves $C \subset S_2$ that are contained in a fiber of π_2 and have negative intersection with the canonical class. The end result is $\pi_3 : S_3 \to B$ where K_{S_3} has non-negative degree on all curves contained in any fiber of π_3.

Step 1.4. Take the relative canonical model. That is, contract all smooth rational curves $C \subset S_3$ that are contained in a fiber of π_3 and have zero intersection with the canonical class. The end result is $\pi_4 : S_4 \to B$ where K_{S_4} has positive degree on all curves contained in any fiber of π_3. Thus K_{S_4} is relatively ample.

Note that S_4 is, in general, not smooth, but has very simple (so called Du Val) singularities.

Step 1.5. Prove that $\pi_4 : S_4 \to B$ is the unique surface containing S^0 that has Du Val singularities and relatively ample canonical class.

Step 1.6. In general, the fibers of π_4 are not reduced and the construction of S_4 does not commute with base change $p : A \to B$. However, if the fibers of π_2 are reduced, then the fibers of π_4 are stable curves and the construction of S_4 does commute with base change. (Assuming only that the fibers of π_4 be reduced would not be enough.)

Step 1.7. Show that if $p : A \to B$ is sufficiently ramified and $T^0 := S^0 \times_B A$ then the analogously constructed $T := T_4 \to A$ satisfies the requirements of (1.17). (Just to be concrete, in characteristic 0, the following ramification condition is sufficient: For every $a \in A$, the ramification index of p at a is divisible by the multiplicity of every irreducible component of $\pi_2^{-1}\bigl(p(a)\bigr)$.)

Secondary case 1.18.2. The generic fiber of $\pi^0 : S^0 \to B^0$ is not normal.

Step 2.0. The generic fiber of $\pi^0 : S^0 \to B^0$ has nodes, and, correspondingly, S^0 has normal crossing singularities along a curve $C^0 \subset S^0$. Let $\bar{S}^0 \to S^0$ be the normalization and $D^0 \subset \bar{S}^0$ the preimage of the double curve. We also keep track of the involution τ^0 of the degree 2 cover $D^0 \to C^0$.

Steps 2.1–7. Run the analog of Steps 1.1–7 for $\bar{S}^0 \to B^0$, with the difference of using

$$\text{(canonical class)} + \text{(birational transform of } D^0\text{)}$$

everywhere instead of the canonical class. The end result is $\pi_T : \bar{T} \to A$ with $D_T \subset T$ the curve corresponding to D^0.

Step 2.8. Show that the involution τ^0 extends to an involution τ_T on D_T. Construct a new, non-normal surface $\sigma : \bar{T} \to T$ such that σ is an isomorphism outside D_T and we identify every point $p \in D_T$ with its image $\tau_T(p)$.

1.19 (Higher dimensional analogs of stable curves of genus ≥ 2). Now we can state the main theses of [**KSB88**] about higher dimensional moduli problems:

Principle 1.19.1. In higher dimensions, we should follow the proof of the Stable reduction theorem (1.17) as outlined in (1.18). The resulting fibers give the right class of stable varieties.

Principle 1.19.2. As in (1.13), a connected k-scheme X is stable iff it satisfies the following two conditions:

(Local property) A restriction on the singularities of X (so-called "semi-log-canonical" singularities).

(Global property) The canonical (or dualizing) sheaf ω_X is ample.

The definition of semi-log-canonical is not important for now (1.41), the key point is that the only global restriction is the ampleness of ω_X.

In general, Step 1.1 of (1.18) is still easy and Step 1.2 uses Hironaka's resolution of singularities. Steps 1.3–5 use Mori's program, also called the minimal model program. When [**KSB88**] was written, the relevant results were only known for families of surfaces, but [**BCHM10**] and [**HX13**] take care of the higher dimensional cases as well.

Steps 1.6–7 need very little change. As a starting point one could use the Semistable reduction theorem [**KKMSD73**], but, as we see in Section 2.4, one can get by without it.

Steps 2.0–8 of the secondary case have not been worked out earlier. Steps 2.0–7 mostly work as before; the relevant results of the minimal model program have been established in [**HX13**].

Step 2.8 turned out to be unexpectedly subtle. It is closely related to some basic questions concerning semi-log-canonical schemes. These were settled in [**Kol16b**] and a detailed treatment was given in [**Kol13c**, Chap.5].

An alternative way to approach this case would be to develop the minimal model program for varieties with normal crossing singularities and apply it directly, without normalizing in Step 2.0. Much of the background for such an approach is worked out in [**Fuj14**]. However, it turns out that the minimal model program fails already for surfaces with normal crossing singularities [**Kol11c**].

1.20 (Moduli functor of stable varieties). In the moduli theory of curves, we go directly from the definition of stable curves over fields to the notion of stable curves over an arbitrary base (1.13). By contrast, for surfaces and in higher dimensions, a major difficulty remains. As we already mentioned in (1.8), not every flat family of stable surfaces can be allowed in a "reasonable" moduli theory. Examples illustrating this are given in Section 1.3. We must restrict to families $S \to T$ where the Hilbert function of the fibers

$$\chi\big(S_t, \mathcal{O}_{S_t}(mK_{S_t})\big)$$

is independent of $t \in T$. The problem is that, for stable varieties, the canonical class K need not be Cartier, and the sheaves $\mathcal{O}_{S_t}(mK_{S_t})$ do not form a flat family over T. It is actually quite difficult to define the right concept. Our final solution of this problem is in Chapter **??**?.

1.21 (Good properties of moduli problems). Let $\mathbf{V}$ be a "reasonable" class of varieties and $\mathcal{V}arieties_{\mathbf{V}}$ the corresponding moduli functor. It is hard to pin down exactly what "reasonable" should mean, but it seems nearly impossible to do anything without the following assumption:

Representability 1.21.0. The functor $\mathcal{V}arieties_{\mathbf{V}}$ is representable by a monomorphism (3.47) if for any flat morphism $X \to S$ there is a monomorphism $S_{\mathbf{V}} \to S$ such that for any $g : T \to S$, the pull-back $X \times_S T \to T$ is in $\mathcal{V}arieties_{\mathbf{V}}(T)$ iff g factors as $g : T \to S_{\mathbf{V}} \to S$.

In many cases, $S_{\mathbf{V}} \to S$ is an open embedding. For instance, being reduced, normal or smooth are all open conditions. On the other hand, being a hyperelliptic curve is not an open condition but it is a locally closed condition.

Representability also implies that membership in $\mathcal{V}arieties_{\mathbf{V}}(T)$ can be tested on 0-dimensional subschemes of T, that is, on spectra of Artin rings. This is the reason why formal deformation theory is such a powerful tool [**Ill71, Art76, Ser06**].

Assume for the moment that there is a coarse moduli space $\mathrm{Moduli}_{\mathbf{V}}$. Our next aim is to understand how to recognize properties of $\mathrm{Moduli}_{\mathbf{V}}$ in terms of the functor $\mathcal{V}arieties_{\mathbf{V}}$.

Let X be a scheme of finite type over a field k. By the valuative criterion of separatedness, X is separated iff the following holds.

Let B be a smooth curve over k and $B^0 \subset B$ an open subset. Then a morphism $\tau^0 : B^0 \to X$ has at most one extension to $\tau : B \to X$.

If $X = \mathrm{Moduli}_{\mathbf{V}}$ is a fine moduli space, then giving a morphism $U \to X$ is equivalent to specifying a proper, flat family $V_U \to U$ whose fibers are in $\mathbf{V}$. Thus the valuative criterion of separatedness translates to functors as follows:

Separatedness 1.21.1. The functor $\mathcal{V}arieties_{\mathbf{V}}$ is separated iff for every smooth curve B and every open subset $B^0 \subset B$, a proper, flat family $\pi^0 : V^0 \to B^0$ whose fibers are in $\mathbf{V}$ has at most one extension to

$$\begin{array}{ccc} V^0 & \subset & V \\ \pi^0 \downarrow & & \downarrow \pi \\ B^0 & \subset & B, \end{array}$$

where $\pi : V \to B$ is also a proper, flat family whose fibers are in $\mathbf{V}$.

We obtain a similar translation of the valuative criterion of properness, but here we have to pay attention to the difference between coarse and fine moduli spaces.

Valuative criterion of properness 1.21.2. The functor $\mathcal{V}arieties_{\mathbf{V}}$ satisfies the valuative criterion of properness iff the following holds:

Let B be a smooth curve, $B^0 \subset B$ an open subset and $\pi^0 : V^0 \to B^0$ a proper, flat family whose fibers are in $\mathbf{V}$. Then there is a finite surjection $p : A \to B$ such that there is an extension

$$\begin{array}{ccccc} V^0 \times_B A & =: & W^0 & \subset & W \\ \downarrow & & \downarrow & & \downarrow \pi_A \\ B^0 \times_B A & =: & A^0 & \subset & A, \end{array}$$

where $\pi_A : W \to A$ is also a proper, flat family whose fibers are in $\mathbf{V}$. (For functors with a fine moduli space, we could take $A = B$, but otherwise a finite base change may be needed.)

It is very convenient to roll these two concepts together. The resulting condition is then exactly the general version of the Stable reduction theorem (1.17).

The valuative criterion of properness implies properness for schemes of finite type, but not in general. The next condition is the functor version of finite type. It ensures that we do not have too many objects to parametrize.

Boundedness 1.21.3. The class of schemes $\mathbf{V}$ is called bounded if there is a flat morphism of schemes of finite type $u : U \to T$ such that for every algebraically closed field K, every K-scheme in $\mathbf{V}$ occurs as a fiber of $U_K \to T_K$. (Some authors also assume that every fiber of $u : U \to T$ is in $\mathbf{V}$.)

How important are these conditions? 1.21.4.

As we already noted, the assumption in this book is that representability (1.21.0) is indispensable.

When separatedness (1.21.1) fails, it usually either fails very badly or it can be restored by a judicious change of the definition; see Section 1.4 for such examples. (Note, however, that most moduli functors of sheaves behave differently. They are not separated but the notions of semi-stability and GIT quotients provide a good method to deal with this. See [**HL97**] for details.)

Properness (1.21.2) is considered a challenge: If a moduli functor does not satisfy the valuative criterion of properness, we should try to enlarge the moduli problem to a proper one.

Finally, boundedness (1.21.3) seems to come automatically, though it can be very hard to prove that it holds. I do not know any natural moduli functor of projective varieties satisfying (1.21.1–2) with a coarse moduli space whose connected components are not of finite type. (In the proper but non-projective setting this can, however, happen. The Hilbert scheme of curves on the Hironaka 3-fold described in [**Har77**, App.B.3.4.1] has a connected component with infinitely many irreducible components, each proper. I do not know any natural moduli functor with a coarse moduli space that has an irreducible component that is not of finite type.)

1.22 (From the moduli functor to the moduli space). Starting with [**Mum65**] and [**Mat64**], much effort was devoted to going from the moduli functor $\mathcal{V}arieties_{\mathbf{V}}$ to the moduli space $\mathrm{Moduli}_{\mathbf{V}}$. In the quasi-projective setting, this was solved in [**Vie95**], but the proofs are quite hard.

The construction of the moduli space as an algebraic space turns out to be much easier, and the general quotient theorems of [**Kol97, KM97**] take care of it completely, see also [**Ols16**].

Once we have a moduli space which is a proper algebraic space, one needs to prove that it is projective. For surfaces this was done in [**Kol90**] and extended to higher dimensions in [**Fuj12**] and [**KP17**].

1.23 (Moduli for varieties of non-general type).

In contrast with varieties of general type, the moduli theory for varieties of non-general type is very complicated.

A general problem, illustrated by Abelian, elliptic and K3 surfaces is that a typical deformation of such an algebraic surface over $\mathbb{C}$ is a non-algebraic complex analytic surface. Thus any algebraic theory captures only a small part of the full analytic deformation theory.

The moduli question for analytic surfaces has been studied, especially for complex tori and K3 surfaces. In both cases it seems that one needs to add some extra structure (for instance, fixing a basis in some topological homology group) in order to get a sensible moduli space. (As an example of what could happen, note that the 3-dimensional space of Kummer surfaces is dense in the 20-dimensional space of all K3 surfaces, cf. [**PŠŠ71**].)

Even if one restricts to the algebraic case, compactifying the moduli space seems rather hopeless. Detailed studies of Abelian varieties and K3 surfaces show that there are many different compactifications depending on additional choices, see [**KKMSD73, AMRT75**].

It is only with the works of [**Ale02**] that a geometrically meaningful compactification of the moduli of principally polarized Abelian varieties became available. This relies on the observation that a pair (A, Θ) consisting of a principally polarized Abelian variety A and its theta divisor Θ behaves as if it were a variety of general type.

1.24 (Further problems). While we provide a solution to the basic general questions of the moduli theory of varieties of general types, there are many unsolved aspects. Some of the main ones are the following.

Problem 1.24.1 (Positive characteristic). Most of our results work only in characteristic 0. This is partly caused by the need for resolution of singularities and minimal model theory. There are, however, many other difficulties that are unsettled in positive characteristic. Even the correct definition of stable families is problematic (1.43). The paper [**Pat14**] makes substantial progress on this.

Problem 1.24.2 (Boundedness). We show that in our moduli spaces, every irreducible component is projective. It is much harder to rule out the possibility of a connected component with infinitely many irreducible components. A solution of this question follows from the deep results of [**HMX14**] that cover a series of interesting conjectures on various numerical invariants satisfying the ascending chain condition. A simpler proof would be very desirable.

Problem 1.24.3 (Effective results). Given a class of varieties of general type, we do not have good general methods to decide which stable varieties occur on the corresponding components of the moduli space. Even bounding basic numerical invariants, for instance the number of irreducible components, seems very hard. The methods in Section **??**? provide an answer in principle, but it does not seem feasible to work it out in practice, save in some very simple cases. A few results are discussed in Section **??**?, but it would be very useful to get much more information.

Problem 1.24.4 (Fine moduli spaces). As we see, stable varieties have finite automorphism groups, and we get a fine moduli space iff the identity is the only automorphism; see Section 1.7. Hence the question: Is there a sensible way to kill automorphisms by additional structures. For curves over $\mathbb{C}$ this is achieved by introducing a "level m structure" for some $m \geq 3$, that is, by fixing an isomorphism $H^1(C, \mathbb{Z}/m) \cong (\mathbb{Z}/m)^{2g}$. For smooth surfaces, similar topological invariants do not seem to be sufficient, but a completely different approach may work.

Problem 1.24.5 (Applications). Many basic questions about smooth curves can be solved by investigating an analogous problem on stable curves, whose geometry is frequently much simpler. There are, so far, few such results in higher dimensions. Some of these are discussed in Section **??**?. For example, [**LP07, PPS09a, PPS09b**] use stable surfaces to construct new examples of smooth surfaces and 4-manifolds.

One problem is that it is not easy to write down stable degenerations, the other is that the stable varieties themselves are rather complicated.

1.2. From smooth curves to canonical models

In the theory of curves, the basic objects are smooth projective curves. We frequently study any other curve by relating it to smooth projective curves. This is why the moduli functor/space of smooth curves is so important.

In higher dimensions, we define the moduli functor of smooth varieties as

$$\mathcal{S}mooth(S) := \left\{ \begin{array}{c} \text{Smooth, proper families } X \to S, \\ \text{modulo isomorphisms over } S. \end{array} \right\}$$

This, however, gives a rather badly behaved and mostly useless moduli functor already for surfaces. First of all, it is very non-separated.

1.25 (Non-separatedness in the moduli of smooth surfaces of general type). We construct two smooth families of projective surfaces $f_i : X^i \to B$ over a pointed smooth curve $b \in B$ such that

(1) all the fibers are smooth, projective surfaces of general type,
(2) $X^1 \to B$ and $X^2 \to B$ are isomorphic over $B \setminus \{b\}$,
(3) the fibers X^1_b and X^2_b are *not* isomorphic.

As the construction shows, this type of behavior happens every time we look at deformations of a surface that contains at least three (-1)-curves.

Let $f : X \to B$ be a smooth family of projective surfaces over a smooth (affine) pointed curve $b \in B$. Let $C_1, C_2, C_3 \subset X$ be three sections of f, all passing through a point $x_b \in X_b$ with independent tangent directions and are disjoint elsewhere.

Set $X^1 := B_{C_1} B_{C_2} B_{C_3} X$, where we first blow-up $C_3 \subset X$, then the birational transform of C_2 in $B_{C_3}X$ and finally the birational transform of C_1 in $B_{C_2}B_{C_3}X$. Similarly, set $X^2 := B_{C_1} B_{C_3} B_{C_2} X$. Since the C_i are sections, all these blow-ups are smooth families of projective surfaces over B.

Over $B \setminus \{b\}$ the curves C_i are disjoint, thus X^1 and X^2 are both isomorphic to $B_{C_1+C_2+C_3}X$, the blow-up of $C_1 + C_2 + C_3 \subset X$.

We claim that, by contrast, the fibers of X^1_b and X^2_b are not isomorphic to each other for a general choice of the C_i.

To see this, choose local analytic coordinates t at $b \in B$ and (x, y, t) at $x_b \in X$. The curves C_i are defined by equations

$$C_i = \bigl(x - a_i t - (\text{higher terms}) = y - b_i t - (\text{higher terms}) = 0\bigr).$$

The blow-up $B_{C_i}X$ is given by

$$B_{C_i}X = \bigl(u_i(x - a_i t - (\text{higher terms})) = v_i(y - b_i t - (\text{higher terms}))\bigr) \subset X \times \mathbb{P}^1_{u_i v_i}.$$

On the fiber over b these give the same blow-up

$$B_{x_b}\bigl(X_b\bigr) = (ux = vy) \subset X_b \times \mathbb{P}^1_{uv}$$

Thus we see that the birational transform of C_j intersects the central fiber $\bigl(B_{C_i}X\bigr)_b = B_{x_b}\bigl(X_b\bigr)$ at the point

$$\frac{u}{v} = \frac{a_j - a_i}{b_j - b_i} \in \{x_b\} \times \mathbb{P}^1_{uv}.$$

The fibers $\bigl(B_{C_2}B_{C_3}X\bigr)_b$ and $\bigl(B_{C_3}B_{C_2}X\bigr)_b$ are isomorphic to each other since they are obtained from $B_{x_b}\bigl(X_b\bigr)$ by blowing up the same point

$$\frac{u}{v} = \frac{a_2 - a_3}{b_2 - b_3} \quad \text{resp.} \quad \frac{u}{v} = \frac{a_3 - a_2}{b_3 - b_2}.$$

When we next blow up the birational transform of C_1 on $\bigl(B_{C_2}B_{C_3}X\bigr)_b$ (resp. on $\bigl(B_{C_3}B_{C_2}X\bigr)_b$) this gives the blow-up of the point

$$\frac{a_1 - a_3}{b_1 - b_3} \quad \text{resp.} \quad \frac{a_1 - a_2}{b_1 - b_2}, \tag{1.25.4}$$

and these are different, unless $C_1 + C_2 + C_3$ is locally planar at x_b.

So far we have seen that the identity $X_b = X_b$ does not extend to an isomorphism between the fibers X^1_b and X^2_b.

If X_b is of general type, then $\operatorname{Aut} X_b$ is finite, hence, to ensure that X^1_b and X^2_b are not isomorphic, we need to avoid finitely many other possible coincidences in (1.25.4).

The main reason, however, why we do not study the moduli functor of smooth varieties up to isomorphism is that, in dimension two, smooth projective surfaces do not form the *smallest* basic class. Given any smooth projective surface S, one can blow up any set of points $Z \subset S$ to get another smooth projective surface $B_Z S$ which is very similar to S. Therefore, the basic object should be not a single smooth projective surface but a whole *birational equivalence class* of smooth projective surfaces. Thus it would be better to work with smooth, proper families $X \to S$ modulo birational equivalence over S. That is, with the moduli functor

$$\mathcal{G}en\mathcal{T}ype_{bir}(S) := \left\{ \begin{array}{c} \text{Smooth, proper families } X \to S, \\ \text{every fiber is of general type,} \\ \text{modulo birational equivalences over } S. \end{array} \right\} \tag{1.25.5}$$

In essence this is what we end up doing, but it is very cumbersome do deal with birational equivalence over a base scheme. Nonetheless, working with birational equivalence classes leads to a separated moduli functor.

PROPOSITION 1.26. *Let $f_i : X^i \to B$ be two smooth families of projective varieties over a smooth curve B. Assume that the generic fibers $X^1_{k(B)}$ and $X^2_{k(B)}$ are birational and the pluricanonical system $\bigl|mK_{X^1_{k(B)}}\bigr|$ is nonempty for some $m > 0$. Then, for every $b \in B$, the fibers X^1_b and X^2_b are birational.*

Proof. Pick a birational map $\phi : X^1_{k(B)} \dashrightarrow X^2_{k(B)}$ and let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of ϕ. Let $Y \to \Gamma$ be the normalization with projections $p_i : Y \to X^i$. Note that both of the p_i are open embeddings on $Y \setminus \bigl(\operatorname{Ex} p_1 \cup \operatorname{Ex} p_2\bigr)$. Thus if we prove that neither $p_1\bigl(\operatorname{Ex} p_1 \cup \operatorname{Ex} p_2\bigr)$ nor $p_2\bigl(\operatorname{Ex} p_1 \cup \operatorname{Ex} p_2\bigr)$ contains a fiber of f_1 or f_2, then $p_2 \circ p_1^{-1} : X^1 \dashrightarrow X^2$ restricts to a birational map $X^1_b \dashrightarrow X^2_b$ for every $b \in B$. (Thus the fiber Y_b contains an irreducible component that is the graph of the birational map $X^1_b \dashrightarrow X^2_b$, but it may have other components too; see (1.28.9).)

We use the canonical class to compare $\operatorname{Ex} p_1$ and $\operatorname{Ex} p_2$. Since the X^i are smooth,

$$K_Y \sim p_i^* K_{X^i} + E_i, \quad \text{where } E_i \geq 0 \text{ and } \operatorname{Supp} E_i = \operatorname{Ex} p_i. \tag{1.26.1}$$

Assume for simplicity that B is affine and let $\operatorname{Bs}\bigl|mK_{X^i}\bigr|$ denote the set-theoretic base locus. By assumption, $\bigl|mK_{X^i}\bigr|$ is not empty and since B is affine, $\operatorname{Bs}\bigl|mK_{X^i}\bigr|$ does not contain any of the fibers of f_i.

Every section of $\mathcal{O}(mK_Y)$ pulls back from X^i, thus

$$\operatorname{Bs}\bigl|mK_Y\bigr| = p_i^{-1}\Bigl(\operatorname{Bs}\bigl|mK_{X^i}\bigr|\Bigr) + \operatorname{Supp} E_i.$$

Comparing these for $i = 1, 2$, we conclude that

$$p_1^{-1}\Bigl(\operatorname{Bs}\bigl|mK_{X^1}\bigr|\Bigr) + \operatorname{Supp} E_1 = p_2^{-1}\Bigl(\operatorname{Bs}\bigl|mK_{X^2}\bigr|\Bigr) + \operatorname{Supp} E_2.$$

Therefore,

$$p_1\bigl(\operatorname{Supp} E_2\bigr) \subset p_1\bigl(\operatorname{Supp} E_1\bigr) + \operatorname{Bs}\bigl|mK_{X^1}\bigr|.$$

Since E_1 is p_1-exceptional, $p_1\bigl(E_1\bigr)$ has codimension ≥ 2 in X^1, hence it does not contain any of the fibers of f_1. We saw that $\operatorname{Bs}\bigl|mK_{X^1}\bigr|$ does not contain any of the fibers either. Thus $p_1\bigl(\operatorname{Ex} p_1 \cup \operatorname{Ex} p_2\bigr)$ does not contain any of the fibers and similarly for $p_2\bigl(\operatorname{Ex} p_1 \cup \operatorname{Ex} p_2\bigr)$. $\square$

REMARK 1.27. A result of [**MM64**] says that, more generally, (1.26) holds as long as the fibers X^i_b are not birationally ruled, that is, not birational to a variety of the form $Z \times \mathbb{P}^1$. The proof of [**MM64**], relies on the study of exceptional divisors over a smooth variety; see [**KSC04**, Sec.4.5] for an overview. Exceptional divisors over a singular variety are much less understood. By contrast, the above proof focuses on the role of the canonical class. It is worthwhile to go back and check that the proof works if the X^i are normal, as long as (1.26.1) holds; the latter is essentially the definition of terminal singularities.

It is precisely the property (1.26.1) and its closely related variants that lead us to the correct class of singular varieties for moduli purposes.

Since it is much harder to work with a whole equivalence class, it would be desirable to find a particularly nice surface in every birational equivalence class. This is achieved by the theory of minimal models of algebraic surfaces. By a result of Enriques (cf. [**BPV84**, III.4.5]), every birational equivalence class of surfaces $\mathbf{S}$ contains a unique smooth projective surface whose canonical class is nef (that is, has nonnegative degree on every effective curve), except when $\mathbf{S}$ contains a ruled surface $C\times\mathbb{P}^1$ for some curve C. This unique surface is called the *minimal model* of $\mathbf{S}$.

It would seem at first sight that (1.26) implies that the moduli functor of minimal models is separated. There is, however, a quite subtle problem.

1.28 (Non-separatedness in the moduli of minimal models). We construct two smooth families of projective surfaces $f_i : X^i \to B$ over a pointed smooth curve $b\in B$ such that

(1) all the fibers are smooth, projective minimal models,
(2) $X^1\to B$ and $X^2\to B$ are isomorphic over $B\setminus\{b\}$,
(3) the fibers X^1_b and X^2_b are isomorphic, but
(4) $X^1\to B$ and $X^2\to B$ are *not* isomorphic.

While it is not clear from our construction, similar problems happen for any smooth family of surfaces where the general fiber has ample canonical class and a special fiber has nef (but not ample) canonical class, see [**Art74, Bri68b, Rei80**].

Let $X_0 := \bigl(f(x_1,\dots,x_4)=0\bigr)\subset\mathbb{P}^3$ be a surface of degree n that has an ordinary double point (10.44) at $p=(0{:}0{:}0{:}1)$ as its sole singularity and contains the pair of lines $(x_1x_2=x_3=0)$. Let g be homogeneous of degree $n-1$ such that x_4^{n-1} appears in it with nonzero coefficient. Consider the family of surfaces

$$X := \bigl(f(x_1,\dots,x_4)+tx_3g(x_1,\dots,x_4)=0\bigr)\subset\mathbb{P}^3_{\mathbf{x}}\times\mathbb{A}^1_t.$$

Note that X_t is smooth for general $t\neq 0$ and X contains the pair of smooth surfaces $(x_1x_2=x_3=0)$.

For $i=1,2$, let $X^i := B_{(x_i,x_3)}X$ denote the blow-up of $(x_i=x_3=0)$ with induced morphisms $\pi_i : X^i\to X$ and $f_i : X^i\to\mathbb{A}^1$. There is a natural birational map $\phi := \pi_2^{-1}\circ\pi_1 : X^1 \dashrightarrow X^2$. Let B_pX denote the blow-up of $p=\bigl((0{:}0{:}0{:}1),0\bigr)$ with exceptional divisor $E\subset B_pX$.

We claim that the following hold.

(5) The $f_i : X^i\to\mathbb{A}^1$ are projective families of surfaces which are smooth over a neighborhood of $(t=0)$.
(6) For $n\geq 5$, the fibers X^i_t have ample canonical class for $t\neq 0$ and nef canonical class for $t=0$.
(7) $X^1\times_X X^2$ is isomorphic to B_pX, hence it is smooth and irreducible.
(8) The map ϕ is an isomorphism over $\mathbb{A}^1\setminus\{0\}$ but it is not an isomorphism over 0.

(9) The fiber of $X^1 \times_X X^2$ over $(t=0)$ has two irreducible components. One of these components is the graph of an isomorphism $X^1_0 \cong X^2_0$. The other component is $E \cong \mathbb{P}^1 \times \mathbb{P}^1$.

(10) Thus $\phi : X^1 \dashrightarrow X^2$ is an isomorphism over $\mathbb{A}^1 \setminus \{0\}$, the $X^i \to \mathbb{A}^1$ have isomorphic fibers over $0 \in \mathbb{A}^1$, but ϕ is not an isomorphism over $\mathbb{A}^1$.

(It is not hard to see that, for general choice of f and g, the X_t have no birational self-maps, thus the only possible isomorphism between X^1 and X^2 would be the identity on X. Thus, by (6), in this case, X^1 and X^2 are not isomorphic to each other.)

Note that $(x_i = x_3 = 0)$ defines a Weil divisor in X which is Cartier outside the point p. Thus all 3 blow-ups are isomorphisms over $X \setminus \{p\}$. This means that all the above claims are local near p.

We prove the claims in (10.45) after choosing better local coordinates near p that make all the assertions (5–10) transparent.

All such problems go away when the canonical class is ample.

PROPOSITION 1.29. *Let $f_i : X^i \to B$ be two smooth families of projective varieties over a smooth curve B. Assume that the canonical classes K_{X^i} are f_i-ample. Let $\phi : X^1_{k(B)} \cong X^2_{k(B)}$ be an isomorphism of the generic fibers.*

Then ϕ extends to an isomorphism $\Phi : X^1 \cong X^2$.

Proof. Let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of ϕ. Let $Y \to \Gamma$ be the normalization, with projections $p_i : Y \to X^i$ and $f : Y \to B$. As in (1.26), we use the canonical class to compare the X^i. Since the X^i are smooth,

$$K_Y \sim p_i^* K_{X^i} + E_i \quad \text{where } E_i \text{ is effective and } p_i\text{-exceptional.} \tag{1.29.1}$$

Since $(p_i)_* \mathcal{O}_Y(mE_i) = \mathcal{O}_{X^i}$ for every $m \geq 0$, we get that

$$\begin{array}{rcll} (f_i)_* \mathcal{O}_{X^i}\big(mK_{X^i}\big) & = & (f_i)_*(p_i)_* \mathcal{O}_Y\big(mp_i^* K_{X^i}\big) & = \\ & = & (f_i)_*(p_i)_* \mathcal{O}_Y\big(mp_i^* K_{X^i} + mE_i\big) & = \\ & = & (f_i)_*(p_i)_* \mathcal{O}_Y\big(mK_Y\big) & = \quad f_* \mathcal{O}_Y\big(mK_Y\big). \end{array}$$

Since the K_{X^i} are f_i-ample, $X^i = \operatorname{Proj}_B \sum_{m \geq 0} (f_i)_* \mathcal{O}_{X^i}\big(mK_{X^i}\big)$. Putting these together, we get the isomorphism

$$\begin{array}{rcll} \Phi : X^1 & \cong & \operatorname{Proj}_B \sum_{m\geq 0} (f_1)_* \mathcal{O}_{X^1}\big(mK_{X^1}\big) & \cong \\ & \cong & \operatorname{Proj}_B \sum_{m\geq 0} f_* \mathcal{O}_Y\big(mK_Y\big) & \cong \\ & \cong & \operatorname{Proj}_B \sum_{m\geq 0} (f_2)_* \mathcal{O}_{X^2}\big(mK_{X^2}\big) & \cong \quad X^2. \qquad \square \end{array}$$

REMARK 1.30. As in (1.27), it is again worthwhile to investigate the precise assumptions behind the proof. The smoothness of the X^i is used only through the pull-back formula (1.29.1), which is weaker than (1.26.1).

If (1.29.1) holds, then, even if the K_{X^i} are not f_i-ample, we obtain an isomorphism

$$\operatorname{Proj}_B \sum_{m\geq 0} (f_1)_* \mathcal{O}_{X^1}\big(mK_{X^1}\big) \cong \operatorname{Proj}_B \sum_{m\geq 0} (f_2)_* \mathcal{O}_{X^2}\big(mK_{X^2}\big). \tag{1.30.1}$$

Thus it is of interest to study objects as in (1.30.1) in general.

Let us start with the absolute case, when X is a smooth projective variety over a field k. Its *canonical ring* is the graded ring

$$R(X, K_X) := \sum_{m\geq 0} H^0\big(X, \mathcal{O}_X(mK_X)\big).$$

In some cases the canonical ring tells us very little about X. For instance, if X is rational or Fano then $R(X, K_X)$ is the base field k and if X is Calabi-Yau then $R(X, K_X)$ is isomorphic to the polynomial ring $k[t]$. One should thus focus on the cases when the canonical ring is large. The following theorem and the resulting definition is due to [**Iit71**]. See [**Laz04**, Sec.2.1.C] for a detailed treatment.

Theorem-Definition 1.31. *For a smooth projective variety X of dimension n, the following are equivalent.*

(1) $h^0\big(X, \mathcal{O}_X(mK_X)\big) \geq \epsilon \cdot m^n$ *for some* $\epsilon > 0$ *and* $m \gg 1$.
(2) $\operatorname{Proj} R(X, K_X)$ *has dimension* n.
(3) *The natural map* $X \dashrightarrow \operatorname{Proj} R(X, K_X)$ *is birational.*

If these hold, then we say that X is of general type.

This enables us to find a distinguished variety in any birational equivalence class.

Definition 1.32 (Canonical models). Let X be a smooth projective variety of general type over a field k such that its canonical ring $R(X, K_X)$ is finitely generated. We define its *canonical model* as

$$X^{\mathrm{can}} := \operatorname{Proj}_k R(X, K_X).$$

If Y is a smooth projective variety birational to X then Y^{can} is isomorphic to X^{can}. Thus X^{can} is also the canonical model of the whole birational equivalence class containing X. (Taking Proj of a non-finitely generated ring may result in a quite complicated scheme. It does not seem profitable to contemplate what would happen in our case.)

Now we know [**BCHM10**] that the canonical ring $R(X, K_X)$ is always finitely generated, thus X^{can} is a projective variety. On the other hand, X^{can} can be singular. Originally this was viewed as a major obstacle but now it seems only as a technical problem.

Definition 1.33 (Canonical class and canonical sheaf). Let X be a smooth variety over a field k. As in [**Sha74**, III.6.3] or [**Har77**, p.180], the *canonical sheaf* of X is $\omega_X := \wedge^{\dim X}\Omega_{X/k}$. Any divisor D such that $\mathcal{O}_X(D) \cong \omega_X$ is called a *canonical divisor*. Their linear equivalence class is called the *canonical class*, denoted by K_X. (Note that both books assume that X is nonsingular. However, they tacitly assume that k is algebraically closed, hence nonsingularity implies smoothness. The definition, however, works over any field k as long as X is smooth over k.)

Let X be a normal variety over a perfect field k. Let $j : X^{\mathrm{sm}} \hookrightarrow X$ be the inclusion of the locus of smooth points. Then $X \setminus X^{\mathrm{sm}}$ has codimension ≥ 2, therefore, restriction from X to X^{sm} is a bijection on Weil divisors and on linear equivalence classes of Weil divisors. Thus there is a unique linear equivalence class K_X of Weil divisors on X such that $K_X|_{X^{\mathrm{sm}}} = K_{X^{\mathrm{sm}}}$. It is called the *canonical class* of X. In general, K_X does not contain any Cartier divisors.

The push-forward $\omega_X := j_*\omega_{X^{\mathrm{sm}}}$ is a rank 1 coherent sheaf on X, called the *canonical sheaf* of X. The canonical sheaf ω_X agrees with the *dualizing sheaf* ω_X° as defined in [**Har77**, p.241]. (Note that [**Har77**] defines the dualizing sheaf only if X is proper. In general, take a normal compactification $\bar{X} \supset X$ and use $\omega_{\bar{X}}^\circ|_X$ instead. For more details, see [**KM98**, Sec.5.5], [**Har66**] or [**Con00**].)

With this definition in place, we can give the following abstract characterization of canonical models.

THEOREM 1.34. *A normal projective variety Y is a canonical model iff*

(1) *$m_0 K_Y$ is Cartier and ample for some $m_0 > 0$, and*
(2) *there is a resolution $f : X \to Y$ (that is, a proper birational morphism where X is smooth) and an effective, f-exceptional divisor E such that*

$$m_0 K_X \sim f^*(m_0 K_Y) + E.$$

Proof. For now we prove only the "if" part since this is what we need for the examples. For the converse, see [**Rei80**] or [**Kol13c**, 1.15].

Note that for any $r > 0$, $f_*\mathcal{O}_X(rE) = \mathcal{O}_Y$ since E is effective and f-exceptional. Thus, by the projection formula,

$$\begin{aligned} H^0\big(X, \mathcal{O}_X(rm_0K_X)\big) &= H^0\big(Y, f_*\mathcal{O}_X(rm_0K_X)\big) \\ &= H^0\big(Y, \mathcal{O}_Y(rm_0K_Y) \otimes f_*\mathcal{O}_X(rE)\big) \\ &= H^0\big(Y, \mathcal{O}_Y(rm_0K_Y)\big). \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{Proj} \textstyle\sum_m H^0\big(X, \mathcal{O}_X(mK_X)\big) &= \operatorname{Proj} \textstyle\sum_r H^0\big(X, \mathcal{O}_X(rm_0K_X)\big) \\ &= \operatorname{Proj} \textstyle\sum_r H^0\big(Y, \mathcal{O}_Y(rm_0K_Y)\big) = Y. \quad \square \end{aligned}$$

This makes it possible to give a local definition of the singularities that occur on canonical models.

DEFINITION 1.35. A normal variety Y has *canonical singularities* if

(1) $m_0 K_Y$ is Cartier for some $m_0 > 0$ and
(2) there is a resolution $f : X \to Y$ and an effective, f-exceptional divisor E such that $m_0 K_X \sim f^*(m_0 K_Y) + E$.

It is easy to show that this is independent of the resolution $f : X \to Y$; see [**Kol13c**, 2.12]. (It is not hard to define canonical singularities without assuming the existence of a resolution as in [**Kol13c**, Sec.2.1] or [**Luo87**].)

Equivalently, Y has canonical singularities iff every point $y \in Y$ has an étale neighborhood which is an open subset on some canonical model.

As an example, consider the cone $C_d(\mathbb{P}^n)$ over the Veronese embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}\big(H^0(\mathbb{P}^n, \mathcal{O}(d))\big)$. It is easy to compute that $C_d(\mathbb{P}^n)$ has a canonical singularity iff $d \le n+1$ and its canonical class is Cartier iff $d | n+1$. (See [**Kol13c**, 3.1] for the case of general cones.)

DEFINITION 1.36 (Moduli of canonical models). The moduli functor of canonical models is

$$\mathcal{C}an\mathcal{M}od(S) := \left\{ \begin{array}{c} \text{Flat, proper families } X \to S, \\ \text{every fiber is a canonical model,} \\ \text{modulo isomorphisms over } S. \end{array} \right\} \tag{1.36.1}$$

This is an improved version of the birational moduli functor $GenType_{bir}$ (1.25.5).

Warning. In retrospect, it seems only by luck that this definition gives the correct functor. See (1.43) and the examples after it.

By a theorem of [**Siu98**], in a smooth, proper family of varieties of general type the canonical rings form a flat family and so do the canonical models. Thus there is a natural transformation T_{CanMod} which, for any reduced scheme S gives a map of sets

$$T_{\mathrm{CanMod}}(S) : GenType_{bir}(S) \to CanMod(S).$$

By definition, if $X_i \to S$ are two smooth, proper families of varieties of general type then $T_{\mathrm{CanMod}}(S)(X_1) = T_{\mathrm{CanMod}}(S)(X_2)$ iff X_1 and X_2 are birational, thus $T_{\mathrm{CanMod}}(S)$ is injective. It is, however, not surjective, but we have the following partial surjectivity statement.

Let $Y \to S$ be a flat family of canonical models. Then there is a dense open subset $S^0 \subset S$ and a smooth, proper family of varieties of general type $Y^0 \to S^0$ such that $T_{\mathrm{CanMod}}(S^0)(Y^0) = \big[Y|_{S^0}\big]$.

1.3. From stable curves to stable varieties

Let C be a stable curve with normalized irreducible components C_i. We frequently view C as an object assembled from the pieces C_i. Note that the restriction of ω_C to C_i is not ω_{C_i}, rather $\omega_{C_i}(P_i)$, where $P_i \subset C_i$ are the preimages of the nodes of C.

Similarly, if X is a scheme with simple normal crossing singularities and normalized irreducible components X_i, then the restriction of ω_X to X_i is not ω_{X_i}, rather $\omega_{X_i}(D_i)$ where $D_i \subset X_i$ is the preimage of $\operatorname{Sing} X$ on X_i.

This suggests that we should develop a theory of "canonical models" where the role of the canonical class is played by a divisor of the form $K_X + D$ where D is a simple normal crossing divisor.

Definition 1.37 (Canonical models of pairs). Let (X, D) be a pair consisting of a smooth projective variety X and a simple normal crossing divisor $D \subset X$. (That is, $D = \sum D_i$ where the D_i are distinct smooth divisors and all intersections are transversal.) We define the *canonical ring* of the pair (X, D) as

$$R(X, K_X + D) := \sum_{m \geq 0} H^0\big(X, \mathcal{O}_X(mK_X + mD)\big).$$

It is conjectured (but known only for $\dim X \leq 4$) that the canonical ring of a pair (X, D) is finitely generated. If this holds then $X^{\mathrm{can}} := \operatorname{Proj}_k R(X, K_X + D)$ is a normal projective variety. Let $D^{\mathrm{can}} \subset X^{\mathrm{can}}$ denote the image of D under the natural birational map $X \dashrightarrow X^{\mathrm{can}}$.

The pair $\big(X^{\mathrm{can}}, D^{\mathrm{can}}\big)$ is called the *canonical model* of (X, D).

The proof of the "if" part of the following characterization goes exactly as in (1.34).

Theorem 1.38. *A pair (Y, B), consisting of a proper normal variety Y and an effective, reduced Weil divisor B, is the canonical model of a simple normal crossing pair iff*

1. *$m_0(K_Y + B)$ is Cartier and ample for some $m_0 > 0$, and*
2. *there is a resolution $f : X \to Y$, an effective, reduced simple normal crossing divisor $D \subset X$ such that $f(D) = B$ and an effective, f-exceptional divisor E such that*

$$m_0(K_X + D) \sim f^*\big(m_0(K_Y + B)\big) + E.$$

REMARK 1.39. Even if $B = 0$, the notion of log canonical model differs from the notion of canonical model (1.34). To see this, let $F_i \subset X$ be the f-exceptional divisors. If $B = 0$, in (1.38.2) we can still take $D = \sum F_i$. Thus (1.38.2) can be rewritten as

$$m_0 K_X \sim f^*\bigl(m_0 K_Y\bigr) + E - m_0 \sum F_i.$$

This looks like (1.34.2), but $E - m_0 \sum F_i$ need not be effective; it can contain divisors with coefficients $\geq -m_0$.

This is the source of some terminological problems. Originally $R(X, K_X + D)$ was called the "log canonical ring" and $\operatorname{Proj}_k R(X, K_X + D)$ the "log canonical model." Since the canonical ring is just the $D = 0$ special case of the "log canonical ring," it seems more convenient to drop the prefix "log." However, log canonical singularities are quite different from canonical singularities, so the "log" cannot be omitted there.

See also (5.8) for other inconsistencies in the standard usage of "canonical model."

As in (1.35), this can be reformulated as a definition. (For now we assume that every irreducible component of B appears in B with coefficient 1; later we also consider cases when the coefficients are rational or real.)

DEFINITION 1.40. Let (Y, B) be a pair consisting of a normal variety Y and a reduced Weil divisor B. Then (Y, B) is *log canonical*, or has *log canonical singularities* iff the condition (1.38.2) is satisfied.

We are now ready to define the higher dimensional analogs of stable curves.

DEFINITION 1.41 (Stable varieties or semi-log-canonical models). Let k be a field and Y a reduced, proper scheme over k. Let $Y_i \to Y$ be the irreducible components of the normalization of Y and $D_i \subset Y_i$ the reduced preimage of the non-normal locus of Y. Then Y is a *semi-log-canonical model* or a *stable variety* iff

(1) at codimension 1 points, Y is either smooth or has a node,
(2) each (Y_i, D_i) is log canonical, and
(3) ω_Y, the canonical or dualizing sheaf of Y (1.33), is ample.

(See (1.83) or [**Kol13c**, 1.41] for the definition of a node in general. Implicit in the definition is that the D_i are divisors and that ω_Y being ample makes sense; the latter is a quite subtle condition. For now we only deal with examples where this is clear.)

We can now state the two cornerstones of the moduli theory of varieties of general type.

PRINCIPLE 1.42. *Stable varieties are the correct higher dimensional analogs of stable curves (1.13).*

PRINCIPLE 1.43. *Flat families of stable varieties $X \to T$ are the correct higher dimensional analogs of flat families of stable curves (1.13) if the canonical sheaves ω_{X_t} are locally free, but* **not** *in general.*

The correct analog will only be defined in Section 2.4 for 1-parameter families, in Section 3.4 over reduced schemes and in Section **??** in general.

I hope that the explanations given so far make (1.42) quite believable. It is more interesting to see examples that support the second assertion of (1.43). The simple fact is that basic numerical invariants, like the self intersection of the canonical class or even the Kodaira dimension fail to be locally constant in flat families of stable varieties, even when the singularities are quite mild. The rest of the section is devoted to such examples.

Jump of K^2 and of the Kodaira dimension

We give examples of flat families of projective surfaces $\{S_t : t \in \mathbb{C}\}$ such that S_t has log canonical singularities for every t (that is, the pair $(S_t, 0)$ has log canonical singularities for every t) but the self intersection of the canonical class $K_{S_t}^2$ varies with t. We also give examples where K_{S_t} is ample for $t = 0$ but not even big for $t \neq 0$. In the examples the S_t are smooth for $t \neq 0$ and S_0 has only quotient singularities. Among log canonical singularities, the quotient singularities are the mildest.

EXAMPLE 1.44 (Degree 4 surfaces in $\mathbb{P}^5$). It is easy to see that there are 2 families of nondegenerate degree 4 smooth surfaces in $\mathbb{P}^5$.

One family consists of Veronese surfaces $\mathbb{P}^2 \subset \mathbb{P}^5$ embedded by $\mathcal{O}(2)$. The general member of the other family is $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^5$ embedded by $\mathcal{O}(2,1)$, special members are embeddings of the ruled surface $\mathbb{F}_2$. The two families are distinct since

$$K_{\mathbb{P}^2}^2 = 9 \quad \text{and} \quad K_{\mathbb{P}^1 \times \mathbb{P}^1}^2 = 8.$$

For both of these surfaces, a smooth hyperplane section is a degree 4 rational normal curve in $\mathbb{P}^4$.

For us the most interesting degree 4 singular surface in $\mathbb{P}^5$ is the cone over the degree 4 rational normal curve in $\mathbb{P}^4$; denote it by $T_0 \subset \mathbb{P}^5$. The minimal resolution of T_0 is the ruled surface $p : \mathbb{F}_4 \to T_0$. Let $E, F \subset \mathbb{F}_4$ be the exceptional curve and the fiber of the ruling. Then $K_{\mathbb{F}_4} = -2E - 6F$ and $p^*(2K_{T_0}) = -3E - 12F$. Thus

$$2\bigl(K_{\mathbb{F}_4} + E\bigr) = p^*(2K_{T_0}) + E$$

shows that T_0 has log canonical singularities. We also get that $K_{T_0}^2 = 9$.

A key feature is that one can write T_0 as a limit of smooth surfaces in two distinct ways, corresponding to the two ways of writing the degree 4 rational normal curve in $\mathbb{P}^4$ as a hyperplane section of a surface. (See [**Kol13c**, 3.9] for a concrete description of these deformations.)

From the first family, we get T_0 as the special fiber of a flat family whose general fiber is $\mathbb{P}^2$. This family is denoted by $\{T_t : t \in \mathbb{C}\}$. From the second family, we get T_0 as the special fiber of a flat family whose general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$. This family is denoted by $\{T'_t : t \in \mathbb{C}\}$. (In general, one needs to worry about the possibility of getting embedded points at the vertex. However, by [**Kol13c**, 3.10], in both cases the special fiber is indeed T_0.)

Note that K^2 is constant in the family $\{T_t : t \in \mathbb{C}\}$ but jumps at $t = 0$ in the family $\{T'_t : t \in \mathbb{C}\}$.

These are, however, families of rational surfaces with negative canonical class, and we are interested in stable varieties.

Next we take a suitable cyclic cover (1.88) of the two families to get similar examples with ample canonical class.

EXAMPLE 1.45 (Jump of Kodaira dimension I).

We give examples of two flat families of projective surfaces S_t and S'_t such that

(1) $S_0 \cong S_0'$ has log canonical singularities and ample canonical class,
(2) S_t is a smooth surface with ample canonical class for $t \neq 0$, and
(3) S_t' is smooth and elliptic with $K^2_{S_t'} = 0$ for $t \neq 0$.

With T_0 as in (1.44), let $\pi_0 : S_0 \to T_0$ be a double cover, ramified along a smooth quartic hypersurface section. Note that $K_{T_0} \sim_{\mathbb{Q}} -\frac{3}{2}H$ where H is the hyperplane class. Thus, by the Hurwitz formula,

$$K_{S_0} \sim_{\mathbb{Q}} \pi_0^*\bigl(K_{T_0} + 2H\bigr) \sim_{\mathbb{Q}} \tfrac{1}{2}\pi_0^* H.$$

So S_0 has ample canonical class and $K^2_{S_0} = 2$. Since π_0 is étale over the vertex of T_0, S_0 has 2 singular points, locally (in the analytic or étale topology) isomorphic to the singularity on T_0. Thus S_0 is a stable surface.

Both of the smoothings in (1.44) lift to smoothings of S_0.

From T_t we get a smoothing S_t where $\pi_t : S_t \to \mathbb{P}^2$ is a double cover, ramified along a smooth octic. Thus S_t is smooth, $K_{S_t} \sim_{\mathbb{Q}} \pi_t^* \mathcal{O}_{\mathbb{P}^2}(1)$ is ample and $K^2_{S_t} = 2$.

From T_t' we get a smoothing S_t' where $\pi_t' : S_t' \to \mathbb{P}^1 \times \mathbb{P}^1$ is a double cover, ramified along a smooth curve of bidegree $(8,4)$. One of the families of lines on $\mathbb{P}^1 \times \mathbb{P}^1$ pulls back to an elliptic pencil on S_t' and $K^2_{S_t'} = 0$. Thus S_t' is not of general type for $t \neq 0$.

Example 1.46 (Jump of Kodaira dimension II). A similar pair of examples is obtained by working with triple covers ramified along a cubic hypersurface section. The family over T_t has ample canonical class and $K^2 = 3$. As before, the family over T_t' is elliptic and so $K^2 = 0$.

Example 1.47 (Jump of Kodaira dimension III).

We construct a flat family of surfaces whose central fiber is the quotient of the square of the Fermat cubic curve by $\mathbb{Z}/3$:

$$S_F^* \cong \bigl(u_1^3 = v_1^3 + w_1^3\bigr) \times \bigl(u_2^3 = v_2^3 + w_2^3\bigr) / \tfrac{1}{3}(1,0,0;1,0,0), \tag{1.47.1}$$

thus it has Kodaira dimension 0. The general fiber is $\mathbb{P}^2$ blown up at 12 points.

In $\mathbb{P}^3$ consider two lines $L_1 = (x_0 = x_1 = 0)$ and $L_2 = (x_2 = x_3 = 0)$. The linear system $\bigl|\mathcal{O}_{\mathbb{P}^2}(2)(-L_1 - L_2)\bigr|$ is spanned by the 4 reducible quadrics $x_i x_j$ for $i \in \{0,1\}$ and $j \in \{2,3\}$. They satisfy a relation $(x_0x_2)(x_1x_3) = (x_0x_3)(x_1x_2)$. Thus we get a morphism

$$\pi : B_{L_1+L_2}\mathbb{P}^3 \to \mathbb{P}^1 \times \mathbb{P}^1$$

which is a $\mathbb{P}^1$-bundle whose fibers are the birational transforms of lines that intersect both of the L_i.

Let $S \subset \mathbb{P}^3$ be a cubic surface such that $\mathbf{p} := S \cap (L_1 + L_2)$ is 6 distinct points. Then we get $\pi_S : B_{\mathbf{p}} S \to \mathbb{P}^1 \times \mathbb{P}^1$.

In general, none of the lines connecting 2 points of $\mathbf{p}$ is contained in S. Thus in this case π_S is a finite triple cover.

At the other extreme we have the Fermat-type surface

$$S_F := \bigl(x_0^3 + x_1^3 = x_2^3 + x_3^3\bigr) \subset \mathbb{P}^3.$$

We can factor both sides and write its equation as $m_1m_2m_3 = n_1n_2n_3$. The 9 lines $L_{ij} := (m_i = n_j = 0)$ are all contained in S_F. Let $L_{ij}' \subset B_{\mathbf{p}} S_F$ denote their birational transforms. Then the self-intersections $\bigl(L_{ij}' \cdot L_{ij}'\bigr)$ equal -3 and π_{S_F} contracts these 9 curves L_{ij}'. Thus the Stein factorization of π_{S_F} gives a triple cover $S_F^* \to \mathbb{P}^1 \times \mathbb{P}^1$ and S_F^* has 9 singular points of type $\mathbb{A}^2/\frac{1}{3}(1,1)$. We see furthermore that

$$-3K_{S_F} \sim \sum_{ij} L_{ij} \quad \text{and} \quad -3K_{B_{\mathbf{P}}S_F} \sim \sum_{ij} L'_{ij}.$$

Thus $-3K_{S_F^*} \sim 0$.

To see the two surfaces denoted by S_F^* are isomorphic, use the map of the surface (1.47.1) to $\mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$(u_1{:}v_1{:}w_1) \times (u_2{:}v_2{:}w_2) \mapsto (v_1{:}w_1) \times (v_2{:}w_2)$$

and the rational map to the cubic surface given by

$$(u_1{:}v_1{:}w_1) \times (u_2{:}v_2{:}w_2) \mapsto \bigl(v_2u_1u_2^2{:}u_1u_2^2{:}v_1u_2^3{:}u_2^3\bigr).$$

EXAMPLE 1.48 (Jump of Kodaira dimension IV). The previous examples are quite typical in some sense. If S_0 is any projective rational surface with quotient singularities, then there is a flat family of surfaces $\{S_t\}$ such that S_t is a smooth rational surface for $t \neq 0$.

To see this, take a minimal resolution $S'_0 \to S_0$. Let H'_0 be the pull-back of an ample Cartier divisor from S_0. Since S'_0 is a smooth rational surface, it can be obtained from a minimal smooth rational surface by blowing up points. We can deform S'_0 by moving these points into general position (and also deforming the minimal smooth rational surface if necessary). Thus we see that if S_0 is singular then a general deformation S'_t of S'_0 is obtained by blowing up points in $\mathbb{P}^2$ in general position. One can see, (cf. [**dF05**, 2.4]) that every smooth rational curve on S'_t with negative self-intersection is a (-1)-curve. In particular, none of the exceptional curves of $S'_0 \to S_0$ lift to S'_t hence H'_t is ample for general t. As before, we get a flat deformation $\{S_t\}$ such that $S_t \cong S'_t$ for $t \neq 0$.

Many recent constructions of surfaces of general type start with a particular rational surface S_0 with quotient singularities and show that it has a flat deformation to a smooth surface with ample canonical class; see [**LP07, PPS09a, PPS09b**]. Thus such an S_0 has flat deformations of general type and also flat deformations that are rational.

EXAMPLE 1.49 (More rational surfaces with ample canonical class). [**Kol08b**, Sec.5] Given natural numbers a_1, a_2, a_3, a_4, consider the surface

$$S = S(a_1,a_2,a_3,a_4) := (x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_4 + x_4^{a_4}x_1 = 0) \subset \mathbb{P}(w_1,w_2,w_3,w_4),$$

where $w'_i = a_{i+1}a_{i+2}a_{i+3} - a_{i+2}a_{i+3} + a_{i+3} - 1$ (with indices modulo 4), and $w_i = w'_i/\gcd(w'_1,w'_2,w'_3,w'_4)$.

It is easy to see that S has only quotient singularities (at the 4 coordinate vertices). It is proved in [**Kol08b**, Thm.39] that S is rational if $\gcd(w'_1,w'_2,w'_3,w'_4) = 1$. (By [**Kol08b**, 38], this happens with probability ≥ 0.75.)

$\mathbb{P}(w_1,w_2,w_3,w_4)$ has isolated singularities iff the $\{w_i\}$ are pairwise relatively prime. (It is easy to see that for $1 \leq a_i \leq N$, this happens for at least $c \cdot N^{4-\epsilon}$ of the 4-tuples.) In this case the canonical class of S is

$$K_S = \mathcal{O}_{\mathbb{P}}\bigl(\prod a_i - 1 - \sum w_i\bigr)|_S.$$

From this it is easy to see that if $a_1,a_2,a_3,a_4 \geq 4$ then K_S is ample and K_S^2 converges to 1 as $a_1,a_2,a_3,a_4 \to \infty$.

1.4. Examples of bad moduli problems

The aim of this section is to present examples of quite reasonable looking moduli problems that turn out to have rather bad properties.

Moduli of hypersurfaces.

The Chow and Hilbert functors describe families of hypersurfaces in a fixed projective space $\mathbb{P}^n$. For many purposes it is more natural to consider the moduli functor of hypersurfaces modulo isomorphisms. We consider what kind of "moduli spaces" one can obtain in various cases.

Definition 1.50 (Hypersurfaces modulo linear isomorphisms).

Over an algebraically closed field k, we consider hypersurfaces $X \subset \mathbb{P}^n_k$ where $X_1, X_2 \subset \mathbb{P}^n_k$ are considered isomorphic if there is an automorphism $\phi \in \operatorname{Aut}(\mathbb{P}^n_k)$ such that $\phi(X_1) = X_2$. (One could also consider hypersurfaces modulo isomorphisms which do not necessarily extend to an isomorphism of the ambient projective space. It is easy to see that smooth hypersurfaces can have such nonlinear isomorphisms only for $(d, n) \in \{(3, 2), (4, 3)\}$. A smooth cubic curve in $\mathbb{P}^2$ has an infinite automorphism group, but only finitely many extend to an automorphism of $\mathbb{P}^2$. Similarly, a smooth quartic surface in $\mathbb{P}^3$ can have an infinite automorphism group (see, for instance, (1.66)), but only finitely many extend to an automorphism of $\mathbb{P}^3$. See also [**SS17, Ogu16**] for further interesting examples of isomorphisms of smooth quartic surfaces in $\mathbb{P}^3$.

Over an arbitrary base scheme S, we consider pairs $(X \subset P)$ where P/S is a $\mathbb{P}^n$-bundle for some n and $X \subset P$ is a closed subscheme, flat over S such that every fiber is a hypersurface. There are two natural invariants, the dimension of P and the degree of X. Thus for any given n, d we get a functor

$$\mathcal{H}yp\mathcal{S}ur_{n,d}(S) := \left\{ \begin{array}{c} \text{Flat families } X \subset P \\ \text{such that } \dim_S P = n,\ \deg X = d, \\ \text{modulo isomorphisms over } S. \end{array} \right\}$$

One can also consider various subfunctors, for instance $\mathcal{H}yp\mathcal{S}ur^{\mathrm{red}}_{n,d}$, $\mathcal{H}yp\mathcal{S}ur^{\mathrm{norm}}_{n,d}$, $\mathcal{H}yp\mathcal{S}ur^{\mathrm{can}}_{n,d}$, $\mathcal{H}yp\mathcal{S}ur^{\mathrm{lc}}_{n,d}$, or $\mathcal{H}yp\mathcal{S}ur^{\mathrm{sm}}_{n,d}$ where we allow only reduced (resp. normal, canonical, log canonical or smooth) hypersurfaces.

Our aim is to investigate what the "coarse moduli spaces" of these functors look like. Our conclusion is that in many cases there can not be any scheme or algebraic space that is a coarse moduli space; any "coarse moduli space" would have to have very strange topology. Assume for simplicity that we work over an infinite field.

Let $\mathcal{H}yp\mathcal{S}ur^{*}_{n,d}$ be any subfunctor of $\mathcal{H}yp\mathcal{S}ur_{n,d}$ and assume that it has a coarse moduli space $\mathrm{HypSur}^{*}_{n,d}$. By definition, the set of k-points of $\mathrm{HypSur}^{*}_{n,d}$ is $\mathcal{H}yp\mathcal{S}ur^{*}_{n,d}(\operatorname{Spec} k)$. We can also get some idea about the Zariski topology of $\mathrm{HypSur}^{*}_{n,d}$ using various families of hypersurfaces.

For instance, we can study the closure $\bar{U}$ of a subset $U \subset \mathrm{HypSur}^{*}_{n,d}(\operatorname{Spec} k)$ using the following observation:

- Assume that there is a flat family of hypersurfaces $\pi : X \to S$ and a dense open subset $S^0 \subset S$ such that $[X_s] \in U$ for every $s \in S^0(k)$. Then $[X_s] \in \bar{U}$ for every $s \in S(k)$.

Next we write down flat families of hypersurfaces $\pi : X \to \mathbb{A}^1$ in $\mathcal{H}yp\mathcal{S}ur^*_{n,d}$ such that for $t \neq 0$ the fibers X_t are isomorphic to each other but X_0 is not isomorphic to them. Such a family corresponds to a morphism $\tau : \mathbb{A}^1 \to \mathrm{HypSur}^*_{n,d}$ such that $\tau\bigl(\mathbb{A}^1 \setminus \{0\}\bigr) = [X_1]$ but $\tau\bigl(\{0\}\bigr) = [X_0]$. This implies that the point $[X_1]$ is not closed and its closure contains $[X_0]$.

This is not very surprising in a scheme, but note that X_1 itself is defined over our base field k, so $[X_1]$ is a k-point. On a k-scheme, k-points are closed. Thus we can conclude that if there is any family as above, the moduli space $\mathrm{HypSur}^*_{n,d}$ can not be a k-scheme or algebraic space.

The simplest way to get such families is by the following construction.

EXAMPLE 1.51 (Deformation to cones). Let $f(x_0, \dots, x_n)$ be a homogeneous polynomial of degree d and $X := (f = 0)$ the corresponding hypersurface. For some $0 \leq i < n$ consider the family of hypersurfaces

$$\mathbf{X} := (f(x_0, \dots, x_i, tx_{i+1}, \dots tx_n) = 0) \subset \mathbb{P}^n \times \mathbb{A}^1_t \tag{1.51.1}$$

with projection $\pi : \mathbf{X} \to \mathbb{A}^1_t$. If $t \neq 0$ then the substitution

$$x_j \mapsto x_j \quad \text{for } j \leq i, \text{ and} \quad x_j \mapsto t^{-1}x_j \quad \text{for } j > i$$

shows that the fiber X_t is isomorphic to X. If $t = 0$ then we get the cone over $X \cap (x_{i+1} = \cdots = x_n = 0)$:

$$X_0 = (f(x_0, \dots, x_i, 0, \dots, 0) = 0) \subset \mathbb{P}^n.$$

Already these simple deformations show that various moduli spaces of hypersurfaces have very few closed points.

COROLLARY 1.52. *The sole closed point of* $\mathrm{HypSur}_{d,n}$ *is* $[(x_0^d = 0)]$.

Proof. Take any $X = (f = 0) \subset \mathbb{P}^n$. After a general change of coordinates, we can assume that x_0^d appears in f with nonzero coefficient. For $i = 0$ consider the family (1.51.1).

Then $X_0 = (x_0^d = 0)$, hence $[X]$ can not be closed point unless $X \cong X_0$. It is quite easy to see that if $X \to S$ is a flat family of hypersurfaces whose generic fiber is a d-fold plane, then every fiber is a d-fold plane. This shows that $[(x_0^d = 0)]$ is a closed point. □

COROLLARY 1.53. *The only closed points of* $\mathrm{HypSur}^{\mathrm{red}}_{d,n}$ *are* $[(f(x_0, x_1) = 0)]$ *where* f *has no multiple roots.*

Proof. If X is a reduced hypersurface of degree d, there is a line that intersects it in d distinct points. We can assume that this is the line $(x_2 = \cdots = x_n = 0)$. For $i = 1$ consider the family (1.51.1).

Then $X_0 = (f(x_0, x_1, 0, \dots, 0) = 0)$ where $f(x_0, x_1)$ has d distinct roots. Since X_0 is reduced, we see that none of the other hypersurfaces correspond to closed points.

It is not obvious that the points corresponding to $(f(x_0, x_1, 0, \dots, 0) = 0)$ are closed, but this can be easily established by studying the moduli of d points in $\mathbb{P}^1$; see [**Mum65**, Chap.3] or [**Dol03**, Sec.10.2]. □

A similar argument establishes the normal case:

COROLLARY 1.54. *The only closed points of* $\mathrm{HypSur}^{\mathrm{norm}}_{d,n}$ *are* $[(f(x_0, x_1, x_2) = 0)]$ *where* $(f(x_0, x_1, x_2) = 0) \subset \mathbb{P}^2$ *is a smooth curve.* □

In the above examples the trouble comes from cones. Cones can be normal, but they are very singular by other measures; they have a singular point whose multiplicity equals the degree of the variety. So one could hope that high multiplicity points cause the problems. This is true to some extent as the next theorems and examples show. For proofs see [**Mum65**, Sec.4.2], [**Dol03**, Sec.10.1], (**??**) and (**??**).

THEOREM 1.55. *Each of the following functors has a coarse moduli space which is a quasi-projective variety.*

(1) *The functor of smooth hypersurfaces* $\mathcal{H}yp\mathcal{S}ur^{\mathrm{sm}}_{d,n}$.
(2) *For* $d \geq n+1$, *the functor* $\mathcal{H}yp\mathcal{S}ur^{\mathrm{can}}_{d,n}$ *of hypersurfaces with canonical singularities.*
(3) *For* $d > n+1$, *the functor* $\mathcal{H}yp\mathcal{S}ur^{\mathrm{lc}}_{d,n}$ *of hypersurfaces with log canonical singularities.*
(4) *For* $d > n+1$, *the functor* $\mathcal{H}yp\mathcal{S}ur^{\mathrm{low-mult}}_{d,n}$ *of those hypersurfaces that have only points of multiplicity* $< \frac{d}{n+1}$.

EXAMPLE 1.56. Consider the family of even degree d hypersurfaces

$$\bigl((x_0^{d/2}+t^dx_1^{d/2})x_1^{d/2}+x_2^d+\cdots+x_n^d=0\bigr)\subset \mathbb{P}^n\times\mathbb{A}^1_t.$$

For $t\neq 0$ the substitution

$$(x_0{:}x_1{:}x_2{:}\cdots x_n)\mapsto (tx_0{:}t^{-1}x_1{:}x_2{:}\cdots x_n).$$

transforms the equation of X_t to

$$X:=\bigl((x_0^{d/2}+x_1^{d/2})x_1^{d/2}+x_2^d+\cdots+x_n^d=0\bigr)\subset \mathbb{P}^n.$$

X has a single singular point which is at $(1{:}0{:}\cdots{:}0)$ and has multiplicity $d/2$.

For $t=0$ we obtain the hypersurface

$$X_0:=\bigl(x_0^{d/2}x_1^{d/2}+x_2^d+\cdots+x_n^d=0\bigr).$$

X_0 has 2 singular points of multiplicity $d/2$, hence it is not isomorphic to X.

Thus we conclude that $[X]$ is not a closed point of the "moduli space" of those hypersurfaces of degree d that have only points of multiplicity $\leq d/2$.

This is especially interesting when $d\leq n$ since in this case X_0 has canonical singularities (1.35).

Thus we see that for $d\leq n$, the functor $\mathcal{H}yp\mathcal{S}ur^{\mathrm{can}}_{d,n}$ parametrizing hypersurfaces with canonical singularities does not have a coarse moduli space. By contrast, for $d>n$ the coarse moduli scheme $\mathrm{HypSur}^{\mathrm{can}}_{d,n}$ exists and is quasi-projective by (1.55).

Other non-separated examples.

The nonseparated examples produced so far all involved ruled or at least uniruled varieties. Next we consider some examples of nonseparatedness where the varieties are not uniruled. The bad behavior is due to the singularities and not to the global structure.

EXAMPLE 1.57 (Double covers of $\mathbb{P}^1$). Let $f(x,y)$ and $g(x,y)$ be two cubic forms without multiple roots, neither divisible by x or y. Consider 2 families of curves

$$\begin{aligned} S_1 &:= \bigl(f(x_1,y_1)g(t^2x_1,y_1)=z_1^2\bigr)\subset \mathbb{P}(1,1,3)\times\mathbb{A}^1 \quad\text{and}\\ S_2 &:= \bigl(f(x_2,t^2y_2)g(x_2,y_2)=z_2^2\bigr)\subset \mathbb{P}(1,1,3)\times\mathbb{A}^1.\end{aligned}$$

Note that $\omega_{S_i/\mathbb{A}^1}$ is relatively ample and the general fiber of $\pi_1 : S_i \to \mathbb{A}^1$ is a smooth curve of genus 2.

The central fibers are $\big(f(x_1, y_1)g(0, y_1) = z_1^2\big)$ resp. $\big(f(x_2, 0)g(x_2, y_2) = z_2^2\big)$. By assumption $g(0, y_1) = a_1 y_1^3$ and $f(x_2, 0) = a_2 x_2^3$ where the $a_i \neq 0$. Setting $z_1 = a_1^{1/2} w_1 y_1$ and $z_2 = a_2^{1/2} w_2 x_2$ gives the normalizations. Hence the central fibers are elliptic curves with a cusp. Their normalization is isomorphic to $\big(f(x_1, y_1)y_1 = w_1^2\big)$ resp. $\big(x_2 g(x_2, y_2) = w_2^2\big)$, and these are, in general, not isomorphic to each other.

This also shows that along the central fibers, the only singularities are at $(1{:}0{:}0; 0)$ and at $(0{:}1{:}0; 0)$. Up to canceling units, the local equations are $g(t^2, y_1) = z_1^2$ resp. $f(x_2, t^2) = z_2^2$. (These are simple elliptic with minimal resolution a single smooth elliptic curve of self intersection -1.) Hence the S_i are normal surfaces, each having 1 simple elliptic singular point.

Finally, the substitution

$$(x_1 : y_1 : z_1; t) = (x_2 : t^2 y_2 : t^3 z_2; t)$$

transforms $f(x_1, y_1)g(t^2 x_1, y_1) - z_1^2$ into

$$f(x_2, t^2 y_2)g(t^2 x_2, t^2 y_2) - t^6 z_2^2 = t^6\big(f(x_2, t^2 y_2)g(x_2, y_2) - z_2^2\big),$$

thus the two families are isomorphic over $\mathbb{A}^1 \setminus \{0\}$

Example 1.58 (Limits of double covers of $\mathbb{P}^3$). Let $a_i(x, y)$ and $b_i(u, v)$ be homogeneous forms of degree n. Consider 2 families of threefolds

$$\begin{aligned} X_1 &:= \big(a_1(x, y) + t^{2n} b_1(u, v)\big)\big(a_2(x, y) + b_2(u, v)\big) = w^2 \subset \mathbb{P}(1^4, n) \times \mathbb{A}^1, \quad \text{and} \\ X_2 &:= \big(a_1(x, y) + b_1(u, v)\big)\big(t^{2n} a_2(x, y) + b_2(u, v)\big) = w^2 \subset \mathbb{P}(1^4, n) \times \mathbb{A}^1. \end{aligned}$$

Claim.

(1) For general a_i, b_i, the central fibers of the $X_i \to \mathbb{A}^1$ are normal. Their singularities are canonical iff $n \leq 3$, and log-canonical iff $n \leq 4$.
(2) The central fibers are of general type if $n \geq 7$, have Kodaira dimension 1 if $n = 5, 6$ and are rationally connected if $n \leq 4$.
(3) The general fibers of $X_i \to \mathbb{A}^1$ have only cA_1-singularities and their canonical class is trivial if $n = 4$ and ample if $n \geq 5$.
(4) The two families are isomorphic over $\mathbb{A}^1 \setminus \{0\}$ but not isomorphic over $\mathbb{A}^1$.

Proof. For general a_i, b_i, the surface $S := (a_2(x, y) + b_2(u, v) = 0) \subset \mathbb{P}^3$ is smooth and $T := (a_1(x, y) = 0)$ has only transverse intersection with it away from the line $L := (x = y = 0)$. The central fiber X_{10} of $X_1 \to \mathbb{A}^1$ is the double cover $\pi : X_{10} \to \mathbb{P}^3$ ramified along $S \cup T$. At a general point of L the function $b_2(u, v)$ is nonzero and the local equation of the double cover can be made into $p^2 = a_1(x, y)$. At special points b_2 can have simple zeros. Here the equation is $p^2 = s \cdot a_1(x, y)$.

Let $g : P' := B_L\mathbb{P}^3 \to \mathbb{P}^3$ denote the blow up with exceptional divisor E. Let $S' \subset P'$ denote the birational transform of S and $T' \subset P'$ the birational transform of T. Note that T' is the union of n disjoint planes from the linear system $M = |g^*\mathcal{O}_{\mathbb{P}^3}(1)(-E)|$ and $S' + T' + E$ is a snc divisor if the a_i, b_i are general. The fiber product $P' \times_{\mathbb{P}^3} X_{10}$ can be realized as a double cover $X_{10}^* \to P'$ ramified along $S' + T' + nE$. This is not normal along E. Its normalization $\pi' : X' \to X_{10}^* \to P'$ is again a double cover that ramifies along $S' + T' + E$ if n is odd and along $S' + T'$ if n is even. Since $S' + T' + E$ is a snc divisor, X_{10}' has only canonical singularities (1.35). Let $g_X : X_{10}' \to X_{10}$ denote the induced morphism.

The canonical classes of X_{10} and of X'_{10} are computed by the Hurwitz formulas

$$K_{X_{10}} \sim \pi^*\mathcal{O}_{\mathbb{P}^3}(n-4) \quad \text{and} \quad K_{X'_{10}} \sim \pi'^*\bigl(g^*\mathcal{O}_{\mathbb{P}^3}(n-4)(-\lfloor \tfrac{n-2}{2}\rfloor E)\bigr).$$

Thus we obtain that

$$K_{X'_{10}} \sim g_X^* K_{X_{10}} - \lfloor \tfrac{n-2}{2}\rfloor \pi'^* E.$$

This shows that X_{10} has canonical singularities if $n \leq 3$ and log canonical singularities if $n = 4$, proving (2). (Note that for $n = 5$ the formula gives $K_{X'_{10}} \sim g_X^* K_{X_{10}} - \pi'^* E$, but π' ramifies along E so $\pi'^* E$ is a divisor with multiplicity 2.)

Furthermore, if $n \geq 7$ then $n - 5 \geq \lfloor \frac{n-2}{2}\rfloor$, thus

$$g^*\mathcal{O}_{\mathbb{P}^3}(n-4)(-\lfloor \tfrac{n-2}{2}\rfloor E) \supset g^*\mathcal{O}_{\mathbb{P}^3}(n-4)(-(n-5)E) = g^*\mathcal{O}_{\mathbb{P}^3}(1)\bigl((n-5)M\bigr),$$

which shows that X'_{10} is of general type.

If $n = 5, 6$ then X'_{10} has Kodaira dimension 1 and $\pi'^* M$ is a pencil of K3 surfaces. For a general plane M in this pencil, we get a double cover ramified along the quintic curve $M \cap S$ plus the line L when $n = 5$. The ramification is along the sextic curve $M \cap S$ when $n = 6$.

The computations for the central fiber of $X_2 \to \mathbb{A}^1$ are the same.

The general fibers of $X_i \to \mathbb{A}^1$ are double covers of $\mathbb{P}^3$ ramified along two smooth surfaces which intersect transversally. This gives the singularities $(p^2 = qr)$. The Hurwitz formula computes the canonical class.

Finally, the substitution

$$(x : y : u : v : w; t) \mapsto (t^2x : t^2y : u : v : t^n w; t)$$

transforms $\bigl(a_1(x,y) + t^{2n} b_1(u,v)\bigr)\bigl(a_2(x,y) + b_2(u,v)\bigr) - w^2$ into

$$\begin{aligned}&\bigl(a_1(t^2x, t^2y) + t^{2n} b_1(u,v)\bigr)\bigl(a_2(t^2x, t^2y) + b_2(u,v)\bigr) - t^{2n}w^2\\ &= t^{2n}\Bigl(\bigl(a_1(x,y) + b_1(u,v)\bigr)\bigl(t^{2n}a_2(x,y) + b_2(u,v)\bigr) - w^2\Bigr).\end{aligned}$$

Let us end our study of hypersurfaces with a different type of example. This shows that the moduli problem for hypersurfaces usually includes smooth limits that are not hypersurfaces. These pose no problem for the general theory, but they show that it is not always easy to see what schemes one needs to include in a moduli space.

EXAMPLE 1.59 (Smooth limits of hypersurfaces). [**Mor75**]

Fix integers $a, b > 1$ and $n \geq 2$. We construct a family of smooth n-folds X_t such that X_t is a smooth hypersurface of degree ab for $t \neq 0$ and X_0 is not isomorphic to a smooth hypersurface.

It is not known if similar examples exist for $n \geq 3$ and $\deg X$ a prime number.

Fix $\mathbb{P}(1^{n+1}, a)$ with coordinates $x_0, \dots, x_n, z$. Let f_a, g_{ab} be general homogeneous forms of degree a (resp. ab) in $x_0, \dots, x_n$. Consider the family of complete intersections

$$X_t := \bigl(tz - f_a(x_0, \dots, x_n) = z^b - g_{ab}(x_0, \dots, x_n) = 0\bigr) \subset \mathbb{P}(1^{n+1}, a).$$

For $t \neq 0$ we can eliminate z to obtain a degree ab smooth hypersurface

$$X_t \cong \bigl(f_a^b(x_0, \dots, x_n) = g_{ab}(x_0, \dots, x_n)\bigr) \subset \mathbb{P}^{n+1}.$$

For $t = 0$ we see that $\mathcal{O}_{X_0}(1)$ is not very ample but realizes X_0 as a b-fold cyclic cover (1.88)

$$X_0 \to \bigl(f_a(x_0, \dots, x_n) = 0\bigr) \subset \mathbb{P}^{n+1}$$

of a degree a smooth hypersurface. In particular, X_0 is not isomorphic to a smooth hypersurface.

1.5. Compactifications of M_g

Here we consider what happens if we try define other compactifications of M_g. First we give a complete study of a compactified moduli functor of genus 2 curves that uses only irreducible curves.

Moduli of genus 2 curves.

Definition 1.60. Let $\mathcal{M}_2^{\mathrm{irr}}$ be the moduli functor of flat families of irreducible curves of arithmetic genus 2 which are either

(1) smooth,
(2) nodal,
(3) rational with 2 cusps or
(4) rational with a triple point whose complete local ring is isomorphic to $\mathbb{C}[[x,y,z]]/(xy,yz,zx)$.

The aim of this subsection is to prove the following. (See [**Mum65**, Chap.3] or [**Dol03**, Sec.10.2] for the relevant background on GIT quotients.)

Proposition 1.61. *Let $\mathcal{M}_2^{\mathrm{irr}}$ be the moduli functor defined above. Then*

(1) *the coarse moduli space M_2^{irr} exists and equals the geometric invariant theory quotient $S^6\mathbb{P}^1/\!/\operatorname{Aut}(\mathbb{P}^1)$, but*
(2) *$\mathcal{M}_2^{\mathrm{irr}}$ is a very bad moduli functor.*

Proof. A smooth curve of genus 2 can be uniquely written as a double cover $\tau : C \to \mathbb{P}^1$, ramified at 6 distinct points $p_1, \dots, p_6 \in \mathbb{P}^1$, up to automorphisms of $\mathbb{P}^1$. Thus, M_2 is isomorphic to the space of 6 distinct points in $\mathbb{P}^1$, modulo the action of $\operatorname{Aut}(\mathbb{P}^1)$. If some of the 6 points coincide, we get singular curves as double covers.

It is easy to see the following (cf. [**Mum65**, Chap.3], [**Dol03**, Sec.10.2]).

(3) A point set is semi-stable iff it does not contain any point with multiplicity ≥ 4. Equivalently, if the corresponding genus 2 cover has only nodes and cusps.
(4) The properly semistable point sets are of the form $3p_1 + p_2 + p_3 + p_4$ where the p_2, p_3, p_4 are different from p_1 but may coincide with each other. Equivalently, the corresponding genus 2 cover has at least one cusp.
(5) Point sets of the form $2p_1 + 2p_2 + 2p_3$ where the p_1, p_2, p_3 are different from each other give the only semistable case when the double cover is reducible. It has two smooth rational components meeting each other at 3 points.

In the properly semistable case, generically the double cover is an elliptic curve with a cusp over p_1. As a special case we can have $3p_1 + 3p_2$, giving as double cover a rational curve with 2 cusps. Note that the curves of this type have a 1-dimensional moduli (the cross ratio of the points p_1, p_2, p_3, p_4 or the j-invariant of the elliptic curve), but they all correspond to the same point in $S^6\mathbb{P}^1/\!/\operatorname{Aut}(\mathbb{P}^1)$. (See (1.57) for an explicit construction.) Our definition (1.60) aims to remedy this non-uniqueness by always taking the most degenerate case; a rational curve with 2 cusps (1.60.3).

In case (5), write the reducible double cover as $C = C_1 + C_2$. The only obvious candidate to get an irreducible curve is to contract one of the two components C_i. We get an irreducible rational curve; denote it by C_j' where $j = 3-i$. Note that C_j' has one singular point which is analytically isomorphic to the 3 coordinate axes in $\mathbb{A}^3$. The resulting singular rational curves C_j' are isomorphic to each other. These are listed in (1.60.4).

Let $p : X \to S$ be any flat family of irreducible, reduced curves of arithmetic genus 2. The trace map (cf. [**BPV84**, III.12.2]) shows that $R^1p_*\omega_{X/S} \cong \mathcal{O}_S$. Thus, by cohomology and base change (cf. [**Har77**, III.12.11]), $p_*\omega_{X/S}$ is locally free of rank 2. Set $P := \mathbb{P}_S\big(p_*\omega_{X/S}\big)$. Then P is a $\mathbb{P}^1$-bundle over S and we have a rational map $\pi : X \to P$. If X_s has only nodes and cusps, then ω_{X_s} is locally free and generated by global sections, thus π is a morphism along X_s.

If X_s is as in (1.60.4), then ω_{X_s} is not locally free and π is not defined at the singular point. $\pi|_{X_s}$ is birational and the 3 local branches of X_s at the singular point correspond to 3 points on $\mathbb{P}\big(H^0(X_s, \omega_{X_s})\big)$.

The branch divisor of π is a degree 6 multisection of $P \to S$, all of whose fibers are stable point sets. Thus we have a natural transformation

$$\mathcal{M}_2^{\mathrm{irr}}(*) \to \operatorname{Mor}\big(*, S^6\mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)\big).$$

We have already seen that we get a bijection

$$\mathcal{M}_2^{\mathrm{irr}}(\mathbb{C}) \cong \big(S^6\mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)\big)(\mathbb{C}).$$

Since $S^6\mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)$ is normal, we conclude that it is the coarse moduli space. This completes the proof of (1.61.1).

The assertion (1.61.2) is more a personal opinion. There are 3 main things "wrong" with the functor $\mathcal{M}_2^{\mathrm{irr}}(*)$. Let us consider them one at a time.

1.61.6 (Stable reduction questions).

At the set-theoretic level, we have our moduli space $M_2^{\mathrm{irr}} = S^6\mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)$, but what about at the level of families?

The first indications are good. Let $\pi_B : S_B \to B$ be a stable family of genus 2 curves. Assume that no fiber has 2 smooth rational components. Let $b_i \in B$ be the points corresponding to fibers with 2 components of arithmetic genus 1. Let $p : A \to B$ be a double cover ramified at the points b_i. Consider the pull-back family $\pi_A : S_A \to A$. Set $a_i = p^{-1}(b_i)$ and let $s_i \in \pi_A^{-1}(a_i)$ be the separating node. Since we took a ramified double cover, each $s_i \in S_A$ is a double point. Thus if we blow up every s_i, the exceptional curve appears in the fiber with multiplicity 1. We can now contract the birational transforms of the elliptic curves to get a family where all these reducible fibers are replaced by a rational curve with 2 cusps. We have proved the following analog of (1.17):

Lemma 1.61.6.1. Let $\pi : S \to B$ be a stable family of genus 2 curves such that no fiber has 2 smooth rational components. Then, after a suitable double cover $A \to B$, the pull-back $S \times_B A$ is birational to another family where each reducible fiber is replaced by a rational curve with 2 cusps. □

This solved our problem for 1-parameter families, but, as it turns out, we have problems over higher dimensional bases. In particular, there is no universal family over any base scheme Y that finitely dominates $S^6\mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)$, not even locally in any neighborhood of the properly semistable point. Indeed, this would give a proper, flat family of curves of arithmetic genus 2 over a 3-dimensional base

$\pi : X \to Y$ where only finitely many of the fibers (the ones over the unique properly semistable point) have cusps. However, the next result shows that there is no such family.

Proposition 1.61.6.2. Let $\pi : X \to Y$ be a proper, flat family of curves of arithmetic genus 2. Assume that X_0 is a rational curve with a cusp for some $0 \in Y$ and that $\dim_0 Y \geq 3$. Then there is a curve $0 \in C \subset Y$ such that X_y has a cusp for every $y \in C$.

Proof. This follows from the deformation theory of the cusp which says that every flat deformation of a cusp is induced by pull-back from the 2-parameter family

$$\begin{array}{ccc} (y^2 = x^3 + ux + v) & \subset & \mathbb{A}^2_{xy} \times \mathbb{A}^2_{uv} \\ p \downarrow & & \downarrow \\ \mathbb{A}^2_{uv} & = & \mathbb{A}^2_{uv}. \end{array}$$

(See Section 10.5 or [**Art76, AGZV85a, Har10**] for introductions.)

Thus our family π gives an analytic morphism $\tau : Y \to \mathbb{A}^2_{uv}$ (defined in some neighborhood of $0 \in Y$) and $C = \tau^{-1}(0,0) \subset Y$ is the required curve along which the fiber has a cusp. $\square$

1.61.7 (Failure of representability).

Following (1.61.6.2), consider the universal deformation of the rational curve with 2 cusps. This is given as

$$\begin{array}{ccc} \bigl(z^2 = (x^3 + uxy^2 + vy^3)(y^3 + syx^2 + tx^3)\bigr) & \subset & \mathbb{P}^2(1,1,3) \times \mathbb{A}^4_{uvst} \\ p \downarrow & & \downarrow \\ \mathbb{A}^4_{uvst} & = & \mathbb{A}^4_{uvst}. \end{array}$$

Let us work in a neighborhood of $(0,0,0,0) \in \mathbb{A}^4$ where the 2 factors $x^3 + uxy^2 + vy^3$ and $y^3 + syx^2 + tx^3$ have no common roots. There are 3 types of fibers of p.

i) $p^{-1}(0,0,0,0)$ is a rational curve with 2 cusps.
ii) $p^{-1}(a,b,0,0)$ and $p^{-1}(0,0,a,b)$ are irreducible with exactly 1 cusp if $(a,b) \neq (0,0)$.
iii) $p^{-1}(a,b,c,d)$ is irreducible with at worst nodes otherwise.

Thus the curves that we allow in our moduli functor $\mathcal{M}_2^{\mathrm{irr}}$ do not form a representable family. Even worse, the subfamily

$$\begin{array}{ccc} \bigl(z^2 = (x^3 + uxy^2 + vy^3)y^3\bigr) & \subset & \mathbb{P}^2(1,1,3) \times \operatorname{Spec} k[[u,v]] \\ p \downarrow & & \downarrow \\ \operatorname{Spec} k[[u,v]] & = & \operatorname{Spec} k[[u,v]] \end{array}$$

is not allowed in our moduli functor $\mathcal{M}_2^{\mathrm{irr}}$, but the family

$$\begin{array}{ccc} \bigl(z^2 = (x^3 + uxy^2 + vy^3)(y^3 + u^n yx^2 + v^n x^3)\bigr) & \subset & \mathbb{P}^2(1,1,3) \times \operatorname{Spec} k[[u,v]] \\ p \downarrow & & \downarrow \\ \operatorname{Spec} k[[u,v]] & = & \operatorname{Spec} k[[u,v]] \end{array}$$

is allowed. Over $\operatorname{Spec} k[u,v]/(u^n, v^n)$ the two families are isomorphic. Since deformation theory is essentially a study of families over Artin rings, this means that the usual methods can not be applied to understand the functor $\mathcal{M}_2^{\mathrm{irr}}$.

1.61.8 (Unusual non-separatedness).

A quite different type of problem arises at the curve corresponding to $2p_1 + 2p_2 + 2p_3$.

Write the double cover as $C = C_1 + C_2$. As before, if we contract one of the two components C_i, we get an irreducible rational curve C'_j where $j = 3 - i$ as in (1.60.4).

Since the curves C'_1 and C'_2 are isomorphic, from the set-theoretic point of view this is a good solution. However, as in (1.28), something strange happens with families. Let $p : S \to \mathbb{A}^1$ be a family of stable curves whose central fiber $S_0 := p^{-1}(0)$ is isomorphic to $C = C_1 + C_2$. We have two ways to construct a family with an irreducible central fiber: contract either of the two irreducible components C_i. Thus we get two families

$$S \xrightarrow{\pi_i} S_i \xrightarrow{p_i} \mathbb{A}^1 \quad \text{with } p_i^{-1}(0) \cong C'_{3-i}.$$

Over $\mathbb{A}^1 \setminus \{0\}$ the two families are naturally isomorphic to $S \to \mathbb{A}^1$, hence to each other, yet this isomorphism does not extend to an isomorphism of S_1 and S_2. Indeed, the closure of the graph of the resulting birational map is given by the image $(\pi_1, \pi_2) : S \to S_1 \times_{\mathbb{A}^1} S_2$. Thus the corresponding moduli functor is not separated.

We claimed above that, by contrast, the coarse moduli space is $\bar{M}_2$, hence separated. A closer study reveals the source of this discrepancy: we have been thinking of schemes instead of algebraic spaces. The occurrence of such problems in moduli theory was first observed by [**Art74**]. The aim of the next paragraph is to show how such examples arise.

1.61.9 (Bug-eyed covers). [**Art74, Kol92a**]

A non-separated scheme always has "extra" points. The typical example is when we take two copies of a scheme $X \times \{i\}$ for $i = 0, 1$, an open dense subscheme $U \subsetneq X$ and glue $U \times \{0\}$ to $U \times \{1\}$ to get $X \amalg_U X$. The non-separatedness arises from having 2 points in $X \amalg_U X$ for each point in $X \setminus U$.

By contrast, an algebraic space can be non-separated by having no extra points, only extra tangent directions. The simplest example is the following.

On $\mathbb{A}^1_t$ consider two equivalence relations. The first is $R_1 \rightrightarrows \mathbb{A}^1$ given by

$$(t_1 = t_2) \cup (t_1 = -t_2) \subset \mathbb{A}^1_{t_1} \times \mathbb{A}^1_{t_2}.$$

Then $\mathbb{A}^1_t / R_1 \cong \mathbb{A}^1_u$ where $u = t^2$.

The second is the étale equivalence relation $R_2 \rightrightarrows \mathbb{A}^1$ given by

$$\mathbb{A}^1 \xrightarrow{(1,1)} \mathbb{A}^1 \times \mathbb{A}^1 \quad \text{and} \quad \mathbb{A}^1 \setminus \{0\} \xrightarrow{(1,-1)} \mathbb{A}^1 \times \mathbb{A}^1.$$

(Note that we take the disconnected union of the two components, instead of their union as 2 lines in $\mathbb{A}^1 \times \mathbb{A}^1$ intersecting at the origin.)

One can also obtain $\mathbb{A}^1_t / R_2$ by taking the quotient of the nonseparated scheme $\mathbb{A}^1 \amalg_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1$ by the (fixed point free) involution that interchanges $(t, 0)$ and $(-t, 1)$.

The morphism $\mathbb{A}^1_t \to \mathbb{A}^1_t / R_2$ is étale, thus $\mathbb{A}^1_t / R_2 \neq \mathbb{A}^1_t / R_1$. Nonetheless, there is a natural morphism

$$\mathbb{A}^1_t / R_2 \to \mathbb{A}^1_t / R_1$$

which is one-to-one and onto on closed points. The difference between the 2 spaces is seen by the tangent vectors at the origin. The tangent space of $\mathbb{A}^1_t / R_2$ at the origin is spanned by $\partial / \partial t$ while the tangent space of $\mathbb{A}^1_t / R_1$ at the origin is spanned by

$$\frac{\partial}{\partial u} = \frac{1}{2t} \cdot \frac{\partial}{\partial t}.$$

Other compactifications of M_g.

While M_g has many compactifications besides $\bar{M}_g$, it is only recently that a systematic search begun for other geometrically meaningful examples. The papers [**Sch91, HH13, Smy13**] contain many examples.

Our attempt to replace the moduli functor of stable curves of genus 2 with another one that parametrizes only irreducible curves was not successful, but some of the problems seemed to have arisen from the symmetry that forced us to make artificial choices.

We can avoid such choices for other values of the genus using the following observation.

Let $\pi : S \to B$ be a flat family of curves with smooth general fiber and reduced special fibers. If $C_b := \pi^{-1}(b)$ is a singular fiber and C_{bi} are the irreducible components of its normalization then

$$\begin{array}{rcl}\sum_i h^1\big(C_{bi}, \mathcal{O}_{C_{bi}}\big) & \leq & h^1\big(C_b, \mathcal{O}_{C_b}\big) = 1 - \chi\big(C_b, \mathcal{O}_{C_b}\big) = 1 - \chi\big(C_{gen}, \mathcal{O}_{C_{gen}}\big) \\ & = & h^1\big(C_{gen}, \mathcal{O}_{C_{gen}}\big),\end{array}$$

where C_{gen} is the general smooth fiber. In particular, there can be at most 1 irreducible component with geometric genus $> \frac{1}{2}g(C_{gen})$.

From this it is easy prove the following:

Let B be a smooth curve and $S^0 \to B^0$ a smooth family of genus g curves over an open subset of B. Then there is at most one normal surface $S \to B$ extending S^0 such that every fiber of $S \to B$ is irreducible and of geometric genus $> \frac{1}{2}g(C_{gen})$.

Moreover, if $S_{stab} \to B$ is a stable family extending S^0 and every fiber of $S_{stab} \to B$ contains an irreducible curve of geometric genus $> \frac{1}{2}g(C_{gen})$, then we obtain S from S_{stab} by contracting all connected components of curves of geometric genus $< \frac{1}{2}g(C_{gen})$ that are contained in the fibers. (It is not hard to show that $S \to B$ exists, at least as an algebraic space.)

In fact, this way we obtain a partial compactification $M_g \subset M'_g$ such that

(1) M'_g parametrizes smoothable irreducible curves of arithmetic genus g and geometric genus $> \frac{1}{2}g$.
(2) Let $M_g \subset M''_g \subset \bar{M}_g$ be the largest open subset parametrizing curves that contain an irreducible component of geometric genus $> \frac{1}{2}g$. Then there is a natural morphism $M''_g \to M'_g$.

So far so good, but, as we see next, we can not extend M'_g to a compactification in a geometrically meaningful way. This happens for every $g \geq 3$; the following example with $g = 13$ is given by simple equations.

This illustrates a general pattern: one can easily propose partial compactifications that work well for some families but lead to contradictions for some others.

EXAMPLE 1.62. Consider the surface $F := \big(x^8 + y^8 + z^8 = u^2\big) \subset \mathbb{P}^3(1,1,1,4)$ and on it the curve $C := F \cap (xyz = 0)$. C has 3 irreducible components $C_x = (x = 0), C_y = (y = 0), C_z = (z = 0)$ which are smooth curves of genus 3. C itself has arithmetic genus 13.

We work with a 3-parameter family of deformations

$$T := \big(xyz - ux^3 - vy^3 - wz^3 = 0\big) \subset F \times \mathbb{A}^3_{uvw}. \tag{1.62.1}$$

For general $uvw \neq 0$ the fiber of the projection $\pi : T \to \mathbb{A}^3$ is a smooth curve of genus 13. If one of the u, v, w is zero, then generically we get a curve with 2 nodes hence with geometric genus 11.

If two of the coordinates are zero, say $v = w = 0$, then we have a family

$$T_x := \big(x(yz - ux^2) = 0\big) \subset F \times \mathbb{A}^1_u.$$

For $u \neq 0$, the fiber $C_{u,0,0}$ has 2 irreducible components. One is $C_x = (x = 0)$, the other is $(yz - tx^2 = 0)$ which is a smooth genus 7 curve.

Thus the proposed rule says that we should contract $C_x \subset C_{u,0,0}$.

Similarly, by working over the v and the w-axes, the rule tells us to contract $C_y \subset C_{0,v,0}$ for $v \neq 0$ and $C_z \subset C_{0,0,w}$ for $w \neq 0$.

It is easy to see that over $\mathbb{A}^3 \setminus \{(0,0,0)\}$ these contractions can be performed (at least among algebraic spaces). Thus we obtain

$$\begin{array}{ccc} T \setminus \{\pi^{-1}(0,0,0)\} & \stackrel{p^0}{\to} & S^0 \\ \pi \downarrow & & \tau^0 \downarrow \\ \mathbb{A}^3 \setminus \{(0,0,0)\} & = & \mathbb{A}^3 \setminus \{(0,0,0)\} \end{array} \tag{1.62.2}$$

where τ^0 is flat with irreducible fibers.

Claim 1.62.3. There is no proper family of curves $\tau : S \to \mathbb{A}^3$ that extends τ^0. (We do not require τ to be flat.)

Proof. Assume to the contrary that $\tau : S \to \mathbb{A}^3$ exists and let

$$\Gamma \subset T \times_{\mathbb{A}^3} S$$

be the closure of the graph of p^0. Since p^0 is a morphism on $T \setminus \{\pi^{-1}(0,0,0)\}$, we see that the first projection $\pi_1 : \Gamma \to T$ is an isomorphism away from $\pi^{-1}(0,0,0)$. Since $T \times_{\mathbb{A}^3} S \to \mathbb{A}^3$ has 2-dimensional fibers, we conclude that $\dim \pi_1^{-1}\big(\pi^{-1}(0,0,0)\big) \leq 2$. T is, however, a smooth 4-fold, hence the exceptional set of any birational map to T has pure dimension 3. Thus $\Gamma \cong T$ and so p^0 extends to a morphism $p : T \to S$.

Now we see that the rule lands us in a contradiction over the origin $(0,0,0)$. Here all 3 components $C_x, C_y, C_z \subset C_{0,0,0} = C$ should be contracted. This is impossible to do since this would give that the central fiber of $S \to \mathbb{A}^3$ is a point.

1.6. More unexpected examples

We start with an example showing that seemingly equivalent moduli problems may lead to different moduli spaces.

Example 1.63. We start with the moduli space P_{n+1} of $n+1$ points in $\mathbb{C}$ up to translations. We can view such a point set as the zeros of a unique polynomial of degree $n+1$ whose leading term is x^{n+1}. We can use a translation to kill the coefficient of x^n and the universal polynomial is then given by

$$x^{n+1} + a_2 x^{n-1} + \cdots + a_{n+1}.$$

Thus $P_{n+1} \cong \mathbb{C}^n$ with coordinates $a_2, \ldots, a_{n+1}$.

Let us now look at those point sets where n of the points coincide. There are 2 ways to formulate this as a moduli problem:

(1) unordered point sets $p_0, \ldots, p_n \in \mathbb{C}$ where at least n of the points coincide, up to translations, or
(2) unordered point sets $p_0, \ldots, p_n \in \mathbb{C}$ plus a point $q \in \mathbb{C}$ such that $p_i = q$ at least n-times, up to translations.

If $n \geq 2$ then q is uniquely determined by the points $p_0, \dots, p_n$, so it would seem that the two formulations are equivalent. We claim, however, that the two versions have non-isomorphic moduli spaces.

If the n-fold point is at t then the corresponding polynomial is $(x-t)^n(x+nt)$. By expanding it we get that

$$a_i = t^i \Big[(-1)^i \tbinom{n}{i} + (-1)^{i-1} n \tbinom{n}{i-1} \Big] \quad \text{for i=2,..., n+1.}$$

This shows that the space $R_{n+1} \subset P_{n+1}$ of polynomials with an n-fold root is a cuspidal rational curve given as the image of the map

$$t \mapsto \Big(a_i = t^i \Big[(-1)^i \tbinom{n}{i} + (-1)^{i-1} n \tbinom{n}{i-1} \Big] : i = 2, \dots, n+1 \Big).$$

So the moduli space R_{n+1} of the first variant (1) is a cuspidal rational curve.

By contrast, the space $\bar{R}_{n+1}$ of the second variant (2) is a smooth rational curve, the isomorphism is given by

$$(p_0, \dots, p_n; q) \mapsto q \in \mathbb{C}.$$

Not surprisingly, the map that forgets the n-fold root gives $\pi : \bar{R}_{n+1} \to R_{n+1}$ which is the normalization map.

Next we have 2 examples of moduli functors that are not representable (1.21.0) yet this does not cause any problems.

Example 1.64. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree 4 with infinite automorphism group (1.66). We claim that $\mathcal{I}sotriv_S(*)$, to be defined in (1.70), is not representable.

Let $\mathbf{S} \to W$ be the universal family of smooth degree 4 surfaces in $\mathbb{P}^3$. The isomorphisms classes of the pairs $\big(S, \mathcal{O}_S(1)\big)$ correspond to the $\operatorname{Aut}(\mathbb{P}^3)$-orbits in W. We see below that the fibers isomorphic to S form countably many $\operatorname{Aut}(\mathbb{P}^3)$-orbits. Thus $\mathcal{I}sotriv_S(*)$ is not representable.

For any $g \in \operatorname{Aut} S$, $g^*\mathcal{O}_S(1)$ gives another embedding of S into $\mathbb{P}^3$. Two such embedding are projectively equivalent iff $g^*\mathcal{O}_S(1) \cong \mathcal{O}_S(1)$, that is, when $g \in \operatorname{Aut}\big(S, \mathcal{O}_S(1)\big)$. The latter can be viewed as the group of automorphisms of $\mathbb{P}^3$ that map S to itself. Thus $\operatorname{Aut}\big(S, \mathcal{O}_S(1)\big)$ is a closed subscheme of the algebraic variety $\operatorname{Aut}(\mathbb{P}^3) \cong \mathrm{PGL}_4$. Since $\operatorname{Aut} S$ is discrete, this implies that $\operatorname{Aut}\big(S, \mathcal{O}_S(1)\big)$ is finite. Hence the fibers of $\mathbf{S} \to W$ that are isomorphic to S lie over countably many $\operatorname{Aut}(\mathbb{P}^3)$-orbits, corresponding to $\operatorname{Aut} S / \operatorname{Aut}\big(S, \mathcal{O}_S(1)\big)$.

Example 1.65. We construct a smooth, proper family of surfaces $X \to C$ over a smooth curve such that

(1) every fiber has nef canonical class,
(2) the generic fiber has ample canonical class,
(3) $X \to C$ is locally projective but
(4) $X \to C$ is not projective.

Start with hypersurfaces of degree $d \geq 5$ in $\mathbb{P}^4$ that contain a fixed 2-plane L. These hypersurfaces form a very ample linear system on the blow-up $B_L\mathbb{P}^4$, hence, by the Lefschetz theorem, the class group of a general $Y \subset \mathbb{P}^4$ is generated by L and the hyperplane class H.

It is easy to see that a general Y has $(d-1)^2$ ordinary double points as its singularities and a general hyperplane containing L intersects Y in $L + S$ where S is also smooth.

The singularities of Y can be resolved either by blowing up L or by blowing up S (10.45). Either of these results in a projective variety, but next we mix these up.

Partition the set of ordinary double points into two nonempty subsets D_1, D_2. Let $Y_1 := B_L(Y \setminus D_2)$ and $Y_2 := B_S(Y \setminus D_1)$. Both of these contain $Y \setminus (D_1 + D_2)$ as an open subset. By gluing them together, we get a proper variety Y^*. We claim that Y^* is not projective.

Indeed, let $E_i \subset Y^*$ be an exceptional curve mapping to a node in D_i. Let $L^* \subset Y^*$ (resp. $H^* \subset Y^*$) denote the birational transforms of L (resp. H). Then, as in (10.45), $L^* \cdot E_1 = +1$, $L^* \cdot E_2 = -1$ and $H^* \cdot E_i = 0$. Thus no linear combination $aL^* + bH^*$ has positive degree on both E_1 and E_2. Since $\operatorname{Pic} Y^*$ is generated by L^* and H^*, this implies that there is no ample divisor on Y^*. Moreover, this also shows that if $X^* \to Y^*$ is a proper birational morphism that is an isomorphism near $E_1 + E_2$ and $X \subset X^*$ is an open set that contains $E_1 + E_2$, then X is not quasi-projective.

It is now easy to construct a family of surfaces as required. Let $H_1, H_2 \subset \mathbb{P}^4$ be general hyperplanes and $Y' := B_{H_1 \cap H_2 \cap Y} Y$ the blow up. The pencil $|H_1, H_2|$ defines a morphism $f' : Y' \to \mathbb{P}^1$. Since the H_i are general, we may assume that there are finite sets $B_0, B_1, B_2 \subset \mathbb{P}^1$ such that the following holds

(5) for $b \notin \cup B_i$, the fiber Y'_b is smooth,
(6) for $b \in B_1$ (resp. $b \in B_2$), the fiber Y'_b has a single node which is at one of the points of D_1 (resp. D_2).

Set $X^* : B_{H_1 \cap H_2 \cap Y} Y^*$ and $f^* : X^* \to Y^* \to \mathbb{P}^1$. Finally let $C := \mathbb{P}^1 \setminus B_0$ and $X := (f^*)^{-1}(C) \subset X^*$ with $f := f^*|_X$.

By the computations of (10.45), $f : X \to C$ is smooth. By construction, f is projective over $C \setminus B_i$ for $i = 1, 2$ but X itself is not quasi-projective.

The following examples are useful in various constructions.

Example 1.66 (Surfaces with infinite discrete automorphism group). Let us start with a smooth genus 1 curve E defined over a field K. Any point $q \in E(K)$ defines an involution τ_q where $\tau_q(p)$ is the unique point such that $p + \tau_q(p) \sim 2q$. (Equivalently, we can set q as the origin, then $\tau_q(p) = -p$.) The first formulation shows that if L/K is a quadratic extension, then any $Q \in E(L)$ also defines an involution τ_Q where $\tau_Q(p)$ is the unique point such that $p + \tau_Q(p) \sim Q$.

Given points $q_1, q_2 \in E(K)$, we see that $p \mapsto \tau_{q_2} \circ \tau_{q_1}(p)$ is translation by $2q_1 - 2q_2$. Similarly, given $Q_i \in E(L_i)$, $p \mapsto \tau_{Q_2} \circ \tau_{Q_1}(p)$ is translation by $Q_1 - Q_2$. Usually these translations have infinite order.

Let now $g : S \to C$ be a smooth, minimal, elliptic surface with generic fiber E over $k(C)$. By the above, any section or double section of g gives an involution of S and two involutions usually generate an infinite group of automorphisms of S.

As a concrete example, let $S \subset \mathbb{P}^3$ be a smooth quartic that contains 3 lines L_i. The pencil of planes through L_1 gives an elliptic fibration and L_2, L_3 are sections. Thus these K3 surfaces usually have an infinite automorphism group.

As another example, let $S \subset \mathbb{P}^3$ be a quartic with a double point $p \in S$. Projecting S from p exhibits the blow-up $B_p S$ as a double cover of $\mathbb{P}^2$, hence we get a Galois involution τ_p. If S has 2 nodes, the two involutions usually generate an infinite group of automorphisms of the minimal resolution of S.

1.7. Coarse and fine moduli spaces

As in (1.7), let $\mathbf{V}$ be a "reasonable" class of projective varieties (or schemes, or ...) and $\mathcal{V}arieties_{\mathbf{V}}$ the corresponding functor. The aim of this section is to study the difference between coarse and fine moduli spaces, mostly through a few examples. We are guided by the following:

PRINCIPLE 1.67. *Let $\mathbf{V}$ be a "reasonable" class as above and assume that it has a coarse moduli space* $\mathrm{Moduli}_{\mathbf{V}}$. *Then* $\mathrm{Moduli}_{\mathbf{V}}$ *is a fine moduli space iff* $\mathrm{Aut}(V)$ *is trivial for every $V \in \mathbf{V}$.*

From the point of view of algebraic stacks, a precise version is given in [**LMB00**, 8.1.1]. Our construction of the moduli spaces in Section **??** also shows that this principle is true for various moduli spaces of polarized varieties.

The rest of the section is devoted to some simple examples illustrating (1.67). The direction $\Rightarrow$ is rather easy to see if $\mathrm{Aut}(V)$ is finite for every $V \in \mathbf{V}$, see (1.70.2). However, (1.67) fails in some cases, as shown by (1.70.3). The direction $\Leftarrow$ is subtler. It again holds for polarized varieties but a precise version needs careful attention to descent theory and the difference between schemes and algebraic spaces.

1.68 (Moduli of varieties without automorphisms). As above, let $\mathbf{V}$ be a "reasonable" class of varieties with a coarse moduli space $\mathrm{Moduli}_{\mathbf{V}}$. Let us make the following

Assumption 1.68.1. $\mathrm{Aut}(V) = \{1\}$ is an open condition in flat families with fibers in $\mathbf{V}$.

If this holds then there is an open subscheme $\mathrm{Moduli}^{\mathrm{rigid}}_{\mathbf{V}} \subset \mathrm{Moduli}_{\mathbf{V}}$ that is a coarse moduli space for varieties in $\mathbf{V}$ without automorphisms. By (1.67) $\mathrm{Moduli}^{\mathrm{rigid}}_{\mathbf{V}}$ should be a fine moduli space. In many important cases $\mathrm{Moduli}^{\mathrm{rigid}}_{\mathbf{V}}$ is dense in $\mathrm{Moduli}_{\mathbf{V}}$, thus one can understand much about the coarse moduli space $\mathrm{Moduli}_{\mathbf{V}}$ by studying the fine moduli space $\mathrm{Moduli}^{\mathrm{rigid}}_{\mathbf{V}}$.

Let $X \to S$ be a flat family with fibers in $\mathbf{V}$ and $\pi : \mathrm{Aut}(X/S) \to S$ the scheme representing automorphisms of the fibers; cf. [**Kol96**, I.1.10]. If $\mathbf{V}$ satisfies the valuative criterion of separatedness (1.21.1) then π is proper. Thus $|\mathrm{Aut}(V)| < \infty$ is an open condition. More careful attention to the scheme structure of the automorphism groups shows that in fact $\mathrm{Aut}(V) = \{1\}$ is an open condition.

The following example, however, shows that (1.68.1) does not hold for all smooth projective surfaces.

Example 1.68.2. Let S be a smooth projective surface such that $G := \mathrm{Aut}(S) = \langle \tau \rangle \cong \mathbb{Z}/p$ has prime order ≥ 3 and there is a τ-fixed point $s \in S$ such that the G action on $\mathbb{P}(T_sS)$ is faithful.

For instance, if $f(x, y, z)$ is a general homogeneous form of degree pd then we can take S to be the degree p cyclic cover $\bigl(u^p = f(x, y, z)\bigr) \subset \mathbb{P}^3(1, 1, 1, d)$ and s to be any branch point.

Take now a smooth (affine) curve $s \in C \subset S$ such that the stabilizer of $T_sC \subset T_sS$ is trivial. For $0 \leq i < p$ let $C_i \subset S \times C$ be the image of $(\tau^i, 1) : C \to S \times C$. By shrinking C we may assume that the C_i intersect only at (s, s).

Let $X_0 \to S \times C$ denote the blow up of C_0. The birational transforms C_i' are disjoint for $0 < i < p$. We can now blow up the C_i' for $0 < i < p$ simultaneously to obtain

$$\pi : X \to S \times C \to C.$$

If $c \neq s$ then the fiber X_c is obtained from S by blowing up the G-orbit of the point $c \in C \subset S$. Thus the G-action on S lifts to a G-action on X_c.

For $c = s$ we get a fiber X_s which is obtained from S in two steps.

First we blow up s to get B_sS with exceptional curve $E \subset B_sS$. The G-action on S lifts to a G-action on B_sS. Second, we blow up the $(p-1)$ intersection points $E \cap C_i'$ for $0 < i < p$ but we do not blow up the point $E \cap C_0'$. There is no G-orbit with $p-1$ elements, thus the G-action on B_sS does not lift to X_s and $\mathrm{Aut}(X_s) = \{1\}$.

Example 1.68.3. A similar jump of the automorphism group also happens for Enriques surfaces. By the works of [**BP83, Dol84, Kon86**], the automorphism group of a general Enriques surface is infinite, but there are special Enriques surfaces with finite automorphism group.

Next we see what goes wrong in the presence of automorphisms. We start with a concrete example.

EXAMPLE 1.69 (Moduli theory of the curve $(z^2 = x^{2n} - 1)$, I.).

A seemingly trivial, but actually quite subtle and revealing, example is the moduli theory of the hyperelliptic curve C, given by a projective equation as

$$C = \left(z^2 = x^{2n} - y^{2n}\right) \subset \mathbb{P}^2(1,1,n).$$

Let k be an algebraically closed field. Following the pattern of (1.9), as a first approximation, our moduli functor should be

$$\mathcal{C}urves_C(T) := \left\{ \begin{array}{c} \text{Smooth families } S \to T \text{ such that} \\ \text{every fiber is isomorphic to } C, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

This is the right definition if T is reduced, but not otherwise, so for now we restrict ourselves to reduced base schemes.

Since the k-points of the coarse moduli space are in one-to-one correspondence with the k-isomorphism classes of objects, a coarse moduli space for $\mathcal{C}urves_C$ has a unique k-point.

The only possible choice for the universal family is now

$$u : C \to \operatorname{Spec} k.$$

Any k-scheme T has a unique morphism $g : T \to \operatorname{Spec} k$ and by pull-back we obtain the trivial family

$$g^*u : C \times T \to T.$$

It is easy to see, however, that for many schemes T, there are other families in $\mathcal{C}urves_C(T)$. Take, for instance, $T = \mathbb{A}^* := \mathbb{A}^1 \setminus \{0\}$ and consider the surface

$$S_1^* := \left(z^2 = x^{2n} - ty^{2n}\right) \subset \mathbb{P}^2(1,1,n)_{xyz} \times \mathbb{A}_t^*.$$

S_1^* is smooth and the fibers of the projection $\pi_1 : S_1^* \to \mathbb{A}^*$ are smooth hyperelliptic curves of genus $n-1$. The substitution $y' := \sqrt[2n]{t} \cdot y$ shows that each geometric fiber is isomorphic to the curve $C := \left(z^2 = x^{2n} - y^{2n}\right) \subset \mathbb{P}^2(1,1,n)$. We claim, however, that, for $n \geq 3$, the family $\pi_1 : S_1^* \to \mathbb{A}^*$ is different from the trivial family $\pi_2 : S_2^* := \left(C \times \mathbb{A}^*\right) \to \mathbb{A}^*$. We can write the latter as

$$S_2^* := \left(z^2 = x^{2n} - y^{2n}\right) \subset \mathbb{P}^2(1,1,n)_{xyz} \times \mathbb{A}_t^*.$$

To see the difference note that a hyperelliptic curve (of genus ≥ 2) has a unique degree 2 map to $\mathbb{P}^1$. In our two families the corresponding maps are the coordinate projection

$$\mathbb{P}^2(1,1,n)_{xyz} \times \mathbb{A}^*_t \to \mathbb{P}^1_{xy} \times \mathbb{A}^*_t$$

restricted to S^*_1 (resp. S^*_2).

The branch curve of $S^*_1 \to \mathbb{P}^1_{xy} \times \mathbb{A}^*_t$ is the irreducible curve

$$B^*_1 := \bigl(x^{2n} - ty^{2n} = 0\bigr) \subset \mathbb{P}^2_{xy} \times \mathbb{A}^*_t,$$

whereas the branch curve of $S^*_2 \to \mathbb{P}^1_{xy} \times \mathbb{A}^*_t$ is the reducible curve

$$B^*_2 := \bigl(x^{2n} - y^{2n} = 0\bigr) \subset \mathbb{P}^2_{xy} \times \mathbb{A}^*_t.$$

Thus the two families are not isomorphic.

We also see that the two families become isomorphic after a finite and surjective base change. Consider the substitution $t = u^{2n}$. By pulling back S^*_1, we get the family

$$T^*_1 := \bigl(z^2 = x^{2n} - u^{2n}y^{2n}\bigr) \subset \mathbb{P}^2(1,1,n)_{xyz} \times \mathbb{A}^*_u.$$

By setting $y_1 := uy$, T^*_1 becomes isomorphic to the trivial family

$$T^*_2 := \bigl(z^2 = x^{2n} - y_1^{2n}\bigr) \subset \mathbb{P}^2(1,1,n)_{xy_1z} \times \mathbb{A}^*_u,$$

which is also obtained by pulling back the trivial family S^*_2 to $\mathbb{A}^*_u$.

We can put these considerations in a somewhat more general setting as follows.

1.70 (Isotrivial families). Let X be a smooth projective variety over $\mathbb{C}$ and assume for simplicity that $\mathrm{Aut}(X)$ is a discrete group. We are interested in the functor, which to a reduced scheme T associates the set

$$\mathcal{I}sotriv_X(T) := \left\{ \begin{array}{c} \text{Smooth families } \mathbf{X} \to T \text{ such that} \\ \text{every fiber is isomorphic to } X, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

More precisely, we should distinguish between the algebraic and the complex analytic versions $\mathcal{I}sotriv^{\mathrm{alg}}_X(*)$ and $\mathcal{I}sotriv^{\mathrm{an}}_X(*)$. It turns out that allowing T to be a complex analytic space is a minor difference, but allowing $\mathbf{X}$ to be complex analytic creates a substantial change. Let us start complex analytically.

Lemma 1.70.1. Assume that $\mathrm{Aut}(X)$ is a discrete group. Then families in $\mathcal{I}sotriv^{\mathrm{an}}_X(T)$ are in one-to-one correspondence with the $\mathrm{Aut}(X)$-conjugacy classes of group homomorphisms $\mathrm{Hom}\bigl(\pi_1(T,t), \mathrm{Aut}(X)\bigr)$.

Proof. Since $\mathrm{Aut}(X)$ is a discrete group, over any contractible subset of T the family has a unique trivialization. Thus, if we fix a point $t \in T$ and an isomorphism $\mathbf{X}_t \cong X$, then the various families are classified by the monodromy representation

$$\rho : \pi_1(T,t) \to \mathrm{Aut}(X).$$

If we do not fix an isomorphism $\mathbf{X}_t \cong X$, then we have to work with conjugacy classes of such homomorphisms. □

It is not hard to go from an analytic classification to an algebraic one.

Lemma 1.70.2. Notation and assumptions as above.

(1) Two such algebraic families $\mathbf{X}_i \to T$ are algebraically isomorphic iff they are analytically isomorphic.
(2) $\mathbf{X} \to T$ is projective iff the image of ρ is finite.

(3) $\mathbf{X} \to T$ is an algebraic space iff $\mathbf{X} \to T$ is projective.

Proof. Assume that $\mathbf{X}_i \to T$ are algebraic and consider the scheme parametrizing relative isomorphisms $\mathrm{Isom}_T(\mathbf{X}_1, \mathbf{X}_2)$ (cf. [**Kol96**, Sec.I.1]). By our assumptions $\mathrm{Isom}_T(\mathbf{X}_1, \mathbf{X}_2) \to T$ is étale, thus it has an algebraic section iff it has an analytic section. This proves (1).

Assume that $\mathbf{X} \to T$, corresponding to $\rho : \pi_1(T, t) \to \mathrm{Aut}(X)$, is projective and let L be a relatively ample divisor on $\mathbf{X}$. Then $c_1\big(L|_X\big) \in H^2(X, \mathbb{Z})$ is invariant under $\operatorname{im} \rho$. For some $d > 0$, the Néron-Severi group $\mathrm{NS}(X)$ is generated by effective divisors of degree $\leq d$ (with respect to $c_1\big(L|_X\big)$). There are only finitely many such divisor classes, hence a finite index subgroup of the image of ρ acts trivially on $\mathrm{NS}(X)$. For any projective variety X, the subgroup $\mathrm{Aut}^\tau(X)$ of $\mathrm{Aut}(X)$ that acts trivially on $\mathrm{NS}(X)$ is an algebraic group (cf. [**Kol96**, I.1.10.2]). Since $\mathrm{Aut}(X)$ is assumed discrete, $\mathrm{Aut}^\tau(X)$ is finite. Thus $\operatorname{im} \rho$ is finite, proving one direction of (2).

Conversely, assume that $G := \operatorname{im} \rho$ is finite and let $T' \to T$ be the étale cover corresponding to G. On the trivial family $X \times T'$ consider the action of G where we act on T' by deck transformations and on X by ρ. The quotient $\mathbf{X} := \big(X \times T'\big)/G$ exists and is projective (cf. [**Kol13c**, 9.29]).

The proof of (3) is left to the reader; we will not use it. $\square$

Corollary 1.70.3. Let X be a smooth projective variety over $\mathbb{C}$ such that $\mathrm{Aut}(X)$ is a discrete group. Then $X \to \operatorname{Spec} \mathbb{C}$ is a fine moduli space for $\mathcal{I}sotriv_X^{\mathrm{an}}(*)$ iff $\mathrm{Aut}(X) = \{1\}$.

Proof. If $\mathrm{Aut}(X) \neq \{1\}$ then there is a nontrivial homomorphism $\mathbb{Z} \to \mathrm{Aut}(X)$. This gives a locally trivial but globally nontrivial complex analytic family over $\mathbb{C}^*$ (or over any elliptic curve) that can not be the pull-back of $X \to \operatorname{Spec} \mathbb{C}$. Conversely, if $\mathrm{Aut}(X) = \{1\}$ then $\mathcal{I}sotriv_X^{\mathrm{an}}(T)$ consists of the trivial family for any T.

Corollary 1.70.4. Let X be a smooth projective variety over $\mathbb{C}$ such that $\mathrm{Aut}(X)$ is discrete and torsion free. Then for any T, the trivial family $X \times T$ gives the only algebraic family in $\mathcal{I}sotriv_X^{\mathrm{alg}}(T)$. In particular, $X \to \operatorname{Spec} \mathbb{C}$ is a fine moduli space for $\mathcal{I}sotriv_X^{\mathrm{alg}}(*)$.

Proof. By our assumption, the only homomorphism $\rho : \pi_1(T, t) \to \mathrm{Aut}(X)$ with finite image is the trivial one. It corresponds to the trivial family $X \times T \to T$. $\square$

The next construction gives such an example that is birational to an Abelian surface.

Example 1.70.5. Let $0 \in E$ be an elliptic curve such that $\mathrm{End}(0 \in E) \cong \mathbb{Z}$, (that is, without complex multiplication). Then the automorphism group of its square is

$$\mathrm{Aut}\big((0,0) \in E \times E\big) \cong GL(2, \mathbb{Z})$$

and the isomorphism is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \big[(x, y) \mapsto (ax + by, cx + dy)\big].$$

Take 3 points $P_1 = (0,0)$, $P_2 = (x_2, 0)$ and $P_3 = (0, x_3)$ where $x_3 \in E$ is 3-torsion and $x_2 \in E$ is non-torsion. It is easy to see that $\{0\} \times E$ (resp. $E \times \{0\}$) is the only elliptic curve in $E\times E$ that contains 2 of the points and their difference is torsion (resp. non-torsion). Thus we conclude that

$$\operatorname{Aut}\bigl(P_1+P_2+P_3, E\times E\bigr)=\left\{\begin{pmatrix}1 & 3m\\ 0 & 1\end{pmatrix} : m\in\mathbb{Z}\right\}.$$

Let now X be the surface obtained from $E\times E$ by blowing up the 3 points P_i. Since the only rational curves on X are the 3 exceptional curves, we conclude that

$$\operatorname{Aut}(X)=\operatorname{Aut}\bigl(P_1+P_2+P_3, E\times E\bigr)\cong\mathbb{Z}.$$

EXAMPLE 1.71 (Moduli theory of the curve $(z^2=x^{2n}-1)$, II.).

Another reincarnation of the phenomenon observed in (1.69) occurs if we notice that C is already defined over $\mathbb{Q}$ and we try to construct the moduli space as $\operatorname{Spec}\mathbb{Q}$.

Over an algebraically closed field, C is isomorphic to any of the curves

$$C_{ab}=\bigl(z^2=ax^{2n}-by^{2n}\bigr)\subset\mathbb{P}^2(1,1,n)\quad\text{for } a,b\neq 0.$$

Over other fields, however, the curves C_{ab} need not be isomorphic. For instance, over $\mathbb{R}$, we can obtain $\bigl(z^2=x^{2n}+y^{2n}\bigr)$ whose set of real points consists of 2 circles, $\bigl(z^2=x^{2n}-y^{2n}\bigr)$ whose set of real points consists of 1 circle and $\bigl(z^2=-x^{2n}-y^{2n}\bigr)$ whose set of real points is empty.

The situation is even worse over $\mathbb{Q}$. For instance, as p runs through all prime numbers, the curves $C_{1p}=\bigl(z^2=x^{2n}-py^{2n}\bigr)$ are pairwise non-isomorphic for $n\geq 4$.

A simple way to see this is to note that the ramification locus of the projection $C_{1p}\to\mathbb{P}^1_{xy}$ is an isomorphism invariant of C_{1p}. In our case, the ramification locus is the scheme $\operatorname{Spec}_{\mathbb{Q}}\mathbb{Q}\bigl(\sqrt[2n]{p}\bigr)$, and these fields are different from each other for different values of p. For instance, the only ramified primes in $\mathbb{Q}\bigl(\sqrt[2n]{p}\bigr)/\mathbb{Q}$ are p and possibly some divisors of $2n$. Thus as p runs through the set of primes not dividing $2n$, we get pairwise non-isomorphic fields and hence non-isomorphic curves C_{1p}.

1.72 (Field of moduli). Let $X\subset\mathbb{P}^n$ be a projective variety defined over some large field, for example $\mathbb{C}$. Any set of defining equations involves only finitely many elements of $\mathbb{C}$, thus X can be defined over a finitely generated subextension of $\mathbb{C}$. It is a natural question to ask: Is there a smallest subfield $K\subset\mathbb{C}$ such that X can be defined by equations over K.

There are three variants for this question.

(1) Fix coordinates on $\mathbb{P}^n$ and view X as a specific subvariety. In this case a smallest subfield exists; see [**Wei46**, Sec.I.7] or [**KSC04**, Sec.3.4]. This is a special case of the existence of Hilbert schemes (1.5).
(2) No embedding of X is fixed. Thus we are looking for a field $K\subset\mathbb{C}$ and a K-variety X_K such that $X\cong(X_K)_{\mathbb{C}}$. We see in (1.75) that this may lead to rather complicated behavior.
(3) As an intermediate choice, fix an embedding $X\hookrightarrow\mathbb{P}^n$ but do not fix the coordinates on $\mathbb{P}^n$. Equivalently, we work with a pair (X,L) where L is a very ample line bundle on X. This is the question that we consider next. Note that, if the canonical line bundle on X is ample or anti-ample, we can harmlessly identify X with the pair $\bigl(X,\mathcal{O}_X(mK_X)\bigr)$ if mK_X is very ample. (There are two further natural variants of this approach. We may decide not to distinguish between the pairs (X,L) and (X,L^m) for $m>0$

or we may identify (X, L) and (X, L') if L is numerically equivalent to L'. Both of these lead to minor technical differences only.)

How is this connected with moduli theory?

Let $\mathbf{V}$ be a class of varieties with a coarse moduli space $\mathrm{Moduli}_{\mathbf{V}}$. Assume that $X \in \mathbf{V}$ can be defined by equations over a field K; that is, there is a K-scheme $X_K \to \operatorname{Spec} K$ whose geometric fiber is isomorphic to X. By the definition of a coarse moduli space, this corresponds to a morphism $\operatorname{Spec} K \to \mathrm{Moduli}_{\mathbf{V}}$. In particular, we get an injection of the residue field of $\mathrm{Moduli}_{\mathbf{V}}$ at $[X]$ into K. Conversely, if $\mathrm{Moduli}_{\mathbf{V}}$ is a fine moduli space, then X can be defined over the residue field of $[X] \in \mathrm{Moduli}_{\mathbf{V}}$. Thus we have proved the following:

Lemma 1.72.4. If $\mathrm{Moduli}_{\mathbf{V}}$ is a fine moduli space then the residue field of $\mathrm{Moduli}_{\mathbf{V}}$ at $[X]$ is the smallest field K such that X can be defined over K as in (1.72.2). □

A consequence is that, for fine moduli spaces, the residue field of $\mathrm{Moduli}_{\mathbf{V}}$ at $[X]$ depends only on X and not on the choice of $\mathbf{V}$.

In general, let us define the *field of moduli* of X as the (function field of) the coarse moduli space of the functor $\mathcal{I}sotriv_X(*)$, where, generalizing the concept in (1.70) from $\mathbb{C}$ to arbitrary fields, for any reduced scheme T we set

$$\mathcal{I}sotriv_X(T) := \left\{ \begin{array}{c} \text{Smooth families } \mathbf{X} \to T \text{ such that} \\ \text{every geometric fiber is isomorphic to } X, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

As we see in (1.75), $\mathcal{I}sotriv_X(*)$ need not have a coarse moduli space. We thus introduce the following variant. For a pair (X, L), where L is an ample line bundle on X, set

$$\mathcal{I}sotriv_{(X,L)}(T) := \left\{ \begin{array}{c} \text{Smooth families } \mathbf{X} \to T \text{ plus a} \\ \text{relatively ample line bundle } \mathbf{L} \text{ such that} \\ \text{every geometric fiber is isomorphic to } (X, L), \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

We see in Section **??** that $\mathcal{I}sotriv_{(X,L)}(*)$ has a coarse moduli space.

In order to avoid some problems with infinite Galois groups (1.75), the following lemma is stated for number fields only.

Lemma 1.72.5. Let X be a smooth projective variety defined over a number field L. For a field K the following are equivalent.

(1) The field of moduli of X is contained in K.
(2) There is a K-scheme T such that $\mathcal{I}sotriv_X(T) \neq \emptyset$.
(3) For any $\sigma \in \mathrm{Gal}(\bar{K}/K)$, the variety X^σ is isomorphic to X over $\bar{K}$. (Here X^σ is obtained by applying σ to a set of defining equations of X.)

Proof. The interesting part is (3) $\Rightarrow$ (2). Choose a finite extension $K(\alpha)/K$ such that $L \subset K(\alpha)$, where α is a root of a polynomial $p(t) \in K[t]$ of degree d. Let

$$f_i(x_0, \dots, x_m) \in K(\alpha)[x_0, \dots, x_m] \ : \ i = 1, \dots, r$$

be defining equations of X (in some projective embedding) over $K(\alpha)$. Since $K(\alpha) = K + \alpha K + \cdots + \alpha^{d-1} K$, we can also think of the f_i as

$$f_i(\alpha, x_0, \dots, x_m) \in K[\alpha, x_0, \dots, x_m],$$

where $\deg_\alpha f_i < d$. Consider now the K-scheme

$$Y_K := \bigl(f_1(t, x_0, \dots, x_m) = \cdots = f_r(t, x_0, \dots, x_m) = p(t) = 0\bigr) \subset \mathbb{P}^m_K \times \mathbb{A}^1_t.$$

The second projection gives $\pi : Y_K \to \operatorname{Spec}_K K[t]/\bigl(p(t)\bigr)$. One of the geometric fibers of π is $X_{\bar{L}}$, the others are its conjugates $X^\sigma_{\bar{L}}$. If (3) holds then $\pi : Y_K \to \operatorname{Spec}_K K(\alpha)$ is an isotrivial family over the K-scheme $\operatorname{Spec}_K K(\alpha)$, which shows (2). $\square$

In (1.74) we construct a hyperelliptic curve whose field of moduli is $\mathbb{Q}$ yet it can not be defined over $\mathbb{R}$. The first such examples are in [**Ear71, Shi72**].

1.73 (Field of moduli for hyperelliptic curves). Let A be a smooth hyperelliptic curve of genus ≥ 2. Over an algebraically closed field, A has a unique degree 2 map to $\mathbb{P}^1$. Let $B \subset \mathbb{P}^1$ be the branch locus, that is, a collection of $2g+2$ points in $\mathbb{P}^1$. If the base field k is not closed, then A has a unique degree 2 map to a smooth genus 0 curve Q. (One can always think of Q as a conic in $\mathbb{P}^2$.) Thus A is defined over a field k iff the pair $(B \subset \mathbb{P}^1)$ can be defined over k.

The latter problem is especially transparent if A is defined over $\mathbb{C}$ and we want to know if it is defined over $\mathbb{R}$ or if its field of moduli is contained in $\mathbb{R}$.

Up to isomorphism, there are 2 real forms of $\mathbb{P}^1$. One is $\mathbb{P}^1$, corresponding to the anti-holomorphic involution $(x{:}y) \mapsto (\bar{x}{:}\bar{y})$, which, after a coordinate change, can also be written as $\sigma_1 : (x{:}y) \mapsto (\bar{y}{:}\bar{x})$. (In the latter form the real points form the unit circle.) The other is the "empty" conic, corresponding to the anti-holomorphic involution $\sigma_2 : (x{:}y) \mapsto (-\bar{y}{:}\bar{x})$. Thus (1.72.5) gives the following.

Lemma 1.73.1. Let $A \to \mathbb{P}^1$ be a smooth hyperelliptic curve of genus ≥ 2 over $\mathbb{C}$ and $B \subset \mathbb{CP}^1$ the branch locus. Then

(1) A can be defined over $\mathbb{R}$ iff there is a $g \in \operatorname{Aut}(\mathbb{CP}^1)$ such that gB is invariant under σ_1 or σ_2.
(2) The field of moduli of A is contained in $\mathbb{R}$ iff there is $h \in \operatorname{Aut}(\mathbb{CP}^1)$ such that hB equals B^{σ_1} or B^{σ_2}.

Note that if $(gB)^\sigma = gB$ then $B^\sigma = \bigl(g^\sigma\bigr)^{-1} gB$ shows that (1) $\Rightarrow$ (2). Conversely, if $B^\sigma = hB$ and we can write $h = \bigl(g^\sigma\bigr)^{-1} g$ then $(gB)^\sigma = gB$.

Example 1.74. Here is an example of a hyperelliptic curve C whose field of moduli is $\mathbb{Q}$ but C can not be defined over $\mathbb{R}$.

Pick $\alpha = a + ib$ where a, b are rational. Consider the hyperelliptic curve

$$C(\alpha) := \Bigl(z^2 - \bigl(x^8 - y^8\bigr)\bigl(x^2 - \alpha y^2\bigr)\bigl(\bar{\alpha} x^2 + y^2\bigr) = 0\Bigr) \subset \mathbb{P}^3(1,1,6).$$

Its complex conjugate is

$$C(\bar{\alpha}) := \Bigl(z^2 - \bigl(x^8 - y^8\bigr)\bigl(x^2 - \bar{\alpha} y^2\bigr)\bigl(\alpha x^2 + y^2\bigr) = 0\Bigr) \subset \mathbb{P}^3(1,1,6).$$

Note that $C(\alpha)$ and $C(\bar{\alpha})$ are isomorphic, as shown by the substitution

$$(x, y, z) \mapsto (iy, x, z).$$

In particular, over the $\mathbb{Q}$-scheme $\operatorname{Spec}_{\mathbb{Q}} \mathbb{Q}[t]/(t^2+1)$ we have a curve

$$C(a,b) := \Bigl(z^2 - \bigl(x^8 - y^8\bigr)\bigl(x^2 - (a+tb)y^2\bigr)\bigl((a-tb)x^2 + y^2\bigr) = 0\Bigr) \subset \mathbb{P}^3(1,1,6)$$

whose geometric fibers are isomorphic to $C(\alpha)$. Thus the field of moduli of $C(\alpha)$ is $\mathbb{Q}$ by (1.72.5).

We claim that, for sufficiently general a, b, the curve $C(\alpha)$ can not be defined over $\mathbb{Q}$, not even over $\mathbb{R}$. By (1.73) we need to show that there is no anti-holomorphic involution that maps the branch locus to itself. In the affine chart $y \neq 0$, the ramification points of $C(\alpha) \to \mathbb{P}^1$ are:

(1) the 8th roots of unity corresponding to $x^8 - y^8$, and
(2) the 4 points $\pm\beta, \pm i/\bar{\beta}$ where $\beta^2 = \alpha$.

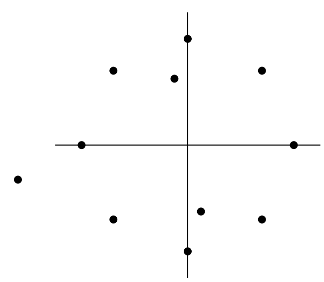
The anti-holomorphic automorphisms of the Riemann sphere map circles to circles. Out of our 12 points, the 8 roots of unity lie on the circle $|z| = 1$, but no other 8 can lie on a circle. Thus any anti-holomorphic automorphism that maps our configuration to itself, must fix the unit circle $|z| = 1$ and map the 8th roots of unity to each other.

The only such anti-holomorphic involutions are

(3) Reflection on the line $\mathbb{R} \cdot \epsilon$ where ϵ is a 16th root of unity, and
(4) $z \mapsto 1/\bar{z}$ or $z \mapsto -1/\bar{z}$.

A short case analysis shows that $C(\alpha)$ is not isomorphic (over $\mathbb{C}$) to a real curve, as long as β^{16} is not a positive real number.

The configuration depicted below shows 12 points $p_1, \dots, p_{12}$ on $\mathbb{C}$ that are invariant under $z \mapsto i/\bar{z}$ but not invariant under any anti-holomorphic involution.



EXAMPLE 1.75. We give an example of a smooth projective surface S such that if S is defined over a field extension $K/\mathbb{C}$ then $\operatorname{trdeg} K \geq 2$ but the intersection of all such fields of definition is $\mathbb{C}$.

Let X be a smooth projective variety such that

(1) $\operatorname{Aut}(X)$ is an infinite discrete group whose general orbit is Zariski dense in X and
(2) $\operatorname{Aut}(X)$ is generated by 2 finite subgroups G_1, G_2.

By (1.70.5), one such example is $B_0(E \times E)$, the blow up of the square of an elliptic curve at a point. There are also K3 surfaces with infinite automorphism group that is generated by 2 involutions (1.66).

Let $\Delta \subset X \times X$ be the diagonal and, using one of the projections, consider the family of smooth varieties

$$f : Y := B_\Delta X \times X \to X.$$

Note that $Y \to X$ is the universal family of the varieties of the form $B_x X$ for $x \in X$. This shows that $f : Y \to X$ can not be obtained by pull-back from any family over a lower dimensional base.

In particular, if $x \in X$ is general, then $\operatorname{Aut}(B_x X) = \mathbb{Z}/2$ if $X = B_0(E \times E)$ and $\operatorname{Aut}(B_x X) = 1$ if X is a K3 surface. The action of $\operatorname{Aut}(X)$ lifts to the diagonal action on Y.

Let $G \subset \operatorname{Aut}(X)$ be a finite subgroup. There is an open subset $U_G \subset X$ such that G operates on U_G without fixed points. Thus $f/G : Y/G \to X/G$ is a family of smooth varieties over U_G/G and $Y|_{U_G} \cong Y/G \times_{X/G} U_G$.

Let $K = \mathbb{C}(X)$ denote the function field of X. The variety we are interested in is Y_K, the generic fiber of $Y \to X$. The above considerations show that Y_K can be defined over $\mathbb{C}(X/G) = K^G$ for every finite subgroup $G \subset \operatorname{Aut}(X)$.

Note that $K = \mathbb{C}(X)$ is a function field of transcendence degree $\dim X$ over $\mathbb{C}$ and so are the subfields K^G. On the other hand, the intersection $K^{G_1} \cap K^{G_2}$ is $\mathbb{C}$. Indeed, any function in $K^{G_1} \cap K^{G_2}$ is constant on every G_1-orbit and also on every G_2-orbit. By assumption (2), it is also constant along every $\operatorname{Aut}(X)$-orbit, hence constant by assumption (1).

This phenomenon is also connected with the behavior of ample line bundles on $\pi_i : Y \to Y/G_i$. Although both of the Y/G_i are projective, there are no ample line bundles L_i on Y/G_i such that $\pi_1^* L_1 \cong \pi_2^* L_2$.

1.8. Singularities of stable varieties

We recall the key definitions and results about singularities of stable varieties. These are treated much more thoroughly in [**Kol13c**]. Here we aim to be concise, discussing all that is necessary for the main results but leaving many details untouched.

Singularities of pairs.

Definition 1.76 (Pairs). We are primarily interested in pairs $\big(X, \Delta\big)$ where X is a normal variety over a perfect field and $\Delta = \sum a_i D_i$ a formal linear combination of prime divisors with rational coefficients. More generally, X can be a pure dimensional, reduced scheme of finite type over a perfect field such that ω_X is locally free outside a subset of codimension ≥ 2 and $\Delta = \sum a_i D_i$ a formal linear combination of prime divisors such that none of the D_i is contained in $\operatorname{Sing} X$. (Even more general scheme cases are discussed in [**Kol13c**, 2.4].)

Definition 1.77 (Discrepancy). Let $\big(X, \Delta\big)$ be a pair as above such that $m(K_X + \Delta)$ is Cartier for some $m > 0$.

Let $f : Y \to X$ be a (not necessarily proper) birational morphism from a normal variety Y, $E \subset Y$ the exceptional locus of f and $E_i \subset E$ the irreducible exceptional divisors. Let $f_*^{-1}\Delta := \sum a_i f_*^{-1} D_i$ denote the birational transform of Δ. Since $Y \setminus E \cong X \setminus f(E)$, there is a natural isomorphism of invertible sheaves

$$\iota_{Y\setminus E} : \omega_Y^{[m]}\big(m f_*^{-1}\Delta\big)|_{Y\setminus E} \cong f^*\big(\omega_X^{[m]}\big(m\Delta\big)\big)|_{Y\setminus E}. \tag{1.77.1}$$

Thus there are rational numbers $a(E_i, X, \Delta)$ such that $m \cdot a(E_i, X, \Delta)$ are integers, and $\iota_{Y\setminus E}$ extends to an isomorphism

$$\iota_Y : \omega_Y^{[m]}\big(m f_*^{-1}\Delta\big) \cong f^*\big(\omega_X^{[m]}\big(m\Delta\big)\big)\big(\textstyle\sum_i m \cdot a(E_i, X, \Delta) E_i\big). \tag{1.77.2}$$

This defines $a(E, X, \Delta)$ for exceptional divisors. Set $a(D, X, \Delta) := -\operatorname{coeff}_D \Delta$ for non-exceptional divisors $D \subset X$.

The rational number $a(E_i, X, \Delta)$ is called the *discrepancy* of E_i with respect to (X, Δ); it depends only on the valuation defined by E_i, not on the choice of f.

Warning about terminology. For most cases of interest to us, $a(E, X, \Delta) \geq -1$. For this reason, some authors use *log discrepancies*, defined as

$$a_\ell(E, X, \Delta) := 1 + a(E, X, \Delta). \tag{1.77.3}$$

Most unfortunately, recently some people started to use $a(E, X, \Delta)$ to denote the log discrepancy, creating ample opportunity for confusion.

DEFINITION 1.78. Let X be a normal variety of dimension ≥ 2 and $\Delta = \sum a_i D_i$ a $\mathbb{Q}$-divisor with $a_i \leq 1$. Assume that $m(K_X + \Delta)$ is Cartier for some $m > 0$. We say that (X, Δ) is

$$\left.\begin{array}{c} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{plt} \\ \textit{dlt} \\ \textit{lc} \end{array}\right\} \quad \text{if } a(E, X, \Delta) \text{ is } \quad \left\{\begin{array}{ll} > 0 & \text{for every exceptional } E, \\ \geq 0 & \text{for every exceptional } E, \\ > -1 & \text{for every } E, \\ > -1 & \text{for every exceptional } E, \\ > -1 & \text{if } \operatorname{center}_X E \subset \text{non-snc}(X, \Delta), \\ \geq -1 & \text{for every } E. \end{array}\right.$$

Here klt is short for *"Kawamata log terminal"*, plt for *"purely log terminal"*, dlt for *"divisorial log terminal"*, lc for *"log canonical"* and non-snc(X, Δ) denotes the set of points where (X, Δ) is not a simple normal crossing pair [**Kol13c**, 1.7].

The simplest examples are given by cones, see (2.34) for some basic results.

CM properties.

Many of the divisorial sheaves on an lc pair are Cohen-Macaulay (CM for short). [**Elk81**] proved that canonical singularities are rational. This was generalized by several authors, the following variant is due to [**KM98**, 5.25] and [**Fuj09**, 4.14]); see also [**Kol13c**, 2.88].

THEOREM 1.79. *Let (X, Δ) be a dlt pair over a field of characteristic 0, L a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor and $D \leq \lfloor \Delta \rfloor$ a reduced $\mathbb{Z}$-divisor. Then*

(1) *$\mathcal{O}_X$ is CM,*
(2) *$\mathcal{O}_X(-D - L)$ is CM,*
(3) *$\omega_X(D + L)$ is CM and*
(4) *if $D + L$ is effective then $\mathcal{O}_{D+L}$ is CM.* □

We will also need the following generalization; see [**Kol11a**] or [**Kol13c**, 7.31].

THEOREM 1.80. *Let (X, Δ) be dlt, D a (not necessarily effective) $\mathbb{Z}$-divisor and $\Delta' \leq \Delta$ an effective $\mathbb{Q}$-divisor on X such that $D \sim_{\mathbb{Q}} \Delta'$. Then $\mathcal{O}_X(-D)$ is CM.*

If (X, Δ) is lc then frequently $\mathcal{O}_X$ is not CM. The following variant of the above theorems, while much weaker, is quite useful. In increasing generality it was proved by [**Ale08, Fuj09, Kol11a**]; see [**Kol13c**, 7.20]. (Even stronger results are proved in [**AH12**].) We state it for semi-log-canonical pairs—to be defined in (1.85)—using the notion of log canonical centers (1.96).

THEOREM 1.81. *Let (X, Δ) be slc and $x \in X$ a point that is not an lc center (1.96). Let D be a $\mathbb{Z}$-divisor such that none of the irreducible components of D are contained in* $\operatorname{Sing} X$. *Assume that there is an effective $\mathbb{Q}$-divisor $\Delta' \leq \Delta$ such that $D \sim_{\mathbb{Q}} \Delta'$. Then*

(1) $\operatorname{depth}_x \mathcal{O}_X(-D) \geq \min\{3, \operatorname{codim}_X x\}$ *and*
(2) $\operatorname{depth}_x \omega_X(D) \geq \min\{3, \operatorname{codim}_X x\}$.

Proof. The first claim is proved in [**Kol13c**, 7.20]. To get the second note that, working locally, $K_X + \Delta \sim_{\mathbb{Q}} 0$, thus $-(K_X + D) \sim_{\mathbb{Q}} \Delta - \Delta'$ and $\Delta - \Delta' \leq \Delta$ is effective. Thus, by the first part, $\omega_X(D) \cong \mathcal{O}_X\big(-(-(K_X + D))\big)$ has depth $\geq \min\{3, \operatorname{codim}_X x\}$. □

Taking $D=0$ gives the following important special case, due to [**Ale08**].

COROLLARY 1.82. *Let (X,Δ) be slc and $x \in X$ a point of codimension ≥ 3 that is not an lc center. Then* $\operatorname{depth}_x \mathcal{O}_X \geq 3$ *and* $\operatorname{depth}_x \omega_X \geq 3$. □

Semi-log-canonical pairs.

DEFINITION 1.83. Let (R,m) be a local k-algebra and $\operatorname{char} k \neq 2$. We say that $\operatorname{Spec} R$ has a *node* if $\hat{R} \cong (R/m)[[x,y]]/(x^2-ay^2)$ for some unit $a \in \hat{R}$. (See [**Kol13c**, 1.41] for the definition of nodes in characteristic 2.)

As a very simple special case of (2.26) or of (10.43), all deformations of a node can be obtained by pull-back from the diagram

$$\begin{array}{ccccc} (xy=0) & \subset & \bigl(xy+t=0\bigr) & \subset & \mathbb{A}^2_{xy}\times\mathbb{A}^1_t \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \in & \mathbb{A}^1_t & = & \mathbb{A}^1_t. \end{array} \tag{1.83.1}$$

If the characteristic is 0 then all non-trivial deformations over $\hat{\mathbb{A}}^1_t$ are of the form

$$\begin{array}{ccccc} (xy=0) & \subset & \bigl(xy+t^n=0\bigr) & \subset & \hat{\mathbb{A}}^2_{xy}\times\hat{\mathbb{A}}^1_t \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \in & \hat{\mathbb{A}}^1_t & = & \hat{\mathbb{A}}^1_t. \end{array} \tag{1.83.2}$$

Thus the total space has canonical singularities; more precisely, Du Val singularities of type A (2.17).

DEFINITION 1.84. Recall that, by Serre's criterion, a scheme X is normal iff it is S_2 and regular at all codimension 1 points. As a weakening of normality, a scheme is called *demi-normal* if it is S_2 and its codimension 1 points are either regular points or nodes.

A 1-dimensional demi-normal variety is a curve C with nodes. It can be thought of as a smooth curve $\bar{C}$ (the normalization of C) together with pairs of points $p_i, p_i' \in \bar{C}$, obtained as the preimages of the nodes. Equivalently, we have the nodal divisor $\bar{D} = \sum_i p_i + p_i'$ on $\bar{C}$ plus a fixed point free involution on $\bar{D}$ given by $\tau : p_i \leftrightarrow p_i'$.

We aim to get a similar description for any demi-normal scheme X. Let $\pi : \bar{X} \to X$ denote the normalization and $D \subset X$ the divisor obtained as the closure of the nodes of X. Set $\bar{D} := \pi^{-1}(D)$ with reduced structure. Then $D, \bar{D}$ are the conductors of π and the induced map $\bar{D} \to D$ has degree 2 over the generic points. This gives a rational involution on $\bar{D}$ which becomes a regular involution on the normalization

$$\tau : \bar{D}^n \to \bar{D}^n. \tag{1.84.1}$$

It is easy to see [**Kol13c**, 5.3] that a demi-normal scheme X is uniquely determined by the triple

$$\bigl(\bar{X}, \bar{D}, \tau\bigr). \tag{1.84.2}$$

However, it is surprising difficult to understand which triples $\bigl(\bar{X}, \bar{D}, \tau\bigr)$ correspond to demi-normal schemes. The solution of this problem in the log canonical case, given in (1.94), is a key result for us.

Let X be a scheme and $j : X^0 \hookrightarrow X$ the largest open set that is demi-normal. If the normalization $\pi : X^{\mathrm{n}} \to X$ is finite (for example, X is excellent) then

$j_*\mathcal{O}_{X^0} \cap \pi_*\mathcal{O}_{X^n}$ is a coherent sheaf of algebras on X. Its spectrum over X is the *demi-normalization* of X, frequently denoted by $X^{\rm dn}$. Thus we have a factorization

$$\pi : X^{\rm n} \to X^{\rm dn} \stackrel{\tau}{\to} X, \tag{1.84.3}$$

$X^{\rm dn}$ is demi-normal and τ is an isomorphism over X^0.

Roughly speaking, the concept of semi-log-canonical is obtained by replacing "normal" with "demi-normal" in the definition of log canonical (1.78).

DEFINITION 1.85. Let X be a demi-normal scheme with normalization $\pi : \bar{X} \to X$ and conductors $D \subset X$ and $\bar{D} \subset \bar{X}$. Let Δ be an effective $\mathbb{Q}$-divisor whose support does not contain any irreducible component of D and $\bar{\Delta}$ the divisorial part of $\pi^{-1}(\Delta)$,

The pair (X, Δ) is called *semi-log-canonical* or *slc* if

(1) $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and
(2) one of the following equivalent conditions holds
  (a) $\big(\bar{X}, \bar{D} + \bar{\Delta}\big)$ is lc, or
  (b) $a(E, X, \Delta) \geq -1$ for every exceptional divisor E over X.

Note that (2.b) is the exact analog of the definition of log canonical given in (1.78). The equivalence of the conditions (2.a) and (2.b) is proved in [**Kol13c**, 5.10].

The discrepancy $a(E, X, \Delta)$ is not defined if $K_X + \Delta$ is not $\mathbb{Q}$-Cartier, thus (1.85.2.b) does not make sense unless (1.85.1) holds. By contrast, (1.85.2.a) makes sense if $K_{\bar{X}} + \bar{D} + \bar{\Delta}$ is $\mathbb{Q}$-Cartier, even if $K_X + \Delta$ is not.

Reid's covering lemma.

This is a method to compare properties of a scheme with properties of its finite ramified covers.

1.86 (Hurwitz formula). The main example is when $\pi : Y \to X$ is a finite, separable morphism between normal varieties of the same dimension but we also need the case when $\pi : Y \to X$ is a finite, separable morphism between demi-normal varieties such that π is étale over the nodes of X. Then

$$K_Y \sim R + \pi^* K_X, \tag{1.86.1}$$

where R is the *ramification divisor* of π. If none of the ramification indices is divisible by the characteristic then $R = \sum_D (e(D) - 1)D$ where $e(D)$ denotes the ramification index of π along the divisor $D \subset Y$.

Note that if π is quasi-étale, that is, étale outside a subset of codimension ≥ 2, then $R = 0$, hence $K_Y \sim \pi^* K_X$.

1.87. Let $\pi : Y \to X$ be a finite, separable morphism as in (1.86) and Δ_X a $\mathbb{Q}$-divisor on X. Set

$$\Delta_Y := -R + \pi^*\Delta_X. \tag{1.87.1}$$

With this choice, (1.86.1) gives that

$$K_Y + \Delta_Y \sim_{\mathbb{Q}} \pi^*\big(K_X + \Delta_X\big). \tag{1.87.2}$$

Reid's covering lemma compares the discrepancies of divisors over X and Y. For precise forms see [**Rei80**], [**KM98**, 5.20] or [**Kol13c**, 2.42-43]. We need the following special cases.

Claim 1.87.3. Using the above notation, assume that Δ_X and Δ_Y are both effective and one of the following holds.

(a) The characteristic is 0,
(b) π is Galois and $\deg \pi$ is not divisible by the residue characteristics, or
(c) $\deg \pi$ is less than the residue characteristics.

Then (X, Δ_X) is klt (resp. lc or slc) iff (Y, Δ_Y) is klt (resp. lc or slc). □

There are 2 cases when (1.87.3) especially simple.

Special case 1.87.4. If π is quasi-étale then $\Delta_Y = \pi^*\Delta_X$, thus we compare (X, Δ_X) and $(Y, \pi^*\Delta_X)$.

Special case 1.87.5. Let D_X be a reduced divisor on X such that π is étale over $X \setminus D_X$. Set $D_Y := \operatorname{red} \pi^*(D_X)$. Then $D_Y + R = \pi^*(D_X)$, thus we compare $(X, D_X + \Delta_X)$ and $(Y, D_Y + \pi^*\Delta_X)$.

We frequently use cyclic covers.

1.88 (Cyclic covers). See [**KM98**, 2.49–52] or [**Kol13c**, Sec.2.3] for details.

Let X be an S_2-scheme, L a rank 1 sheaf that is locally free in codimension 1 and s a section of $L^{[n]}$, where, as usual, the bracket denotes that we take the double dual of the usual tensor power. These data define a *cyclic cover* or μ_n *cover* $\pi : Y \to X$ such that

$$\pi_*\mathcal{O}_Y = \textstyle\sum_{i=0}^{m-1} L^{[-i]}$$

and

$$\pi_*\omega_{Y/C} \cong \mathcal{H}om_X\bigl(\pi_*\mathcal{O}_Y, \omega_{X/C}\bigr) = \textstyle\sum_{i=0}^{m-1} L^{[i]} \hat{\otimes} \omega_{X/C},$$

where $\hat{\otimes}$ denotes the double dual of the usual tensor product. The morphism π is étale over $x \in X$ iff L is locally free at x and $s(x) \neq 0$. Thus π is quasi-étale iff s is a nowhere zero section, hence $L^{[n]} \cong \mathcal{O}_X$.

Adjunction and the different.

Adjunction is a classical method that allows induction on the dimension by lifting information from divisors to the ambient variety.

Definition 1.89 (Poincaré residue map). Let X be a (pure dimensional) CM scheme and $S \subset X$ a subscheme of pure codimension 1. By applying $\mathcal{H}om(\ , \omega_X)$ to the exact sequence

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

we get the short exact sequence

$$0 \to \omega_X \to \omega_X(S) \overset{\mathcal{R}_S}{\to} \omega_S \to 0. \tag{1.89.1}$$

The map $\mathcal{R}_S : \omega_X(S) \to \omega_S$ is called the *Poincaré residue map.*

By taking tensor powers, we get maps

$$\mathcal{R}_S^{\otimes m} : \bigl(\omega_X(S)\bigr)^{\otimes m} \to \omega_S^{\otimes m},$$

but, if $m(K_X + S)$ and mK_S are Cartier for some $m > 0$ then we really would like to get a corresponding map between the locally free sheaves

$$\omega_X^{[m]}(mS)|_S \overset{???}{\dashrightarrow} \omega_S^{[m]}. \tag{1.89.2}$$

There is no such map in general and one needs a correction term.

Definition 1.90 (Different). Let X be a demi-normal variety over a perfect field, S a reduced divisor on X and Δ a $\mathbb{Q}$-divisor on X. We assume that there are no coincidences, that is, the irreducible components of $\operatorname{Supp} S, \operatorname{Supp}\Delta$ and $\operatorname{Sing} X$ are all different from each other. Let $\bar S \to S$ denote the normalization.

Then there is a closed subscheme $Z \subset S$ of codimension 1 such that $S \setminus Z$ and $X \setminus Z$ are both smooth along $S \setminus Z$, $\pi : (\bar S \setminus \pi^{-1}Z) \to (S\setminus Z)$ is an isomorphism and $\operatorname{Supp}\Delta \cap S \subset Z$.

Thus the Poincaré residue map (1.89) gives an isomorphism

$$\mathcal{R}^m_{S\setminus Z} : \pi^*\omega_X^{[m]}(mS+m\Delta)|_{(\bar S\setminus \pi^{-1}Z)} \cong \omega_{\bar S}^{[m]}|_{(\bar S\setminus\pi^{-1}Z)}.$$

Thus, if $m(K_X+S+\Delta)$ is Cartier then there is a unique (not necessarily effective) divisor $\Delta_{\bar S}$ on $\bar S$ supported on $\pi^{-1}Z$ such that $\mathcal{R}^m_{S\setminus Z}$ extends to an isomorphism

$$\mathcal{R}^m_{\bar S} : \left(\pi^*\omega_X^{[m]}(mS+m\Delta)\right) \cong \omega_{\bar S}^{[m]}\left(\Delta_{\bar S}\right). \tag{1.90.1}$$

We formally divide by m and define the *different* of Δ on $\bar S$ as the $\mathbb{Q}$-divisor

$$\operatorname{Diff}_{\bar S}(\Delta) := \tfrac{1}{m}\Delta_{\bar S}. \tag{1.90.2}$$

We can write (1.90.1) in terms of $\mathbb{Q}$-divisors as

$$(K_X+S+\Delta)|_{\bar S} \sim_{\mathbb{Q}} K_{\bar S} + \operatorname{Diff}_{\bar S}(\Delta). \tag{1.90.3}$$

Note that (1.90.3) has the disadvantage that it indicates only that the two sides are $\mathbb{Q}$-linearly equivalent, whereas (1.90.1) is a *canonical* isomorphism.

For simplicity, the above definition is stated only for the cases that we mainly use. We will occasionally need that if $(X, S+\Delta)$ is lc (or slc), then the obvious modification of the definition gives $\operatorname{Diff}_S \Delta$ and the two versions are related by the expected formula

$$\operatorname{Diff}_{\bar S}(\Delta) + K_{\bar S/S} = \pi^* \operatorname{Diff}_S(\Delta). \tag{1.90.4}$$

See [**Kol13c**, 4.2] for this result and for the most general setting where the different can be defined. The following basic properties of the different are proved in [**Kol13c**, 4.4–8].

Proposition 1.91. *Using the notation of* (1.90) *write* $\operatorname{Diff}_{\bar S}(\Delta) = \sum d_i V_i$ *where* $V_i \subset \bar S$ *are prime divisors. Then the following hold.*

(1) *If* $(X, S+\Delta)$ *is lc (or slc) then* $\left(\bar S, \operatorname{Diff}_{\bar S}(\Delta)\right)$ *is lc.*
(2) *If* $\operatorname{coeff}_D \Delta \in \{1, \frac12, \frac23, \frac34, \dots\}$ *for every prime divisor D then the same holds for* $\operatorname{Diff}_{\bar S}(\Delta)$.
(3) *If S is Cartier outside a codimension 3 subset then* $\operatorname{Diff}_{\bar S}(\Delta) = \pi^*\Delta$.
(4) *If K_X+S and D are both Cartier outside a codimension 3 subset then* $\operatorname{Diff}_{\bar S} D$ *is a $\mathbb{Z}$-divisor and* $\left(K_X+S+D\right)|_{\bar S} \sim K_{\bar S} + \operatorname{Diff}_{\bar S} D$.

The following facts about codimension 1 behavior of the different can be proved by elementary but somewhat lengthy computations; see [**Kol13c**, 2.31, 2.36].

Lemma 1.92. *Let S be a normal surface, $E \subset S$ a reduced curve and $\Delta = \sum d_i D_i$ an effective $\mathbb{Q}$-divisor. Assume that $0 \le d_i \le 1$ and $D_i \not\subset \operatorname{Supp} E$ for every i. Let $\pi : F \to E$ denote the normalization and let $x \in F$ be a point.*

(1) *If E is singular at $\pi(x)$ then* $\operatorname{coeff}_x \operatorname{Diff}_F(\Delta) \ge 1$ *and equality holds iff E has a node at $\pi(x)$ and $\pi(x) \notin \operatorname{Supp}\Delta$.*
(2) *If $\pi(x) \in D_i$ then* $\operatorname{coeff}_x \operatorname{Diff}_F(\Delta) \ge d_i$. □

The next Theorem—whose first part is proved in [**Kol92b**, 17.4] and second part in [**Kaw07**]—is frequently referred to as *adjunction* if we assume something about X and obtain conclusions about S, or *inversion of adjunction* if we assume something about S and obtain conclusions about X. See [**Kol13c**, 4.9] for a proof of a more precise version. The last cases uses the notions of minimal log discrepancy and log centers to be discussed in (1.95).

THEOREM 1.93. *Let X be a normal variety over a field of characteristic 0 and S a reduced divisor on X with normalization $\pi_S : \bar{S} \to S$. Let Δ be an effective $\mathbb{Q}$-divisor that has no irreducible components in common with S. Assume that $K_X + S + \Delta$ is $\mathbb{Q}$-Cartier. Then*

(1) *$\big(\bar{S}, \operatorname{Diff}_{\bar{S}}(\Delta)\big)$ is klt iff $\big(X, S + \Delta\big)$ is plt in a neighborhood of S and*
(2) *$\big(\bar{S}, \operatorname{Diff}_{\bar{S}}(\Delta)\big)$ is lc iff $\big(X, S + \Delta\big)$ is lc in a neighborhood of S.*
(3) *For any irreducible subset $Z \subset \bar{S}$ we have*
$$\operatorname{mld}\big(Z, \bar{S}, \operatorname{Diff}_{\bar{S}}(\Delta)\big) \le \operatorname{mld}\big(\pi_S(Z), X, S + \Delta\big),$$
provided the latter is ≤ 1. □

Characterization of slc pairs.

Let (X, Δ) be an slc pair. Let $\pi : \bar{X} \to X$ be the normalization, $\bar{D} \subset \bar{X}$ the conductor, $\bar{\Delta}$ the divisorial part of $\pi^{-1}(\Delta)$ and τ the involution on $\bar{D}^n$ constructed in (1.84). Thus we obtain a map
$$(X, \Delta) \mapsto \big(\bar{X}, \bar{D} + \bar{\Delta}, \tau\big)$$
from slc pairs to lc pairs with the extra involution on $\bar{D}^n$. As we noted in (1.84.2), this map is an injection. That is, $\big(\bar{X}, \bar{D} + \bar{\Delta}, \tau\big)$ uniquely determines (X, Δ). The following theorem, proved in [**Kol16b**] and [**Kol13c**, 5.13], describes the image.

THEOREM 1.94. *Over a field of characteristic 0, normalization gives a one-to-one correspondence:*
$$\left\{\begin{array}{c}\textit{Proper slc pairs}\\ (X,\Delta)\textit{ such that}\\ K_X+\Delta\textit{ is ample.}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{Proper lc pairs }\big(\bar{X},\bar{D}+\bar{\Delta}\big)\textit{ plus}\\ \textit{a generically fixed point free}\\ \textit{involution }\tau\textit{ of }\big(\bar{D}^{\mathrm{n}},\operatorname{Diff}_{\bar{D}^{\mathrm{n}}}\bar{\Delta}\big)\\ \textit{such that }K_{\bar{X}}+\bar{D}+\bar{\Delta}\textit{ is ample.}\end{array}\right\} \quad \square$$

Minimal log discrepancy and log centers.

DEFINITION 1.95. Let (X, Δ) be an slc pair and $W \subset X$ an irreducible subset. The *minimal log discrepancy* of W is defined as the infimum of the numbers $1 + a(E, X, \Delta)$ where E runs through all divisors over X such that $\operatorname{center}_X(E) = W$. It is denoted by
$$\operatorname{mld}(W, X, \Delta) \quad \text{or by} \quad \operatorname{mld}(W) \tag{1.95.1}$$
if the choice of (X, Δ) is clear. Note that if W is an irreducible divisor on X and $W \not\subset \operatorname{Sing} X$ then
$$\operatorname{mld}(W, X, \Delta) = 1 - \operatorname{coeff}_W \Delta. \tag{1.95.2}$$
If $W \subset X$ is a closed subset with irreducible components W_i then we set
$$\operatorname{mld}(W, X, \Delta) = \max_i \operatorname{mld}(W_i, X, \Delta). \tag{1.95.3}$$

If (X, Δ) is slc then, by definition, $\operatorname{mld}(W, X, \Delta) \ge 0$ for every W. The subvarieties with $\operatorname{mld}(W, X, \Delta) = 0$ play a key role in understanding (X, Δ).

DEFINITION 1.96. Let (X, Δ) be an slc pair. An irreducible subvariety $W \subset X$ is a *log canonical center* or *lc center* of (X, Δ) if $\operatorname{mld}(W, X, \Delta) = 0$. Equivalently, if there is a divisor E over X such that $a(E, X, \Delta) = -1$ and $\operatorname{center}_X E = W$. Log canonical centers have many useful properties.

(1) There are only finitely many lc centers. Their union is the *non-klt locus* of (X, Δ), denoted by $\operatorname{nklt}(X, \Delta)$.
(2) Any union of lc centers is seminormal and Du Bois; see (1.98.1–2).
(3) Any intersection of lc centers is also a union of lc centers; see [**Amb03, Fuj09, Amb11**] or (1.98.4).
(4) If (X, Δ) is snc then the lc centers of (X, Δ) are exactly the *strata* of $\Delta_{=1}$, that is, the irreducible components of the various intersections $D_{i_1} \cap \cdots \cap D_{i_s}$ where the D_{i_k} appear in Δ with coefficient 1, see [**Kol13c**, 2.11]. More generally, this also holds if (X, Δ) is dlt; see [**Fuj07**, Sec.3.9] or [**Kol13c**, 4.16].
(5) At codimension 2 normal points, $\operatorname{nklt}(X, \Delta)$ is either smooth or has a node; see [**Kol13c**, 2.31].

DEFINITION 1.97. Let (X, Δ) be an slc pair. An irreducible subvariety $W \subset X$ is a *log center* of (X, Δ) if $\operatorname{mld}(W, X, \Delta) < 1$.

Building on earlier results of [**Amb03, Fuj09, Amb11**], part 1 of the following theorem is proved in [**KK10**] and the rest in [**Kol14**]; see also [**Kol13c**, Chap.7].

THEOREM 1.98. *Let (X, Δ) be an slc pair and $Z, W \subset X$ closed, reduced subsets.*

(1) *If* $\operatorname{mld}(Z, X, \Delta) = 0$ *then Z is Du Bois (cf. (2.63) or* [**Kol13c**, 6.32]*).*
(2) *If* $\operatorname{mld}(Z, X, \Delta) < \frac{1}{6}$ *then Z seminormal (3.24).*
(3) *If* $\operatorname{mld}(Z, X, \Delta) + \operatorname{mld}(W, X, \Delta) < \frac{1}{2}$ *then $Z \cap W$ is reduced.*
(4) $\operatorname{mld}(Z \cap W, X, \Delta) \leq \operatorname{mld}(Z, X, \Delta) + \operatorname{mld}(W, X, \Delta)$. $\square$

CHAPTER 2

One-parameter families

In [**Kol13c**] we studied in detail canonical and semi-log-canonical models, especially their singularities; a summary of the main results is given in Section 1.8. These are the objects that correspond to the points in a moduli functor/stack of canonical and semi-log-canonical models. We start the study of the general moduli problem with 1-parameter families. That is, we investigate the moduli functor/stack of semi-log-canonical models over 1-dimensional regular schemes.

In traditional moduli theory, for instance for curves, smooth varieties or sheaves, the description of all families over 1-dimensional regular schemes pretty much completes the story: the definitions and theorems have obvious generalizations to families over an arbitrary base. The best examples are the valuative criteria of separatedness and properness; we discussed these in (1.21.1–2). In our case, however, much remains to be done in order to work over arbitrary base schemes.

Two notions of locally stable or semi-log-canonical families are introduced in Section 2.1. Their equivalence is proved in characteristic 0, but remains open in general. For surfaces, one can give a rather complete étale-local description of all locally stable families; this is worked out in Section 2.2.

A series of higher dimensional examples is presented in Section 2.3. These show that stable degenerations of smooth projective varieties can get rather complicated.

Next we turn to global questions and define our main objects, stable families in Section 2.4. The main result says that stable families satisfy the valuative criteria of separatedness and properness.

Cohomological properties of stable families are studied in Section 2.5. In particular, we show that in a proper locally stable family $f : X \to C$, the basic numerical invariants $h^i(X_c, \mathcal{O}_{X_c})$ and $h^i(X_c, \omega_{X_c})$ are independent of $c \in C$. We also show that being CM is deformation invariant.

In the next two sections we turn to a key problem of the theory: Understanding the difference between the divisor theoretic and the scheme theoretic restriction of divisors, equivalently, the role of embedded points. The general theory is outlined in Section 2.6. Then in Section 2.7 we show that if all the coefficients of the boundary divisor are $> \frac{1}{2}$ then we need not worry about embedded points in moduli questions. We see in Chapter ?? that, in this case, all variants of the moduli functor/stack all agree with each other.

In order to get the stronger form of the local stability criterion we prove several Grothendieck–Lefschetz-type theorems in Sections 2.8–2.9, building on the techniques of Section 2.6.

2.1. Locally stable families

Following the pattern established in Section 1.3, we expect that the definition of a stable family $f : (X, \Delta) \to S$ consists of some local conditions describing the singularities of f and a global condition, that $K_X + \Delta$ be f-ample. We are now ready to formulate the correct local condition, at least for 1-parameter families.

Assumptions. While the basic definitions (2.1–2.2) are formulated for arbitrary schemes, the results of this Section are known only in characteristic 0.

DEFINITION 2.1. Let C be a regular 1-dimensional scheme. A *family of varieties* over C is a flat morphism $f : X \to C$ whose fibers are pure dimensional and geometrically reduced. For $c \in C$, let $X_c := f^{-1}(c)$ denote the fiber of f over c.

A *family of pairs* over C is a family of varieties $f : X \to C$ plus a $\mathbb{Q}$-divisor Δ on X such that, for every $c \in C$, the support of Δ does not contain any irreducible component of X_c and none of the irreducible components of $X_c \cap \operatorname{Supp} \Delta$ is contained in $\operatorname{Sing} X_c$. The latter condition holds if the fibers are slc pairs and it tuns out to be technically crucial, so it is much easier to assume it from the beginning.

The assumptions imply that X is regular at the generic points of $X_c \cap \operatorname{Supp} \Delta$, thus Δ is a $\mathbb{Q}$-Cartier divisor at the generic points of $X_c \cap \operatorname{Supp} \Delta$. In particular, $\Delta_c := \Delta|_{X_c}$ is a well defined $\mathbb{Q}$-divisor on X_c. Thus the pair-fibers (X_c, Δ_c) make sense.

DEFINITION 2.2. Let $f : (X, \Delta) \to C$ be a family of pairs over a regular 1-dimensional scheme. We say that $f : (X, \Delta) \to C$ is *locally stable* or *semi-log-canonical* (usually abbreviated as *slc*) if $(X, X_c + \Delta)$ is semi-log-canonical for every closed point $c \in C$.

Since X_c is a Cartier divisor, this implies that (X, Δ) is semi-log-canonical, hence X is demi-normal.

Warning. While the definition is made for arbitrary regular 1-dimensional schemes C, not much is known in positive and mixed characteristics, see (2.14).

As we noted in Section 1.3, usually (2.2) can not be reformulated as a condition about the fibers of f only. (Significant exceptions are discussed in (2.5) and (2.7).) The following result, however, comes close to achieving this.

THEOREM 2.3. *Let C be a smooth curve over a field of characteristic zero and $f : (X, \Delta) \to C$ a family of pairs over C with Δ effective. For any $c \in C$ and $p \in X_c := f^{-1}(c)$ the following are equivalent.*

1. *$f : (X, \Delta) \to C$ is locally stable in an open neighborhood of p in X.*
2. *$K_{X/C} + \Delta$ is $\mathbb{Q}$-Cartier at p and $\big(X_c, \Delta_c\big)$ is semi-log-canonical in an open neighborhood of p in X_c.*
3. *$K_{X/C} + \Delta$ is $\mathbb{Q}$-Cartier at p and $\big(\bar{X}_c, \operatorname{Diff}_{\bar{X}_c}(\Delta)\big)$ is log canonical in an open neighborhood of $\pi^{-1}(p)$ in $\bar{X}_c$, where $\pi : \bar{X}_c \to X_c$ denotes the normalization.*

While it is hard to see how (2.2) could be generalized to families over higher dimensional bases, the variants (2.3.2–3) make sense in general. This observation leads to the general definition of our moduli functor in Chapters **??**?.

Proof. If (3) holds then inversion of adjunction (1.93) shows that $(X, X_c + \Delta)$ is semi-log-canonical in a neighborhood of p and, by [**Kol13c**, 4.10] this continues to hold if we vary the fiber X_c. Thus (3) $\Rightarrow$ (1) and the converse also holds since (1.93) works both ways.

Since X_c is a Cartier divisor in X, the restriction $\Delta|_{X_c}$ equals the different $\operatorname{Diff}_{X_c}(\Delta)$ by (1.91). Furthermore, by (1.90.4)

$$K_{\bar{X}_c} + \operatorname{Diff}_{\bar{X}_c}(\Delta) = \pi^*\big(K_{X_c} + \operatorname{Diff}_{X_c}(\Delta)\big).$$

Thus it would seem that (1.85) says that (2) $\Leftrightarrow$ (3).

This is almost the case, except that in order to apply (1.85) we need to know that X_c is demi-normal.

By assumption X_c is geometrically reduced and an easy local computation shows that X_c is either smooth or has nodes at codimension 1 points; see [**Kol13c**, 2.33]. Thus it remains to prove that X_c is S_2.

This is actually quite subtle, with at least three different proofs, all of which provide valuable insight.

First, if the generic fiber is klt, then, by (2.13), (X, Δ) is klt. Thus X is CM by (1.79) and so is every fiber X_c. In general, however, (X, Δ) is not klt and X is not CM. However, CM is much more than we need.

The second method looks carefully at what weaker versions of CM would still imply that the fibers are S_2. Since the X_c are Cartier divisors in X, it is enough to prove that X is S_3. As noted in [**Kol13c**, 3.6], X is not S_3 in general; fortunately this is not a problem for us. If $g \in C$ is the generic point, then a local ring of X_g is also a local ring of X, hence X_g is S_2 if X is S_2. Therefore $\big(X_g, \Delta_g\big)$ is semi-log-canonical. If $c \in C$ is a closed points and $p \in X_c$ has codimension ≥ 2 then $p \in X$ has codimension ≥ 3, thus $\operatorname{depth}_p \mathcal{O}_X \geq 3$ by (1.82), hence $\operatorname{depth}_p \mathcal{O}_{X_c} \geq 2$. Thus again X_c is S_2.

Third, we know that X_c is a Cartier divisor on a demi-normal scheme. A local version of the Enriques-Severi-Zariski lemma, proved in [**Gro68**, XIII.2.1], says that if D is a Cartier divisor on an S_2 scheme and $p \in D$ has codimension ≥ 2 then $\hat{D}_p \setminus \{p\}$ is connected, where $\hat{D}_p$ denotes the completion of D at p. Thus X_c has this local connectedness property.

Furthermore, X_c is the union of log canonical centers of $\big(X, X_c+\Delta\big)$. Therefore, X_c is seminormal by (1.98.2). These two observations together imply that X_c is S_2, hence demi-normal. $\square$

Remark 2.4. We prove in (3.62) that if $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier then, for any given $m \in \mathbb{Z}$, $m(K_{X/S} + \Delta)$ is Cartier near X_s iff $m(K_s + \Delta_s)$ is Cartier.

2.5 (When is $K_{X/C}+\Delta$ automatically $\mathbb{Q}$-Cartier?). In (2.3.2–3) we make a fiber-wise assumption (that (X_c, Δ_c) be slc) and a global assumption (that $K_{X/C} + \Delta$ be $\mathbb{Q}$-Cartier).

If the latter condition is automatic, then we have a fiber-wise stability criterion. Section 1.3 contains examples of families of surfaces with quotient singularities where $K_{X/C}$ is not $\mathbb{Q}$-Cartier but the situation gets better in dimension ≥ 3.

We prove in (2.85) that if (X_c, Δ_c) is dlt and there is a subset $Z \subset X_c$ such that $K_{X/C} + \Delta$ is $\mathbb{Q}$-Cartier on $X \setminus Z$ and $\dim Z \leq \dim X_c - 3$, then $K_{X/C} + \Delta$ is $\mathbb{Q}$-Cartier everywhere.

The main aim of Section 2.8 is to show that this holds even if (X_c, Δ_c) is slc.

2.6 (The relative dualizing sheaf I). Let $f : (X, \Delta) \to C$ be locally stable. The relative dualizing sheaf $\omega_{X/C} \cong \mathcal{O}_C\big(K_{X/C}\big)$ exists. (Since ω_C is locally free, we can define it as $\omega_{X/C} := \omega_X \otimes f^*\omega_C^{-1}$. A more conceptual construction will be given in (2.70).)

By (1.89) for $c \in C$ there is a Poincaré residue (or adjunction) map

$$\mathcal{R} : \omega_{X/C}|_{X_c} \to \omega_{X_c}. \tag{2.6.1}$$

The map exists for any flat morphism $f : X \to C$ and general duality theory implies that it is an isomorphism if the fibers are CM. It is, however, not an isomorphism in general but we prove in (2.69) that for locally stable morphisms the adjunction map is an isomorphism. Thus $\omega_{X/C}$ can be thought of as a flat family of the dualizing sheaves of the fibers.

The isomorphism in (2.6.1) is easy to prove if the fibers are dlt or if $K_{X/C}$ is $\mathbb{Q}$-Cartier (2.76.1). A proof for slc fibers, following [**Kol11a**] and [**Kol13c**, 7.22], follows directly from (1.82).

The general case, when C is replaced by an arbitrary base scheme, is quite subtle. The known proofs use the Du Bois property of X_c. The projective case was proved in [**KK10**] and the quasi-projective one in [**KK17**]. However, neither of these proofs works for complex analytic morphisms. We discuss these in Section 2.5.

It is also worth noting that the powers of the Poincaré residue map

$$\mathcal{R}^m : \omega_{X/C}^{[m]}|_{X_c} \to \omega_{X_c}^{[m]} \tag{2.6.2}$$

are isomorphisms for locally stable maps if $\Delta = 0$, but not in general; see (2.76.1) and (2.41).

Note that if ω_{X_c} is locally free then (2.6.1) implies that $\omega_{X/C}$ is also locally free along X_c. Thus (2.69) and (2.3) imply the following.

Corollary 2.7 (Deformations if K_{X_c} is Cartier). *Let $f : X \to C$ be a flat morphism of finite type over a field of characteristic 0 such that X_c is slc and ω_{X_c} is locally free for some $c \in C$. Then $\omega_{X/C}$ is locally free near X_c and f is locally stable near X_c.* □

Note that (2.7) is a special property of slc varieties. Analogous claims fail both for normal varieties (2.42) and for pairs (X, D). To see the latter, consider a flat family X_c of smooth quadrics in $\mathbb{P}^3$ becoming a quadric cone for $c = 0$. Let $D_c \subset X_c$ be two disjoint lines that degenerate to a pair of distinct lines on X_0. Then K_{X_c}, D_c are both Cartier divisors for every c, but on the total space X they give a divisor $K_X + D$ that is not even $\mathbb{Q}$-Cartier.

If X_c is canonical then K_{X_c} is Cartier in codimension 2. We can thus use (2.7) in codimension 2 and then (2.5) in higher codimensions obtain the next result.

Corollary 2.8 (Deformations if X_c is canonical). *Let $f : X \to C$ be a flat morphism of finite type over a field of characteristic 0 such that X_c is canonical for some $c \in C$. Then f is locally stable near X_c.* □

Permanence properties.

Proposition 2.9. *Let C be a smooth curve over a field of characteristic zero and $g : C' \to C$ be a quasi-finite morphism. If $f : (X, \Delta) \to C$ is locally stable then so is the pull-back*

$$g^*f : (X', \Delta') := \bigl(X \times_C C', \Delta \times_C C'\bigr) \to C'.$$

Proof. We may assume that $g : (c', C') \to (c, C)$ is a finite, local morphism, étale away from c'. Set $D := X_c$ and $D' := X'_{c'}$. By (1.87.5), $(X, D + \Delta)$ is lc iff $(X', D' + \Delta')$ is. The rest follows from (2.3). □

The following result shows that one can usually reduce questions about locally stable families to the special case when X is normal.

PROPOSITION 2.10. *Let C be a smooth curve over a field of characteristic zero and $f:(X,\Delta)\to C$ a family of pairs over C. Assume that X is demi-normal and let $\pi:\bar X\to X$ denote the normalization with conductor $\bar D\subset\bar X$ (1.84).*

(1) *If $f:(X,\Delta)\to C$ is locally stable then so is $f\circ\pi:\bigl(\bar X,\bar D+\bar\Delta\bigr)\to C$.*
(2) *If $K_X+\Delta$ is $\mathbb{Q}$-Cartier and $f\circ\pi:\bigl(\bar X,\bar D+\bar\Delta\bigr)\to C$ is locally stable then so is $f:(X,\Delta)\to C$.*

Proof. Fix a closed point $c\in C$. By (1.94) or [**Kol13c**, 5.38], if $K_X+\Delta$ is $\mathbb{Q}$-Cartier, then $\bigl(X,X_c+\Delta\bigr)$ is slc iff $\bigl(\bar X,\bar X_c+\bar D+\bar\Delta\bigr)$ is lc. □

The next result allows us to pass to hyperplane sections. This is quite useful in proofs that use induction on the dimension.

PROPOSITION 2.11 (Bertini theorem for local stability). *Let $f:(X,\Delta)\to C$ be locally stable and $H\in|H|$ a general divisor in a base point free linear system on X. Then the following morphisms are also locally stable.*

(1) *$f:(X,H+\Delta)\to C$,*
(2) *$f|_H:(H,\Delta|_H)\to C$ and*
(3) *the composite $f\circ\pi:\bigl(Y,\pi^{-1}(\Delta)\bigr)\to C$ where $\pi:Y\to X$ is a μ_m-cover ramified along H; see (1.88).*

Proof. As we noted in (2.2), we can assume that X is normal. Let $p:Y\to X$ be a log resolution of (X,Δ) such that

$$p^{-1}(\operatorname{Supp}\Delta)+\operatorname{Ex}(p)+(\text{any fiber of } f\circ p)$$

is an snc divisor [**Kol13c**, 10.46]. Pick $H\in|H|$ such that $p^{-1}(H)=p_*^{-1}(H)$ and

$$p^{-1}(H)+p^{-1}(\operatorname{Supp}\Delta)+\operatorname{Ex}(p)+(\text{any fiber of } f\circ p)$$

is an snc divisor. Then every exceptional divisor of p has the same discrepancy with respect to $(X,X_c+\Delta)$ and $(X,X_c+H+\Delta)$. Therefore, $(X,X_c+H+\Delta)$ is slc for every $c\in C$. Thus $f:(X,H+\Delta)\to C$ is locally stable, proving (1). By adjunction, this implies that $(H,H_c+\Delta|_H)$ is slc for every $c\in C$, proving (2). By (1.87),

$$\bigl(Y,Y_c+\pi^{-1}(\Delta)\bigr)\quad\text{is slc}\quad\Leftrightarrow\quad\Bigl(X,X_c+\bigl(1-\tfrac{1}{m}\bigr)H+\Delta\Bigr)\quad\text{is slc.}$$

The latter holds since even $(X,X_c+H+\Delta)$ is slc for every $c\in C$. □

2.12 (Inverse Bertini theorem, weak form). Inversion of adjunction (1.93) implies that if $f|_H:(H,\Delta|_H)\to C$ is locally stable then $f:(X,H+\Delta)\to S$, and hence also $f:(X,\Delta)\to S$, are locally stable in a neighborhood of H. A much stronger result will be proved in (5.6).

The following simple result shows that if $f:(X,\Delta)\to C$ is locally stable, then (X,Δ) behaves as if it were *canonical*, as far as divisors over closed fibers are concerned. In some situations, for instance in (2.47), this is a very useful observation, but at other times the technical problems caused by log canonical centers in the generic fiber are hard to overcome.

PROPOSITION 2.13. *Let $f:(X,\Delta)\to C$ be a locally stable morphism. Let E be a divisor over X such that* $\operatorname{center}_X E\subset X_c$ *for some closed point $c\in C$. Then* $a(E,X,\Delta)\geq 0$. *Therefore every log center of (X,Δ) dominates C. In particular, if the generic fiber is klt (resp. canonical) then (X,Δ) is also klt (resp. canonical).*

Proof. Since $(X, X_c + \Delta)$ is semi-log-canonical, $a(E, X, X_c + \Delta) \geq -1$. Let $\pi : Y \to X$ be a proper birational morphism such that E is a divisor on Y and let b_E denote the coefficient of E in $\pi^*(X_c)$. Then b_E is an integer and it is positive since $\operatorname{center}_X E \subset X_c$. Thus,

$$a(E, X, \Delta) = a(E, X, X_c + \Delta) + b_E \geq -1 + b_E \geq 0.$$

In particular, none of the log centers of (X, Δ) are contained in X_c. □

2.14 (Some results in positive characteristic). As we already noted, very few of the previous theorems are known in positive characteristic, but the following partial results are sometimes helpful.

(2.14.1) Let (X, Δ) be a pair and $g : Y \to X$ a smooth morphism. By [**Kol13c**, 2.14.2], if (X, Δ) is slc, lc, klt, ... then so is $(Y, g^*\Delta)$.

(2.14.2) As a special case of [**Kol13c**, 2.14.4] we see that if (X, Δ) is slc then, for every smooth curve C, the trivial family $(X, \Delta) \times C \to C$ is locally stable.

(2.14.3) The proof of (2.13) works in any characteristic. Applying this to a trivial family will have interesting consequences in (2.48).

(2.14.4) Let (X_i, Δ_i) be two pairs that are slc, lc, klt, ... Then their product $\big(X_1 \times X_2, X_1 \times \Delta_2 + \Delta_1 \times X_2\big)$ is also slc, lc, klt, ... This is a generalization of (2.14.2) and can be proved by the same method as in [**Kol13c**, 2.14.2], using [**Kol13c**, 2.22].

(2.14.5) Assume that $f : (X, \Delta) \to C$ is locally stable and let $g : C' \to C$ be a tamely ramified morphism. Then the pull-back

$$g^*f : \big(X \times_C C', \Delta \times_C C'\big) \to C'$$

is also locally stable. This follows from (1.87.3) as in (2.9); see [**Kol13c**, 2.42] for details.

(2.14.6) Neither the wildly ramified nor the inseparable case of (2.14.5) is known.

Other deformations of ω.

The dualizing sheaf plays a very special role in algebraic geometry, thus it is natural to focus on understanding the powers of the relative dualizing sheaf. [**LN16**] studies other deformations of ω that behave as well as one would expect from locally stable families. The next result, closely related to [**LN16**, 7.18], says that the relative dualizing sheaf is the "best" deformation of the dualizing sheaf of a fiber.

Proposition 2.15. *Let C be a smooth curve over a field of characteristic 0 and $f : X \to C$ a flat morphism. Assume that X_0 is slc and there is a rank 1, reflexive sheaf L on X and a restriction morphism $\Re_L : L|_{X_0} \to \omega_{X_0}$ such that its reflexive powers*

$$\Re_L^{[m]} : L^{[m]}|_{X_0} \to \omega_{X_0}^{[m]} \tag{2.15.1}$$

are isomorphisms for every m. Then

$$\Re^{[m]} : \omega_{X/C}^{[m]}|_{X_0} \to \omega_{X_0}^{[m]} \tag{2.15.2}$$

is an isomorphism for every m.

Proof. Let n be the smallest positive integer such that $\omega_{X_0}^{[n]}$ is locally free. By assumption, then $L^{[n]}$ is also locally free. The question is local, we may thus assume that X is local, hence $L^{[n]}$ is free. By (1.88) we can take a cyclic cover $\pi : Y \to X$ such that $\pi_*\mathcal{O}_Y = \sum_{i=0}^{n-1} L^{[-i]}$ and

$$\pi_*\omega_{Y/C} \cong \mathcal{H}om_X\bigl(\pi_*\mathcal{O}_Y, \omega_{X/C}\bigr) = \textstyle\sum_{i=0}^{n-1} L^{[i]} \mathbin{\hat{\otimes}} \omega_{X/C}.$$

The resulting $g : Y \to C$ is flat, Y_0 is slc by (1.87.3) and ω_{Y_0} is locally free. Therefore $\omega_{Y/C}$ is locally free by (2.7), hence free since Y is semilocal. Thus $\pi_*\omega_{Y/C} \cong \pi_*\mathcal{O}_Y$ and so one of the summands $L^{[i]} \mathbin{\hat{\otimes}} \omega_{X/C}$ is free. Restriction to X_0 tells us that in fact $i = n-1$. Next note that

$$\begin{array}{rcl} \omega_{X/C} & \cong & \omega_{X/C} \mathbin{\hat{\otimes}} L^{[n-1]} \mathbin{\hat{\otimes}} L \mathbin{\hat{\otimes}} L^{[-n]} \\ & \cong & L \mathbin{\hat{\otimes}} \bigl(\omega_{X/C} \mathbin{\hat{\otimes}} L^{[n-1]}\bigr) \mathbin{\hat{\otimes}} L^{[-n]} \\ & \cong & L \otimes \bigl(\omega_{X/C} \mathbin{\hat{\otimes}} L^{[n-1]}\bigr) \otimes L^{[-n]}\bigr) \cong L, \end{array}$$

where in the last line we changed to the usual tensor product since the tensor product of a reflexive sheaf and of a line bundle is reflexive. Thus (2.15.2) is obtained from (2.15.1) by tensoring with a line bundle. $\square$

2.2. Locally stable families of surfaces

In this section we develop a rather complete local picture of slc families of surfaces. That is, we start with a pointed, local slc pair $(x \in X_0, \Delta_0)$ and aim to describe all locally stable deformations over local schemes $0 \in S$

$$\begin{array}{ccc} (X_0, \Delta_0) & \subset & (X_S, \Delta_S) \\ \downarrow & & \downarrow \\ 0 & \in & S. \end{array}$$

In the study of singularities it is natural to work étale-locally. That is, two pointed schemes $(x_1 \in X_1)$ and $(x_2 \in X_2)$ are considered the same if there is a third pointed scheme $(x_3 \in X_3)$ and a strictly étale morphisms of pointed schemes

$$(x_1 \in X_1) \stackrel{\pi_1}{\leftarrow} (x_3 \in X_3) \stackrel{\pi_2}{\rightarrow} (x_2 \in X_2),$$

where an étale morphism is called *strictly étale* if the induced maps on the residue fields $\pi_i^* : k(x_i) \to k(x_3)$ are isomorphisms. We will mostly work over algebraically closed fields and then strictness is automatic.

Since we have not yet defined the notion of a locally stable family in general, we concentrate on the case when S is the spectrum of a DVR.

We start by recalling the classification of lc surface singularities. This has a long history, starting with [**DV34**]. For simplicity we work over an algebraically closed field. It turns out that lc surface singularities have a very clear description using their dual graphs and this is independent of the characteristic. (By contrast, the equations of the singularities depend on the characteristic.)

Definition 2.16 (Dual graph). Let $(0 \in S)$ be a normal surface singularity over an algebraically closed field and $f : S' \to S$ the minimal resolution with irreducible exceptional curves $\{C_i\}$. We associate to this a *dual graph* $\Gamma = \Gamma(0 \in S)$ whose vertices correspond to the C_i. We use the negative of the self-intersection number $(C_i \cdot C_i)$ to represent a vertex and connect two vertices C_i, C_j by r edges iff $(C_i \cdot C_j) = r$. In the lc cases, the C_i are almost always smooth rational curves and $(C_i \cdot C_j) \le 1$, so we get a very transparent picture.

The *intersection matrix* of the resolution is $\bigl(-(C_i\cdot C_j)\bigr)$. This matrix is positive definite (essentially by the Hodge index theorem). Its determinant is denoted by

$$\det(\Gamma) := \det\bigl(-(C_i \cdot C_j)\bigr).$$

For example, if $\Gamma = \{\ 2 \text{ --- } 2 \text{ --- } 2\ \}$ then

$$\det(\Gamma) = \det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = 4.$$

Let B be a curve on S and B_i the local analytic branches of B that pass through $0 \in S$. The *extended dual graph* (Γ, B) has an additional vertex for each B_i, represented by $\bullet$, and it is connected to C_j by r edges if $\bigl(f_*^{-1}B_i \cdot C_j\bigr) = r$.

Next we list the dual graphs of all lc pairs $(0 \in S, B)$, starting with the terminal and canonical ones. For proofs see [**Ale93**] or [**Kol13c**, Sec.3.3].

2.17 (List of canonical surface singularities I).

Case 1 (Terminal). $(0 \in S, B)$ is terminal iff $B = \emptyset$ and S is smooth at 0.

Case 2 (Canonical). $(0 \in S, B)$ is canonical iff either B and S are both smooth at 0 or $B = \emptyset$ and Γ is one of the following. The corresponding singularities are called *Du Val* singularities or *rational double points* or *simple* surface singularities. See [**Dur79**] for more information. The equations below are correct only in characteristic zero; see [**Art77**] for the general case.

A_n: $x^2 + y^2 + z^{n+1} = 0$, with $n \geq 1$ curves in the dual graph:

$$2 \text{ --- } 2 \text{ --- } \cdots \text{ --- } 2 \text{ --- } 2$$

D_n: $x^2 + y^2z + z^{n-1} = 0$, with $n \geq 4$ curves in the dual graph:

$$\begin{array}{ccccccccc} & & 2 & & & & & & \\ & & | & & & & & & \\ 2 & \text{---} & 2 & \text{---} & \cdots & \text{---} & 2 & \text{---} & 2 \end{array}$$

E_6: $x^2 + y^3 + z^4 = 0$, with 6 curves in the dual graph:

$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & & | & & & & \\ 2 & \text{---} & 2 & \text{---} & 2 & \text{---} & 2 & \text{---} & 2 \end{array}$$

E_7: $x^2 + y^3 + yz^3 = 0$, with 7 curves in the dual graph:

$$\begin{array}{ccccccccccc} & & & & 2 & & & & & & \\ & & & & | & & & & & & \\ 2 & \text{---} & 2 & \text{---} & 2 & \text{---} & 2 & \text{---} & 2 & \text{---} & 2 \end{array}$$

E_8: $x^2 + y^3 + z^5 = 0$, with 8 curves in the dual graph:

$$\begin{array}{ccccccccccccc} & & & & 2 & & & & & & & & \\ & & & & | & & & & & & & & \\ 2 & \text{---} & 2 & \text{---} & 2 & \text{---} & 2 & \text{---} & 2 & \text{---} & 2 & \text{---} & 2 \end{array}$$

Before moving to the plt cases, it is best to fix our terminology.

DEFINITION 2.18. A connected graph is a *twig* if all vertices have ≤ 2 edges. Thus such a graph is of the form

$$c_1 \text{ --- } c_2 \text{ --- } \cdots \text{ --- } c_n$$

A connected graph is a *tree with 1 fork* if there is a vertex (called the root) with 3 edges and all other vertices have ≤ 2 edges. Thus such a dual graph is of the form

$$\begin{array}{ccccc} \Gamma_1 & \text{---} & c_0 & \text{---} & \Gamma_2 \\ & & \vert & & \\ & & \Gamma_3 & & \end{array}$$

where each Γ_i is twig joined to c_0 at an end vertex. We will mainly be interested in the cases when $\det(\Gamma) \in \{2, 3, 4, 5, 6\}$. These are

$$\begin{array}{lcl} \det(\Gamma) = 2 & \Leftrightarrow & \Gamma \text{ is } 2 \\ \det(\Gamma) = 3 & \Leftrightarrow & \Gamma \text{ is } 3 \text{ or } 2 - 2, \\ \det(\Gamma) = 4 & \Leftrightarrow & \Gamma \text{ is } 4 \text{ or } 2 - 2 - 2, \\ \det(\Gamma) = 5 & \Leftrightarrow & \Gamma \text{ is } 5 \text{ or } 2 - 2 - 2 - 2 \text{ or } 2 - 3 \text{ or } 3 - 2, \\ \det(\Gamma) = 6 & \Leftrightarrow & \Gamma \text{ is } 6 \text{ or } 2 - 2 - 2 - 2 - 2. \end{array}$$

2.19 (List of log canonical surface singularities II).

CASE 3 (Purely log terminal). The names below reflect that, at least in characteristic 0, these singularities are obtained as the quotient of $\mathbb{C}^2$ by the indicated type of group. See [**Bri68a**] for details.

Subcase 3.1 (Cyclic quotient). B is smooth at 0 (or empty) and (Γ, B) is

$$\bullet \text{ --- } c_1 \text{ --- } \cdots \text{ --- } c_n \quad \text{or} \quad c_1 \text{ --- } \cdots \text{ --- } c_n$$

Subcase 3.2 (Dihedral quotient).

$$\begin{array}{ccccccc} & & & & & & 2 \\ & & & & & / & \\ c_1 & \text{---} & \cdots & \text{---} & c_n & & \\ & & & & & \backslash & \\ & & & & & & 2 \end{array}$$

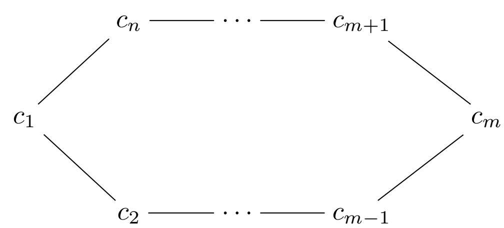
Subcase 3.3 (Other quotient). The dual graph is a tree with 1 fork (2.18) with 3 possibilities for $\big(\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3)\big)$:

(Tetrahedral) (2,3,3)
(Octahedral) (2,3,4)
(Icosahedral) (2,3,5).

CASE 4 (Log canonical with $B = 0$).

Subcase 4.1 (Simple elliptic). There is a unique exceptional curve E, it is smooth and of genus 1. If the self-intersection $r := -(E^2)$ is ≥ 3 then the singularity is isomorphic to the cone over the elliptic normal curve $E \subset \mathbb{P}^{r-1}$ of degree r.

Subcase 4.2 (Cusp). The dual graph is a circle of smooth rational curves



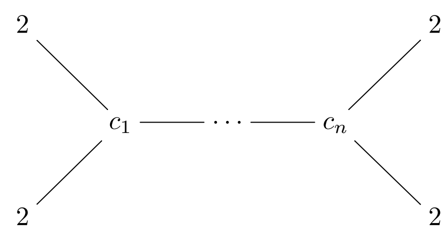
The cases $n = 1, 2$ are exceptional. For $n = 2$ we have 2 smooth rational curves meeting at 2 points and for $n = 1$ the unique exceptional curve is a rational curve with a single node. We can draw the dual graphs as

$$c_1 =\!=\!= c_2 \quad \text{and} \quad c_1.$$

For example the dual graphs of the three singularities $\big(z(xy - z^2) = x^4 + y^4\big)$, $\big(z^2 = x^2(x + y^2) + y^7\big)$ and $\big(z^2 = x^2(x^2 + y^2) + y^5\big)$ are

$$3 =\!=\!= 4, \quad 1 \quad \text{and} \quad 2.$$

Subcase 4.3 ($\mathbb{Z}/2$-quotient of a cusp).



(For $n = 1$ it is a $\mathbb{Z}/2$-quotient of a simple elliptic singularity.)

Subcase 4.4 (Simple elliptic quotient). The dual graph is a tree with 1 fork (2.18) with 3 possibilities for $\big(\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3)\big)$:

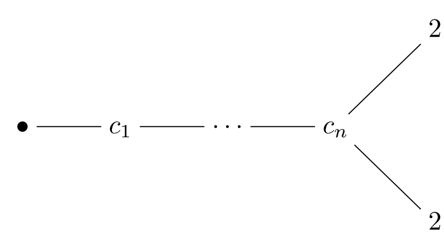
($\mathbb{Z}/3$-quotient) (3,3,3)
($\mathbb{Z}/4$-quotient) (2,4,4)
($\mathbb{Z}/6$-quotient) (2,3,6).

Case 5 (Log canonical with $B \neq 0$).

Subcase 5.1 (Cyclic). B has 2 smooth branches meeting transversally at 0 and (Γ, B) is

$$\bullet \text{ --- } c_1 \text{ --- } \cdots \text{ --- } c_n \text{ --- } \bullet$$

Subcase 5.2 (Dihedral).



2.20 (List of semi-log-canonical surface singularities III). The dual graphs are very similar to the previous ones but there are two possible changes due to the double curve of the surface S passing through the chosen point $0 \in S$.

In the normal case, the local picture represented by an edge is

$$(xy = 0) \subset \mathbb{A}^2, \quad \text{denoted by} \quad \circ \;—\; \circ,$$

where $(x = 0)$ and $(y = 0)$ are the exceptional curves meeting at the origin. We can now have a non-normal variant

$$(xy = z = 0) \subset (xy = 0) \subset \mathbb{A}^3, \quad \text{denoted by} \quad \circ \;\overset{\mathbf{d}}{—}\; \circ,$$

where $(x = z = 0)$ and $(y = z = 0)$ are the exceptional curves and $(x = y = 0)$ the double curve of the surface.

The local picture represented by a $\bullet$ and an edge was

$$(xy = 0) \subset \mathbb{A}^2, \quad \text{denoted by} \quad \bullet \;—\; \circ,$$

where $(x = 0)$ is a component of B and $(y = 0)$ an exceptional curve. We can now have a non-normal variant where (as long as char $\neq 2$) we create a pinch point by identifying the points $(0, y) \leftrightarrow (0, -y)$. The local equation is

$$(xy = z = 0) \subset (z^2 = xy^2) \subset \mathbb{A}^3, \quad \text{denoted by} \quad \mathbf{p} \;—\; \circ,$$

where $(y = z = 0)$ is the double curve of the surface and $(x = z = 0)$ an exceptional curve.

Case 6 (Semi-plt).

Subcase 6.1 (Higher pinch points). These are obtained from the cyclic dual graph of (2.19.Case.3.1) by replacing $\bullet \;—\; \circ$ by $\mathbf{p} \;—\; \circ$.

The simplest one is the pinch point, whose dual graph is $\mathbf{p} \;—\; 1$. The equation of the pinch point is $(x^2 = zy^2)$; it is its own semi-resolution.

As another example, start with the A_n singularity $(xy = z^{n+1})$ and pinch it along the line $(x = z = 0)$. The dual graph is

$$\mathbf{p} \;—\; 2 \;—\; \cdots \;—\; 2$$

with 2 occurring n-times. As a subring of $k[x, y, z]/(xy - z^{n+1})$ the coordinate ring is generated by (x, z, y^2, xy, yz) but $xy = z^{n+1}$. Thus $u_1 = x, u_2 = z, u_3 = y^2, u_4 = yz$ gives an embedding into $\mathbb{A}^4$. The image is a triple point whose equations can be written as

$$\operatorname{rank}\begin{pmatrix} u_2^n & u_4 & u_3 \\ u_1 & u_2^2 & u_4 \end{pmatrix} \leq 1.$$

Subcase 6.2. The dual graph is

$$\Gamma_1 \;\overset{\mathbf{d}}{—}\; \Gamma_2$$

where the Γ_i are twigs such that $\det(\Gamma_1) = \det(\Gamma_2)$. Note that here we allow $\Gamma_i = \{1\}$ and $1 \;\overset{\mathbf{d}}{—}\; 1$ corresponds to $(xy = 0) \subset \mathbb{A}^3$. Similarly $2 \;\overset{\mathbf{d}}{—}\; 2$ corresponds to

$$(x_1 y - z_1^2 = x_2 = z_2 = 0) \cup (x_2 y - z_2^2 = x_1 = z_1 = 0) \subset \mathbb{A}^5.$$

(It is a good exercise to check that if $\det(\Gamma_1) \neq \det(\Gamma_2)$ then the canonical class of the resulting surface is not $\mathbb{Q}$-Cartier. The case $2 \;\overset{\mathbf{d}}{—}\; 1$ is easy to compute by hand. The key in general is to compute the different on the double curve; see [**Kol13c**, 5.18] for details.)

Case 7 (Slc and $K_S + B$ Cartier).

Subcase 7.1 (Degenerate cusp). Here $B = 0$ and these are obtained from the dual graph of a cusp (2.19.Case.4.2) by replacing some of the edges $\circ$ — $\circ$ with $\circ \overset{\mathbf{d}}{—} \circ$.

The cases $n = 1, 2$ are again exceptional. For $n = 2$ we can replace either of the edges $\circ$ — $\circ$ with $\circ \overset{\mathbf{d}}{—} \circ$. For example, $(z^2 = x^2y^2)$ and $(z^2 = x^2y^2 + y^5)$ correspond to the dual graphs

$$1 \overset{\mathbf{d}}{\underset{\mathbf{d}}{=\!=}} 1 \quad \text{and} \quad 2 \overset{\mathbf{d}}{=\!=} 2.$$

For $n = 1$ the unique exceptional curve is a rational curve with a single node. We can think of the dual graph as

$$\mathbf{d} \big(c_1.$$

For example the singularities $\big(z^2 = x^2(x + y^2)\big)$ and $\big(z^2 = x^2(x^2 + y^2)\big)$ give the dual graphs

$$\mathbf{d} \big(1 \quad \text{and} \quad \mathbf{d} \big(2.$$

Subcase 7.2. These are obtained from the cyclic dual graph of (2.19.Case.5.1) by replacing some of the edges $\circ$ — $\circ$ with $\circ \overset{\mathbf{d}}{—} \circ$.

Case 8 (Slc and $2(K_S + B)$ Cartier).

Subcase 8.1. Here $B = 0$ and these are obtained from the dual graph of a $\mathbb{Z}/2$-quotient of a cusp (2.19.Case.4.3) by replacing some of the horizontal edges $\circ$ — $\circ$ with $\circ \overset{\mathbf{d}}{—} \circ$.

Subcase 8.2. These are obtained from the cyclic dual graph of (2.19.Case.5.1) by replacing at least one of $\bullet$ — $\circ$ by $\mathbf{p}$ — $\circ$ and replacing some of the edges $\circ$ — $\circ$ with $\circ \overset{\mathbf{d}}{—} \circ$.

Subcase 8.3. These are obtained from the dihedral dual graph of (2.19.Case.5.2) by replacing $\bullet$ — $\circ$ by $\mathbf{p}$ — $\circ$ and replacing some of the horizontal edges $\circ$ — $\circ$ with $\circ \overset{\mathbf{d}}{—} \circ$.

This completes the list of all slc surface singularities and now we turn to describing their locally stable deformations. An slc surface can be singular along a curve and the transversal hyperplane sections are nodes. Thus first we need to understand their deformations. In codimension 1 we have nodes; their deformations are described in (1.83).

The situation is much more complicated for surfaces, so we start with the case $\Delta_0 = 0$. It would be natural to first try to understand all flat deformations of $(x \in X_0)$ and then decide which of these are locally stable. However, in many interesting cases, flat deformations are rather complicated, but a good description of all locally stable deformations can be obtained by relating them to locally stable deformations of certain cyclic covers of X (1.88).

PROPOSITION 2.21. *Let k be a field and (X, D) a local, slc scheme over k with D reduced. Assume that $\omega_X^{[m]}(mD) \cong \mathcal{O}_X$ for some $m \geq 1$ that is not divisible by* $\operatorname{char} k$ *and let $\pi : (\tilde{X}, \tilde{D}) \to (X, D)$ be a corresponding μ_m-cover (1.88). Let R be a complete DVR with residue field k and set $S = \operatorname{Spec} R$.*

Taking μ_m-invariants establishes a bijection between

(1) *flat, affine, slc morphisms $\tilde{f} : (\tilde{X}_S, \tilde{D}_S) \to S$ such that $(\tilde{X}_0, \tilde{D}_0) \cong (\tilde{X}, \tilde{D})$ plus a μ_m-action on $(\tilde{X}_S, \tilde{D}_S)$ extending the μ_m-action on $(\tilde{X}, \tilde{D})$ and*
(2) *flat, affine, slc morphisms $f : (X_S, D_S) \to S$ such that $(X_0, D_0) \cong (X, D)$.*

Note that $\omega_{\tilde{X}}(\tilde{D})$ is locally free, and, in many cases, this makes $(\tilde{X}, \tilde{D})$ much simpler than (X, D). This reduction step is especially useful when $D = 0$, in which case $\omega_{\tilde{X}}$ is locally free. As we saw in (2.7), then all flat deformations of $\tilde{X}$ are slc. For surfaces, this leads to an almost complete description of all slc deformations.

ASIDE 2.22 (Deformations of quotients). Let $\tilde{X}$ be a scheme and G a finite group acting on it. The proof of (2.21) shows that G-equivariant deformations of $\tilde{X}$ always induce flat deformations of $X := \tilde{X}/G$ provided the characteristic does not divide $|G|$.

The converse is, however, quite subtle and usually deformations of X are not related to any deformation of $\tilde{X}$. As an example, consider the family $(xy - z^n - tz^m = 0)$ for $m < n$. For $t = 0$ the fiber is isomorphic to $\mathbb{C}^2/\mathbb{Z}_{n+1}$ and for $t \neq 0$ the fiber has a singularity (analytically) isomorphic to $\mathbb{C}^2/\mathbb{Z}_{m+1}$. There is no relation between the corresponding degree $n+1$ cover of the central fiber and the (local analytic) degree $m+1$ cover of a general fiber.

However, if G acts freely outside a subset of codimension ≥ 3 and $\tilde{X}$ is S_3, then every deformation of X arises from a deformation of $\tilde{X}$ [**Kol95a**, 12.7].

The following two examples show that the codimension ≥ 3 condition is not enough, not even for μ_m-covers.

1. Let E be an elliptic curve and S a K3 surface with a fixed point free involution τ. Set $Y = E \times S$ and $X = Y/\sigma$ where σ is the involution $(-1, \tau)$. Note that $p : Y \to X$ is an étale double cover, $h^1(Y, \mathcal{O}_Y) = 1$ and $h^1(X, \mathcal{O}_X) = 0$. Let H_X be a smooth ample divisor on X and H_Y its pull-back to Y. Consider the cones and general projections

$$\begin{array}{ccc} C_a(Y, H_Y) & \xrightarrow{p_C} & C_a(X, H_X) \\ \pi_Y \downarrow & & \downarrow \pi_X \\ \mathbb{A}^1 & = & \mathbb{A}^1 \end{array}$$

Since $h^1(X, \mathcal{O}_X) = 0$, the central fiber of π_X is the cone over H_X. By contrast, the central fiber F_0 of π_Y is not S_2 since $h^1(Y, \mathcal{O}_Y) \neq 0$ (see, for example [**Kol13c**, 3.10]). Thus, although the normalization of F_0 is the cone over H_Y, it is not isomorphic to it.

2. Let $g : X \to B$ be a smooth projective morphism to a smooth curve and H an ample line bundle on X. For large enough m and for every $r \in \mathbb{N}$, the direct images $g_*\mathcal{O}_X(rmH)$ commute with base change, hence the cones $C_a(X_b, \mathcal{O}_{X_b}(mH|_{X_b}))$ form a flat family.

The cones $C_a(X_b, \mathcal{O}_{X_b}(H|_{X_b}))$ are μ_m-covers of the cones $C_a(X_b, \mathcal{O}_{X_b}(mH|_{X_b}))$, but they form a flat family only if $g_*\mathcal{O}_X(rH)$ commutes with base change for every

r. That is, we get the required examples whenever $H^0\big(X_b, \mathcal{O}_{X_b}(H|_{X_b})\big)$ jumps for special values of b. The latter is easy to arrange, even on a family of smooth curves, as long as $\deg H|_{X_b} < 2g-2$.

2.23 (Proof of (2.21)). Let us start with $f: \big(X_S, D_S\big) \to S$. Since $\omega_{X_S}^{[m]}(mD_S)$ is locally free, the restriction map

$$\omega_{X_S}^{[m]}(mD_S) \twoheadrightarrow \omega_{X_0}^{[m]}(mD_0) \cong \mathcal{O}_{X_0}$$

is surjective. Since X_S is affine, the constant 1 section lifts back to a nowhere zero section $s: \mathcal{O}_{X_S} \cong \omega_{X_S}^{[m]}(mD_S)$. Let $\tilde{f}: \big(\tilde{X}_S, \tilde{D}_S\big) \to S$ be the corresponding μ_m-cover (1.88).

The pull back of the canonical class is computed by the Hurwitz formula (1.86.1). Since $\big(X_S, D_S + X_0\big)$ is slc, $\big(\tilde{X}_S, \tilde{D}_S + \tilde{X}_0\big)$ is also slc by (1.87.3). Thus $\tilde{f}$ is also locally stable. By (2.3), this implies that $\tilde{X}_0$ is S_2, hence it agrees with the μ_m-cover of (X_0, D_0).

To see the converse, let $g: Y \to S$ be any flat, affine morphism and G a reductive group (or group scheme) acting on Y with quotient $g/G: Y/G \to S$. Then $(g/G)_*\mathcal{O}_{Y/G} = \big(g_*\mathcal{O}_Y\big)^G$ is a direct summand of $g_*\mathcal{O}_Y$, hence g/G is also flat. Taking invariants commutes with base change since G is reductive. This shows that (1) $\Rightarrow$ (2). $\square$

Assumptions. For the rest of this Section, we work in characteristic 0.

2.24 (Classification plan). We establish an étale-local description of all slc deformations of surface singularities in four steps.

(1) Classify all slc surface singularities $(0, \tilde{S})$ with $\omega_{\tilde{S}}$ locally free.
(2) Classify all flat deformations of these $(0, \tilde{S})$.
(3) Classify all μ_m-actions on these surfaces and decide which ones correspond to our μ_m-covers.
(4) Describe the μ_m-actions on the miniversal deformation spaces of these $(0, \tilde{S})$.

(Almost everything works in general as long as the characteristic does not divide m but very little is has been proved otherwise.)

The first task was already accomplished in (2.17–2.20); we have Du Val singularities (2.17.Case.2), simple elliptic singularities and cusps (2.19.Cases.4.1–2) and degenerate cusps (2.20.Case.6). We can thus proceed to the next step (2.24.2).

2.25 (Deformations of slc surface singularities with K_S Cartier).

(Du Val singularities.) It is easy to work out the miniversal deformation space from the equations and (2.26). For each of the A_n, D_n, E_n cases the dimension of the miniversal deformation space is exactly n. For instance, for A_n we get

$$\begin{array}{ccccc} (xy + z^{n+1} = 0) & \subset & \big(xy + z^{n+1} + \sum_{i=0}^{n-1} t_i z^i = 0\big) & \subset & \mathbb{A}^3_{xyz} \times \mathbb{A}^n_{\mathbf{t}} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \in & \mathbb{A}^n_{\mathbf{t}} & = & \mathbb{A}^n_{\mathbf{t}}. \end{array}$$

(Elliptic/cusp/degenerate cusp.) Let $(0 \in S)$ be one of these singularities and C_i the exceptional curves of the minimal (semi)resolution. Set $m = -(\sum C_i)^2$ and write $(0 \in S_m)$ to indicate such a singularity.

(1) If $m = 1, 2, 3$ then $(0 \in S_m)$ is (isomorphic to) a singular point on a surface in $\mathbb{A}^3$ [**Sai74, Lau77**]. Their deformations are completely described by (2.26).
(2) If $m = 4$ then $(0 \in S_4)$ is (isomorphic to) a singular point on a surface in $\mathbb{A}^4$ that is a complete intersection of 2 hypersurfaces. The miniversal deformation space of a complete intersection can be described in a manner similar to (2.26); see [**Art76, Loo84, Har10**].
(3) If $m = 5$ then the deformations are completely described by the method of [**BE77**]; see [**Har10**, Sec.9].
(4) If $m \geq 3$ and $(0 \in S_m)$ is simple elliptic, then it is (isomorphic to) the singular point of a projective cone $\bar{S}_m \subset \mathbb{P}^m$ over an elliptic normal curve $E_m \subset \mathbb{P}^{m-1}$. By [**Pin74**, Sec.9], every deformation of $(0 \in S_m)$ is the restriction of a deformation of $\bar{S}_m \subset \mathbb{P}^m$. In particular, any smoothing corresponds to a smooth surface of degree m in $\mathbb{P}^m$. The latter have been fully understood classically: these are the del Pezzo surfaces embedded by $|-K|$. In particular, a simple elliptic $(0 \in S_m)$ is smoothable only for $m \leq 9$ [**Pin74**, Sec.9].
(5) The $m = 9$ case is especially interesting. Given an elliptic curve E, a degree 9 embedding $E_9 \hookrightarrow \mathbb{P}^8$ is given by global sections of a line bundle L_9 of degree 9 on E. Embeddings of E onto $\mathbb{P}^3$ are given by lines bundles L_3 of degree 3. If we take $(E \hookrightarrow \mathbb{P}^2)$ given by L_3 and then embed $\mathbb{P}^2$ into $\mathbb{P}^9$ by $\mathcal{O}_{\mathbb{P}^2}(3)$, then E is mapped to E_9 iff $L_3^{\otimes 3} \cong L_9$. For a fixed L_9 this gives 9 choices of L_3. Thus a given $E_9 \hookrightarrow \mathbb{P}^8$ is a hyperplane section of a $\mathbb{P}^2 \hookrightarrow \mathbb{P}^9$ in 9 different ways. Correspondingly, the deformation space $(0 \in S_9)$ has 9 smoothing components. (This was overlooked in [**Pin74**, Sec.9].) The automorphism group of $(0 \in S_9)$ permutes these 9 components. See [**LW86**, Sec.6] for another description.
(6) For $m \geq 6$ the deformation theory of cusps is much harder. A full description was given only recently by [**GHK15**].
(7) Degenerate cusps are all smoothable [**Ste98**].

2.26 (Deformations of hypersurface singularities). For general references, see [**Art76, Loo84**].

Let $0 \in X \subset \mathbb{A}^n_{\mathbf{x}}$ be a hypersurface singularity defined by an equation $\bigl(f(\mathbf{x}) = 0\bigr)$. Choose polynomials p_i that give a basis of

$$k[[x_1, \dots, x_n]] / \Bigl(f, \tfrac{\partial f}{\partial x_1}, \dots, \tfrac{\partial f}{\partial x_n}\Bigr). \tag{2.26.1}$$

If $(0 \in X)$ is an isolated singularity, then the quotient has finite length, say N. In this case, the miniversal deformation of $(0 \in X)$ is given by

$$\begin{array}{ccccc} X & \subset & \bigl(f(\mathbf{x}) + \sum_i t_i p_i(\mathbf{x}) = 0\bigr) & \subset & \mathbb{A}^n_{\mathbf{x}} \times \mathbb{A}^N_{\mathbf{t}} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \in & \mathbb{A}^N_{\mathbf{t}} & = & \mathbb{A}^N_{\mathbf{t}}. \end{array}$$

In particular, the miniversal deformation space $\operatorname{Def}(X)$ is smooth.

If the quotient in (2.26.1) has infinite length, then it is best to think of the resulting infinite dimensional deformation space as an inverse system of deformations over Artin rings whose embedding dimension goes to infinity.

The next step (2.24.3) in the classification is to describe all μ_m-actions, but it is more transparent to consider reductive commutative groups. These are of the form

$G \times \mathbb{G}_m^r$ where G is a finite, commutative group and $\mathbb{G}_m = \mathrm{GL}(1)$ the multiplicative group of scalars, cf. [**Hum75**, Sec.16].

2.27 (Commutative groups acting on Du Val singularities).
The action of a reductive commutative group on $\mathbb{A}^n$ can be diagonalized. Thus let $S \subset \mathbb{A}^3$ be a Du Val singularity which is invariant under a diagonal group action on $\mathbb{A}^3$. It is easy to work through any one of the standard classification methods (for instance, the one in [**KM98**, 4.24]) to obtain the following normal forms. In each case we describe first the maximal connected group actions and then the maximal non-connected group actions.

(Main series: $\mathbb{G}_m$-actions)

A_n: $(xy + z^{n+1} = 0)$ and $\mathbb{G}_m^2$ acts with character $(1, -1, 0), (0, n+1, 1)$.
D_n: $(x^2 + y^2 z + z^{n-1} = 0)$ and $\mathbb{G}_m$ acts with character $(n-1, n-2, 2)$.
E_6: $(x^2 + y^3 + z^4 = 0)$ and $\mathbb{G}_m$ acts with character $(6, 4, 3)$.
E_7: $(x^2 + y^3 + yz^3 = 0)$ and $\mathbb{G}_m$ acts with character $(9, 6, 4)$.
E_8: $(x^2 + y^3 + z^5 = 0)$ and $\mathbb{G}_m$ acts with character $(15, 10, 6)$.

(Twisted versions: $\mu_r \times \mathbb{G}_m$-actions)

A_n: $(x^2 + y^2 + z^{n+1} = 0)$. If $n+1$ is odd then $\mathbb{G}_m$ acts with character $(n+1, n+1, 2)$ and μ_2 acts with character $(0, 1, 0)$. If $n+1$ is even then $\mathbb{G}_m$ acts with character $(\frac{n+1}{2}, \frac{n+1}{2}, 1)$ and μ_2 acts with character $(0, 1, 0)$.
D_n: $(x^2 + y^2 z + z^{n-1} = 0)$, $\mathbb{G}_m$ acts with character $(n-1, n-2, 2)$ and μ_2 acts with character $(1, 1, 0)$.
D_4: $(x^2 + y^3 + z^3 = 0)$, $\mathbb{G}_m$ acts with character $(3, 2, 2)$ and μ_3 acts with character $(0, 1, 0)$.
E_6: $(x^2 + y^3 + z^4 = 0)$ and $\mathbb{G}_m$ acts with character $(6, 4, 3)$ and μ_2 acts with character $(1, 0, 0)$.

EXAMPLE 2.28 (Locally stable deformations of surface quotient singularities). Let $(0 \in S)$ be a surface quotient singularity with Du Val cover $(0 \in \tilde{S}) \to (0 \in S)$. By (2.21), the classification of locally stable deformations of all such $(0 \in S)$ is equivalent to classifying all cyclic group actions on Du Val singularities $(0 \in \tilde{S})$ that are free outside the origin and whose action on $\omega_{\tilde{S}} \otimes k(0)$ is faithful. This is straightforward, though somewhat tedious, using (2.27). Alternatively, one can use the classification of finite subgroups of GL(2) as in [**Bri68a**].

Thus the miniversal locally stable deformation space, which we denote by $\mathrm{Def}_{\mathrm{qG}}(S)$ (6.4), is the fixed point set of the corresponding cyclic group action on $\mathrm{Def}(\tilde{S})$, hence it is also smooth.

A_n-series: $\big(xy + z^{n+1} = 0\big)/\frac{1}{m}(1, (n+1)c - 1, c)$ for any m where $((n+1)c - 1, m) = 1$. These are equivariantly smoothable only if $m | (n+1)c$.
D_n-series: $\big(x^2 + y^2 z + z^{n-1} = 0\big)/\frac{1}{2k+1}(n-1, n-2, 2)$ where $(2k+1, n-2) = 1$. These are not equivariantly smoothable, but, for instance, if $2k+1 | n-1$, they deform to the quotient singularity $\mathbb{A}^2/\frac{1}{2k+1}(-1, 2)$.
E_6-series: $\big(x^2 + y^3 + z^4 = 0\big)/\frac{1}{m}(6, 4, 3)$ for $(m, 6) = 1$. For $m > 1$ all equivariant deformations are trivial, save for $m = 5$, when there is a 1-parameter family $\big(x^2 + y^3 + z^4 + \lambda yz = 0\big)/\frac{1}{5}(1, 4, 3)$.
E_7-series: $\big(x^2 + y^3 + yz^3 = 0\big)/\frac{1}{m}(9, 6, 4)$ for $(m, 6) = 1$. For $m > 1$ all equivariant deformations are trivial, save for $m = 5$ and $m = 7$, when

there are 1-parameter families $\left(x^2+y^3+yz^3+\lambda xz=0\right)/\frac{1}{5}(4,1,4)$ and $\left(x^2+y^3+yz^3+\lambda z=0\right)/\frac{1}{7}(2,6,4)$.

E_8**-series:** $\left(x^2+y^3+z^5=0\right)/\frac{1}{m}(15,10,6)$ for $(m,30)=1$. For $m>1$ all equivariant deformations are trivial, save for $m=7$, when there is a 1-parameter family $\left(x^2+y^3+z^5+\lambda yz=0\right)/\frac{1}{7}(1,3,6)$.

A_n**-twisted:** $\left(x^2+y^2+z^{n+1}=0\right)/\frac{1}{4m}(n+1,n+1+2m,2)$ for any $(2m,n+1)=1$. These are never equivariantly smoothable.

D_4**-twisted:** $\left(x^2+y^3+z^3=0\right)/\frac{1}{18k+9}(9k+6,1,6k+4)$. All equivariant deformations are trivial.

Example 2.29 (Quotients of simple elliptic and cusp singularities).

Let $(0\in S)$ be a simple elliptic, cusp or degenerate cusp singularity with minimal resolution (or semi-resolution) $f:T\to S$ and exceptional curves $C=\sum C_i$. Then $\omega_T(C)\cong f^*\omega_S$, which gives a canonical isomorphism

$$\omega_S\otimes k(0)\cong H^0(C,\omega_C).$$

Since C is either a smooth elliptic curve or a cycle of rational curves, $\operatorname{Aut}(C)$ is infinite but a finite index subgroup acts trivially on $H^0(C,\omega_C)$.

For cusps and for most simple elliptic singularities this leaves only μ_2-actions. The corresponding quotients are listed in (2.19.Case.4.3). When the elliptic curves have extra automorphisms, one can have μ_3,μ_4 and μ_6-actions. These were enumerated in (2.19.Case.4.4).

The following is one of the simplest degenerate cusp quotients.

Example 2.30 (Deformations of the double pinch point). Let $(0\in S)$ be the *double pinch point* singularity, defined by $\left(\bar S=\mathbb{A}^2,\bar D=(xy=0),\tau=(-1,-1)\right)$.

Here ω_S is not locally free but $\omega_S^{[2]}$ is and one can write S as the quotient

$$S=\tilde S/\tfrac{1}{2}(1,1,1)\quad\text{where}\quad \tilde S=\left(z^2-x^2y^2=0\right)\subset\mathbb{A}^3.$$

A local generator of $\omega_{\tilde S}$ is given by $z^{-1}dx\wedge dy$, which is anti-invariant. Thus ω_S has index 2 and $\tilde S\to S$ is the index 1 cover. Thus every locally stable deformation of S is obtained as the μ_2-quotient of an equivariant deformation of $\tilde S$. By (2.26) the miniversal deformation space is given by

$$\Bigl(z^2-x^2y^2+u_0+u_1xy+\sum_{i\ge 1}v_ix^{2i}+\sum_{j\ge 1}w_jy^{2j}=0\Bigr)/\tfrac{1}{2}(1,1,1).$$

When $u_0=u_1=v_1=w_1=0$, we get equimultiple deformations to μ_2-quotients of cusps with minimal resolution

$$\begin{array}{ccccccccccccc}
2 & & & & & & & & & & & & 2\\
 & \diagdown & & & & & & & & & & \diagup & \\
 & & 2 & -\cdots- & 2 & - & 3 & - & 2 & -\cdots- & 2 & & \\
 & \diagup & & & & & & & & & & \diagdown & \\
2 & & & & & & & & & & & & 2
\end{array}$$

The slc deformations of pairs (X,Δ) are more complicated, even if Δ is a $\mathbb{Z}$-divisor. One difficulty is that $\omega_S(D)$ is locally free for every pair

$$(S,D):=\left(\mathbb{A}^2,(xy=0)\right)/\tfrac{1}{n}(1,q)$$

since $\frac{dx}{x} \wedge \frac{dy}{y}$ is invariant. Thus we would need to describe the deformations of every such pair (S, D) by hand. The following is one of the simplest examples, and it already shows that the answer is likely to be subtle.

EXAMPLE 2.31 (Deformations of $\bigl(\mathbb{A}^2, (xy=0)\bigr)/\frac{1}{n}(1,1)$).

Flat deformations of the quotient singularity $H_n := \mathbb{A}^2/\frac{1}{n}(1,1)$ are quite well understood; see [**Pin74**]. H_n can be realized as the affine cone over the rational normal curve $C_n \subset \mathbb{P}^n$ and all local deformations are induced by deformations of the projective cone $C_p\bigl(C_n\bigr) \subset \mathbb{P}^{n+1}$. If $n \neq 4$ then the deformation space is irreducible and the smooth surfaces in it are minimal ruled surfaces of degree n in $\mathbb{P}^{n+1}$. We describe these completely below. (For $n=4$ there is another component, corresponding to the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.)

Since $(xy)^{-1}dx\wedge dy$ is invariant under the group action, it descends to a 2-form on H_n with poles along the curve $D_n := (xy=0)/\frac{1}{n}(1,1)$. Thus $K_{H_n} + D_n \sim 0$ and the pair (H_n, D_n) is lc. Our aim is to understand which deformations of H_n extend to a deformation of the pair (H_n, D_n).

Claim 2.31.1. Fix $n \geq 7$ and let $\pi : X \to \mathbb{A}^1$ be a general smoothing of H_n. Then the divisor D_n can not be extended to a divisor D_X such that $\pi : \bigl(X, D_X\bigr) \to \mathbb{A}^1$ is locally stable.

However, there are special smoothings $\pi : X' \to \mathbb{A}^1$ for which such a divisor D'_X exists.

Proof. For $m \in \mathbb{N}$, let $\mathbb{F}_m$ denote the ruled surface $\operatorname{Proj}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(-m)\bigr)$. Let $E_m \subset \mathbb{F}_m$ denote the section with self intersection $-m$ and $F \subset \mathbb{F}_m$ denote a fiber. Note that $K_{\mathbb{F}_m} \sim -\bigl(2E_m + (m+2)F\bigr)$.

For $a \geq 1$ set $A_{ma} := E + (m+a)F$. Then A_{ma} is very ample with self intersection $n := m+2a$ and it embeds $\mathbb{F}_m$ into $\mathbb{P}^{n+1}$ as a surface of degree n. Denote the image by S_{ma}. A general hyperplane section of S_{ma} is a rational normal curve $C_n \subset \mathbb{P}^n$. Consider the affine cones $X_{ma} := C_a\bigl(S_{ma}\bigr)$ and $H_n := C_a\bigl(C_n\bigr)$. We can choose coordinates such that

$$X_{ma} \subset \mathbb{A}^{n+2}_{x_1,\dots,x_{n+2}} \quad \text{and} \quad H_n = (x_{n+2} = 0).$$

The last coordinate projection gives $\pi : X_{ma} \to \mathbb{A}^1$ which is a flat deformation (in fact a smoothing) of H_n. By [**Kol13c**, 3.14.5]

$$\begin{aligned} H^0\bigl(X_{ma}, \mathcal{O}_{X_{ma}}(-K_{X_{ma}})\bigr) &= \textstyle\sum_{i\in\mathbb{Z}} x_0^i \cdot H^0\bigl(S_{ma}, \mathcal{O}_{S_{ma}}(-K_{S_{ma}} + iA_{ma})\bigr) \\ &= \textstyle\sum_{i\in\mathbb{Z}} x_0^i \cdot H^0\bigl(S_{ma}, \mathcal{O}_{S_{ma}}\bigl((2+i)E_m + (m+2+im+ia)F\bigr)\bigr). \end{aligned}$$

The lowest degree terms in the sum depend on m and a. For $i < -2$, we get 0. For $i = -2$ we have

$$H^0\bigl(S_{ma}, \mathcal{O}_{S_{ma}}\bigl((2-m-2a)F\bigr)\bigr) = H^0\bigl(S_{ma}, \mathcal{O}_{S_{ma}}\bigl((2-n)F\bigr)\bigr).$$

This is 0, unless $n=2$, that is, when X is the quadric cone in $\mathbb{A}^3$. Then D_2 is a Cartier divisor H_2 and so every deformation of H_2 extends to a deformation of the pair (H_2, D_2). Thus assume next that $n \geq 3$.

For $i = -1$ we have the summand

$$H^0\bigl(S_{ma}, \mathcal{O}_{S_{ma}}\bigl(E_m + (2-a)F\bigr)\bigr).$$

This is again zero if $a \geq 3$, but for $a=1$ we get a pencil $|E_m + F|$ (whose members are pairs of intersecting lines) and for $a=2$ we get a unique member E_m (which is a smooth conic in $\mathbb{P}^{n+1}$). This shows the following.

Claim 2.31.2. For $a = 1, 2$ and any $m \geq 0$, the anticanonical class of the 3-fold X_{ma} contains a (possibly reducible) quadric cone $D \subset X_{ma}$ and $\pi : \big(X_{ma}, D\big) \to \mathbb{A}^1$ is locally stable. $\square$

For $a \geq 3$, we have to look at the next term

$$H^0\big(S_{ma}, \mathcal{O}_{S_{ma}}\big(2E_m + (m+2)F\big)\big)$$

for a nonzero section. The corresponding linear system consists of reducible curves of the form $E_m + G_m$ where $G_m \in |E_m + (m+2)F|$. These curves have 2 nodes and arithmetic genus 1. Let $B \subset X_{ma}$ denote the cone over any such curve. Then $\big(X_{ma}, B\big)$ is log canonical but $\pi : \big(X_{ma}, B\big) \to \mathbb{A}^1$ is not locally stable since the restriction of B to H_n consists of $n+2$ lines through the vertex. Thus we have proved:

Claim 2.31.3. For $a \geq 3$ and any $m \geq 0$, the anticanonical class of X_{ma} does not contain any divisor D for which $\pi : \big(X_{ma}, D\big) \to \mathbb{A}^1$ is locally stable. $\square$

Note finally that the surfaces S_{ma} with $n = m + 2a$ form an irreducible family. General points correspond to the largest possible value $a = \lfloor (n-1)/2 \rfloor$. The surfaces with $a \leq 2$ correspond to a closed subset, which is a 2-dimensional subspace of the versal deformation space of H_n.

2.3. Examples of locally stable families

The aim of this section is to investigate, mostly through examples, fibers of locally stable morphisms. If (S, Δ) is slc then, for any smooth curve C, the projection $\pi : (S \times C, \Delta \times C) \to C$ is locally stable with fiber (S, Δ). Thus, in general we can only say that fibers of locally stable morphisms are exactly the slc pairs.

The question becomes, however, quite interesting, if we look at special fibers of locally stable morphisms whose general fibers are "nice," for instance smooth or canonical. The main point is thus to probe the difference between arbitrary snc singularities and those snc singularities that occur on locally stable degenerations of smooth varieties. We focus on two main questions.

QUESTION 2.32. Let $f : X \to T$ be a locally stable morphism over a pointed curve $(0 \in T)$ such that X_t is smooth for $t \neq 0$.

(1) Is X_0 CM?
(2) Are the irreducible components of X_0 CM?
(3) Is the normalization of X_0 CM?

QUESTION 2.33. Let $f : (X, \Delta) \to T$ be a locally stable morphism over a pointed curve $(0 \in T)$ such that X_t is smooth and Δ_t is snc for $t \neq 0$.

(1) Do the supports of $\{\Delta_t : t \in T\}$ form a flat family of divisors?
(2) Are the sheaves $\mathcal{O}_{X_0}(mK_{X_0} + \lfloor m\Delta_0 \rfloor)$ CM?
(3) Do the sheaves $\{\mathcal{O}_{X_t}(mK_{X_t} + \lfloor m\Delta_t \rfloor) : t \in T\}$ form a flat family?

A normal surface is always CM, and the (local analytic) irreducible components of an slc surface are CM. The latter follows from the classification of slc surfaces given in [**Kol13c**, Sec.2.2]. Starting with dimension 3, there are lc singularities that are not CM. The simplest examples are cones over Abelian varieties; see (2.34). On the other hand, we noted in (1.79) that canonical and log terminal singularities are CM and rational in characteristic 0.

Let us note next that the answer to (2.32.1) is positive, that is, X_0 is CM. Indeed, X is canonical by (2.13) and hence CM by (1.79). Therefore X_0 is also CM. A more complete answer to (2.32.1), without assuming that X_t is smooth or canonical for $t \neq 0$, is given in (2.68).

For locally stable families of pairs, the boundary provides additional sheaves whose CM properties are important to understand; this motivates (2.33). Unlike for (2.32), the answer to all of these is negative already for surfaces. The first convincing examples were discovered by Hassett (2.39). As a consequence, we see that we can not think of the deformations of (S, Δ) as a flat deformation of S and a flat deformation of Δ that are compatible in certain ways. In general it is imperative to view (S, Δ) as a single object. See, however, Section 2.7 for many cases where viewing (S, Δ) as a pair does work well.

Our examples will be either locally or globally cones and we need some basic information about them.

2.34 (Cones). Let X be a projective scheme with an ample line bundle L. The *affine cone* over X with conormal bundle L is

$$C_a(X, L) := \operatorname{Spec}_k \textstyle\sum_{m\geq 0} H^0(X, L^m).$$

Away from the vertex the cone is locally isomorphic to $X \times \mathbb{A}^1$, but the vertex is usually more complicated. The following results are quite straightforward, see [**Kol13c**, Sec.3.1] or details.

Let X be a projective variety with rational singularities over a field of characteristic 0 and L an ample line bundle on X.

(1) If $-K_X$ is ample then $C_a(X, L)$ is CM and has rational singularities.
(2) If $-K_X$ is nef (for instance, X is Calabi-Yau), then
  (a) $C_a(X, L)$ is CM iff $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$, and
  (b) $C_a(X, L)$ has rational singularities iff $H^i(X, \mathcal{O}_X) = 0$ for $0 < i \leq \dim X$.

Next let (X, Δ) be a projective, slc pair and L an ample Cartier divisor on X. Let $\Delta_{C_a(X,L)}$ denote the $\mathbb{Q}$-divisor corresponding Δ on $C_a(X, L)$. Assume that $K_X + \Delta \sim_{\mathbb{Q}} r \cdot L$ for some $r \in \mathbb{Q}$. Then $\bigl(C_a(X, L), \Delta_{C_a(X,L)}\bigr)$ is

(3) terminal iff $r < -1$ and (X, Δ) is terminal,
(4) canonical iff $r \leq -1$ and (X, Δ) is canonical,
(5) klt iff $r < 0$ (that is, $-(K_X + \Delta)$ is ample) and (X, Δ) is klt,
(6) dlt iff either $r < 0$ and (X, Δ) is dlt or $(X, \Delta) \cong \bigl(\mathbb{P}^n, (\prod x_i = 0)\bigr)$ and the cone is $\bigl(\mathbb{A}^{n+1}, (\prod x_i = 0)\bigr)$.
(7) lc iff $r \leq 0$ (that is, $-(K_X + \Delta)$ is nef) and (X, Δ) is lc,
(8) semi-log-canonical iff $r \leq 0$ and X is semi-log-canonical.

Example 2.35 (Counter example to (2.32.2)). Let $Q_0 \subset \mathbb{P}^4$ be the singular quadric $(xy - uv = 0)$. Let $|A|$ and $|B|$ be the two families of planes on Q_0 and H the hyperplane class. Let $S_1 \in |2A + H|$ be a general member. Note that S_1 is smooth away from the vertex of Q_0 and at the vertex it has 2 local analytic components intersecting at a single point. In particular, S_1 is non-normal and non-CM. (The easiest way to see these is to blow up a plane $B_1 \in |B|$. Then $B_{B_1}Q_0 \to Q_0$ is a small resolution whose exceptional set E is a smooth rational curve. The birational transform of $|2A + H|$ is a very ample linear system whose general member is a smooth surface that intersects E in 2 points. This is the normalization of the surface S_1.)

Let B_1, B_2 be planes in the other family. Then $X_0 := S_1 + B_1 + B_2 \sim 3H$, thus X_0 is a $(2) \cap (3)$ complete intersection in $\mathbb{P}^4$. We can thus write X_0 as the limit of a smooth family of $(2) \cap (3)$ complete intersections X_t. The general X_t is a smooth K3 surface.

On the other hand, X_0 can also be viewed as a general member of a flat family whose special fiber is $A_1 + A_2 + B_1 + B_2 + H$. The latter is slc by (2.34), thus X_0 is also slc. Hence $\{X_t : t \in T\}$ is a locally stable family such that X_t is a smooth K3 surface for $t \neq 0$. Moreover, the irreducible component $S_1 \subset X_0$ is not CM.

In this case, the source of the problem is easy to explain. At its singular point, S_1 is analytically reducible. The local analytic branches of S_1 and the normalization of S_1 are both smooth.

One can, however, modify this example to get analytically irreducible non-CM examples, albeit in dimension 3. To see this, let

$$Y_0 := C(X_0) = C(S_1) + C(B_1) + C(B_2) \subset \mathbb{P}^5$$

be the cone over X_0. It is still a $(2) \cap (3)$ complete intersection, thus we can write Y_0 as the limit of a smooth family of $(2) \cap (3)$ complete intersections Y_t. The general Y_t is a smooth Fano 3-fold.

By (2.34), Y_0 is slc, thus $\{Y_t : t \in T\}$ is a stable family such that Y_t is a smooth 3-fold for $t \neq 0$. Since S_1 is irreducible, the cone $C(S_1)$ is analytically irreducible at its vertex. It is non-normal along a line and non-CM.

One can check that the normalization of $C(S_1)$ is CM.

EXAMPLE 2.36 (Counter example to (2.32.3)). As in (2.35), let $Q_0 \subset \mathbb{P}^4$ be the singular quadric $(xy - uv = 0)$. On it, take a divisor

$$D_0 := A_1 + A_2 + \tfrac{1}{2}(B_1 + \cdots + B_4) + \tfrac{1}{2}H_4$$

where the A_i are planes in one family, the B_i are planes in the other family and H_4 is a general quartic section.

Note that (Q_0, D_0) is lc (2.34) and $2D_0$ is an octic section of Q_0. We can thus write (Q_0, D_0) as the limit of a family (Q_t, D_t) where Q_t is a smooth quadric and $2D_t$ a smooth octic hypersurface section of Q_t.

Let us now take the double covers of Q_t ramified along $2D_t$ (1.88) We get a family of $(2) \cap (8)$ complete intersections $X_t \subset \mathbb{P}(1^5, 4)$. The general X_t is smooth with ample canonical class. The special fiber is irreducible, slc, but not normal along $A_1 + A_2$, which is the union of 2 planes meeting at a point.

Let $\pi : \bar{X}_0 \to Q_0$ denote the projection of the normalization of X_0. Then

$$\pi_* \mathcal{O}_{\bar{X}_0} = \mathcal{O}_{Q_0} + \mathcal{O}_{Q_0}(4H - A_1 - A_2).$$

It is easy to compute that $\mathcal{O}_{Q_0}(4H - A_1 - A_2)$ is not CM (see, for instance, [**Kol13c**, 3.15]), so we conclude that $\bar{X}_0$ is not CM.

It is also interesting to note that the preimage of $A_1 + A_2$ in $\bar{X}_0$ is the union of 2 elliptic cones meeting at their common vertex. These are quite complicated lc centers.

EXAMPLE 2.37 (Counter example to (2.32.2–3)). Here is an example of a locally stable family of smooth projective varieties $\{Y_t : t \in T\}$ such that

(1) the canonical class K_{Y_t} is ample and Cartier for every t,

(2) Y_0 is slc and CM,
(3) the irreducible components of Y_0 are normal, but
(4) one of the irreducible components of Y_0 is not CM.

Let Z be a smooth Fano variety of dimension $n \geq 2$ such that $-K_Z$ is very ample, for instance $Z = \mathbb{P}^2$. Set $X := \mathbb{P}^1 \times Z$ and view it as embedded by $|-K_X|$ into $\mathbb{P}^N$ for suitable N. Let $C(X) \subset \mathbb{P}^{N+1}$ be the cone over X.

Let $M \in |-K_Z|$ be a smooth member and consider the following divisors in X:

$$D_0 := \{(0:1)\} \times Z, \ D_1 := \{(1:0)\} \times Z \quad \text{and} \quad D_2 := \mathbb{P}^1 \times M.$$

Note that $D_0 + D_1 + D_2 \sim -K_X$. Let $E_i \subset C(X)$ denote the cone over D_i. Then $E_0 + E_1 + E_2$ is a hyperplane section of $C(X)$ and $\big(C(X), E_0 + E_1 + E_2\big)$ is lc by (2.34).

For some $m > 0$, let $H_m \subset C(X)$ be a general intersection with a degree m hypersurface. Then

$$\big(C(X), E_0 + E_1 + E_2 + H_m\big)$$

is snc outside the vertex and is lc at the vertex. Set $Y_0 := E_0 + E_1 + E_2 + H_m$. Since $\mathcal{O}_{C(X)}(Y_0) \sim \mathcal{O}_{C(X)}(m+1)$, we can view Y_0 as an slc limit of a family of smooth hypersurface sections $Y_t \subset C(X)$.

The cone over X is CM by (2.34), hence its hyperplane section $E_0+E_1+E_2+H_m$ is also CM. However, E_2 is not CM. To see this, note that E_2 is the cone over $\mathbb{P}^1 \times M$ and, by the Küneth formula,

$$H^i(\mathbb{P}^1 \times M, \mathcal{O}_{\mathbb{P}^1 \times M}) = H^i(M, \mathcal{O}_M) = \begin{cases} k & \text{if } i = 0, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus E_2 is not CM by (2.34).

Example 2.38 (Easy counter examples to (2.33)). There are some obvious problems with all of the questions in (2.33) if the D_t contain divisors with different coefficients. For instance, let C be a smooth curve and $D', D'' \subset \mathbb{A}^1 \times C =: S$ two sections of the 1st projection π_1. Set $D := \frac{1}{2}(D' + D'')$. Then

$$\pi_1 : \big(S, D\big) \to \mathbb{A}^1$$

is a stable family of 1-dimensional pairs. For general t, the sections D', D'' intersect C_t at two different points and then $\mathcal{O}_{C_t}(K_{C_t} + \lfloor D_t \rfloor) \cong \mathcal{O}_C(K_C)$. If, however, D', D'' intersect C_t at the same point $p_t \in C_t$, then $\mathcal{O}_{C_t}(K_{C_t} + \lfloor D_t \rfloor) \cong \mathcal{O}_C(K_C)(p_t)$.

Similarly, the support of D_t is 2 points for general t but only 1 point for special values of t.

In 1-dimension one can correct for these problems by a more careful book keeping of the different parts of the divisor D_t. However, starting with dimension 2, no correction seems possible, except when all the coefficients are $> \frac{1}{2}$ (2.82).

The following example is due to Hassett (unpublished).

Example 2.39 (Counter example to (2.33.1–3)). We start with the already studied example of deformations of the cone $S \subset \mathbb{P}^5$ over the degree 4 rational normal curve (1.44), but here we add a boundary to it. Fix $r \geq 1$ and let D_S be the sum of $2r$ lines. Then $(S, \frac{1}{r} D_S)$ is lc and $\big(K_S + \frac{1}{r} D_S\big)^2 = 4$.

As in (1.44), there are two different deformations of the pair (S, D_S).

(2.39.1) First, set $P := \mathbb{P}^2$ and let D_P be the sum of r general lines. Then $(P, \frac{1}{r}D_P)$ is lc (even canonical if $r \geq 2$) and $\bigl(K_P + \frac{1}{r}D_P\bigr)^2 = 4$. The usual smoothing of $S \subset \mathbb{P}^5$ to the Veronese surface gives a family $f : (X, D_X) \to \mathbb{P}^1$ with general fiber (P, D_P) and special fiber (S, D_S). We can concretely realize this as deforming $(P, D_P) \subset \mathbb{P}^5$ to the cone over a general hyperplane section. Note that for any general D_S there is a choice of lines D_P such that the above limit is exactly D_S.

The total space (X, D_X) is the cone over (P, D_P) (blown up along curve) and X is $\mathbb{Q}$-factorial. Thus by (1.79) the structure sheaf of an effective divisor on X is CM.

In particular, D_S is a flat limit of D_P. Since the D_P is a plane curve of degree r, we conclude that

$$\chi(\mathcal{O}_{D_S}) = \chi(\mathcal{O}_{D_P}) = -\frac{r(r-3)}{2}.$$

(2.39.2) Second, set $Q := \mathbb{P}^1 \times \mathbb{P}^1$ and let A, B denote the classes of the 2 rulings. Let D_Q be the sum of r lines from the A-family. Then $(Q, \frac{1}{r}D_Q)$ is canonical and $\bigl(K_Q + \frac{1}{r}D_Q\bigr)^2 = 4$. The usual smoothing of $S \subset \mathbb{P}^5$ to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $H := A + 2B$ gives a family $g : (Y, D_Y) \to \mathbb{P}^1$ with general fiber (Q, D_Q) and special fiber (S, D_S). We can concretely realize this as deforming $(Q, D_Q) \subset \mathbb{P}^5$ to the cone over a general hyperplane section.

The total space (Y, D_Y) is the cone over (Q, D_Q) (blown up along curve) and Y is not $\mathbb{Q}$-factorial. However, $K_Q + \frac{1}{r}D_Q \sim_{\mathbb{Q}} -H$, thus $K_Y + \frac{1}{r}D_Y$ is $\mathbb{Q}$-Cartier and $(Y, S + \frac{1}{r}D_Y)$ is lc by inversion of adjunction (1.93) and so is $(Y, \frac{1}{r}D_Y)$.

In this case, however, D_S is not a flat limit of D_Q for $r > 1$. This follows, for instance, from comparing their Euler characteristic:

$$\chi(\mathcal{O}_{D_S}) = -\frac{r(r-3)}{2} \quad \text{and} \quad \chi(\mathcal{O}_{D_Q}) = r.$$

(2.39.3) Because of their role in the canonical ring, we are also interested in the sheaves $\mathcal{O}(mK + \lfloor \frac{m}{r} D \rfloor)$.

Let H_P be the hyperplane class of $P \subset \mathbb{P}^5$ (that is, 2 times a line $L \subset P$) and write $m = br + a$ where $0 \leq a < r$. Then

$$mK_P + \lfloor \tfrac{m}{r} D_P \rfloor + nH_P \sim (2n - 2m - a)L,$$

and hence

$$\begin{aligned} \chi(P, \mathcal{O}_P(mK_P + \lfloor \tfrac{m}{r} D_P \rfloor + nH_P)) &= \tbinom{2n-2m-a+2}{2} \\ &= \tbinom{2n-2m+2}{2} - a(2n - 2m + 1) + \tbinom{a}{2}. \end{aligned}$$

Again by (1.79) all divisorial sheaves on X are CM. Thus the restriction of $\mathcal{O}_X(mK_X + \lfloor \frac{m}{r} D_X \rfloor)$ to the central fiber S is $\mathcal{O}_S(mK_S + \lfloor \frac{m}{r} D_S \rfloor)$. In particular,

$$\chi(S, \mathcal{O}_S(mK_S + \lfloor \tfrac{m}{r} D_S \rfloor + nH_S)) = \binom{2n - 2m + 2}{2} - a(2n - 2m + 1) + \binom{a}{2}.$$

The other deformation again behaves differently. Write $m = br + a$ where $0 \leq a < r$. Then, for $H_Q \sim A + 2B$, we see that

$$mK_Q + \lfloor \tfrac{m}{r} D_Q \rfloor + nH_Q \sim (n - m - a)A + (2n - 2m)B,$$

and therefore

$$\chi(Q, \mathcal{O}(mK_Q + \lfloor \tfrac{m}{r} D_Q \rfloor + nH_Q) = \binom{2n - 2m + 2}{2} - a(2n - 2m + 1).$$

From this we conclude that the restriction of $\mathcal{O}_Y(mK_Y+\lfloor mD_Y\rfloor)$ to the central fiber S agrees with $\mathcal{O}_S(mK_S+\lfloor mD_S\rfloor)$ only if $a\in\{0,1\}$, that is when $m\equiv 0,1 \mod r$. The if part was clear from the beginning. Indeed, if $a=0$ then $\mathcal{O}_Y(mK_Y+\lfloor mD_Y\rfloor)=\mathcal{O}_Y(mK_Y+mD_Y)$ is locally free and if $a=1$ then

$$\mathcal{O}_Y(mK_Y+\lfloor mD_Y\rfloor)=\mathcal{O}_Y(K_Y)\otimes\mathcal{O}_Y\bigl((m-1)K_Y+(m-1)D_Y\bigr)$$

is $\mathcal{O}_Y(K_Y)$ tensored with a locally free sheaf. Both of these commute with restrictions.

In the other cases we only get an injection

$$\mathcal{O}_Y(mK_Y+\lfloor mD_Y\rfloor)|_S\hookrightarrow\mathcal{O}_S(mK_S+\lfloor mD_S\rfloor)$$

whose quotient is a torsion sheaf of length $\binom{a}{2}$ supported at the vertex.

Example 2.40 (Counter example to (2.33.1)). As in (2.37), let Z be a smooth Fano variety of dimension $n\geq 2$ such that $-K_Z$ is very ample. Set $X:=\mathbb{P}^1\times Z$ but now view it as embedded by global sections of $\mathcal{O}_{\mathbb{P}^1}(1)\otimes\mathcal{O}_Z(-K_Z)$ into $\mathbb{P}^N$ for suitable N. Let $C(X)\subset\mathbb{P}^{N+1}$ be the cone over X.

Fix $r\geq 1$ and let D_r be the sum of r distinct divisors of the form $\{\text{point}\}\times Z\subset X$. Let $H\subset X$ be a general hyperplane section. Then $H\sim_{\mathbb{Q}}-\bigl(K_X+\frac{1}{r}D_r\bigr)$, that is, $(X,\frac{1}{r}D_r)$ is (numerically) anticanonically embedded. Thus, by (2.34), $\bigl(C(H),\frac{1}{r}C(H\cap D_r)\bigr)$ is lc and there is a locally stable family with general fiber $(X,\frac{1}{r}D_r)$ and special fiber $\bigl(C(H),\frac{1}{r}C(H\cap D_r)\bigr)$.

However, $C(H\cap D_r)$ is not a flat deformation of D_r. Indeed, if $D_{ri}(\cong Z)$ is any irreducible component of D_r, then $C(H\cap D_{ri})$ is a flat deformation of D_{ri}. Thus $\amalg_i C(H\cap D_{ri})$ is a flat deformation of $D_r=\amalg_i D_{ri}$. Note further that $\amalg_i C(H\cap D_{ri})$ is the normalization of $C(H\cap D_r)$, and the normalization map is $r:1$ over the vertex of the cone. Thus

$$\begin{array}{rcl}\chi\bigl(D_r,\mathcal{O}_{D_r}\bigr) & = & \sum_i\chi\bigl(D_{ri},\mathcal{O}_{D_{ri}}\bigr)=\\ & = & \sum_i\chi\bigl(C(H\cap D_{ri}),\mathcal{O}_{C(H\cap D_{ri})}\bigr)\\ & \geq & \chi\bigl(C(H\cap D_r),\mathcal{O}_{C(H\cap D_r)}\bigr)+(r-1).\end{array}$$

Therefore $C(H\cap D_r)$ can not be a flat deformation of D_r for $r>1$. We pick up at least $r-1$ embedded points.

Example 2.41 (Counter example to (2.33.3)). Set $X:=C_a\bigl(\mathbb{P}^1\times\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1,a)\bigr)$ for some $0<a<n+1$. Let $D\subset X$ be the cone over a smooth divisor in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1,n+1-a)|$. Then (X,D) is canonical and K_X+D is Cartier.

Let $\pi:(X,D)\to\mathbb{A}^1$ be a general projection. Then π is locally stable and its central fiber is the cone $X_0=C_a\bigl(H,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1,a)|_H\bigr)$ where $H\in|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1,a)|$ is a smooth divisor.

We claim that if $2a>n+1$ then $r_m:\omega^{[m]}_{X/\mathbb{A}^1}\big|_{X_0}\to\omega^{[m]}_{X_0}$ is not surjective for $m\gg 1$.

Indeed, we can write this map as

$$\sum_{r\geq 0}H^0\bigl(\mathbb{P}^1\times\mathbb{P}^n,\mathcal{O}(r-2m,ra-(n+1)m)\bigr)\to\sum_{r\geq 0}H^0\bigl(H,\mathcal{O}(r-2m,ra-(n+1)m)|_H\bigr)$$

and r_m is surjective iff

$$H^1\bigl(\mathbb{P}^1\times\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(r-2m,ra-(n+1)m)\bigr)=0$$

for every $r\geq -1$. Choose $r=2m-2$. Then, by the Küneth formula, this group is

$$H^1\bigl(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-2)\bigr)\otimes H^0\bigl(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(m(2a-n-1)-2a)\bigr).$$

Since $2a-n-1>0$, this is nonzero for $m\geq 2a$.

The following example, related to [**Pat13**], shows that the relative dualizing sheaf does not commute with base change in general.

EXAMPLE 2.42. We give an example of a flat family of normal varieties $Y\to U$ such that ω_{Y_0} is locally free for some $0\in U$ yet $\omega_{Y/U}$ is not locally free along Y_0.

We start with a smooth, projective variety X such that $H^1(X,\mathcal{O}_X)\neq 0$ but $H^0(X,\omega_X)=H^1(X,\omega_X)=0$. For example, we can take $X=C\times\mathbb{P}^n$ where C is a smooth curve of genus >0 and $n\geq 2$.

Let L_0 be a very ample line bundle such that $L_0\otimes\omega_X$ is ample. All line bundles algebraically equivalent to L_0 are parametrized by $\operatorname{Pic}^L(X)$.

Pick a smooth divisor $D\subset X$ linearly equivalent to L_0. Our example will be the family of cones

$$Y_L:=\operatorname{Spec}_k \textstyle\sum_m H^0\bigl(D,(L\otimes\omega_X)^m|_D\bigr),$$

parametrized by a suitable open set $[L_0]\in U\subset\operatorname{Pic}^L(X)$.

The Y_L form a flat family iff the $h^0\bigl(D,(L\otimes\omega_X)^m|_D\bigr)$ are all constant on U. To compute these, consider the exact sequence

$$0\to(L\otimes\omega_X)^m(-D)\to(L\otimes\omega_X)^m\to(L\otimes\omega_X)^m|_D\to 0.$$

Since $(L\otimes\omega_X)^m(-D)$ is numerically equivalent to $\omega_X\otimes(L\otimes\omega_X)^{m-1}$, its higher cohomologies vanish. Thus $h^0\bigl(D,(L\otimes\omega_X)^m|_D\bigr)$ is independent of L for $m\geq 2$. If $m=1$ then $(L_0\otimes\omega_X)(-D)\cong\omega_X$ and we assumed that $H^0(X,\omega_X)=H^1(X,\omega_X)=0$. Thus $H^0\bigl(D,(L\otimes\omega_X)|_D\bigr)=H^0\bigl(X,L\otimes\omega_X\bigr)=0$ holds for all L in a neighborhood of $[L_0]$; this conditions defines our U.

The cones Y_L form the fibers of a flat morphism $Y\to U$. By [**Kol13c**, 3.14.4], ω_{Y_L} is locally free iff $L=L_0$. Thus $\omega_{Y/U}$ is not locally free along Y_{L_0} yet $\omega_{Y_{L_0}}$ is locally free.

2.4. Stable families

Next we define the notion of stable families over a regular 1-dimensional base scheme and establish, in characteristic 0, the valuative criteria of separatedness and properness.

DEFINITION 2.43. Let $f:(X,\Delta)\to C$ be a family of pairs (2.1) over a regular 1-dimensional scheme C. We say that $f:(X,\Delta)\to C$ is *stable* if

(1) f is locally stable (2.2),
(2) f is proper and
(3) $K_{X/C}+\Delta$ is f-ample.

Note that if f is locally stable then $K_X+\Delta$ is $\mathbb{Q}$-Cartier, so f-ampleness makes sense. As we remarked in (2.2), if C is a over a field of characteristic zero, then being stable is preserved by base change $C'\to C$. This is expected to hold in general, but it is not known. See (2.58) for an important special case.

More generally, whenever the notion of local stability is defined later over a scheme S, then $f:(X,\Delta)\to S$ is called *stable* if the above 3 conditions are satisfied. (Thus we have to make sure that local stability implies that $K_{X/S}+\Delta$ makes sense and is $\mathbb{Q}$-Cartier.)

The relationship between locally stable morphisms and stable morphisms parallels the connection between smooth varieties and their canonical models.

PROPOSITION 2.44. *Let $f:\bigl(Y,\Delta_Y\bigr)\to B$ be a locally stable proper morphism over a 1-dimensional regular scheme B. Let $g:\bigl(X,\Delta_X\bigr)\to B$ be the canonical model of f. Then $f:\bigl(X,\Delta_X\bigr)\to B$ is stable.*

Furthermore, if B is over a field of characteristic zero, then taking the canonical model commutes with flat base changes $\pi: B'\to B$.

Proof. First, $K_X+\Delta_X$ is g-ample by definition (1.37) and $\bigl(X,\Delta_X\bigr)$ is lc.

Let $b\in B$ be any closed point and Y_b (resp. X_b) the fibers over b. Since f is locally stable, $\bigl(Y,Y_b+\Delta_Y\bigr)$ is lc. Since any fiber is f-linearly trivial, we conclude using [**Kol13c**, 1.28]. that $\bigl(X,X_b+\Delta_X\bigr)$ is also lc. Thus g is locally stable, hence stable.

By definition $X=\operatorname{Proj}_B\sum_{m\geq 0}f_*\mathcal{O}_Y(mK_Y+\lfloor m\Delta_Y\rfloor)$ and taking f_* commutes with flat base change. In characteristic zero, being locally stable commutes with base change (2.2), which shows the last assertion. ☐

As discussed in (1.21), the separatedness and properness criteria of the (not yet defined) moduli functor/stack of stable morphisms involve the extensions of a stable family defined over an open subset $C^0\subset C$ to a stable family defined over C.

2.45 (Separatedness and Properness). Let C be a regular 1-dimensional scheme, $C^0\subset C$ an open and dense subscheme and $f^0:(X^0,\Delta^0)\to C^0$ a stable morphism. We aim to prove the following two properties.

Separatedness: $f^0:(X^0,\Delta^0)\to C^0$ has at most one extension to a stable morphism $f:(X,\Delta)\to C$.

Properness: There is a finite surjection $\pi: B\to C$ such that the pull back
$$\pi^*f^0:(X^0\times_C B,\Delta^0\times_C B)\to\pi^{-1}(C^0)$$
extends to a stable morphism $f_B:(X_B,\Delta_B)\to B$.

Next we show that both of these hold for stable morphisms in characteristic 0. In both cases the proof relies on theorems which, for later applications, we state in rather general forms.

Proof of separatedness.

We start with a variant of (1.29) which holds over arbitrary base schemes and then conclude that separatedness holds for stable morphisms.

THEOREM 2.46. *Let $f_i:\bigl(X^i,\Delta^i\bigr)\to B$ be two proper morphisms from slc pairs to an irreducible, Noetherian scheme B. Assume that*

(1) *every irreducible component of X^i dominates B,*
(2) *every divisor E^i over X^i satisfying $a\bigl(E^i,X^i,\Delta^i\bigr)<0$ dominates B and*
(3) *$K_{X^i}+\Delta^i$ is f_i-ample.*

Then every isomorphism of the generic fibers
$$\phi:\bigl(X^1_{k(B)},\Delta^1_{k(B)}\bigr)\cong\bigl(X^2_{k(B)},\Delta^2_{k(B)}\bigr)$$
extends to an isomorphism
$$\Phi:\bigl(X^1,\Delta^1\bigr)\cong\bigl(X^2,\Delta^2\bigr).$$

Proof. Let $\Gamma\subset X^1\times_B X^2$ be the closure of the graph of ϕ and $\Gamma'\subset\Gamma$ the union of those irreducible components that dominate B. Let $Y\to\Gamma'$ be a partial normalization (2.49) that is an isomorphism over the generic point of B and such that every irreducible component of the non-normal locus of Y dominates B. Let $p_i : Y \to X^i$ and $f : Y \to B$ be the projections. The p_i are isomorphisms over the generic point of B by construction.

Let $\bar{X}^i \to X^i$ denote the normalization with conductor $\bar{D}^i \subset \bar{X}^i$. Since $a\big(\bar{D}^i_j, X^i, \Delta^i\big) = -1$ for every irreducible component $\bar{D}^i_j \subset \bar{D}^i$, we see that the $\bar{D}^i_j$ dominate B. Thus the p_i are isomorphisms over the nodes of X^i, hence outside a codimension ≥ 2 subset of X^i.

As in (1.29), we use the log canonical class to compare the X^i. If F^i is an irreducible component of Δ^i then $a\big(F^i, X^i, \Delta^i\big) = -\operatorname{coeff}_{F^i} \Delta^i < 0$, thus every irreducible component of Δ^i dominates B by assumption (2). In particular, $(p_1)^{-1}_* \Delta^1 = (p_2)^{-1}_* \Delta^2$; let us denote this divisor by Δ_Y. Write

$$K_Y + \Delta_Y \sim p_i^*\big(K_{X^i} + \Delta^i\big) + E_i, \tag{2.46.4}$$

where E_i is p_i-exceptional and does not dominate B. Hence E_i is effective by assumption (2). Choose $m \geq 0$ sufficiently divisible. Then $(p_i)_* \mathcal{O}_Y(mE_i) = \mathcal{O}_{X^i}$ since the p_i are isomorphisms over the nodes of X^i. Therefore

$$\begin{aligned} (f_i)_* \mathcal{O}_{X^i}\big(mK_{X^i} + m\Delta^i\big) &= (f_i)_*(p_i)_* \mathcal{O}_Y\big(mp_i^*(K_{X^i} + \Delta^i)\big) \\ &= (f_i)_*(p_i)_* \mathcal{O}_Y\big(mp_i^*(K_{X^i} + \Delta^i) + mE_i\big) \\ &= (f_i)_*(p_i)_* \mathcal{O}_Y\big(mK_Y + m\Delta_Y\big) \\ &= f_* \mathcal{O}_Y\big(mK_Y + m\Delta_Y\big). \end{aligned}$$

Since the $K_{X^i} + \Delta^i$ are f_i-ample, $X^i = \operatorname{Proj}_B \sum_{r\geq 0} (f_i)_* \mathcal{O}_{X^i}\big(rmK_{X^i} + rm\Delta^i\big)$. Putting these together, we get the isomorphism

$$\begin{aligned} \Phi : X^1 &\cong \operatorname{Proj}_B \textstyle\sum_{r\geq 0} (f_1)_* \mathcal{O}_{X^1}\big(rmK_{X^1} + rm\Delta^1\big) \cong \\ &\cong \operatorname{Proj}_B \textstyle\sum_{r\geq 0} f_* \mathcal{O}_Y\big(rmK_Y + rm\Delta_Y\big) \cong \\ &\cong \operatorname{Proj}_B \textstyle\sum_{r\geq 0} (f_2)_* \mathcal{O}_{X^2}\big(rmK_{X^2} + rm\Delta^2\big) \cong X^2. \qquad \square \end{aligned}$$

Corollary 2.47 (Separatedness for stable maps). *Let $f_i : \big(X^i, \Delta^i\big) \to B$ be two stable morphisms over a 1-dimensional regular scheme B. Let*

$$\phi : \big(X^1_{k(B)}, \Delta^1_{k(B)}\big) \cong \big(X^2_{k(B)}, \Delta^2_{k(B)}\big)$$

be an isomorphism of the generic fibers. Then ϕ extends to an isomorphism

$$\Phi : \big(X^1, \Delta^1\big) \cong \big(X^2, \Delta^2\big).$$

Proof. Observe that (2.46.1) holds since the f_i are flat, (2.46.2) was proved in (2.13), and (2.46.3) holds by definition. Thus (2.46) implies (2.47). $\square$

The following is another consequence of (2.47). In characteristic 0 it can be proved in other ways as well, see [**Uen75**, Sec.14].

Corollary 2.48. *Let (X, Δ) be a stable pair over a field k of arbitrary characteristic. Then $\operatorname{Aut}(X, \Delta)$ is finite.*

Proof. Choose m such that $m(K_X + \Delta)$ is very ample. Then $\operatorname{Aut}(X, \Delta)$ is identified with the closed subgroup of $\operatorname{PGL}\big(H^0\big(X, \mathcal{O}_X(mK_X + m\Delta)\big)\big)$ that stabilizes (X, Δ). Thus $\operatorname{Aut}(X, \Delta)$ is a linear algebraic group, hence an affine variety.

Let T be the spectrum of a DVR over k with generic point t_g and $\phi_g : t_g \to \operatorname{Aut}(X, \Delta)$ a morphism. We can view it as an isomorphism between the generic fibers of two trivial families $(X, \Delta) \times T \to T$. The trivial families are stable by (2.14.2), hence, by (2.47), ϕ_g extends to an isomorphism

$$\Phi : (X, \Delta) \times T \to (X, \Delta) \times T.$$

This is exactly the valuative criterion of properness for $\operatorname{Aut}(X, \Delta)$. Thus $\operatorname{Aut}(X, \Delta)$ is both affine and proper, hence finite. □

2.49 (Partial normalization). Let Y be a reduced scheme with finite normalization $\pi : \bar{Y} \to Y$. Let $D \subset Y$ be the reduced conductor. Let $D_1 \subset D$ be the union of some of the irreducible components and $D_2 \subset D$ the union of the others. We construct a partial normalization

$$\bar{Y} \xrightarrow{\pi_2} Y' \xrightarrow{\pi_1} Y$$

such that $\pi_1 : Y' \to Y$ is an isomorphism over $Y \setminus D_2$ and $\pi_2 : \bar{Y} \to Y'$ is an isomorphism over $Y' \setminus \pi_1^{-1}(D_1)$.

Note that $Y \setminus D_2$ and $\bar{Y} \setminus \pi^{-1}(D_1)$ naturally glue together to a scheme $p : W \to Y \setminus (D_1 \cap D_2)$. Let F denote the push-forward of $p_* \mathcal{O}_W$ to Y. Then F is a coherent sheaf of algebras (10.16) and $Y' := \operatorname{Spec}_Y F$ has the required properties.

Proof of properness.

The following result verifies the valuative criterion of properness for slc morphisms.

Theorem 2.50 (Properness for stable maps). *Let C be a smooth curve over a field of characteristic 0 and $C^0 \subset C$ an open and dense subset. Let $f^0 : (X^0, \Delta^0) \to C^0$ be a stable morphism.*

Then there is a finite surjection $\pi : B \to C$ such that the pull back

$$f_B^0 := \pi^* f^0 : (X^0 \times_C B, \Delta^0 \times_C B) \to \pi^{-1}(C^0)$$

extends to a stable morphism $f_B : (X_B, \Delta_B) \to B$.

Proof. We closely follow the steps of the proof for curves outlined in (1.17).

We begin with the case when X^0 is normal. Start with $f^0 : (X^0, \Delta^0) \to C^0$ and extend it to a proper flat morphism $f_1 : (X_1, \Delta_1) \to C$ where X_1 is normal. In general (X_1, Δ_1) is no longer lc.

By [**Kol13c**, 10.46]. there is a log resolution $g_1 : Y_1 \to X_1$ such that $\left(g_1^{-1}\right)_* \Delta_1 + \operatorname{Ex}(g_1) + Y_{1c}$ is an snc divisor for every $c \in C$. In general, the fibers of $f_1 \circ g_1 : Y_1 \to C$ are not reduced, hence $g_1 : \left(Y_1, \left(g_1^{-1}\right)_* \Delta_1 + \operatorname{Ex}(g_1)\right) \to C$ is not locally stable.

Let B be a smooth curve and $\pi : B \to C$ a finite surjection. Let $X_2 \to X_1 \times_C B$ and $Y_2 \to Y_1 \times_C B$ denote the normalizations and $g_2 : Y_2 \to X_2$ the induced morphism. Let Δ_2 be the pull back of $\Delta_1 \times_C B$ to X_2.

Note that

$$f_2 \circ g_2 : \left(Y_2, \left(g_2^{-1}\right)_* \Delta_2 + \operatorname{Ex}(g_2)\right) \to B$$

is a log resolution over the points where π is étale, but Y_2 need not be smooth everywhere. However, by (2.52), $\left(Y_2, \left(g_2^{-1}\right)_* \Delta_2 + \operatorname{Ex}(g_2) + \operatorname{red} Y_{2b}\right)$ is lc for every $b \in B$.

By (2.53), one can choose $\pi : B \to C$ such that every fiber of $f_2 \circ g_2$ is reduced. With such a choice, $f_2 \circ g_2$ is locally stable.

If the generic fiber (X_g^0, Δ_g^0) is klt, then, using (2.13) and after shrinking C^0, we may assume that (X^0, Δ^0) is klt. Pick $0 < \epsilon \ll 1$. Then $\left(Y_2, \Delta_2 + (1-\epsilon) \operatorname{Ex}(g_2)\right)$ is also klt and so by [**Kol13c**, 1.30.5] it has a canonical model $f_B : (X_B, \Delta_B) \to B$ which is stable by (2.44).

We are almost done, except that, by construction, $f_B : (X_B, \Delta_B) \to B$ is isomorphic to the pull-back of $f^0 : (X^0, \Delta^0) \to C^0$ only over a possibly smaller dense open subset. However, by (2.47), this implies that this isomorphism holds over the entire C^0.

The argument is the same if (X^0, Δ^0) is lc, but we need to take the canonical model of $\big(Y_2, \Delta_2 + \operatorname{Ex}(g_2)\big)$. The latter is dlt but not klt. Here we rely on [**HX16**]; see also [**Kol13c**, 1.30.7].

Next we show how the semi-log-canonical case can be reduced to the log canonical case, by again following the steps outlined in (1.17).

Let $\bar{X}^0 \to X^0$ be the normalization with conductor $\bar{D}^0 \subset \bar{X}^0$. As we noted in (2.2), we get a stable morphism

$$\bar{f}^0 : (\bar{X}^0, \bar{\Delta}^0 + \bar{D}^0) \to C^0. \tag{2.50.4}$$

By the already completed normal case, we get $B \to C$ such that the pull back of (2.50.4) extends to a stable morphism

$$\bar{f}_B : \big(\bar{X}_B, \bar{\Delta}_B + \bar{D}_B\big) \to B. \tag{2.50.5}$$

Finally, (2.56) shows that (2.50.5) is the normalization of a stable morphism $f_B : (X_B, \Delta_B) \to B$ which is the required extension of the pull-back of $f^0 : (X^0, \Delta^0) \to C^0$. $\square$

We have used the following 3 lemmas during the proof. The first one will be strengthened in (2.81).

LEMMA 2.51. *Let B be a smooth curve over a field of characteristic 0 and $f : (X, D+\Delta) \to B$ a locally stable (resp. stable) morphism where D is a $\mathbb{Z}$-divisor. Let $n : D^n \to D$ be the normalization. Then $f \circ n : (D^n, \operatorname{Diff}_{D^n} \Delta) \to B$ is also locally stable (resp. stable).*

Proof. For any $b \in B$, the fiber X_b is a Cartier divisor thus

$$\operatorname{Diff}_{D^n}(\Delta + X_b) = \big(\operatorname{Diff}_{D^n} \Delta\big) + X_b|_{D^n} = \big(\operatorname{Diff}_{D^n} \Delta\big) + D^n_b.$$

Together with adjunction (1.93), this shows that $f_D : (D^n, \operatorname{Diff}_{D^n} \Delta) \to B$ is locally stable. Since $D^n \to D$ is finite and

$$K_{D^n} + \operatorname{Diff}_{D^n} \Delta \sim_{\mathbb{Q}} n^*(K_X + D + \Delta),$$

we see that if $K_X + D + \Delta$ is f-ample then $K_{D^n} + \operatorname{Diff}_{D^n}$ is $f \circ n$-ample. Hence if f is stable then so is $f \circ n : (D^n, \operatorname{Diff}_{D^n} \Delta) \to B$. $\square$

LEMMA 2.52. *Let C be a smooth curve over a field of characteristic 0, $f : X \to C$ a flat morphism and Δ a $\mathbb{Q}$-divisor on X. Assume that $(X, \operatorname{red} X_c + \Delta)$ is lc for every $c \in C$. Let B be a smooth curve, $g : B \to C$ a quasi-finite morphism, $g_Y : Y \to X \times_C B$ the normalization and $\Delta_Y := g_Y^* \Delta$.*

Then $(Y, \operatorname{red} X_b + \Delta_Y)$ is lc for every $b \in B$.

Proof. Pick $c \in C$ and let $b_i \in B$ be its preimages. By the Hurwitz formula

$$K_Y + \Delta_Y + \textstyle\sum_i \operatorname{red} Y_{b_i} = g_X^*(K_X + \Delta + \operatorname{red} X_c).$$

By assumption, $(X, \Delta + \operatorname{red} X_c)$ is lc for every $c \in C$. Hence, by (1.87.3), $(Y, \Delta_Y + \sum_i \operatorname{red} Y_{b_i})$ is also lc. $\square$

LEMMA 2.53. *Let $f : X \to T$ be a flat morphism from a normal scheme to a 1-dimensional regular scheme T. Let S be another 1-dimensional regular scheme and $\pi : S \to T$ a quasi-finite morphism. Let $Y \to X \times_T S$ be the normalization and $f_Y : Y \to S$ the projection. Assume that*

(1) *for every $s \in S$, the multiplicity of every irreducible component of $X_{\pi(s)}$ divides the ramification index of π at s and*
(2) *π is tamely ramified everywhere.*

Then every fiber of $f_Y : Y \to S$ is reduced.

Proof. The claim is local, so pick points $0_S \in S$ and $0_T := \pi(0_S) \in T$.

We want to study how the multiplicities of the irreducible components of the fiber over 0_T change under base extension. We can focus on one such irreducible component and pass to any open subset of X that is not disjoint from the chosen component. We can thus think of X as a hypersurface $X \subset \mathbb{A}^n_T$ defined by an equation $f \in \mathcal{O}_T[x_1, \dots, x_n]$. The central fiber X_0 is defined by $\bar{f} = 0$ where $\bar{f}$ is the mod t reduction of f. By focusing at a generic point of X_0, after an étale coordinate change we may assume that $\bar{f} = x_1^m$ where m is the multiplicity of X_0. We can thus write $f = x_1^m - t \cdot u(\mathbf{x}, t)$. Since X is normal (hence regular) at the generic point of X_0, we see that u is not identically zero along X_0.

Let s be a local coordinate at 0_S. We can write $\pi^* t = s^e v(s)$ where e is the ramification index of π at 0_S and v is a unit at 0_S. Consider now the fiber product $X_S := X \times_T S \to S$. It is defined by the equation

$$x_1^m = s^e \cdot u\bigl(\mathbf{x}, s^e v(s)\bigr) \cdot v(s).$$

Note that X_S is not normal along $(s = x_1 = 0)$ if $m, e > 1$.

We construct its normalization by repeatedly blowing up. This is especially simple if e is a multiple of m. Write $e = md$ and set $x_1' := x s^{-d}$. Then we get $Y \subset \mathbb{A}^n_S$ (with coordinates $x_1', x_2, \dots, x_n$) defined by

$${x_1'}^m = u\bigl(x_1' s^d, x_2, \dots, x_n, s^e v(s)\bigr) \cdot v(s)$$

and the central fiber Y_0 is defined by the equation

$${x_1'}^m = u\bigl(0, x_2, \dots, x_n, 0\bigr) \cdot v(0),$$

where the right hand side is not identically zero.

If the characteristic of $k(0_S)$ does not divide m, then the projection $Y_0 \to \mathbb{A}^{n-1}_{x_2,\dots,x_n}$ is generically étale and Y_0 is smooth at its generic points. In this case, Y is the normalization of X_S (at least generically along Y_0) and the central fiber of $Y \to S$ has multiplicity 1. □

Note that the proof of (2.53) does not work if the characteristic of $k(0_S)$ divides m. Then $Y_0 \to \mathbb{A}^{n-1}_{x_2,\dots,x_n}$ is inseparable. If $u\bigl(0, x_2, \dots, x_n, 0\bigr)$ is not a pth power over the algebraic closure of $k(0_S)$, then Y_0 is geometrically integral, hence generically nonsingular. In this case, Y is the normalization of X_S and the central fiber of $Y \to S$ has multiplicity 1.

However, if $u\bigl(0, x_2, \dots, x_n, 0\bigr)$ is a pth power, then Y_0 is not generically reduced. In this case Y need not be normal, and further blow-ups may be needed to reach the normalization. In any case, usually one does not get a reduced fiber. The situation seems rather complicated, even for families of curves [**AW71**]. A weaker result is in (2.62).

Gluing of slc pairs.

At the end of the proof of (2.50) we needed to reconstruct an slc pair form its normalizations. The technical background for this is discussed in [**Kol13c**, Chaps. 5 and 9]. In the current setting we aim to compactify an slc pair (X^0, Δ^0) by first normalizing it, then obtaining an lc compactification of the normalization and finally descending the lc compactification to a compactification of (X^0, Δ^0). A precise version of this process is the following.

2.54 (Compactification problem for slc pairs). Consider a diagram

$$\begin{array}{ccc} \big(\bar X^0, \bar\Delta^0 + \bar D^0\big) & \stackrel{\bar\iota}{\hookrightarrow} & \big(\bar X, \bar\Delta + \bar D\big) \\ \pi^0 \downarrow & & \\ (X^0, \Delta^0) & & \end{array} \tag{2.54.1}$$

where (X^0, Δ^0) is an slc pair, π^0 its normalization and $\bar\iota$ an open embedding with dense image. We say that (2.54.1) defines a partial compactification of (X^0, Δ^0) if (2.54.1) can be extended to a diagram

$$\begin{array}{ccc} \big(\bar X^0, \bar\Delta^0 + \bar D^0\big) & \stackrel{\bar\iota}{\hookrightarrow} & \big(\bar X, \bar\Delta + \bar D\big) \\ \pi^0 \downarrow & & \downarrow \pi \\ (X^0, \Delta^0) & \stackrel{\iota}{\hookrightarrow} & (X, \Delta) \end{array} \tag{2.54.2}$$

where (X, Δ) is demi-normal, π is its normalization and ι an open embedding.

Note that (X, Δ) is unique. [**Kol13c**, Sec.9.4] contains a series of examples where (X, Δ) does not exist.

THEOREM 2.55. *Let (X^0, Δ^0) be an slc pair over a field of characteristic 0 and consider a diagram (2.54.1). Let $n : \bar D^n \to \bar D$ denote the normalization and assume that the involution τ^0 on $(\bar D^0)^n$ extends to an involution τ on $\bar D^n$.*

(1) *If none of the lc centers of $\big(\bar X, \bar\Delta + \bar D\big)$ is disjoint from $\bar X^0$ then (2.54.1) has an extension to a diagram (2.54.2).*
(2) *If none of the log centers of $\big(\bar X, \bar\Delta + \bar D\big)$ is disjoint from $\bar X^0$ then (X, Δ) is slc.*

Proof. Our aim is to construct (X, Δ) as the geometric quotient [**Kol13c**, 9.4] by the gluing relation generated by the relation $(n, n \circ \tau) : \bar D^n \rightrightarrows \bar X$ as in [**Kol13c**, 5.31].

We assume that $\big(\bar X, \bar\Delta + \bar D\big)$ is lc. By assumption none of the lc centers of $\big(\bar X, \bar\Delta + \bar D\big)$ is contained in $\bar X \setminus \bar X^0$ and, over $\bar X^0$, we have a finite equivalence relation whose quotient is X^0. Thus [**Kol13c**, 9.55] implies that $(n, n\circ\tau)$ generates a finite equivalence relation on $\big(\bar X, \bar\Delta + \bar D\big)$. Therefore, by [**Kol13c**, 5.33], there is a demi-normal scheme (X, Δ) that contains (X^0, Δ^0) as an open subscheme, proving (1).

By inversion of adjuntion (1.93), every irreducible component of $\operatorname{Diff}_{\bar D^n}$ lies over a log center of $\big(\bar X, \bar\Delta + \bar D\big)$. Thus if none of the log centers of $\big(\bar X, \bar\Delta + \bar D\big)$ is disjoint from $\bar X^0$ then none of the irreducible components of $\operatorname{Diff}_{\bar D^n}$ is disjoint from $\bar X^0$. Thus $\operatorname{Diff}_{\bar D^n}$ is $\bar\tau$-invariant and therefore [**Kol13c**, 5.38] shows that (X, Δ) is slc. □

COROLLARY 2.56. *Let B be a smooth curve over a field of characteristic 0 and $B^0 \subset B$ a dense open subset. Let $f^0 : (X^0, \Delta^0) \to B^0$ be a stable morphism. Let $\bar X^0 \to X^0$ be the normalization with conductor $\bar D^0 \subset \bar X^0$.*

Assume that $\bar f^0 : (\bar X^0, \bar\Delta^0 + \bar D^0) \to B^0$ extends to a stable morphism $\bar f : \big(\bar X, \bar\Delta + \bar D\big) \to B$.

Then $f^0 : (X^0, \Delta^0) \to B^0$ also extends to a stable morphism $f : (X, \Delta) \to B$.

Proof. Let $n : \bar{D}^n \to \bar{D}$ denote the normalization. By (2.51),

$$\bar{f} \circ n : \left(\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}\right) \to B$$

is also stable. In particular, by (2.47) the involution τ^0 of $(\bar{D}^0)^n$ extends to an involution τ on $\bar{D}^n$.

By (2.13), none of the log centers of $\left(\bar{X}, \bar{\Delta} + \bar{D}\right)$ is contained in $\bar{X} \setminus \bar{X}^0$. Thus the assumption of (2.55.2) hold, hence we get (X, Δ). Finally f^0 extends to a proper morphism $f : (X, \Delta) \to B$ by the universal property of geometric quotients. □

Base change in positive characteristic.

As we noted in (2.14), it is not known whether being locally stable commutes with base change in positive characteristic. However, the next result shows that this holds for all families obtained as in (2.50).

THEOREM 2.57. *Let $h : C' \to C$ be a quasi-finite morphisms of regular schemes of dimension 1 and $f : X \to C$ a proper morphism from a regular scheme X to C whose fibers are simple normal crossing divisors. Then $X' := X \times_C C'$ has canonical singularities and*

$$\textstyle\sum_{m \geq 0} f'_* \omega_{X'/C'}^{\otimes m} \cong h^* \sum_{m \geq 0} f_* \omega_{X/C}^{\otimes m}. \tag{2.57.1}$$

Proof. Note that (2.57.1) is just the claim that push forward commutes with flat base change $h : C' \to C$. The substantial part is the assertion that X' has canonical singularities, hence the proj of $\sum_{m \geq 0} f'_* \omega_{X'/C'}^{\otimes m}$ is also the relative canonical model of any resolution of X'.

Pick a a point $x \in X$ and set $c = f(x)$. We may assume that C and C' are the spectra of a DVRs with local parameters t and s. Thus the Henselisation of (x, X) can be given as a hypersurface

$$(x_1 \cdots x_m = t) \subset \left(\mathbf{A}^n_C, 0\right), \tag{2.57.2}$$

where $\mathbf{A}^n_C$ denotes the Henselisation of $\mathbb{A}^n_C$ at $(0, 0)$.

If $h^* t = \phi(s)$ then (x', X') can be given as a hypersurface

$$(x_1 \cdots x_m = \phi(s)) \subset \left(\mathbf{A}^n_{C'}, 0\right). \tag{2.57.3}$$

Thus the main claim is that the singularity defined by (2.57.3) is canonical.

If we are over a field then (2.57.3) defines a toric singularity and we are done, essentially as in (4.90). We check below that although there is no torus action on the base C, we can compute the simplest blow-ups suggested by toric geometry and everything works out as expected.

(Note, however, that although the pair $\left(\mathbb{A}^n_k, (x_1 \cdots x_n = 0)\right)$ is lc, this is not a completely toric question. We need to understand all exceptional divisors over $\mathbb{A}^n_k$, not just the toric ones; see [**Kol13c**, 2.11].) □

LEMMA 2.58. *Let T be a DVR with local parameter t and residue field k and $\mathbf{A}^n_T$ the Henselisation of $\mathbb{A}^n_T$ at $(0, 0)$. Let $m \leq n$ and e be natural numbers and ϕ a regular function on $\mathbf{A}^n_T$. Set*

$$X := X(m, n, e, \phi) = \left(x_1 \cdots x_m = t^e + t^{e+1} \phi(x_1, \ldots, x_n)\right) \subset \left(\mathbf{A}^n_T, 0\right) \tag{2.58.1}$$

and let D be the divisor $(t = 0) \subset X$. Then the pair (X, D) is log canonical and X is canonical, near the origin.

Proof. If $\operatorname{char} k = 0$, this immediately follows from (2.9), so the main point is that it also holds for any DVR.

If $m = 0$ or $e = 0$ then X is empty and we are done. Otherwise we can set $x'_m := x_m(1+t\phi)^{-1}$ to get the simpler equation $x_1 \cdots x_m = t^e$. For inductive purposes we introduce a new variable s and work with the more general systems

$$\begin{aligned} &X := \bigl(x_1 \cdots x_m - s^e = x_{m+1} \cdots x_{m+r} s - t = 0\bigr) \subset \bigl(\mathbf{A}^{n+1}_T, 0\bigr) \\ &D := (t = 0), \quad \text{where } 0 \le r \le n - m. \end{aligned} \tag{2.58.2}$$

The case $r = 0$ corresponds to (2.58.1). We use induction on m and e.

Let E be an exceptional divisor over X and v the corresponding valuation. Assume first that $v(x_1) \ge v(s)$. We blow up $(x_1 = s = 0)$. In the affine chart where $x'_1 := x_1/s$ we get the new equations

$$x'_1 x_2 \cdots x_m - s^{e-1} = x_{m+1} \cdots x_{m+r} s - t = 0 \tag{2.58.2}$$

defining (X', D'). A local generator of $\omega_{X/T}(D)$ is

$$\frac{1}{t} \cdot \frac{dx_2 \wedge \cdots \wedge dx_n}{x_2 \cdots x_{m+r}}, \tag{2.58.3}$$

which is unchanged by pull-back.

Such operations reduce e, until we reach a situation where $v(x_i) < v(s)$ for every i. If $v(x_i) = 0$ for some i and $i \neq m$ then x_i is nonzero at the generic point of $\operatorname{center}_X E$. Thus we can set $x'_m := x_i x_m$ and reduce the value of m. Thus we may assume that $v(x_i) > 0$ for $i = 1, \dots, m$. Since $\sum v(x_i) = e \cdot v(s)$, we conclude that $e < m$. If $e \ge 2$ then we may assume that $v(x_e)$ is the smallest. Set $x'_i = x_i/x_e$ for $i = 1, \dots, e-1$ and $s' := s/x_e$. We get new equations

$$x'_1 \cdots x'_{e-1} x_{e+1} \cdots x_m - (s')^e = x_e x_{m+1} \cdots x_{m+r} s' - t = 0 \tag{2.58.4}$$

defining (X', D') and the value of m dropped. The pull-back of the form (2.58.3) is

$$\begin{aligned} &\frac{1}{t} \cdot \frac{d(x_e x'_2) \wedge \cdots \wedge d(x_e x'_{e-1}) \wedge dx_e \wedge \cdots \wedge dx_n}{(x_e x'_2) \cdots (x_e x'_{e-1}) x_e \cdots x_{m+r}} \\ &= \frac{1}{t} \cdot \frac{dx'_2 \wedge \cdots dx'_{e-1} \wedge dx_e \wedge \cdots \wedge dx_n}{x'_2 \cdots x'_{e-1} x_e \cdots x_{m+r}}, \end{aligned} \tag{2.58.5}$$

which is again a local generator of $\omega_{X'/T}(D')$.

Eventually we reach the situation where $e = 1$. We can now eliminate s and, after setting $r + m \mapsto m$, rewrite the system as

$$\begin{aligned} &X := \bigl(x_1 \cdots x_m = t\bigr) \subset \bigl(\mathbf{A}^n_T, 0\bigr) \\ &D := (t = 0). \end{aligned} \tag{2.58.6}$$

Now X is regular, this case was treated in [**Kol13c**, 2.11]. □

Other extension theorems.

We discuss a collection of other results about extending 1-parameter familes of varieties or pairs. These can be useful in many situations.

2.59 (Extending a stable family without base change).

Let C be a smooth curve over a field of characteristic 0, $C^0 \subset C$ an open and dense subscheme and $f^0 : (X^0, \Delta^0) \to C^0$ a stable morphism. Here we consider the question of how to extend f^0 to a proper morphism $f : X \to C$ in a "nice" way without a base change. For simplicity assume that X^0 is normal.

As in (1.17), we can take any extension of f^0 to a proper morphism $f_1 : X_1 \to C$, then take a log resolution of $(X_2, \Delta_2) \to (X_1, \Delta_1)$ and finally the canonical model of (X_2, Δ_2) using [**Kol13c**, 1.30.7] We have proved:

Claim 2.59.1. There is a unique extension $f : (X, \Delta) \to C$ such that (X, Δ) is lc and $K_X + \Delta$ is f-ample. □

This model has the problem that its fibers over the points $C \setminus C^0 =: \{c_1, \dots, c_r\}$ can be pretty complicated. A slight twist improves the fibers considerably. Instead of starting with the above (X_1, Δ_1), we take a log resolution $\bigl(X_2, \Delta_2 + \sum \operatorname{red} X_{2,c_i}\bigr)$ of $\bigl(X_1, \Delta_1 + \sum \operatorname{red} X_{1,c_i}\bigr)$ and its canonical model over C. We need to apply [**Kol13c**, 1.30.7] to $\bigl(X_2, \Delta_2 + \sum \operatorname{red} X_{2,c_i} - \epsilon \sum X_{2,c_i}\bigr)$ and use [**Kol13c**, 1.28] to obtain the following.

Claim 2.59.2. There is a unique extension $f : (X, \Delta) \to C$ such that $(X, \Delta + \sum \operatorname{red} X_{c_i})$ is lc and $K_X + \Delta + \sum \operatorname{red} X_{c_i}$ is f-ample. By adjunction, in this case $\bigl(\operatorname{red} X_{c_i}, \operatorname{Diff} \Delta\bigr)$ is slc. □

A variant of this starts with any extension (X_1, Δ_1) and then takes a dlt modification of $(X_1, \Delta_1 + \sum \operatorname{red} X_{1,c_i})$ as in [**Kol13c**, 1.36].

Claim 2.59.3. There is a dlt modification $(Y^0, \Delta^0_Y) \to (X^0, \Delta^0)$ and an extension of it to $g : (Y, \Delta_Y) \to C$ such that $(Y, \Delta + \sum \operatorname{red} Y_{c_i})$ is dlt. □

Taking a minimal model of the above $g : (Y, \Delta_Y) \to C$ yields another useful version.

Claim 2.59.4. There is a dlt modification $(Y^0, \Delta^0_Y) \to (X^0, \Delta^0)$ and an extension of it to $g : (Y, \Delta_Y) \to C$ such that $(Y, \Delta + \sum \operatorname{red} Y_{c_i})$ is dlt and $K_X + \Delta + \sum \operatorname{red} X_{c_i}$ is f-nef. □

Finally, if we are willing to change X^0 drastically, [**Kol13c**, 10.46] gives the following.

Claim 2.59.5. There is a log resolution $(Y^0, \Delta^0_Y) \to (X^0, \Delta^0)$ and an extension of it to $g : (Y, \Delta_Y) \to C$ such that $(Y, \Delta_Y + \operatorname{red} Y_c)$ is snc for every $c \in C$. □

Let us also mention the following very strong variant of (2.59.5), traditionally called the “semi-stable reduction theorem.” We do not use it, and one of the points of our proof of (2.50) was to show that the much easier (2.52) and (2.53) are enough for our purposes.

THEOREM 2.60. [**KKMSD73**] *Let C be a smooth curve over a field of characteristic 0, $f : X \to C$ a flat morphism of finite type and D a divisor on X. Then there is a smooth curve B, a finite surjection $\pi : B \to C$ and a log resolution $g : Y \to X \times_C B$ such that for every $b \in B$,*

(1) *$g_*^{-1}(D \times_C B) + \operatorname{Ex}(g) + Y_b$ is an snc divisor and*
(2) *Y_b is reduced.* □

The positive or mixed characteristic analogs of (2.60) are not known, but the following result on “semi-stable alterations” holds in general.

THEOREM 2.61. [**dJ96**, Sec.6] *Let T be a 1-dimensional regular scheme, $f : X \to T$ a flat morphism of finite type whose generic fiber is geometrically reduced. Then there is a 1-dimensional regular scheme S, a finite surjection $\pi : S \to T$ and* *a generically finite, separable, proper morphism $g : Y \to X \times_T S$ such that for every $s \in S$, Y_s is a reduced snc divisor.* □

The following variant of (2.53) is an easy consequence of (2.61).

COROLLARY 2.62. *Let $f : X \to T$ be a flat morphism of finite type from a pure dimensional scheme to a 1-dimensional regular scheme T. Then there is a 1-dimensional regular scheme S and a finite morphism $\pi : S \to T$ such that every fiber of the projection of the normalization $\overline{X \times_T S} \to S$ is generically reduced.* □

2.5. Cohomology of the structure sheaf

In studying moduli questions, it is very useful to know that certain numerical invariants are locally constant. In this section we study the deformation invariance of (the dimension of) certain cohomology groups. The key to this is the Du Bois property of slc pairs. The definition of Du Bois singularities is rather complicated, but fortunately for our applications we need to know only the following two facts.

2.63 (Properties of Du Bois singularities). Let M be a complex analytic variety. Since constant functions are analytic, there is an injection of sheaves $\mathbb{C}_M \hookrightarrow \mathcal{O}_M^{\mathrm{an}}$. Taking cohomologies we get

$$H^i(M, \mathbb{C}) \to H^i\big(M, \mathcal{O}_M^{\mathrm{an}}\big).$$

If X is projectve over $\mathbb{C}$ and X^{an} the corresponding analytic variety, then, by the GAGA theorems (cf. [**Ser56**] or [**Har77**, App.B]), $H^i\big(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}}\big) \cong H^i(X, \mathcal{O}_X)$.

If X is also smooth, Hodge theory tells us that

$$H^i(X^{\mathrm{an}}, \mathbb{C}) \to H^{0,i}(X^{\mathrm{an}}, \mathbb{C}) \cong H^i(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}}) \cong H^i(X, \mathcal{O}_X)$$

is surjective. Du Bois singularities were essentially defined to preserve this surjectivity [**DB81, Ste83**]. (There does not seem to be a good definition of Du Bois singularities in positive characteristic.) Thus we have the following.

Property 2.63.1. Let X be a proper variety over $\mathbb{C}$ with Du Bois singularities. Then the natural maps

$$H^i\big(X^{\mathrm{an}}, \mathbb{C}\big) \to H^i\big(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}}\big) \cong H^i(X, \mathcal{O}_X).$$

are surjective.

Next we need to know which singularities are Du Bois. Over a field of characteristic 0, rational singularities are Du Bois; see [**Kol95b**, 12.9] and [**Kov99**] but for our applications the key result is the following. The normal case is proved in [**KK10**] and extended to the non-normal case in [**Kol13c**, 6.32].

Property 2.63.2. Let (X, Δ) be an slc pair over $\mathbb{C}$. Then X has Du Bois singularities.

These are the only facts we need to know about Du Bois singularities.

The main use of (2.63.1) is through the following base-change theorem, due to [**DJ74, DB81**].

THEOREM 2.64. *Let S be a Noetherian scheme over a field of characteristic 0 and $f : X \to S$ a flat, proper morphism. Assume that the fiber X_s is Du Bois for some $s \in S$. Then there is an open neighborhood $s \in S^0 \subset S$ such that, for all i,*

(1) *$R^i f_* \mathcal{O}_X$ is locally free and compatible with base change over S^0 and*

(2) *$s \mapsto h^i(X_s, \mathcal{O}_{X_s})$ is a locally constant function on S^0.*

Proof. By Cohomology and Base Change [**Har77**, III.12.11], the theorem is equivalent to proving that the restriction maps

$$\phi_s^i : R^i f_* \mathcal{O}_X \to H^i(X_s, \mathcal{O}_{X_s}) \tag{2.64.3}$$

are surjective for every i. By the Theorem on Formal Functions [**Har77**, III.11.1], it is enough to prove this when S is replaced by any 0-dimensional scheme S_n whose closed point is s.

Thus assume form now on that we have a flat, proper morphism $f_n : X_n \to S_n$, $s \in S_n$ is the only closed point and X_s is Du Bois. Then $H^0\big(S_n, R^i f_* \mathcal{O}_X\big) = H^i\big(X_n, \mathcal{O}_{X_n}\big)$, hence we can identify the ϕ_s^i with the maps

$$\psi^i : H^i\big(X_n, \mathcal{O}_{X_n}\big) \to H^i\big(X_s, \mathcal{O}_{X_s}\big). \tag{2.64.4}$$

By the Lefschetz principle we may assume that $k(s) \cong \mathbb{C}$ and then both sides of (2.64.4) are unchanged if we replace X_n by the corresponding analytic space X_n^{an}. Let $\mathbb{C}_{X_n}$ (resp. $\mathbb{C}_{X_s}$) denote the sheaf of locally constant functions on X_n (resp. X_s) and $j_n : \mathbb{C}_{X_n} \to \mathcal{O}_{X_n}$ (resp. $j_s : \mathbb{C}_{X_s} \to \mathcal{O}_{X_s}$) the natural inclusions. We have a commutative diagram

$$\begin{array}{ccc} H^i\big(X_n, \mathbb{C}_{X_n}\big) & \stackrel{\alpha^i}{\to} & H^i\big(X_s, \mathbb{C}_{X_s}\big) \\ j_n^i \downarrow & & \downarrow j_s^i \\ H^i\big(X_n, \mathcal{O}_{X_n}\big) & \stackrel{\psi^i}{\to} & H^i\big(X_s, \mathcal{O}_{X_s}\big). \end{array}$$

Note that α^i is an isomorphism since the inclusion $X_s \hookrightarrow X_n$ is a homeomorphism and j_s^i is surjective since X_s is Du Bois. Thus ψ^i is also surjective. $\square$

Definition 2.65. A scheme Y is said to be *potentially slc* if for every point $y \in Y$ there is an effective $\mathbb{R}$-divisor Δ_y on Y such that (Y, Δ_y) is slc at y.

Let $f : X \to S$ be a flat morphism. We say that f has *potentially slc fibers over closed points* if the fiber X_s is potentially slc for every closed point $s \in S$.

One can similarly define the notion *potentially klt,* and so on.

In our final applications, the Δ_s usually come as the restriction of a global divisor Δ to X_s, but here we do not assume this.

If (X_s, Δ_s) is semi-log-canonical then X_s is Du Bois by (2.63.2), hence (2.64) implies the following.

Corollary 2.66. *Let S be a Noetherian scheme over a field of characteristic 0 and $f : X \to S$ a proper and flat morphism with potentially slc fibers over closed points. Then, for all i,*

(1) *$R^i f_* \mathcal{O}_X$ is locally free and compatible with base change and*
(2) *if S is connected, then $h^i\big(X_s, \mathcal{O}_{X_s}\big)$ is independent of $s \in S$.* $\square$

We can derive from (2.66) similar results for other line bundles. A line bundle L on X is called *f-semi-ample* if there is an $m > 0$ such that L^m is f-generated by global sections. That is, the natural map $f^*\big(f_*(L^m)\big) \to L^m$ is surjective. Equivalently, L^m is the pull-back of a relatively ample line bundle by a morphism $X \to Y$.

Corollary 2.67. *Let S be a Noetherian, connected scheme over a field of characteristic 0 and $f : X \to S$ a proper and flat morphism with potentially slc* *fibers over closed points. Let L be an f-semi-ample line bundle on X. Then, for all i,*

(1) $R^i f_*\big(L^{-1}\big)$ *is locally free and compatible with base change and*
(2) $h^i\big(X_s, L^{-1}_{X_s}\big)$ *is independent of $s \in S$.*

Proof. The question is local on S, thus we may assume that S is local with closed point s. Chose $m > 0$ such that L^m is f-generated by global sections. Since S is affine, L^m is generated by global sections. By (2.11), there is a finite morphism $\pi : Y \to X$ such that $\pi_*\mathcal{O}_Y = \sum_{r=0}^{m-1} L^{-r}$ and $f \circ \pi : \big(Y, \pi^{-1}\Delta\big) \to S$ also has potentially slc fiber over s. Thus, by (2.66),

$$R^i(f \circ \pi)_*\mathcal{O}_Y = \textstyle\sum_{r=0}^{m-1} R^i f_*\big(L^{-r}\big)$$

is locally free and compatible with arbitrary base change. Thus the same holds for every summand. □

Corollary 2.68. [**KK10**] *Let S be a Noetherian, connected scheme over a field of characteristic 0 and $f : X \to S$ a projective and flat morphism with potentially slc fibers over closed points. Then, if one fiber of f is CM then all fibers of f are CM.*

For arbitrary flat morphisms $\pi : X \to S$, the set of points $x \in X$ such that the fiber $X_{\pi(x)}$ is CM at x is open (10.2), but usually not closed. (Many such examples can be constructed using [**Kol13c**, 3.9–11].) If π is proper, then the set $\big\{s \in S : X_s \text{ is CM}\big\}$ is open in S (10.3). Thus the key point of (2.68) is to show that, in our case, this set is also closed.

More generally, under the assumptions of (2.68), if one fiber of f is S_k for some k then all fibers of f are S_k, see [**KK10**, 1.3].

Note that we assume that f is projective, not just proper. This is almost certainly an artifice of the proof.

Proof. Let L be an f-ample line bundle on X. If X_s is CM for some $s \in S$, then, by [**KM98**, 5.72] $H^i\big(X_s, L^{-r}_{X_s}\big) = 0$ for $r \gg 1$ and $i < \dim X_s$. Thus by (2.67), the same vanishing holds for every $s \in S$. Hence, using [**KM98**, 5.72] in the other direction, we conclude that X_s is CM for every $s \in S$. □

The next theorem implies that $\omega_{X/S}$ exists and commutes with base change for locally stable morphisms. For projective morphisms it was proved in [**KK10**], the general case is settled in [**KK17**].

Theorem 2.69. *Let S be a Noetherian scheme over a field of characteristic 0 and $f : X \to S$ a flat morphism of finite type with potentially slc fibers over closed points. Then $\omega_{X/S}$ exists and is compatible with base change. That is, for any $g : T \to S$ the natural map*

$$g_X^*\omega_{X/S} \to \omega_{X_T/T} \quad \textit{is an isomorphism,} \tag{2.69.1}$$

where $g_X : X_T := X \times_S T \to X$ is the first projection.

We give a detailed proof of the projective case below; this is sufficient for almost all applications in this book. For the general case we refer to [**KK17**].

The existence of $\omega_{X/S}$ is easy and, as we see in (2.70.1–3), it holds under rather weak restrictions. Compatibility with base change is not automatic; see [**Pat13**] and (2.42) for some examples.

As we explain in (2.70.4–5), once the definition of $\omega_{X/S}$ is set up right, (2.69) becomes an easy consequence of (2.67).

2.70 (The relative dualizing sheaf II). The best way to define the relative dualizing sheaf is via general duality theory as in [**Har66, AK70, Con00**]. It is, however, worthwhile to observe that a slight modification of the treatment in [**Har77**] gives the relative dualizing sheaf in the following cases.

Assumptions. S is an arbitrary Noetherian scheme and $f : X \to S$ a projective morphism of pure relative dimension n (3.34).

Weak duality for $\mathbb{P}^n_S$ 2.70.1. Let $P = \mathbb{P}^n_S$ with projection $g : P \to S$ and set $\omega_{P/S} := \wedge^n \Omega_{P/S}$.

The proof of [**Har77**, III.7.1] shows that there is a natural isomorphism, called the *trace map*, $t : R^n g_* \omega_{P/S} \cong \mathcal{O}_S$ and for any coherent sheaf F on X there is a natural isomorphism

$$g_* \mathcal{H}om_P\big(F, \omega_{P/S}\big) \cong \mathcal{H}om_S\big(R^n g_* F, \mathcal{O}_S\big).$$

Note that if S is a point then $g_* \mathcal{H}om_P = \mathrm{Hom}_P$, thus we recover the usual formulation of [**Har77**, III.7.1].

Construction of $\omega_{X/S}$ 2.70.2. Let $f : X \to S$ be a projective morphism of pure relative dimension n. We construct $\omega_{X/S}$ first locally over S. Once we establish weak duality, the proof of [**Har77**, III.7.2] shows that a relative dualizing sheaf is unique up to unique isomorphism, hence the local pieces glue together to produce $\omega_{X/S}$. Working locally over S we can assume that there is a finite morphism $\pi : X \to P = \mathbb{P}^n_S$. Set

$$\omega_{X/S} := \mathcal{H}om_P\big(\pi_* \mathcal{O}_X, \omega_{P/S}\big). \tag{2.70.2.a}$$

If f is flat with CM fibers over S then $\pi_* \mathcal{O}_X$ is locally free and so is $\pi_* \omega_{X/S}$. Thus $\omega_{X/S}$ is also flat over S with CM fibers and it commutes with base change. We discuss a local version of this in (2.70.7).

Weak duality for X/S 2.70.3. Let $f : X \to S$ be a projective morphism of pure relative dimension n (3.34). Use [**Har77**, Exrc.III.6.10] to show that there is a trace map

$$t : R^n f_* \omega_{X/S} \to \mathcal{O}_S$$

and for any coherent sheaf F on X there is a natural isomorphism

$$f_* \mathcal{H}om_X\big(F, \omega_{X/S}\big) \cong \mathcal{H}om_S\big(R^n f_* F, \mathcal{O}_S\big).$$

If F is locally free, this is equivalent to the isomorphism

$$f_*\big(\omega_{X/S} \otimes F^{-1}\big) \cong \mathcal{H}om_S\big(R^n f_* F, \mathcal{O}_S\big).$$

(Note that $M \mapsto \mathcal{H}om_S(M, \mathcal{O}_S)$ is a duality for locally free coherent $\mathcal{O}_S$-sheaves but not for all coherent sheaves. In particular, the torsion in $R^n f_* F$ is invisible on the left hand side $f_*\big(\omega_{X/S} \otimes F^{-1}\big)$. Duality over a base scheme is less symmetric than over a field.)

Flatness of $\omega_{X/S}$ 2.70.4. Let L be relatively ample on X/S. By the proof of [**Har77**, III.9.9] $\omega_{X/S}$ is flat over S iff $f_*\big(\omega_{X/S} \otimes L^m\big)$ is locally free for $m \gg 1$; see also (3.44). If this holds then $\omega_{X/S}$ is the coherent $\mathcal{O}_X$-sheaf associated to

$$\sum_{m \geq m_0} f_*\big(\omega_{X/S} \otimes L^m\big)$$

as a module over the $\mathcal{O}_S$-algebra $\sum_{m\geq 0} f_*\big(L^m\big)$.

Applying weak duality with $F = L^{-m}$ we see that these hold if $R^n f_*\big(L^{-m}\big)$ is locally free for $m \gg 1$. The latter is satisfied in two important cases.

(a) $f : X \to S$ is flat with CM fibers. Then $R^i f_*\big(L^{-m}\big) = 0$ for $i < n$ and $m \gg 1$, hence $R^n f_*\big(L^{-m}\big)$ is locally free of rank $(-1)^n \chi\big(X_s, L^{-m}\big)$ for $m \gg 1$.
(b) $f : X \to S$ is flat with potentially slc fibers. Then $R^n f_*\big(L^{-m}\big)$ is locally free for $m \geq 0$ by (2.67).

Base change properties of $\omega_{X/S}$ 2.70.5. Let $f : X \to S$ be a projective morphism of pure relative dimension n. We claim that the following are equivalent.

(a) $\omega_{X/S}$ commutes with base change as in (2.69.1).
(b) $R^n f_*\big(L^{-m}\big)$ is locally free for $m \gg 0$.

To see this first note that (2.70.3–4) show that $\omega_{X/S}$ commutes with base change iff $\mathcal{H}om_S\big(R^n f_*\big(L^{-m}\big), \mathcal{O}_S\big)$ is locally free and commutes with base change for $m \gg 0$. Finally show that a coherent sheaf M is locally free iff $\mathcal{H}om_S\big(M, \mathcal{O}_S\big)$ is locally free and commutes with base change.

Warning on general duality 2.70.6. If F is locally free, then we get a natural pairing

$$R^i f_*\big(\omega_{X/S} \otimes F^{-1}\big) \times R^{n-i} f_*\big(F\big) \to R^n f_* \omega_{X/S} \to \mathcal{O}_S,$$

but this is not a perfect pairing, not even if $f : X \to S$ is smooth.

If f is CM, one should not expect this pairing to be perfect unless both sheaves on the left are locally free and commute with base change.

More on the CM case 2.70.7. Let $f : X \to S$ be a projective morphism of pure relative dimension n. We already noted in (2.70.2) that if f is flat with CM fibers over S then the same holds for $\omega_{X/S}$. We consider what happens of f is not everywhere CM. By (10.2) there is a largest open subset $X^{\mathrm{cm}} \subset X$ such that $f|_{X^{\mathrm{cm}}}$ is flat with CM fibers. Assume for simplicity that $X_s \cap X^{\mathrm{cm}}$ is dense in X_s and $s \in S$ is local. Then, for every $x \in X_s \cap X^{\mathrm{cm}}$ one can choose a finite morphism $\pi : X \to P = \mathbb{P}^n_S$ such that $\pi^{-1}\big(\pi(x)\big) \subset X^{\mathrm{cm}}$. Thus $\pi_* \mathcal{O}_X$ is locally free at $\pi(x)$ and so is $\pi_* \omega_{X/S}$. Thus we have proved that the restriction of $\omega_{X/S}$ to X^{cm} is

(a) flat over S with CM fibers and
(b) commutes with base change.

This is actually true for all finite type morphisms, one just needs to find a local analog of the projection π (see Section 10.7) and show that (2.70.2.a) holds if π is finite; see [**Con00**] for details.

COROLLARY 2.71. *Let S be a Noetherian scheme over a field of characteristic 0 and $f : X \to S$ a proper and flat morphism with potentially slc fibers over closed points. Let L be an f-semi-ample line bundle on X. Then, for all i,*

(1) $R^i f_*\big(\omega_{X/S} \otimes L\big)$ *is locally free and compatible with base change and*
(2) $h^i\big(X_s, \omega_{X_s} \otimes L_s\big)$ *is independent of $s \in S$.*

In particular, for $L = \mathcal{O}_X$ we get that

(3) $R^i f_* \omega_{X/S}$ *is locally free and compatible with base change and*
(4) $h^i\big(X_s, \omega_{X_s}\big)$ *is independent of $s \in S$.*

If the fibers X_s are CM, then $H^i\big(X_s, \omega_{X_s}\otimes L_s\big)$ is dual to $H^{n-i}\big(X_s, L_s^{-1}\big)$ and (2.71) follows from (2.67). If the fibers X_s are not CM, the relationship between (2.71) and (2.67) is not so clear.

Proof. Let us start with the case $i=0$. By weak duality (2.70.3),

$$f_*\big(\omega_{X/S}\otimes L\big) \cong \operatorname{Hom}_S\big(R^n f_*\big(L^{-1}\big), \mathcal{O}_S\big),$$

where $n=\dim(X/S)$. By (2.67), $R^n f_*\big(L^{-1}\big)$ is locally free and compatible with base change, hence so is $f_*\big(\omega_{X/S}\otimes L\big)$. Thus (2.71.1) holds for $i=0$. Next we use this and induction on n to get the $i>0$ cases.

Choose M very ample on X such that $R^i f_*\big(\omega_{X/S}\otimes L\otimes M\big)=0$ for $i>0$, and this also holds after any base change. Working locally on S, as in the proof of (2.67), let $H\subset X$ be a general member of $|M|$ such that $H\to S$ is also flat with potentially slc fibers (2.11). The push forward of the sequence

$$0\to \omega_{X/S}\otimes L\to \omega_{X/S}\otimes L\otimes M\to \omega_{H/S}\otimes L\to 0$$

gives isomorphisms

$$R^i f_*\big(\omega_{X/S}\otimes L\big)\cong R^{i-1} f_*\big(\omega_{H/S}\otimes L\big)\quad \text{for } i\geq 2.$$

Using induction, these imply that (2.71.1) holds for $i\geq 2$.

The beginning of the push-forward is an exact sequence

$$0\to f_*\big(\omega_{X/S}\otimes L\big)\to f_*\big(\omega_{X/S}\otimes L\otimes M\big)\to f_*\big(\omega_{H/S}\otimes L\big)\to R^1 f_*\big(\omega_{X/S}\otimes L\big)\to 0.$$

We already proved that the first 3 terms are locally free. In general, this does not imply that the last term is locally free, but this implication holds if S is the spectrum of an Artin ring (2.72).

In general, pick any point $s\in S$ with maximal ideal sheaf m_s. Set $A_n := \mathcal{O}_{s,S}/m_s^n$ and $X_n := \operatorname{Spec}\big(\mathcal{O}_X/f^*m_s^n\big)$. By the above considerations,

$$H^1\big(X_n, (\omega_{X/S}\otimes L)|_{X_n}\big)$$

is a free A_n-module and the restriction maps

$$H^1\big(X_n, (\omega_{X/S}\otimes L)|_{X_n}\big)\otimes_{A_n} k(s)\to H^1\big(X_s, \omega_{X_s}\otimes L_s\big)$$

are isomorphisms. By the Theorem on Formal Functions [**Har77**, III.11.1], this implies that $R^1 f_*\big(\omega_{X/S}\otimes L\big)$ is locally free and commutes with base change. $\square$

2.72. Let (A,m) be a local Artin ring. Let F be a free A-module and $j: A\hookrightarrow F$ an injection. We claim that $j(A)$ is a direct summand of F. Indeed, let $r\geq 1$ be the smallest natural number such that $m^rA=0$. Note that $m^{r-1}m=0$. If $j(A)\subset mF$ then $m^{r-1}A=0$, a contradiction. Thus $j(A)$ is a direct summand of F. By induction this shows that any injection between free A-modules is split. This also implies that if

$$0\to M_1\to\cdots\to M_n\to 0$$

is an exact sequence of A-modules and all but one of them are free then they are all free.

2.6. Families of divisors I

Assumptions. In this Section we work with arbitrary schemes.

We saw in (2.69) that for locally stable morphisms $g : (X, \Delta) \to C$ the relative dualizing sheaf $\omega_{X/C}$ commutes with base change. We also saw in (2.41) that its powers $\omega_{X/C}^{[m]}$ usually do not commute with base change. Here we consider this question for a general divisor D: What does it mean to restrict a divisor D on X to a fiber X_c and how are the two sheaves $\mathcal{O}_X(D)|_{X_c}$ and $\mathcal{O}_{X_c}\big(D|_{X_c}\big)$ related?

2.73 (One-parameter families of divisors). Let T be a regular 1-dimensional scheme and $f : X \to T$ a flat, proper morphism. For simplicity assume for now that X is normal. Let D be an effective Weil divisor on X. Under what conditions can we view D as giving a "reasonable" family of Weil divisors on the fibers of f?

We can view D as a subscheme of X and, if $\operatorname{Supp} D$ does not contain any irreducible component of any fiber X_t, then $f|_D : D \to T$ is flat, hence the fibers D_t form a flat family of subschemes of pure codimension 1 of the fibers X_t. The D_t may have embedded points, ignoring them gives a well defined effective Weil divisor on the fiber X_t. Let us denote it temporarily by $[D_t]$. Understanding the difference between the subscheme D_t and the divisor $[D_t]$ is the key to dealing with many issues. As a rule of thumb, D defines a "nice" family of divisors iff $D_t = [D_t]$ for every t.

It can happen that D_t is contained in $\operatorname{Sing} X_t$ for some t. These are the cases when the correspondence between Weil divisors and rank 1 reflexive sheaves breaks down. Fortunately, this does not happen for locally stable families. That is, we can restrict to the cases when X_t is smooth at all generic points of D_t.

It is now time to drop the normality assumption, and work with divisors in the following more general setting. (Further generalizations will be considered in Sections 5.8 and 9.4.)

(1) T is a regular, 1-dimensional, irreducible scheme and $f : X \to T$ is a flat, pure dimensional morphism whose fibers are reduced and S_2.
(2) D is a Weil divisor on X such that $\operatorname{Supp} D$ contains neither an irreducible component of a fiber X_t nor a codimension 1 irreducible component of $\operatorname{Sing} X_t$.
(3) These imply that there is a closed subscheme $Z \subset X$ such that $D|_{X \setminus Z}$ is a Cartier divisor and $\operatorname{codim}_{X_t}\big(X_t \cap Z\big) \geq 2$ for every $t \in T$.

Under these conditions, $[D_t]$ is defined as the unique Weil divisor on X_t that agrees with the restriction of the Cartier divisor $D|_{X \setminus Z}$ to $X_t \setminus Z$.

If D is effective, it can be identified with a subscheme of pure codimension 1 of X and then D_t denotes the fiber of this subscheme over $t \in T$. As we noted before, D_t and $[D_t]$ differ only in the former possibly having some embedded points.

PROPOSITION 2.74. *Notation and assumptions as in* (2.73.1–3) *with D effective. Let $0 \in T$ be a closed point and $g \in T$ the generic point. The following conditions are equivalent.*

(1) *$\mathcal{O}_D$ has depth ≥ 2 at every point of $X_0 \cap Z$.*
(2) *D_0 has no embedded points.*
(3) *$D_0 = [D_0]$.*
(4) *$\mathcal{O}_X(-D)$ has depth ≥ 3 at every point of $X_0 \cap Z$.*
(5) *$\mathcal{O}_X(-D)|_{X_0}$ is S_2.*

(6) *The restriction map* $r_0 : \mathcal{O}_X(-D)|_{X_0} \to \mathcal{O}_{X_0}\bigl(-[D_0]\bigr)$ *is an isomorphism.*

If f *is projective and* $\mathcal{O}_X(1)$ *is* f*-ample then these are also equivalent to:*

(7) $\chi\bigl(X_0, \mathcal{O}_{X_0}(-[D_0])(m)\bigr) = \chi\bigl(X_g, \mathcal{O}_{X_g}(-D_g)(m)\bigr)$ *for all* $m \in \mathbb{Z}$.

If $\dim(X_0 \cap Z) = 0$ *then these are further equivalent to:*

(8) $\chi\bigl(X_0, \mathcal{O}_{X_0}(-[D_0])\bigr) = \chi\bigl(X_g, \mathcal{O}_{X_g}(-D_g)\bigr)$.

Proof. Let t be a local coordinate at $0 \in T$. Then f^*t is not a zero divisor on $\mathcal{O}_D$ and $\mathcal{O}_{D_0} = \mathcal{O}_D/(t_0)$. Thus (1) $\Leftrightarrow$ (2) and (3) is just a reformulation of (2). A similar argument gives that (4) $\Leftrightarrow$ (5). Since $\mathcal{O}_X(-D)$ is S_2, r_0 is an injection and an isomorphism outside Z. Since $\mathcal{O}_{X_0}\bigl(-[D_0]\bigr)$ is S_2 by definition, it is the S_2-hull of $\mathcal{O}_X(-D)|_{X_0}$; see (9.12.4). Thus r_0 is surjective $\Leftrightarrow$ r_0 is an isomorphism $\Leftrightarrow$ $\mathcal{O}_X(-D)|_{X_0}$ is S_2. This proves (5) $\Leftrightarrow$ (6).

Since $\mathcal{O}_X$ has depth ≥ 3 at every codimension ≥ 2 point of X_0, the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

and an easy lemma (2.78) shows that (1) $\Leftrightarrow$ (4). Since $\mathcal{O}_X(-D)$ is flat over T,

$$\begin{aligned} \chi\bigl(X_g, \mathcal{O}_{X_g}(-D_g)(m)\bigr) &= \chi\bigl(X_g, \mathcal{O}_X(m) \otimes \mathcal{O}_X(-D)|_{X_g}\bigr) \\ &= \chi\bigl(X_0, \mathcal{O}_X(m) \otimes \mathcal{O}_X(-D)|_{X_0}\bigr). \end{aligned}$$

Therefore the difference of the two sides in (7) is $\chi\bigl(X_0, \mathcal{O}_{X_0}(m) \otimes Q\bigr)$ where $Q :=$ coker r_0. Thus $Q = 0$ iff equality holds in (7), hence (6) $\Leftrightarrow$ (7).

If $\dim(X_0 \cap Z) = 0$ then Q has 0-dimensional support, thus

$$\chi\bigl(X_0, \mathcal{O}_{X_0}(m) \otimes Q\bigr) = \chi(X_0, Q) = H^0(X_0, Q),$$

so, in this case, (7) is equivalent to (8). □

Note that (2.74) shows that one can go rather freely between effective divisors and their ideal sheaves when studying restrictions. Much of the above results on ideal sheaves generalize to arbitrary sheaves; these are worked out in Sections 5.8 and 9.4.

As shown by (2.74.4), the conditions (2.74) are all preserved by linear equivalence. However, they are not preserved by sums of divisors.

Example 2.75. Consider a family of smooth quadrics $Q \subset \mathbb{P}^3 \times \mathbb{A}^1$ degenerating to the quadric cone Q_0. Take four families of lines L^i, M^i such that $L^1_0, L^2_0, M^1_0, M^2_0$ are 4 distinct lines in Q_0 and $L^1_c \neq L^2_c$ are in one family of lines on Q_c and $M^1_c \neq M^2_c$ are in the other family for $c \neq 0$. Note that

$$\bigl(Q, \tfrac{1}{2}(L^1 + L^2 + M^1 + M^2)\bigr) \to \mathbb{A}^1$$

is a locally stable family.

Each of the 4 families of lines L^i, M^i is a flat family of Weil divisors.

For pairs of lines, flatness is more complicated. $L^1 + L^2$ is *not* a flat family (the flat limit has an embedded point at the vertex) but $L^i + M^j$ is a flat family for every i, j. The union of any 3 of them, for instance $L^1 + L^2 + M^1$ is again a flat family and so is $L^1 + L^2 + M^1 + M^2$.

The situation looks even more complicated if we choose $L^1_0 = M^1_0$ and $L^2_0 = M^2_0$.

Next we give examples of divisors and divisorial sheaves that satisfy the equivalent conditions of (2.74). We state them using the equivalent form (2.74.6).

PROPOSITION 2.76. *Let $f : (X, \Delta) \to C$ be a locally stable morphism to a smooth curve defined over a field of characteristic 0 and $c \in C$ a closed point.*

(1) *If $\Delta = 0$ then, for every $m \in \mathbb{Z}$, $\omega^{[m]}_{X/C}\big|_{X_c} \cong \omega^{[m]}_{X_c}$.*
(2) *If $m\Delta$ is a $\mathbb{Z}$-divisor then*
$$\big(\omega^{[m]}_{X/C}(m\Delta)\big)\big|_{X_c} \cong \omega^{[m]}_{X_c}\big(m\Delta|_{X_c}\big).$$
(3) *If $m\Delta$ is a $\mathbb{Z}$-divisor then*
$$\big(\omega^{[m+1]}_{X/C}(m\Delta)\big)\big|_{X_c} \cong \omega^{[m+1]}_{X_c}\big(m\Delta|_{X_c}\big).$$
(4) *Assume that $\Delta = \sum\big(1 - \frac{1}{r_i}\big)D_i$ for some $r_i \in \mathbb{N}$. Then, for every $m \in \mathbb{Z}$,*
$$\big(\omega^{[m]}_{X/C}(\lfloor m\Delta\rfloor)\big)\big|_{X_c} \cong \omega^{[m]}_{X_c}\big(\lfloor m\Delta\rfloor|_{X_c}\big).$$
(5) *Assume that $\Delta = \sum c_i D_i$ and $1 - \frac{1}{m} \le c_i \le 1$ for every i. Then*
$$\big(\omega^{[m]}_{X/C}(\lfloor m\Delta\rfloor)\big)\big|_{X_c} \cong \omega^{[m]}_{X_c}\big(\lfloor m\Delta\rfloor|_{X_c}\big).$$

Proof. Let D be a Weil divisor on X as in (2.73.2–4)). Assume that there is an effective $\mathbb{Q}$-divisor $\Delta' \le \Delta$ and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L such that $D \sim_{\mathbb{Q}} \Delta' + L$. Then $\mathcal{O}_X(-D)$ satisfies the equivalent conditions of (2.74) by (1.81).

In cases (1–2) we can take $\Delta' = 0$ and $L := -m\big(K_{X/C} + \Delta\big)$ and in case (3) we use $\Delta' = \Delta$ and $L := -(m+1)\big(K_{X/C} + \Delta\big)$.

Finally in cases (4–5) we employ $\Delta' = m\Delta - \lfloor m\Delta\rfloor$ and $L := -m\big(K_{X/C} + \Delta\big)$. The assumptions on the coefficients of Δ ensure that $\Delta' \le \Delta$. (Note that if $m\Delta - \lfloor m\Delta\rfloor \le \Delta$ for every m then in fact every coefficient of Δ is of the form $1 - \frac{1}{r}$ for some $r \in \mathbb{N}$.) □

These results are close to being optimal. For instance, under the assumptions of (2.76.2), if n is different from m and $m+1$ then the two sheaves
$$\big(\omega^{[n]}_{X/C}(mD)\big)\big|_{X_c} \quad \text{and} \quad \omega^{[n]}_{X_c}\big(mD|_{X_c}\big)$$
are frequently different, see (2.39.3). In general, as shown by (2.41), even the two sheaves $\big(\omega^{[m]}_{X/C}\big)\big|_{X_c}$ and $\omega^{[m]}_{X_c}$ can be different if $\Delta \ne 0$. However, it is likely that there are further results similar to (2.76). The following would be especially interesting.

QUESTION 2.77. Let $f : (X, \Delta) \to C$ be locally stable and defined over a field of characteristic 0. Assume that $\Delta = \sum c_i D_i$ and $\frac{1}{2} \le c_i \le 1$ for every i. Is it true that, for every m,
$$\big(\omega^{[m]}_{X/C}(\lfloor m\Delta\rfloor)\big)\big|_{X_c} \cong \omega^{[m]}_{X_c}\big(\lfloor m\Delta\rfloor|_{X_c}\big)?$$

The next lemma is quite straightforward; see [**Kol13c**, 2.60] for details.

LEMMA 2.78. *Let X be a scheme and $0 \to F' \to F \to F'' \to 0$ a sequence of coherent sheaves on X that is exact at $x \in X$.*

(1) *If* $\operatorname{depth}_x F \ge r$ *and* $\operatorname{depth}_x F'' \ge r - 1$ *then* $\operatorname{depth}_x F' \ge r$.
(2) *If* $\operatorname{depth}_x F \ge r$ *and* $\operatorname{depth}_x F' \ge r - 1$ *then* $\operatorname{depth}_x F'' \ge r - 1$. □

2.7. Boundary with coefficients $> \frac{1}{2}$

Assumptions. In this Section we work with varieties over a field of characteristic 0.

2.79 (Boundaries and embedded points). Consider a locally stable morphism $f : (X, \Delta = \sum a_i D^i) \to C$ to a smooth curve C. It is very tempting to think of each fiber $\bigl(X_c, \Delta_c\bigr)$ as a compound object $\bigl(X_c, D^i_c : i \in I, a_i : i \in I\bigr)$ consisting of the scheme X_c, the divisors $D^i_c := D^i|_{X_c}$ and their coefficients a_i. Two problems make this simple picture questionable.

(1) Different D^i_c may have an irreducible component E_c in common. Our definition of the fiber says that we should treat E_c as a divisor with coefficient $\sum_{i\in I} \operatorname{coeff}_E D^i_c$. The individual D^i_c do not seem to be part of the data any more.

(2) The D^i_c may have embedded points. Do we ignore them or do we take them into consideration?

One could hope that the first problem (2.79.1) is just a matter of book-keeping, but this does not seem to be the case, as shown by the examples (2.75). Similar examples were given in (2.39). In both cases the coefficients in Δ were $\leq \frac{1}{2}$.

The aim of this section is to show that these examples were optimal; the problems (2.79.1–2) do not occur if the coefficients in Δ are all $> \frac{1}{2}$. We start with the case when the coefficients are 1.

Given a locally stable map $f : (X, \Delta) \to C$ it is not true that the lc centers of the fibers (X_c, Δ_c) form a flat family. Indeed, there are many cases when the generic fiber is smooth but a special fiber is not klt. However, as we show next, the specialization of an lc center on the generic fiber becomes a union of lc centers on a special fiber. Set theoretically this follows from adjunction (1.93) and (1.98.4), but now we prove this even scheme theoretically.

THEOREM 2.80. *Let C be a smooth curve over a field of characteristic 0, $f : (X, \Delta) \to C$ a locally stable morphism and $Z \subset X$ any union of lc centers of (X, Δ). Then $f|_Z : Z \to C$ is flat with reduced fibers and for every $c \in C$, the fiber Z_c is a union of lc centers of (X_c, Δ_c) (scheme theoretically).*

Proof. Z is reduced, and by (2.13), every irreducible component of Z dominates C. Thus $f|_Z : Z \to C$ is flat. We can write its fibers as $Z_c = X_c \cap Z$. Since $X_c + Z$ is a union of lc centers of $(X, X_c + \Delta)$, it is seminormal (1.98.2) and $X_c \cap Z$ is reduced by (1.98.3). The last claim follows from (1.96). □

In the divisorial case we can say more.

COROLLARY 2.81. *Let C be a smooth curve over a field of characteristic 0 and $f : (X, \Delta) \to C$ a locally stable (resp. stable) morphism. Let $\{D_i : i \in I\}$ be irreducible components of $\lfloor \Delta \rfloor$ and set $D := \cup_{i\in I} D_i$. Then*

$$f|_D : \bigl(D, \operatorname{Diff}_D(\Delta - D)\bigr) \to C$$

is locally stable (resp. stable).

Proof. We have already proved in (2.51) that we have a locally stable (resp. stable) morphism on the normalization of D. Using (2.10) it remains to prove that D is deminormal. The fibers of $f|_D : D \to C$ are reduced, hence S_1, so D is S_2. In codimension 1 D has only nodes by (1.96.5). hence it is demi-normal. □

In general, when the coefficient is $> \frac{1}{2}$, let us start with a simple result.

LEMMA 2.82. *Let $f : (X, \sum_{i\in I} a_i D^i) \to C$ be a locally stable family over a smooth curve over a field of characteristic 0.*

(1) *If $a_i > \frac{1}{2}$ then every irreducible component of D^i_c has multiplicity 1.*
(2) *If $a_i + a_j > 1$ then the divisors D^i_c and D^j_c have no irreducible components in common.*

Proof. By (2.3), $\bigl(X_c, \Delta_c\bigr)$ is slc, hence every component of Δ_c appears with coefficient ≤ 1. For a divisor $E \subset X_c$,

$$\operatorname{coeff}_E\bigl(\Delta|_{X_c}\bigr) = \textstyle\sum_{i\in I} a_i \cdot \operatorname{coeff}_E\bigl(D^i_c\bigr).$$

This shows both (1) and (2). □

The next result of [**Kol14**] solves the embedded point problem (2.79.2) when all the occurring coefficients are $> \frac{1}{2}$.

THEOREM 2.83. *Let $f : (X, \Delta = \sum_{i\in I} a_i D_i) \to C$ be a locally stable morphism to a smooth curve over a field of characteristic 0. Let $J \subset I$ be any subset such that $a_j > \frac{1}{2}$ for every $j \in J$ and set $D_J := \cup_{j\in J} D_j$. Then*

(1) *$f|_{D_J} : D_J \to C$ is flat with reduced fibers,*
(2) *D is S_2 and*
(3) *$\mathcal{O}_X(-D)$ is S_3.*

Proof. Note that each D_i is a log center of (X, Δ) (1.97) and $\operatorname{mld}(D_i, X, \Delta) = 1 - a_i$ by (1.95.2). Thus $\operatorname{mld}(D_J, X, \Delta) < \frac{1}{2}$.

Let X_c be any fiber of f. Then $(X, X_c + \Delta)$ is slc and

$$\operatorname{mld}(D_i, X, X_c + \Delta) = \operatorname{mld}(D_i, X, \Delta) < \tfrac{1}{2},$$

since none of the D_i is contained in X_c. Each irreducible component of X_c is a log canonical center of $(X, X_c + \Delta)$ (1.96), thus $\operatorname{mld}(X_c, X, X_c + \Delta) = 0$. Therefore, $\operatorname{mld}(D_J, X, X_c + \Delta) + \operatorname{mld}(X_c, X, X_c + \Delta) < \frac{1}{2}$.

We can apply (1.98.3) to $(X, X_c + \Delta)$ with $W = D_J$ and $Z = X_c$ to conclude that $X_c \cap D_J$ is is reduced. This proves (1) which in turn implies (2–3) by (2.74). □

2.8. Grothendieck–Lefschetz-type theorems

The following theorem was conjectured in [**Kol13a**] and proved there in the lc case. For normal schemes the proof is given in [**BdJ14**], aside from possible p-torsion in characteristic $p > 0$. The general case is established in [**Kol16a**].

THEOREM 2.84. *Let $(x \in X)$ be an excellent, local scheme of pure dimension ≥ 4 over a field such that $\operatorname{depth}_x \mathcal{O}_X \geq 3$. Let $x \in D \subset X$ be a Cartier divisor. Then the restriction map*

$$r^X_D : \operatorname{Pic}^{\mathrm{loc}}(x, X) \to \operatorname{Pic}^{\mathrm{loc}}(x, D) \quad \textit{is an injection.}$$

Recall that the *local Picard group* $\operatorname{Pic}^{\mathrm{loc}}(x, X)$ is defined as $\operatorname{Pic}(X \setminus \{x\})$.

We only discuss the situation when X is normal and essentially of finite type over a field k; this is the only case that we use in this book. The non-normal case is reduced to the normal one in [**Kol16a**]. The general setting can be reduced to the finite type cases by a short but subtle approximation argument; see [**BdJ14**, Sec.1.4] for details.

The argument in this section proves a weaker version: the kernel of r_D^X : $\operatorname{Pic}^{\mathrm{loc}}(x,X) \to \operatorname{Pic}^{\mathrm{loc}}(x,D)$ is torsion. That is, if L is a line bundle on $U := X \setminus \{x\}$ such that $L_D := L|_{U\cap D} \cong \mathcal{O}_{U\cap D}$ then $L^m \cong \mathcal{O}_U$ for some $m > 0$. Then we show in the next section that in fact L is trivial.

The proof is somewhat roundabout. The main step is to prove a variant of (2.84) in characteristic p; see (2.88). Then in (2.89) we reduce everything to positive characteristic and lift back to characteristic 0 using (2.85).

During the proof we need several general results on cohomology groups of sheaves over quasi affine schemes, these are recalled in (10.18).

Our discussions present these steps in the reverse order. The reason is that the proof of (2.85) is the simplest, showing the key ideas. The proof of (2.88) follows the same path but with several technical detours.

THEOREM 2.85. *Let $(x \in X)$ be a local scheme such that* $\operatorname{depth}_x \mathcal{O}_X \geq 4$. *Let $x \in D \subset X$ be a Cartier divisor. Set $U := X \setminus \{x\}$ and $U_D := D \setminus \{x\}$. Let L be a coherent, rank 1, S_2 sheaf on U such that $L_D := L|_{U_D} \cong \mathcal{O}_{U_D}$. Then $L \cong \mathcal{O}_U$.*

Proof. Let t be a defining equation of D and consider the exact sequence

$$0 \to L \stackrel{t}{\to} L \stackrel{r}{\to} L_D \cong \mathcal{O}_{U_D} \to 0.$$

Take cohomologies to get

$$\begin{array}{ccccccc} H^0(U,L) & \stackrel{t}{\to} & H^0(U,L) & \stackrel{r}{\to} & H^0(U_D, L_D \cong \mathcal{O}_{U_D}) & \to \\ H^1(U,L) & \stackrel{t}{\to} & H^1(U,L) & \to & H^1(U_D, L_D \cong \mathcal{O}_{U_D}). & \end{array} \tag{2.85.1}$$

In order to prove that the second row of (2.85.1) is identically zero, we start on the right hand side. The cohomology sequence of

$$0 \to \mathcal{O}_U \stackrel{t}{\to} \mathcal{O}_U \to \mathcal{O}_{U_D} \to 0$$

contains the piece

$$H^1(U, \mathcal{O}_U) \to H^1(U_D, \mathcal{O}_{U_D}) \to H^2(U, \mathcal{O}_U). \tag{2.85.2}$$

Since $\operatorname{depth}_x \mathcal{O}_X \geq 4$, (10.18.2–3) imply that $H^i(U, \mathcal{O}_U) = 0$ for $1 \leq i \leq 2$, hence the two sides of (2.85.2) are 0. Thus $H^1(U_D, \mathcal{O}_{U_D}) = 0$ and so the second line of (2.85.1) shows that $t : H^1(U,L) \to H^1(U,L)$ is surjective. By (10.18.7), $H^1(U,L)$ has finite length since $\dim U \geq 3$, hence

$$H^1(U,L) \stackrel{t}{\to} H^1(U,L) \quad \text{is an isomorphsm.}$$

(By (10.18.5), multiplication by t is nilpotent on $H^i(U, \mathcal{O}_U)$ for $i > 0$, thus in fact $H^1(U,L) = 0$.)

Thus (2.85.1) shows that the restriction map

$$r : H^0(U,L) \to H^0(U_D, L_D) \cong H^0(U_D, \mathcal{O}_{U_D}) \quad \text{is surjective.}$$

In particular, the constant 1 section of $H^0(U_D, \mathcal{O}_{U_D})$ lifts to a section $s \in H^0(U,L)$. Thus $L \cong \mathcal{O}_U$ by (2.86). $\square$

LEMMA 2.86. *Let X be a pure dimensional, S_2 scheme, $D \subset X$ a Cartier divisor and $W \subset D$ a subscheme such that* $\operatorname{codim}_D W \geq 2$. *Let L be a rank 1, torsion free sheaf on X that is locally free along $D \setminus W$ and s a section of L such that $s|_{D\setminus W}$ is nowhere zero. Then L is trivial and s is nowhere zero in a neighborhood of D.*

Proof. The section s gives an exact sequence

$$0 \to \mathcal{O}_X \stackrel{s}{\to} L \to Q \to 0.$$

By (9.7) every associated prime of Q has codimension 1 in X by (9.7). Thus $D \cap \operatorname{Supp} Q$ has codimension 1 in D. Therefore D is disjoint from $\operatorname{Supp} Q$ and L is trivial on $X \setminus \operatorname{Supp} Q$. □

The next example, following [**BdJ14**] and [**Kol13a**, 12], shows that (2.85) fails if $\operatorname{depth}_x X = 3$; see also (2.42). As we see afterwards, one can say more if L is locally free on U.

Example 2.87. Let (A, Θ) be a principally polarized Abelian variety over a field k. Let $C_a(A, \Theta)$ be the affine cone over A with vertex v. It is easy to compute that $\operatorname{depth}_v C_a(A, \Theta) = 2$, see [**Kol13c**, 3.12]. Set $X := C_a(A, \Theta) \times \operatorname{Pic}^0(A)$ with $f : X \to \operatorname{Pic}^0(A)$ the second projection. Since $L(\Theta)$ has a unique section for every $L \in \operatorname{Pic}^0(A)$, there is a unique divisor D_A on $A \times \operatorname{Pic}^0(A)$ whose restriction to $A \times \{[L]\}$ is the above divisor. By taking the cone we get a divisor D_X on X.

For $L \in \operatorname{Pic}^0(A)$, let $D_{[L]}$ denote the restriction of D_X to the fiber $C_a(A, \Theta) \times \{[L]\}$ of f. We see that

(1) $D_{[L]}$ is Cartier iff $L \cong \mathcal{O}_A$.
(2) $mD_{[L]}$ is Cartier iff $L^m \cong \mathcal{O}_A$.
(3) $D_{[L]}$ is not $\mathbb{Q}$-Cartier for very general $L \in \operatorname{Pic}^0(A)$.

The next result proves that, at least in characteristic p, the kernel of the restriction map between the local Picard groups is torsion.

Theorem 2.88. [**BdJ14**] *Let $(x \in X)$ be a normal, excellent, local scheme of characteristic $p > 0$ and dimension ≥ 4. Let $x \in D \subset X$ be a Cartier divisor. Set $U := X \setminus \{x\}$ and $U_D := D \setminus \{x\}$.*

Let L be a line bundle on U such that $L_D := L|_{U_D} \cong \mathcal{O}_{U_D}$. Then $L^m \cong \mathcal{O}_U$ for some $m > 0$.

Proof. In order to emphasize the similarities, we follow the proof of (2.85) as closely as possible, even though this is somewhat repetitive. In a few places, we need to add technical details to establish results that were obvious under the assumptions of (2.85).

Our main effort goes to proving that there is a normal scheme V and a finite, surjective morphism $\pi : V \to U$ such that $\pi^* L \cong \mathcal{O}_V$.

We do not know a priori which finite surjective morphism to take, so we work with their direct limit. That is, let $\mathcal{O}_X^+$ denote the normalization of $\mathcal{O}_X$ in an algebraic closure of the function field of X. We view $\mathcal{O}_X^+$ as a quasi coherent sheaf of X. Note that $\mathcal{O}_X^+$ is the direct limit of the structure sheaves of the normalizations of X in finite degree algebraic extensions of its function field. Set $\mathcal{O}_U^+ := \mathcal{O}_X^+|_U$.

The key result we use is that $\mathcal{O}_X^+$ is CM, which is a hard theorem due to [**HH92**]. This is the only point where we use that the characteristic is positive. (In characteristic 0 the sheaves $\mathcal{O}_X^+$ are never CM if $\dim X \geq 3$.)

We use Grothendieck's characterization of CM sheaves by local cohomology groups (see [**Gro67**, Sec.3] or [**BH93**, 3.5.7]):

$$H^j_x\big(X, \mathcal{O}_X^+\big) = 0 \quad \text{for } j < \dim X. \tag{2.88.1}$$

As in (10.18.2–3) this implies that

$$H^i\big(U, \mathcal{O}_U^+\big) = 0 \quad \text{for } 1 \le i \le \dim X - 2. \tag{2.88.2}$$

This is the only consequence of the CM property we use.

Let $(t = 0)$ be an equation of D and consider the exact sequence

$$0 \to L \stackrel{t}{\to} L \stackrel{r}{\to} L_D \cong \mathcal{O}_{U_D} \to 0. \tag{2.88.3}$$

If the constant 1 section of $L_D \cong \mathcal{O}_{U_D}$ can be lifted to a section of L, then $L \cong \mathcal{O}_U$ and we are done. This holds if $H^1(U, L) = 0$, but the latter usually fails. However, as we noted before, we need this only after some finite base change. The groups $H^1(V, \pi^* L)$ usually do not vanish for any finite cover $\pi : V \to U$, but, rather surprisingly, vanishing holds for their direct limit.

Thus we tensor (2.88.3) with $\mathcal{O}_U^+$ to get

$$0 \to L \otimes \mathcal{O}_U^+ \stackrel{t}{\to} L \otimes \mathcal{O}_U^+ \stackrel{r}{\to} \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+ \to 0. \tag{2.88.4}$$

While $\mathcal{O}_U^+$ is not flat, it is torsion free, so (2.88.4) is still left exact. Next we take cohomologies to get

$$\begin{array}{ccccccc} H^0\big(U, L \otimes \mathcal{O}_U^+\big) & \stackrel{t}{\to} & H^0\big(U, L \otimes \mathcal{O}_U^+\big) & \stackrel{r}{\to} & H^0\big(U_D, \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+\big) & \to \\ H^1\big(U, L \otimes \mathcal{O}_U^+\big) & \stackrel{t}{\to} & H^1\big(U, L \otimes \mathcal{O}_U^+\big) & \to & H^1\big(U_D, \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+\big) & \end{array} \tag{2.88.5}$$

Note that all of these cohomology groups are naturally $H^0(X, \mathcal{O}_X)$-modules. (They are even $H^0(X, \mathcal{O}_X^+)$-modules, but we will not use this richer structure.)

Our next aim is to prove that the second row of (2.88.3) is identically zero. Again we start on the right hand side. The cohomology sequence of

$$0 \to \mathcal{O}_U^+ \stackrel{t}{\to} \mathcal{O}_U^+ \to \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+ \to 0$$

contains the piece

$$H^1\big(U, \mathcal{O}_U^+\big) \to H^1\big(U_D, \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+\big) \to H^2\big(U, \mathcal{O}_U^+\big).$$

The two sides are 0 by (2.88.2) since $\dim X \ge 4$. Thus $H^1\big(U_D, \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+\big) = 0$ and so

$$H^1\big(U, L \otimes \mathcal{O}_U^+\big) \stackrel{t}{\to} H^1\big(U, L \otimes \mathcal{O}_U^+\big) \quad \text{is surjective.}$$

Equivalently, $H^1\big(U, L \otimes \mathcal{O}_U^+\big)$ is t-divisible. (This does not yet imply vanishing since $H^1\big(U, L \otimes \mathcal{O}_U^+\big)$ is not a coherent $\mathcal{O}_X$-module.)

Next we establish that $H^1\big(U, L \otimes \mathcal{O}_U^+\big)$ is killed by t^r for $r \gg 1$. Together with t-divisibility, this proves that $H^1\big(U, L \otimes \mathcal{O}_U^+\big) = 0$.

Here we use that L is locally free. Since U is quasi affine, for every point $x' \in U$ there is a global section g of L not vanishing at x'. This gives an exact sequence

$$0 \to \mathcal{O}_U \stackrel{g}{\to} L \to Q_g \to 0$$

where multiplication by g kills Q_g. Tensoring with $\mathcal{O}_X^+$ and taking cohomologies we get

$$H^1\big(U, \mathcal{O}_U^+\big) \stackrel{g}{\to} H^1\big(U, L \otimes \mathcal{O}_U^+\big) \to H^1\big(U, Q_g \otimes \mathcal{O}_U^+\big).$$

The group on the left is 0 by (2.88.2) and the one on the right is killed by g. Thus multiplication by g kills $H^1\big(U, L \otimes \mathcal{O}_U^+\big)$. Using this for every $x' \in U$, we get that the annihilator of $H^1\big(U, L \otimes \mathcal{O}_U^+\big)$ is an $m_{x,X}$-primary ideal. In particular, t^r is in the annihilator of $H^1\big(U, L \otimes \mathcal{O}_U^+\big)$ for $r \gg 1$.

These observations together imply that $H^1\big(U, L\otimes \mathcal{O}_U^+\big) = 0$. Thus (2.88.3) shows that the restriction map

$$r : H^0\big(U, L\otimes \mathcal{O}_U^+\big) \to H^0\big(U_D, \mathcal{O}_{U_D}\otimes \mathcal{O}_U^+\big) \quad \text{is surjective.}$$

In particular, the constant 1 section of $H^0\big(U_D, \mathcal{O}_{U_D}\otimes \mathcal{O}_U^+\big)$ lifts to a section $s \in H^0\big(U, L\otimes \mathcal{O}_U^+\big)$.

Since $\mathcal{O}_U^+$ is the direct limit of the structure sheaves of the normalizations of $U \subset X$ in finite degree algebraic extensions, we conclude that there is a normal scheme V and a finite, surjective morphism $\pi : V \to U$ such that the constant 1 section of

$$H^0\big(V_D, \pi^* L_D \cong \pi^*\mathcal{O}_{U_D} = \mathcal{O}_{V_D}\big)$$

lifts to a section

$$s_V \in H^0\big(V, \pi^* L\big).$$

By (2.86) this implies that $\pi^* L$ is a trivial line bundle on V.

Taking the norm (cf. [**Kol13c**, 2.40]) then gives that

$$\mathcal{O}_U \cong \operatorname{norm}_{V/U} \mathcal{O}_V \cong \operatorname{norm}_{V/U} \pi^* L \cong L^{\deg V/U}. \quad \square$$

Next we prove a weaker version of (2.84), which, as we noted in the discussion after the statement, can be used to settle the general case as well.

Proposition 2.89. *Let $(x \in X)$ be a normal, local scheme of finite type over a field k of characteristic 0. Let $x \in D \subset X$ be a Cartier divisor. Set $U := X \setminus \{x\}$ and $U_D := D \setminus \{x\}$. Let L be a line bundle on U such that $L_D := L|_{U_D} \cong \mathcal{O}_{U_D}$. Assume that* $\dim X \geq 4$ *and* $\operatorname{depth}_x \mathcal{O}_X \geq 3$.

Then $L^m \cong \mathcal{O}_U$ for some $m > 0$.

Proof. We use reduction to positive characteristic; see for instance [**KM98**, p.14] for a more detailed exposition.

There is a finitely generated $\mathbb{Z}$-algebra $R \subset k$, an R-scheme of finite type X^R, a section $x^R \subset X^R$, a Cartier divisor $x^R \subset D^R \subset X^R$, a line bundle L^R on $U^R := X^R \setminus x^R$ such that $L^R_D := L^R|_{U^D} \cong \mathcal{O}_{U_D}$ where $U^R_D := U^R \cap X^D$. Furthermore, after base change to k and localizing at x we recover the original X, $x \in D \subset X$ and L.

The assumptions of (2.89) are open in families, hence, after inverting finitely many elements of R, we may assume that the following holds.

Let $P \subset R$ be any prime ideal and X^P, $x^P \in D^P \subset X^P$ and L^P the fiber over $\operatorname{Spec} R/P$ of X^R, $x^R \in D^R \subset X^R$ and L^R. Then, after localizing at a generic point of x^P, the assumptions of (2.89) are satisfied, except that $k(P)$ need not have characteristic 0. (The only non-obvious assertion is that normality and the depth are preserved. For these see [**Gro60**, IV.12.1.6] or (10.2).)

Choose P to be a minimal prime ideal sitting over a prime $p \in \mathbb{Z}$ and localize R at P and X^R at a generic point of x^P. Set $T := \operatorname{Spec} R_P$. Denote the closed point of T by p. We thus have

(1) a scheme X^T that is flat over T with normal fibers,
(2) a section $x : T \to X^T$ such that $\operatorname{depth}_{x_p} \mathcal{O}_{X^T_p} = \operatorname{depth}_x \mathcal{O}_X$,
(3) a relative Cartier divisor $x(T) \subset D^T \subset X^T$,
(4) a line bundle L^T on $U^T := X^T \setminus x(T)$ such that the restriction of L^T to $U^T_p := U^T \cap X_p$ is trivial and

over the generic point of T we recover the original $x \in D \subset X$ and L.

Since the closed fiber has positive characteristic, we can apply (2.88) to show that that the restriction of $(L^T)^m$ to U_p^T is trivial for some $m > 0$. Then we use (2.85) to obtain that $(L^T)^m$ is trivial on U^T, hence also on the generic fiber. Thus L^m is trivial for some $m > 0$; proving the assertion in the first case. □

2.9. Torsion in Grothendieck–Lefschetz-type theorems

Here we complete most of the proof of (2.84) that the kernel is trivial. We proved in (2.88) that the kernel of the restriction map $\mathrm{Pic}^{\mathrm{loc}}(x, X) \to \mathrm{Pic}^{\mathrm{loc}}(x, D)$ is torsion. Now we aim to prove that it is also torsion free. For prime to $\mathrm{char}\, k(x)$ torsion this is proved in [**Gro68**, XIII]. In the global setting a short proof is given in [**Kol16a**], we recall it in (5.54). Using a suitable compactification, this implies the local version for schemes that are essentially of finite type of fields. The general case is due to [**dJ15**].

THEOREM 2.90. *Let $(x \in X)$ be a Noetherian, local scheme and $x \in D \subset X$ a Cartier divisor. Assume that* $\mathrm{depth}_x \mathcal{O}_X \geq 3$. *Then the kernel of the restriction map*

$$\ker\bigl[r_D^X : \mathrm{Pic}^{\mathrm{loc}}(x, X) \to \mathrm{Pic}^{\mathrm{loc}}(x, D)\bigr] \quad \textit{is torsion free.}$$

Note that, unlike the previous results, this works already when the dimension is ≥ 3.

Here we discuss two proofs of the weaker claim that the kernel does not contain m-torsion if $\mathrm{char}\, k(x) \nmid m$, which is enough for all applications in this book. The general case is postponed to Section 5.8.

2.91 (Proof of (2.90) in characteristic 0). Set $U := X \setminus \{x\}$ and $U_D := U \cap D$. Let L be a line bundle on U. Assume that $L|_{U_D} \cong \mathcal{O}_{U_D}$ and $L^m \cong \mathcal{O}_U$ for some m that is not divisible by $\mathrm{char}\, k(x)$. We prove that $L \cong \mathcal{O}_U$.

Let m denote the smallest natural number such that $L^m \cong \mathcal{O}_U$. This isomorphism gives a cyclic cover $\pi : \tilde{X} \to X$ that is étale over U (1.88). If $L_D \cong \mathcal{O}_{U_D}$ then $\pi^{-1}(D)$ is geometrically reducible and its irreducible components meet only at $\pi^{-1}(x)$. [**Gro68**, XIII.2.1] shows that this is impossible if $\dim X \geq 3$ and $m \geq 2$. □

Next we discuss a relative variant of (2.90) with an infinitesimal proof, which is basically just an adaptation of the method of [**Gro68**, XIII.2.1].

THEOREM 2.92. *Let (s, S) be a local scheme, $f : X \to S$ a flat morphism and $x \in X_s$ a point such that* $\mathrm{depth}_x X_s \geq 2$. *Set $U := X \setminus \{x\}$ and let L be a line bundle on U. Assume that $L|_{U_s}$ is trivial and L^m is trivial for some m not divisible by* $\mathrm{char}\, k(s)$. *Then L is trivial.*

Proof. Set $S_n := \mathrm{Spec}_S \mathcal{O}_S/m_s^n$. First we use (2.93) to show that the restriction of L to $X_n := X \times_S S_n$ is trivial for every n. Then (2.94.1) implies that L itself is trivial. □

LEMMA 2.93. *Let (A, m) be a local Artin k-algebra and $J \subset m$ an ideal such that $mJ = 0$. Let $f : X \to \mathrm{Spec}\, A$ be a flat morphism and $x \in X$ a point such that* $\mathrm{depth}_x \mathcal{O}_{X_0} \geq 2$. *Then the kernel of the restriction map*

$$\ker\bigl[\mathrm{Pic}^{\mathrm{loc}}(x, X) \to \mathrm{Pic}^{\mathrm{loc}}(x, X_J)\bigr]$$

is a k-vector space (possibly infinite dimensional).

In particular, if L is an element of the kernel and L^m is trivial for some m not divisible by $\operatorname{char} k$ *then L is trivial.*

Proof. We have an exact sequence

$$0 \to J \otimes_k \mathcal{O}_{U_0} \xrightarrow{\tau} \mathcal{O}_U^* \to \mathcal{O}_{U_J}^* \to 1$$

where $\tau(g) = 1 + g$ for a local section g of $J\mathcal{O}_U \cong J \otimes_k \mathcal{O}_{U_0}$.

A global section of $\mathcal{O}_{U_J}^*$ extends to a global section of $\mathcal{O}_{X_J}$ since $\operatorname{depth}_x \mathcal{O}_{X_J} = \operatorname{depth}_x \mathcal{O}_{X_0} \geq 2$ and then it lifts to a section of $\mathcal{O}_X$, which is necessarily nowhere zero by (2.86). Thus the cohomology sequence gives

$$0 \to J \otimes_k H^1(U, \mathcal{O}_{U_0}) \to \operatorname{Pic}^{\mathrm{loc}}(x, X) \to \operatorname{Pic}^{\mathrm{loc}}(x, X_J). \quad \square$$

PROPOSITION 2.94. *Let (s, S) be a local scheme with maximal ideal m. Let $f : X \to S$ be a flat morphism with S_2-fibers, $X_n := \operatorname{Spec}_X \mathcal{O}_X/m^{n+1}\mathcal{O}_X$ the nth infinitesimal neighborhood of $X_0 := X_s$ and $Z \subset X$ a subscheme that is finite over S with natural injections $j : X \setminus Z \hookrightarrow X$ and $j_n : X_n \setminus Z_n \hookrightarrow X_n$. Let L be an invertible sheaf on $X \setminus Z$ and $L_n := L|_{X_n \setminus Z_n}$. Assume that one of the following holds.*

(1) *$(j_n)_*(L_n)$ is locally free for every $n \geq 0$.*
(2) *$(j_0)_*(L_0)$ is locally free and $R^1(j_0)_*(L_0) = 0$.*

Then j_*L is invertible in a neighborhood of Z_0.

Proof. We may assume that $\mathcal{O}_S$ is m-adically complete and, possibly after passing to a smaller neighborhood of Z_0, we may assume that f is affine and $(j_0)_*(L_0) \cong \mathcal{O}_{X_0}$. For every n we have an exact sequence

$$0 \to (m_0^n/m_0^{n+1}) \otimes L_0 \to L_n \to L_{n-1} \to 0.$$

Pushing it forward we get an exact sequence

$$\begin{aligned} 0 \to (m_0^n/m_0^{n+1}) \otimes (j_0)_*(L_0) \to (j_n)_*(L_n) \xrightarrow{r_n} (j_{n-1})_*(L_{n-1}) \to \\ \to (m_0^n/m_0^{n+1}) \otimes R^1(j_0)_*(L_0). \end{aligned}$$

If $(j_n)_*(L_n)$ is locally free then so is its restriction to X_{n-1} and r_n gives a map of locally free sheaves

$$\bar{r}_n : (j_n)_*(L_n)|_{X_{n-1}} \to (j_{n-1})_*(L_{n-1})$$

that is an isomorphism on $X_{n-1} \setminus Z_{n-1}$. Since $\operatorname{depth}_{Z_{n-1}} X_{n-1} \geq 2$, this implies that $\bar{r}_n$ is an isomorphism and so r_n is surjective. The vanishing of $R^1(j_0)_*(L_0)$ also implies that r_n is surjective. Thus each $(j_n)_*(L_n)$ is locally free along X_n and the constant 1 section of $(j_0)_*(L_0) \cong \mathcal{O}_{X_0}$ lifts back to a nowhere zero global section of $\varprojlim (j_n)_*(L_n)$. Hence $\varprojlim (j_n)_*(L_n) \cong \mathcal{O}_X$ by (2.86).

Furthermore, we have a natural map $j_*L \to \varprojlim (j_n)_*(L_n) \cong \mathcal{O}_X$ that is an isomorphism on $X \setminus Z$. Since $\operatorname{depth}_Z j_*L \geq 2$, this implies that $j_*L \cong \mathcal{O}_X$. $\square$

The next example shows that going from formal triviality to triviality is not automatic.

EXAMPLE 2.95. Let $(e, E) \cong (e, E')$ be an elliptic curve. Set $X := (E \setminus \{e\}) \times E'$ and $p : X \to E'$ the second projection. Let $\Delta \subset X$ be the diagonal and $L = \mathcal{O}_X(\Delta)$.

For $p \in E' \setminus \{e\}$ the line bundle $L|_{X_p}$ is a nontrivial element of

$$\operatorname{Pic}(X_p \setminus \{e\}) \cong \operatorname{Pic}(E \setminus \{e\}) \cong \mathbf{Pic}^\circ(E).$$

but $L|_{X_e}$ is trivial.

For $m \in \mathbb{N}$ let $X_m \subset X$ denote the mth infinitesimal thickening of the fiber $X_1 := X_e$. We have exact sequences

$$H^1\big(X_1, \mathcal{O}_{X_1}\big) \to H^1\big(X_{m+1}, \mathcal{O}^*_{X_{m+1}}\big) \to H^1\big(X_m, \mathcal{O}^*_{X_m}\big) \to H^2\big(X_1, \mathcal{O}_{X_1}\big).$$

Since $X_1 \cong E \setminus \{e\}$ is affine, this shows that

$$\mathrm{Pic}(X_m \setminus \{e\}) \cong \mathrm{Pic}(E \setminus \{e\}) \cong \mathbf{Pic}^{\circ}(E).$$

Thus $L|_{X_m}$ is trivial for every m.

CHAPTER 3

Families of stable varieties

We have defined stable and locally stable families over 1-dimensional regular schemes in Sections 2.1 and 2.4. The first task in this Chapter is to define these notions for families over more general base schemes. It turns out that this is much easier if we assume that the base scheme is reduced and there is no boundary divisor Δ. Since this case is of considerable interest, we treat it here before delving into the general setting in the next Chapter. While restricting to the special case saves quite a lot of foundational work, the key parts of the proofs of the main theorems stay the same. To avoid repetition, we outline the proofs here but leave the detailed discussions to Chapter 4.

In Section 3.1 we review the theory of Chow varieties and Hilbert schemes. In general these suggest different answers to what a "family of varieties" or a "family of divisors" should be. The main conclusions, (3.11) and (3.13), can be summarized in the following principles.

- A *family of S_2 varieties* should be a **flat** morphism $f : X \to S$ whose geometric fibers are reduced, connected and satisfy Serre's condition S_2.
- Flatness is **not** the right condition for the canonical divisors of the fibers.

Note that both stability and local stability should be preserved by pull-back. Together with the earlier definitions for 1-parameter families given in (2.2) and (2.43), we get necessary conditions for a family to be stable or locally stable. The next definition declares these conditions to be also sufficient.

Temporary Definition 3.1. Let S be reduced scheme and $f : X \to S$ a flat morphism $f : X \to S$ whose geometric fibers are reduced and S_2.

Then $f : X \to S$ is called *stable* (resp. *locally stable*) iff the family obtained by base change $f_T : X_T \to T$ is stable (resp. locally stable) whenever T is the spectrum of a DVR and $T \to S$ a morphism.

As we mentioned above, these conditions are clearly necessary, but it seems quite surprising that this definition works, and we see in Section 4.1 that for locally stable families of *pairs* one needs to make further assumptions about the boundary divisor. We establish the equivalence of (3.1) with the more traditional definitions in (3.68).

Let now $f : X \to S$ be a projective family of S_2 varieties. It turns out that, starting in relative dimension 3, the set of points

$$\{s \in S : X_s \text{ is semi-log-canonical}\}$$

is not even locally closed; see (3.72) for an example. In order to describe the situation, in Section 3.2 we study functors that are representable by a locally closed decomposition.

We start the study of families of non-Cartier divisors in Section 3.3. As we noted above, this is one of the key new technical issues of the theory.

In Section 3.4 we use a representablility theorem (3.67) to clarify the definition of stable and locally stable families, the main result (3.68) gives 5 equivalent definitions of local stability. In Section 3.5 we bring these results together to prove the first main theorem of the chapter.

THEOREM 3.2 (Local stability is representable). *Let S be a reduced scheme over a field of characteristic 0 and $f: X \to S$ a projective family of S_2 varieties. Then there is a locally closed partial decomposition (3.48) $j: S^{\mathrm{ls}} \to S$ such that the following holds.*

Let W be any reduced scheme and $q: W \to S$ a morphism. Then the family obtained by base change $f_W: X_W \to W$ is locally stable iff q factors as $q: W \to S^{\mathrm{ls}} \to S$.

Stability is an open condition for a locally stable morphism, thus (3.2) implies that stability is also representable, see (3.74).

Next we turn to the moduli functor $\mathcal{SV}^{\mathrm{red}}$ that associates to a reduced scheme S the set of all stable families $f: X \to S$, up-to isomorphism. (Here SV stands for stable varieties and the superscript $^{\mathrm{red}}$ indicates that we work with reduced schemes.) In order to get a moduli space of finite type, we fix the relative dimension n and the volume $v = \mathrm{vol}(K_{X_s}) := \big(K_{X_s}^n\big)$ of the fibers. This gives the subfunctor

$$\mathcal{SV}^{\mathrm{red}}(n, v) : \{\text{reduced } S\text{-schemes}\} \to \{\text{sets}\}.$$

We can now state the second main theorem of this Chapter.

THEOREM 3.3 (Existence of reduced moduli spaces). *Let S be a base scheme of characteristic 0 and fix n, v. Then the functor $\mathcal{SV}^{\mathrm{red}}(n, v)$ has a coarse moduli space*

$$\mathrm{SV}^{\mathrm{red}}(n, v) \to S,$$

which is a reduced and separated algebraic space that satisfies the valuative criterion of properness.

COMPLEMENT 3.4. *We see later that $\mathrm{SV}^{\mathrm{red}}(n, v)$ is proper and it is the reduced subscheme of the "true" moduli space $\mathrm{SV}(n, v)$ of stable varieties.*

3.1. Chow varieties and Hilbert schemes

What is a good family of algebraic varieties? Historically 2 answers emerged to this question. The first one originates with Cayley [**Cay62**], with a detailed presentation given in [**HP47**, Chap.X]. The corresponding moduli space is usually called the Chow variety. The second one is due to Grothendieck [**Gro62a**]; it is the theory of Hilbert schemes. For both of them see [**Kol96**, Chap.I], [**Ser06**] or the original sources for details.

For the purposes of the following general discussion, a variety is a proper, geometrically reduced and pure dimensional k-scheme.

The theory of Chow varieties suggests the following.

DEFINITION 3.5 (Cayley-Chow variant). A *Cayley-Chow family* of varieties is a proper, pure dimensional (3.34) morphism $f: X \to S$ whose fibers X_s are generically reduced and $\mathrm{red}(X_s)$ is geometrically reduced for every $s \in S$. (This is called an algebraic family of varieties in [**Har77**, p.263].) More general Cayley-Chow families are defined in (3.19).

It seems hard to make a precise statement but one can think of Cayley-Chow families as being "topologically flat." That is, any topological consequence of flatness also holds for Cayley-Chow families. This holds for the Zariski topology but also for the Euclidean topology if we are over $\mathbb{C}$.

There are 2 disadvantages of Cayley-Chow families. First, basic numerical invariants, for example the arithmetic genus of curves can jump in a Cayley-Chow family. Second, the topological nature of the definition implies that we completely ignore the nilpotent structure of S. In fact, it really does not seem possible to define what a Cayley-Chow family should be over an Artinian base scheme S.

The theory of Hilbert schemes was introduced to solve these problems. It suggest the following definition.

DEFINITION 3.6 (Hilbert-Grothendieck variant). A *Hilbert-Grothendieck family* of varieties is a proper, flat morphisms $f : X \to S$ whose fibers X_s are geometrically reduced and pure dimensional.

Note that every Hilbert-Grothendieck family is also a Cayley-Chow family and technically it is much better to have a Hilbert-Grothendieck family than a Cayley-Chow family. However, there are many Cayley-Chow families that are not flat.

3.7 (Universal families). Both Cayley-Chow and Hilbert-Grothendieck families are preserved by pull-backs thus they form a functor. In both cases this functor has a fine moduli space if we work with families that are subvarieties of a given scheme Y/S.

Let us thus fix a scheme Y that is projective over a base scheme S. For general existence questions the key case is $Y = \mathbb{P}^N_S$. For any closed subscheme $Y \subset \mathbb{P}^N_S$, the Chow variety (resp. the Hilbert scheme) of Y is naturally a subvariety (resp. subscheme) of the Chow variety (resp. the Hilbert scheme) of $\mathbb{P}^N_S$ and the corresponding universal family is obtained by restriction. (See (3.23) or [**Kol96**, Secs.I.5] for some cases when Y/S is not projective.)

Chow variety 3.7.1. (See (3.15–3.23) or [**Kol96**, Sec.I.3] for details and (3.14) for comments on seminormality.) There is a seminormal S-scheme $\operatorname{Chow}^\circ(Y/S)$ and a universal family

$$\operatorname{Univ}^\circ(Y/S) \to \operatorname{Chow}^\circ(Y/S) \tag{3.7.1.1}$$

that represents the functor $\mathcal{C}how^\circ(Y/S)$ of Cayley-Chow subfamilies of Y over seminormal S-schemes. That is, given a seminormal S-scheme $q : T \to S$,

$$\mathcal{C}how^\circ(Y/S)(T) := \left\{ \begin{array}{c} \text{closed subvarieties } X \subset Y \times_S T \text{ such that} \\ X \to T \text{ is a Cayley-Chow family of varieties} \end{array} \right\}. \tag{3.7.1.2}$$

($\operatorname{Chow}^\circ(Y/S)$ is the "open" part of the full $\operatorname{Chow}(Y/S)$, to be defined in (3.21).) If we also fix a relatively very ample line bundle $\mathcal{O}_Y(1)$ then we can write

$$\operatorname{Chow}^\circ(Y/S) = \amalg_n \operatorname{Chow}^\circ_n(Y/S) = \amalg_{n,d} \operatorname{Chow}^\circ_{n,d}(Y/S), \tag{3.7.1.3}$$

where $\operatorname{Chow}^\circ_n$ parametrizes varieties of dimension n and $\operatorname{Chow}^\circ_{n,d}$ parametrizes varieties of dimension n and of degree d. Each $\operatorname{Chow}^\circ_{n,d}(Y/S)$ is of finite type but usually still reducible.

Hilbert scheme 3.7.2. (See [**Kol96**, Sec.I.1] or [**Ser06**] for details.) There is an S-scheme $\operatorname{Hilb}^\circ(Y/S)$ and a universal family

$$\operatorname{Univ}^\circ(Y/S) \to \operatorname{Hilb}^\circ(Y/S) \tag{3.7.2.1}$$

that represents the functor of Hilbert-Grothendieck families

$$\mathcal{H}\!ilb^\circ(Y/S)(T) := \left\{ \begin{array}{c} \text{closed subschemes } X \subset Y \times_S T \text{ such that} \\ X \to T \text{ is a flat family of varieties} \end{array} \right\}. \tag{3.7.2.2}$$

More generally, there is an S-scheme $\operatorname{Hilb}(Y/S)$ and a universal family

$$\operatorname{Univ}(Y/S) \to \operatorname{Hilb}(Y/S) \tag{3.7.2.3}$$

that represents the functor

$$\mathcal{H}\!ilb(Y/S)(T) := \left\{ \begin{array}{c} \text{closed subschemes } X \subset Y \times_S T \\ \text{such that } X \to T \text{ is flat} \end{array} \right\}. \tag{3.7.2.4}$$

We can write

$$\operatorname{Hilb}(Y/S) = \amalg_n \operatorname{Hilb}_n(Y/S) = \amalg_H \operatorname{Hilb}_H(Y/S), \tag{3.7.2.5}$$

where where Hilb_n parametrizes subschemes of (not necessarily pure) dimension n and Hilb_H parametrizes subschemes with Hilbert polynomial $H(t)$. Each $\operatorname{Hilb}_H(Y/S)$ is projective but usually still reducible.

3.8 (Comparing Chow and Hilb). Given a subscheme $X \subset Y$ of dimension $\leq n$, we get an n dimensional cycle $[X] = \sum_i m_i[X_i]$ where X_i are the n-dimensional associated primes and m_i is the length of $\mathcal{O}_X$ at the generic point of X_i. (Thus we completely ignore the lower dimensional associated primes.)

If $m_i = 1$ for every i then $[X] = \sum_i [X_i]$ can be identified with a point in $\operatorname{Chow}^\circ(Y/S)$. In order to make this map everywhere defined, we need to extend the notion of Cayley-Chow families to allow fibers that are formal linear combinations of varieties; see (3.15–3.21) for details. The end result is an everywhere defined map $\operatorname{Hilb}_n(Y/S) \dashrightarrow \operatorname{Chow}_n(Y/S)$. Since $\operatorname{Hilb}_n(Y/S)$ is a scheme but $\operatorname{Chow}_n(Y/S)$ is a seminormal variety, it is better to think of it as a morphism defined on the seminormalization

$$\Re^H_C : \operatorname{Hilb}_n(Y/S)^{\rm sn} \to \operatorname{Chow}_n(Y/S). \tag{3.8.1}$$

This is a very complicated morphism. As written, its fibers have infinitely many irreducible components for $n \geq 1$ since we can just add disjoint 0-dimensional subschemes to any variety $X \subset Y$ to get new subschemes with the same underlying variety. Even if we restrict to pure dimensional subschemes we get fibers with infinitely many irreducible components. This happens for instance for the fiber over $m[L] \in \operatorname{Chow}_{1,m}(\mathbb{P}^3)$ where $L \subset \mathbb{P}^3$ is a line and $m \geq 2$.

It is much more interesting to understand what happens on

$$\overline{\operatorname{Hilb}}{}^\circ_n(Y/S) := \text{ closure of } \operatorname{Hilb}^\circ_n(Y/S) \text{ in } \operatorname{Hilb}_n(Y/S). \tag{3.8.2}$$

That is, $\overline{\operatorname{Hilb}}{}^\circ_n(Y/S)$ parametrizes n-dimensional subschemes that occur as limits of varieties. It turns out that the restriction of the Hilbert-to-Chow map

$$\Re^H_C : \overline{\operatorname{Hilb}}{}^\circ_n(Y/S)^{\rm sn} \to \operatorname{Chow}_n(Y/S) \tag{3.8.3}$$

is a local isomorphism at many points. For smooth varieties this is quite clear from the definition of Chow-forms. Classical writers seem to have been fully aware of various equivalent versions, but I did not find an explicit formulation. The normal case, due to [**Hir58**], is more subtle and in fact quite surprising; see [**Har77**, III.9.11] for its usual form and (10.69) for a stronger version. These imply the following comparison of Hilbert schemes and Chow varieties.

THEOREM 3.9. *Using the notation of (3.8) let $s \in S$ be a point and $X_s \subset Y_s$ a normal, projective subvariety of dimension n. Then the Hilbert-to-Chow morphism*

$$\Re_C^H : \overline{\mathrm{Hilb}}_n^\circ(Y/S)^{\mathrm{sn}} \to \mathrm{Chow}_n(Y/S)$$

is a local isomorphism over $[X_s] \in \mathrm{Chow}_n(Y/S)$. □

Informally speaking, for normal varieties the Cayley-Chow theory is equivalent to the Hilbert-Grothendieck theory, at least over seminormal base schemes.

By contrast, $\mathrm{Hilb}(Y/S)$ and $\mathrm{Chow}(Y/S)$ are different near the class of a singular curve. For example, let $C \subset \mathbb{P}^3$ be a planar, nodal cubic. Then $[C] \in \mathrm{Chow}_1(\mathbb{P}^3)$ is contained in 2 different irreducible components of $\mathrm{Chow}_1(\mathbb{P}^3)$ but only in 1 irreducible component of $\mathrm{Hilb}_1(\mathbb{P}^3)$. A general member of one component is a planar, smooth cubic. This component parametrizes flat deformations. A general member of the other component is a smooth, rational, non-planar cubic. The arithmetic genus jumps, so these deformations are not flat. Thus we see that $\Re_C^H$ is not a local isomorphism over $[C] \in \mathrm{Chow}_1(\mathbb{P}^3)$, but this is explained by the change of the genus. It turns out that once we correct for the genus change, (3.9) becomes stronger.

DEFINITION 3.10. Let $X \subset \mathbb{P}^N$ be a closed subscheme of pure dimension n. Let $X \cap L$ denote the intersection of X with $n-1$ general hyperplanes. Then

$$1 - \chi\bigl(X \cap L, \mathcal{O}_{X \cap L}\bigr)$$

is independent of L. It is called the *sectional genus* of X. (The sectional genus is a linear combination of the 2 highest coefficients of the Hilbert polynomial of X. Knowing the degree of X and its sectional genus is equivalent to knowing the 2 highest coefficients of its Hilbert polynomial.)

It is easy to see that the sectional genus is a constructible and upper semicontinuous function on $\mathrm{Chow}_n^\circ(Y/S)$. (A more general assertion is proved in Section 5.4.) Thus there are locally closed subschemes $\mathrm{Chow}_{n,*,g}^\circ(Y/S) \subset \mathrm{Chow}_n^\circ(Y/S)$ that parametrize geometrically reduced cycles with sectional genus g; see (3.48). (The $*$ stands for the degree which we ignore in these formulas. Also, one can not define the sectional genus for cycles with multiplicities (3.15) though this can easily be corrected.) We can now define the Chow variety parametrizing families with locally constant sectional genus as

$$\mathrm{Chow}_n^{\mathrm{sg}}(Y/S) := \amalg_{n,g} \mathrm{Chow}_{n,*,g}^\circ(Y/S)^{\mathrm{sn}},$$

the disjoint union of the seminormalizations of the $\mathrm{Chow}_{n,*,g}^\circ(Y/S)$.

The sectional genus is constant in a flat family, and we get the following strengthening of (3.9).

THEOREM 3.11. *Using the notation of (3.8) let $s \in S$ be a point and $X_s \subset Y_s$ a geometrically reduced, pure dimensional, projective, S_2 subvariety of dimension n. Then the Hilbert-to-Chow map*

$$\Re_C^H : \overline{\mathrm{Hilb}}_n^\circ(Y/S)^{\mathrm{sn}} \to \mathrm{Chow}_n^{\mathrm{sg}}(Y/S)$$

is a local isomorphism over $[X_s] \in \mathrm{Chow}_n^{\mathrm{sg}}(Y/S)$.

We can informally summarize these considerations as follows.

PRINCIPLE 3.12. *For reduced, pure dimensional, projective, S_2 varieties the Cayley-Chow theory is equivalent to the Hilbert-Grothendieck theory over seminormal base schemes, once we correct for the sectional genus.*

We are studying not just varieties but semi-log-canonical pairs (X, Δ). The underlying variety is deminormal, hence geometrically reduced and S_2. Thus (3.12) says that even if we start with the more general Cayley-Chow families, we end up with flat morphisms $f : X \to S$ with S_2 fibers. The latter is a class that is well behaved over arbitrary base schemes.

However, the divisorial part is harder to understand. Although we have seen only a few examples supporting it, the following counterpart of (3.12) turns out to give the right picture.

PRINCIPLE 3.13. *For stable families of semi-log-canonical pairs (X, Δ) the Hilbert-Grothendieck theory is optimal for the underlying variety X but the Cayley-Chow theory is the "right" one for the divisorial part $K_X + \Delta$.*

Summary of the theory of Chow varieties.

We recall the basic properties of Chow varieties; see [**HP47**, Chap.X] or [**Kol96**, Secs.I.3–4] for details.

3.14 (Comment on seminormality). Hilbert schemes work well over any base scheme, but in [**Kol96**] the theory of Cayley–Chow families is developed only over seminormal bases. In characteristic 0 it might be possible to work over reduced base schemes (see [**Bar75**] for key special cases) but examples of Nagata [**Nag55**] suggest that in positive characteristic the restriction to seminormal bases may be necessary. Thus, in what follows, we outline the theory of Chow varieties over seminormal bases only.

DEFINITION 3.15. Let X be a proper scheme over a field k. A *d-cycle* is a formal, finite linear combination $Z := \sum_i m_i[V_i]$ where $m_i \in \mathbb{Z}$ and the V_i are d-dimensional irreducible, reduced subschemes. We usually tacitly assume that the V_i are distinct and $m_i \neq 0$. Then the V_i are called the *irreducible components* of Z and the m_i the *multiplicities.* Z is called *reduced* if all its multiplicities equal 1. A d-cycle is called *effective* if $m_i \geq 0$ for every i.

If L is an ample line bundle on X then the *degree* of a d-cycle is defined as $\deg_L Z := \sum_i m_i \deg_L V_i = \sum_i m_i(L^d \cdot V_i)$.

3.16 (Chow forms). Fix a projective space $\mathbb{P}^n$ with dual projective space $\check{\mathbb{P}}^n$. That is, points in $\check{\mathbb{P}}^n$ are hyperplanes in $\mathbb{P}^n$.

Let $X \subset \mathbb{P}^n$ be an irreducible, reduced, closed subvariety of dimension d. Then

$$Ch(X) := \{(H_0, \dots, H_d) \in (\check{\mathbb{P}}^n)^{d+1} : X \cap H_0 \cap \dots \cap H_d \neq \emptyset\} \tag{3.16.1}$$

is a hypersurface in $(\check{\mathbb{P}}^n)^{d+1}$ of multidegree $(\deg X, \dots, \deg X)$. The equation of $Ch(X)$ is called the *Chow form* of X; we denote it by $ch(X)$. It is a multihomogeneous polynomial of multidegree $(\deg X, \dots, \deg X)$. It is not hard to check that $ch(X)$ determines X uniquely. If $Z = \sum_i m_i[X_i]$ is an effective d-cycle then set

$$ch(Z) := \textstyle\prod_i ch(X_i)^{m_i}. \tag{3.16.2}$$

The fundamental observation of the theory is the following.

Claim 3.16.3. If k is a perfect field then $Ch(Z)$ uniquely determines Z.

In general we turn this into a definition and say that 2 cycles Z_1, Z_2 are *essentially the same* iff $Ch(Z_1) = Ch(Z_2)$; see [**Kol96**, I.4.1] for some examples.

The method of Chow varieties uses the Chow form $Ch(Z)$ to describe the cycle Z. This is satisfactory if $\operatorname{char} k = 0$, but it runs into difficulties if $\operatorname{char} k > 0$.

DEFINITION 3.17 (Chow field). Let k be a field and $Z \subset \mathbb{P}^n$ a d-cycle defined over a field extension K/k.

Using the coordinates $\{x_{ij} : 0 \le i \le n, 0 \le j \le d\}$ on the factors of $(\check{\mathbb{P}}^n)^{d+1}$ the Chow form is a multihomogeneous polynomial

$$ch(Z) := \textstyle\sum_{\mathbf{r}} c(\mathbf{r}) \prod_{ij} x_{ij}^{r_{ij}} \tag{3.17.1}$$

where $\mathbf{r} = (r_{ij})$ is a $(d+1) \times (n+1)$ matrix whose rows sum to $\deg Z$. Note that the $c(\mathbf{r})$ are only determined up to a constant factor, so the important numbers are their quotients. Together they generate the *Chow field* of Z

$$k^{\mathrm{ch}}(Z) := k\Bigl(\tfrac{c(\mathbf{r}_1)}{c(\mathbf{r}_2)} : c(\mathbf{r}_2) \ne 0\Bigr). \tag{3.17.2}$$

Thus $k^{\mathrm{ch}}(Z)$ is the smallest field over which the hypersurface $Ch(Z)$ can be defined.

3.18 (Field of definition and problems in positive characteristic). There are 2 special features of the behavior of cycles under field extensions K/k that cause many problems in positive characteristic. While we are mainly interested in characteristic 0, they effect the formulation of several of the general theorems.

First, let V be an irreducible, reduced subscheme. If K/k is separable then V_K is reduced. Thus if $\operatorname{char} k = 0$ then $Z_{\bar{k}}$ has the same multiplicities as Z but if $\operatorname{char} k = p > 0$ then the multiplicities of $Z_{\bar{k}}$ can be a p-power times the multiplicities of Z.

Second, let Z_1, Z_2 be d-cycles and assume that $(Z_1)_K = (Z_2)_K$. If K/k is separable then $Z_1 = Z_2$ but not in general. In fact, Z_1, Z_2 are essentially the same iff $(Z_1)_K = (Z_2)_K$ for some purely inseparable field extension K/k.

More generally, let X be a k-variety and $\bar{Z}$ a cycle defined over the algebraic closure $\bar{k}$. We say that $\bar{Z}$ can be *defined over* a subfield $k \subset k' \subset \bar{k}$ (or that k' is a *field of definition* of SZ) if there is a cycle Z' on $X_{k'}$ such that $(Z')_{\bar{k}} = \bar{Z}$.

It turns out that intersection of all fields of definition is the Chow field of $\bar{Z}$; see [**Kol96**, I.4.5]. The ideal situation is when the Chow field is also a field of definition. The following is proved in [**Kol96**, I.3.5].

Claim 3.18.1. Let X be a k-variety and $\bar{Z}$ a cycle defined over $\bar{k}$. Then $\bar{Z}$ can be defined over $k^{\mathrm{ch}}(\bar{Z})$ in any of the following cases.

(a) $\operatorname{char} k = 0$,
(b) $\operatorname{char} k > 0$ is relatively prime to the geometric multiplicities of $\bar{Z}$ or
(c) $\bar{Z}$ is a divisor and X is smooth at all generic points of $\bar{Z}$. □

The problems of Chow varieties with multiplicities are already apparent for 0-cycles. Consider $\mathbb{P}^n_{\mathbf{x}}$ and use $z_0, \dots, z_n$ as coordinates on $\check{\mathbb{P}}^n$. If $\mathbf{a} = (a_0{:}\cdots{:}a_n)$ is a single point then (3.16.1) gives that

$$ch(\mathbf{a}) = a_0 z_0 + a_1 z_1 + \cdots + a_n z_n,$$

and if $A = \sum_j m_j \mathbf{a}_j = \sum_j m_j (a_{0j}{:}\cdots{:}a_{nj})$ then

$$ch(A) = \textstyle\prod_j (a_{0j} z_0 + \cdots + a_{nj} z_n)^{m_j}. \tag{3.18.2}$$

Multiplicity p problem 3.18.3. Let k be a field of characteristic $p > 0$ and set $\mathbf{a} := (a_0^{1/p}:\cdots:a_n^{1/p})$. Then

$$Ch\big(p\mathbf{a}\big) = a_0 z_0^p + a_1 z_1^p + \cdots + a_n z_n^p$$

is defined over k but if the a_i are a p-basis of k/k^p (and $n \geq 2$) then there is no k-subscheme $W \subset \mathbb{P}^n$ such that $[W] = p\mathbf{a}$. Thus $p\mathbf{a}$ is a k-point on $\mathrm{Chow}(\mathbb{P}^n)$ but there is no 0-cycle defined over k that corresponds to it.

In order to state the general case, we need some notation. Let R be a ring, $I \subset R$ an ideal and $m \in \mathbb{N}$. Consider the ideal $I^{[m]} := (r^m : r \in I)$. If R is a k-algebra and $\operatorname{char} k = 0$ then $I^{[m]} = I^m$, if $\operatorname{char} k = p > 0$ and $m = p^c$ then $I^{[p^c]}$ is called the *Frobenius power* of I. The other cases are also interesting, but they do not seem to have a standard name.

Going back to Chow varieties, let $Z \subset \mathbb{P}^n$ be a geometrically integral subscheme with Cayley–Chow hypersurface $Ch(Z)$. Let $Z' \subset \mathbb{P}^n$ be any pure subscheme (9.2) such that $[Z'] = mZ$. Then we obtain that $Ch(Z') = m \cdot Ch(Z)$. Thus Cayley–Chow theory does not distinguish such subschemes Z' from each other and the only natural subscheme that one can associate to $m \cdot Ch(Z)$ is

$$\operatorname{Spec}_{\mathbb{P}^n}\big(\operatorname{pure}(\mathcal{O}_{\mathbb{P}^n}/I_Z^{[m]})\big) \subset \mathbb{P}^n$$

where pure( ) denotes the maximal pure quotient (9.2). Assume next that we have $X \subset \mathbb{P}^n$ and $Z \subset X$ is a divisor such that X is smooth at the generic point of Z. In this very special case,

$$mZ := \operatorname{pure}\Big(X \cap \operatorname{Spec}_{\mathbb{P}^n}\big(\operatorname{pure}(\mathcal{O}_{\mathbb{P}^n}/I_Z^{[m]})\big)\Big) \subset X \tag{3.18.4}$$

is the unique pure subscheme $Z' \subset X$ such that $Ch(Z') = m \cdot Ch(Z)$. In particular, under these conditions, mZ is defined over the Chow field $k^{\mathrm{ch}}(mZ)$. This explains (3.18.1.c).

DEFINITION 3.19 (Cayley-Chow families). Let X be a proper scheme over S and $V \subset X$ an irreducible, reduced subscheme. We would like to understand when can we view

$$g := f|_V : V \to S$$

as a “good” family of d-cycles. An obvious necessary condition is that the fibers V_s have should have pure dimension d, thus we get d-cycles $[V_s]$.

A “good” family should commute with base change. With this in mind, let T be the spectrum of a DVR and $h : T \to S$ a morphism that maps the generic point of T to a generic point of S and the closed point $0 \in T$ to $s \in S$. By pull-back we get $V_T \subset X_T$. The subscheme V_T can have embedded points along the special fiber. We correct this problem by passing to $\operatorname{pure}(V_T)$ (9.2). Now we have $\operatorname{pure}(V_T) \subset X_T$ and $\operatorname{pure}(V_T)$ is flat over T. Thus the fiber over $0 \in T$ gives the “correct” cycle over $s \in S$ by (3.19.2). We denote it by

$$\lim_{h \to s}(V/S) := \big[\big(\operatorname{pure}(V_T)\big)_0\big]. \tag{3.19.1}$$

Keep in mind that $\lim_{h \to s}(V/S)$ is not a cycle on X_s but on $X_s \times \operatorname{Spec} K$ where K is the residue field of T and K is almost always a non-algebraic extension of $k(s)$.

We say that the family of cycles $g : V \to S$ satisfies the *field of definition condition* if for every $s \in S$ there is a d-cycle on X_s—denoted by $g^{[-1]}(s)$ and called the *Cayley-Chow fiber*—such that $\lim_{h \to s}(V/S) = g^{[-1]}(s)_K$ for every $T \to S$ as above. If $g^{[-1]}(s)$ is defined only over the purely inseparable closure of $k(s)$, then we say that V is a *well defined family of cycles.* The two notions agree in characteristic 0.

If V is flat over S then it is also satisfies the field of definition condition by [**Kol96**, I.3.15], but the proof actually shows the following stronger variant.

Claim 3.19.2. Let S be a reduced scheme and $f : V \to S$ a proper morphism. Assume that f is flat at all generic points of V_s for every $s \in S$. Then $f : V \to S$ satisfies the field of definition condition and $g^{[-1]}(s) = [V_s]$. $\square$

If V is not generically flat over S then the $[V_s]$ do not form a "good" family, see (3.22) for a rather typical example.

A key observation is that, over normal base schemes, the obvious dimension restrictions are enough to get a well defined family of cycles. The following is proved in [**Kol96**, I.3.17].

THEOREM 3.20. *Let S be a normal scheme, $f : X \to S$ a proper morphism and $V \subset X$ a closed subscheme such that $g := f|_V : V \to S$ has pure relative dimension d. Then V is a well defined family of d-cycles.* $\square$

DEFINITION 3.21 (Chow varieties). Let X be a proper scheme over a base scheme S and $g : Z \to S$ a family of cycles that satisfies the field of definition condition.

Let S' be a seminormal scheme and $q : S' \to S$ a morphism. There is a unique cycle, denoted by $q^{[*]}Z$, whose support is $S' \times_S \operatorname{Supp} Z$ and whose fiber over a point $s' \in S'$ is $Z_{k(s')}$; see [**Kol96**, I.3.18] for details. This cycle also satisfies the field of definition condition and it is called the *Cayley-Chow pull-back* of Z.

If S is over a field of characteristic 0 then the field of definition condition holds for every well defined family of cycles. Thus the Cayley-Chow pull-back is always defined and we get a functor on seminormal schemes. If X/S is projective then the universal family over the *Chow variety*

$$\operatorname{Univ}_{\mathrm{chow}}(X/S) \to \operatorname{Chow}(X/S) \tag{3.21.1}$$

represents this functor; see [**Kol96**, Sec.I.3] for details.

In general, given a point $P \in \operatorname{Chow}(X/S)$, its residue field $k(P)$ is the Chow field of the corresponding cycle. Thus the field of definition condition holds only if the cycle is defined over its Chow field. Such cases are listed in (3.18.1). (If these are not satisfied, then there are at least 3 sensible variants of the Chow functor; see [**Kol96**, Sec.I.4] for details.)

The Chow variety is not really a variety; it has infinitely many irreducible components. In order to get something of finite type, fix a relatively ample line bundle L on X. The degree of a cycle is locally constant function on $\operatorname{Chow}(X/S)$ by [**Kol96**, I.3.12], thus we can write

$$\operatorname{Chow}(X/S) = \amalg_{n,d} \operatorname{Chow}_{n,d}(X/S), \tag{3.21.2}$$

where $\operatorname{Chow}_{n,d}(X/S)$ parametrizes cycles of dimension n and degree d. The schemes $\operatorname{Chow}_{n,d}(X/S)$ are projective over S, though usually disconnected.

If $X \to S$ has pure relative dimension m then we are especially interested in

$$\operatorname{WDiv}(X/S) := \operatorname{Chow}_{m-1,*}(X/S), \tag{3.21.3}$$

which parametrizes Weil divisors on the fibers of $X \to S$.

EXAMPLE 3.22. Set $X = (xy = uv) \subset \mathbb{A}^4$ and $D = (x = u = 0) \cup (y = v = 0)$. Let $f : X \to \mathbb{A}^2$ be the map $(x, y, u, v) \mapsto (x + y, u + v)$. Note that f is flat and the central fiber is a pair of intersecting lines $(x^2 = u^2) \subset \mathbb{A}^2$.

Away from the origin, the fibers of $f|_D$ consist of 2 points. In order to compute the scheme theoretic fiber of $f|_D$ over the origin, note that the ideal sheaf of D is (xy, xv, uy, uv). Thus the scheme theoretic fiber is given by

$$k[x, y, u, v]/(xy, xv, uy, uv, x + y, u + v).$$

This is easily seen to have length 3, with $1, x, u$ serving as a basis.

3.23 (Non-projective cases). Let Y be an arbitrary scheme over S. We define $\mathcal{H}ilb(Y/S)(T)$ as the set if all subschemes $X \subset Y \times_S T$ that are proper and flat over T. [**Art69**] proves that if $Y \to S$ is locally of finite presentation then the Hilbert functor is represented by a morphism $\mathrm{Hilb}(Y/S) \to S$ that is also locally of finite presentation. However, in general $\mathrm{Hilb}(Y/S)$ is not a scheme but an algebraic space over S. More generally, $Y \to S$ is allowed to be an algebraic space.

Most likely similar results hold for $\mathrm{Chow}(Y/S)$ but I am not aware of complete references. See [**Kol96**, Sec.I.5] for further discussions.

Summary of seminormality and weak normality.

(For more details see [**Kol96**, Sec.I.7.2] and [**Kol13c**, Sec.10.2].)

Normalization is a very useful operation that can be used to improve a scheme X. However, the normalization $X^n \to X$ usually creates new points and this makes it harder to relate X and X^n. The notion of seminormalization intends to do as much of the normalization as possible, without creating new points.

For example, the normalization of the higher cusps $C_{2m+1} := (x^2 = y^{2m+1})$ is

$$\pi_{2m+1} : \mathbb{A}^1_t \to C_{2m+1} \quad \text{given by} \quad t \mapsto (t^{2m+1}, t^2).$$

The map π_{2m+1} is a homeomorphism, so it is also the seminormalization. By contrast, the normalization of the higher tacnode $C_{2m} := (x^2 = y^{2m})$ is

$$\pi_{2m} : \mathbb{A}^1_t \times \{\pm 1\} \to C_{2m} \quad \text{given by} \quad t \mapsto (\pm t^m, t).$$

The map π_{2m} is not a homeomorphism since $(0, 0) \in C_{2m}$ has 2 preimages, $(0, 1)$ and $(0, -1)$. The seminormalization of C_{2m} is

$$\tau_{2m} : C_2 \cong (s^2 = t^2) \to C_{2m} \quad \text{given by} \quad (s, t) \mapsto (s^m, t).$$

These examples lead to a general definition of seminormalization and seminormal schemes, but they do not adequately show how complicated seminormal schemes are in higher dimensions.

DEFINITION 3.24. A finite morphism of schemes $g : X' \to X$ is called a *partial seminormalization* if X' is reduced and for every point $x \in X$, the induced map $g^* : k(x) \to k\big(\operatorname{red} g^{-1}(x)\big)$ is an isomorphism. (For finite type schemes over a field of characteristic 0, it is enough to assume this for closed points only.)

A partial seminormalization is birational and it is dominated by the normalization $n : X^{\mathrm{n}} \to X$ of $\operatorname{red} X$. If $\tau : X^{\mathrm{n}} \to X$ is finite (which holds for excellent schemes) then there is a unique largest partial seminormalization $\pi : X^{\mathrm{sn}} \to X$, called the *seminormalization* of X.

To be more explicit, $\pi_* \mathcal{O}_{X^{\mathrm{sn}}}$ is the subsheaf of $\tau_* \mathcal{O}_{X^{\mathrm{n}}}$ consisting of those sections ϕ such that for every point $x \in X$ with preimage $\bar{x} := \operatorname{red} \tau^{-1}(x)$ we have

$$\phi|_{\bar{x}} \in \operatorname{im}\big[\tau^* : k(x) \to k(\bar{x})\big]. \tag{3.24.1}$$

A scheme X is called *seminormal* if its seminormalization $\pi : X^{\mathrm{sn}} \to X$ is an isomorphism. A seminormal scheme is reduced.

It is easy to see that an open subscheme of a seminormal scheme is also seminormal. Furthermore, being seminormal is a local property. That is, X is seminormal $\Leftrightarrow$ it is covered by seminormal open subschemes $\Leftrightarrow$ every local ring of X is seminormal.

Seminormality is a quite useful notion but it is not always easy to use. A major difficulty is that an irreducible component of a seminormal scheme need not be seminormal. In fact, by [**Kol13c**, 10.12], every reduced and irreducible affine scheme that is smooth in codimension 1 occurs as an irreducible component of a seminormal complete intersection scheme.

However, a major advantage of seminormality over normality is that seminormalization $X \mapsto X^{\mathrm{sn}}$ is a functor from the category of excellent schemes to the category of excellent seminormal schemes. It is thus reasonable to expect that taking the coarse moduli space commutes with seminormalization. This is indeed the case for coarse moduli spaces satisfying the following mild condition.

DEFINITION 3.25. Let $\mathcal{M}$: (schemes) $\to$ (sets) be a functor with coarse moduli space M. We say that $\mathcal{M}$ has *enough 1-parameter families* if the following holds.

Let T be the spectrum of a DVR and $\phi : T \to M$ a morphism. Then there is a spectrum of a DVR T', a finite morphism $\pi : T' \to T$ and $F \in \mathcal{M}(T')$ such that $\phi \circ \pi : T' \to M$ is the moduli map of F.

PROPOSITION 3.26. *Let* $\mathcal{M}$: (*schemes*) $\to$ (*sets*) *be a functor defined on excellent schemes over a field of characteristic 0. Assume that* $\mathcal{M}$ *has a coarse moduli space* M *and enough 1-parameter families.*

Then M^{sn} *is the coarse moduli space for its restriction to the category* $\mathrm{Sch}^{\mathrm{sn}}$ *of excellent seminormal schemes*

$$\mathcal{M}^{\mathrm{sn}} := \mathcal{M}|_{\mathrm{Sch}^{\mathrm{sn}}} : (\textit{seminormal schemes}) \to (\textit{sets}).$$

Proof. Since seminormalization is a functor, every morphism $W \to M$ lifts to $W^{\mathrm{sn}} \to M^{\mathrm{sn}}$. Thus we have a natural transformation $\Phi : \mathcal{M}^{\mathrm{sn}} \to \mathrm{Mor}\big(*, M^{\mathrm{sn}}\big)$.

Assume that M' is a seminormal scheme and we have another natural transformation $\Psi : \mathcal{M}^{\mathrm{sn}} \to \mathrm{Mor}\big(*, M'\big)$. Thus we get a natural transformation to the product $M^{\mathrm{sn}} \times M'$; let $Z \subset M^{\mathrm{sn}} \times M'$ denote the set-theoretic image. Since M^{sn} is a coarse moduli space, the coordinate projection $Z \to M^{\mathrm{sn}}$ is geometrically bijective. Since $\mathcal{M}$ has enough 1-parameter families, $Z \to M^{\mathrm{sn}}$ is a universal homeomorphism by (3.28). Thus $Z \to M^{\mathrm{sn}}$ is an isomorphism since M^{sn} is seminormal and the characteristic is 0.

Thus we get a morphism $M^{\mathrm{sn}} \to M'$ and Ψ factors through Φ. $\square$

EXAMPLE 3.27. Let $\mathcal{D}$ be any diagram of schemes with direct limit $\lim \mathcal{D}$. Since seminormalization is a functor, we get a a diagram $\mathcal{D}^{\mathrm{sn}}$ and a natural morphism $\lim(\mathcal{D}^{\mathrm{sn}}) \to (\lim \mathcal{D})^{\mathrm{sn}}$. However, this need not be an isomorphism.

To get such an example, let k be an infinite field and consider the diagram of all maps $\phi_a : \operatorname{Spec} k[x] \to \operatorname{Spec} k[(x-a)^2, (x-a)^3]$ for $a \in k$.

If $\operatorname{char} k = 0$ the direct limit is $\operatorname{Spec} k$. After seminormalization, the maps ϕ_a become isomorphisms $\phi_a^{\mathrm{sn}} : \operatorname{Spec} k[x] \cong \operatorname{Spec} k[x]$ and now the direct limit is $\operatorname{Spec} k[x]$.

If $\operatorname{char} k = p > 0$ then $x^p - a^p = (x-a)^p \in k[(x-a)^2, (x-a)^3]$ shows that the direct limit is $\operatorname{Spec} k[x^p]$. After seminormalization, the direct limit is again $\operatorname{Spec} k[x]$.

LEMMA 3.28. *Let $g : X \to S$ be a morphism of schemes and $Z \subset X$ a subset. Then Z is a subscheme and $g|_Z : Z \to S$ is a universal homeomorphism iff the following hold.*

(1) *Every geometric point $\tau : s \to S$ has a unique lifting $\tau_X : s \to X$ whose image is in Z.*
(2) *Let T be the spectrum of a DVR and $\phi : T \to S$ a morphism. Then there is a spectrum of a DVR T', a finite morphism $\pi : T' \to T$ and a lifting $(\phi \circ \pi)_X : T' \to X$ whose image is in Z.*

Proof. By assumption (1), $g|_Z : Z \to S$ is a universal bijection. Let $s_g \in S$ be a generic point and $z_g \in Z$ its preimage in X. We claim that $\bar{z}_g \subset Z$. For any $z_0 \in \bar{z}_g$ there is a DVR T and a morphism $T \to X$ that maps the generic point to z_g and the closed point to z_0. We apply (2) to $g \circ \tau$ to conclude that $z_0 \in Z$.

Thus Z is the union of all $\bar{z}_g$ hence Zariski closed and $g|_Z : Z \to S$ is a finite, universal bijection, hence a homeomorphism. □

In positive characteristic there is a more restrictive variant, called weak normalization. Most of the important results about Chow varieties hold only over weakly normal base schemes. Unfortunately, weak normalization is *not* a functor, leading to various technical problems; see, for instance [**Kol96**, Sec.I.4].

DEFINITION 3.29. A finite morphism of schemes $g : X' \to X$ is called a *partial weak normalization* if X' is reduced, for every generic point $x \in X$, the induced map $g^* : k(x) \to k\big(\operatorname{red} g^{-1}(x)\big)$ is an isomorphism and for every point $x \in X$, the induced map $g^* : k(x) \to k\big(\operatorname{red} g^{-1}(x)\big)$ is a purely inseparable extension.

If the normalization $\tau : X^{\mathrm{n}} \to X$ is finite then there is a unique largest partial weak normalization $\pi : X^{\mathrm{wn}} \to X$, called the *weak normalization* of X.

A scheme X is called *weakly normal* if its weak normalization $\pi : X^{\mathrm{wn}} \to X$ is an isomorphism.

It is clear that (normal) $\Rightarrow$ (weakly normal) $\Rightarrow$ (seminormal) and if X is a scheme over a field of characteristic 0 then (weakly normal) $\Leftrightarrow$ (seminormal). More generally, the latter also holds if the residue fields of all non-generic points are perfect. In practice this happens in a few additional cases only; namely when X is a 1-dimensional scheme over a perfect field or it is finite over $\operatorname{Spec} \mathbb{Z}$.

The following example illustrates some of the differences between weak and seminormality.

EXAMPLE 3.30. Let $g(t) \in k[t]$ be a polynomial without multiple factors and set $C_g := \operatorname{Spec}_k\big(k + g \cdot k[t]\big)$. Then C_g is an integral curve whose normalization is $\mathbb{A}^1$. We can think of C_g as obtained from $\mathbb{A}^1$ by identifying all roots of g.

If g is separable then C_g is seminormal and weakly normal. If g is irreducible and purely inseparable then C_g is seminormal but not weakly normal; the weak normalization of C_g is $\mathbb{A}^1$.

The problem is that, in the latter case, C_g looks and behaves very much like a (higher) cusp. For example if $\operatorname{char} k = 2$ and $g = t^2 - a$ then $k + g \cdot k[t]$ is generated by $x := t^2 - a$ and $z := t(t^2 - a)$ and they satisfy the equation $z^2 = x^3 + ax^2$. If we adjoin $\alpha = \sqrt{a}$ to k and set $y := z - \alpha x$ then this becomes $y^2 = x^3$.

Both of the properties ascend for flat morphisms in a strong form. For normality this is classical; see for instance [**Mat86**, 23.9]. The other cases are proved in [**GT80, Man80**] for N-1 schemes and the general version is in [**Kol16c**, 37].

THEOREM 3.31. *Let $f : Y \to X$ be a flat morphism of Noetherian schemes with geometrically reduced fibers. Assume that X and the geometric generic fibers are normal (resp. seminormal, weakly normal). Then Y is also normal (resp. seminormal, weakly normal).* $\square$

3.2. Representable properties

Let $f : X \to S$ be a morphism and $\mathcal{P}$ a property of schemes that is invariant under base field extensions. One can then consider the set

$$S(\mathcal{P}) := \{s \in S : X_s \text{ satisfies } \mathcal{P}\}.$$

Note that $S(\mathcal{P})$ depends on $f : X \to S$, so we use the notation $S(\mathcal{P}, X/S)$ if the choice of $f : X \to S$ is not clear.

In nice situations, $S(\mathcal{P})$ is an open or closed or at least locally closed subset of S. For example satisfying Serre's condition S_m is an open condition for proper, flat morphisms by (10.3) and being singular is a closed condition.

Similarly, if $f : X \to S$ is a proper morphism of relative dimension 1 then

$$S(\text{stable}) := \{s \in S : X_s \text{ is a stable curve}\}$$

is an open subset of S. However, we see in (3.72) that if $f : X \to S$ is a proper, flat morphism of relative dimension ≥ 3 then

$$S(\text{stable}) := \{s \in S : X_s \text{ is a stable variety}\}$$

is not even a locally closed subset of S in general.

We already noted in Section 1.3 that flat morphisms with stable fibers do not give the right moduli problem in higher dimensions and one should look at stable families instead. Thus our main interest is not in the set $S(\text{stable})$ but in the class of morphisms $q : T \to S$ for which the pulled-back family $f_T : X_T \to T$ is stable. We then hope to prove that this happens in a predictable way. The following definition formalizes this.

DEFINITION 3.32. Let $\mathcal{P}$ be a property of morphisms that is preserved by pull-back. That is, if $X \to S$ satisfies $\mathcal{P}$ and $q : T \to S$ is a morphism then we get $f_T : X_T \to T$ that also satisfies $\mathcal{P}$. Depending on the situation, pull–back can mean the usual fiber product $X_T := X \times_S T$ or a modified version of it, like the Cayley-Chow pull-back of cycles (3.21), the S_2 pull-back defined in (3.52) or the divisorial pull-back to be defined in (4.25).

We can associate to $\mathcal{P}$ the functor of $\mathcal{P}$-pull-backs defined for morphisms $W \to S$ by setting

$$Property(\mathcal{P})(W) := \begin{cases} 1 & \text{if } X_W \to W \text{ satisfies } \mathcal{P}, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.32.1}$$

Thus a morphism $i_P : S^P \to S$ *represents* $\mathcal{P}$-pull-backs iff the following hold.

(2) $f^P : X^P := X_{S^P} \to S^P$ satisfies $\mathcal{P}$ and
(3) if $f_W : X_W \to W$ satisfies $\mathcal{P}$ then q factors as $q : T \to S^P \to S$, and the factorization is unique.

It is also of interest to understand what happens if we focus on special classes of bases. Let $\mathcal{R}$ be a property of schemes. We say that $i_P : S^P \to S$ *represents* $\mathcal{P}$-pull-backs for $\mathcal{R}$-schemes if S^P satisfies $\mathcal{R}$ and (3) holds whenever W satisfies $\mathcal{R}$. In this section we are mostly interested in the properties $\mathcal{R} =$ (reduced), $\mathcal{R} =$ (seminormal) and $\mathcal{R} =$ (normal).

If (3) holds for all $T =$ (spectrum of a field) then $i_P : S^P \to S$ is geometrically injective (3.47). If (3) holds for all schemes then i_P is a monomorphism (3.47).

In many cases of interest $\mathcal{P}$ is invariant under base field extensions. That is, if K/k is a field extension then $f : X \to \operatorname{Spec} k$ satisfies $\mathcal{P}$ iff $f_K : X_K \to \operatorname{Spec} K$ satisfies $\mathcal{P}$. If this holds then $i_P : S^P \to S$ is also residue field preserving (3.47).

If $X \to S$ is projective then we are frequently able to prove that $i_P : S^P \to S$ is a locally closed partial decomposition (3.48).

If $i_P : S^P \to S$ represents $\mathcal{P}$-pull-backs and i_P is of finite type (this will always be the case for us) then

$$S(\mathcal{P}) = \{s : X_s \text{ satisfies } \mathcal{P}\} = i_P(S^P)$$

is a constructible subset of S. Constructibility is much weaker than representability but we will frequently need constructibility in our proofs of representability.

Let us give 2 basic examples of representable properties.

EXAMPLE 3.33. Let $f : X \to S$ be a proper morphism and $m \in \mathbb{N}$. We claim that both of the following functors are representable by a locally closed decomposition for reduced schemes but not representable for all schemes.

(1) The functor of pull-backs whose fibers have pure dimension m.
(2) The functor of pull-backs of pure relative dimension m (3.34).

A typical example illustrating the difference is the following. Set

$$S := (xy = 0) \subset \mathbb{A}^2 \quad \text{and} \quad X := (x = u = 0) \cap (y = uv + x = 0) \subset \mathbb{A}^4.$$

All fibers of the coordinate projection $\pi : X \to S$ have pure dimension 1. However π does not have pure relative dimension 1 since base change to $(x = 0) \subset S$ results in 2 irreducible components of $(x = 0) \times_S X$, one of dimension 2 and one of dimension 1. The pull-backs of pure relative dimension 1 are represented by the locally closed decomposition

$$(y = 0) \amalg \bigl((x = 0) \setminus \{(0, 0)\}\bigr) \to S.$$

To see the claims let n be the maximum fiber dimension of f. Then

$$X^{(n)} := \{x \in X : \dim_x X_{f(x)} = n\}$$

is a closed subset of X. Then $S^c := f\bigl(X^{(n)}\bigr)$ is a closed subset of S and $S^0 := f\bigl(X \setminus X^{(n)}\bigr)$ is an open subset of S. Thus $S^c \setminus S^0$ is a closed subset parametrizing fibers of pure dimension n and $S^0 \setminus S^c$ is an open subset parametrizing fibers of dimension $< n$.

Let T be a reduced scheme and $q : T \to S$ a morphism such that $f_T : X_T \to T$ has fibers of pure dimension m. If $m = n$ then q factors through the closed immersion $(S^c \setminus S^0) \hookrightarrow S$, otherwise q factors through the open immersion $(S^0 \setminus S^c) \hookrightarrow S$. We can continue with $S^0 \setminus S^c$ to obtain $S^{\mathrm{md}} \to S$ representing pull-backs whose fibers have pure dimension m.

It is also clear that if $f_T : X_T \to T$ has pure relative dimension m then $T \to S$ factors through $S^{\mathrm{md}} \to S$. Thus it is enough to prove (2) in case all fibers of

$f : X \to S$ have pure dimension m and construct $S^{\mathrm{pmd}} \to S$ representing the second functor.

Let $\pi : \bar{S} \to S$ denote the normalization. By base change we get $\bar{f} : \bar{X} := X \times_S \bar{S} \to \bar{S}$. Let $W \subset \bar{X}$ denote the union of those irreducible components of $\bar{X}$ that do not dominate any irreducible component of $\bar{S}$. By (3.34.2) $\bar{f}$ has pure relative dimension m over $\bar{S} \setminus \bar{f}(W)$. Furthermore, if T is the spectrum of a DVR and $h : T \to \bar{S}$ is a morphism whose generic point maps to $\bar{S} \setminus \bar{f}(W)$, then the pull-back to T gives a morphism of pure relative dimension m iff $h(T) \subset \bar{S} \setminus \bar{f}(W)$. Thus

$$S \setminus \pi\bigl(\bar{f}(W)\bigr) \hookrightarrow S$$

are those connected components of S^{pmd} whose image is not contained in $\pi\bigl(\bar{f}(W)\bigr)$. We can now restrict to $\pi\bigl(\bar{f}(W)\bigr) \subset S$ and finish by Noetherian induction. $\square$

Finally observe that the definition is not set up right to work with non-reduced schemes. For example, consider $f : X := B_p\mathbb{A}^2 \to S := \mathbb{A}^2$. The only 1-dimensional fiber is over p. However, if $T \hookrightarrow \mathbb{A}^2$ is any artinian scheme supported at the origin then $f_T : X_T \to T$ has purely 1-dimension fibers but $T \hookrightarrow \mathbb{A}^2$ does not factor through $\{p\} \hookrightarrow \mathbb{A}^2$. Working with the completion $\hat{\mathbb{A}}^2_p$ of $\mathbb{A}^2$ at p gives a scheme that represents our functor over Artin schemes. However, we do not have a family of pure relative dimension 1 over $\hat{\mathbb{A}}^2_p$ itself.

3.34 (Pure dimensional morphisms). A finite type morphism $f : X \to S$ is said to have *pure relative dimension* n if for every integral scheme T and every $h : T \to S$, every irreducible component of $X \times_S T$ has dimension $\dim T + n$. We also say that f is *pure dimensional* if it is pure of relative dimension n for some n. It is enough to check this property for all cases when T is the spectrum of a DVR.

Applying the definition when T is a point shows that if f has pure relative dimension n then every fiber of f has pure dimension n, but the converse does not always hold. For instance, let C be a curve and $\pi : \bar{C} \to C$ the normalization. If C is nodal then π does not have pure relative dimension 0; if C is cuspidal then it does. However, the converse does hold in several important cases.

Claim 3.34.1. Let $f : X \to S$ be a finite type morphism whose fibers have pure dimension n. Then f has pure relative dimension n iff it is universally open. Thus both properties hold if f is flat.

Proof. Both properties can be checked after base change to spectra of DVRs. In the latter case the equivalence is clear and flatness implies both. $\square$

The following is called *Chevalley's criterion,* see [**Gro60**, IV.14.4.1].

Claim 3.34.2. Let $f : X \to S$ be a finite type morphism whose fibers have pure dimension n. Assume that S is normal (or geometrically unibranch) and X is irreducible. Then f is universally open.

Proof. By an easy limit argument, it is enough to check openness after base change for finite type, affine morphisms $S' \to S$; see [**Gro60**, IV.8.10.1]. We may thus assume that $S' \subset \mathbb{A}^n_S$ for some n. The restriction of an open morphism to the preimage of a closed subset is also open, thus it is enough to show that the natural morphism $f^{(n)} : \mathbb{A}^n_Y \to \mathbb{A}^n_S$ is open for every n. If S is normal then so is $\mathbb{A}^n_S$, thus it is enough to show that, under the assumptions of (3.34.2), the map f is open.

To see openness, let $U \subset X$ be an open set and $x \in U$ a closed point. We need to show that $f(U)$ contains an open neighborhood of $s := f(x)$. Let $x \in W \subset X$ be an irreducible component of a complete intersection of n Cartier divisors such that x is an isolated point of $W \cap X_s$. It is enough to prove that $f(U \cap W)$ contains an open neighborhood of s. After extending $W \to S$ to a proper morphism and Stein factorization, we are reduced to showing that (3.34.2) holds for finite morphisms.

In this case $f(U)$ is constructible, hence open iff it is closed under generalization. The latter holds by the going-down theorem; see for instance [**AM69**, 5.16]. □

Ultimately we are interested in pairs (X, Δ), so let us see first what to do with families of pairs (X, D) where D is a single divisor. For this to make sense we need to know what a divisor is. By any definition, the support of a divisor has pure codimension 1 on a pure dimensional scheme.

DEFINITION 3.35. Let X be a scheme and $D \subset X$ a closed subset. A finite type morphism $f : (X, D) \to S$ is called a family of *pure dimensional pairs* of relative dimension n if $f : X \to S$ has pure relative dimension n and $f|_D : D \to S$ has pure relative dimension $n - 1$.

Arguing as in (3.33) we get the following.

LEMMA 3.36. *Let X be a scheme, $D \subset X$ a closed subset and $f : (X, D) \to S$ a proper morphism. Then the functor of pull-backs that are pure of relative dimension n is representable by a locally closed decomposition for reduced schemes.* □

REMARK 3.37. As we noted in (3.34.1), being pure dimensional is an open property for flat, proper morphisms. Thus, using (3.43) we obtain that for any projective morphism $f : X \to S$ we have a locally closed partial decomposition

$$f^{\mathrm{fp}} : S^{\mathrm{fp}} \to S$$

that represents flat and pure dimensional pull-backs of f. Next let $\mathcal{P}$ be a property that implies flat and pure dimensional. Assume that $q : T \to S$ is a morphism such that $f_T : X_T \to T$ satisfies $\mathcal{P}$. Then $f_T : X_T \to T$ is also flat and pure dimensional, hence $q : T \to S$ factors through f^{fp}. This shows that

$$S^{\mathrm{P}} = (S^{\mathrm{fp}})^{\mathrm{P}}.$$

In particular, if we want to prove that $S^{\mathrm{P}} \to S$ exists for all projective morphisms, then it is enough to show that it exists for all flat, pure dimensional and projective morphisms. More generally, if $\mathcal{P}_1 \Rightarrow \mathcal{P}_2$ and S^{P_2} exists then

$$S^{\mathrm{P}_1} = (S^{\mathrm{P}_2})^{\mathrm{P}_1}. \tag{3.37.1}$$

3.38 (Simultaneous normalization). Sometimes it is best to focus not on a property of a morphism but on a property of its "improvement." We say that $f : X \to S$ has *simultaneous normalization* if there is a finite morphism $\pi : \bar{X} \to X$ such that $\pi_s : \bar{X}_s \to X_s$ is the normalization for every $s \in S$ and $f \circ \pi : \bar{X} \to S$ is flat.

For example, consider the family of quadrics

$$X := \left(x_0^2 - x_1^2 + u_2 x_2^2 + u_3 x_3^2 = 0\right) \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{A}^2_{\mathbf{u}}.$$

Then the functor of simultaneous normalizations is represented by

$$\{(0,0)\} \amalg \left(\mathbb{A}^2_{\mathbf{u}} \setminus \{(0,0)\}\right) \to \mathbb{A}^2_{\mathbf{u}}.$$

In general, we have the following result, due to [**CHL06, Kol11b**].

Theorem 3.38.1. Let $f : X \to S$ be a proper morphism whose fibers X_s are generically geometrically reduced. Then there is a morphism $\pi : S^n \to S$ such that for any $g : T \to S$, the fiber product $X \times_S T \to T$ has a simultaneous normalization iff g factors through $\pi : S^n \to S$. □

Valuative criteria.

If a property $\mathcal{P}$ is representable, then we can check it using one of the following criteria. They are especially useful to us since we already understand slc morphisms over DVR's but not over higher dimensional bases.

PROPOSITION 3.39 (Valuative criterion, global version). *Let $\mathcal{P}$ be a property of morphisms that is preserved by base change and is invariant under base field extensions. Let $f : X \to S$ be a proper morphism and assume the following.*

(1) *$\mathcal{P}$ is representable by a finite type morphism $j : S^P \to S$ for seminormal schemes.*
(2) *$f_T : X_T \to T$ satisfies $\mathcal{P}$ whenever T is the spectrum of a DVR and $q : T \to S$ a morphism.*

Then j is a partial seminormalization.

Proof. By (2) the morphism $q : T \to S$ factors through S^P for every DVR. Thus $S^P \to S$ is proper and surjective. For any point $s \in S$ the constant morphism $\operatorname{Spec} k(s)[[t]] \to \operatorname{Spec} k(s) \hookrightarrow S$ uniquely factors through $S^P \to S$. Thus $S^P \to S$ is a partial seminormalization. □

PROPOSITION 3.40 (Valuative criterion, local version). *Let $\mathcal{P}$ be a property of morphisms that is preserved by base change and is invariant under base field extensions. Let $f : X \to S$ be a proper morphism and assume the following.*

(1) *$0 \in S$ is local.*
(2) *$\mathcal{P}$ is representable by a finite type morphism $j : S^P \to S$ for seminormal schemes.*
(3) *There are local morphisms $q_i : (0_i \in T_i) \to (0 \in S)$ such that the $p_{T_i} : X_{T_i} \to T_i$ satisfy $\mathcal{P}$ and $\cup_i q_i(T_i)$ is dense in S.*

Then j is a partial seminormalization.

Proof. By definition, the $q_i : T_i \to S$ factor as

$$q_i : T_i \xrightarrow{q_i'} S^P \xrightarrow{i_P} S$$

and $q_i'(0_i) = i_P^{-1}(0)$. Let $S' \subset S^P$ be the closure of $\cup_i q_i'(T_i) \subset S^P$. Then $i_P : S' \to S$ is a finite, local, geometrically injective morphism with a dense image and $k\bigl(i_P^{-1}(0)\bigr) = k(0)$. So $S' \to S$ is a partial seminormalization. □

A partial seminormalization that is also a locally closed partial decomposition is a closed embedding by (3.49), thus we obtain the following variant.

COMPLEMENT 3.41. *If in (3.39) or (3.40) we also assume that $j : S^P \to S$ is a locally closed partial decomposition then the induced map $S^P \to \operatorname{red} S$ is an isomorphism.* □

The next lemma shows that, once we have a candidate for $S^P \to S$, it is frequently enough to check condition (3.32.3) for DVR's.

LEMMA 3.42. *Let W be a seminormal scheme, $g : W \to S$ a morphism and $i : S' \to S$ a geometrically injective morphism. The following are equivalent.*

(1) *$g : W \to S$ factors through S'.*
(2) *Every composite $g \circ q : T \to W \to S$ factors through S' where T is the spectrum of a DVR and $q : T \to W$ is a morphism.*

Proof. It is clear that (1) $\Rightarrow$ (2). To see the converse consider $i_W : W \times_S S' \to W$. It is geometrically injective and (2) shows that it is proper and surjective. For any point $w \in W$ the constant morphism $\operatorname{Spec} k(w)[[t]] \to \operatorname{Spec} k(w) \hookrightarrow W$ factors through i_W. Thus i_W is an isomorphism since W is seminormal. $\square$

Note that in most cases (2) $\Rightarrow$ (1) holds even of i is not geometrically injective, but the following example should be kept in mind. Let S be the triangle $(xyz = 0) \subset \mathbb{P}^2$ and $S' \to S$ a nontrivial étale cover of it. Then every $T \to S$ lifts (non-uniquely) to $T \to S'$ but the identity $S \cong S$ does not lift to $S \to S'$.

Flatness is representable.

Let $f : X \to S$ be a morphism and F a coherent sheaf on X. Given any $q : W \to S$, we get

$$\begin{array}{ccccc} X \times_S W & =: & X_W & \xrightarrow{q_X} & X \\ & & f_W \downarrow & & \downarrow f \\ & & W & \xrightarrow{q} & S. \end{array}$$

The *functor of flat pull-backs* of F is defined as

$$Flat(F)(W) := \begin{cases} 1 & \text{if } q_X^*F \text{ is flat over } W, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

One of the most useful representation theorems is the following. The projective case is proved in [**Mum66**, Lect.8], the proper case is more subtle [**Art69**].

Theorem 3.43 (Flattening decomposition theorem). *Let $f : X \to S$ be a proper morphism and F a coherent sheaf on X. Then the functor of flat pull-backs $Flat(F)(*)$ is represented by a finite type monomorphism $i^{\text{flat}} : S^{\text{flat}} \to S$.*

If f is projective then i^{flat} is a locally closed decomposition. $\square$

One can frequently check flatness using the following numerical criterion which is proved, but not fully stated, in [**Har77**, III.9.9]. (See also (9.55) for a more precise variant of the last part.)

Theorem 3.44. *Let $f : X \to S$ be a projective morphism with relatively ample $\mathcal{O}_X(1)$ and F a coherent sheaf on X. The following are equivalent.*

(1) *F is flat over S.*
(2) *$f_*\big(F(m)\big)$ is locally free for $m \gg 1$.*

If S is reduced then these are also equivalent to the following.

(3) *$s \mapsto \chi\big(X_s, F_s(m)\big)$ is a locally constant function on S.* $\square$

Corollary 3.45. *Using the notation of (3.44) assume that S is reduced. Then F is flat over S iff q_X^*F is flat over T whenever T is the spectrum of a DVR and $q : T \to S$ a morphism.* $\square$

The local version of (3.45) is also true, but its proof is harder, see [**Gro60**, IV.11.6, IV.11.8].

THEOREM 3.46. *Let S be a reduced scheme, $f : X \to S$ a morphism of finite type and F a coherent sheaf on X. Let $x \in X$ be a point and $s = f(x)$. Then F is flat over S at x iff $q_X^* F$ is flat over T along $q_X^{-1}(x)$ for every local morphism $q : (0,T) \to (s,S)$ from the spectrum of a DVR to S.*

Moreover, it is enough to check this for finitely many local morphisms $q_i : (0,T_i) \to (s,S)$ whose images together dominate S. □

Locally closed decompositions.

DEFINITION 3.47. A morphism $p : X \to Y$ is *geometrically injective* if for every geometric point $\bar{y} \to Y$ the fiber $X \times_Y \bar{y}$ consists of at most 1 point.

Equivalently, for every point $y \in Y$, its preimage $p^{-1}(y)$ is either empty or a single point and $k\bigl(p^{-1}(y)\bigr)$ is a purely inseparable extension of $k(y)$.

If, furthermore, $k\bigl(p^{-1}(y)\bigr)$ equals $k(y)$ then we say that p is *residue field preserving.* The two notions are equivalent in characteristic 0.

A morphism of schemes $f : X \to Y$ is a *monomorphism* if for every scheme Z the induced map of sets $\operatorname{Mor}(Z,X) \to \operatorname{Mor}(Z,Y)$ is an injection.

A monomorphism is geometrically injective. The normalization of the cusp $\pi : \operatorname{Spec} k[t] \to \operatorname{Spec} k[t^2,t^3]$ is geometrically injective but not a monomorphism. The problem is with the fiber over the origin, which is $\operatorname{Spec} k[t]/(t^2) \cong \operatorname{Spec} k[\epsilon]$ (where $\epsilon^2 = 0$). The 2 maps $g_i : \operatorname{Spec} k[\epsilon] \to \operatorname{Spec} k[t]$ given by $g_0^*(t) = 0$ and $g_1^*(t) = \epsilon$ are different but $\pi \circ g_0 = \pi \circ g_1$. A similar argument shows that a morphism is a monomophism iff it is geometrically injective and unramified; see [**Gro60**, IV.17.2.6].

We will usually need to understand when certain natural maps between moduli spaces are monomorphisms. As the above example shows, this requires understanding the corresponding functors over $\operatorname{Spec} k[\epsilon]$ for all fields k.

See (1.63) for an example that is geometrically injective but, unexpectedly, not a monomorphism.

A closed, open or locally closed embedding is a monomorphism. A typical example of a monomorphism that is not a locally closed embedding is the normalization of the node with a point missing, that is $\mathbb{A}^1 \setminus \{-1\} \to (y^2 = x^3 + x^2)$ given by $(t \mapsto (t^2-1, t^3-t)$.

The following property is frequently useful.

Claim 3.47.1. A proper monomorphism $f : X \to Y$ is a closed embedding.

Proof. A proper monomorphism is injective on geometric points, hence finite. Thus it is a closed embedding iff $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is onto. By the Nakayama lemma this is equivalent to $f_y : f^{-1}(y) \to y$ being an isomorphism for every $y \in f(X)$. By passing to geometric points, we are down to the case when $Y = \operatorname{Spec} k$, k is algebraically closed and $X = \operatorname{Spec} A$ where A is an Artin k-algebra. If $A \neq k$ then there are at least 2 different k maps $A \to k[\epsilon]$, thus $\operatorname{Spec} A \to \operatorname{Spec} k$ is not a monomorphism. □

DEFINITION 3.48. A morphism $g : X \to Y$ is a *locally closed embedding* if it can be factored as $g : X \to Y^0 \hookrightarrow Y$ where $X \to Y^0$ is a closed embedding and $Y^0 \hookrightarrow Y$ is an open embedding.

A monomorphism $g : X \to Y$ is is called a *locally closed partial decomposition* of Y if the restriction of g to every connected component $X_i \subset X$ is a locally closed embedding.

If g is also surjective, it is called a *locally closed decomposition* of Y.

As a key example, let $h : Y \to \mathbb{Z}$ be an upper semi continuous function and set $Y_i := \{y \in Y : h(y) = i\}$. Then $\amalg_i Y_i \to Y$ defines a locally closed decomposition.

Sometimes our proofs apply only to the seminormalization $\amalg_i Y_i^{\mathrm{sn}} \to Y$ of a locally closed (partial) decomposition. We call this a *seminormal locally closed (partial) decomposition.*

PROPOSITION 3.49 (Valuative criterion of locally closed embedding). *For a geometrically injective morphism of finite type $f : X \to Y$ the following are equivalent.*

(1) *$f(X) \subset Y$ is locally closed and $X \to f(X)$ is finite.*
(2) *Let T be the spectrum of a DVR and $g : T \to Y$ a morphism such that $g(T) \subset f(X)$. Then there is a spectrum of a DVR T' and a finite morphism $\pi : T' \to T$ such that $g \circ \pi$ lifts to $g'_X : T' \to X$.*
(3) *The previous condition holds for those $g : T \to Y$ that map the generic point of T to a generic point of $f(X)$.*

If f is a monomorphism then these are further equivalent to

(4) *f is a locally closed embedding.*

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Next assume (3).

A geometrically injective morphism of finite type is quasi-finite, hence, by Zariski's main theorem, there is a finite morphism $\bar{f} : \bar{X} \to Y$ extending f. Set $Z := \bar{X} \setminus X$.

If $Z \neq \bar{f}^{-1}\bar{f}(Z)$ then there are points $z \in Z$ and $x \in X$ such that $\bar{f}(z) = \bar{f}(x)$. Let T be the spectrum of a DVR and $h : T \to \bar{X}$ a morphism which maps the closed point to z and the generic point of T to a generic point of X. Set $g := \bar{f} \circ h$. Then $g(T) \subset f(X)$ and the only lifting of g to $T \to \bar{X}$ is h, but $h(T) \not\subset X$.

Thus $Z = \bar{f}^{-1}\bar{f}(Z)$ hence $X \to Y \setminus \bar{f}(Z)$ is proper, proving (1). A proper monomorphism is an isomorphism by (3.47.1), showing the equivalence with (4). $\square$

3.3. Divisorial sheaves

We frequently have to deal with divisors $D \subset X$ that are not Cartier, hence the corresponding sheaves $\mathcal{O}_X(D)$ are not locally free. Understanding families of such sheaves is a key aspect of the moduli problem. Many of the results proved here are developed for arbitrary coherent sheaves in Chapter 9.

DEFINITION 3.50 (Divisorial sheaves). A coherent sheaf F on a scheme X is called a *divisorial sheaf* if F is S_2 and there is a closed subset $Z \subset X$ of codimension ≥ 2 such that $F|_{X \setminus Z}$ is locally free of rank 1.

Set $U := X \setminus Z$ and let $j : U \hookrightarrow X$ denote the natural injection. Then $F = j_*(F|_U)$ by (9.7), thus F is uniquely determined by $F|_U$. The prime examples we have in mind are the following.

Let X be a normal scheme and D a Weil divisor on X. Then $\mathcal{O}_X(D)$ is a divisorial sheaf and we can take $Z = \operatorname{Sing} X$.

Let X be a demi-normal scheme. Then ω_X is a divisorial sheaf and we can take Z to be the non-nodal locus of X.

If $\dim X = 1$ then $Z = \emptyset$ and a divisorial sheaf is the same as an invertible sheaf.

We are mostly interested in the cases when X itself is demi-normal, but the definition makes sense in general, although with unexpected properties. For example, $\mathcal{O}_X$ is a divisorial sheaf iff X is S_2.

Definition 3.51 (Mostly flat families of divisorial sheaves). Let $f : X \to S$ be a morphism. A coherent sheaf F is called a *mostly flat family of divisorial sheaves* if there is a closed subscheme $Z \subset X$ with complement $U := X \setminus Z$ such that

(1) $Z \cap X_s$ has codimension ≥ 2 in X_s for every $s \in S$,
(2) $f|_U : U \to S$ is flat over S with pure, S_2 fibers,
(3) $F|_U$ is locally free of rank 1 and
(4) $\operatorname{depth}_Z F \geq 2$.

The last assumption and (9.7) imply that $F = j_*(F|_U)$. Furthermore, if G is a coherent sheaf that satisfies (1–3) then $j_*(G|_U)$ satisfies (1–4). (This needs a mild technical condition which holds if X is excellent, see (10.16).) We call $j_*(G|_U)$ the *relative hull* of G and denote it by G^H. (Hulls of more general sheaves will be defined and studied in Chapter 9.) The natural map

$$r^G : G \to j_*\bigl(G|_U\bigr) = G^H \tag{3.51.5}$$

is an isomorphism iff $\operatorname{depth}_Z G \geq 2$.

If $\dim X/S = 1$ then $Z = \emptyset$ and a mostly flat family of divisorial sheaves is the same as a flat family of invertible sheaves.

Definition 3.52 (S_2 pull-back). Let $f : X \to S$ be a morphism and F a mostly flat family of divisorial sheaves on X. If $q : W \to S$ is any morphism then we get

$$\begin{array}{cccccc} X \times_S W & =: & X_W & \xrightarrow{q_X} & X \\ & & f_W \downarrow & & \downarrow f \\ & & W & \xrightarrow{q} & S. \end{array} \tag{3.52.1}$$

Thus we also have $U_W := q_X^{-1}(U)$ with injection $j_W : U_W \hookrightarrow X_W$, $Z_W := q_X^{-1}(Z)$ and $F_W := q_X^* F$. Note that F_W satisfies the conditions (3.51.1–3), so its hull $F_W^H := (F_W)^H$ is a mostly flat family of divisorial sheaves. We call F_W^H the S_2 *pull-back* of F. (If confusion is likely, we use $(F_W)^H$ to denote the hull of the pull-back and $(F^H)_W$ to denote pull-back of the hull F^H.) As in (3.51.5), we are especially interested in the map

$$r_W^F : F_W = q_X^* F \to (j_W)_*\bigl(F_W|_{U_W}\bigr) = F_W^H. \tag{3.52.2}$$

We have already encountered these maps in (2.74.6) when $W = \{s\}$ is a point

$$r_s^F : F_s \to (j_s)_*\bigl(F|_{U_s}\bigr) = F_s^H. \tag{3.52.3}$$

A mostly flat family of divisorial sheaves F is called a *flat family of divisorial sheaves* if it satisfies the following equivalent conditions.

(4) F is flat over S and the maps r_W^F defined in (3.52.2) are isomorphisms for every $q : W \to S$.
(5) The maps r_s^F in (3.52.3) are surjective for every closed point $s \in S$.

It is clear that (4) $\Rightarrow$ (5) and the converse is proved in (10.68).

The following two observations are useful.

(6) If $g \in S$ is a generic point then F_g is S_2, hence r_g^F is an isomorphism by (9.7). Thus F is a flat family of divisorial sheaves over some dense, open subset $S^0 \subset S$ by (10.2).
(7) If F is a flat family of divisorial sheaves then every pull-back of it is also a flat family of divisorial sheaves and there is no need to take the hull of the pull-back.

For applications the key point is to understand when a mostly flat family of divisorial sheaves is a flat family of divisorial sheaves. The main result is the following.

THEOREM 3.53. *Let S be a reduced scheme, $f : X \to S$ a projective morphism with relatively ample $\mathcal{O}_X(1)$ and L a mostly flat family of divisorial sheaves on X. Then L is a flat family of divisorial sheaves iff $s \mapsto \chi\bigl(X_s, L_s^H(*)\bigr)$ is locally constant on S.*

Remark 3.53.1. Recall that by (3.44) a coherent sheaf G is flat over S iff $s \mapsto \chi\bigl(X_s, G_s(*)\bigr)$ is locally constant on S. However, the assumptions of (3.53) are quite different. First, L is not assumed to be flat over S and L_s^H is not assumed to be the fiber of L over s. In fact, usually there is no coherent sheaf on X whose fiber over s is isomorphic to L_s^H for every $s \in S$.

Nonetheless, at the end the key point is to compare the Euler characteristic of the sheaves

$$r_s^L : L_s \to L_s^H$$

occurring in (3.52.3); see also (2.74). The map r_s^L is an isomorphism over U_s, but both its kernel and the cokernel can be nontrivial and they have opposite contributions to the Euler characteristic.

Remark 3.53.2. In this section we prove only the special case when S and U are seminormal. Note that if S is seminormal and the fibers of $f|_U$ are seminormal, then U is seminormal by (3.31). So this covers the main cases that we are interested in. It is actually possible to push the methods of this section to give a complete proof of (3.53). However, we will state and prove an even more general result in (9.56).

Proof. We check in (3.56.4) that there is a locally closed decomposition $i : S' \to S$ such that $(i_X^* L)^H$ is a flat family of divisorial sheaves. Thus the L_s^H can be viewed as fibers of a single coherent sheaf $(i_X^* L)^H$. In particular, there is a common $m \in \mathbb{N}$ such that $s \mapsto h^0\bigl(X_s, L_s^H(m)\bigr)$ is a locally constant function on S and $L_s^H(m)$ is generated by its global sections for every $s \in S$. Thus (3.57) shows that L is a flat family of divisorial sheaves. $\square$

The main application of (3.53) is the following. Note that the results proved so far give only a seminormal locally closed decomposition in (3.54) and a seminormal locally closed partial decomposition in (3.55). In both cases we need (9.56) for the full results.

THEOREM 3.54. *Let S be a reduced scheme, $f : X \to S$ a projective morphism with relatively ample $\mathcal{O}_X(1)$ and L a mostly flat family of divisorial sheaves on X. Then there is a locally closed decomposition $j : S^{\mathrm{H-flat}} \to S$ with the following property.*

Let W be a reduced scheme and $q : W \to S$ a morphism. Then L_W^H is a flat family of divisorial sheaves on X_W iff q factors as $q : W \to S^{\mathrm{H-flat}} \to S$.

Proof of (3.53) $\Rightarrow$ (3.54). We prove in (3.56.3) that $s \mapsto \chi\bigl(X_s, L_s^H(*)\bigr)$ is a constructible and upper semi-continuous function on S. Thus its level sets $S_\chi \subset S$ are locally closed. We claim that

$$S^{\mathrm{H-flat}} = \amalg_\chi S_\chi.$$

To see this note first that if L_W^H is a flat family of divisorial sheaves then $w \mapsto \chi\big(X_{q(w)}, L^H_{q(w)}(*)\big)$ is a locally constant function on W, so q factors through j.

Conversely, by construction $s \mapsto \chi(X_s, L_s^H(*))$ is a locally constant function on $\amalg_\chi S_\chi$, so $(j_X^* L)^H$ is a flat family of divisorial sheaves by (3.53), hence the same holds for every pull-back of it. $\square$

The following consequence is especially important.

COROLLARY 3.55. *Let S be a reduced scheme, $f : X \to S$ a flat, projective morphism with S_2 fibers and relatively ample $\mathcal{O}_X(1)$. Let L be a mostly flat family of divisorial sheaves on X. Then there is a locally closed partial decomposition $j : S^{\rm inv} \to S$ with the following property.*

Let W be a reduced scheme and $q : W \to S$ a morphism. Then L_W^H is a flat family of invertible sheaves on X_W iff q factors as $q : W \to S^{\rm inv} \to S$.

Proof. For flat morphisms with S_2 fibers a flat family of invertible sheaves is also a flat family of divisorial sheaves. Thus if L_W^H is a flat family of invertible sheaves then q factors through $S^{\rm H-flat} \to S$. So, by (3.37.1),

$$S^{\rm inv} = \big(S^{\rm H-flat}\big)^{\rm inv}.$$

For a flat family of sheaves being invertible is an open condition, thus $S^{\rm inv}$ is an open subscheme of $S^{\rm H-flat}$. $\square$

Next we establish the two results that we used in the proof of (3.53).

LEMMA 3.56. *Let $f : X \to S$ be a proper morphism and L a mostly flat family of divisorial sheaves. Then*

(1) *$s \mapsto h^0(X_s, L_s^H)$ is a constructible and upper semi-continuous function.*

Furthermore, if $\mathcal{O}_X(1)$ is relatively ample then

(2) *$s \mapsto \chi\big(X_s, L_s^H(t)\big)$ for $t \gg 1$ and*

(3) *$s \mapsto \chi\big(X_s, L_s^H(*)\big)$*

are also constructible and upper semi-continuous, where for polynomials we use the ordering $f(*) \preceq g(*) \iff f(t) \le g(t)\ \forall t \gg 1$.

Remark 3.56.3. If a coherent sheaf F is flat over S then $s \mapsto h^0\big(X_s, F_s\big)$ is constructible and upper semi-continuous on S. However, as in (3.53.3), our assumptions are different since L is not flat over S and L_s^H is not the fiber of L over s.

Proof. In order to prove constructibility, we may replace S by a locally closed decomposition of it. Such a decomposition is provided by the following result, which is a weak version of (3.54).

Claim 3.56.4. There is a locally closed decomposition $i : S' \to S$ such that $(i_X^* L)^H$ is a flat family of divisorial sheaves.

To prove this, note that the generic fibers L_g are S_2, hence, by (10.3), there is a dense open subset $S^0 \subset S$ such that every fiber over S^0 is S_2. Thus L is a flat family of divisorial sheaves over S^0. We can now replace S by $S^0 \amalg (S \setminus S^0)$ and finish by Noetherian induction. $\square$

After replacing S by S', we may assume that L is flat over S. Then (3.56.1–3) become the usual constructibility and upper semi-continuity claims for coherent sheaves that are flat over S.

A constructible function is upper semicontinuous iff it is upper semicontinuous after base change to any DVR. Thus we may assume from now on that $S = T$ is the spectrum of a DVR with closed point 0 and generic point g.

In this case L is flat over T and S_2. Thus its central fiber L_0 is S_1. In particular the restriction map (3.52.3) $r_0^F : L_0 \to L_0^H$ is an injection and we get that

$$h^0\big(X_g, L_g\big) \le h^0\big(X_0, L_0\big) \le h^0\big(X_0, (L_0)^H\big). \tag{3.56.5}$$

This proves (1) and using it for $t \gg 1$ gives (2). Finally (3) is equivalent to (2) by definition. □

The next result roughly says that flatness of H^0 implies flatness for globally generated shaves.

Proposition 3.57. *Let S be a seminormal scheme, $f : X \to S$ be a projective morphism with seminormal fibers and L a mostly flat family of divisorial sheaves on X. Assume that*

(1) *$s \mapsto h^0(X_s, L_s^H)$ is a locally constant function on S and*
(2) *L_s^H is generated by its global sections for every $s \in S$.*

Then L is a flat family of divisorial sheaves.

Idea of the proof. For $s \in S$ we look at the sequence of maps

$$L_s \to L_s/\operatorname{tors}(L_s) \hookrightarrow L_s^H.$$

We hope to prove by semicontinuity of H^0 that

$$h^0(X_g, L_g) \overset{?}{\le} h^0\big(X_s, L_s/\operatorname{tors}(L_s)\big), \tag{3.57.3}$$

where $g \in S$ is a generic point. Combining it with an obvious inequality we get that

$$h^0(X_g, L_g) \overset{?}{\le} h^0\big(X_s, L_s/\operatorname{tors}(L_s)\big) \le h^0\big(X_s, L_s^H\big). \tag{3.57.4}$$

We assumed that the 2 ends are equal, hence

$$H^0\big(X_s, L_s/\operatorname{tors}(L_s)\big) = H^0\big(X_s, L_s^H\big). \tag{3.57.5}$$

Since L_s^H is generated by its global sections, this implies that $L_s/\operatorname{tors}(L_s) = L_s^H$ and the rest follows.

The problem with this is that (3.57.3) fails in general; see (3.58) for a not very convincing example, (3.59) and (10.67.4) for better ones. However, the above argument works if S is the spectrum of a DVR and this is enough to cobble the proof together if S is seminormal.

Proof. If $\dim X/S \le 1$ then L is flat over S by definition. Thus assume from now on that $\dim X/S \ge 2$.

We may assume that S is local, in particular $s \mapsto h^0(X_s, L_s^H)$ is a constant function. Let $C \subset X$ be a general complete intersection curve of sufficiently ample divisors. Then C is disjoint from the non-flat locus $Z \subset X$, hence $L|_C$ is flat over S. Furthermore, by repeatedly applying the Enriques-Severi-Zariski lemma (9.9), we know that the restriction maps

$$H^0(X_s, L_s^H) \to H^0\big(C_s, (L_s^H)|_{C_s}\big) = H^0\big(C_s, L|_{C_s}\big) \tag{3.57.6}$$

are isomorphisms. Thus $s \mapsto h^0(C_s, L|_{C_s})$ is a constant function on S, hence by Grauert's theorem $(f|_C)_*(L|_C)$ is locally free. Our aim is to prove that the restriction map

$$f_*L \to (f|_C)_*(L|_C) \tag{3.57.7}$$

is an isomorphism. If this holds then $(f_*L)_s = h^0(C_s, L|_{C_s}) = H^0(X_s, L_s^H)$, hence by pull-back, we get natural maps

$$\mathcal{O}_{X_s} \otimes_{k(s)} H^0(X_s, L_s^H) \cong \mathcal{O}_{X_s} \otimes_{k(s)} (f_*L)_s \to L_s \xrightarrow{r_s^L} L_s^H. \tag{3.57.8}$$

We assumed that the composite is a surjection, thus r_s^L is also surjective. Thus, by (3.52.5), L is a flat family of divisorial sheaves.

The map $f_*L \to (f|_C)_*(L|_C)$ is an injection by construction, thus we need to show that it is surjective. Pick any section $\sigma_C \in H^0\big(C, L|_C\big)$.

By (3.56.4) there is a locally closed decomposition $\amalg_i S_i \to S$ such that each $L_i := (L|_{X_i})^H$ is a flat family of divisorial sheaves where $X_i := S_i \times_S X$ and $C_i := S_i \times_S C$. Over each S_i we can use (9.9) to conclude that

$$(f_i)_*L_i \to (f|_{C_i})_*(L|_{C_i}) \quad \text{is an isomorphism.} \tag{3.57.9}$$

Thus $\sigma_C|_{C_i}$ can be lifted back to a section σ_i of L_i. We need to prove that these σ_i glue together to a global section σ_U of $F|_U$. Since $F = j_*(F|_U)$, it then extends to a global section σ of F.

We aim to do this geometrically, thus we think of $F|_U$ as a line bundle over U. Then $\amalg_i \sigma_i$ is a constructible subset of $F|_U$ and the projection $\pi : \amalg_i \sigma_i \to U$ is a locally closed decomposition. We aim to prove that it is an isomorphism.

If U is seminormal, then (3.49) reduces this to the case when $S = T$ is the spectrum of a DVR.

Thus assume that $S = T$ with closed point 0 and generic point g. We can now follow the path outlined in (3.57.3–5). In this case L is flat over T and S_2. Thus its central fiber L_0 is S_1. In particular the restriction map (3.52.3) $r_0^F : L_0 \to L_0^H$ is an injection and we get that

$$h^0\big(X_g, L_g^H\big) = h^0\big(X_g, L_g\big) \le h^0\big(X_0, L_0\big) \le h^0\big(X_0, L_0^H\big). \tag{3.57.10}$$

We assumed that the 2 sides are equal, hence $h^0\big(X_0, L_0\big) = h^0\big(X_0, L_0^H\big)$. Since L_0^H is generated by global sections, the latter can happen only if $L_0 = L_0^H$. Thus L is a flat family of divisorial sheave hence, as in (3.57.9), (9.9) shows that σ_C lifts to σ. □

Example 3.58. Let C be the triangle $(xyz = 0) \subset \mathbb{P}^2$ and L_C a nontrivial degree 0 line bundle on C. Set $S' := C \times \mathbb{A}^1$ and L' the pull-back of L_C to S'. Set

$$S := \big(C \times \{0\}\big) \cup \big((x=0) \times \mathbb{A}^1\big) \subset S' \quad \text{and} \quad L := L'|_S.$$

Compute that

$$H^0(S_c, L_c) = \begin{cases} 0 & \text{if} \quad c = 0 \quad \text{and} \\ 1 & \text{if} \quad c \neq 0. \end{cases}$$

Set $T' := C \times (st = 0) \subset \mathbb{P}^2_{xyz} \times \mathbb{A}^2_{st}$ and M' the pull-back of L_C to T'. Set

$$T := \big((xy=0) \times (s=0)\big) \cup \big((yz=0) \times (t=0)\big) \subset T' \quad \text{and} \quad M := M'|_T.$$

Compute that

$$H^0(T_c, M_c) = \begin{cases} 0 & \text{if} \quad c = (0,0) \quad \text{and} \\ 1 & \text{if} \quad c \neq (0,0). \end{cases}$$

EXAMPLE 3.59. Let C be a smooth, projective curve of genus 2 and J the degree 2 component of $\operatorname{Pic}(C)$. Let $0 \in J$ denote the class of K_C. On $\pi : C \times J \to J$ consider the Poincaré bundle P. We claim that $\pi_* P$ is a line bundle on J but

$$r_0^P : (\pi_* P)_0 \to H^0(C, P_0) = H^0(C, \omega_C) \quad \text{is the 0 map.}$$

To see this note that if $\deg L = 2$ then

$$h^0(C, L) = \begin{cases} 2 & \text{if } L \cong \omega_C \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Thus $\pi_* P$ is a line bundle on $J \setminus \{0\}$. It is also reflexive, hence a line bundle. Therefore P is represented by a unique divisor $D \subset C \times J$. Our claim says that D contains $C \times \{0\}$.

For a general $z \in J$ the fiber D_z consists of the zero set of the unique section of P_z. As $z \to 0$, the fibers D_z converge to a fiber of the hyperelliptic map $C \to \mathbb{P}^1$. However, converging from different directions yields different fibers. Thus indeed D contains $C \times \{0\}$.

Taking the cone over $C \times J \to J$ with ample line bundle P gives a family of surfaces $f : X \to J$ and a mostly flat family of divisorial sheaves L where the map

$$f_* L \to H^0\big(X_0, L_0/\operatorname{tors}(L_0)\big)$$

in (3.57.3) is the zero map.

It is also clear that similar examples arise from other flat families of ample line bundles P on a flat family of varieties $X \to B$ where $b \mapsto h^0(X_b, P_b)$ jumps on a codimension ≥ 2 subset of B.

COROLLARY 3.60. *Let $f : X \to S$ be a flat, proper morphism with S_2 fibers such that $H^0(X_s, \mathcal{O}_{X_s}) \cong k(s)$ for every $s \in S$. Let L be mostly flat family of divisorial sheaves on X such that $L_s^H \cong \mathcal{O}_{X_s}$ for every $s \in S$. Then there is a line bundle L_S on S such that $L \cong f^* L_S$.*

Proof. By (3.57) L is a flat family of divisorial sheaves. Hence, by Grauert's theorem, $f_* L$ is locally free of rank 1 and $L \cong f^*(f_* L)$. □

LEMMA 3.61. *Let $f : X \to S$ be a flat, proper morphism with S_2 fibers such that $H^0(X_s, \mathcal{O}_{X_s}) \cong k(s)$ for every $s \in S$. Let L, M be mostly flat families of divisorial sheaves on X. Then*

$$\operatorname{Isom}_S(L, M) := \{s \in S : L_s^H \cong M_s^H\} \subset S \quad \textit{is locally closed.}$$

Proof. After replacing L by $\mathcal{H}om(M, L)$ and M by $\mathcal{O}_X$, we need to prove that

$$S^{\text{triv}} := \{s \in S : L_s^H \cong \mathcal{O}_{X_s}\} \subset S \quad \text{is locally closed.} \tag{3.61.1}$$

By (3.56.1), $s \mapsto h^0(X_s, L_s^H)$ is an upper semicontinuous function on S. If $L_s^H \cong \mathcal{O}_{X_s}$ then $h^0(X_s, L_s^H) = 1$, we can thus harmlessly replace S by the locally closed subset where $h^0(X_s, L_s^H) = 1$. Thus we may assume from now on that $h^0(X_s, L_s^H) = 1$ for all $s \in S$.

For every $s \in S$ we have a natural map $\mathcal{O}_{X_s} \to L_s^H$, let $Z_s \subset X_s$ denote the support of its cokernel. Note that $Z_s \subset X_s$ has pure codimension 1 by (9.7). We claim that $Z := \cup_{s\in S} Z_s \subset X$ is a closed subset. Once this is proved then $S^{\mathrm{triv}} = S \setminus f(Z)$.

First we prove that Z is constructible. By (3.56.4) we can write S as a finite union of locally closed subsets $S_i \subset S$ such that $L_i^H := \bigl(L|_{X_i}\bigr)^H$ is flat over S_i with fibers L_s^H for each i. Thus $Z_i := \cup_{s\in S_i} Z_s$ is the support of the cokernel of the natural map

$$f^* f_*\bigl(L_i^H\bigr) \to L_i^H,$$

hence closed. Thus Z is constructible. A constructible set is closed iff it is closed under specialization, thus it is enough to prove that Z is closed after base change to the spectrum of a DVR T with closed and generic points $0, g \in T$. So we have $g : X_T \to T$ and L_T^H. Now $g_*(L_T^H)$ is a line bundle and we claim that

$$Z_0 \cup Z_g = Z_T := \operatorname{Supp} \operatorname{coker}\bigl[g^* g_*\bigl(L_T^H\bigr) \to L_T^H\bigr].$$

This is clear over the generic fiber hence it remains to compare $(Z_T)_0$ and Z_0. The two can differ only along the support of the cokernel of the restriction map $L_T^H|_{X_0} \to L_0^H$. This has codimension ≥ 2, so $(Z_T)_0 = Z_0$ since both sides have pure codimension 1. $\square$

PROPOSITION 3.62. *Let $f : X \to S$ a flat, finite type morphism with S_2 fibers and L a mostly flat family of divisorial sheaves on X such that $L^{[m]}$ is locally free for some $m > 0$. Then L is locally free iff L_s is locally free for every $s \in S$.*

Proof. Assume first that f is projective. By (3.53) we need to check that $s \mapsto \chi\bigl(X_s, L_s^H(*)\bigr)$ is locally constant on S. It is enough to check this over DVR's, which was done in (2.92).

In general, let $U \subset X$ denote the largest open subset where L is locally free. If $U \neq X$, pick a generic point $x \in X \setminus U$ and set $s := f(x)$. Since L is mostly free, x has codimension ≥ 2 in X_s, hence $\operatorname{depth}_x X_s \geq 2$. Now we use (4.36) and (2.92) to get that L is locally free at x, a contradiction. $\square$

Given a mostly flat family of divisorial sheaves L on $f : X \to S$, we will be interested in the set

$$\{s \in S : L_s^{[m_s]} \text{ is locally free for some } m_s > 0\}.$$

We see in (4.15) that this set is not constructible in general. The following lemma is sometimes useful.

LEMMA 3.63. *Let $f : X \to S$ be a flat, finite type morphism with S_2 fibers and L a mostly flat family of divisorial sheaves on X. Assume that $L_s^{[m_s]}$ is locally free for some $m_s > 0$ for every $s \in S$. Then there is a common $m > 0$ such that $L_s^{[m]}$ is locally free for every $s \in S$.*

Note that we do not claim that $L^{[m]}$ itself is locally free.

Proof. Let $g \in S$ be a generic point. Then $L_g^{[m_g]}$ is locally free for some $m_g \in \mathbb{N}$, thus the same holds in an open neighborhood of $g \in S$. We finish by Noetherian induction. $\square$

3.64 (Hilbert function of divisorial sheaves). Let X be a proper scheme of dimension n and L, M line bundles on X. The Hirzebruch-Riemann-Roch theorem computes $\chi(X, L\otimes M^r)$ as a polynomial of r. Its leading terms are

$$\chi(X, L\otimes M^r) = \frac{r^n}{n!}(M^n) + \frac{r^{n-1}}{2(n-1)!}\Bigl(\bigl(\tau_1(X)+2L\bigr)\cdot M^{n-1}\Bigr) + \dots \tag{3.64.1}$$

Assume next that L is a torsion free sheaf that is locally free outside a subset $Z\subset X$ of codimension ≥ 2. By blowing up L we get a proper birational morphism $\pi: X'\to X$ and a line bundle L' such that $\pi_*L' = L$. Thus we can compute $\chi(X, L\otimes M^r)$ as $\chi(X', L'\otimes\pi^*M^r)$, modulo an error term which involves the sheaves $R^i\pi_*L'$. These may be hard to control, but they are supported on Z, hence the $\chi(X, R^i\pi_*L'\otimes M^r)$ all have degree $\leq n-2$. Thus we again obtain the HRR formula (3.65.1). If X is deminormal then $\tau_1(X) = -K_X$, hence we get the usual form

$$\chi(X, L\otimes M^r) = \frac{r^n}{n!}(M^n) - \frac{r^{n-1}}{2(n-1)!}\bigl((K_X-2L)\cdot M^{n-1}\bigr) + \dots \tag{3.64.2}$$

If, in addition, $L^{[m]}$ is locally free for some $m>0$, then applying (3.65.2) to $L\mapsto L^{[a]}$ for all $0\leq a<m$ and $M = L^{[m]}$ we end up with the expected formula

$$\chi(X, L^{[r]}) = \frac{r^n}{n!}(L^n) - \frac{r^{n-1}}{2(n-1)!}\bigl(K_X\cdot L^{n-1}\bigr) + (\text{lower order terms}). \tag{3.64.3}$$

Note further that (3.64.2) shows that $\chi(X, L^{[r]})$ is a polynomial on any translate of $m\mathbb{Z}$. We can thus write

$$\chi(X, L^{[r]}) = \frac{r^n}{n!}(L^n) - \frac{r^{n-1}}{2(n-1)!}\bigl(K_X\cdot L^{n-1}\bigr) + \textstyle\sum_{i=0}^{n-2} a_i(r) r^i, \tag{3.64.4}$$

where the $a_i(r)$ are periodic functions that depend on X and L.

3.65 (Hilbert function of slc varieties). Let X be a proper, slc variety of dimension n. We are especially interested in

$$\chi(X, r) := \chi\bigl(X, \omega_X^{[r]}\bigr), \tag{3.65.1}$$

which we call the *Hilbert function* of X. (Note that one could also call $r\mapsto h^0\bigl(X, \omega_X^{[r]}\bigr)$ the Hilbert function. The problem in our case is that (3.65.1) is not a polynomial, thus it would be misleading to call it a Hilbert polynomial. For stable varieties the two variants differ only for $r=1$, see (3.65.3).)

By (3.64.4) we can write the Hilbert function as

$$\chi(X, r) = \frac{r^n}{n!}(K_X^n) - \frac{r^{n-1}}{2(n-1)!}(K_X^n) + \textstyle\sum_{i=0}^{n-2} a_i(r) r^i, \tag{3.65.2}$$

where the $a_i(r)$ are periodic functions with period $=$ index(X), that depend on X.

If ω_X is ample and the characteristic is 0, then a singular version of Kodaira's vanishing theorem [**Fuj14**, 1.9] implies that, for $r\geq 2$,

$$\begin{aligned} h^i\bigl(X, \omega_X^{[r]}\bigr) &= 0, \quad \text{hence} \\ h^0\bigl(X, \omega_X^{[r]}\bigr) &= \chi\bigl(X, \omega_X^{[r]}\bigr). \end{aligned} \tag{3.65.3}$$

3.4. Local stability over reduced schemes

DEFINITION 3.66 (Relative canonical class). Let $f : X \to S$ be a flat, projective family of demi-normal varieties. The relative dualizing sheaf $\omega_{X/S}$ was constructed in (2.70).

Let $Z \subset X$ the subset where the fibers are neither smooth nor nodal and set $U := X \setminus Z$. Then $f|_U$ is flat with CM, even Gorenstein fibers. Thus, by (2.70.7), $\omega_{U/S}$ is locally free, commutes with base change and $X_s \cap Z$ has codimension ≥ 2 for every fiber X_s. Thus $\omega_{X/S} = j_*\omega_{U/S}$, hence $\omega_{X/S}$ is a divisorial sheaf. The corresponding divisor class is denoted by $K_{X/S}$.

We define the reflexive powers of $\omega_{X/S}$ by the formula

$$\omega_{X/S}^{[m]} := j_*\big(\omega_{U/S}^m\big). \tag{3.66.1}$$

Thus $\omega_{X/S}^{[m]} \cong \mathcal{O}_X(mK_{X/S})$. In particular, $mK_{X/S}$ is Cartier iff $\omega_{X/S}^{[m]}$ is locally free.

All these hold for finite type morphisms by (2.70.7).

If the fibers of $f : X \to S$ are slc then $\omega_{X/S}$ is a flat family of divisorial sheaves by (2.69). However, its reflexive powers are usually only mostly flat over S. Applying (3.55) to $\omega_{X/S}^{[m]}$ gives the following, which turns out to be the key to our treatment of local stability over reduced schemes.

COROLLARY 3.67. *Let S be a reduced scheme, $f : X \to S$ a projective family of demi-normal varieties and fix $m \in \mathbb{Z}$. Then there is a locally closed decomposition $j : S^{[\mathrm{m}]} \to S$ such that the following holds.*

Let W be any reduced scheme and $q : W \to S$ a morphism. Then $\omega_{X_W/W}^{[m]}$ is a flat family of divisorial sheaves iff q factors as $q : W \to S^{[\mathrm{m}]} \to S$. □

In applications of (3.67) a frequent problem is that $S^{[\mathrm{m}]}$ depends on m, even if we choose m to be large and divisible; see (3.71) for such an example.

We are now ready to prove that the temporary definition (3.1) of local stability over reduced schemes is equivalent to the more conceptual one, which is (3.68.1).

THEOREM 3.68 (Local stability over reduced schemes). *Let S be a reduced scheme and $f : X \to S$ a flat family of deminormal varieties. Then the following are equivalent.*

(1) *$\omega_{X/S}^{[m]}$ is a flat family of divisorial sheaves for every $m \in \mathbb{Z}$ and the fibers X_s are slc for all points $s \in S$.*
(2) *$\omega_{X/S}^{[m]}$ is a flat family of invertible sheaves for some $m > 0$ and the fibers X_s are slc for all points $s \in S$.*
(3) *$K_{X/S}$ is $\mathbb{Q}$-Cartier and the fibers X_s are slc for all points $s \in S$.*
(4) *$K_{X/S}$ is $\mathbb{Q}$-Cartier and X_s is slc for all closed points $s \in S$.*
(5) *$f_T : X_T \to T$ is locally stable (2.2) whenever T is the spectrum of a DVR and $q : T \to S$ is a morphism.*

Proof. Assume (1) and pick $s \in S$. Since X_s is slc, $\omega_{X_s}^{[m_s]}$ is locally free for some $m_s > 0$. In a flat family of sheaves being invertible is an open condition, thus $\omega_{X/S}^{[m_s]}$ is a flat family of invertible sheaves in an open neighborhood $X_s \subset U_s \subset X$. Finitely many of these U_{s_i} cover X, and then $m = \operatorname{lcm}\{m_{s_i}\}$ works for (2). Assertions (2) and (3) say the same using different terminology and (3) $\Rightarrow$ (4) is clear.

To see the converse, let $s \in S$ be a non-closed point. Choose a spectrum of a DVR T and a morphism $q : T \to S$ that maps the generic point to s and the closed point of T to a closed point of S. We get $f_T : X_T \to T$ such that $K_{X_T/T}$ is $\mathbb{Q}$-Cartier and the special fiber is slc. Thus the generic fiber is also slc by (2.3), hence X_s is slc. This shows that (4) $\Rightarrow$ (3).

If $K_{X/S}$ is $\mathbb{Q}$-Cartier then so is any pull-back $K_{X_T/T}$. Thus (4) $\Rightarrow$ (5) also follows from (2.3).

It remains to show that (5) $\Rightarrow$ (1). If (5) holds then all fibers are slc and we need to prove that $\omega_{X/S}^{[m]}$ is a flat family of divisorial sheaves. This is a local question on S, hence we may assume that $(0 \in S)$ is local.

Let us discuss first the case when f is projective. By (3.67) the property

$$\mathcal{P}^{[m]}(W) := \bigl(\omega_{X_W/W}^{[m]} \text{ is a flat family of divisoral sheaves}\bigr)$$

is representable by a locally closed decomposition $i_m : S^{[m]} \to S$. We aim to prove that i_m is an isomorphism.

For each generic point $g_i \in S$ choose a local morphism $q_i : (0_i \in T_i) \to (0 \in S)$ that maps the generic point $t_i \in T_i$ to g_i. By assumption $X_{T_i} \to T_i$ is locally stable, hence $\omega_{X_{T_i}/T_i}^{[m]}$ is a flat family of divisorial sheaves by (2.76.1). Thus g_i factors through $i_m : S^{[m]} \to S$. Therefore $i_m : S^{[m]} \to S$ is an isomorphism by (3.41).

We postpone the discussion of the non-projective case to (4.45). □

Corollary 3.69. *Let $f : X \to S$ be a flat morphism with demi-normal fibers such that $K_{X/S}$ is $\mathbb{Q}$-Cartier. Then*

$$S^* := \{s : X_s \textit{ is slc }\} \subset S \quad \textit{is open.} \tag{3.69.1}$$

Proof. A set $U \subset S$ is open iff it is closed under generalization and U contains a dense open subset of $\bar{s}$ for every $s \in U$.

For S^*, the first of these follows from (2.3). In order to see the second, assume first that X_s is lc. Then mK_{X_s} is Cartier for some $m > 0$ hence $mK_{X/S}$ is Cartier over an open neighborhood of $s \in U_s \subset \bar{s}$. Next consider a log resolution $p_s : Y_s \to X_s$. It extends to a simultaneous log resolution $p^0 : Y^0 \to X^0$ over a suitable $U_s^0 \subset \bar{s}$. Thus, if $E^0 \subset Y^0$ is any exceptional divisor, then $a(E_t, X_t) = a(E^0, X^0) = a(E_s, X_s)$ for every $t \in U_s^0$. This shows that all fibers over U_s^0 are lc.

If X_s is not normal, one can use either a simultaneous semi-log resolution [**Kol13c**, Sec.10.4] or normalize first, apply the above argument and descend to X, essentially by definition (1.85). □

The following is a direct consequence of (3.62).

Corollary 3.70. *Let S be a reduced scheme and $f : X \to S$ a locally stable morphism. Then $\omega_{X/S}^{[m]}$ is locally free along X_s iff $\omega_{X_s}^{[m]}$ is locally free.* □

Example 3.71. Following [**Pat13**], we give an example of a flat family of normal varieties $Y \to U$ such that ω_{Y_0} is locally free for some $0 \in U$ yet $\omega_{Y/U}$ is not locally free along Y_0. Furthermore, $\{u : K_{Y_u} \text{ is } \mathbb{Q}\text{-Cartier}\}$ is a countable dense subset of U.

We start with a smooth, projective variety X such that $H^1(X, \mathcal{O}_X) \neq 0$ but $H^0(X, \omega_X) = H^1(X, \omega_X) = 0$. For example, we can take $X = C \times \mathbb{P}^n$ where C is a smooth curve of genus > 0 and $n \geq 2$.

Let L_0 be a very ample line bundle such that $L_0 \otimes \omega_X$ is ample. All line bundles algebraically equivalent to L_0 are parametrized by $\mathbf{Pic}^L(X)$, a connected component of $\mathbf{Pic}(X)$.

Choose a smooth divisor $D \subset X$ linearly equivalent to L_0. Our example will be the family of cones

$$Y_L := \operatorname{Spec}_k \textstyle\sum_m H^0\bigl(D, (L \otimes \omega_X)^m|_D\bigr),$$

parametrized by a suitable open set $[L_0] \in U \subset \mathbf{Pic}^L(X)$.

The Y_L form a flat family iff the $h^0\bigl(D, (L \otimes \omega_X)^m|_D\bigr)$ are all constant on U. To compute these, consider the exact sequence

$$0 \to (L \otimes \omega_X)^m(-D) \to (L \otimes \omega_X)^m \to (L \otimes \omega_X)^m|_D \to 0.$$

By Kodaira vanishing, the higher cohomologies of the first 2 sheaves vanish, except for $L \otimes \omega_X(-D)$. We assumed that $H^0(X, \omega_X) = H^1(X, \omega_X) = 0$. Thus, by semicontinuity,

$$H^0\bigl(X, L \otimes \omega_X(-D)\bigr) = H^1\bigl(X, L \otimes \omega_X(-D)\bigr) = 0$$

for all L in a neighborhood of $[L_0]$; this conditions defines our U. (If $X = C \times \mathbb{P}^n$ then actually $U = \mathbf{Pic}^L(X)$.) Hence $h^0\bigl(D, L \otimes \omega_X|_D\bigr)$ is independent of L for $[L] \in U$. The cones Y_L are the fibers of a flat morphism $Y \to U$.

By [**Kol13c**, 3.14.4], ω_{Y_L} is locally free iff ω_D is a power of $L \otimes \omega_X|_D$ and the latter is isomorphic to $\omega_D \otimes \bigl(L \otimes L_0^{-1}\bigr)$. Thus ω_{Y_L} is locally free iff $L|_D \cong L_0|_D$. By the Lefschetz theorem this holds iff $L \cong L_0$. Thus $\omega_{Y/U}$ is not locally free along Y_{L_0} yet $\omega_{Y_{L_0}}$ is locally free.

Similarly we get that $\omega_{Y_L}^{[m]}$ is locally free iff $L^m \cong L_0^m$, so $\bigl\{L : K_{Y_L} \text{ is } \mathbb{Q}\text{-Cartier}\bigr\}$ is a countable dense subset of U.

3.5. Stability is representable I

We start with an example showing that being locally stable is not an open condition, not even a locally closed one.

Example 3.72. In $\mathbb{P}^5_{\mathbf{x}} \times \mathbb{A}^2_{st}$ consider the family of varieties given by the equations

$$X := \left(\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \le 1\right).$$

We claim that the fibers X_{st} are normal, projective with rational singularities and for every s, t the following equivalences hold:

(3.72.1) X_{st} is lc $\Leftrightarrow$ X_{st} is klt $\Leftrightarrow$ $K_{X_{st}}$ is $\mathbb{Q}$-Cartier $\Leftrightarrow$ $3K_{X_{st}}$ is Cartier $\Leftrightarrow$ either $(s,t) = (0,0)$ or $st \neq 0$.

All these become clear once we show that there are 3 types of fibers.

(3.72.2) If $st \neq 0$ then after a linear coordinate change we get that

$$X_{11} \cong X_{st} \cong \left(\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_4 & x_5 & x_3 \end{pmatrix} \le 1\right).$$

This is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$, hence smooth. The self-intersection of its canonical class is -54.

(3.72.3) If $s = t = 0$ then we get the fiber

$$X_{00} := \left(\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \le 1\right).$$

This is the cone (with $\mathbb{P}^1$ as vertex) over the rational normal curve $C_3 \subset \mathbb{P}^3$. The singularity along the vertex-line is isomorphic to $\mathbb{A}^2/\frac{1}{3}(1,1) \times \mathbb{A}^1$, hence log terminal. The canonical class of X_{00} is $-\frac{8}{3}H$ where H is the hyperplane class and its self-intersection is $-512/9 < -54$.

(3.72.4) Otherwise either s or t (but not both) are zero. After possibly permuting s, t and a linear coordinate change we get the fiber

$$X_{01} \cong X_{0t} \cong \left(\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_4 & x_3 \end{pmatrix} \leq 1\right).$$

This is the cone over the degree 3 surface $S_3 \cong \mathbb{F}_1 \hookrightarrow \mathbb{P}^4$. Its canonical class is not $\mathbb{Q}$-Cartier at the vertex, so this is not lc.

Thus the best one can hope for is that local stability is representable. From now on the base scheme is assumed to be over a field of characteristic 0. (See (4.58) for a list of problems in positive characteristic.)

THEOREM 3.73 (Local stability is representable). *Let S be a reduced scheme over a field of characteristic 0 and $f : X \to S$ a projective morphism. Then there is a locally closed partial decomposition $j : S^{\mathrm{ls}} \to S$ such that the following holds.*

Let W be any reduced scheme and $q : W \to S$ a morphism. Then the family obtained by base change $f_W : X_W \to W$ is locally stable iff q factors as $q : W \to S^{\mathrm{ls}} \to S$.

Outline of proof. We start with some easy reduction steps to achieve that $f : X \to S$ is flat with demi-normal fibers of pure relative dimension n for some n. This part of the argument works over any base scheme S.

We repeatedly apply (3.37.1) to various properties that are weaker than local stability. Each time we obtain that it is enough to prove (3.73) for morphisms that satisfy some additional properties.

Being flat is representable by (3.43), thus we may assume from now on that $f : X \to S$ is flat. By (3.34) we may also assume that it has pure relative dimension n. For flat morphisms being demi-normal is an open condition by (10.41), hence we may assume that $f : X \to S$ is flat and its fibers are demi-normal of pure relative dimension n.

Now we come to a surprisingly subtle part of the argument. By definition, if X_s is slc then K_{X_s} is $\mathbb{Q}$-Cartier, thus the next natural step would be to consider the following.

Question 3.73.1. Is $\{s \in S : K_{X_s} \text{ is } \mathbb{Q}\text{-Cartier}\}$ a constructible subset of S?

We see in (3.71) that this is not the case, not even for families of normal varieties. We thus need to jump ahead to deal with the slc property and prove that

$$\{s \in S : X_s \text{ is slc}\} \quad \text{is a constructible subset of } S. \tag{3.73.2}$$

Actually, it is better to establish a different version that controls the denominators of the canonical divisors. Recall that the *index* of an slc variety Y, denoted by $\operatorname{index}(Y)$, is the smallest positive natural number m such that mK_Y is Cartier.

The key property turns out to be the following, which is an immediate consequence of (4.48). This allows us to avoid the dependence on the extra constant m when applying (3.67). (This result can also be thought of as a local variant of [**HMX14**].)

Lemma 3.73.3. Let $f : X \to S$ be a flat, proper family of demi-normal varieties. Then

$$\{\mathrm{index}(X_s) : X_s \text{ is slc}\} \quad \text{is a finite set.} \qquad \square$$

Let now m be a common multiple of the indices of slc fibers and apply (3.67) to get a locally closed partial decomposition $j : S^{[\mathrm{m}]} \to S$ and

$$X^{[\mathrm{m}]} := X \times_S S^{[\mathrm{m}]} \to S^{[\mathrm{m}]} \quad \text{such that} \quad mK_{X^{[\mathrm{m}]}/S^{[\mathrm{m}]}} \quad \text{is Cartier.}$$

Let $q : W \to S$ be a morphism such that $q_W : X_W \to W$ is locally stable. Then every fiber of q_W is slc, hence mK_{X_w} is Cartier for every $w \in W$ by our choice of m. Therefore $mK_{X_W/W}$ is Cartier by (3.70). Thus q factors as $q : W \to S^{[\mathrm{m}]} \to S$. Therefore, by (3.37.1),

$$S^{\mathrm{ls}} = (S^{[\mathrm{m}]})^{\mathrm{ls}}.$$

Thus it is sufficient to prove (3.73) in case $f : X \to S$ is flat with demi-normal fibers of pure relative dimension n and $K_{X/S}$ is $\mathbb{Q}$-Cartier. In this case, S^{ls} is an open subscheme of S by (3.69). $\square$

As in (3.1), a proper morhism $f : X \to S$ is called stable iff it is locally stable and $K_{X/S}$ is $\mathbb{Q}$-Cartier and f-ample. Since ampleness is an open condition for a $\mathbb{Q}$-Cartier divisor, (3.73) implies the following.

Corollary 3.74 (Stability is representable). *Let S be a reduced scheme over a field of characteristic 0 and $f : X \to S$ a projective morphism. Then there is a locally closed partial decomposition $j : S^{\mathrm{stab}} \to S$ such that the following holds.*

Let W be any reduced scheme and $q : W \to S$ a morphism. Then the family obtained by base change $f_W : X_W \to W$ is stable iff q factors as $q : W \to S^{\mathrm{stab}} \to S$. $\square$

3.6. Moduli spaces of stable varieties I

Let C be a smooth projective curve of genus $g \geq 2$. Then ω_C^r is very ample for $r \geq 3$ and any basis of its global sections gives an embedding

$$C \hookrightarrow \mathbb{P}^{r(2g-2)-g}.$$

The same holds for stable curves. Thus all stable curves of genus g appear in the Chow variety or Hilbert scheme of $\mathbb{P}^{5g-6}$. This makes it possible to construct the moduli space of curves of genus g as the quotient of an open subset of $\mathrm{Chow}(\mathbb{P}^{5g-6})$ or of $\mathrm{Hilb}(\mathbb{P}^{5g-6})$ by $\mathrm{Aut}(\mathbb{P}^{5g-6})$. In this approach to the moduli of curves we deal with 3 types of objects.

- Proper curves.
- Proper curves with a very ample line bundle.
- Proper curves with an embedding into a projective space.

If a proper curve C is stable then we can choose ω_C as the ample line bundle, but for stable pairs we choose (some multiple of) $K_C + \Delta$ and other situations may dictate different choices. Thus it is useful to understand the situation where the ample line bundle can be arbitrary.

If L is a very ample line bundle on C then any basis of its global sections gives an embedding $C \hookrightarrow \mathbb{P}^N$ where $N = h^0(C, L) - 1$. Different bases lead to different embeddings, but they differ from each other by the action of $\mathrm{Aut}(\mathbb{P}^N)$ only.

Conversely, given $C \hookrightarrow \mathbb{P}^N$, the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ gives a very ample line bundle $\mathcal{O}_C(1)$ on C.

We follow the same general path in higher dimensions. We define the functors that correspond to the above 3 set-ups, but first we need some comments about ampleness.

3.75 (Ampleness conditions). Let X be a proper scheme over a field k and L a line bundle on X. The most important positivity notion is *ampleness,* but in connection with projective geometry the notion of *very ampleness* seems more relevant. If L is ample then L^r is very ample for $r \gg 1$ and there are numerous Matsusaka-type theorems that give effective control of the smallest such r [**Mat72, LM75, KM83**]. In practice, this will not be a difficulty for us.

A problem with very ampleness is that it is not open in flat families (X_s, L_s). Thus one needs to consider stronger variants. The two most frequently needed additional conditions are the following.

(1) $H^i(X, L) = 0$ for $i > 0$.
(2) $H^0(X, L)$ generates the ring $\sum_{r \geq 0} H^0(X, L^r)$.

These are connected by the notion of Castelnuovo-Mumford regularity; see [**Laz04**, Sec.I.8] for details.

For our purposes the relevant issue is (1). Thus we say that a line bundle L is *strongly ample* if it is very ample and $H^i(X, L) = 0$ for $i > 0$.

Let $f : X \to S$ be a proper, flat morphism and L a line bundle on X. We say that L is *strongly f-ample* or *strongly ample over S* if L is strongly ample on the fibers. Equivalently, if $R^i f_* L = 0$ for $i > 0$ and L is f-very ample. Thus $f_* L$ is locally free and we get an embedding $X \hookrightarrow \mathbb{P}_S(f_* L)$.

Definition 3.76 (Main moduli functors). We define the most important moduli functors and their variants. We use calligraphic letters to denote the functor and roman letters to denote the corresponding stack or moduli space.

As we discussed above, in the polarized and embedded cases we always use a strong polarization or embedding. Thus we build this into the definitions and use a superscript ^{s} as a reminder.

(3.76.1) The functor of *stable varieties*, denoted by $\mathcal{SV}(*)$ associates to a reduced scheme S the isomorphism classes of all stable morphisms $f : X \to S$. Note that while defining $\mathcal{SV}$ for fields was easy (1.41), it took considerable work to define $\mathcal{SV}$ over reduced schemes (3.68) and we still have not defined it over arbitrary schemes. As a reminder, we use $\mathcal{SV}^{\mathrm{red}}$ to denote the restriction of $\mathcal{SV}$ to the category of reduced schemes. The corresponding moduli space is denoted by SV.

Let S be a connected, reduced scheme and $f : X \to S$ a stable morphism. By (3.68), the Hilbert function (3.65) $\chi_s(r) := \chi\big(X_s, \omega_{X_s}^{[r]}\big)$ is independent of $s \in S$. Thus the moduli stack of stable varieties decomposes as a disjoint union

$$\mathrm{SV} = \amalg_\chi \mathrm{SV}(\chi), \tag{3.76.1.a}$$

where $\mathcal{SV}(\chi)(*)$ is the functor of those families of stable varieties whose Hilbert function is χ and $\mathrm{SV}(\chi)$ is its moduli stack. It is thus sufficient to construct $\mathcal{SV}(\chi)$ for any given Hilbert function χ.

The most important numerical invariants of the Hilbert function are the dimension $n = \dim X \in \mathbb{N}$ and the volume $v = (K_X^n) \in \mathbb{Q}$. We will also use the decomposition

$$\mathrm{SV} = \amalg_{n,v} \mathrm{SV}(n,v). \tag{3.76.1.b}$$

(3.76.2) While our main interest is the functor $\mathcal{SV}$ and its moduli space SV, many related functors and spaces appear during the proofs and also in their own right. The most important one is the functor of *locally stable varieties*, denoted by $\mathcal{LSV}(*)$.

(3.76.3) The functor of *strongly polarized (locally) stable varieties* is denoted by $\mathcal{P}^s\mathcal{SV}(*)$ (resp. $\mathcal{P}^s\mathcal{LSV}(*)$). If k is an algebraically closed field then $\mathcal{P}^s\mathcal{SV}(\operatorname{Spec} k)$ (resp. $\mathcal{P}^s\mathcal{LSV}(\operatorname{Spec} k)$) is the set of isomorphism classes of pairs (X, L) where X is a (locally) stable, proper k-variety and L a strongly ample line bundle on X (3.75).

There seems to be only one sensible way to extend this definition to a functor over arbitrary schemes, but it takes some work; see (3.77) and (3.78) for details.

(3.76.4) A closely related variant is the functor of *strongly embedded (locally) stable varieties*, denoted by $\mathcal{E}^s\mathcal{SV}(*)$ (resp. $\mathcal{E}^s\mathcal{LSV}(*)$). It associates to a reduced scheme S the set of all subschemes $X \subset \mathbb{P}^N_S$ for which the coordinate projection $\pi_S : X \to S$ is a (locally) stable morphism and $\mathcal{O}_X(1)$ is strongly ample over S. Thus $\big(X, \mathcal{O}_X(1)\big) \to S$ is a polarized (locally) stable family.

We are also interested in those case when the polarization is given by the relative canonical sheaf.

(3.76.5) The functor of *m-canonically strongly polarized stable varieties*, denoted by $\mathcal{SV}_m(*)$ associates to a reduced scheme S the isomorphism classes of all stable morphisms $f : X \to S$ for which $\omega_{X/S}^{[m]}$ is locally free and strongly f-ample. Since being locally free and strongly relatively ample are open conditions, we get open substacks

$$\mathrm{SV}_m \subset \mathrm{SV} \quad \text{and} \quad \mathrm{SV}_m(\chi) \subset \mathrm{SV}(\chi). \tag{3.76.5.a}$$

Observe that if $m_1 \mid m_2$ then $\mathrm{SV}_{m_1} \subset \mathrm{SV}_{m_2}$,

$$\mathrm{SV} = \textstyle\bigcup_m \mathrm{SV}_m \quad \text{and} \quad \mathrm{SV}(\chi) = \textstyle\bigcup_m \mathrm{SV}_m(\chi). \tag{3.76.5.b}$$

(Note that its locally stable version does not make sense since $f : X \to S$ is stable iff it is locally stable and $\omega_{X/S}$ is f-ample.)

(3.76.6) The functor of *m-canonically strongly embedded stable varieties*, denoted by $\mathcal{CE}^s\mathcal{SV}_m(*)$ associates to a reduced scheme S the isomorphism classes of all closed subschemes $X \subset \mathbb{P}^N_S$ for which the coordinate projection $\pi_S : X \to S$ is a stable morphism and $\omega_{X_s}^{[m]} \cong \mathcal{O}_{X_s}(1)$ for every $s \in S$.

Next we construct the moduli space of stable varieties SV by first studying the moduli space of embedded schemes and the corresponding moduli stack of polarized schemes. The general theory works for varieties as well as schemes (with one extra condition), so we work out the general setting. Using (3.74) this gives the moduli space of embedded stable varieties and the moduli stack of polarized stable varieties. Finally using (3.61) we get the moduli space of canonically embedded stable varieties and the moduli space of stable varieties.

Moduli of polarized schemes.

We discuss the construction of the moduli stack of polarized schemes. The method is quite typical of the subject. First we parametrize objects with some of additional structure; in our case a choice of a basis in $H^0(X, L)$. Then we take quotient by $\mathrm{PGL}\big(H^0(X, L)\big)$ to get rid of the choice of the basis.

DEFINITION 3.77 (Polarizations). A *polarized scheme* is a pair (X, L) consisting of a projective scheme X plus an ample line bundle L on X.

A *polarized family of schemes* over a scheme S consist of a flat family of projective schemes $f : X \to S$ plus a relatively ample line bundle L on X. We are interested only in the relative behavior of L, thus two families (X, L) and (X, L') are considered equivalent if there is a line bundle M on S such that $L \cong L' \otimes f^*M$. There are some quite subtle issues with this in general [**Ray70**], but if S is reduced and $H^0(X_s, \mathcal{O}_{X_s}) \cong k(s)$ for every $s \in S$ (for example, the fibers of f are geometrically reduced and connected) then, by Grauert's theorem, $L \cong L' \otimes f^*M$ for some M iff $L|_{X_s} \cong L'|_{X_s}$ for every $s \in S$. (See (3.78) for further comments on this.)

For technical reasons it is more convenient do deal with the cases when, in addition, L is relatively very ample and $R^if_*L = 0$ for $i > 0$; this can always be achieved by replacing L with a high enough power L^m. We call such a polarization *strongly ample* or a *strong polarization.* Thus we let

$$S \mapsto \mathcal{P}^s\mathcal{S}ch^{\mathrm{zar}}(n, N)(S) \tag{3.77.1}$$

denote the functor of strongly polarized schemes that associates to a scheme S the equivalence classes of all $f : (X, L) \to S$ such that

(2) f is flat, proper, of pure relative dimension n,
(3) $H^0(X_s, \mathcal{O}_{X_s}) \cong k(s)$ for every $s \in S$,
(4) L is strongly f-ample (3.75) and
(5) f_*L is locally free of rank $N + 1$.

(Since L is flat over S, strong f-ampleness, that is, the vanishing of the R^if_*L, implies that f_*L is locally free.) The meaning of the superscript $^{\mathrm{zar}}$ is explained in the next paragraph.

It is frequently more convenient to fix not just $n = \dim X$ and $N = h^0(X, L)-1$ but the whole Hilbert polynomial $\chi(X, r) := \chi(X, L^r)$. This leads to the functor

$$S \mapsto \mathcal{P}^s\mathcal{S}ch^{\mathrm{zar}}(\chi)(S). \tag{3.77.6}$$

REMARK 3.78 (Pre-polarization). The above definition of polarization is geometrically clear but it does not have the sheaf property. In analogy with the notion of a presheaf, we could define a *pre-polarization* of a projective morphism $f : X \to S$ to consist of

(1) an open cover $\cup_i U_i \to S$ and
(2) relatively ample line bundles L_i on $X_i := X \times_S U_i$

such that, for every i, j, the restrictions of L_i and L_j to $X_{ij} := X \times_S U_i \times_S U_j$ are identified as in (3.77). (That is, there are line bundles M_{ij} on $U_i \times_S U_j$ such that $L_i|_{X_{ij}} \cong L_j|_{X_{ij}} \otimes f_{ij}^*M_{ij}$.)

Pre-polarizations form a pre-sheaf and the "right" notion of polarization should be a global section of the corresponding sheaf. The definition in (3.77) uses the Zariski topology. Later we see that, for the moduli space of varieties, it is most natural to use the étale topology. (For arbitrary polarized schemes one needs even finer topologies, see [**Ray70**].)

char 0 only??

A simple example to keep in mind is the following. Consider

$$X := \big(x^2+sy^2+tz^2=0\big) \subset \mathbb{P}^2_{xyz} \times \big(\mathbb{A}^2_{st} \setminus (st=0)\big),$$

with coordinate projection to $S := \mathbb{A}^2_{st} \setminus (st=0)\big)$. The fibers are all smooth conics. In the analytic or étale topology there is a pre-polarization whose restriction to each fiber is a degree 1 line bundle but there is no such line bundle on X. However, $\mathcal{O}_{\mathbb{P}^2}(1)$ gives a line bundle on X whose restriction to each fiber has degree 2.

The latter turns out to be true in general: a suitable power of a pre-polarization gives an actual polarization (**??**). So at the end this distinction does not matter much for us, but for the correct functorial notion, we need to define the functors

$$S \mapsto \mathcal{P}^s\mathcal{S}ch(n,N)(S) \quad \text{and} \quad S \mapsto \mathcal{P}^s\mathcal{S}ch(\chi)(S) \tag{3.78.3}$$

which is the sheafification of the Zariski version of the corresponding functors of polarized schemes (3.77.1) and (3.77.6) in the étale topology.

Definition 3.79 (Embedded schemes). Fix a base scheme B and a projective space $\mathbb{P}^N_B$ over it. Over the Hilbert scheme there is a universal family, hence we get

$$\operatorname{Univ}(\mathbb{P}^N_B) \subset \mathbb{P}^N_B \times \operatorname{Hilb}(\mathbb{P}^N_B), \tag{3.79.1}$$

and $\mathcal{O}_{\mathbb{P}^N}(1)$ gives a polarization of $\operatorname{Univ}(\mathbb{P}^N_B) \to \operatorname{Hilb}(\mathbb{P}^N_B)$. Let

$$\mathrm{E^sSch}(n,\mathbb{P}^N_B) \subset \operatorname{Hilb}(\mathbb{P}^N_B) \tag{3.79.2}$$

denote the open subset parametrizing embedded subschemes of pure dimension n that are linearly normal and satisfy the conditions (3.77.2–5). The universal family restricts to

$$\operatorname{Univ}(n,\mathbb{P}^N_B) \to \mathrm{E^sSch}(n,\mathbb{P}^N_B) \tag{3.79.3}$$

The corresponding functor $\mathcal{E}^s\mathcal{S}ch(n,\mathbb{P}^N_B)$ associates to a scheme $S \to B$ the set of all flat families of closed subschemes of pure dimension n of $\mathbb{P}^N_S$

$$f : \big(X \subset \mathbb{P}^N_S; \mathcal{O}_X(1)\big) \to S, \tag{3.79.4}$$

where $\mathcal{O}_X(1)$ is strongly ample. Together with linear normality the latter condition is equivalent to $R^i\pi_*\mathcal{O}_{\mathbb{P}^N}(1) \cong R^if_*\mathcal{O}_X(1)$ for $i \geq 0$, where $\pi : \mathbb{P}^N_S \to S$ is the natural projection.

Equivalently, we can view $\mathcal{E}^s\mathcal{S}ch(n,\mathbb{P}^N)$ as parametrizing objects

$$\Big(f : (X;L) \to S; \phi \in \operatorname{Isom}_S\big(\mathbb{P}_S(f_*L), \mathbb{P}^N_S\big)\Big) \tag{3.79.5}$$

consisting of a strongly polarized, flat families of purely n-dimensional schemes plus an isomorphism $\phi : \mathbb{P}_S(f_*L) \cong \mathbb{P}^N_S$. We call the latter a *projective framing* of f_*L or of L. We can summarise these discussions as follows.

As before, we can also fix the Hilbert polynomial χ of X and consider the subschemes

$$\mathrm{E^sSch}(\chi) \subset \operatorname{Hilb}(\mathbb{P}^N_B) \tag{3.79.6}$$

where $N = \chi(1) - 1$.

Proposition 3.80. *Let B be a scheme and fix n, N. Then*

$$\operatorname{Univ}(n,\mathbb{P}^N_B) \to \mathrm{E^sSch}(n,\mathbb{P}^N_B)$$

constructed in (3.79) represents the functor of polarized schemes with a projective framing. That is, for every scheme S over B, pull-back gives a one-to-one correspondence between

(1) $\operatorname{Mor}_B\bigl(S, \mathrm{E^sSch}(n, \mathbb{P}^N_B)\bigr)$ *and*
(2) *flat families of purely n-dimensional schemes with a strong polarization $f:(X;L)\to S$ such that $H^0(X_s,\mathcal{O}_{X_s})\cong k(s)$ for every $s\in S$ and f_*L is locally free of rank $N+1$, plus an isomorphism $\mathbb{P}_S(f_*L)\cong\mathbb{P}^N_S$.* □

3.81 (Boundedness conditions). The schemes $\mathrm{E^sSch}(n, \mathbb{P}^N_B)$ have infinitely many irreducible components since we have not fixed the Hilbert polynomial of X. Since the Hilbert polynomial is a locally constant function on $\mathrm{E^sSch}(n, \mathbb{P}^N_B)$, its level sets give a decomposition

$$\mathrm{E^sSch}(n, \mathbb{P}^N_B) = \amalg_\chi \mathrm{E^sSch}\bigl(\chi(*)\bigr), \tag{3.81.1}$$

where $N=\chi(1)-1$. By the theory of Hilbert schemes, the spaces $\mathrm{E^sSch}\bigl(\chi\bigr)$ are quasiprojective, though usually nonprojective, reducible and disconnected; see [**Gro62a**], [**Kol96**, Chap.I] or [**Ser06**].

The general correspondence between the moduli of polarized varieties and the moduli of embedded varieties (**??**?) gives now the following.

Corollary 3.82. *Let B be a scheme and fix n, N. Then the stack*

$$\bigl[\mathrm{E^sSch}(n, \mathbb{P}^N_B)/\mathrm{PGL}_{N+1}(\mathcal{O}_B)\bigr]$$

represents the functor $\mathcal{P}^s\mathcal{S}ch(n,N)$ defined in (3.78.3). □

The spaces $\mathrm{E^sSch}(n, \mathbb{P}^N_B)$ parametrize rather complicated subschemes. Our main interest is in stable families. Applying (3.82) to the universal family over $\mathrm{E^sSch}(n, \mathbb{P}^N_B)$ gives the following.

Corollary 3.83. *Let B be a scheme and fix n, N. Then $\mathcal{E}^s\mathcal{SV}^{\mathrm{red}}(n, \mathbb{P}^N_B)(*)$ is represented by a locally closed partial decomposition*

$$j : \mathrm{E^sSV^{red}}(n, \mathbb{P}^N_B) \to \mathrm{E^sSch}(n, \mathbb{P}^N_B). \quad \square$$

The stacks $\bigl[\mathrm{E^sSV^{red}}(n, \mathbb{P}^N_B)/\mathrm{PGL}_{N+1}(\mathcal{O}_B)\bigr]$ parametrize pairs (X,L) where X is a stable variety and L a strongly ample line bundle. We aim to parametrize stable varieties, so we need to make a canonical choice for L. For curves this was given by $L:=\omega_C^3$, but in higher dimensions we run into a problem.

3.84 (Canonical polarization). In higher dimensions we aim to follow the method outlined for curves at the beginning of the section.

This approach works well for canonical models of surfaces of general type. If S is such a canonical model then ω_S is an ample line budle and ω_S^r is very ample for $r\geq 5$ by [**Bom73, Eke88**] Thus again we get an embedding of S into a projective space whose dimension depends only on the coefficients of the Hilbert polynomial $\chi(\omega_S^r)$, namely (K_S^2) and $\chi(\mathcal{O}_S)$.

The situation is, however, more complicated for stable surfaces. These can have singularities where ω_S is not locally free. Even worse, for any $m\in\mathbb{N}$ there are stable surfaces S_m such that $\omega_{S_m}^{[m]}$ is not locally free at some point $x_m\in S_m$. Thus every section of $\omega_{S_m}^{[m]}$ vanishes at x_m and $H^0\bigl(X,\omega_{S_m}^{[m]}\bigr)$ does not define an embedding of S_m.

In higher dimensions, even canonical models of varieties of general type can have singularities where $\omega_X^{[m]}$ is not locally free.

Here we skirt this problem by fixing $m>0$ and aiming to construct a moduli space for those stable varieties for which $\omega_S^{[m]}$ is locally free and strongly ample.

Then we need to show that if m is sufficiently divisible (depending on other numerical invariants), then $\omega_S^{[m]}$ works for all stable varieties.

Fix a Hilbert polynomial h. We have a universal family $\operatorname{Univ}(h) \to \mathrm{E^sSV^{red}}(h)$ paranetrizing strongly embedded stable varieties with Hilbert polynomial h. On $\operatorname{Univ}(h)$ we have the ample line bundle $\mathcal{O}(1)$ and the mostly flat divisorial sheaf $\omega^{[m]}$, where ω denotes the relative dualizing sheaf of $\operatorname{Univ}(h) \to \mathrm{E^sSV^{red}}(h)$. We can next apply (3.61) to this setting to obtain the following.

COROLLARY 3.85. *Let B be a scheme. Fix $\chi : \mathbb{Z} \to \mathbb{Z}$, $m \in \mathbb{N}$ and set $h(t) := \chi(mt)$. Then $\mathcal{CE}^s\mathcal{SV}_m^{\mathrm{red}}(\chi)(*)$ is represented by a locally closed subscheme*

$$\mathrm{CE^sSV}_m^{\mathrm{red}}(\chi) \hookrightarrow \mathrm{E^sSV^{red}}(h).$$

Therefore the quotient stack

$$\big[\mathrm{CE^sSV}_m^{\mathrm{red}}(h)/\mathrm{PGL}_{N+1}(\mathcal{O}_B)\big], \quad \textit{where} \quad N = \chi(m) - 1,$$

represents the functor $\mathcal{SV}_m^{\mathrm{red}}(\chi)$ defined in (3.76). □

We can now combine (3.85) and (3.76.5.b) with the results of Section 2.4 to obtain the following restatement of (3.3).

THEOREM 3.86. *Let B be a scheme over a field of charactertsic 0 and $\chi : \mathbb{Z} \to \mathbb{Z}$ a function. Then $\mathcal{SV}^{\mathrm{red}}(\chi)(*)$, the functor if stable families with Hilbert function χ over reduced schemes, has a coarse moduli space $\mathrm{SV}(\chi)^{\mathrm{red}} \to B$ which is an algebraic space. Furthermore*

1. $\mathrm{SV^{red}}(\chi)$ *is separated,*
2. $\mathrm{SV^{red}}(\chi)$ *satisfies the valuative criterion of properness and*
3. $\mathrm{SV^{red}}(\chi)$ *is the directed union of its open subspaces $\mathrm{SV}_m^{\mathrm{red}}(\chi)$ which are of finite type over B.* □

COMPLEMENT 3.87. *We see later that in fact*

1. $\mathrm{SV^{red}}(\chi) \to B$ *is projective,*
2. $\mathrm{SV^{red}}(\chi) = \mathrm{SV}_m(\chi)$ *for some m depending on χ and*
3. $\mathrm{SV^{red}}(\chi) = \mathrm{red}\big(\mathrm{SV}(\chi)\big)$.

CHAPTER 4

Families over reduced base schemes

So far we have identified stable pairs (X, Δ) as the basic objects of our moduli problem, defined stable and locally stable families of pairs over 1-dimensional regular schemes in Chapter 2 and in Chapter 3 we treated families of varieties over reduced base schemes. Here we unite the two by discussing stable and locally stable families over reduced base schemes. Some of the final results apply only over seminormal base schemes.

After stating the main results in Section 4.1 we give a series of examples in Section 4.2. The technical core of the chapter is the treatment of various notions of families of divisors given in Section 4.3. The behavior of generically $\mathbb{Q}$-Cartier divisors is studied in Section 4.4.

In Section 4.5 we finally define stable and locally stable families over reduced base schemes and prove that local stability is a representable property.

The moduli space of polarized schemes marked with divisors is constructed in Section 4.6.

In Section 4.7 we bring all of these results together to construct the seminormal moduli space of stable pairs. The proofs are worked out for excellent base schemes over a field of characteristic 0. The main results should all hold over positive and mixed characteristic bases, but very few of the proofs apply in general.

Families over a smooth base scheme are especially well behaved; their properties are discussed in the short Section 4.8.

Assumptions. In the foundational Sections 4.1–4.6 we work with arbitrary schemes, but, for the applications to stable morphisms presented in Sections 4.5–4.8, we need to assume that the base scheme is over a field of characteristic 0.

4.1. Statement of the main results

In the study of locally stable families of pairs over reduced base schemes the key step is to give the "correct" definition for the divisor component for families of pairs.

TEMPORARY DEFINITION 4.1. A *family of pairs* (with $\mathbb{Z}$-coefficients) of dimension n over a reduced scheme is an object

$$f : (X, D) \to S \tag{4.1.1}$$

consisting of a morphism of schemes $f : X \to S$ and an effective "divisor" D satisfying the following properties.

4.1.2 (Flatness for X). *The morphism $f : X \to S$ is flat, of pure relative dimension n and with geometrically reduced fibers.* This is the expected condition from the point of view of moduli theory, following the Principles (3.12) and (3.13). (Note, however, that (S, Δ) slc does not imply that X is slc, so maybe we are just lucky that this is the right condition. Later we will consider some cases where f is assumed to be flat only outside a codimension ≥ 2 set on each fiber.)

4.1.3 (Equidimensionality for $\operatorname{Supp} D$). *The nonempty fibers of* $\operatorname{Supp} D \to S$ *have pure dimension* $n-1$. This implies that every irreducible component of $\operatorname{Supp} D$ dominates an irreducible component of S and $\operatorname{Supp} D$ does not contain any irreducible component of any fiber of f. If S is normal then this condition holds iff $\operatorname{Supp} D \to S$ has pure relative dimension $n-1$ by (3.34.2), but in general our assumption is weaker. We noted in (2.39) that $D \to S$ need not be flat for locally stable families. So we start with the above weak assumption and strengthen it later as needed.

So far we have not said what a "divisor" is. Working on a normal variety X, by an effective "divisor" D we usually mean either a Weil divisor or a divisorial subscheme, that is, a pure, codimension 1 subscheme. The two versions are equivalent since X is regular at the generic points of D; see (4.16) for details. If (X, Δ) is an slc pair, then X is smooth at all generic points of $\operatorname{Supp} \Delta$. So if D is an effective divisor supported on $\operatorname{Supp} \Delta$, the 2 viewpoints are again interchangeable.

It turns out that such generic smoothness is a crucial condition technically and it is very hard to do anything without it. So we make it part of the definition for families of pairs.

4.1.4 (Generic smoothness along D). *The morphism f is smooth at generic points of* $X_s \cap \operatorname{Supp} D$ *for every* $s \in S$. Equivalently, for each $s \in S$, none of the irreducible components of $X_s \cap \operatorname{Supp} D$ is contained in $\operatorname{Sing}(X_s)$.

This means that from now on we can identify effective Weil divisors with divisorial subschemes. The usual notation uses Weil divisors.

After these preliminary, mostly obvious assumptions, now we come to the heart of the matter.

4.1.5 (Fibers are well defined). *For every geometric point $\tau : s \to S$, there is a "sensible" way to define the fiber (X_s, D_s).*

More generally, we would like the notion of families of pairs to give a functor, so for any morphism $g : W \to S$ we need to define the pulled-back family. We have a fiber product diagram

$$\begin{array}{ccc} X \times_S W & \stackrel{g_X}{\to} & X \\ f_W \downarrow & & \downarrow f \\ W & \stackrel{g}{\to} & S. \end{array} \tag{4.1.5.a}$$

It is clear that we should take $X_W := X \times_S W$ with morphism $f_W : X_W \to W$. The definition of D_W is more subtle since pull-backs of Weil divisors can not be defined in general.

The most naive definition of the *divisorial pull-back* is the following. Let $Z \subset X$ be a subscheme and $h : Y \to X$ a morphism. First take the scheme theoretic inverse image $h^{-1}(Z)$, which is a subscheme of Y, and then consider either the divisorial subscheme $\operatorname{Div}\bigl(h^{-1}(Z)\bigr)$ or the Weil divisor $\operatorname{Weil}\bigl(h^{-1}(Z)\bigr)$ associated to it (4.16.5). (These 2 are equivalent if Y is regular at all codimension 1 generic points of $h^{-1}(Z)$.) We can thus start with D, view it as a divisorial subscheme $D \subset X$ and then set

$$D_W := g^{[*]}(D) := \operatorname{Div}\bigl(g_X^{-1}(D)\bigr) \quad \text{or} \quad \operatorname{Weil}\bigl(g_X^{-1}(D)\bigr). \tag{4.1.5.b}$$

Note that condition (4.1.4) is crucial here in identifying the two versions with each other.

Warning. Note that, in general, $\mathcal{O}_{D_W} \neq g_X^*\mathcal{O}_D$ and $\mathcal{O}_{X_W}(-D_W) \neq g_X^*\mathcal{O}_X(-D)$, so when the scheme structure is crucial, we (aim to) carefully distinguish these objects. Divisorial pull-back does not preserve linear equivalence, it is not even additive.

4.1.6 (Well defined families of pairs I). *We say that* $f : (X, D) \to S$ *is a* well defined family *if it satisfies the assumptions (4.1.2–4) and the divisorial pull-back defined in (4.1.5.b) is a functor for reduced schemes. That is*

$$h^{[*]}\bigl(g^{[*]}(D)\bigr) = (g \circ h)^{[*]}(D) \tag{4.1.6.a}$$

for all morphisms of reduced schemes $h : T \to W$ *and* $g : W \to S$.

In any concrete situation the conditions (4.1.2–4) should be easy to check but (4.1.6) requires computing $g^{[*]}(D)$ for all morphisms $W \to S$. It turns out that (4.1.6) is automatic in many cases and it is frequently easy to check. Over normal base schemes we have the following, which is an immediate consequence of (4.21).

THEOREM 4.2. *Let* $f : (X, D) \to S$ *be a family of pairs satisfying the conditions (4.1.2–4). If* S *is normal then it is a well defined family of pairs. That is, (4.1.6) also holds.*

Over non-normal bases the situation is more complicated. First we show that (4.1.6) is equivalent to several other natural conditions. The common theme is that we need to understand only the codimension 1 behavior of $f : (X, D) \to S$. These results are proved in (4.26) and (4.28).

THEOREM 4.3. *Let* $f : (X, D) \to S$ *be a family of pairs satisfying the conditions (4.1.2–4) over a reduced scheme* S. *Viewing* D *as a divisorial subscheme, the following are equivalent.*

(1) *The family is well defined (4.1.6).*
(2) D *is a Cartier divisor on* X, *locally at the generic points of* $X_s \cap \operatorname{Supp} D$ *for every* $s \in S$.
(3) $D \to S$ *is flat at the generic points of* $X_s \cap \operatorname{Supp} D$ *for every* $s \in S$.

Furthermore, if S *is seminormal then these are further equivalent to*

(4) D *is a well defined family of Weil divisors that satisfies the field of definition condition (3.19).*

Next we turn to the case that we are really interested in, when the boundary Δ is a $\mathbb{Q}$ or $\mathbb{R}$-divisor. Since the divisorial pull-back is not additive, we can apply it to $\Delta = \sum_i a_i D^i$ in 2 basic ways.

DEFINITION 4.4 (Divisorial pull-back). Let $f : (X, \Delta) \to S$ be a family of pairs over a reduced scheme S satisfying the conditions (4.1.2–4), where $\Delta = \sum a_i D^i$ is an effective Weil $\mathbb{Q}$-divisor and the D^i are irreducible Weil divisors. Let $g : W \to S$ be a morphism from a reduced scheme W to S. There are several ways to define the *divisorial pull-back* of Δ.

4.4.1 (Component-wise definition). For each D^i set $D^i_W := \operatorname{Weil}\bigl(g_X^{-1}(D^i)\bigr)$. It is a sum of irreducible Weil divisors $D^i_W = \sum_j c_{ij} D^{ij}_W$ and then the pulled-back family can be defined as

$$g^{[*]}_{cw}(X, \Delta) := \bigl(X_W, \Delta_W := \textstyle\sum_{ij} a_i c_{ij} D^{ij}_W\bigr).$$

Over a normal base this definition works, essentially by (4.2). However, otherwise it frequently gives the "wrong" fiber, and, in most cases, the following variant works better.

4.4.2 (Common denominator definition). Choose a common denominator N for the numbers a_i. Then $N\Delta$ is an effective $\mathbb{Z}$-divisor. There is thus a unique divisorial subscheme $D_N \subset X$ such that $N\Delta = \operatorname{Weil}(D_N)$. Then the pulled-back family can be defined as

$$g_N^{[*]}(X, \Delta) := \big(X_W, \Delta_W := \tfrac{1}{N} \operatorname{Weil}\big(g_X^{-1}(D_N)\big)\big).$$

If g is flat on a dense open subset of W then (4.4.1–2) give the same pull-back, but otherwise, already when the base scheme S is a reduced curve, everything can go wrong with these definitions. That is, they differ from each other, (4.4.2) does depend on the choice of N and neither one is functorial in general; see Section 4.2 for such examples.

We can now formalize the functoriality condition.

4.4.3 (Well defined families of pairs II). *Let $f : \big(X, \Delta\big) \to S$ be a family of pairs that satisfies the assumptions (4.1.2–4). Let $g \mapsto g^{[*]}$ denote one of the pull-back constructions defined in (4.4.1–2). Then $f : \big(X, \Delta\big) \to S$ is called a* well defined family *with pull-back $g^{[*]}$ if*

$$h^{[*]}\big(g^{[*]}(X, \Delta)\big) = (g \circ h)^{[*]}(X, \Delta) \tag{4.4.3.a}$$

for all morphisms of reduced schemes $h : T \to W$ and $g : W \to S$.

Note that this definition gives several flavors of "well defined families," depending on whether we use (4.4.1) or (4.4.2). In the latter case the choice of N is also an issue. For now we leave the precise choice open.

The next result gives necessary and sufficient criteria by comparing the fibers over geometric points $\tau : s \to S$ with the fibers over geometric points of the normalization $\bar{S} \to S$ for all possible liftings $\bar{\tau} : s \to \bar{S}$ of τ.

THEOREM 4.5. *Let S be a reduced, excellent scheme and $f : (X, \Delta) \to S$ a projective family of pairs satisfying the assumptions (4.1.2–4). Then the family is well defined—that is, (4.4.3) holds—in the following cases.*

(1) *S is normal. Moreover, in this case both definitions (4.4.1–2) give the correct pull-back; we denote it by $g^{[*]}(X, \Delta)$.*

Otherwise let $\bar{S} \to S$ be the normalization and $\bar{f} : \big(\bar{X}, \bar{\Delta}\big) \to S$ the corresponding family. Note that $\bar{g}^{[*]}(X_{\bar{S}}, \Delta_{\bar{S}})$ is defined for every lifting $\bar{g} : W \to \bar{S}$ by (1).

(2) *S is weakly normal (3.29) and the fiber $\bar{\tau}^{[*]}(X_{\bar{S}}, \Delta_{\bar{S}})$ is independent of the lifting $\bar{\tau} : s \to \bar{S}$ for every geometric point $\tau : s \to S$. Moreover, (4.4.2) gives the correct pull-back, independent of N, but (4.4.1) need not.*

(3) *S is a reduced scheme over a field of characteristic 0 and $\tau_N^{[*]}(X, \Delta) \cong \bar{\tau}^{[*]}(X_{\bar{S}}, \Delta_{\bar{S}})$ holds for every geometric point $\tau : s \to S$ and for every lifting $\bar{\tau} : s \to \bar{S}$. Moreover, (4.4.2) gives the correct pull-back, independent of N, but (4.4.1) need not.*

(4) *S is a reduced scheme and $\tau_N^{[*]}(X, \Delta) \cong \bar{\tau}^{[*]}(X_{\bar{S}}, \Delta_{\bar{S}})$ holds for every geometric point $\tau : s \to S$ and for every lifting $\bar{\tau} : s \to \bar{S}$. However, this condition depends on the largest power of* $\operatorname{char} k(s)$ *that divides N. Once*

a given value N yields well-defined pull-backs, every multiple of N gives the same pull-back.

The key step in the proof of part (1) is (4.21) while part (2) is established in (4.28). Part (3) is proved in (4.37) and (4.40). Part (4) follows from (4.26). Section 4.2 contains a series of examples which show that all parts of (4.5) are optimal.

We have defined stable and locally stable families over a DVR in (2.2), and being locally stable should be preserved by pull-back. We can thus define these notions in general by imposing the following valuative criterion.

TEMPORARY DEFINITION 4.6. Let S be reduced scheme over a field of characteristic 0 and $f:\big(X,\Delta\big)\to S$ a well defined family of pairs as in (4.4), using the common denominator variant for pull-back (4.4.2). Note that by (4.5.3), this is independent of the choice of the common denominator N.

Then $f:\big(X,\Delta\big)\to S$ is called *stable* (resp. *locally stable*) iff the family obtained by base change $f_T:\big(X_T,\Delta_T\big)\to T$ is stable as in (2.43) (resp. locally stable as in (2.2)) whenever T is the spectrum of a DVR and $T\to S$ a morphism.

Let now $f:\big(X,\Delta\big)\to S$ be a family of pairs. It turns out that, starting in relative dimension 3, the set of points

$$\big\{s\in S:\big(X_s,\Delta_s\big)\text{ is semi-log-canonical}\big\}$$

is neither open nor closed; see (3.72) for an example. Thus the strongest result one can hope for is the following.

THEOREM 4.7 (Local stability is representable). *Let S be a reduced, excellent scheme over a field of characteristic 0 and $f:\big(X,\Delta\big)\to S$ a well-defined, projective family of pairs using the common denominator variant for pull-back (4.4.2). Then there is a locally closed partial decomposition $j:S^{\mathrm{ls}}\to S$ such that the following holds.*

Let W be any reduced scheme and $q:W\to S$ a morphism. Then the family obtained by base change $f_W:\big(X_W,\Delta_W\big)\to W$ is locally stable iff q factors as $q:W\to S^{\mathrm{ls}}\to S$.

A stable morphism is locally stable and stability is an open condition for a locally stable morphism. Thus (4.7) implies the following.

COROLLARY 4.8 (Stability is representable). *Using the notation and assumptions as in (4.7), there is a locally closed partial decomposition $j:S^{\mathrm{stab}}\to S$ such that the following holds.*

Let W be any reduced scheme and $q:W\to S$ a morphism. Then the family obtained by base change $f_W:\big(X_W,\Delta_W\big)\to W$ is stable iff q factors as $q:W\to S^{\mathrm{stab}}\to S$. □

Next we turn to the moduli functor $\mathcal{SP}^{\mathrm{sn}}$ that associates to a seminormal scheme S the set of all stable families $f:(X,\Delta)\to S$, up-to isomorphism. (Here SP stands for stable pairs and the superscript $^{\mathrm{sn}}$ indicates that we work with seminormal schemes.) In order to get a moduli space of finite type, we fix the relative dimension n of the fibers, a common denominator m for all coefficients occurring in Δ and the volume $v=\mathrm{vol}(K_{X_s}+\Delta_s):=\big((K_{X_s}+\Delta_s)^n\big)$ of the fibers. This gives the subfunctor

$$\mathcal{SP}^{\mathrm{sn}}(n,m,v):\{\text{seminormal }S\text{-schemes}\}\to\{\text{sets}\}.$$

By [**HMX14**], there is an $M = M(n,m,v)$ such that $M(K_{X_s}+\Delta_s)$ is Cartier for every $(X_s,\Delta_s)\in \mathcal{SP}^{\rm sn}(n,m,v)(\text{point})$, but for now we just add M as a new constraint that we suppress in the notation. We can now state the second main theorem of this Chapter.

THEOREM 4.9 (Existence of seminormal moduli spaces). *Let S be an excellent base scheme of characteristic 0 and fix n, m, v. Then the functor $\mathcal{SP}^{\rm sn}(n,m,v)$ has a coarse moduli space*

$$\mathrm{SP}^{\rm sn}(n,m,v)\to S$$

that is a seminormal scheme, whose irreducible components are proper over S.

Moreover—though this can be made precise only later—the space $\mathrm{SP}^{\rm sn}(n,m,v)$ is the seminormalization of the "true" moduli space $\mathrm{SP}(n,m,v)$ of stable pairs.

4.2. Examples

We start with a series of examples related to (4.5).

EXAMPLE 4.10. Let $S=(xy=0)\subset\mathbb{A}^2$ and $X=(xy=0)\subset\mathbb{A}^3$. Consider the divisors $D_x:=\big(y=z-1=0\big)$ and $D_y:=\big(x=z+1=0\big)$. We get a family

$$f:\big(X, D_x+D_y\big)\to S \tag{4.10.1}$$

that satisfies the assumptions (4.1.2–4).

We compute the "fiber" of the above family over the origin in 3 different ways and get 3 different results.

First restrict the family to the x-axis. The pull back of X becomes the plane $\mathbb{A}^2_{xz}$. The divisor D_x pulls back to $(z-1=0)$ but the pull back of the ideal sheaf of D_y is the maximal ideal $(x, z+1)$. It has no divisorial part, so restriction to the x-axis gives the pair

$$\big(\mathbb{A}^2_{xz}, (z-1=0)\big)\to\mathbb{A}^1_x. \tag{4.10.2}$$

Similarly, restriction to the y-axis gives the pair

$$\big(\mathbb{A}^2_{yz}, (z+1=0)\big)\to\mathbb{A}^1_y. \tag{4.10.3}$$

If we restrict these to the origin, we get

$$\big(\mathbb{A}^1_{z}, (z-1=0)\big)\quad\text{and}\quad\big(\mathbb{A}^1_{z}, (z+1=0)\big). \tag{4.10.4}$$

Finally, if we restrict to the origin of S in one step then we get the pair

$$\big(\mathbb{A}^1_{z}, (z-1=0)+(z+1=0)\big). \tag{4.10.5}$$

Thus we have 3 different pairs in (4.10.4–5) that can claim to be the fiber of (4.10.1) over the origin.

In the above example the problem is visibly set-theoretic, but there can be problems even when the set theory works out. For example, with X, S as above, consider the family

$$f:\big(X, \tfrac12 D'_x+\tfrac12 D'_y\big)\to S, \tag{4.10.6}$$

where $D'_x:=\big(y=(z-1)(z^2+2z+1-x)=0\big)$ and $D'_y:=\big(x=(z+1)(z^2-2z+1-y)=0\big)$. Computing as above we again get 3 different pairs as in (4.10.4–5) that can claim to be the fiber of (4.10.6) over the origin:

$$\big(\mathbb{A}^1_{z}, \tfrac12 P+Q\big), \big(\mathbb{A}^1_{z}, P+\tfrac12 Q\big)\quad\text{and}\quad\big(\mathbb{A}^1_{z}, P+Q\big), \tag{4.10.7}$$

where $P:=(z-1=0)$ and $Q:=(z+1=0)$.

In the above example the problem is that the restrictions of D'_x and D'_y to the z-axis have different multiplicities. The next example shows that even when the multiplicities are the same, there can be scheme theoretic problems.

EXAMPLE 4.11. Set $X = (x^2 - y^2 = u^2 - v^2) \subset \mathbb{A}^4$, $D = (x - u = y - v = 0) \cup (x + u = y + v = 0)$ and $f : (X, D) \to \mathbb{A}^2_{uv}$ the coordinate projection. The irreducible components of D intersect only at the origin and D is not Cartier there.

Let L_c be the line $(v = cu)$ for some $c \neq \pm 1$. Restricting the family to L_c we get $X_c = (x^2 - y^2 = (1 - c^2)u^2) \subset \mathbb{A}^3$ and the divisor becomes $D_c = (x - u = y - cu = 0) \cup (x + u = y + cu = 0)$. Observe that D_c is a Cartier divisor with defining equation $cx = y$. (Note that base change does not commute with union, so $D \times_{\mathbb{A}^2} L_c$ has an embedded point at the origin.)

Thus although D is not Cartier at the origin, after base change to a general line we get a Cartier divisor. For all of these base changes, D_c has multiplicity 2 at the origin. However, the origin is a singular point of the fiber, and if we restrict D_c to the fiber over the origin, the resulting scheme structure varies with c.

This would be a very difficult problem to deal with, but for a stable pair (X, Δ) we are in a better situation since the irreducible components of Δ are not contained in $\operatorname{Sing} X$.

EXAMPLE 4.12. Let B be a smooth projective curve of genus ≥ 1 with an involution σ and $b_1, b_2 \in B$ a pair of points interchanged by σ. Let C' be another smooth curve with two points $c'_1, c'_2 \in C'$. Start with the trivial family $(B \times C', \{b_1\} \times C' + \{b_1\} \times C') \to C'$ and then identify $c'_1 \sim c'_2$ and $(b, c'_1) \sim (\sigma(b), c'_2)$ for every $b \in B$. We get an étale locally trivial stable morphism $(S, D_1 + D_2) \to C$. Here C is a nodal curve with node $\tau : \{c\} \to C$. The fiber over the node is $\big(B, [b_1] + [b_2]\big)$.

However, the fiber of each D_i over c is $[b_1] + [b_2]$, hence the component-wise pull-back (4.4.1) is $\tau^{[*]}_{cw} = \big(B, 2[b_1] + 2[b_2]\big)$.

EXAMPLE 4.13. Set $C := (xy(x - y) = 0) \subset \mathbb{A}^2_{xy}$ and $X := (xy(x - y) = 0) \subset \mathbb{A}^3_{xyz}$. For any $c \in k$ consider the divisor

$$D_c := (x = z = 0) + (y = z = 0) + (x - y = z - cx = 0).$$

The pull-back of D_c to any of the irreducible components of X is Cartier, it intersects the central fiber at the origin of the z-axis and with multiplicity 1. Nonetheless, we claim that D_c is Cartier only for $c = 0$.

Indeed, assume that $h(x, y, z) = 0$ is a local equation of D_c. Then $h(x, 0, z) = 0$ is a local equation of the x-axis and $h(0, y, z) = 0$ is a local equation of the y-axis. Thus $h = az + (\text{higher terms})$. Restricting to the $(x - y = 0)$ plane we get that $c = 0$.

Note also that if $\operatorname{char} k = 0$ and $c \neq 0$ then no multiple of D_c is a Cartier divisor. To see this note that if $f(x, y, z) = 0$ is a local defining equation of mD_c on X then $\partial^{m-1} f / \partial z^{m-1}$ vanishes on D_c. Its restriction to the z-axis vanishes at the origin with multiplicity 1. We proved above that this is not possible.

The situation is different if $\operatorname{char} k > 0$, see (4.41).

EXAMPLE 4.14. Consider the cusp $C := (x^2 = y^3) \subset \mathbb{A}^2_{xy}$ and the trivial curve family $Y := C \times \mathbb{A}^1_z \to C$. Let $D \subset Y$ be the Cartier divisor given by the equation $y = z^2$. Then $D \to C$ is flat of degree 2. Furthermore, D is reducible with irreducible components $D^{\pm} :=$ image of $t \mapsto (t^3, t^2, \pm t)$.

Note that $D^{\pm} \cong \mathbb{A}^1_t$ and the projections $D^{\pm} \to C$ corresponds to the ring extension $k[t^3, t^2] \hookrightarrow k[t]$. Thus the projections $D^{\pm} \to C$ are not flat and the fiber of $D^{\pm} \to C$ over the origin has length 2.

Thus if we compute the fiber of $D = D^+ \cup D^- \to C$ over the origin $(0,0) \in C$ using the common denominator $N = 1$ as in (4.4.2), then we get the point $(0,0,0)$ with multiplicity 2. However, if we compute the fiber component-wise (4.4.2) then we get the point $(0,0,0)$ with multiplicity 4.

Thus $(Y, D^+) \to C$ is not stable but it becomes stable after pull-back to C^n.

Arguing as in (4.13) shows that the $D^{\pm}$ are not $\mathbb{Q}$-Cartier in characteristic 0. The situation is again more complicated if $\operatorname{char} k > 0$, see (4.42).

The next examples discuss the variation of the $\mathbb{Q}$-Cartier property in families of divisors. We discuss some positive results in Section 4.5.

EXAMPLE 4.15. Let $C \subset \mathbb{P}^2$ be a smooth cubic curve and $S_C \subset \mathbb{P}^3$ the cone over it. For $p \in C$ let $L_p \subset S_C$ denote the ruling over p. Note that L_p is $\mathbb{Q}$-Cartier iff p is a torsion point, that is, $3m[p] \sim \mathcal{O}_C(m)$ for some $m > 0$. The latter is a countable dense subset of the moduli space of the lines $\operatorname{Chow}_{1,1}(S_C) \cong C$.

In the above example the surface is not $\mathbb{Q}$-factorial and the curve L_p is sometimes $\mathbb{Q}$-Cartier, sometimes not. Next we give a similar example of a flat family of lc surfaces $S \to B$ such that $\{b : S_b \text{ is } \mathbb{Q}\text{-factorial}\} \subset B$ is a countable set of points. Thus being $\mathbb{Q}$-factorial is not a constructible condition.

Let $C \subset \mathbb{P}^2$ be a smooth cubic curve. Pick 11 points $P_1, \dots .P_{11} \in C$ and set $P_{12} = -(P_1 + \cdots + P_{11})$. Then there is a quartic curve D such that $C \cap D = P_1 + \cdots + P_{12}$. Thus the linear system $\left|\mathcal{O}_{\mathbb{P}^2}(4)(-P_1 - \cdots - P_{12})\right|$ blows up the points P_i and contracts C. Its image is a degree 4 surface $S = S(P_1, \dots .P_{11})$ in $\mathbb{P}^3$ with a single simple elliptic singularity. If $C = (f_3(x,y,z) = 0)$ and $D = (f_4(x,y,z) = 0)$ then

$$S \cong \bigl(f_3(x,y,z)w + f_4(x,y,z) = 0\bigr) \subset \mathbb{P}^3.$$

At the point $(x = y = z = 0)$ the singularity of S is analytically isomorphic to the cone S_C and S is smooth elsewhere iff the points $P_1, \dots .P_{12}$ are distinct. If this holds then the class group of S is generated by the image L of a line in $\mathbb{P}^2$ and the images $E_1, \dots, E_{12}$ of the 12 exceptional curves. They satisfy a single relation $3L = E_1 + \cdots + E_{12}$. Note that E_i is $\mathbb{Q}$-Cartier iff P_i is a torsion point.

If we vary $P_1, \dots .P_{11} \in C$ we get a flat family of lc surfaces parametrized by

$$\pi : \mathbf{S} \to C^{11} \setminus (\text{diagonals}),$$

with universal divisors $\mathbf{E}_i \subset \mathbf{S}$. We see that

(1) $E_i(P_1, \dots .P_{11})$ is $\mathbb{Q}$-Cartier iff P_i is a torsion point and
(2) $S(P_1, \dots .P_{11})$ is $\mathbb{Q}$-factorial iff P_i is a torsion point for every i.

4.3. Families of divisors II

At least 3 different notions of effective divisors are commonly used in algebraic geometry and our discussions in Section 4.1 show that a 4th variant is also necessary.

4.16 (Four notions of divisors). Let X be an arbitrary scheme.

(1) An effective *Cartier divisor* is a subscheme $D \subset X$ such that, for every $x \in D$, the ideal sheaf of $\mathcal{O}_X(-D)$ is locally generated by a non-zerodivisor $s_x \in \mathcal{O}_{x,X}$, called a *local equation* of D.

(2) A *divisorial subscheme* is a subscheme $D \subset X$ such that $\mathcal{O}_D$ has no embedded points and $\operatorname{Supp} D$ has pure codimension 1 in X.
(3) A divisorial subscheme is called an effective *generically Cartier divisor* if it is Cartier at its generic points.
(4) A *Weil divisor* (in traditional terminology) is a formal, finite linear combination $D = \sum_i m_i D_i$ where $m_i \in \mathbb{Z}$ and the D_i are integral subschemes of codimension 1 in X. We say that D is effective if $m_i \geq 0$ for every i.

If A is an abelian group then an *Weil A-divisor* is a formal, finite linear combination $D = \sum_i a_i D_i$ where $a_i \in A$. We will only use the cases $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Thus Weil $\mathbb{Z}$-divisor = traditional Weil divisor; we use the terminology "Weil $\mathbb{Z}$-divisor" if the coefficient group is not clear. (A Weil $\mathbb{Z}$-divisor is called an integral Weil divisor by some authors, but the latter could also mean the Weil divisor corresponding to an integral subscheme of codimension 1.)

Note that usually divisorial subschemes and Weil divisors are used only when X is irreducible or at least pure dimensional, but the definition makes sense in general.

If X is smooth then the 4 variants are equivalent to each other, but in general they are different.

Usually we think of Cartier divisor as the most restrictive notion. If X is S_2 then every effective Cartier divisor is a divisorial subscheme, but this does not hold if X is not S_2. This is good to keep in mind but it will not be a problem for us.

Let $W \subset X$ be a closed subscheme. We can associate to it both a divisorial subscheme and a Weil divisor by the rules

$$\begin{array}{rcl} \operatorname{Div}(W) & := & \mathcal{O}_W/(\text{torsion in codimension} \geq 2) \quad \text{and} \\ \operatorname{Weil}(W) & := & \sum_i \operatorname{length}_{g_i}\bigl(\mathcal{O}_{g_i,W}\bigr) \cdot [D_i], \end{array} \tag{4.16.5}$$

where in the first case we take the quotient by the subsheaf of those sections whose support has codimension ≥ 2 in X and in the second case $D_i \subset \operatorname{Supp} W$ are the irreducible components of codimension 1 in X and $g_i \in D_i$ the generic points. In particular, this associates a Weil divisor to any effective Cartier divisor or divisorial subscheme.

Thus, if X is S_2 then we have the basic relations among effective divisors

$$\begin{pmatrix} \text{Cartier} \\ \text{divisors} \end{pmatrix} \subset \begin{pmatrix} \text{generically} \\ \text{Cartier divisors} \end{pmatrix} \subset \begin{pmatrix} \text{divisorial} \\ \text{subschemes} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{Weil} \\ \text{divisors} \end{pmatrix}.$$

Assume next that X is regular at a codimension 1 point $g \in X$. Then $\mathcal{O}_{g,X}$ is a DVR, hence an ideal in it is uniquely determined by its colength. Thus, if X is a normal scheme then we get the stronger relationships among effective divisors

$$\begin{pmatrix} \text{generically} \\ \text{Cartier divisors} \end{pmatrix} = \begin{pmatrix} \text{divisorial} \\ \text{subschemes} \end{pmatrix} = \begin{pmatrix} \text{Weil} \\ \text{divisors} \end{pmatrix}. \tag{4.16.6}$$

We are mainly interested in slc pairs (X, Δ), thus the underlying schemes X are deminormal but not normal. Fortunately, X is smooth at the generic points of Δ. Thus for our purposes we can always imagine that the identifications (4.16.6) hold.

Convention 4.16.7. Let X be a scheme and $W \subset X$ subscheme. Assume that X is regular at all 1-dimensional generic points of W. Then we will frequently identify $\operatorname{Div}(W)$, the divisorial subscheme associated to W and $\operatorname{Weil}(W)$, the Weil divisor associated to W and denote this common object by $[W]$.

We can thus usually harmlessly identify divisorial subschemes and Weil divisors. However—and this is one of the basic difficulties of the theory—it is quite problematic to keep the identification between *families* of divisorial subschemes and *families* of Weil divisors.

Corresponding to the 4 notions of divisors, there are 4 main notions of families of divisors. In order to avoid further complications, assume that we have a flat, pure-dimensional morphism $f : X \to S$ with S_2 fibers.

Relative Weil divisors.

DEFINITION 4.17. Let $f : X \to S$ a morphism whose fibers have pure dimension n. A Weil divisor $W = \sum m_i W_i$ is called a *relative Weil divisor* if the fibers of $f|_{W_i} : W_i \to f(W_i)$ have pure dimension $n - 1$ for every i.

We are interested in defining the divisorial fibers of $W \to S$. A typical example is (3.22), where the multiplicity of the scheme-theoretic fiber jumps over the origin. It is, however, quite natural to say that the "correct" fiber is the origin with multiplicity 2, the only problem we have is that scheme theory miscounts the multiplicity. The following theorem, proved in [**Kol96**, 3.17], says that this is indeed frequently the case. As with many results about Chow varieties, all the essential ideas are in [**HP47**, Chap.X].

THEOREM 4.18. *Let S be a normal scheme, $f : X \to S$ a projective morphism and $Z \subset X$ a closed subscheme such that $f|_Z : Z \to S$ has pure relative dimension m. Then there is a section $\sigma_Z : S \to \mathrm{Chow}_m(X/S)$ with the following properties.*

(1) *Let $g \in S$ be the generic point. Then $\sigma_Z(g) = [Z_g]$, the cycle associated to the generic fiber of $f|_Z : Z \to S$ as in (3.8).*
(2) $\mathrm{Supp}\big(\sigma_Z(s)\big) = \mathrm{Supp}(Z_s)$ *for every $s \in S$.*
(3) $\sigma_Z(s) = [Z_s]$ *if $f|_Z$ is flat at all generic points of Z_s.*
(4) $s \mapsto \big(\sigma_Z(s) \cdot L^m\big)$ *is a locally constant function of $s \in S$, for any line bundle L on X.*

Example (4.10) shows that (4.18) does not hold if S is only seminormal. The notion of well-defined families of algebraic cycles (3.19) is designed to avoid similar problems, leading to the definition of the Chow functor; see [**Kol96**, Sec.I.3–4] for details.

Flat families of divisorial subschemes.

Let $X \to S$ be a morphism and $D \subset X$ a subscheme. If $\mathrm{Supp}\, D$ does not contain any irreducible component of a fiber X_s, then $\mathcal{O}_{D\cap X_s}/(\text{torsion in codimension} \geq 2)$ is (the structure sheaf of) a divisorial subscheme of X_s. This notion, however, frequently does not have good continuity properties, as illustrated by (3.22).

We would like to have a notion of flat families of divisorial subschemes where both the structure sheaf $\mathcal{O}_D$ and the ideal sheaf $\mathcal{O}_X(-D)$ are well behaved. This seems possible only if $X \to S$ is well behaved, but then the two aspects turn out to be equivalent.

DEFINITION–LEMMA 4.19. Let $f : X \to S$ be a flat morphism of pure relative dimension n with S_2-fibers and $D \subset X$ a closed subscheme of relative dimension $n - 1$ over S. We say that $f|_D : D \to S$ is a *flat family of divisorial subschemes* if the following equivalent conditions hold.

(1) $f|_D : D \to S$ is flat with pure fibers of dimension $n-1$.
(2) $\mathcal{O}_X(-D)$ is flat over S with S_2 fibers.

Proof. We have a surjection $\mathcal{O}_X \to \mathcal{O}_D$ and if both of these sheaves are flat then so is the kernel $\mathcal{O}_X(-D)$. If the kernel is flat then $\mathcal{O}_{X_s}(-D_s) \cong \mathcal{O}_X(-D)|_{X_s}$ is also the kernel of $\mathcal{O}_{X_s} \to \mathcal{O}_{D_s}$. Since $\mathcal{O}_{X_s}$ is S_2, we see that $\mathcal{O}_{X_s}(-D_s)$ is S_2 iff $\mathcal{O}_{D_s}$ is pure of dimension $n-1$.

Conversely, assume (2). For any $T \to S$ the pull-back map $q_T^*\mathcal{O}_X(-D) \to q_T^*\mathcal{O}_X$ is an isomorphism over $X_T \setminus D_T$. Since $\mathcal{O}_X(-D)$ is flat with S_2 fibers, $q_T^*\mathcal{O}_X(-D)$ does not have any sections supported on D_T. Thus the pulled-back sequence

$$0 \to q_T^*\mathcal{O}_X(-D) \to q_T^*\mathcal{O}_X \to q_T^*\mathcal{O}_D \to 0$$

is exact. Therefore $\operatorname{Tor}_1^S(\mathcal{O}_T, \mathcal{O}_D) = 0$ hence $\mathcal{O}_D$ is flat over S and we already noted that then it has pure fibers of dimension $n-1$. □

Relative Cartier divisors.

Definition–Lemma 4.20. Let $f : X \to S$ be a morphism, $x \in X$ a point such that f is flat at x and set $s := f(x)$. A subscheme $D \subset X$ is a *relative Cartier divisor* at $x \in X$ if the following equivalent conditions hold.

(1) D is flat over S at x and $D_s := D|_{X_s}$ is a Cartier divisor on X_s at x.
(2) D is a Cartier divisor on X at x and a local equation $g_x \in \mathcal{O}_{x,X}$ of D restricts to a non-zerodivisor on the fiber X_s.
(3) D is a Cartier divisor on X at x and it does not contain any irreducible component of X_s that passes through x.

If these hold for all $x \in D$ then D is a *relative Cartier divisor.* If $f : X \to S$ is also proper then the functor of relative Cartier divisors is represented by an open subscheme of the Hilbert scheme $\operatorname{CDiv}(X/S) \subset \operatorname{Hilb}(X/S)$; see [**Kol96**, I.1.13] for the easy details.

If (2) holds then D is flat by (4.19). The other nontrivial claim is that (1) implies that D is a Cartier divisor on X at x. We may assume that $(x \in X)$ is local. A defining equation g_s of D_s lifts to an equation g of D. We have the exact sequence

$$0 \to I_D/(g) \to \mathcal{O}_X/(g) \to \mathcal{O}_D \to 0.$$

Here $\mathcal{O}_X/(g)$ and $\mathcal{O}_D$ are both flat, hence so is $I_D/(g)$. Restricting to X_s we get

$$0 \to \bigl(I_D/(g)\bigr)_s \to \mathcal{O}_{X_s}/(g_s) \xrightarrow{\cong} \mathcal{O}_{D_s} \to 0.$$

Thus $I_D/(g) = 0$ by the Nakayama lemma and so g is a defining equation of D. □

Relative Cartier divisors form a very well behaved class, but in applications we frequently have to handle 2 problems. It is not always easy to see which divisors are Cartier and we also need to deal with divisors that are not Cartier.

On a smooth variety every divisor is Cartier, thus if X itself is smooth then a divisor D is relatively Cartier iff its support does not contain any of the fibers. In the relative setting, we usually focus on properties of the morphism f. Thus we would like to prove similar results for smooth morphisms. The next result says that this is indeed true if S is normal. Note that by (4.24) normality is a necessary assumption.

THEOREM 4.21. *Let S be a normal scheme, $f : X \to S$ a smooth morphism and D a Weil divisor on X. Assume that D does not contain any irreducible component of a fiber. Then D is a Cartier divisor, hence a relative Cartier divisor.*

Proof. The question is local, so pick $x \in X$ and set $s = f(x)$.

We start with the special case when $k(x) = k(s)$ and f has relative dimension 1. Then $f|_D : D \to S$ is quasi-finite at x, so f is flat at x by (10.53). Thus D is a relative Cartier divisor at x by (4.20.1).

Next assume that $k(x) = k(s)$ still holds but f has relative dimension $n > 1$. Since f is smooth, over a neighborhood of x it can be written as a composite

$$f : (x, X) \xrightarrow{\tau} \big((0, s), \mathbb{A}^n_S\big) \xrightarrow{\pi} S,$$

where τ is étale and π is the structure projection. Composing with any of the coordinate projections we factor f as

$$f : (x, X) \xrightarrow{g} \big((0, s), \mathbb{A}^{n-1}_S\big) \to S,$$

where g is smooth of relative dimension 1. If D does not contain the fiber of g passing through x then D is a Cartier divisor by the already discussed 1-dimensional case.

We have an étale morphism $\tau_s : X_s \to \mathbb{A}^n_s$. If $k(s)$ is infinite and $L \subset \mathbb{A}^n_s$ a general line through the origin then $\pi_s^{-1}(L) \not\subset D_s$. Thus if we choose the projection $\mathbb{A}^n_S \to \mathbb{A}^{n-1}_S$ to have kernel L over s then the argument proves that D is a Cartier divisor at x.

If $k(s)$ is finite then consider the trivial lifting $f^{(1)} : X \times \mathbb{A}^1 \to S \times \mathbb{A}^1$. By the previous argument $D \times \mathbb{A}^1$ is a Cartier divisor at the generic point of $\{x\} \times \mathbb{A}^1$, hence D is a Cartier divisor at x by (4.22).

Finally (10.47) shows how to reduce the general case when $k(x) \neq k(s)$ to the special case where $k(x) = k(s)$ by a simple base change. $\square$

LEMMA 4.22. *Let $(R, m_R) \to (S, m_S)$ be a flat extension of local rings and $I_R \subset R$ an ideal. Then I_R is principal iff $I_R S$ is principal.*

Proof. One direction is clear. Conversely, assume that $I_R S$ is principal, thus $I_R S / m_S I_R S \cong S/m_s$. Let $r_1, \dots, r_n$ be generators of I_R. They also generate $I_R S$ hence at least one of them, say r_1, is not contained in $m_S I_R S$. Thus $(r_1) \subset I_R$ is a sub-ideal such that $r_1 S = I_R S$. Since $(R, m_R) \to (S, m_S)$ is faithfully flat, this implies that $(r_1) = I_R$. $\square$

As a consequence of (4.21), we obtain another variant of (3.18.1); see also (4.18).

COROLLARY 4.23. *Let S be a normal scheme, $f : X \to S$ a proper morphism of pure relative dimension n and $D \subset X$ a closed subscheme such that $g := f|_D : D \to S$ has pure relative dimension $n-1$. Assume furthermore that, for some $s \in S$, the morphism f is smooth at all generic points of D_s.*

Then $[D_s] = g^{[-1]}(s)$, the Cayley-Chow fiber of g over s (3.19).

Proof. By (4.21) g is flat at all generic points of D_s. Thus $[D_s] = g^{[-1]}(s)$ by (3.19.2). $\square$

EXAMPLE 4.24. We give 2 examples showing that in (4.21) we do need normality of S.

Set $S_n := \operatorname{Spec} k[x, y]/(xy)$ and $X_n = \operatorname{Spec} k[x, y, z]/(xy)$. Then (x, z) defines a Weil divisor which is not Cartier.

Set $S_c := \operatorname{Spec} k[x^2, x^3]$ and $X_c = \operatorname{Spec} k[x^2, x^3, y]$. Then $(y^2 - x^2, y^3 - x^3)$ defines a Weil divisor which is not Cartier.

Relative generically Cartier divisors.

DEFINITION 4.25. Let $f : X \to S$ be a morphism. A subscheme $D \subset X$ is a *relative, generically Cartier, effective divisor* or a *family of generically Cartier, effective divisors* over S if there is an open subset $U \subset X$ such that

(1) f is flat over U with S_2 fibers,
(2) $\operatorname{codim}_{X_s}(X_s \setminus U) \geq 2$ for every $s \in S$,
(3) $D|_U$ is a relative Cartier divisor (4.20) and
(4) D is the closure of $D|_U$.

If $U \subset X$ denotes the largest open set with these properties then $Z := X \setminus U$ is the *non-Cartier locus* of D.

Thus $\mathcal{O}_X(mD)$ is a mostly flat family of divisorial sheaves on X (3.51) for any $m \in \mathbb{Z}$. Conversely, if L is a mostly flat family of divisorial sheaves on X and h a global section of it that does not vanish on any irreducible component of any fiber then $(h = 0)$ is a family of generically Cartier, effective divisors over S.

Let $q : W \to S$ be any morphism. We have a fiber product diagram

$$\begin{array}{ccc} X_W & \stackrel{q_X}{\to} & X \\ f_W \downarrow & & \downarrow f \\ W & \stackrel{q}{\to} & S. \end{array} \tag{4.25.5}$$

Then $q_X^*\big(D|_U\big)$ is a well defined relative Cartier divisor on $U_T := q_X^{-1}(U)$; let $D_T \subset X_T$ denote its closure. It agrees with the divisorial pull-back of D defined in (4.1.5.b) Since the pull-back of Cartier divisors is functorial, this shows that a a family of relative, generically Cartier divisors is a well defined family of divisors.

The next result shows that the converse is also true.

THEOREM 4.26. *Let S be a reduced scheme and $f : (X, D) \to S$ a projective family of pairs satisfying the assumptions (4.1.2–4). Then the following are equivalent.*

(1) *The family is well defined (4.1.6) using the divisorial pull-back $g^{[*]}$.*
(2) *D is a relative, generically Cartier divisor on X.*
(3) *$g : D \to S$ is flat at generic points of D_s for every $s \in S$.*

Proof. All 3 conditions can be checked on a general relative hyperlane section of X; see (4.32), (4.31) and (10.46).

Thus we may assume that $X \to S$ has relative dimension 1, hence f is smooth along $\operatorname{Supp} D$. We view D as a divisorial subscheme of X. We show that all 3 conditions are equivalent to the following.

(4) The function $s \mapsto \dim_{k(s)} \mathcal{O}_{D_s}$ is locally constant on S.

Applying (4.27) to $f_*\mathcal{O}_D$ we see that (4) holds iff $\mathcal{O}_D$ is flat over S. By (4.20) the latter holds iff D a relative Cartier divisor. Thus (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

Let $\tau_s : s \to S$ be a geometric point. By construction, $\deg \tau_s^{[*]} D = \dim_{k(s)} \mathcal{O}_{D_s}$, thus $s \mapsto \deg \tau^{[*]}\Delta$ is locally constant iff (4) holds.

Finally, let T be the spectrum of a DVR and $h : T \to S$ a morphism that maps the closed point to $s \in S$ and the generic point to a generalization g of s. Then

$h^{[*]}D$ is flat over T of degree $\deg_{k(g)} \mathcal{O}_{D_g}$. Thus if $\bar{\tau}_s : s \to T$ is a lifting of τ then

$$\deg \bar{\tau}_s^{[*]} h^{[*]} D = \deg_{k(g)} \mathcal{O}_{D_g}.$$

This shows that $(X, D) \to S$ is a well defined family of 0-cycles iff $s \mapsto \deg \tau^{[*]} D$ is locally constant, proving that (1) $\Leftrightarrow$ (4). □

LEMMA 4.27. *A coherent sheaf F on a reduced scheme S is locally free iff $s \mapsto \dim_{k(s)} F_s$ is locally constant.* □

The following important result says that, over seminormal base schemes, the three equivalent notions of a "good" relative family of divisors discussed in (4.26) also coincide with the Cayley-Chow theoretic variant.

THEOREM 4.28. *Let S be a seminormal scheme, $f : X \to S$ a projective morphism of pure relative dimension n and $D \subset X$ a closed subscheme such that $g := f|_D : D \to S$ has pure relative dimension $n - 1$. Assume further that f is smooth at generic points of D_s for every $s \in S$. The following are equivalent.*

1. *D is a well defined family of Weil divisors that satisfies the field of definition condition (3.19).*
2. *D is relatively Cartier at general points of D_s for every $s \in S$.*

Proof. (2) $\Rightarrow$ (1) follows from (3.19.2).

In order to prove that (1) $\Rightarrow$ (2) we may assume that S is local. Then (4.31) allows us to pass to a general hypersurface section. This reduces everything to the case $n = 1$ and then f is smooth along D by assumption.

If S is normal then (4.21) shows that D is relative Cartier divisor and we are done. In the seminormal case essentially the same proof as in (4.21) works but we need to use different references. First (10.54.3) shows that D is flat over S and then D is relatively Cartier by (4.20.1). □

We are now ready to show that the universal family of Weil divisors is a relative, generically Cartier family, at least over the open set where we avoid bad singularities.

DEFINITION 4.29. Let $f : X \to S$ be a flat, projective morphism of pure relative dimension n. As in (3.21.3), $\pi : \mathrm{Univ}(X/S) \to \mathrm{WDiv}(X/S)$ denotes the universal family of relative Weil divisors. By our conventions, $\mathrm{WDiv}(X/S)$ is seminormal and it parametrizes pairs (X_s, D_s) where $s \in S$ is a point and $[D_s]$ is a Weil divisor on X_s.

Let $\mathrm{WDiv}^{\mathrm{gs}}(X/S) \subset \mathrm{WDiv}(X/S)$ be the set of pairs (X_s, D_s) such that X_s is smooth at all generic points of D_s (the superscript stands for generically smooth). Set $\mathrm{Univ}^{\mathrm{gs}}(X/S) := \pi^{-1}\big(\mathrm{WDiv}^{\mathrm{gs}}(X/S)\big)$.

The following consequence of (4.28) is a crucial ingredient in the construction of the moduli space of stable pairs.

COROLLARY 4.30. *Let $f : X \to S$ be a flat, projective morphism of pure relative dimension n. Then*

1. $\mathrm{WDiv}^{\mathrm{gs}}(X/S)$ *is an open subscheme of* $\mathrm{WDiv}(X/S)$,
2. $\pi : \mathrm{Univ}^{\mathrm{gs}}(X/S) \to \mathrm{WDiv}^{\mathrm{gs}}(X/S)$ *is a well defined family of cycles that satisfies the field of definition condition (3.21),*
3. $\mathrm{Univ}^{\mathrm{gs}}(X/S) \subset X \times_S \mathrm{WDiv}^{\mathrm{gs}}(X/S)$ *is a generically Cartier family of divisors over* $\mathrm{WDiv}^{\mathrm{gs}}(X/S)$ *and*

(4) *the scheme theoretic fibers of* $\operatorname{Univ}^{\mathrm{gs}}(X/S) \to \operatorname{WDiv}^{\mathrm{gs}}(X/S)$ *represent the Cayley-Chow fibers.*

Proof. The first claim follows from the upper semicontinuity of the fiber dimension of

$$\operatorname{Supp}(\operatorname{Univ}(X/S)) \cap \bigl(\operatorname{WDiv}(X/S) \times_S \operatorname{Sing}(f)\bigr) \to \operatorname{WDiv}(X/S)$$

and the second is a combination of (3.21) and (3.18.1.c), see also (3.18.6–7) for more details. The third is a special case of (4.28) while the last part follows from (3.19.2). □

We have used two Bertini-type results. The first is an immediate consequence of (10.46) and the second follows from (10.12.1).

PROPOSITION 4.31. *Let* $(0 \in S)$ *be a local scheme,* $X \subset \mathbb{P}^N_S$ *a quasi-projective* S*-scheme with fibers of pure dimension* ≥ 2 *and* $D \subset X$ *a relative divisorial subscheme. Then* D *is a generically Cartier family of divisors on* X *iff* $D|_H$ *is a generically Cartier family of divisors on* $X \cap H$ *for general* $H \in |\mathcal{O}_{\mathbb{P}^N_S}(1)|$. □

LEMMA 4.32. *Let* $f : \bigl(X, \Delta\bigr) \to S$ *be a family of pairs that satisfies the assumptions (4.1.2–4) and* X *excellent. For a morphism* $g : T \to S$ *let* $g^{[*]}$ *denote one of the pull-back constructions defined in (4.4.1–2). Then there are finitely many points* $\{x_i : i \in I\}$ *of* X *(depending on* X, D *and* g*) such that if* $H \subset X$ *is a relative Cartier divisor that does not contain any of the points* x_i *then*

$$\bigl(g^{[*]}(X, D)\bigr)|_{H_T} \cong g^{[*]}\bigl(H, D|_H\bigr)$$

where H_T *denotes the preimage of* H *in* $X \times_S T$. □

Representability for divisorial pull-backs.

Let $f : (X, D) \to S$ be a family of generically Cartier divisors . We study those morphisms $q : W \to S$ for which the divisorial pull-back D_W is flat or relatively Cartier. We prove that in both cases the corresponding functor is representable by a locally closed partial decomposition $S' \to S$. This, however, does not hold for $\mathbb{Q}$-Cartier divisorial pull-backs (4.15). To remedy this we introduce the notion of numerically $\mathbb{Q}$-Cartier divisors later in (4.52).

Since the distinction is important, in the remainder of this subsection we use D_s to denote the fiber of D over $s \in S$ and $[D_s] = \operatorname{pure}(D_s)$ to denote the pure fiber, as in (4.16.7). Thus $[D_s] = \operatorname{Div}(D_s)$ in the notation of (4.16.5).

The first result is another version of (3.53). See (9.56) for a common generalization of both.

THEOREM 4.33. *Let* S *be a reduced scheme,* $f : X \to S$ *a flat, projective morphism with* S_2 *fibers and* $D \subset X$ *a family of generically Cartier divisors. Then there is a locally closed decomposition (3.48)* $i : S^{\mathrm{dflat}} \to S$ *such that for every* S*-scheme* $q : W \to S$*, the divisorial pull-back* $f_W : (X_W, D_W) \to W$ *is a flat family of divisorial subschemes (4.19) iff* q *factors as* $q : W \to S^{\mathrm{dflat}} \to S$.

Remark 4.33.1. As in (3.53.1), note that under the above assumptions the subset of S

$$\bigl\{s : D \text{ is flat along } \operatorname{Supp} D_s\bigr\}$$

is open, but we want the "corrected" restrictions $[D_s]$ (4.16.5) to form a flat family.

Proof. First we claim that $s \mapsto \chi\bigl(X_s, \mathcal{O}_{[D_s]}(*)\bigr)$ is constructible and upper semicontinuous on S. For this we may replace S with its seminormalization, hence the claim follows from (3.56). Thus we get a locally closed decomposition $j : S' \to S$ such that $s \mapsto \chi\big(X_s, \mathcal{O}_{[D_s]}(*)\big)$ is locally constant on S'.

If $f_W : (X_W, D_W) \to W$ is a flat family of divisorial subschemes then $W \to S$ factors through $j : S' \to S$. Thus it remains to prove that $D' := j_X^{[*]} D \subset X' := X \times_S S'$ is a flat family of divisorial subschemes. The latter follows from (4.34). $\square$

PROPOSITION 4.34. *Let $f : X \to S$ be a flat, projective morphism with S_2 fibers and $D \subset X$ a family of generically Cartier divisors. Assume in addition that S is reduced and $\mathcal{O}_X(1)$ is relatively ample. The following are equivalent.*

1. *$f : (X, D) \to S$ is a flat family of divisorial subschemes,*
2. *$s \mapsto \chi\big(X_s, \mathcal{O}_{X_s}(-[D_s])(*)\big)$ is locally constant on S and*
3. *$s \mapsto \chi\big(X_s, \mathcal{O}_{[D_s]}(*)\big)$ is locally constant on S.*

Proof. The last two assertions are equivalent since

$$\chi\big(X_s, \mathcal{O}_{X_s}(-[D_s])(*)\big) = \chi\big(X_s, \mathcal{O}_{X_s}(*)\big) - \chi\big(X_s, \mathcal{O}_{[D_s]}(*)\big).$$

If (1) holds then the $\mathcal{O}_{[D_s]}$ are fibers of the flat sheaf $\mathcal{O}_D$, hence (1) $\Rightarrow$ (3).

To see the converse, we may as well assume that $\chi\big(X_s, \mathcal{O}_{[D_s]}(*)\big)$ is independent of $s \in S$, call it $p(*)$. Let $\mathrm{Hilb}_p(X/S)$ denote the Hilbert scheme of X/S parametrizing subschemes that are flat over S with Hilbert polynomial $p(*)$. Set

$$Z := \{[D_s] : s \in S\} \subset \mathrm{Hilb}_p(X/S).$$

We claim that Z is a closed subscheme. To see this note first that Z is a constructible subset of $\mathrm{Hilb}_p(X/S)$. Indeed, by (10.3) $D \to S$ is flat with S_2 fibers over an open subset $S^0 \subset S$. Thus $D_s = [D_s]$ for $s \in S^0$ and so $\mathcal{O}_D$ defines a section $S^0 \to \mathrm{Hilb}_p(X^0/S^0)$ whose image equals $Z \cap \mathrm{Hilb}_p(X^0/S^0)$, where $X^0 := f^{-1}(S^0)$. Noetherian induction now shows that Z is constructible.

A constructible subset is closed iff it is closed under specialization. This reduces our claim to showing that $Z \times_S T$ is closed whenever $q : T \to S$ is the spectrum of a DVR mapping to S. Set $B := q_X^{[*]}(D)$. In this case B is flat over T and the restriction map

$$r_0^D : \mathcal{O}_{B_0} \to \mathcal{O}_{[B_0]} \quad \text{is surjective.}$$

Let Q denote its kernel. We compute that

$$\begin{array}{rcl} p(*) = \chi\big(X_g, \mathcal{O}_{B_g}(*)\big) & = & \chi\big(X_0, \mathcal{O}_{B_0}(*)\big) \\ & = & \chi\big(X_0, \mathcal{O}_{[B_0]}(*)\big) + \chi\big(X_0, Q(*)\big) \\ & = & p(*) + \chi\big(X_0, Q(*)\big). \end{array}$$

Thus $\chi\big(X_0, Q(*)\big) \equiv 0$ and hence $Q = 0$. Therefore $Z \times_S T$ is the image of the section of $\mathrm{Hilb}_p(X/T) \to T$ defined by B, hence closed.

This shows that $Z \subset \mathrm{Hilb}_p(X/S)$ is a closed subscheme. Since $\mathrm{Hilb}_p(X/S) \to S$ is proper, so is its restriction $\tau : Z \to S$. We claim that τ is an isomorphism.

The universal family over $\mathrm{Hilb}_p(X/S)$ restricts to a subscheme $U_Z \subset X_Z$ that is flat over Z. Furthermore, U_Z is a subscheme of D_Z and

$$U_Z|_{X_z} = [D_{\tau(z)}] \times_{\operatorname{Spec} k(s)} \operatorname{Spec} k(z) \quad \text{for every } z \in Z.$$

This shows that $[D_s] \in \mathrm{Hilb}_p(X/S)$ is the unique point of the fiber $\tau^{-1}(s)$, so τ is geometrically injective. It remains to show that the fibers of τ are reduced.

Fix $s \in S$ and write $F = \tau^{-1}(s)$. Then $F \to \{s\}$ is flat, so

$$U_Z|_F = \operatorname{pure}\big(D_s \times_s F\big) = \operatorname{pure}\big(D_s\big) \times_s F.$$

Thus $U_Z|_F$ is the constant family $[D_s] \times_s F$. Thus the corresponding map $F \to \mathrm{Hilb}(X/S)$ is constant. By construction F is a subscheme of $\mathrm{Hilb}(X/S)$, so F is a point. Thus $Z \cong S$ and $U_Z = D$ is flat over S. $\square$

The representability of Cartier pull-backs now follows easily. Example (4.11) shows that (4.33) and (4.35) both can fail if D is not a generically Cartier family.

COROLLARY 4.35. *Let $f : X \to S$ be a flat, projective morphism with S_2 fibers and D a family of generically Cartier, not necessarily effective divisors on X. Then there is a locally closed partial decomposition $i : S^{\mathrm{car}} \to S$ such that for every reduced S-scheme $q : W \to S$, the divisorial pull-back $D_W \subset X_W$ is relatively Cartier iff q factors as $q : W \to S^{\mathrm{car}} \to S$.*

As shown by (3.72), in general $i\big(S^{\mathrm{car}}\big) \subset S$ is neither open nor closed.

Proof. An effective, relatively Cartier family is also a flat family of divisors, and we proved in (4.33) that flat divisorial pull-backs are represented by a locally closed decomposition $S^{\mathrm{dflat}} \to S$. As we noted in (3.37.1), we can replace S by S^{dflat} and henceforth consider only the special case when D is flat over S.

For a flat family of divisorial subschemes being Cartier is an open condition, thus S^{car} is an open subset of S^{dflat}. $\square$

THEOREM 4.36 (Valuative criterion for Cartier divisors). *Let S be a reduced, excellent scheme, $f : X \to S$ a flat morphism of finite type with S_2 fibers and D a family of generically Cartier divisors on X. Then following are equivalent.*

(1) *D is a relatively Cartier divisor.*
(2) *For every morphism $q : T \to S$ from the spectrum of a DVR to S, the divisorial pull-back $D_T \subset X_T$ is a relatively Cartier divisor.*

Proof. It is clear that (1) implies (2). Thus assume that (2) holds.

Assume first that f is projective. Consider the locally closed embedding $i : S^{\mathrm{car}} \to S$ given by (4.35). Since every $q : T \to S$ factors through $i : S^{\mathrm{car}} \to S$, we see that i is proper and surjective, hence an isomorphism.

Consider next the case when f is non-projective. Pick any point $x \in X$ and its image $s := f(x)$. Let $\hat{S}$ denote the completion of S at s; it is reduced since S is excellent. Then D is Cartier at x iff this holds after base change to $\hat{S}$. Thus it is enough to show that (2) $\Rightarrow$ (1) whenever S is complete.

Now we use (9.61) to get $i : S^u \hookrightarrow S$. Let $(0, T)$ be the spectrum of a DVR and $q : (0, T) \to (0, S)$ a local morphism. By assumption $\mathcal{O}_{X_T}(D_T)$ is locally free, hence it is flat with S_2 fibers over T. Thus $\mathcal{O}_{X_T}(D_T)$ is a universal hull by (9.26). By assumption (2) we obtain that q factors through S^u. Since this holds for every $q : T \to S$, we again conclude that $S^u = S$. $\square$

4.4. Generically $\mathbb{Q}$-Cartier divisors

In the study of lc and slc pairs, $\mathbb{Q}$-Cartier divisors are more important than Cartier divisors. Unfortunately, as (4.15–3.71) show, having $\mathbb{Q}$-Cartier divisorial pull-backs is not a representable condition in general. See, however, (4.51) for a positive result.

We have also seen many examples of Weil $\mathbb{Z}$-divisors that are $\mathbb{Q}$-Cartier but not Cartier. By contrast, we show that if a relative Weil $\mathbb{Z}$-divisor is generically $\mathbb{Q}$-Cartier then it is generically Cartier, at least in characteristic 0.

One of the main consequences of this is that the common denominator pull-back $g_N^{[*]}(X,\Delta)$ defined in (4.4.2) is independent of the denominator N, at least in characteristic 0.

Let $f:(X,D)\to S$ be a family of pairs and D a relative Weil $\mathbb{Z}$-divisor on X.

Since we are interested in generic properties, we can focus on a generic point x of $D\cap X_s$. If the assumption (4.1.4) holds then f is smooth at x. Thus we may as well assume that f is smooth (but not proper).

If S is normal then D is a Cartier divisor by (4.21), thus here our main interest is in those cases where S is reduced but not normal. Examples (4.10) and (4.12) show that then D need not be Cartier in general. However, the next result shows that if some multiple of D is Cartier, then so is D, at least in characteristic 0. This also completes the proof of (4.5.3).

Positive characteristic counter examples are given in (4.14) and (4.41).

Proposition 4.37. *Let S be a reduced scheme, $f:X\to S$ a smooth morphism and D a relative Weil $\mathbb{Z}$-divisor on X. Assume that mD is Cartier at a point $x\in X$ and* $\operatorname{char} k(x)\nmid m$. *Then D is Cartier at x.*

Proof. Using (10.47), it is enough to prove this when $k(x)=k(f(x))$. We may also assume that $f:(x,X)\to(s,S)$ is a local morphism of local, henselian schemes and $k(x)=k(s)$ is perfect. By noetherian induction we may assume that D is Cartier on $X\setminus\{x\}$. By (1.88) $mD\sim 0$ determines a cyclic cover $\tilde X\to X$ that is étale over $X\setminus\{x\}$ whenever $\operatorname{char} k(x)\nmid m$. In our case $\tilde X\to X$ is trivial by (4.38) hence D is Cartier at x. $\square$

Lemma 4.38. *Let $f:(x,X)\to(s,S)$ be a smooth, local morphism of henselian, local schemes. Assume that $k(x)=k(s)$ is perfect, f has relative dimension ≥ 1 and S has dimension ≥ 1. Then $\hat\pi_1\bigl(X\setminus\{x\}\bigr)=1$.*

Proof. Over $\mathbb{C}$, a topological proof is given in (4.39). A similar algebraic argument is the following.

Set $X^0:=X\setminus\{x\}$ and let $\tilde X^0\to X^0$ be a finite étale cover. Let (t,T) be the spectrum of a DVR and $(t,T)\to(s,S)$ a local morphism that maps the generic point of T to a generic point of S. By pull-back we get a finite étale cover $\tilde X^0_T\to X^0_T$. Since X_T is regular and of dimension ≥ 2, the purity of branch loci implies that $\tilde X^0_T\to X^0_T$ is trivial. In particular, $\tilde X^0\to X^0$ is trivial on every irreducible component $X^0_i\subset X^0$.

Thus $\tilde X^0\to X^0$ is also trivial on X^0_s. If $\tilde X^0_s=\cup_j Z^0_j$ then, for every i,j there is a unique irreducible component $\tilde X^0_{ij}\subset\tilde X^0$ that dominates X^0_i and contains Z^0_j. Furthermore $\tilde X^0_j:=\cup_i\tilde X^0_{ij}$ is a connected component of $\tilde X^0$ that maps isomorphically onto X^0. Thus $\tilde X^0\to X^0$ is trivial. $\square$

4.39 (Links and smooth morphisms). Let $f:X\to S$ be a smooth morphism of complex spaces of relative dimension $n\geq 1$. We describe the topology of the link of a point $x\in X$ in terms of the topology of the link of $s:=f(x)\in S$.

We can write $S\subset\mathbb{C}^N_{\mathbf{z}}$ such that s is the origin and $X=S\times\mathbb{C}^n_{\mathbf{t}}$ where x is the origin. Intersecting S with a sphere of radius ϵ centered at s we get L_S, the link of $s\in S$. The intersection of S with the corresponding ball of radius ϵ is homeomorphic to the cone C_S over L_S.

The link L_X of $x \in X$ can be obtained as the intersection of X with the level set $\max\{\sum|z_i|^2, \sum|t_j|^2\} = \epsilon^2$. Thus L_X is homeomorphic to the amalgamation of

$$\begin{array}{lcll} L_S \times \mathbb{D}^{2n} & = & \big\{(\mathbf{z},\mathbf{t}) : \sum|z_i|^2 = \epsilon^2,\ \sum|t_j|^2 \le \epsilon^2\big\} & \text{and of} \\ C_S \times \mathbb{S}^{2n-1} & = & \big\{(\mathbf{z},\mathbf{t}) : \sum|z_i|^2 \le \epsilon^2,\ \sum|t_j|^2 = \epsilon^2\big\}, & \text{glued along} \\ L_S \times \mathbb{S}^{2n-1} & = & \big\{(\mathbf{z},\mathbf{t}) : \sum|z_i|^2 = \epsilon^2,\ \sum|t_j|^2 = \epsilon^2\big\}. & \end{array}$$

Note that $\pi_1(L_S) = \pi_1\big(L_S \times \mathbb{D}^{2n}\big)$ injects into $\pi_1\big(L_S \times \mathbb{S}^{2n-1}\big)$, but the latter gets killed in the cone $C_S \times \mathbb{S}^{2n-1}$. Thus L_X is simply connected for $n \ge 1$.

The cohomology of L_X can be computed from the Mayer-Vietoris sequence. Using that $H^i\big(L_S \times \mathbb{D}^{2n}, \mathbb{Z}\big) = H^i\big(L_S, \mathbb{Z}\big)$ and $H^i\big(C_S \times \mathbb{S}^{2n-1}, \mathbb{Z}\big) = H^i\big(\mathbb{S}^{2n-1}, \mathbb{Z}\big)$, for H^2 the key pieces are

$$\begin{array}{lllll} & \to & H^1\big(L_S, \mathbb{Z}\big) + H^1\big(\mathbb{S}^{2n-1}, \mathbb{Z}\big) & \to & H^1\big(L_S \times \mathbb{S}^{2n-1}, \mathbb{Z}\big) \\ \to\ H^2\big(L_X, \mathbb{Z}\big) & \to & H^2\big(L_S, \mathbb{Z}\big) + H^2\big(\mathbb{S}^{2n-1}, \mathbb{Z}\big) & \to & H^2\big(L_S \times \mathbb{S}^{2n-1}, \mathbb{Z}\big). \end{array}$$

If $n \ge 2$ then this gives that $H^2\big(L_X, \mathbb{Z}\big) = 0$. If $n = 1$ then, using the Künneth formula we get that

$$H^2\big(L_X, \mathbb{Z}\big) \cong H^0\big(L_S, \mathbb{Z}\big)/\mathbb{Z}. \tag{4.39.1}$$

We have thus proved the following.

Claim 4.39.2. $f : X \to S$ be a smooth morphism of complex spaces, L_X the link of a point $x \in X$ and $s := f(x)$. Assume that $\dim_x X > \dim_s S \ge 1$.

Then L_X is simply connected. Furthermore, $H^2(L_X, \mathbb{Z}) = 0$ iff the link of $s \in S$ is connected. □

The next result can be used to understand those divisors that become Cartier after pull-back to the seminormalization.

Lemma 4.40. *Let $\pi : (s' \in S') \to (s \in S)$ be a local morphism of reduced, local schemes that is an isomorphism outside the closed points and such that $k(s') = k(s)$. Let $X \to S$ be a flat morphism, $x \in X_s$ a point and $x' \in X' := X \times_S S'$ its preimage. Consider the pull-back map $\pi^* : \operatorname{Pic}^{\mathrm{loc}}(x, X) \to \operatorname{Pic}^{\mathrm{loc}}(x', X')$. Then*

(1) π^* *is an injection if* $\operatorname{depth}_x X_s \ge 2$*,*
(2) $\ker \pi^*$ *is a (possibly infinite dimensional) $k(s)$-vector space if* $\operatorname{char} k(s) = 0$ *and*
(3) $\ker \pi^*$ *is a (possibly infinite dimensional) unipotent group in general.*

Proof. Let $m \subset \mathcal{O}_S$ and $m' \subset \mathcal{O}_{S'}$ denote the maximal ideals and $I' \subset \mathcal{O}_{S'}$ the conductor ideal. By assumption $\mathcal{O}_{S'}/I'$ is artinian, thus there is an $r \in m' \setminus I'$ such that $m'r \in I'$. Set $\mathcal{O}_{S_1} := \langle \mathcal{O}_S, r\rangle$. Since $I' \subset \mathcal{O}_S$, we see that $\mathcal{O}_{S_1}/\mathcal{O}_S \cong k(s)$. Iterating this procedure eventually ends with $S' = S_n$ for some n. Thus it is enough to prove the lemma in the special case when $\mathcal{O}_{S'}/\mathcal{O}_S \cong k(s)$, hence $m'/m \cong k$ and $m \cdot \mathcal{O}_{S'} \subset \mathcal{O}_S$.

Set $U := X \setminus \{x\}$ and $U' := X' \setminus \{x'\}$. We claim that there is an exact sequence

$$1 \to \mathcal{O}_U^* \to \mathcal{O}_{U'}^* \stackrel{\tau}{\to} \mathcal{O}_{U_0} \to 0, \tag{4.40.1}$$

defined as follows. Since $X_s \cong X'_{s'}$, for any local section h' of $\mathcal{O}_{U'}^*$, there is a local section h of $\mathcal{O}_U^*$ such that $h'|_{X'_{s'}} = h|_{X_s}$. Now we set

$$\tau : h' \mapsto \tfrac{h'}{h} - 1 \in m'\mathcal{O}_{U'}/m\mathcal{O}_U \cong \mathcal{O}_{U_0}.$$

To see that it is well defined, note that any other choice of h is of the form $h/(1+g)$ where $g \in m\mathcal{O}_U$. Then

$$\tfrac{h'}{h/(1+g)} - 1 = \tfrac{h'}{h} - 1 + g\big(\tfrac{h'}{h} - 1\big) + g$$

and the last 2 terms are contained in $m\mathcal{O}_U$. From (4.40.1) we get the exact sequence

$$H^0\big(U', \mathcal{O}^*_{U'}\big) \xrightarrow{r} H^0\big(U_0, \mathcal{O}_{U_0}\big) \to \operatorname{Pic}^{\mathrm{loc}}(x, X) \to \operatorname{Pic}^{\mathrm{loc}}(x', X'). \tag{4.40.2}$$

We claim that

$$\operatorname{coker}(r) = H^0\big(U_0, \mathcal{O}_{U_0}\big)/H^0\big(X_0, \mathcal{O}_{X_0}\big).$$

Indeed, let ϕ_0^* be any global section of $\mathcal{O}_{U_0}$ that extends to a global section ϕ_0 of $\mathcal{O}_{X_0}$. Then it lifts to a section ϕ of $m'\mathcal{O}_{X'}$. Now $h' := 1 + \phi$ and $g := 1$ show that $\tau(h') = \phi_0^*$.

Finally note that $\operatorname{depth}_x X_s \geq 2$ iff $H^0\big(U_0, \mathcal{O}_{U_0}\big) = H^0\big(X_0, \mathcal{O}_{X_0}\big)$. Otherwise their quotient is a $k(s)$-vector space that is finite dimensional if $\dim X_0 = 2$ but infinite dimensional if $\dim X_0 = 1$. An extension of $k(s)$-vector spaces is a $k(s)$-vector space if $\operatorname{char} k(s) = 0$ and a unipotent group in general; cf. [**Bor91**, §10]. □

Positive characteristic examples.

We show by a series of examples that p-torsion is a frequently occurring problem in positive characteristic.

EXAMPLE 4.41. In (4.13) we studied the trivial family $X := (xy(x-y) = 0) \subset \mathbb{A}^3_{xyz}$ over the curve $C := (xy(x-y) = 0) \subset \mathbb{A}^2_{xy}$. We proved that for $c \neq 0$ the divisor $D_c := (x = z = 0) + (y = z = 0) + (x - y = z - cx = 0)$ is not Cartier, yet its pull-back to the normalization is Cartier.

Here we note that if $\operatorname{char} k = p > 0$ then $z^p - c^p xy^{p-1} = 0$ shows that pD_c is a Cartier divisor.

EXAMPLE 4.42. In (4.14) we considered the cusp $C := (x^2 = y^3) \subset \mathbb{A}^2_{xy}$, the trivial curve family $Y := C \times \mathbb{A}^1_z \to C$ and the Weil divisor $D^+ :=$ image of $t \mapsto (t^3, t^2, t)$.

We proved that D^+ is not $\mathbb{Q}$-Cartier in characteristic 0 but the equation

$$pD^+ = \big(xy^{(p-3)/2} = z^p\big)$$

shows that it is $\mathbb{Q}$-Cartier in characteristic $p > 0$.

The next example shows that the previous ones are rater typical.

EXAMPLE 4.43. Let k be an algebraically closed field of characteristic $p > 0$ and B, C smooth curves over k. Let Δ be an effective divisor on $B \times C$.

Let $\tau_C : C \to C'$ be any birational, universal homeomorphism. Taking product by B we get a homeomorphism $\tau : B \times C \to B \times C'$ and hence a divisor $\Delta' := \tau_*\Delta$ on $B \times C'$. We claim that Δ' is $\mathbb{Q}$-Cartier. Indeed, since τ is a homeomorphism, it factors through a power of the Frobenius

$$1_B \times F_{p^m} : B \times C \xrightarrow{\tau} B \times C' \xrightarrow{\tau'} B \times C,$$

see, for instance, [**Kol97**, Sec.6]. Since $(1_B \times F_{p^m})^*\Delta = p^m\Delta$, we see that $(\tau')^*\Delta = p^m\Delta'$ hence $p^m\Delta'$ is Cartier.

A typical local example of this with concrete equations is the following. Consider the higher cusp $C_n := (x^2 = y^{2n+1}) \subset \mathbb{A}^2_{xy}$ and the trivial curve family $Y_n := C_n \times \mathbb{A}^1_z \to C_n$. The normalization is $\mathbb{A}^1_s \times \mathbb{A}^1_z$ and $(s, z) \mapsto (s^{2n+1}, s^2, z)$.

Next consider the Weil divisor $D_n \subset Y_n$ which is the image of the map $t \mapsto (t^{2n+1}, t^2, t) \in Y_n$. Its preimage in the normalization is the image of the diagonal map $t \mapsto (t, t) \in \mathbb{A}^1_s \times \mathbb{A}^1_z$.

Choose $m > 0$ such that $p^m \geq 2n+1$ and set $c := \frac{1}{2}(p^m - 2n - 1)$. We claim that $p^m D_n$ is a Cartier divisor with equation $z^{p^m} - xy^c = 0$. Indeed, pulling back to the normalization we get

$$z^{p^m} - s^{2n+1}(s^2)^c = (z-s)^{p^m}.$$

A similar seminormal example is the following.

EXAMPLE 4.44. Let k be a field of characteristic $p > 0$ and $a \in k \setminus k^p$. Set $R := k + (x^p - a)k[x] \subset k[x]$. As we noted in (3.30), R is seminormal but not weakly normal.

The ideal $(z - x)$ is the kernel of the diagonal morphism $d : k[z, x] \to k[t]$ that sends $z \mapsto t, x \mapsto t$. Let $P \subset R[z]$ denote the kernel of its restriction $d_R : R[z] \mapsto k[t]$. Then P is not a principal ideal but its pth symbolic power $P^{(p)}$ is generated by $z^p - a - (x^p - a) \in R[z]$.

4.5. Stability is representable II

Assumption. In this Section we work over a field of characteristic 0.

Let $f : (X, \Delta) \to S$ be a well defined family of pairs (4.4.3) using the common denominator definition of the divisorial pull-back (4.4.2). By (4.3.3), the pull-back does not depend on the choice of the common denominator since we are in characteristic 0.

In (3.68) we gave 5 equivalent definitions locally stable families of varieties. No we extend these to families of pairs. The main difference is that the natural analog of (3.68.1) is no longer equivalent to the others; see Section 2.6 for some case when it is.

DEFINITION–THEOREM 4.45. Let S be a reduced scheme and $f : (X, \Delta) \to S$ a projective, well defined family of pairs. Then $f : (X, \Delta) \to S$ is *locally stable* or *slc* if the following equivalent conditions hold.

(1) $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier and the fibers (X_s, Δ_s) are slc for all points $s \in S$.
(2) $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier and (X_s, Δ_s) is slc for all closed points $s \in S$.
(3) $f_T : (X_T, \Delta_T) \to T$ is locally stable whenever T is the spectrum of a DVR and $q : T \to S$ is a morphism.

Proof. The arguments are essentially the same as in (3.68). It is clear that (1) $\Rightarrow$ (2). The converse and (2) $\Rightarrow$ (3) both follow from (2.3) after base change.

If (3) holds then all fibers are slc. In particular, $m_s K_{X_s} + m_s \Delta_s$ is Cartier for some $m_s > 0$ for every $s \in S$. By (3.63) there is a common m such that $mK_{X_s} + m\Delta_s$ is Cartier for every $s \in S$. Let T be the spectrum of a DVR mapping to S. Then $K_{X_T/T} + \Delta_T$ is $\mathbb{Q}$-Cartier by assumption, thus $mK_{X_T/T} + m\Delta_T$ is Cartier by (2.92) or the stronger (2.90). Finally the Valuative criterion for Cartier divisors (4.36) shows that $mK_X + m\Delta$ is Cartier. □

We can now state the main result of this section which can be thought of as a local variant of [**HMX14**]. Eventually we remove the reduced assumption in (**??**?).

THEOREM 4.46. *Let $f : (X, \Delta) \to S$ be a projective, well defined family of pairs. Then the functor of locally stable divisorial pull-backs is represented by a locally closed partial decomposition $i^{\mathrm{lst}} : S^{\mathrm{lst}} \to S$ for reduced schemes.*

As in (3.1), a proper morphism $f:(X,\Delta)\to S$ is called stable iff it is locally stable and $K_{X/S}+\Delta$ is $\mathbb{Q}$-Cartier and f-ample. Since ampleness is an open condition for a $\mathbb{Q}$-Cartier divisor, (4.46) implies analogous result for stable morphisms.

COROLLARY 4.47. *Let $f:(X,\Delta)\to S$ be a projective, well defined family of pairs. Then the functor of stable divisorial pull-backs is represented by a locally closed partial decomposition $i^{\rm stab}:S^{\rm stab}\to S$ for reduced schemes.* □

We start the proof of (4.46), which will be completed in (4.50), with a weaker version.

LEMMA 4.48. *Let $f:(X,\Delta)\to S$ be a proper, well defined family of pairs. Then there is a finite collection of locally closed subschemes $S_i\subset S$ such that*

(1) *$f_i: X_i := X\times_S S_i\to S_i$ is locally stable for every i,*
(2) *$K_{X_i/S_i}+\Delta_{S_i}$ is $\mathbb{Q}$-Cartier and*
(3) *the fiber (X_s,Δ_s) is slc iff $s\in\cup_i S_i$.*

In particular, $\{s:(X_s,\Delta_s)$ is slc$\}\subset S$ is constructible.

Proof. Being demi-normal is an open condition by (10.41) and slc implies demi-normal by definition. Thus we may assume that all fibers are demi-normal and S is irreducible with generic point g. Throughout the proof we use $S^0\subset S$ to denote a dense open subset which we shrink whenever necessary.

First we treat morphisms whose generic fiber X_g is normal.

Case 1: (X_g,Δ_g) is lc. Then $m(K_{X_g}+\Delta_g)$ is Cartier for some $m>0$ hence $m(K_{X/S}+\Delta)$ is Cartier over an open neighborhood of g. Next consider a log resolution $p_g:Y_g\to X_g$. It extends to a simultaneous log resolution $p^0:Y^0\to X^0$ over a suitable $S^0\subset S$. Thus, if $E^0\subset Y^0$ is any exceptional divisor, then $a(E_s,X_s,\Delta_s)=a(E^0,X^0,\Delta^0)=a(E_g,X_g,\Delta_g)$. This shows that all fibers over S^0 are lc.

Case 2: (X_g,Δ_g) is not lc. Note that the previous argument works if $K_{X_g}+\Delta_g$ is $\mathbb{Q}$-Cartier. Indeed, then there is divisor E with $a(E_g,X_g,\Delta_g)<-1$ and this shows that $a(E_s,X_s,\Delta_s)<-1$ for $s\in S^0$. However when $K_{X_g}+\Delta_g$ is not $\mathbb{Q}$-Cartier then the discrepancy $a(E_g,X_g,\Delta_g)$ is not defined. We could try to prove that $K_{X_s}+\Delta_s$ is not $\mathbb{Q}$-Cartier for $s\in S^0$ but this is not true in general; see (4.15).

Thus we use the notion of numerically Cartier divisors (4.52) instead. If $K_{X_g}+\Delta_g$ is not numerically Cartier then, by (4.55), $K_{X_s}+\Delta_s$ is also not numerically Cartier over an open subset $S^0\ni g$. Thus (X_s,Δ_s) is not lc for $s\in S^0$.

If $K_{X_g}+\Delta_g$ is numerically Cartier then the notion of discrepancy makes sense (4.52) and, again using (4.55), the above arguments show that if (X_g,Δ_g) is numerically lc (resp. not numerically lc) then the same holds for (X_s,Δ_s) for s in a suitable open subset $S^0\ni g$. We complete Case 2 by noting that being numerically lc is equivalent to being lc by (4.53).

An alternate approach to the previous case is the following. By (4.54) the log canonical modification (5.16) $\pi_g:(Y_g,\Theta_g)\to(X_g,\Delta_g)$ exists and it extends to a simultaneous log canonical modification $\pi:(Y,\Theta)\to(X,\Delta)$ over an open subset $S^0\subset S$. By the arguments of Case 1, (Y_s,Θ_s) is lc for $s\in S^0$ and the relative ampleness of the log canonical class is also an open condition. Thus $\pi_s:(Y_s,\Theta_s)\to(X_s,\Delta_s)$ is the log canonical modification for $s\in S^0$. By assumption π_g is not an isomorphism, so none of the π_s are isomorphisms. Therefore none of the fibers over S^0 are lc.

If X_g is not normal, the proofs mostly work the same using a simultaneous semi-log resolution [**Kol13c**, Sec.10.4]. However, for Case 2 it is more convenient to use the following argument.

Let $\pi_g : \bar{X}_g \to X_g$ denote the normalization. Over an open subset $S^0 \ni g$ it extends to a simultaneous normalization $\big(\bar{X}, \bar{D} + \bar{\Delta}\big) \to S$. If $\big(\bar{X}_g, \bar{D}_g + \bar{\Delta}_g\big)$ is not lc then $\big(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s\big)$ is not lc for $s \in S^0$, hence (X_s, Δ_s) is not slc, essentially by definition; see [**Kol13c**, 5.10].

Using the already settled normal case, it remains to deal with the situation when $(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s)$ is lc for every $s \in S^0$. By [**Kol13c**, 5.38], (X_s, Δ_s) is slc iff $\operatorname{Diff}_{\bar{D}_s^n} \bar{\Delta}_s$ is τ_s-invariant. The different can be computed on any log resolution as the intersection of the birational transform of $\bar{D}_s$ with the discrepancy divisor. Thus $\operatorname{Diff}_{\bar{D}_s^n} \bar{\Delta}_s$ is also locally constant over an open set S^0. Therefore, if $\operatorname{Diff}_{\bar{D}_g^n} \bar{\Delta}_g$ is not τ_g-invariant then $\operatorname{Diff}_{\bar{D}_s^n} \bar{\Delta}_s$ is also not τ_s-invariant for $s \in S^0$. Hence (X_s, Δ_s) is not slc for every $s \in S^0$.

In both cases we complete the proof by Noetherian induction. $\square$

The following consequence of (4.48) is quite useful, though it could have been proved before it as in (3.69).

COROLLARY 4.49. *Let $f : (X, \Delta) \to S$ be a proper, well defined family of pairs. Assume in addition that $\omega_{X/S}^{[m]}(m\Delta)$ is locally free for some $m > 0$. Then $\{s : (X_s, \Delta_s)$ is slc $\} \subset S$ is open.*

Proof. By (4.48) this set is constructible. A constructible set $U \subset S$ is open iff it is closed under generalization, that is, $x \in U$ and $x \in \bar{y}$ implies that $y \in U$. This follows from (2.3). $\square$

4.50 (Proof of (4.46)). Let $S_i \subset S$ be as in (4.48). By restriction we get slc families $f_i : (X_i, \Delta_i) \to S_i$. In particular, there is an $m_i > 0$ such that $m_i\big(K_{X_i/S_i} + \Delta_i\big)$ is Cartier. Let m be a common multiple of the m_i. Then $m(K_{X_s} + \Delta_s)$ is Cartier whenever (X_s, Δ_s) is slc.

We apply (4.35) to the family $f : \big(X, m(K_{X/S} + \Delta)\big) \to S$ to obtain $S^{\mathrm{car}} \to S$ such that, for every seminormal S-scheme $q : T \to S$, the pulled-back divisor $m\big(K_{X_T/T} + \Delta_T\big)$ is Cartier iff q factors as $q : T \to S^{\mathrm{car}} \to S$.

Assume now that $f_T : (X_T, \Delta_T) \to T$ is slc. Then, as we noted in (3.62), $m\big(K_{X_T/T} + \Delta_T\big)$ is Cartier, hence q factors through $S^{\mathrm{car}} \to S$. As we observed in (3.37), this implies that $S^{\mathrm{slc}} = (S^{\mathrm{car}})^{\mathrm{slc}}$. By definition $K_{X^{\mathrm{car}}/S^{\mathrm{car}}} + \Delta$ is $\mathbb{Q}$-Cartier, thus (4.49) implies that $S^{\mathrm{slc}} = (S^{\mathrm{car}})^{\mathrm{slc}}$ is an open subscheme of S^{car}. $\square$

We showed in (4.15) that being $\mathbb{Q}$-Cartier is not a constructible condition. The next result shows that the situation is better for boundary divisors of lc pairs.

COROLLARY 4.51. *Let $f : (X, \Delta) \to S$ be a proper, flat family of pairs with slc fibers. Let D be an effective divisor on X. Assume that*

(1) *either* $\operatorname{Supp} D \subset \operatorname{Supp} \Delta$,
(2) *or* $\operatorname{Supp} D$ *does not contain any of the log canonical centers of any of the fibers (X_s, Δ_s).*

Then $\{s : D_s$ is $\mathbb{Q}$-Cartier$\} \subset S$ is constructible.

Proof. Choose $0 < \epsilon \ll 1$. In the first case $(X_s, \Delta_s - \epsilon D_s)$ is slc iff D_s is $\mathbb{Q}$-Cartier. In the second case $(X_s, \Delta_s + \epsilon D_s)$ is slc iff D_s is $\mathbb{Q}$-Cartier. Thus, in both cases, (4.48) implies our claim. □

Numerically $\mathbb{Q}$-Cartier divisors.

DEFINITION 4.52. Let $g : Y \to S$ be a proper morphism. A $\mathbb{Q}$-Cartier divisor D is called *numerically q-trivial* if $(C \cdot D) = 0$ for every curve $C \subset Y$ that is contracted by g.

Let X be a normal scheme and $p : Y \to X$ a resolution. A $\mathbb{Q}$-divisor D on X is called *numerically $\mathbb{Q}$-Cartier* if there is a p-exceptional $\mathbb{Q}$-divisor E_D such that $E_D + p_*^{-1}D$ is numerically p-trivial. (See (4.57) for a different variant of this definition.)

If $g : X \to S$ is proper then a numerically $\mathbb{Q}$-Cartier divisor D is called *numerically q-trivial* if $E_D + p_*^{-1}D$ is numerically $g \circ p$-trivial on Y.

Being numerically $\mathbb{Q}$-Cartier is preserved by q-linear equivalence. Indeed, if $D_1 \sim_{\mathbb{Q}} D_2$ then there is a function f such that $(f) = mD_1 - mD_2$ for some $m > 0$. Thus $(f \circ p) = mp_*^{-1}D_1 - mp_*^{-1}D_2 + E_f$ where E_f is p-exceptional. Hence $E_{D_1} - E_{D_2} = \frac{1}{m}E_f$. We can thus define when a linear equivalence class $|D|$ is numerically $\mathbb{Q}$-Cartier, though the divisors E_D depend on $D \in |D|$. It is easy to see that these notions are independent of the resolution.

For $K_X + \Delta$ we can make a canonical choice. Thus we see that $K_X + \Delta$ is numerically $\mathbb{Q}$-Cartier iff there is a p-exceptional $\mathbb{Q}$-divisor $E_{K+\Delta}$ such that $E_{K+\Delta} + K_Y + p_*^{-1}\Delta$ is numerically p-trivial

If $K_X + \Delta$ is numerically $\mathbb{Q}$-Cartier then one can define the *discrepancy* of any divisor E over X by

$$a(E, X, \Delta) := a(E, Y, E_{K+\Delta} + p_*^{-1}\Delta).$$

We can thus define when a pair (X, Δ) is *numerically lc*. This concept was useful in the proof of (4.48). There are many divisors that are numerically $\mathbb{Q}$-Cartier but not $\mathbb{Q}$-Cartier, however, the next result says that the notion of numerically lc pairs does not give anything new.

THEOREM 4.53. [**HX16**, 1.6] *A numerically lc pair is lc.*

Outline of proof. This is surprisingly complicated, using many different ingredients. For clarity, let us concentrate on a very special case when (X, Δ) is dlt, except at a single point $x \in X$. All the key ideas appear in this case but we avoid a more technical inductive argument.

Let $f : (Y, E + \Delta_Y) \to (X, \Delta)$ be a $\mathbb{Q}$-factorial, dlt modification (as in [**Kol13c**, 1.34]) where E is the exceptional divisor dominating x and Δ_Y is the birational transform of E. Let $\Delta_E := \operatorname{Diff}_E \Delta_Y$. Then (E, Δ_E) is a semi-dlt pair such that $K_E + \Delta_E$ is numerically trivial. Next we need a global version of the theorem.

Claim 4.53.1. Let (E, Δ_E) is a semi-slc pair such that $K_E + \Delta_E$ is numerically trivial. Then $m(K_E + \Delta_E) \sim 0$ for some $m > 0$.

The first general proof is in [**Gon13**], but special cases go back to [**Kaw85, Fuj00**]. We discuss a very special case: E is smooth and $\Delta = 0$. The following argument is from [**CKP12, Kaw13**].

We assume that $\mathcal{O}_E(K_E) \in \operatorname{Pic}^\tau(E)$ but after passing to an étale cover of E we have that $\mathcal{O}_E(K_E) \in \operatorname{Pic}^\circ(E)$. Note that $H^n\big(E, \mathcal{O}_E(K_E)\big) = 1$ where $n = \dim E$.

Next we use a theorem of [**Sim93**] which says that the cohomology groups of line bundles in Pic° jump along torsion translates of subtori. However

$$H^n\big(E, L\big) = 1 \Leftrightarrow H^0\big(E, L^{-1}(K_E)\big) = 1 \Leftrightarrow L \cong \mathcal{O}_E(K_E).$$

Thus $\mathcal{O}_E(K_E)$ is a torsion element of $\operatorname{Pic}^\circ(E)$. $\square$

It remains to lift information from the exceptional divisor E to the dlt model Y. To this end consider the exact sequence

$$0 \to \mathcal{O}_Y\big(m(K_Y{+}E{+}\Delta_Y){-}E\big) \to \mathcal{O}_Y\big(m(K_Y{+}E{+}\Delta_Y)\big) \to \mathcal{O}_E\big(m(K_E{+}\Delta_E)\big) \to 0.$$

Note that $m(K_Y + E + \Delta_Y) - E - (K_Y + \Delta_Y) \sim_{\mathbb{Q},f} 0$, thus

$$R^1 f_*\big(\mathcal{O}_Y\big(m(K_Y + E + \Delta_Y) - E\big)\big) = 0$$

by [**Kol13c**, 10.38.1] (or the even stronger [**Fuj14**, 1.10]). Hence a nowhere zero global section of $\mathcal{O}_E\big(m(K_E + \Delta_E)\big)$ lifts back to a global section of $\mathcal{O}_Y\big(m(K_Y + E + \Delta_Y)\big)$ that is nowhere zero near E. Thus $\mathcal{O}_X\big(m(K_X + \Delta)\big) \cong f_*\mathcal{O}_Y\big(m(K_Y + E + \Delta_Y)\big)$ is free in a neighborhood of x. $\square$

The following was used to give an alternate proof of one of the steps in (4.48). C. Xu pointed out that it can be proved using the arguments of [**OX12**].

THEOREM 4.54. *Let X be a normal variety and Δ a boundary such that $K_X+\Delta$ is numerically $\mathbb{Q}$-Cartier. Then (X, Δ) has a log canonical modification (5.15).* $\square$

The advantage of the concept of numerically $\mathbb{Q}$-Cartier divisors is that we have better behavior in families.

PROPOSITION 4.55. *Let $f : X \to S$ be a proper morphism with normal fibers over a field of characteristic 0 and D a generically Cartier family of divisors on X. Then*

$$\{s \in S : D_s \text{ is numerically } \mathbb{Q}\text{-Cartier}\}$$

is a constructible subset of S.

Proof. Let $g \in S$ be a generic point. We show that if D_g is numerically $\mathbb{Q}$-Cartier (resp. not numerically $\mathbb{Q}$-Cartier) then the same holds for all D_s in an open neighborhood $s \in S^0 \subset S$. Then we finish by Noetherian induction.

To see our claim, consider a log resolution $p_g : Y_g \to X_g$. It extends to a simultaneous log resolution $p^0 : Y^0 \to X^0$ over a suitable open neighborhood $g \in S^0 \subset S$.

If D_g is numerically $\mathbb{Q}$-Cartier then there is a p_g-exceptional $\mathbb{Q}$-divisor E_g such that $E_g + (p_g)_*^{-1} D_g$ is numerically p_g-trivial. This E_g extends to a p-exceptional $\mathbb{Q}$-divisor E and $E + p_*^{-1} D$ is numerically p-trivial over an open neighborhood $g \in S^0 \subset S$ by (4.56). Thus D_s is numerically $\mathbb{Q}$-Cartier for $s \in S^0$.

Assume next that D_g is not numerically $\mathbb{Q}$-Cartier. Let E_g^i be the p-exceptional divisors. Then there are proper curves $C_g^j \subset Y_g$ that are contracted by p_g and such that $(p_g)_*^{-1} D_g$, viewed as a linear function on $\oplus_j \mathbb{R}[C_g^j]$, is linearly independent of the E_g^i. Both the divisors E_g^i and the curves C_g^j extend to give divisors E_s^i and the curves C_s^j over an open neighborhood $g \in S^0 \subset S$. Thus $(p_s)_*^{-1} D_s$, viewed as a linear function on $\oplus_j \mathbb{R}[C_s^j]$, is linearly independent of the E_s^i, hence D_s is not numerically $\mathbb{Q}$-Cartier for $s \in S^0$. $\square$

LEMMA 4.56. *Let $p : Y \to X$ be a morphism of proper S-schemes and L a line bundle on Y. Then*

$$S^{\mathrm{nt}} := \{s \in S : L_s \text{ is numerically } p_s\text{-trivial}\}$$

is an open subset of S.

Proof. Let us start with the special case when $X = S$.

Let $g \in S$ be a generic point. If L_g is not numerically trivial then there is a curve $C_g \subset Y_g$ that is contracted by p_g and such that $(C_g \cdot L_g) \neq 0$. Let $s \in S$ be any specialization of g and C_s the specialization of C_g. Then $(C_s \cdot L_s) = (C_g \cdot L_g) \neq 0$ shows that L_s is also not numerically trivial.

If L_g is numerically trivial then L_g^m is algebraically equivalent to 0 by [**Laz04**, I.4.38] for some $m > 0$. We can spread out this algebraic equivalence to obtain that there is an open subset $g \in S^0 \subset S$ such that L is algebraically (and hence numerically) equivalent to 0 on all fibers of p over S^0.

Applying this to $Y \to X$ shows that

$$X^{\mathrm{nt}} := \{x \in X : L_x \text{ is numerically trivial on } Y_x\}$$

is an open subset of X. Thus

$$S^{\mathrm{nt}} = S \setminus \pi_X\bigl(X \setminus X^{\mathrm{nt}}\bigr)$$

is an open subset of S, where $\pi_X : X \to S$ is the structure map. $\square$

REMARK 4.57. On a normal surface every $\mathbb{Q}$-divisor is numerically $\mathbb{Q}$-Cartier. This observation was used in [**Mum61**] to define intersection numbers of divisors on normal surfaces but in higher dimensions one needs a different version of numerically $\mathbb{Q}$-Cartier in order to define intersection numbers with curves.

Let X be a proper and normal variety. Let us say that a divisor D on X is *strongly numerically $\mathbb{Q}$-Cartier* if there is a p-exceptional $\mathbb{Q}$-divisor E_D such that $E_D + p_*^{-1}D$ is *strongly numerically p-trivial,* that is, $\bigl(Z \cdot (E_D + p_*^{-1}D)\bigr) = 0$ for every (not necessarily effective) 1-cycle Z on Y such that $p_*[Z] = 0$.

For example, let $E \subset \mathbb{P}^2$ be a smooth cubic and $S \subset \mathbb{P}^3$ the cone over it. For $p \in E$ let $L_p \subset S$ denote the line over p. Set $X := S \times E$ and consider the divisors D_1, swept out by the lines $L_{p_0} \times \{p\}$ for some fixed $p_0 \in E$, and D_2, swept out by the lines $L_p \times \{p\}$ for $p \in E$. Then $D_1 - D_2$ is numerically Cartier but not strongly numerically Cartier. To see the latter compute that if F is the exceptional divisor obtained by blowing up the singular locus then $F \cdot (D_1' - D_2')^2 = -2$ where D_i' denotes the birational transform of D_i.

One can define the intersection of a 1-cycle $W \subset X$ with a strongly numerically $\mathbb{Q}$-Cartier divisor D by the formula

$$(D \cdot W) = p_*\bigl(W_Y \cdot (E_D + p_*^{-1}D)\bigr),$$

where $W_Y \subset Y$ is any 1-cycle such that $p_*(W_Y) = W$. (We can set $W_Y = p_*^{-1}W$ if the latter is defined but there is always a 1-cycle W_Y such that $p_*(W_Y) = dW$ for some $d > 0$.)

If D is $\mathbb{Q}$-Cartier outside a finite set of points then D is strongly numerically $\mathbb{Q}$-Cartier iff it is numerically $\mathbb{Q}$-Cartier and this case can be understood in terms of the local Picard groups of X as follows.

Assume that $\dim X \geq 3$ and D is Cartier except at a point $x \in X$. There is a local Picard scheme $\mathbf{Pic}^{\mathrm{loc}}(x, X)$ which is an extension of a finitely generated local

Néron-Severi group with a connected algebraic group $\mathbf{Pic}^{\mathrm{loc}-\circ}(x, X)$; see [**Bou78**] or [**Kol16a**] for details. Then D is numerically $\mathbb{Q}$-Cartier iff $[D] \in \mathbf{Pic}^{\mathrm{loc}-\tau}(x, X)$ where $\mathbf{Pic}^{\mathrm{loc}-\tau}(x, X)/\mathbf{Pic}^{\mathrm{loc}-\circ}(x, X)$ is the torsion subgroup of the local Néron-Severi group.

The local Picard scheme also exists in positive characteristic, thus one can turn the above equivalence into a definition of numerically $\mathbb{Q}$-Cartier divisors in positive characteristic. However, it is not clear how to prove various theorems, including (4.55), using this definition.

Over $\mathbb{C}$ one can also consider those line bundles on the smooth locus of X that extend to a topological line bundle on X. It is not clear how this notion compares with the above algebraic ones.

4.58 (Comments on positive characteristic). There are numerous problems with the arguments of this section in positive characteristic.

To start with, the 3 versions of the basic definition (4.45) are nor known to be equivalent. Clearly (4.45.1) implies the other two, we can adopt it as the definition in general.

The discussion in (10.40) has gaps in characteristic 2, but these can be fixed.

Case 1 of the proof of (4.48) uses generic smoothness. For surfaces the structure of slc singularities in any characteristic is worked out in [**Kol13c**, Sec.3.3], and this can be used instead of generic smoothness. Probably one can do something similar in higher dimensions as well. We also use many properties of the different that are not known in positive characteristic.

The discussion on numerically $\mathbb{Q}$-Cartier divisors uses resolution of singularities. A treatment without resolution would be desirable.

Finally (4.53) is also not known in positive characteristic.

4.6. Varieties marked with divisors

We now start to construct the seminormalization of the moduli space of stable pairs $\bigl(X, \Delta = \sum_i a_i D_i\bigr)$, by constructing a moduli space of varieties with distinguished Weil divisors. The reasons for working with seminormalization are discussed in (4.63).

DEFINITION 4.59. Let k be a field. A *variety marked with divisors* or a *marked variety* over k is an object $(X, D^1, \dots, D^m)$ consisting of

(1) a pure dimensional, geometrically connected, geometrically reduced k-scheme X that satisfies Serre's condition S_2 and
(2) effective Weil $\mathbb{Z}$-divisors $D^1, \dots, D^m$ on X such that none of the irreducible components of the D^i are contained in $\operatorname{Sing} X$.

As we noted in (4.16.7), we can identify each D^i with a divisorial subscheme, which we also denote by D^i.

DEFINITION 4.60. Let S be a reduced scheme. A *family of varieties marked with divisors* over S is a compound object $f : (X, D^1, \dots, D^m) \to S$ where

(1) $f : X \to S$ is a pure dimensional, flat morphism with geometrically connected, geometrically reduced, S_2 fibers and
(2) the D^i are well defined families of divisors on X (4.1.6),

such that the fibers $(X_s, D^1_s, \dots, D^m_s)$ satisfy (4.59.1–2) for every $s \in S$. (Here D^i_s denotes the divisorial restriction as in (4.1.5).)

Note that $(X, D^1, \dots, D^m)$ is not a variety marked with divisors unless S itself is a variety that is S_2.

Aside. The flatness assumption on f is, to some extent, a matter of choice. Many of the basic results hold provided the non-flat locus of f has codimension ≥ 2 in each fiber. On the other hand, working with non-flat morphisms is technically much harder and we saw in (3.11) that flatness holds for locally stable morphisms.

DEFINITION 4.61. If we have both marking divisors and a polarization as in (3.77), we get the notion of a *polarized variety marked with divisors* or a *polarized, marked variety* consisting of a projective variety marked with divisors $(X, D^1, \dots, D^m)$ plus an ample line bundle L on X. Similarly, we have *polarized families of varieties marked with divisors* over a reduced scheme S. These are written in the form

$$f : (X; D^1, \dots, D^m; L) \to S. \tag{4.61.1}$$

As before, for technical reasons it is more convenient do deal with strong polarizations. Thus we let

$$S \mapsto \mathcal{P}^s\mathcal{MV}(n, m, N)(S) \tag{4.61.2}$$

denote the functor, sheafified as in (3.78.3), that associates to a reduced scheme S the set of all families of projective varieties with a strong polarization and marked with divisors

$$f : (X; D^1, \dots, D^m; L) \to S \tag{4.61.3}$$

for which f_*L is locally free of rank $N+1$. (Since L is flat over S, the vanishing of the R^if_*L implies that f_*L is locally free.)

DEFINITION 4.62 (Embedded marked varieties). The functor of strongly embedded marked varieties is denoted by

$$S \mapsto \mathcal{E}^s\mathcal{MV}(n, m, \mathbb{P}^N)(S). \tag{4.62.1}$$

It associates to a reduced scheme S the set of all families of closed subschemes of a given $\mathbb{P}^N_S$ marked with divisors

$$f : \bigl(X \subset \mathbb{P}^N_S; D^1, \dots, D^m; \mathcal{O}_X(1)\bigr) \to S \tag{4.62.2}$$

where $\mathcal{O}_X(1)$ is strongly ample. The latter condition is equivalent to

$$R^i\pi_*\mathcal{O}_{\mathbb{P}^N}(1) \cong R^if_*\mathcal{O}_X(1) \quad \text{for} \quad i \geq 0, \tag{4.62.3}$$

where $\pi : \mathbb{P}^N_S \to S$ is the natural projection.

Equivalently, we can view $\mathcal{E}^s\mathcal{MV}(n, m, \mathbb{P}^N)$ as parametrizing objects

$$\Bigl(f : (X; D^1, \dots, D^m; L) \to S; \phi \in \operatorname{Isom}_S\bigl(\mathbb{P}_S(f_*L), \mathbb{P}^N_S\bigr)\Bigr) \tag{4.62.4}$$

consisting of a strongly polarized family of varieties marked with divisors plus a projective framing $\phi : \mathbb{P}_S(f_*L) \cong \mathbb{P}^N_S$.

4.63 (Comment on seminormality). Hilbert schemes work well over any base scheme, but in [**Kol96**] the theory of Cayley–Chow families is developed only over seminormal bases. In characteristic 0 it might be possible to work over reduced base schemes (see [**Bar75**] for key special cases) but examples of Nagata [**Nag55**] suggest that in positive characteristic the restriction to seminormal bases may be necessary.[1] Thus, in what follows, we work with the above functors over seminormal bases. We indicate this by adding a superscript $^{\text{sn}}$ to the notation.

[1] The situation is not clear to me. JK

Thus, for example, we let

$$S \mapsto \mathcal{P}^s\mathcal{MV}^{\mathrm{sn}}(n,m,N)(S) \tag{4.63.1}$$

denote the restriction of the functor $\mathcal{P}^s\mathcal{MV}(n,m,N)$ to seminormal schemes S.

4.64 (Universal family of embedded marked varieties). Let B be a base scheme (which we may as well assume seminormal for what follows). Fix a projective space $\mathbb{P}^N_B$ and integers $n \geq 1$ and $m \geq 0$. In analogy with the Hilbert scheme and the Chow variety, we construct a universal family of embedded varieties marked with divisors in $\mathbb{P}^N_B$.

Start with the universal family over the Hilbert scheme

$$\mathrm{Univ}_n(\mathbb{P}^N_B) \to \mathrm{Hilb}_n(\mathbb{P}^N_B). \tag{4.64.1}$$

We are interested in varieties, these are parametrized by an open subset $\mathrm{Hilb}^\circ_n(\mathbb{P}^N_B) \subset \mathrm{Hilb}_n(\mathbb{P}^N_B)$. As before, it is more convenient to work with the smaller open subset where the varieties are geometrically connected, geometrically reduced, S_2 and $\mathcal{O}_X(1)$ is strongly ample (3.77). That is, we assume that

$$H^i\bigl(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)\bigr) \cong H^i\bigl(X, \mathcal{O}_X(1)\bigr) \quad \text{for} \quad i \geq 0. \tag{4.64.2}$$

These conditions give an open subset $\mathrm{E^sV}(n,\mathbb{P}^N_B) \subset \mathrm{Hilb}^\circ_n(\mathbb{P}^N_B) \subset \mathrm{Hilb}_n(\mathbb{P}^N_B)$. So far $\mathrm{E^sV}(n,\mathbb{P}^N_B)$ is a scheme, but at the next step it needs to be seminormal. So we take base change to its seminormalization to obtain

$$\mathrm{Univ}(n,\mathbb{P}^N_B)^{\mathrm{sn}} \to \mathrm{E^sV}(n,\mathbb{P}^N_B)^{\mathrm{sn}}. \tag{4.64.3}$$

(Note that the universal family is obtained by base change, so its total space need not be seminormal since the fibers are not assumed seminormal.)

So far we have the universal family for the underlying varieties. Next take the universal family of well defined families of Weil divisors (3.21)

$$\mathrm{WDiv}\bigl(\mathrm{Univ}(n,\mathbb{P}^N_B)^{\mathrm{sn}}/\,\mathrm{E^sV}(n,\mathbb{P}^N_B)^{\mathrm{sn}}\bigr). \tag{4.64.4}$$

Warning. This space is defined and is known to be universal only for families over seminormal bases. This is why we can not yet work over reduced base schemes.

By (4.30) the latter has an open subset

$$\mathrm{WDiv}^{\mathrm{gs}}\bigl(\mathrm{Univ}(n,\mathbb{P}^N_B)^{\mathrm{sn}}/\,\mathrm{E^sV}(n,\mathbb{P}^N_B)^{\mathrm{sn}}\bigr) \tag{4.64.5}$$

parametrizing pairs (X,D) for which X is generically smooth along $\operatorname{Supp} D$. Furthermore, by (4.30.3), there is a universal family of generically Cartier divisors

$$\mathrm{Univ}^{\mathrm{gs}}\bigl(\mathrm{Univ}(n,\mathbb{P}^N_B)^{\mathrm{sn}}/\,\mathrm{E^sV}(n,\mathbb{P}^N_B)^{\mathrm{sn}}\bigr) \to \mathrm{WDiv}^{\mathrm{gs}}\bigl(\mathrm{Univ}(n,\mathbb{P}^N_B)^{\mathrm{sn}}/\,\mathrm{E^sV}(n,\mathbb{P}^N_B)^{\mathrm{sn}}\bigr).$$

Finally take the fiber product of m copies of

$$\mathrm{WDiv}^{\mathrm{gs}}\bigl(\mathrm{Univ}(n,\mathbb{P}^N_B)^{\mathrm{sn}}/\,\mathrm{E^sV}(n,\mathbb{P}^N_B)^{\mathrm{sn}}\bigr) \longrightarrow \mathrm{E^sV}(n,\mathbb{P}^N_B)^{\mathrm{sn}}$$

and then seminormalize to obtain the moduli space of embedded marked varieties with a strong polarization

$$\mathrm{E^sMV}^{\mathrm{sn}}(n,m,\mathbb{P}^N_B) \to B. \tag{4.64.6}$$

Over $\mathrm{E^sMV}^{\mathrm{sn}}(n,m,\mathbb{P}^N_B)$ we have a universal family of strongly polarized varieties marked with divisors

$$\mathbf{F} : \bigl(\mathbf{X}, \mathbf{D}^1, \dots, \mathbf{D}^m; \mathbf{L}\bigr) \to \mathrm{E^sMV}^{\mathrm{sn}}(n,m,\mathbb{P}^N_B), \tag{4.64.7}$$

where we really should have written

$$\Big(\mathbf{X}(n,m,\mathbb{P}^N_B),\mathbf{D}^1(n,m,\mathbb{P}^N_B),\dots,\mathbf{D}^m(n,m,\mathbb{P}^N_B);\mathbf{L}(n,m,\mathbb{P}^N_B)\Big)$$

but the latter is rather cumbersome.

It is clear from the construction that the spaces $\mathrm{E}^{\mathrm{s}}\mathrm{MV}^{\mathrm{sn}}(n,m,\mathbb{P}^N_B)$ parametrize polarized families of varieties marked with divisors, where the varieties are equipped with an extra framing.

Proposition 4.65. *Let B be a seminormal scheme and fix n,m,N. Then the scheme of embedded, marked varieties $\mathrm{E}^{\mathrm{s}}\mathrm{MV}^{\mathrm{sn}}(n,m,\mathbb{P}^N_B)$ constructed in (4.64) represents the functor $\mathcal{E}^s\mathcal{MV}^{\mathrm{sn}}(n,m,\mathbb{P}^N)$, defined in (4.62), for seminormal schemes. That is, for every seminormal scheme S over B, there is a natural one-to-one correspondence between*

1. $\mathrm{Mor}_B\big(S,\mathrm{E}^{\mathrm{s}}\mathrm{MV}^{\mathrm{sn}}(n,m,\mathbb{P}^N_B)\big)$ *and*
2. *families of n-dimensional, geometrically connected, geometrically reduced, S_2 varieties, with a strong polarization and marked with generically Cartier divisors $f:(X;D^1,\dots,D^m;L)\to S$, such that f_*L is locally free of rank $N+1$, plus a projective framing $\mathbb{P}_S(f_*L)\cong\mathbb{P}^N_S$.* □

4.66 (Boundedness conditions). The schemes $\mathrm{E}^{\mathrm{s}}\mathrm{MV}^{\mathrm{sn}}(n,m,\mathbb{P}^N_B)$ have infinitely many irreducible components since we have not fixed the degrees of X and of the divisors D^i. Set

$$\deg_L(X;D^1,\dots,D^m):=\big(\deg_L X,\deg_L D^1,\dots,\deg_L D^m\big)\in\mathbb{N}^{m+1}. \tag{4.66.1}$$

This multidegree is a locally constant function on $\mathrm{E}^{\mathrm{s}}\mathrm{MV}^{\mathrm{sn}}(n,m,\mathbb{P}^N_B)$ by (3.21), hence its level sets give a decomposition

$$\mathrm{E}^{\mathrm{s}}\mathrm{MV}^{\mathrm{sn}}(n,m,\mathbb{P}^N_B)=\amalg_{\mathbf{d}\in\mathbb{N}^{m+1}}\mathrm{E}^{\mathrm{s}}\mathrm{MV}^{\mathrm{sn}}_{\mathbf{d}}(n,m,\mathbb{P}^N_B). \tag{4.66.2}$$

The spaces $\mathrm{E}^{\mathrm{s}}\mathrm{MV}^{\mathrm{sn}}_{\mathbf{d}}(n,m,\mathbb{P}^N_B)$ are quasi-projective, though usually non-projective, reducible and disconnected.

The general correspondence between the moduli of polarized varieties and the moduli of embedded varieties (**??**?) gives now the following.

Corollary 4.67. *Let B be a seminormal scheme and fix n,m,N. Then the stack $\big[\mathrm{E}^{\mathrm{s}}\mathrm{MV}^{\mathrm{sn}}(n,m,\mathbb{P}^N_B)/\mathrm{PGL}_{N+1}(\mathcal{O}_B)\big]$ represents the functor $\mathcal{P}^s\mathcal{MV}^{\mathrm{sn}}(n,m,N)$, defined in (4.61.4) and (4.63.1) for seminormal schemes.* □

Already for 1-dimensional marked pairs, the stacks $\big[\mathrm{E}^{\mathrm{s}}\mathrm{MV}^{\mathrm{sn}}/\mathrm{PGL}\big]$ are very complicated. They are not separated and have high dimensional stabilizers. Next we show that the substacks of stable pairs are much better behaved.

4.7. Moduli of marked slc pairs I

4.68. So far we have studied slc pairs (X,Δ) but did not worry too much about how Δ was written as a sum of divisors. As long as we look at a single variety, we can write Δ uniquely as $\sum a_iD_i$ where the D_i are prime divisors and there is usually not much reason to do anything else.

However, the situation changes when we look at families. Assume for instance that we have an slc family over an irreducible base $f:(X,\Delta)\to S$ with generic point $g\in S$. Then the natural approach is to write $\Delta_g=\sum a_iD^i_g$ where the D^i_g are prime divisors on the generic fiber X_g. For any other point $s\in S$ this gives a decomposition $\Delta_s = \sum a_i D_s^i$, where D_s^i is the specialization of D_g^i. Note that the D_s^i need not be prime divisors. They can have several irreducible components with different multiplicities and two different D_s^i, D_s^j can have common irreducible components. Thus $\Delta_s = \sum a_i D_s^i$ is not the "standard" way to write Δ_s.

Let us now turn this around. We fix a proper slc pair (X_0, Δ_0) and aim to understand all deformations of it. A first suggestion could be the following:

Naive definition 4.68.1. An slc deformation of (X_0, Δ_0) over a local scheme $(0 \in S)$ is a proper slc morphism $f : (X, \Delta) \to S$ whose central fiber $(X, \Delta)_0$ is isomorphic to (X_0, Δ_0).

As an example of this, start with $\bigl(\mathbb{P}^1_{xy}, (x = 0)\bigr)$. Pick any $n \geq 1$ and polynomials $a_i(t)$ that vanish at $t = 0$. Then

$$\Bigl(\mathbb{P}^1_{xy} \times \mathbb{A}^1_t, \tfrac{1}{n}\bigl(x^n + a_{n-1}(t)x^{n-1}y + \cdots + a_0(t)y^n = 0\bigr)\Bigr) \tag{4.68.2}$$

is a deformation of $\bigl(\mathbb{P}^1_{xy}, (x = 0)\bigr)$ over $\mathbb{A}^1_t$ by the naive definition (4.68.1). We can eliminate the $a_{n-1}(t)x^{n-1}y$ term, hence we get a deformation space of dimension $n - 1$. Letting n vary results in an infinite dimensional deformation space.

For a fixed n and general choice of the a_i, the polynomial in (4.68.2) is irreducible over $k(t)$, thus our recipe above says that we should write $\Delta = \frac{1}{n} D_g$ (where D_g is irreducible) and then the special fiber is written as $(x = 0) = \frac{1}{n}(x^n = 0)$.

The situation becomes even less clear if we take 2 deformations as in (4.68.2) for 2 different values n, m and glue them together over the origin to get a family of pairs over the nodal curve $(st = 0)$. The family is locally stable. One side suggests that the fiber over the origin should be $\frac{1}{n}(x^n = 0)$, the other side that it should be $\frac{1}{m}(x^m = 0)$.

This suggests that, at last over non-normal base schemes, some bookkeeping is necessary to control the multiplicities. On the positive side, once we control how a given $\mathbb{Q}$-divisor Δ is written as a linear combination of $\mathbb{Z}$-divisors, we obtain finite dimensional moduli spaces. This leads to the following definition.

DEFINITION 4.69 (Marked slc pairs). Fix a finite index set I and real numbers $0 < a_i \leq 1$ for $i \in I$. A *marked slc pair* with *coefficient vector* $\{a_i : i \in I\}$ consists of

(1) an slc pair (X, Δ) plus
(2) a way of writing $\Delta = \sum a_i D_i$, where the D_i are effective $\mathbb{Z}$-divisors on X.

We also call $\sum a_i D_i$ a *marking* of Δ. We allow the D_i to be empty; this has the advantage that the restriction of a marking to an open subset is again is marking. However in other contexts this is not natural and I will probably sometimes forget about empty divisors.

Observe that $\Delta = \sum a_i D_i$ and $\Delta = \sum (\frac{1}{2} a_i)(2D_i)$ are different as markings. This seems rather pointless for one pair but, as we observed in (4.68), it is a meaningful distinction when we consider deformations of a pair.

Note that, for a given (X, Δ), markings are combinatorial objects that are not constrained by the geometry of X. If $\Delta = \sum_i b_i B_i$ and the B_i are distinct prime divisors, then the markings correspond to ways of writing the vector $(b_1, \ldots, b_r)$ as a positive linear combination of nonnegtive integral vectors.

Comments. Working with such markings is a rather natural thing to do. For example, plane curves C of degree d can be studied using the log-CY pair $\bigl(\mathbb{P}^2, \Delta_C :=$

$\frac{3}{d}C\big)$ as in [**Hac04**]. Thus, even if C is reducible, we want to think of the $\mathbb{Q}$-divisor Δ_C as $\frac{3}{d}C$; hence as a marked divisor with $I := \{1\}$ and $a_1 = \frac{3}{d}$. Similarly, in most cases when we choose the boundary divisor Δ, it has a natural marking.

However, when a part of Δ is forced upon us, for instance coming from the exceptional divisor of a resolution, there is frequently no natural marking, though usually it is possible to choose a marking that works well enough.

In some cases the marking is determined by Δ. If $a_i > \frac{1}{2}$ for every i then there is at most 1 way of writing $\Delta = \sum a_i D_i$, where the D_i are effective $\mathbb{Z}$-divisors. However, if we allow $a_i = \frac{1}{2}$ then a divisor D can have 3 different markings: $[D]$, $\frac{1}{2}[2D]$ or $\frac{1}{2}[D] + \frac{1}{2}[D]$. The smaller the a_i, the more markings are possible.

If I is a finite set then a divisor Δ has only finitely many possible markings. More generally, this also holds if I is infinite but the numbers $\{a_i\}$ satisfy the strong descending chain condititION (there is no infinite sequence $a_{i_1} \geq a_{i_2} \geq \cdots$ where the indices $i_1, i_2, \dots$ are all different) and we ignore empty divisors.

Definition 4.70 (Marked, locally stable families). Fix a finite index set I and rational (or real) numbers $0 < a_i \leq 1$ for $i \in I$. A *marked, locally stable family of pairs* over a reduced scheme S and with coefficient vector $\mathbf{a} := \{a_i : i \in I\}$ is a family of varieties marked with divisors $f : (X; D^i : i \in I) \to S$ (4.60) such that

$$f : \big(X, \textstyle\sum a_i D^i\big) \to S \quad \text{is locally stable (4.45).} \tag{4.70.1}$$

Note that by (4.60.2) the D^i are assumed to be well defined families of divisors. In analogy with (4.61.2) we have the functor

$$S \mapsto \mathcal{MLSP}(\mathbf{a})(S) \tag{4.70.2}$$

that associates to a reduced scheme S the set of all proper, marked families of pairs $f : (X; D^i : i \in I) \to S$ for which $f : \big(X, \sum a_i D^i\big) \to S$ is locally stable.

We are mostly interested in the subfunctor of marked, stable pairs

$$S \mapsto \mathcal{MSP}(\mathbf{a})(S) \tag{4.70.3}$$

where, in addition, $K_{X/S} + \sum a_i D^i$ is assumed relatively ample.

Definition 4.71 (Polarized, marked, locally stable families). If we also have a polarization, we get a *polarized, marked, locally stable family of pairs*

$$f : (X; D^i : i \in I; L) \to S. \tag{4.71.1}$$

In analogy with (4.61.2) we have the functor

$$S \mapsto \mathcal{P}^s\mathcal{MLSP}(n, \mathbf{a}, N)(S) \tag{4.71.2}$$

that associates to a reduced scheme S the set of all strongly polarized, marked, locally stable families of pairs

$$f : (X; D^i : i \in I; L) \to S \tag{4.71.3}$$

for which f_*L is locally free of rank $N + 1$. As in (3.78.3), we also need to sheafify in the étale topology.

As in (4.62.1), we can also define the functor of strongly embedded, marked, locally stable families

$$S \mapsto \mathcal{E}^s\mathcal{MLSP}(n, \mathbf{a}, \mathbb{P}^N)(S). \tag{4.71.4}$$

REMARK 4.72. There are some subtle aspects of the notion of marked, slc families of pairs.

First we claim that every locally stable family of pairs can be viewed as a marked family. Indeed, let $f:(X,\Delta)\to S$ be a locally stable family of pairs. Write $\Delta=\sum_i b_iB_i$ where the B_i are distinct prime divisors. Assume that the b_i are rational and let N be their smallest common denominator. Then $D=\sum_i(Nb_i)B_i$ is a generically Cartier family of divisors over S. Thus $\Delta=\frac{1}{N}D$ give a marking of (X,Δ); we call this the *natural marking.* (We discuss real coefficients in (**??**?).)

If S is normal then, by (4.21), a marking of (X,Δ) is the same as a marking of the generic fiber (X_g,Δ_g), hence markings are combinatorial objects, corresponding to ways of writing the coefficient vector $(b_1,\dots,b_r)$ as a positive linear combination of nonnegtive integral vectors.

However, if S is not normal, then the geometry of (X,Δ) constrains the allowable markings. The reason for this is that each D^i is generically $\mathbb{Q}$-Cartier. In particular, if S is connected then $\operatorname{Supp} D^i$ dominates S for every i. For example, consider the slc family of pairs

$$S:=(st=0)\subset\mathbb{A}^2_{st},\ X:=\mathbb{P}^1_{xy}\times S,\ \Delta:=\tfrac{1}{n}B_1+\tfrac{1}{m}B_2,$$

where $B_1:=(s=x^n-ty^n=0)$ and $B_2:=(t=x^m-sy^m=0)$. Here $D_1=B_1, D_2=B_2$ is not an allowed marking since the B_i are not $\mathbb{Q}$-Cartier. In fact, the only possible marking is the natural one

$$\Delta=\tfrac{(n,m)}{nm}\bigl(\tfrac{m}{(n,m)}B_1+\tfrac{n}{(n,m)}B_2\bigr),$$

and its obvious relatives of the form $\Delta=\sum_j a_j(mB_1+nB_2)$.

As another example, let C be a nodal curve with normalization (C',p,q). Fix 4 points $a_1,\dots,a_4$ on $\mathbb{P}^1$. Let $D_1',D_2'\subset C'\times\mathbb{P}^1$ be two curves such that $D_1'+D_2'$ has simple normal crossings only, $D_i'\to C'$ have degree 2, D_1' meets $\mathbb{P}^1_p$ (resp. $\mathbb{P}^1_q$) in the points a_1,a_2 (resp. a_1,a_3) while D_2' meets $\mathbb{P}^1_p$ (resp. $\mathbb{P}^1_q$) in the points a_3,a_4 (resp. a_2,a_4). We can now glue $\mathbb{P}^1_p$ to $\mathbb{P}^1_q$ to get a locally stable family $f:(C\times\mathbb{P}^1,D_1+D_2)\to C$. Note that D_1+D_2 is a Cartier divisor, but neither D_1 nor D_2 is $\mathbb{Q}$-Cartier. Thus the only possible marking is the natural marking $D=D_1+D_2$ (and its obvious variations).

4.73 (Universal family of embedded, marked slc pairs). Fix a base scheme B, the ambient projective space $\mathbb{P}^N_B$, the dimension n and the number of divisors m. By (4.65) there is a universal family

$$\mathbf{F}:\bigl(\mathbf{X},\mathbf{D}^i:i\in I;\mathcal{O}_{\mathbf{X}}(1)\bigr)\to \mathrm{E^sMV^{sn}}(n,|I|,\mathbb{P}^N_B)$$

parametrizing embedded marked varieties of dimension n with a strongly ample polarization. As we noted in (4.63), for now it is necessary to work with the seminormalization.

Let us also fix a coefficient vector $\mathbf{a}:=(a_i:i\in I)$. Since each $\mathbf{D}^i$ is a family of generically Cartier divisors (4.25), we can apply (4.46) to obtain a seminormal, locally closed partial decomposition

$$j:\mathrm{E^sMLSP^{sn}}(n,\mathbf{a},\mathbb{P}^N_B)\longrightarrow \mathrm{E^sMV^{sn}}(n,|I|,\mathbb{P}^N_B)$$

which represents the functor of locally stable pull-backs for seminormal schemes. We have thus proved the following.

THEOREM 4.74. *Fix a base scheme B, the dimension n, the embedding dimension N and the coefficient vector $\mathbf{a}$. Then $\mathrm{E^sMLSP^{sn}}(n, \mathbf{a}, \mathbb{P}^N_B)$ represents the functor $\mathcal{E}^s\mathcal{MLSP}^{\mathrm{sn}}(n, \mathbf{a}, \mathbb{P}^N)$ over seminormal base schemes.* $\square$

General results about the moduli of polarized and embedded varieties (**??**?) give the following.

COROLLARY 4.75. *Fix a base scheme B, the dimension n, the embedding dimension N and the coefficient vector $\mathbf{a}$. Then the stack*

$$\left[\mathrm{E^sMLSP^{sn}}(n, \mathbf{a}, \mathbb{P}^N_B)/\mathrm{PGL}_B(N+1)\right]$$

represents the functor $\mathcal{P}^s\mathcal{MLSP}^{\mathrm{sn}}(n, \mathbf{a}, N)$ over seminormal base schemes. $\square$

4.76 (Hilbert function of a pair). Let (X, Δ) be a proper, slc pair. Its *canonical algebra* is

$$R(X, K_X + \Delta) := \textstyle\sum_r H^0\big(X, \omega_X^{[r]}(\lfloor r\Delta\rfloor)\big). \tag{4.76.1}$$

(Observe that the use of rounding down ensures that we do get a ring.) As in (3.65), the *Hilbert function* of (X, Δ) is the function

$$r \mapsto \chi(X, \Delta, r) := \chi\big(X, \omega_X^{[r]}(\lfloor r\Delta\rfloor)\big). \tag{4.76.2}$$

One problem with this is that, unlike for the $\Delta = 0$ case, the Hilbert function of the fibers is *not* locally constant for stable morphisms $f : (X, \Delta) \to S$ for every value of r; see (2.39) for an example. However, (2.76.2) says that deformation invariance holds for certain values of r.

Claim 4.76.3. Fix a rational coefficient vector $\mathbf{a} = (a_i : i \in I)$ and let $\delta(\mathbf{a})$ denote the smallest common denominator of the a_i. Let $f : (X, \sum a_i D_i) \to S$ be a locally stable family of marked slc pairs. Then the Hilbert function of the fibers $s \mapsto \chi(X_s, \Delta_s, r)$ is locally constant on S whenever $\delta(\mathbf{a}) \mid r$. $\square$

This leads us to define the open subfunctor

$$\mathcal{MSP}(\mathbf{a}, \chi)(*) \subset \mathcal{MSP}(\mathbf{a})(*) \tag{4.76.3}$$

of those families $f : (X, \sum a_i D_i) \to S$ for which

$$\chi\big(X, \omega_X^{[r]}(\lfloor r\Delta\rfloor)\big) = \chi(r) \quad \text{for every} \quad r \in \delta(\mathbf{a})\mathbb{Z}.$$

Then the stack of marked, stable pairs with coefficient vector $\mathbf{a}$ is decomposed as a disjoint union

$$\mathrm{MSP}(\mathbf{a}) = \amalg_\chi \mathrm{MSP}(\mathbf{a}, \chi), \tag{4.76.5}$$

where χ runs through all functions $\chi : \delta(\mathbf{a})\mathbb{Z} \to \mathbb{Z}$.

As in (3.76.5), for technical reasons we also introduce the open subfunctors

$$\mathcal{MSP}_m(\mathbf{a}, \chi)(*) \subset \mathcal{MSP}(\mathbf{a}, \chi)(*), \tag{4.76.6}$$

consisting of those families $f : (X, \sum a_i D_i) \to S$ for which $\omega_{X/S}^{[m]}(m \sum a_i D_i)$ is locally free and strongly f-ample.

Observe that, as in (3.76.5.b), if $m_1 \mid m_2$ then $\mathrm{MSP}_{m_1}(\mathbf{a}, \chi) \subset \mathrm{MSP}_{m_2}(\mathbf{a}, \chi)$ and

$$\mathrm{MSP}(\mathbf{a}, \chi) = \textstyle\bigcup_m \ \mathrm{MSP}_m(\mathbf{a}, \chi). \tag{4.76.7}$$

As in (3.83), fix the Hilbert polynomial h. Then we have a universal family $\mathrm{Univ}(h) \to \mathrm{E^sSV^{sn}}(h)$ parametrizing strongly embedded, marked stable varieties with Hilbert polynomial h. On $\mathrm{Univ}(h)$ we have the ample line bundle $\mathcal{O}(1)$ and the mostly flat divisorial sheaf $\omega^{[m]}(\lfloor m\Delta \rfloor)$, where ω denotes the relative dualizing sheaf of $\mathrm{Univ}(h) \to \mathrm{E^sSV^{sn}}(h)$. We can next apply (3.61) to this setting to obtain the following.

Corollary 4.77. *Fix a base scheme B, a rational coefficient vector $\mathbf{a} = (a_i : i \in I)$, a function $\chi : \delta(\mathbf{a})\mathbb{Z} \to \mathbb{Z}$ and m. Set $h(t) := \chi(mt)$. Then $\mathcal{C}^s\mathcal{MSP}^{\mathrm{sn}}_m(\mathbf{a}, \chi)(*)$ is represented by the seminormalization of a locally closed subscheme*

$$\mathrm{CEMSP}^{\mathrm{sn}}_m(\mathbf{a}, \chi) \to \mathrm{EMSP}^{\mathrm{sn}}(\mathbf{a}, h).$$

Therefore the quotient stack

$$\big[\mathrm{CEMSP}^{\mathrm{sn}}_m(\mathbf{a}, \chi)/\mathrm{PGL}_{N+1}(\mathcal{O}_B)\big], \quad \textit{where} \quad N = \chi(m) - 1,$$

represents the functor $\mathcal{MSP}^{\mathrm{sn}}_m(\mathbf{a}, \chi)$ defined in (4.76.6). □

Note that the properness of the PGL_{N+1}-action (**??**?) is equivalent to the following immediate consequence of (2.46) about the scheme of relative automorphisms $\mathrm{Aut}_S(X, \Delta) \to S$ [**Kol96**, I.1.10.2].

Proposition 4.78. *Let $f : (X, \Delta) \to S$ be a stable morphism. Then the induced map $\mathrm{Aut}_S(X, \Delta) \to S$ is finite.* □

We can now combine (4.77) and (4.76.7) with the results of Section 2.4 to obtain the following restatement of (4.9).

Theorem 4.79. *Fix a base scheme B, a rational coefficient vector $\mathbf{a} = (a_i : i \in I)$ and a function $\chi : \delta(\mathbf{a})\mathbb{Z} \to \mathbb{Z}$. Then $\mathcal{MSP}^{\mathrm{sn}}(\mathbf{a}, \chi)(*)$, the functor of marked, stable pairs with Hilbert function χ over seminormal schemes, has a coarse moduli space $\mathrm{MSP}(\mathbf{a}, \chi)^{\mathrm{sn}} \to B$. Furthermore*

1. *$\mathrm{MSP}^{\mathrm{sn}}(\mathbf{a}, \chi)$ is separated,*
2. *$\mathrm{MSP}^{\mathrm{sn}}(\mathbf{a}, \chi)$ satisfies the valuative criterion of properness and*
3. *$\mathrm{MSP}^{\mathrm{sn}}(\mathbf{a}, \chi)$ is the directed union of its open subspaces $\mathrm{MSP}^{\mathrm{sn}}_m(\mathbf{a}, \chi)$, which are of finite type over B.* □

Complement 4.80. *We see later that in fact*

1. *$\mathrm{MSP}^{\mathrm{sn}}(\mathbf{a}, \chi) \to B$ is projective,*
2. *$\mathrm{MSP}^{\mathrm{sn}}(\mathbf{a}, \chi) = \mathrm{MSP}^{\mathrm{sn}}_m(\mathbf{a}, \chi)$ for some m depending on χ and*
3. *$\mathrm{MSP}^{\mathrm{sn}}(\mathbf{a}, \chi) = \mathrm{red}\big(\mathrm{MSP}(\mathbf{a}, \chi)\big)$.*

Pairs with marked points.

So far we have studied varieties with marked divisors on them. Next we investigate what happens if we also mark some points. For curves the points are also divisors and they interact with the log canonical structure. By contrast, in dimension ≥ 2, the points and the log canonical structure are independent of each other. This makes the resulting notion much less interesting. However, it gives a quick and convenient way to rigidify slc pairs, and this turns out to be useful in Section 5.9.

Definition 4.81. A flat family of *r-pointed* schemes is a flat morphism $f : X \to S$ plus r sections $\sigma_i : S \to X$. This gives a functor of r-pointed schemes.

We are interested in r-pointed stable pairs. If we fix a rational coefficient vector $\mathbf{a} = (a_i : i \in I)$ and a Hilbert function $\chi : \delta(\mathbf{a})\mathbb{Z} \to \mathbb{Z}$, we get the functor $\mathcal{M}p\mathcal{SP}^{\rm sn}(\mathbf{a}, \chi, r)(*)$.

By (4.78) this has an open subfunctor $\mathcal{M}p\mathcal{SP}^{\rm sn}_{\rm rigid}(\mathbf{a}, \chi, r)(*)$ parametrizing *rigid* objects, that is marked pairs $(X, \Delta, x_1, \dots, x_r)$ where the subgroup scheme of $\operatorname{Aut}(X, \Delta)$ that fixes the points $x_1, \dots, x_r$ is trivial.

Proposition 4.82. *Using the notation of (4.79) and (4.81), the functor of marked, r-pointed stable pairs $\mathcal{M}p\mathcal{SP}^{\rm sn}(\mathbf{a}, \chi, r)(*)$ has a coarse moduli space*

$$\operatorname{MpSP}(\mathbf{a}, \chi, r)^{\rm sn} \to B,$$

which is separated and satisfies the valuative criterion of properness.

Moreover, $\mathcal{M}p\mathcal{SP}^{\rm sn}_{\rm rigid}(\mathbf{a}, \chi, r)(*)$ has a fine moduli space with a universal family

$$\operatorname{Univ}(\mathbf{a}, \chi, r)^{\rm sn}_{\rm rigid} \to \operatorname{MpSP}(\mathbf{a}, \chi, r)^{\rm sn}_{\rm rigid}.$$

Proof. We closely follow the proof of (4.79) and first we construct the moduli of r-pointed, embedded pairs. This is easy to do. By (4.77) we already have $\operatorname{CEMSP}^{\rm sn}_m(\mathbf{a}, \chi)$ with a universal family

$$\operatorname{Univ} \to \operatorname{CEMSP} := \operatorname{CEMSP}^{\rm sn}_m(\mathbf{a}, \chi).$$

Thus the functor of m-canonically, strongly embedded, marked, r-pointed stable pairs is represented by the r-fold fiber product

$$\operatorname{CEMpSP}^{\rm sn}_m(\mathbf{a}, \chi, r) = \operatorname{Univ} \times_{\rm CEMSP} \times \cdots \times_{\rm CEMSP} \times \operatorname{Univ} \to \operatorname{CEMSP}.$$

As in (4.79) the quotient stacks

$$\big[\operatorname{CEMpSP}^{\rm sn}_m(\mathbf{a}, \chi, r)/\operatorname{PGL}_{N+1}(\mathcal{O}_B)\big], \quad \text{where} \quad N = \chi(m) - 1,$$

form open substacks of $\operatorname{MpSP}(\mathbf{a}, \chi, r)^{\rm sn}$.

The existence of a universal family over $\operatorname{MpSP}(\mathbf{a}, \chi, r)^{\rm sn}_{\rm rigid}$ is a general property of quotients (**??**?). $\square$

4.8. Stable families over smooth base schemes

All the results of the previous sections apply to families $p : (X, \Delta) \to S$ over a smooth base scheme, but the smooth case has other interesting features as well. One can then obtain results about families over other base schemes by working over a resolution of singularities of the base. The following can be viewed as a direct generalization of (2.3).

Theorem 4.83. *Let $(0 \in S)$ be a smooth, local scheme and $D_1 + \cdots + D_r \subset S$ an snc divisor such that $D_1 \cap \cdots \cap D_r = \{0\}$. Let $p : (X, \Delta) \to (0 \in S)$ be a pure dimensional morphism and Δ a $\mathbb{Q}$-divisor on X such that $\operatorname{Supp}\Delta \cap \operatorname{Sing} X_0$ has codimension ≥ 2 in X_0. The following are equivalent.*

(1) *$p : (X, \Delta) \to S$ is slc.*
(2) *$K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier, p is flat and (X_0, Δ_0) is slc.*
(3) *$K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier, X is S_2 and $\big(\operatorname{pure}(X_0), \Delta_0\big)$ is slc.*
(4) *$\big(X, \Delta + p^*D_1 + \cdots + p^*D_r\big)$ is slc.*

Proof. (1) $\Rightarrow$ (2) holds by definition and (2) $\Rightarrow$ (3) since both S and X_0 are S_2 (9.5). If (3) holds then (10.69) shows that p is flat and X_0 is pure, hence (3) $\Rightarrow$ (2). Next we show that (2) $\Leftrightarrow$ (4) using induction on r. Both implications are trivial if $r = 0$.

Assume (4). Then $K_X + \Delta + p^*D_1 + \cdots + p^*D_r$ is $\mathbb{Q}$-Cartier at x hence so is $K_X + \Delta$. Set $D_Y := p^*D_r$. Adjunction (1.93) shows that

$$\bigl(D_Y, \Delta|_{D_Y} + p^*D_1|_{D_Y} + \cdots + p^*D_{r-1}|_{D_Y}\bigr)$$

is slc at x, hence $\bigl(X_0, \Delta_0\bigr)$ is slc at x by induction. The local equations of the p^*D_i form a regular sequence at x by (4.87), hence p is flat at x.

Conversely, assume that (2) holds. By induction

$$\bigl(D_Y, \Delta|_{D_Y} + p^*D_1|_{D_Y} + \cdots + p^*D_{r-1}|_{D_Y}\bigr)$$

is slc at x hence inversion of adjunction (1.93) shows that $\bigl(X, \Delta + p^*D_1 + \cdots + p^*D_r\bigr)$ is slc at x. □

COROLLARY 4.84. *Let S be a smooth scheme and $p : (X, \Delta) \to S$ a morphism. Then $p : (X, \Delta) \to S$ is locally stable iff the pair $(X, \Delta + p^*D)$ is slc for every snc divisor $D \subset S$.* □

COROLLARY 4.85. *Let S be a smooth, irreducible scheme and $p : (X, \Delta) \to S$ a locally stable morphism. Then every log center of (X, Δ) dominates S.*

Proof. Let E be a divisor over X such that $a(E, X, \Delta) < 0$ and let $Z \subset S$ denote the image of E in S. If $Z \neq S$ then, possibly after replacing S by an open subset, we may assume that Z is contained in a smooth divisor $D \subset S$. Thus $(X, \Delta + p^*D)$ is slc by (4.84). However, $a(E, X, \Delta + p^*D) \le a(E, X, \Delta) - 1 < -1$, a contradiction. □

COROLLARY 4.86. *Let S be a smooth scheme and $p : (X, \Delta) \to S$ a projective, locally stable morphism. Let $p^w : (X^w, \Delta^w) \to S$ denote a weak canonical model and $p^c : (X^c, \Delta^c) \to S$ the canonical model of $p : (X, \Delta) \to S$ (cf.* [**KM98**, 3.50] *or* [**Kol13c**, 1.19]*). Then*

(1) *$p^w : (X^w, \Delta^w) \to S$ is locally stable and*
(2) *$p^c : (X^c, \Delta^c) \to S$ is stable.*

Proof. Let $D \subset S$ be an snc divisor. By (4.84) $(X, \Delta + p^*D)$ is lc and $p^w : \bigl(X^w, \Delta^w + (p^*D)^w\bigr) \to S$ is also a weak canonical model over S by [**Kol13c**, 1.28]. Thus $\bigl(X^w, \Delta^w + (p^*D)^w\bigr)$ is also slc. Next we claim that $(p^*D)^w = (p^w)^*D$. This is clear away from the exceptional set of ϕ^{-1} which has codimension ≥ 2 in X^w. Thus $(p^*D)^w$ and $(p^w)^*D$ are 2 divisors that agree outside a codimension ≥ 2 subset, hence they agree. Now we can use (4.84) again to conclude that $p^w : (X^w, \Delta^w) \to S$ is locally stable.

A weak canonical model is a canonical model iff $K_{X^w/S} + \Delta^w$ is p^w-ample and the latter is also what makes a locally stable morphism stable. □

LEMMA 4.87. *Let $\bigl(y \in Y, \Delta + D_1 + \cdots + D_r\bigr)$ be slc. Assume that the D_i are Cartier divisors with local equations $(s_i = 0)$. Then the s_i form a regular sequence.*

Proof. We use induction on r. Since Y is S_2, s_r is a non-zerodivisor at y. By adjunction $\bigl(y \in D_r, \Delta|_{D_r} + D_1|_{D_r} + \cdots + D_{r-1}|_{D_r}\bigr)$ is slc, hence the restrictions $s_1|_{D_r}, \dots, s_{r-1}|_{D_r}$ form a regular sequence at x. Thus $s_1, \dots, s_r$ is a regular sequence at y. □

The following result of [**Kar00**] is a generalization of (2.50) from 1-dimensional to higher dimensional bases. As we see later (**??**?), it implies that every irreducible component of the moduli space of stable pairs is proper.

THEOREM 4.88. *Let U be a k-variety and $f_U : (X_U, \Delta_U) \to U$ a stable morphism. Then there is projective, generically finite, dominant morphism $\pi : V \to U$ and a compactification $V \hookrightarrow \bar{V}$ such that the pull-back $(X_U, \Delta_U) \times_U V$ extends to a stable morphism $f_{\bar{V}} : (X_{\bar{V}}, \Delta_{\bar{V}}) \to \bar{V}$.*

Proof. We may assume that U is irreducible with generic point g.

Assume first that the generic fiber of f_U is geometrically irreducible. Let $(Y_g, \Delta_g^Y) \to (X_g, \Delta_g)$ be a log resolution. It extends to a simultaneous log resolution $(Y_{U_0}, \Delta_{U_0}^Y) \to (X_{U_0}, \Delta_{U_0})$ over an open subset $U_0 \subset U$. By (4.89.2) there is a projective, generically finite, dominant morphism $\pi : V_0 \to U_0$ and a compactification $V_0 \hookrightarrow \bar{V}$ such that the pull-back $(Y_{U_0}, \Delta_{U_0}^Y) \times_{U_0} V_0$ extends to a locally stable morphism $g_{\bar{V}} : (Y_{\bar{V}}, \Delta_{\bar{V}}^Y) \to \bar{V}$.

We can harmlessly replace $\bar{V}$ by a resolution of it. Thus we may assume that $\bar{V}$ is smooth and that there is an open subset $V \subset \bar{V}$ such that the rational map $\bar{\pi}|_V : V \dashrightarrow U$ is a proper morphism.

Since $g_{\bar{V}}$ is a projective, locally stable morphism, the relative canonical model $f_{\bar{V}} : (X_{\bar{V}}, \Delta_{\bar{V}}) \to \bar{V}$ of $g_{\bar{V}} : (Y_{\bar{V}}, \Delta_{\bar{V}}^Y) \to \bar{V}$ exists by [**HX13**] and it is stable by (4.86.2).

By construction $(X_{\bar{V}}, \Delta_{\bar{V}})$ and $(X_U, \Delta_U) \times_U V$ are isomorphic over $V_0 \subset V$, but (2.46) implies that in fact they are isomorphic over V. This completes the case when the generic fiber of f_U is geometrically irreducible.

In general, we can first pull back everything to the Stein factorization of $X^n \to U$ where X^n is the normalization. Thus we may assume that every irreducible component of the generic fiber of f_U is geometrically irreducible. The previous step now gives $f_{\bar{V}} : (X_{\bar{V}}^n, \Delta_{\bar{V}}^n) \to \bar{V}$. Finally (4.85) shows that (2.55) applies and we get $f_{\bar{V}} : (X_{\bar{V}}, \Delta_{\bar{V}}) \to \bar{V}$. $\square$

4.89 (Weakly semistable reduction). A higher dimensional generalization of the Semistable reduction theorem of [**KKMSD73**] (see also (2.60)) is proved in [**AK00**]. The general question is the following.

Problem 4.89.1. Let $f : X \to S$ be a morphism. Find a proper, dominant, generically finite morphism $S' \to S$ and a proper morphism $f' : X' \to S'$ such that X' is birational to the main component of $X \times_S S'$, and the local structure of f' is as "nice" as possible.

If $\dim S = 1$ then, as shown by (2.60), we can achieve that f' is flat and its fibers are reduced snc divisors but a similar expectation would be overly optimistic if $\dim S > 1$ [**Kar00**].

The methods of [**KKMSD73**] are toric, and this suggests to look for a generalization where f' is toric. However, if S is not unirational then one certainly can not take S' to be toric, so the best one can hope in this direction is that f' is toroidal, where a map $p : U \to V$ is *toroidal* if for each point $u \in U$ with image $v := p(u)$, the map of formal completions $\hat{p} : \hat{U}_u \to \hat{V}_v$ is isomorphic to the formal completion of a toric morphism. The following is proved in [**AK00**, Thm.0.3 and Lem.6.1]; the last claim follows from (4.90).

Theorem 4.89.2. Let S be a scheme of finite type over a field of characteristic 0 and $f : X \to S$ a proper morphism. Then there is a proper, dominant, generically finite morphism $S' \to S$ and a proper morphism $f' : X' \to S'$ such that

(a) X' is birational to the main components of $X \times_S S'$,

(b) f' is flat with reduced fibers,
(c) f' is toroidal,
(d) S' is smooth, X' is canonical, $K_{X'}$ is Cartier and
(e) f' is locally stable. □

Note on terminology. Such morphism are called weakly semistable in [**AK00**]; this is a much stronger condition than being locally stable. The terminology of [**AK00**] does not match ours.

4.90 (Toric varieties). (See [**Ful93**] or [**Oda88**] for introductions to toric varieties.)

Let X be a normal, toric variety and D the sum of the torus-invariant Weil divisors. Then $K_X + D \sim 0$ and (X, D) is lc. Furthermore the log centers are exactly the torus-invariant irreducible subvarieties.

To prove these, we may assume that the base field is algebraically closed. Thus $X \setminus D \cong \mathbb{G}_m^n$ and

$$\sigma_X := \tfrac{dz_1}{z_1} \wedge \cdots \wedge \tfrac{dz_n}{z_n}$$

is a torus invariant n-form with simple poles along D. This shows that $K_X + D \sim 0$.

Next let $p : (Y, D_Y) \to (X, D)$ be a toric resolution of (X, D). Since $K_Y + D_Y \sim 0 \sim p^*(K_X + D)$, we conclude that all exceptional divisors have discrepancy -1. Thus (X, D) is lc and the log centers are exactly the images of the exceptional divisors. The latter are the torus-invariant irreducible subvarieties.

CHAPTER 5

Numerical flatness and stability criteria

The aim of this chapter is to prove several characterizations of stable and locally stable families $f : (X, \Delta) \to S$. An earlier result, established in (3.68), has two assumptions:

- every fiber $\big(X_s, \Delta_s\big)$ is semi-log-canonical and
- $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier.

In many applications the first of these is given but the second one can be quite subtle.

Note that such difficulties arise already for surfaces, even if $\Delta = 0$. Indeed, we saw in Section 1.3 that there are flat, projective families $g : X \to C$ of surfaces with quotient singularities that are not locally stable. In these cases every fiber is log terminal but $K_{X/C}$ is not $\mathbb{Q}$-Cartier although its restriction to every fiber $K_{X/C}|_{X_c} = K_{X_c}$ is $\mathbb{Q}$-Cartier.

In all the examples in Section 1.3, this unexpected behavior coincides with a jump in the self-intersection number of the canonical class of the fiber. Our aim is to prove that this is always the case, as shown by the following simplified version of the main theorem.

Theorem 5.1 (Numerical criterion of stability, weak form). *Let S be a connected, reduced scheme over a field of characteristic 0 and $f : X \to S$ a proper morphism of pure relative dimension n. Assume that all fibers are semi-log-canonical with ample canonical class K_{X_c}. Then*

(1) *$s \mapsto \big(K^n_{X_s}\big)$ is an upper semicontinuous function on S and*
(2) *$f : X \to S$ is stable iff the above function is constant.*

If $f : X \to S$ is stable then $K_{X/S}$ is $\mathbb{Q}$-Cartier, hence $\big(K^n_{X_s}\big)$ is clearly independent of $s \in S$, but the converse is surprising. General theory says that stability holds iff the Hilbert function $\chi\big(X_s, \mathcal{O}_{X_s}(mK_{X_s})\big)$ is independent of $s \in S$. Thus (5.1.2) asserts that if the leading coefficient of the Hilbert function is independent of s then the same holds for the whole Hilbert function. We collect many similar results in this chapter.

The main theorems are stated in Section 5.1 and related results on simultaneous canonical models and modifications are discussed in Section 5.2. The key claim is that, for families of slc pairs, local stability can fail only in relative codimension two and it can be characterized by the constancy of just 1 intersection number. A similar numerical condition characterizes Cartier divisors on flat families.

A series of examples in Section 5.3 shows that the assumptions of the theorems are likely to be optimal in characteristic 0. All the results are expected to hold in positive and mixed characteristic as well, but very few of the proofs apply to these cases. Numerical criteria for stability in codimension ≤ 1 are discussed in Section 5.4.

For all of the main theorems the key step is to establish them for families over smooth curves. This is done in Sections 5.5–2.8. The numerical criterion of global stability and a weaker version of local stability are derived in Section 5.5. The existence of simultaneous canonical models is studied in Section 5.6 and we treat simultaneous canonical modifications in Section 5.7.

Going from families over smooth curves to families over higher dimensional singular bases turns out to be quite quick, but several of the arguments, presented in Section 5.9, rely heavily on the techniques and results of Chapters **??**? and 9.

Assumptions. For all the main theorems of this Chapter we work with varieties over a field of characteristic 0 but the background results worked out in Sections 5.8–2.8 are established for excellent schemes.

5.1. Statements of the main theorems

We develop a series of criteria to characterize locally stable (4.45) or stable (2.43) morphisms using a few, simple, numerical invariants of the fibers.

We follow the general set-up of (5.1) but we strengthen it in 3 ways:

• We add a boundary divisor Δ.

• We assume only that f is flat in codimension 1 on each fiber. The reason for this is that many natural constructions (for instance flips, taking cones or ramified covers) do not preserve flatness. Thus we frequently end up with morphisms that are not known to be flat everywhere. This is rarely a problem when the base space is a smooth curve, but it becomes a serious issue over higher dimensional singular bases.

• We deal with local stability as well. A weak variant, involving several intersection numbers, is quite similar to the global case but the sharper form requires different considerations.

For the main results of this Chapter we work with the following set-up, which is a slight generalization of (3.51) and (4.1).

Assumption 5.2. Let $f : X \to S$ be a proper morphism of pure relative dimension n (3.34) and $Z \subset X$ a closed subset with complement $U := X \setminus Z$ such that the following hold.

(1) $\operatorname{codim}_{X_s}(Z \cap X_s) \geq 2$ for every $s \in S$,
(2) $f|_U : U \to S$ is flat and
(3) $\operatorname{depth}_Z X \geq 2$.

Sheaf versions of these assumptions are studied in Section 5.8.

Given $f : X \to S$ and $U = X \setminus Z$ as above, we also consider effective $\mathbb{Q}$-divisors $\Delta = \sum b_i B_i$ on X where the B are generically Cartier divisors (4.25). In applications to moduli problems we usually know that

(4) $f|_U : (U, \Delta|_U) \to S$ is locally stable.

In this case let $\pi_s : \big(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s\big) \to (X_s, \Delta_s)$ denote the normalization of a fiber where $\bar{D}_s \subset \bar{X}_s$ is the conductor (1.84). Thus

$$K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s \sim_{\mathbb{Q}} \pi_1^*\big(K_{X_s} + D_s + \Delta_s\big) \tag{5.2.5}$$

and it makes sense to ask whether $\big(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s\big)$ is lc or not.

We aim to give numerical criteria applicable to any morphism satisfying (5.2.1–4). The first such result generalizes (5.1) to pairs.

THEOREM 5.3 (Numerical criterion of stability). *We use the notation of (5.2). In addition to (5.2.1–3) assume that S is a reduced scheme over a field of* char 0. *Assume further that*

(1) $f|_U : \bigl(U, \Delta|_U\bigr) \to S$ *is locally stable,*
(2) $\bigl(X_g, \Delta_g\bigr)$ *is slc for all generic points* $g \in S$,
(3) *every fiber has lc normalization* $\pi_s : \bigl(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s\bigr) \to \bigl(X_s, \Delta_s\bigr)$ *and*
(4) $K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s$ *is ample for every* $s \in S$.

Then

(5) $s \mapsto \bigl(K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s\bigr)^n$ *is an upper semicontinuous function on* S *and*
(6) $f : (X, \Delta) \to S$ *is stable iff the above function is locally constant.*

The local stability version of (5.3) is the following.

THEOREM 5.4 (Numerical criterion of local stability). *We use the notation of (5.2). In addition to (5.2.1–3) assume that S is a reduced scheme over a field of* char 0 *and H is a relatively ample Cartier divisor class on X. Assume further that*

(1) $f|_U : \bigl(U, \Delta|_U\bigr) \to S$ *is locally stable,*
(2) $\bigl(X_g, \Delta_g\bigr)$ *is slc for all generic points* $g \in S$ *and*
(3) *every fiber has lc normalization* $\pi_s : \bigl(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s\bigr) \to \bigl(X_s, \Delta_s\bigr)$.

Then

(4) $s \mapsto \bigl(\pi_s^* H^{n-2} \cdot (K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s)^2\bigr)$ *is upper semicontinuous and*
(5) $f : (X, \Delta) \to S$ *is locally stable iff the above function is locally constant.*

Under the assumptions of (5.4) the functions $\bigl(\pi_s^* H^n\bigr)$ and $\bigl(\pi_s^* H^{n-1} \cdot (K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s)\bigr)$ are always locally constant but the functions $\bigl(\pi_s^* H^{n-i} \cdot (K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s)^i\bigr)$ are neither upper nor lower semicontinuous for $i \geq 3$.

A key part of the proof of (5.4) is to show that local stability is essentially a 2-dimensional question. The following is a strong form of this claim.

THEOREM 5.5 (Local stability is automatic in codimension ≥ 3). [**Kol13a**] *Using the notation and assumptions of (5.2.1–3) let S be a reduced scheme of* char 0. *Assume that*

(1) $\operatorname{codim}_{X_s}(Z \cap X_s) \geq 3$ *for every* $s \in S$,
(2) $f|_U : \bigl(U, \Delta|_U\bigr) \to S$ *is locally stable,*
(3) $\bigl(X_g, \Delta_g\bigr)$ *is slc for all generic points* $g \in S$ *and*
(4) *every fiber has lc normalization* $\pi_s : \bigl(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s\bigr) \to \bigl(X_s, \Delta_s\bigr)$.

Then $f : (X, \Delta) \to S$ *is locally stable.*

One can also restate this as a converse of the Bertini-type result (2.11).

COROLLARY 5.6. *Notation and assumptions as in (5.4). Assume in addition that the relative dimension is* $n \geq 3$ *and* $f|_H : (H, \Delta|_H) \to S$ *is locally stable where* $H \subset X$ *is a relatively ample Cartier divisor. Then* $f : (X, \Delta) \to S$ *is also locally stable.* □

Comment. As we noted in (2.12), (1.93) implies that $f : (X, H + \Delta) \to S$, and hence also $f : (X, \Delta) \to S$, are locally stable in a neighborhood of H. The unexpected new claim is that local stability holds everywhere.

A variant of (5.3) holds for arbitrary divisors and for non-slc fibers but we have to assume that f is flat with S_2 fibers. On the other hand, this holds over any field.

We state the general form treated in [**Kol16a**] but in this book we prove only the special case when $f : X \to S$ has normal fibers.

THEOREM 5.7 (Numerical criterion for relative line bundles). [**Kol16a**] *Let S be a reduced scheme over a field, $f : X \to S$ a flat, proper morphism of pure relative dimension n with S_2 fibers and $Z \subset X$ a closed subset such that $\operatorname{codim}_{X_s}(Z \cap X_s) \geq 2$ for every $s \in S$. Let H be an f-ample line bundle on X.*

Let L_U be an invertible sheaf on $U := X \setminus Z$ and assume that, for every $s \in S$, the restriction $L_U|_{U_s}$ extends to an invertible sheaf L_s on X_s. Then

(1) *$s \mapsto \big(H_s^{n-2} \cdot L_s^2\big)$ is an upper semicontinuous function on S and*
(2) *L_U extends to an invertible sheaf L on X iff the above function is locally constant.*

Furthermore, if L_s is ample for every s then

(3) *$s \mapsto (L_s^n)$ is an upper semicontinuous function on S and*
(4) *L_U extends to an f-ample invertible sheaf L on X iff the above function is locally constant.*

Most likely (5.7) holds over any reduced scheme S, but a key step (2.84) is known only over fields.

5.2. Simultaneous canonical models and modifications

We also aim to get numerical criteria for the existence of simultaneous canonical models and canonical modifications. That is, given a morphism $f : X \to S$, we would like to know when the canonical models (or the canonical modifications) of the fibers form a flat family; see (5.9) and (5.16) for the precise definitions.

There are two distinct definitions of canonical models.

DEFINITION 5.8 (Canonical models). Let (X, Δ) be a proper lc pair such that $K_X + \Delta$ is big. As usual (see [**KM98**, 3.50] or [**Kol13c**, 1.19]) its *canonical model* is the unique lc pair (X^c, Δ^c) such that $K_{X^c} + \Delta^c$ is ample and

$$\textstyle\sum_{m\geq 0} H^0\big(X, \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)\big) \cong \sum_{m\geq 0} H^0\big(X^c, \mathcal{O}_{X^c}(mK_{X^c} + \lfloor m\Delta^c \rfloor)\big).$$

There is a natural birational map

$$\phi : (X, \Delta) \dashrightarrow (X^c, \Delta^c). \tag{5.8.1}$$

On the other hand, if X is a proper variety with arbitrary singularities, then one frequently defines the *canonical model* of X as the canonical model of a resolution of X. We denote the latter variant by X^{cr}.

More generally, let X be a proper, pure dimensional scheme over a field. The *canonical model of resolutions* of X, denoted by X^{cr}, is obtained as follows. We start with a resolution $X^r \to \operatorname{red} X$ and then take the disjoint union of the canonical models of those irreducible components that are of general type. With a slight abuse of terminology, there is a natural map

$$\phi : X \dashrightarrow X^{\mathrm{cr}} \tag{5.8.2}$$

which is birational on the general type components and not defined on the others.

If X has log canonical singularities then both variants are defined but note that $X^c \cong X^{\mathrm{cr}}$ iff X has only canonical singularities.

Aside. One can also define the canonical model of resolutions of a pair (X, Δ) as long as none of the irreducible components of the boundary Δ is contained in

$\operatorname{Sing} X$. We start with a resolution $p : X' \to X$ such that $p_*^{-1}\Delta$ has smooth support and then take the canonical model of $(X', p_*^{-1}\Delta)$ to get $\bigl(X^{\mathrm{cr}}, \Delta^{\mathrm{cr}}\bigr)$.

We see in (5.21) that this notion does not seem to behave well in families.

Definition 5.9 (Simultaneous canonical model). Let $f : (X, \Delta) \to S$ be a morphism as in (5.2) such that every fiber has log canonical normalization $\pi_s : \bigl(\bar X_s, \bar\Delta_s\bigr) \to \bigl(X_s, \Delta_s\bigr)$. Its *simultaneous canonical model* is a diagram

$$\begin{array}{ccccc} X & & \overset{\phi}{\dashrightarrow} & & X^{\mathrm{sc}} \\ f & \searrow & & \swarrow & f^{\mathrm{sc}} \\ & & S & & \end{array} \tag{5.9.1}$$

where $f^{\mathrm{sc}} : \bigl(X^{\mathrm{sc}}, \Delta^{\mathrm{sc}}\bigr) \to S$ is stable and

$$\phi_s \circ \pi_s : \bigl(\bar X_s, \bar\Delta_s\bigr) \dashrightarrow \bigl(X^{\mathrm{sc}}_s, \Delta^{\mathrm{sc}}_s\bigr)$$

is the canonical model, as in (5.8.1), for every $s \in S$.

Comments. Note that we do not add the conductor of π_s to $\bar\Delta_s$. If the fibers are normal in codimension 1 then the reduced conductor is 0, hence the above notion is the only sensible one. In general, however, one has a choice and the simultaneous slc model, to be defined in (5.44), may be a better concept.

For a pure dimensional proper morphism $f : X \to S$ the *simultaneous canonical model of resolutions* $f^{\mathrm{scr}} : X^{\mathrm{scr}} \to S$ is defined analogously. Here we require that $\phi_s : X_s \dashrightarrow X^{\mathrm{scr}}_s$ be the canonical model of resolutions (5.8.2) for every $s \in S$.

We give criteria for the existence of simultaneous canonical models in terms of the volume (10.29) of the canonical class of the fibers. Note that if Y is a proper scheme of dimension n then $\operatorname{vol}(K_{Y^r})$ is independent of the choice of the resolution $Y^r \to Y$ and it equals the self-intersection number $\bigl((K_{Y^{\mathrm{cr}}})^n\bigr)$. Similarly, if (Y, Δ) is log canonical then $\operatorname{vol}(K_Y + \Delta) = \bigl((K_{Y^{\mathrm{c}}} + \Delta^{\mathrm{c}})^n\bigr)$.

Theorem 5.10 (Numerical criterion for simultaneous canonical models I). *Let S be a seminormal scheme of* char 0 *and $f : X \to S$ a proper morphism of pure relative dimension n. Then*

(1) *$s \mapsto \operatorname{vol}(K_{X^r_s})$ is a lower semicontinuous function on S and*
(2) *$f : X \to S$ has a simultaneous canonical model of resolutions iff this function is locally constant (and positive).*

This is a surprising result on two accounts. First, cohomology groups almost always vary *upper* semicontinuously; the *lower* semicontinuity in this setting was first observed and proved in [**Nak86, Nak87**]. Second, usually it is easy to generalize similar proofs from smooth varieties to klt or lc pairs, but here adding any boundary can ruin the argument and the conclusion as show by the Examples 5.21–5.23.

Example (5.20) shows that S needs to be seminormal.

The following is a similar result for normal lc pairs, but the lower semicontinuity of (5.10) changes to upper semicontinuity.

Theorem 5.11 (Numerical criterion for simultaneous canonical models II). *We use the notation of (5.2). In addition to (5.2.1–3) assume that S is a seminormal scheme of* char 0*. Assume furthermore that* reduced enough??

(1) *$f|_U : U \to S$ is smooth (hence every fiber is irreducible),*
(2) *every fiber has lc normalization $\pi_s : \bigl(\bar X_s, \bar\Delta_s\bigr) \to \bigl(X_s, \Delta_s\bigr)$ and*
(3) *the canonical models $\phi_s : \bigl(\bar X_s, \bar\Delta_s\bigr) \dashrightarrow \bigl(X^{\mathrm{c}}_s, \Delta^{\mathrm{c}}_s\bigr)$ exist.*

Then

(4) $s \mapsto \operatorname{vol}(K_{\bar{X}_s} + \bar{\Delta}_s)$ *is an upper semicontinuous function on* S *and*
(5) $f : (X, \Delta) \to S$ *has a simultaneous canonical model iff this function is locally constant.*

One should think of (5.11) as a generalization of (5.3) but there are differences. In (5.11) we allow only fibers that are smooth in codimension 1 and S is assumed seminormal, not just reduced. (The extra assumption (3) is expected to hold always.) However, the key difference is in the proofs given in Section 5.9. While the proof of (5.3) uses only the basic theory of hulls and husks, we rely on the existence of moduli spaces of pairs in order to establish (5.11).

Both (5.10) and (5.11) apply to $f : X \to S$ iff the normalizations of the fibers have canonical singularities. In this case f is locally stable (2.8) and the plurigenera—and hence the volume—are locally constant [**Siu98, Kaw99**].

A key ingredient of the proof of (5.10–5.11) is the following characterization of canonical models. We prove a more general version of it in (10.34).

Proposition 5.12. *Let* X *be a smooth proper variety of dimension* n*. Let* Y *be a normal, proper variety birational to* X *and* D *an effective* $\mathbb{Q}$*-divisor on* Y *such that* $K_Y + D$ *is* $\mathbb{Q}$*-Cartier, nef and big. Then*

(1) $\operatorname{vol}(K_X) \le \operatorname{vol}(K_Y + D) = (K_Y + D)^n$ *and*
(2) *equality holds iff* $D = 0$ *and* Y *has canonical singularities (thus* Y *is a weak canonical model (cf.* [**Kol13c**, 1.19]*) of* X*).*

For surfaces, the existence criterion of simultaneous canonical modifications is proved in [**KSB88**, Sec.2]. In higher dimensions we need to work with a sequence of intersection numbers and with their lexicographic ordering.

Definition 5.13. Let X be a proper scheme of dimension n and A, B $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Their *sequence of intersection numbers* is

$$I(A, B) := \big((A^n), \dots, (A^{n-i} \cdot B^i), \dots, (B^n)\big) \in \mathbb{Q}^{n+1}.$$

Definition 5.14. The *lexicographic* ordering of length $n+1$ real sequences is denoted by

$$(a_0, \dots, a_n) \preceq (b_0, \dots, b_n).$$

This holds if either $a_i = b_i$ for every i or there is an $r \le n$ such that $a_i = b_i$ for $i < r$ but $a_r < b_r$. For polynomials we define an ordering

$$f(t) \preceq g(t) \;\Leftrightarrow\; f(t) \le g(t) \;\forall t \gg 0.$$

Note that

$$\textstyle\sum_i a_i t^{n-i} \preceq \sum_i b_i t^{n-i} \Leftrightarrow (a_0, \dots, a_n) \preceq (b_0, \dots, b_n).$$

If we have proper schemes X, X' of dimension n and $\mathbb{Q}$-Cartier divisors A, B on X and A', B' on X' then

$$I(A, B) \preceq I(A', B') \;\Leftrightarrow\; (mA + B)^n \le (mA' + B')^n \;\forall m \gg 0.$$

We will consider functions that associate a sequence or a polynomial to all points of a scheme X. Using the above definitions it makes sense to ask if such a function is *lexicographically upper/lower semicontinuous* or not.

Definition 5.15 (Canonical and log canonical modification). Let Y be a scheme over a field k. (We allow Y to be reducible and non-reduced but in applications usually pure dimensional.) Its *canonical modification* $p : Y^{\mathrm{can}} \to Y$ is the canonical modification of its normalization $\pi : \bar{Y} \to Y$; that is, Y^{can} has canonical singularities, $Y^{\mathrm{can}} \to \bar{Y}$ is proper, birational and $K_{Y^{\mathrm{can}}}$ is ample over $\bar{Y}$.

Let Δ be an effective divisor on Y. We define the *canonical modification* $p : \big(Y^{\mathrm{can}}, \Delta^{\mathrm{can}}\big) \to (Y, \Delta)$ as the canonical modification of the normalization $\big(\bar{Y}, \bar{\Delta} := \pi^*\Delta\big)$, provided that it makes sense. That is, the pull-back $\pi^*\Delta$ should be defined as in (5.2) and its irreducible components should have coefficient ≤ 1. If these hold then $\bar{p} : Y^{\mathrm{can}} \to \bar{Y}$ is the unique proper, birational morphism such that $\Delta^{\mathrm{can}} := (\bar{p})^{-1}_*\bar{\Delta}$, $\big(Y^{\mathrm{can}}, \Delta^{\mathrm{can}}\big)$ is canonical and $K_{Y^{\mathrm{can}}} + \Delta^{\mathrm{can}}$ is ample over $\bar{Y}$; see [**Kol13c**, 1.31].

The *log canonical modification* $p : \big(Y^{\mathrm{lc}}, \Delta^{\mathrm{lc}}\big) \to (Y, \Delta)$ is defined similarly. The change is that $\big(Y^{\mathrm{lc}}, \Delta^{\mathrm{lc}} + E^{\mathrm{lc}}\big)$ is log canonical and $K_{Y^{\mathrm{can}}} + \Delta^{\mathrm{can}} + E^{\mathrm{lc}}$ is ample over $\bar{Y}$ where E^{lc} denotes the reduced exceptional divisor of $\bar{p}$.

Log canonical modifications are conjectured to exist. Currently this is known when $K_Y + \Delta$ is $\mathbb{Q}$-Cartier; see [**OX12**], (4.54) or [**Kol13c**, 1.32].

Definition 5.16 (Simultaneous canonical modification). Let $f : X \to S$ be a morphism of pure relative dimension n and $\Delta = \sum a_i D_i$ a generically $\mathbb{Q}$-Cartier effective divisor on Y. A *simultaneous canonical modification* is a proper morphism $p : (Y, \Delta^Y) \to (X, \Delta)$ such that $f \circ p : (Y, \Delta^Y) \to S$ is locally stable and

$$p_s : \big(Y_s, (\Delta^Y)_s\big) \to (X_s, \Delta_s)$$

is the canonical modification for every $s \in S$.

A *simultaneous log canonical modification* is defined analogously.

In the following result we definitely need to assume that the base scheme is seminormal; see (5.24) for some examples.

Theorem 5.17 (Numerical criterion for simultaneous canonical modification). *We use the notation of (5.2). In addition to (5.2.1–4) assume that S is a seminormal scheme of* char 0 *and H is a relatively ample Cartier divisor class on X. For $s \in S$ let $\pi_s : \big(X_s^{\mathrm{can}}, \Delta_s^{\mathrm{can}}\big) \to (X_s, \Delta_s)$ denote the canonical modification of the fiber (X_s, Δ_s). Then*

(1) *$s \mapsto I\big(\pi_s^* H_s, K_{X_s^{\mathrm{can}}} + \Delta_s^{\mathrm{can}}\big)$ is a lexicographically lower semicontinuous function on S and*
(2) *$f : (X, \Delta) \to S$ has a simultaneous canonical modification iff this function is locally constant.*

There is also a similar condition for simultaneous log canonical and semi-log-canonical modifications (5.45) but these only apply when $K_{X/S} + \Delta$ is $\mathbb{Q}$-Cartier.

5.3. Examples

Here we present a series of examples that show that the assumptions of the Theorems in Sections 5.1–5.2 are close to being optimal except that the characteristic 0 assumption is probably superfluous.

Examples related to Theorems 5.3, 5.4 and 5.7. ‘

The following is the simplest example illustrating the difference between being Cartier and fiber-wise Cartier.

EXAMPLE 5.18. Consider the family of quadrics

$$X = (x^2 - y^2 + z^2 - t^2w^2 = 0) \subset \mathbb{P}^3_{xyzw} \times \mathbb{A}_t \quad \text{and} \quad D = (x - y = z - tw = 0).$$

Here X_0 is a quadric cone and X_t is a smooth quadric for $t \neq 0$. The divisor D is Cartier, except at the origin, where it is not even $\mathbb{Q}$-Cartier. However D_0 is a line on a quadric cone, hence $2D_0 = (x - y = 0)$ is Cartier. It is easy to compute that

$$L = \mathcal{O}_X(-2D) = (x - y, z - tw)^2 \cdot \mathcal{O}_X$$

is locally free outside the origin, not locally free at the origin but the hull of its restriction

$$L_0^H := \mathcal{O}_{X_0}(-2D_0) = (x - y) \cdot \mathcal{O}_{X_0}$$

is locally free. The natural restriction map gives an identification

$$\mathcal{O}_X(-2D)|_{X_0} = (x, y, z) \cdot \mathcal{O}_{X_0}(-2D_0) \subset \mathcal{O}_{X_0}(-2D_0).$$

Note that the self-intersection number of the fibers of D also jumps. For $t \neq 0$ we have $(D_t^2) = 0$ but $(D_0^2) = 1/2$.

It is harder to get examples where the self-intersections in (5.7) are locally constant yet the divisor is not Cartier, but, as we see next, this can happen even for the canonical class. Thus in (5.7) one needs to assume that the fibers of f are S_2 and in (5.3) that the fibers are slc.

EXAMPLE 5.19. (See (2.34) or [**Kol13c**, 3.8–14] for the notation and basic results on cones.) Let $X \subset \mathbb{P}^N$ be a smooth, projective variety of dimension n and $L_X = \mathcal{O}_X(1)$. Let $C(X) := C_p(X, L_X)$ denote the projective cone over X with vertex v and natural ample line bundle $L_{C(X)}$. Let $H \subset X$ be a smooth hyperplane section and $C(H) := C_p(H, L_H)$ the projective cone over H. Note that

$$\left(L_X^n\right) = \left(L_{C(X)}^{n+1}\right) = \left(L_H^{n-1}\right) = \left(L_{C(H)}^n\right).$$

The canonical class of $C(X)$ is Cartier iff $K_X \sim mc_1(L_X)$ for some $m \in \mathbb{Z}$. In this case $K_{C(X)} \sim (m - 1)c_1(L_{C(X)})$.

We can think of H as sitting in $X \subset C(X)$. The pencil of hyperplanes containing $H \subset C(X)$ gives a morphism of the blow-up $p : Y := B_H C(X) \to \mathbb{P}^1$ such that $Y_t \cong X$ for $t \neq 0$ and the normalization $\bar{Y}_0$ of Y_0 is isomorphic to $C(H)$. However, if $H^1(X, \mathcal{O}_X) \neq 0$ then Y_0 is not normal. For instance, this happens if X is the product of non-hyperelliptic curves of genus ≥ 2 with its canonical embedding. Thus, if these hold, then

(1) Y_t is smooth and K_{Y_t} is ample for $t \neq 0$,
(2) $K_{\bar{Y}_0}$ is locally free and ample,
(3) the normalization $\bar{Y}_0 \to Y_0$ is an isomorphism except at v,
(4) $\left(K_{Y_t}^n\right) = \left(K_{\bar{Y}_0}^n\right)$ (where $n = \dim X$) yet
(5) Y_0 is not normal.

This shows that (5.3) needs some assumptions about the singularities of the normalizations of the fibers. However, in this example K_Y is Cartier.

One can get another example where the canonical class of the total space is not Cartier as follows.

We can also obtain the family $Y \to \mathbb{P}^1$ by starting with $X \times \mathbb{P}^1$, blowing up $H \times \{0\}$ and contracting the birational transform of $X \times \{0\}$. This construction shows that if

Assume next that $\mathrm{Pic}(X)$ is positive dimensional. After a suitable base change $(c \in C) \to (0 \in \mathbb{P}^1)$, there is a line bundle M_C on $X \times C$ that is trivial on $X \times \{c\}$ but only numerically trivial on $X \times \{c'\}$ for general $c' \in C$. After blowing up $H \times \{c\}$ and contracting the birational transform of $X \times \{c\}$ we get $(v \in Y) \to (c \in C)$ and a line bundle M on $Y \setminus \{v\}$ such that M is trivial on $Y_c \setminus \{v\}$ but only numerically trivial on $Y_{c'}$ for general $c' \in C$.

Let $Z \to Y$ be a double cover ramified along a general section of $M^2 \otimes p_1^* L_X^{2m}$ for $m \gg 1$. Then we get a morphism of a normal variety Z to a smooth curve $p_Z : Z \to C$ such that

(6) Z_t is smooth and K_{Z_t} is ample for $t \neq 0$,
(7) $K_{\bar{Z}_0}$ is locally free and ample,
(8) the normalization $\bar{Z}_0 \to Z_0$ is an isomorphism except at a point v,
(9) $\left(K_{Z_t}^n\right) = \left(K_{\bar{Z}_0}^n\right)$ yet
(10) K_Z is not Cartier at v.

Examples related to Theorems 5.10 and 5.11.

The next example shows that (5.10) fails if S is not seminormal.

EXAMPLE 5.20. Let S be a local, reduced but not seminormal scheme with seminormalization $S' \to S$. Choose an embedding of S' into the moduli space of automorphism-free curves of genus g for some g. Let $p' : X' \to S'$ be the resulting smooth family. This induces a family $p : X' \to S' \to S$ that satisfies the assumptions of (5.10). However, there is no simultaneous canonical model since $p' : X' \to S'$ does not descend to $p : X \to S$.

The next examples show that there does not seem to be a log version of (5.10) for families with reducible fibers, not even for families of curves.

EXAMPLE 5.21. Let $g : S \to C$ be a smooth family of curves and $D_i \subset S$ a set of n disjoint sections. Set $\Delta := \sum d_i D_i$. Pick a point $0 \in C$, the fiber over it is $(S_0, \sum d_i[p_i])$ where $p_i = S_0 \cap D_i$. The "log volume" is $2g(S_0) - 2 + \sum d_i$.

Let $\pi : S^1 \to S$ be the blow up of all the points p_i with exceptional curves E_i and set $\Delta^1 := \pi_*^{-1}\Delta$. The central fiber of $g^1 : (S^1, \Delta^1) \to C$ is $(S_0^1, 0) + \sum_i (E_i, d_i[p_i'])$. Its normalization consists of S_0 (with no boundary points) and $E_i \cong \mathbb{P}^1$, each with one marked point of multiplicity d_i. Thus the "log volume" of the central fiber is now $2g(S_0) - 2$; the effect of the boundary vanished.

One can try to compensate for this, as in (5.30), by adding the double point divisor $\bar{D}_0$. This variant of the "log volume" is now $2g(S_0) - 2 + n$. This formula remembers only the number of the sections, not their coefficients. Even worse, we can blow up m other points on S_0, then the "log volume" formula gives $2g(S_0) - 2 + n + m$.

In general, there does not seem to be a sensible and birationally invariant way do define the "log volume" of degenerations. For families of curves one can use the degree of the log canonical class; this gives negative contribution for some of the components. I do not know whether something similar can be done in higher dimension or not.

The next series of examples shows that, even for locally stable morphisms, the canonical models of the fibers need not form a flat family.

EXAMPLE 5.22. Let $f : X \to B$ be a locally stable family of surfaces. Assume for simplicity that the fibers have only quotient singularities.

Let $g : X \to Z$ be a flipping contraction. (For concrete examples, see [**KM98**, 2.7] or the exhaustive list in [**KM92**].) Thus there is a closed point $0 \in B$ such that g is an isomorphism over $B \setminus \{0\}$. Over the special point we have a birational contraction $g_0 : X_0 \to Z_0$ that contracts an irreducible curve $C \subset X_0$ to a point. Moreover $(C \cdot K_{X_0}) = (C \cdot K_X) < 0$, thus Z_0 is again log terminal and the contraction $g_0 : X_0 \to Z_0$ is a step in the MMP for X_0.

However, since $g : X \to Z$ a flipping contraction, the special fiber of the flip $g^+ : X^+ \to Z$ is another surface $X_0^+ \to Z_0$ with a new exceptional curve $C^+ \subset X_0^+$ such that $\bigl(C^+ \cdot K_{X_0^+}\bigr) = (C^+ \cdot K_{X^+}) > 0$. Thus X_0^+ is not the canonical model of X_0 and $X_0 \dashrightarrow X_0^+$ is not even a correct step of the minimal model program.

It is easy to write down examples when $g^+ : X^+ \to Z$ is the canonical model of $g : X \to Z$; thus we get many examples without simultaneous canonical models.

In the above examples X_0 is log terminal but never canonical. There are further counter examples when (X, Δ) is canonical but $\Delta \neq 0$.

EXAMPLE 5.23. Set $Y := (xy + z^2 - s^2) \subset \mathbb{A}^4$ and $X := B_{(x,z-s)}Y$ with 4th projection $\pi : X \to \mathbb{A}^1_s$. The central fiber X_0 is the minimal resolution of the quadric cone $Y_0 := (xy+z^2) \subset \mathbb{A}^3$ with exceptional curve $E_0 \subset X_0$. Let D_1 be the birational transform of $(y = z + s = 0) \subset Y$. Note that D_1 is smooth but $D_1|_{X_0} = E_0 + L_0$ where L_0 denotes the birational transform of the line $L_Y := (y = z + s = s = 0)$. Thus $\bigl(X_0, \epsilon D_1|_{X_0}\bigr)$ is canonical if $\epsilon \leq \frac{1}{2}$ and terminal if $\epsilon < \frac{1}{2}$. Furthermore,

$$(E_0 \cdot D_1)_X = (E_0 \cdot E_0)_{X_0} + (E_0 \cdot L_0)_{X_0} = -2 + 1 = -1.$$

For any $0 < \epsilon \leq 1$ the canonical model of $(X, \epsilon D_1) \to Y$ is given by the flop $X^+ := B_{(x,z+s)}Y$. Note that $X_0^+ \cong X_0$ and under this isomorphism $D_1^+|_{X_0^+} = L_0$. Thus E_0 is not contained in $\operatorname{Supp} D_1^+$.

Therefore $\bigl(X_0^+, \epsilon D_1^+|_{X_0^+}\bigr)$ is its own canonical model, but the canonical model of $\bigl(X_0, \epsilon D_1|_{X_0}\bigr)$ is $\bigl(Y_0, \epsilon L_Y\bigr)$.

We see that the map $X_0 \dashrightarrow X_0^+$ is the identity, the problem is the unexpected change in the boundary divisor D_1.

One can obtain from the above local example a global one as follows. Compactify Y as

$$\bar{Y} := (xy + z^2 - t^2 s^2) \subset \mathbb{P}^3_{xyzt} \times \mathbb{A}^1_s$$

with $\bar{\pi} : \bar{Y} \to \mathbb{A}^1_s$ the projection. Set $\bar{X} := B_{(x,z-ts)}\bar{Y}$ and let $\bar{D}_1$ be the birational transform of $(y = z+ts = 0) \subset \bar{Y}$. Let $\bar{D}_2, \dots, \bar{D}_5$ denote the pull-back of 4 general hyperplanes in $\mathbb{P}^3$. Fix $0 < \epsilon < \frac{1}{8}$ and consider

$$\bigl(\bar{X}, \bar{\Delta} := 4\epsilon D_1 + \bigl(\tfrac{1}{2} - \tfrac{\epsilon}{4}\bigr)\bigl(\bar{D}_2 + \cdots + \bar{D}_5\bigr)\bigr).$$

Every fiber of $\bar{X} \to \mathbb{A}^1_s$ is terminal.

The central fiber is is the minimal resolution of the quadric cone. Since the pull-back of the hyperplane class is $E_0 + 2L_0$, the boundary divisor $\bar{\Delta}_0$ is linearly equivalent to

$$4\epsilon(E_0 + L_0) + (2 - \epsilon)(E_0 + 2L_0) = \bigl(2E_0 + 4L_0\bigr) + 2\epsilon E_0 + \epsilon\bigl(E_0 + 2L_0\bigr).$$

The canonical class of the quadric is -2(hyperplane class), thus we get that

$$K_{\bar{X}_0} + \bar{\Delta}_0 \sim 2\epsilon E_0 + \epsilon\bigl(E_0 + 2L_0\bigr),$$

thus $\bigl(\bar{X}_0, \bar{\Delta}_0\bigr)$ is of general type.

The general fiber is a smooth quadric; choose the two families of lines A, B such that D_1 restricts to A. Then the boundary divisor $\bar{\Delta}_g$ is linearly equivalent to

$$4\epsilon A + (2-\epsilon)(A+B) = 2(A+B) + 2\epsilon A - 2\epsilon B.$$

Therefore

$$K_{\bar{X}_g} + \bar{\Delta}_g \sim 2\epsilon A - 2\epsilon B,$$

hence its Kodaira dimension is $-\infty$.

Examples related to Theorem 5.17.

In (5.17) the base scheme is assumed to be seminormal. The reason for this is that canonical modifications do have unexpected infinitesimal deformations.

Example 5.24 (Deformation of canonical modifications). We give an example of a normal, projective variety with isolated singularities and canonical modification $X^{\mathrm{c}} \to X$ such that the trivial deformation of X can be lifted to a nontrivial deformation of X^{c}.

Consider the hypersurface

$$X := X_{n,r} := \bigl(x_1^r + \cdots + x_n^r + x_{n+1}^{r+1} = 0\bigr) \subset \mathbb{A}_k^{n+1}.$$

It has an isolated singularity at the origin which is canonical iff $r \leq n$.

Let $p : Y := B_0 X \to X$ denote the blow-up of the origin. Then Y is smooth and, for $r > n$, it is the canonical modification of X.

We claim that $p : Y \to X$ has a nontrivial deformation over $X \times_k \operatorname{Spec} k[\epsilon]$. The trivial deformation is obtained by blowing up

$$(x_1 = \cdots = x_{n+1} = 0) \subset X \times_k \operatorname{Spec} k[\epsilon].$$

The nontrivial deformation is obtained by blowing up

$$Z := (x_1 = \cdots = x_n = x_{n+1} - \epsilon = 0) \subset X \times_k \operatorname{Spec} k[\epsilon].$$

We need to check that X is equimultiple along the blow-up center. This is more transparent if we introduce a new coordinate $y := x_{n+1} - \epsilon$. Then the equations become

$$Z := (x_1 = \cdots = x_n = y = 0) \subset \bigl(x_1^r + \cdots + x_n^r + y^{r+1} + (r+1)\epsilon y^r = 0\bigr),$$

thus $X \times_k \operatorname{Spec} k[\epsilon]$ is clearly equimultiple along Z.

The following examples show that the existence of simultaneous canonical modifications is more complicated for pairs.

Example 5.25. In $\mathbb{P}^2$ consider a line $L \subset \mathbb{P}^2$ and a family of degree 8 curves C_t such that C_0 has 4 nodes on L plus an ordinary 6-fold point outside L and C_t is smooth and tangent to L at 4 points for $t \neq 0$.

Let $\pi_t : S_t \to \mathbb{P}^2$ denote the double cover of $\mathbb{P}^2$ ramified along C_t. Note that $K_{S_t} = \pi_t^* \mathcal{O}(1)$, thus $(K_{S_t}^2) = 2$. For each t, the preimage $\pi_t^{-1}(L)$ is a union of 2 curves $D_t + D_t'$. Our example is the family of pairs (S_t, D_t). We claim that

(1) there is a log canonical modification $\bigl(S_t^{\mathrm{lc}}, D_t^{\mathrm{lc}}\bigr) \to (S_t, D_t)$ for every t and
(2) $\bigl(K_{S_t^{\mathrm{lc}}} + D_t^{\mathrm{lc}}\bigr)^2 = 1$ for every t yet

(3) there is no simultaneous log canonical modification.

If $t \neq 0$ then S_t is smooth and D_t is smooth. Furthermore D_t, D'_t meet transversally at 4 points, thus $(D_t \cdot D'_t) = 4$. Using $\bigl((D_t + D'_t)^2\bigr) = 2$, we obtain that $(D_t^2) = -3$. Thus $(K_{S_t} + D_t)^2 = 1$.

If $t = 0$ then S_0 is singular at 5 points. D_0, D'_0 meet transversally at 4 singular points of type A_1, thus $(D_0 \cdot D'_0) = 2$. This gives that $(D_0^2) = -1$. Thus $(K_{S_0} + D_0)^2 = 3$. The pair (S_0, D_0) is lc away from the preimage of the 6-fold point. Let $q : T_0 \to S_0$ denote the minimal resolution of this point. The exceptional curve E is smooth, has genus 2 and $(E^2) = -2$. Thus $K_{T_0} = q^* K_{S_0} - 2E$ hence $(T_0, E + D_0)$ is the log canonical modification of (S_0, D_0) and

$$(K_{T_0} + E + D_0)^2 = \bigl(q^* K_{S_0} - E + D_0\bigr)^2 = (K_{S_0} + D_0)^2 + (E^2) = 1.$$

Thus $\bigl(K_{S_t^{\rm lc}} + D_t^{\rm lc}\bigr)^2 = 1$ for every t.

Nonetheless, the log canonical modifications do not form a flat family. Indeed, such a family would be a family of surfaces with ordinary nodes, so the relative canonical class would be a Cartier divisor. However, $(K_{S_t}^2) = 2$ for $t \neq 0$ but $(K_{T_0}^2) = \bigl(q^* K_{S_0} - 2E\bigr)^2 = -6$.

EXAMPLE 5.26. We start with a family of quadric surfaces $Q_t \subset \mathbb{P}^3$ where Q_0 is a cone and Q_t is smooth for $t \neq 0$. We take 6 families of lines L_t^i such that for $t = 0$ we have 6 distinct lines and for $t \neq 0$ two of them L_t^1, L_t^2 are from one ruling of the quadric, the other 4 from the other ruling.

Finally S_t denotes the double cover of Q_t ramified along the 6 lines $L_t^1 + \cdots + L_t^6$.

For $t \neq 0$ the surface S_t has ordinary nodes and $(K_{S_t}^2) = 0$.

For $t = 0$ the surface S_0 has a unique singular point. Its minimal resolution $q : T_0 \to S_0$ is a double cover of $\mathbb{F}_2$ ramified along 6 fibers. Thus $(K_{T_0}^2) = -4$. Thus the canonical modifications do not form a flat family. The log canonical modification of S_0 is (T_0, E_0) where E_0 is the q-exceptional curve. Thus $(K_{T_0} + E_0)^2 = 0$.

The numerical condition is satisfied but the log canonical modifications do not form a flat family since $T_0 = S_0^{\rm lc}$ is smooth but $S_t^{\rm lc} = S_t$ is singular for $t \neq 0$.

However, there is a flat family that is a weaker variant of a simultaneous log canonical modification.

This is obtained by replacing the singular quadric Q_0 with its resolution $Q'_0 \cong \mathbb{F}_2$. Let $E \subset \mathbb{F}_2$ denote the -2-section and $|F|$ the ruling. One can arrange that L_t^1, L_t^2 degenerate to $F^i + E$ for $F^i \in |F|$ and the others degenerate to fibers F^j. This way a flat limit of the double covers S_t is obtained as the double cover of $\mathbb{F}_2$ ramified along $F^1 + \cdots + F^6 + 2E$. This is a semi-log-canonical surface whose normalization is the log canonical modification of S_0.

5.4. Stability criteria in codimension 1

As a preliminary step we characterize those morphisms that are locally stable in codimension ≤ 1 on each fiber. In applications this is rarely an issue but it is instructive to see which arguments work or fail.

EXAMPLE 5.27. Let $f : X \to C$ be a projective morphism from a normal variety of dimension $n + 1$ to a smooth curve and H a relatively ample divisor class on X. We would like to understand when f is stable in terms of numerical invariants of the normalizations of the fibers $\pi_c : \bar{X}_c \to X_c$. The simplest invariant is the self-intersection number $(\pi_c^* H)^n$ which describes the codimension 0 behavior of f. It is clear that $c \mapsto (\pi_c^* H)^n$ is a

(1) lower semicontinuous function on C and
(2) it is locally constant iff the fibers are generically reduced.

The following result is a reformulation of [**Kol96**, I.6.5].

THEOREM 5.28 (Smoothness criterion in codimension 0). *Let S be a weakly-normal scheme, $f : X \to S$ a projective morphism of pure relative dimension n (3.34) and H an f-ample divisor class. Assume that X is reduced and for $s \in S$ let $\pi_s : \bar{X}_s \to X_s$ denote the normalization of the fiber. Then $s \mapsto (\pi_s^* H)^n$ is a lower semicontinuous function on S and it is locally constant iff there is a closed subset $Z_1 \subset X$ such that*

(1) $\dim_s(Z_1 \cap X_s) \leq n-1$ *for every* $s \in S$ *and*
(2) $f : (X \setminus Z_1) \to S$ *is smooth.* □

The following example illustrates the necessity of the assumptions.

EXAMPLE 5.29. Consider the family of conics

$$S := (x^2 + ty^2 + tz^2 = 0) \subset \mathbb{P}^2_{xyz} \times \mathbb{A}^1_t \quad \text{with} \quad H := \mathcal{O}_S(1).$$

For $t \neq 0$ the fiber is a smooth conic and $\deg\bigl(H|_{S_t}\bigr) = 2$. For $t = 0$ the fiber is a double line whose normalization is a line $\bar{S}_0$ and $\deg\bigl(H|_{\bar{S}_0}\bigr) = 1$. Projection to $\mathbb{A}^1_t$ is not smooth along the fiber S_0.

Let us now take two disjoint copies $X := S_1 \amalg S_2$ and map them to the corresponding coordinate axes $C = (t_1 t_2 = 0) \subset \mathbb{A}^2$ to get $f : X \to C$.

For $c \neq (0,0)$ the fiber is a smooth conic and $\deg\bigl(H|_{X_c}\bigr) = 2$. For $c = (0,0)$ the fiber is a disjoint union of two double lines whose normalization is a disjoint union of two lines. Thus $\deg\bigl(H|_{\bar{X}_{(0,0)}}\bigr) = 2$. The degree of the reduced fibers is always 2 yet $f : X \to C$ is not smooth along the central fiber $X_{(0,0)}$.

Note that although every fiber of f has dimension 1, f does not have pure relative dimension 1, thus (5.28) is not contradicted.

One can hope that, similarly, the codimension 1 behavior of f is described by $\bigl((\pi_s^* H)^{n-1} \cdot K_{\bar{X}_s}\bigr)$. We can think of intersecting with $(\pi_s^* H)^{n-1}$ as restricting to the complete intersection of $n-1$ very ample divisors $H_i \subset X$. Thus, for all practical purposes, we may work with a normal surface S and a flat, proper family of curves $f : S \to T$ where T is the spectrum of a DVR with closed point 0 and generic point t.

It is again easy to see that $\deg K_{\bar{S}_0} \leq \deg K_{S_t}$ and, if the generic fiber has genus ≥ 2 then $f : S \to T$ is smooth iff equality holds. (Note that rational or elliptic curves can degenerate to a multiple rational or elliptic curve, thus the genus ≥ 2 assumption is necessary.)

In order to characterize stability, we need to take the singularities of the central fiber into account. In the stable case the correct formula adds 1 for each point on $\bar{S}_0$ whose image is a singular point of the fiber S_0. So let us define the divisor of singularities $\bar{D}_0 \subset \bar{S}_0$ as the sum of all the points $p \in \bar{S}_0$ such that $\operatorname{red} S_0$ is singular at $\pi_0(p)$. Let us say that the fiber S_0 is *pre-stable* if $K_{\bar{S}_0} + \bar{D}_0$ is ample. We have the following stability criterion.

LEMMA 5.30. *Let S be a normal surface, $(0, T)$ the spectrum of a DVR and $f : S \to T$ a proper morphism with pre-stable central fiber. Then*

(1) $\deg\bigl(K_{\bar S_0} + \bar D_0\bigr) \le \deg\bigl(K_{S_t}\bigr)$ *and*
(2) $f : S \to T$ *is stable iff equality holds.*

Proof. In order to prove these write $S_0 = \sum e_i E_i$ with normalizations $\pi_i : F_i \to E_i$. The key is to understand that we have to work with the divisor $K_S + \sum E_i$. We use that on a normal surface intersection numbers make sense for arbitrary divisors by [**Mum61**]. Since $\sum E_i$ is disjoint from the generic fiber,

$$\begin{array}{rcl} \deg\bigl(K_{S_t}\bigr) & = & \bigl(K_S \cdot S_t\bigr) = \bigl((K_S + \sum E_i)\cdot S_t\bigr) \\ & = & \sum_j e_j\bigl((K_S + \sum E_i)\cdot E_j\bigr) \\ & = & \sum_j e_j \deg \pi_j^*(K_S + \sum E_i). \end{array} \tag{5.30.3}$$

As in (1.90) we can write

$$\pi_j^*(K_S + \sum E_i) = K_{F_j} + \operatorname{Diff}_{F_j}\bigl(\sum_{i\neq j} E_i\bigr).$$

Using (1.92) we obtain that

$$\deg \pi_j^*(K_S + \sum E_i) \ge \deg\bigl(K_{F_j} + \bar D_j\bigr) \tag{5.30.4}$$

and equality holds iff all singularities of $\operatorname{red} S_0$ are ordinary nodes. We can thus continue the inequalities (5.30.3) to get that

$$\begin{array}{rcl} \deg\bigl(K_{S_t}\bigr) & = & \sum_j e_j \deg \pi_j^*(K_S + \sum E_i) \\ & \ge & \sum_j e_j \deg\bigl(K_{F_j} + \bar D_j\bigr) \\ & \ge & \sum_j \deg\bigl(K_{F_j} + \bar D_j\bigr) \\ & = & \deg\bigl(K_{\bar S_0} + \bar D_0\bigr), \end{array} \tag{5.30.5}$$

and the first inequality is an equality iff all the singularities of $\operatorname{red} S_0$ are ordinary nodes and the second inequality is an equality iff S_0 is reduced. □

Although (5.30) is promising, the normality assumption on S makes dimension induction difficult and the following example shows that its natural analog fails if T is replaced by a nodal curve.

EXAMPLE 5.31. Let $p(x)$ be a polynomial of degree $2d$ without multiple roots and pick $a_1 \neq a_2$ that are not roots of p. Let C_i denote the compactification (smooth at infinity) of the singular hyperelliptic curve $\bigl(y^2 = (x - a_i)^4 p(x)\bigr)$ and C_0 the compactification of $\bigl(y^2 = (x-a_1)^4(x-a_2)^4 p(x)\bigr)$. (Thus C_i has a tacnode at $x = a_i$ for $i = 1, 2$, C_0 has 2 tacnodes and they are smooth elsewhere.) For $i = 1, 2$ let $\pi_i' : S_i' \to \mathbb{A}^1_{t_i}$ be a general smoothing of C_i. The generic fiber has genus $d+1$. There are natural finite maps $C_i \to C_0$ that pinch the points $\bigl(a_{3-i}, \pm(a_i - a_{3-i})^2\sqrt{p(a_i)}\bigr)$ together. Doing the same pinching on S_i' we get surfaces $\pi_i : S_i \to \mathbb{A}^1_{t_i}$ where the central fiber is C_0 (plus some embedded points). We can identify the reduced central fibers to get a reducible surface $S = S_1 \amalg_{C_0} S_2$ and a morphism $\pi : S \to (t_1 t_2 = 0) \subset \mathbb{A}^2_{t_1 t_2}$. Note that although the S_i are not seminormal, S itself is seminormal and satisfies Serre's condition S_2.

The normalization $\bar C_0$ of C_0 has genus $d-1$. The divisor of singularities consists of the 4 preimages of the points $(x = a_i)$. Thus for $t \neq (0, 0)$ we have

$$\deg(K_{S_t}) = 2d \quad \text{and} \quad \deg\bigl(K_{\bar S_{(0,0)}} + \bar D_{(0,0)}\bigr) = 2d - 4 + 4 = 2d.$$

Hence the numerical stability criterion of (5.30) does not extend to seminormal, S_2 surfaces over nodal curves.

A satisfactory analog of (5.30) over higher dimensional normal bases is proved in [**Kol11b**, 14–15].

We can thus expect that, for families that are locally stable in codimension 1, there are results connecting the intersection numbers $\bigl((\pi_0^*H)^{n-i}\cdot(K_{\bar X_0}+\bar D_0)^i\bigr)$ with the higher codimension behavior of f. There are two surprising twists.

- The *lower* semicontinuity in (5.28) and (5.30) switches to *upper* semicontinuity.
- In most cases we need only *one* more intersection number to take care of all codimensions.

5.5. Deformations of slc pairs

So far we have focused on locally stable deformations of slc pairs. The next result, due to [**KSB88**], connects arbitrary flat deformations (X_t,Δ_t) of an slc pair (X_0,Δ_0) to locally stable deformations of a suitable birational model $f_0:\bigl(Y_0,\Delta_0^Y\bigr)\to(X_0,\Delta_0)$. We then compare various numerical invariants of (X_0,Δ_0) and of (X_t,Δ_t) by going through $\bigl(Y_0,\Delta_0^Y\bigr)$. This implies a weaker version of (5.4).

THEOREM 5.32. [**KSB88**] *Let (X,Δ) be a normal pair and $g:X\to C$ a flat morphism of pure relative dimension n to a smooth pointed curve $(0\in C)$. Assume that the fiber X_0 is nodal in codimension 1 and its normalization $\bigl(\bar X_0,\operatorname{Diff}_{\bar X_0}\Delta\bigr)$ is lc. Let $f:(Y,\Delta^Y+Y_0+E)\to(X,X_0+\Delta)$ be the log canonical modification as in (5.15) where Δ^Y+Y_0 is the birational transform of $\Delta+X_0$ and E is f-exceptional. Then, possibly after shrinking $(0\in C)$, the following hold.*

(1) *f is small (that is, $E=0$) and $f\bigl(\operatorname{Ex}(f)\bigr)$ is precisely the locus where g is not locally stable.*
(2) *$g\circ f:(Y,\Delta^Y)\to C$ is locally stable.*
(3) *For every f_0-exceptional divisor $F\subset Y_0$, the divisor Y_0 is normal at the generic point of F and $a\bigl(F,\bar X_0,\operatorname{Diff}_{\bar X_0}\Delta\bigr)<0$.*

Furthermore, if $K_X+\Delta$ is $\mathbb{Q}$-Cartier on the generic fiber then

(4) *f is an isomorphism over $C\setminus\{0\}$ and*
(5) *g is locally stable over $C\setminus\{0\}$.*

Proof. Let $\pi_X:\bar X_0\to X_0$ and $\pi_Y:\bar Y_0\to Y_0$ be the normalizations. Then f_0 lifts to $\bar f_0:\bar Y_0\to\bar X_0$. Write $K_{\bar Y_0}+\Delta_{\bar Y_0}\sim_{\mathbb{Q}}\bar f_0^*\bigl(K_{\bar X_0}+\operatorname{Diff}_{\bar X_0}\Delta\bigr)$. By adjunction,

$$\begin{aligned}\pi_Y^*\bigl(K_Y+\Delta^Y+Y_0+E\bigr)\ &\sim_{\mathbb{Q}}\ K_{\bar Y_0}+\operatorname{Diff}_{\bar Y_0}\bigl(\Delta^Y+E\bigr)\\ &\sim_{\mathbb{Q}}\ \bar f_0^*\bigl(K_{\bar X_0}+\operatorname{Diff}_{\bar X_0}\Delta\bigr)+\bigl(\operatorname{Diff}_{\bar Y_0}\bigl(\Delta^Y+E\bigr)-\Delta_{\bar Y_0}\bigr).\end{aligned}$$

Since X_0 has only nodes at codimension 1 points, X is canonical at codimension 1 points of X_0 (1.83) and f is an isomorphism near these points. Thus $\operatorname{Diff}_{\bar Y_0}\bigl(\Delta^Y+E\bigr)-\Delta_{\bar Y_0}$ is $\bar f_0$-exceptional and $\bar f_0$-ample. By [**KM98**, 3.39] this implies that every $\bar f_0$-exceptional divisor appears in $\operatorname{Diff}_{\bar Y_0}\bigl(\Delta^Y+E\bigr)-\Delta_{\bar Y_0}$ with strictly negative coefficient.

Every divisor in $Y_0\cap E$ appears in $\operatorname{Diff}_{\bar Y_0}\bigl(\Delta^Y+E\bigr)$ with coefficient ≥ 1 by (1.92). On the other hand, since $\bigl(\bar X_0,\operatorname{Diff}_{\bar X_0}\Delta\bigr)$ is lc by assumption, every exceptional divisor appears in $\Delta_{\bar Y_0}$ with coefficient ≤ 1. Thus $Y_0\cap E=\emptyset$ and, possibly after shrinking C, there are no exceptional divisors in $f:Y\to X$, hence f is small.

Thus Y_0 is a complete fiber of $g\circ f:Y\to C$. Since (Y,Δ^Y+Y_0) is lc, this implies that $g\circ f:(Y,\Delta^Y)\to C$ is locally stable in a neighborhood of Y_0 by (2.3).

Let $\bar{F} \subset \bar{Y}_0$ be any $\bar{f}_0$-exceptional divisor. Since it appears in $\operatorname{Diff}_{\bar{Y}_0}(\Delta^Y) - \Delta_{\bar{Y}_0}$ with negative coefficient, it must appear in $\Delta_{\bar{Y}_0}$ with positive coefficient and in $\operatorname{Diff}_{\bar{Y}_0}(\Delta^Y)$ with coefficient < 1. By (1.92) the latter implies that Y_0 is smooth at the generic point of $\pi_Y(\bar{F})$, proving (3).

Finally let $x \in X \setminus X_0$ be a point where $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Since f is small, $K_Y + \Delta^Y \sim_{\mathbb{Q}} f^*(K_X + \Delta)$ over a neighborhood of x. Since $K_Y + \Delta^Y$ is f-ample, f is an isomorphism over a neighborhood of x. This proves the second assertion in (1) and also completes (4–5). □

5.33 (Proof of (5.3)). We prove (5.3) when the base S is the spectrum of a DVR. By (4.45), this implies the case when S is higher dimensional, provided f is assumed to be flat with S_2 fibers.

As a preliminary step, we replace (X, Δ) by its normalization. This leaves the assumptions and the numerical conclusion unchanged. By (2.56), a demi-normal pair $(X, \Delta) \to C$ with slc generic fibers is slc iff its normalization is lc. Thus the conclusion is also unchanged.

Thus assume that X is normal. The conclusions are local on C, so pick a point $0 \in C$ and let $f : (Y, \Delta^Y + Y_0) \to (X, X_0 + \Delta)$ be the log canonical modification as in (5.32). Let $\pi_Y : \bar{Y}_0 \to Y_0$ be the normalization and $\bar{f}_0 : \bar{Y}_0 \to \bar{X}_0$ the induced birational morphism. We apply (10.30.3–4) to

$$D_Y := K_{\bar{Y}_0} + \operatorname{Diff}_{\bar{Y}_0} \Delta^Y \quad \text{and} \quad D_X := K_{\bar{X}_0} + \operatorname{Diff}_{\bar{X}_0} \Delta = K_{\bar{X}_0} + \bar{D}_0 + \bar{\Delta}_0.$$

The assumptions are satisfied since

$$(\bar{f}_0)_*\bigl(K_{\bar{Y}_0} + \operatorname{Diff}_{\bar{Y}_0} \Delta^Y\bigr) = K_{\bar{X}_0} + \operatorname{Diff}_{\bar{X}_0} \Delta$$

and $K_{\bar{Y}_0} + \operatorname{Diff}_{\bar{Y}_0} \Delta^Y$ is $\bar{f}_0$-ample. Using the volume of divisors (10.29), this implies that

$$\bigl(K_{\bar{X}_0} + \operatorname{Diff}_{\bar{X}_0} \Delta\bigr)^n = \operatorname{vol}\bigl(K_{\bar{X}_0} + \operatorname{Diff}_{\bar{X}_0} \Delta\bigr) \geq \operatorname{vol}\bigl(K_{\bar{Y}_0} + \operatorname{Diff}_{\bar{Y}_0} \Delta^Y\bigr)$$

and equality holds iff $\bar{f}_0$ is an isomorphism. Furthermore, since $K_Y + \Delta^Y$ is $\mathbb{Q}$-Cartier,

$$\operatorname{vol}\bigl(K_{\bar{Y}_0} + \operatorname{Diff}_{\bar{Y}_0} \Delta^Y\bigr) \geq \operatorname{vol}\bigl(K_{\bar{Y}_c} + \Delta^Y|_{\bar{Y}_c}\bigr) = \bigl(K_{\bar{Y}_c} + \bar{\Delta}_c\bigr)^n$$

for general $c \neq 0$ and $(\bar{Y}_c, \bar{\Delta}_c) = (X_c, \Delta_c)$ by (5.32.4). Combining the inequalities shows that

$$\bigl(K_{\bar{X}_0} + \bar{D}_0 + \bar{\Delta}_0\bigr)^n \geq \bigl(K_{\bar{X}_c} + \Delta_c\bigr)^n \quad \text{for general } c \neq 0$$

and equality holds iff $\bar{f}_0$, and hence f, are isomorphisms over $0 \in C$. □

The same method can be used to prove a weaker version of the numerical criterion of local stability over smooth curves. This establishes (5.4) for families of surfaces over a smooth curve. I do not know how to use these methods to complete the proof of (5.4) for higher dimensional families. We will derive (5.4) from (5.7) instead; see (5.52) for the key step.

Proposition 5.34 (Weak numerical criterion of local stability). *Let C be a smooth curve of* char 0 *and $f : (X, \Delta) \to C$ a morphism satisfying the assumptions (5.4.1–3). Then*

(1) $c \mapsto I\bigl(\pi_c^* H, K_{\bar{X}_c} + \bar{D}_c + \bar{\Delta}_c\bigr)$ *is lexicographically upper semicontinuous and*
(2) $f : (X, \Delta) \to C$ *is locally stable iff the above function is locally constant.*

Note that the first two numbers in the sequence $I\big(\pi_c^*H, K_{\bar X_c}+\bar D_c+\bar\Delta_c\big)$ equal $\big(H^n\cdot X_c\big)$ and $\big(H^{n-1}\cdot(K_X+\Delta)\cdot X_c\big)$, hence they are always locally constant. The first interesting number is $\big(\pi_c^*H^{n-2}\cdot(K_{\bar X_c}+\bar D_c+\bar\Delta_c)^2\big)$ which is thus an upper semicontinuous function on C by (1).

Proof. As in (5.33) we may assume that X is normal. Let $f:(Y,\Delta^Y+Y_0)\to(X,X_0+\Delta)$ be the log canonical modification and $\bar f_0:\bar X_0\to\bar Y_0$ the induced birational morphism between the normalizations. Here we apply (10.30.1–2) to $K_{\bar Y_0}+\operatorname{Diff}_{\bar Y_0}\Delta^Y$ and $K_{\bar X_0}+\operatorname{Diff}_{\bar X_0}\Delta$ to obtain that

$$I\big(\pi_0^*H, K_{\bar X_0}+\operatorname{Diff}_{\bar X_0}\Delta\big)\succeq I\big(\bar f_0^*\pi_0^*H, K_{\bar Y_0}+\operatorname{Diff}_{\bar Y_0}\Delta^Y\big),$$

and equality holds iff $\bar f_0$ is an isomorphism. Since $K_Y+\Delta^Y$ is a $\mathbb{Q}$-Cartier divisor,

$$I\big(\bar f_0^*\pi_0^*H, K_{\bar Y_0}+\operatorname{Diff}_{\bar Y_0}\Delta^Y\big)=I\big(\pi_c^*H, K_{\bar Y_c}+\Delta^Y|_{\bar Y_c}\big)=I\big(\pi_c^*H, K_{\bar X_c}+\bar\Delta_c\big)$$

for general $c\neq 0$. Thus

$$I\big(\pi_0^*H, K_{\bar X_0}+\bar D_0+\bar\Delta_0\big)\succeq I\big(\pi_c^*H, K_{\bar X_c}+\bar\Delta_c\big)\quad\text{for general } c\neq 0$$

and equality holds iff $\bar f_0$, and hence f, are isomorphisms. $\square$

5.35 (Start of the proof of (5.5)). We prove (5.5) when the base S is the spectrum of a DVR. By (4.45), this implies the case when S is higher dimensional, provided f is assumed to be flat with S_2 fibers.

As in (5.33) we may assume that X is normal. Thus, for suitable $m>0$ we have $U:=X\setminus\{x\}$ and a line bundle $L:=\mathcal{O}_U(mK_U+m\Delta)$ whose restriction to $U_D:=U\cap X_0=X_0\setminus\{x\}$ is trivial. We can not apply (2.89) since $\operatorname{depth}_x\mathcal{O}_X$ may be only 2.

However, we are in the situation studied in (5.32), hence there is a proper, birational, small morphism $f:Y\to X$ such that $K_Y+\Delta^Y$ is $\mathbb{Q}$-Cartier and f-ample.

For the rest of the argument it is not important that we are dealing with $K_X+\Delta$. Thus, for suitable $m>0$ we have an f-ample line bundle $M:=\mathcal{O}_X(mK_Y+m\Delta^Y)$ on Y such that $M|_{f^{-1}(U)}\cong L$. The key new additional information is that $\dim f^{-1}(x)\le\dim X-2$.

Next we use reduction modulo p as in (2.89) but we have to keep track of $f:Y\to X$ as well. In our case, in addition to (2.89.1–4) we also have a proper, birational morphism $f^T:Y^T\to X^T$ that is an isomorphism over U^T and an f^T-ample line bundle M^T such that $M^T|_{U^T}\cong L^T$ and $\dim(f_p^T)^{-1}(x_p)=\dim f^{-1}(x)$.

As before, (2.88) shows that $(L_p^T)^m\cong\mathcal{O}_{U_p^T}$ for some $m>0$. Thus $(M_p^T)^m$ and $\mathcal{O}_{Y_p^T}\cong\big((f_p^T)^*\mathcal{O}_{X_p^T}\big)^m$ are 2 invertible sheaves on Y_p^T that are isomorphic over the open subset $Y_p^T\setminus(f_p^T)^{-1}(x_p)$. If $(f_p^T)^{-1}(x_p)$ has codimension ≥ 2 then $(M_p^T)^m\cong\mathcal{O}_{Y_p^T}$. Since M_p^T is f_p^T-ample, this is only possible is f_p^T is an isomorphism. Then f^T and hence f are also isomorphisms and so $M^m\cong\mathcal{O}_Y$ shows that $L^m\cong\mathcal{O}_U$. $\square$

5.6. Simultaneous canonical models

In this section we consider the existence of simultaneous canonical models.

5.36 (Proof of (5.10) over curves). Let B be a smooth curve of char 0 and $f:X\to B$ a morphism of pure relative dimension n.

First we prove that $b\mapsto\operatorname{vol}(K_{X_b^r})$ is a lower semicontinuous function on B.

If we replace X by a resolution $X^r \to X$ then $\operatorname{vol}(K_{X^r_b})$ is unchanged for general fibers and it can only increase for special fibers. There are two possible sources for an increase. First, the resolution may introduce new divisors of general type. Second, if X is not normal, an irreducible component of a fiber may be replaced by a finite cover of it. The latter increases the volume by (10.36).

Thus it is enough to check lower semicontinuity when X is smooth and all fibers are snc.

There is nothing to prove if the volume of the general fiber is 0, hence we may assume that general fibers are of general type.

Let F be the union of all singular fibers and $f^c : X^c \to B$ the relative canonical model of $(X, \operatorname{red} F) \to B$ as in (2.59.2). An irreducible component $E \subset F$ may get contracted. However, when this happens, then $K_E + (F - E)|_E = (K_X + F)|_E$, and hence also K_E, are ≤ 0 on the fibers of the contraction. Such divisors contribute 0 to the volume. Thus we can check lower semicontinuity on $f^c : X^c \to B$.

Pick $b \in B$, let $\sum e_i E_i := X^c_b$ denote the fiber over b and $\pi_i : \bar{E}_i \to E_i$ the normalizations. As in (1.90) write $\pi_i^*(K_{X^c} + \operatorname{red} F^c) = K_{\bar{E}_i} + \bar{D}_i$ where $\bar{D}_i = \operatorname{Diff}_{\bar{E}_i}\bigl(\sum_{j \neq i} E_j\bigr)$. Let $g \in B$ be a point not contained in $f^c(F)$. Then F^c is disjoint from X^c_g and we have

$$\begin{array}{rcl}\bigl(K_{X^c_g}\bigr)^n & = & \bigl((K_{X^c} + \operatorname{red} F^c)^n \cdot X^c_g\bigr)\\ & = & \bigl((K_{X^c} + \operatorname{red} F^c)^n \cdot X^c_b\bigr)\\ & = & \sum_i e_i \bigl(K_{\bar{E}_i} + \bar{D}_i\bigr)^n \geq \sum_i \bigl(K_{\bar{E}_i} + \bar{D}_i\bigr)^n.\end{array}$$

Next we use (5.12) to obtain that $\bigl(K_{\bar{E}_i} + \bar{D}_i\bigr)^n \geq \bigl(K_{E_i^{\mathrm{cr}}}\bigr)^n$. Putting these together we see that

$$\operatorname{vol}(K_{X^r_g}) = \bigl(K_{X^c_g}\bigr)^n \geq \sum_i \bigl(K_{E_i^{\mathrm{cr}}}\bigr)^n = \operatorname{vol}(K_{X^r_b}),$$

proving the lower semicontinuity assertion. Furthermore, by (5.12), equality holds iff $D_i = 0$, the E_i have canonical singularities and $e_i = 1$ for every i. If $D_i = 0$ then E_i is the only irreducible component of its fiber by (1.92). Thus X^c_b is reduced, irreducible and has canonical singularities. We can now use either [**Kol13c**, 1.28] or (2.8) to conclude that $f^c : X^c \to B$ is also the relative canonical model of $f : X \to B$, hence a simultaneous canonical model.

□

Next we prove (5.11) when the base is a smooth curve.

5.37 (Proof of (5.11) over curves). Let B be a smooth curve over a field of char 0 and $f : (X, \Delta) \to B$ a flat morphism whose fibers are irreducible and smooth outside a codimension ≥ 2 subset. We may replace X by its normalization. Thus we may assume to start with that X is normal and then the generic fiber is lc.

Assume first that f is locally stable. We prove that $b \mapsto \operatorname{vol}(K_{X_b} + \Delta_b)$ is an upper semicontinuous function on S and $f : (X, \Delta) \to B$ has a simultaneous canonical model iff this function is locally constant.

To see these let $f^c : (X^c, \Delta^c) \to B$ denote the canonical model of $f : (X, \Delta) \to B$, it exists by [**Kol13c**, 1.30.7]. For every $b \in B$ we need to understand the difference between

- $\bigl((X^c)_b, (\Delta^c)_b\bigr)$, the fiber of f^c over b and
- $\bigl((X_b)^c, (\Delta_b)^c\bigr)$, the canonical model of the fiber (X_b, Δ_b) of f over b.

These two are the same for general $g \in B$ but they can be different for some special points in B.

Let $\phi : X \dashrightarrow X^c$ denote the natural birational map. Since the fibers of f are irreducible, they can not be contracted, thus ϕ induces birational maps $\phi_b : X_b \dashrightarrow (X^c)_b$. Let Z_b denote the normalization of the closure of the graph of ϕ_b with projections $X_b \xleftarrow{g} Z_b \xrightarrow{h} (X^c)_b$. The key computation, done in (5.39), shows that

$$g^*\bigl(K_{X_b} + \Delta_b\bigr) \sim_{\mathbb{Q}} h^*\bigl(K_{(X^c)_b} + (\Delta^c)_b\bigr) + F_b \tag{5.37.1}$$

where F_b is effective. This implies that

$$\operatorname{vol}(K_{X_b}+\Delta_b) = \operatorname{vol}\bigl(g^*(K_{X_b}+\Delta_b)\bigr) \geq \operatorname{vol}\bigl(h^*(K_{(X^c)_b}+(\Delta^c)_b)\bigr) = \operatorname{vol}\bigl(K_{(X^c)_b}+(\Delta^c)_b\bigr).$$

Note further that since $f^c : (X^c, \Delta^c) \to B$ is flat and $K_{X^c} + \Delta^c$ is f^c-ample, its restriction to different fibers have the same volume. Thus

$$\operatorname{vol}\bigl(K_{(X^c)_b} + (\Delta^c)_b\bigr) = \operatorname{vol}\bigl(K_{(X^c)_g} + (\Delta^c)_g\bigr) = \operatorname{vol}(K_{X_g} + \Delta_g)$$

for generic $g \in B$. Putting the two together shows that

$$\operatorname{vol}(K_{X_b} + \Delta_b) \geq \operatorname{vol}(K_{X_g} + \Delta_g) \tag{5.37.2}$$

and, by (10.37), equality holds iff F_b is h-exceptional, in which case $\bigl((X^c)_b, (\Delta^c)_b\bigr)$ is the canonical model of (X_b, Δ_b). This proves both claims.

In the general case, when $f : (X, \Delta) \to B$ is not locally stable, we first use (5.32) to construct $h : (\bar{X}, \bar{\Delta}) \to (X, \Delta)$ such that the composite $f \circ h : (\bar{X}, \bar{\Delta}) \to B$ is locally stable. Thus (5.37.2) applies and we get that

$$\operatorname{vol}(K_{\bar{X}_b} + \bar{\Delta}_b) \geq \operatorname{vol}(K_{X_g} + \Delta_g). \tag{5.37.3}$$

Note that $h_b : (\bar{X}_b, \bar{\Delta}_b) \to (X_b, \Delta_b)$ is birational by (5.32) and $K_{\bar{X}_b}+\bar{\Delta}_b$ is h_b-ample. Thus (10.30.1) implies that

$$\operatorname{vol}\bigl(X_b, \Delta_b\bigr) \geq \operatorname{vol}(\bar{X}_b, \bar{\Delta}_b). \tag{5.37.4}$$

Putting (5.37.3) and (5.37.4) together shows the upper semicontinuity of the volume.

It remains to show that if equality holds in (5.37.3) and (5.37.4) then there is a simultaneous canonical model. We already proved that if equality holds in (5.37.3) then $f \circ h : (\bar{X}, \bar{\Delta}) \to B$ has a simultaneous canonical model $\bigl(\bar{X}^c_b, \bar{\Delta}^c_b\bigr)$. Next we show that if equality holds in (5.37.4) then $\bigl(\bar{X}^c_b, \bar{\Delta}^c_b\bigr)$ is also the simultaneous canonical model of $f : (X, \Delta) \to B$. Equivalently, that $(\bar{X}_b, \bar{\Delta}_b)$ and (X_b, Δ_b) have isomorphic canonical models. The latter follows from (10.37) but it can also be obtained by applying the simpler (10.30) to the (normalization of the closure of the) graph of $\bigl(\bar{X}_b, \bar{\Delta}_b\bigr) \dashrightarrow \bigl(X^c_b, \Delta^c_b\bigr)$. $\square$

Looking at the above proof shows that the existence of simultaneous canonical models is part of the following more general problem.

QUESTION 5.38. Let $(X, D + \Delta)$ be an lc pair and $\bigl(X^c, D^c + \Delta^c\bigr)$ its canonical model. What is the relationship between

- the canonical model of $(D, \operatorname{Diff}_D \Delta)$ and
- $\bigl(D^c, \operatorname{Diff}_{D^c} \Delta^c\bigr)$?

The following simple example shows that these two are usually different. Start with a smooth variety X', a smooth divisor $D' \subset X'$ and another smooth divisor $C' \subset D'$. Assume that $K_{X'} + D'$ is ample. Set $X := B_{C'}X'$ with exceptional divisor E and let $D \subset X$ denote the birational transform of D'.

Then $(X, D+E)$ is an lc pair whose canonical model is (X', D') and $(D', 0)$ is its own canonical model.

However, $(D, \operatorname{Diff}_D E) \cong (D', C')$ is different from $(D', 0)$.

Note that for suitable choices we can arrange that $K_{D'} + C'$ is ample (in which case (D', C') is its own canonical model) or that $K_{D'} + C'$ is negative on C' (in which case the canonical model of (D', C') is obtained by contracting C' to a point).

First we prove that, under fairly general conditions, the expected discrepancy inequalities hold; cf. [**KM98**, 3.38] or [**Kol13c**, 1.19 and 1.22]. Then, under much more stringent restrictions, we establish the existence of simultaneous canonical models by studying how the boundary divisor changes as we run the minimal model program.

LEMMA 5.39. *Let $(X, D+\Delta)$ be lc where D is a reduced Weil divisor and $\Delta = \sum a_i D_i$ is a $\mathbb{Q}$-divisor. Let $f : X \to S$ be a proper morphism and*

$$\begin{array}{ccccc} X & & \overset{\phi}{\dashrightarrow} & & X^w \\ f & \searrow & & \swarrow & f^w \\ & & S & & \end{array} \tag{5.39.1}$$

a weak canonical model of $f : (X, D+\Delta) \to S$. If none of the irreducible components of D are contracted by ϕ then we obtain the diagram

$$\begin{array}{ccccc} D & & \overset{\phi_D}{\dashrightarrow} & & D^w \\ f|_D & \searrow & & \swarrow & f^w|_{D^w} \\ & & S & & \end{array} \tag{5.39.2}$$

where ϕ_D is birational. Then

(3) *$a(E, D, \operatorname{Diff}_D \Delta) \le a\big(E, D^w, \operatorname{Diff}_{D^w} \Delta^w\big)$ for every divisor E over D and*
(4) *$\big(\phi_D\big)_* \operatorname{Diff}_D \Delta \ge \operatorname{Diff}_{D^w} \Delta^w$.*

Proof. Let Y be the normalization of the main component of the fiber product $X \times_S X^w$ with projections $X \overset{g}{\leftarrow} Y \overset{h}{\to} X^w$. By definition,

$$g^*(K_X + D + \Delta) \sim_{\mathbb{Q}} h^*\big(K_{X^w} + D^w + \Delta^w\big) + F \tag{5.39.5}$$

where F is effective. Let D_Y denote the birational transform of D on Y. Restricting to D_Y we get

$$(g|_{D_Y})^*(K_D + \operatorname{Diff}_D \Delta) \sim_{\mathbb{Q}} (h|_{D_Y})^*\big(K_{D^w} + \operatorname{Diff}_{D^w} \Delta^w\big) + F|_{D_Y}$$

and $F|_{D_Y}$ is also effective. This proves (3) and (4) is a special case. $\square$

This looks very promising, since (5.39.3) is the main inequality that we require for weak canonical models, see [**Kol13c**, 1.19]. There are, however, two problems.

(5.39.6) Although ϕ is a contraction, as in (5.23), this does not imply that ϕ_D is a contraction. If $G^w \subset D^w$ is a “new” divisor, then G^w is an exceptional divisor over D and the coefficient of G^w in $\operatorname{Diff}_{D^w} \Delta^w$ is compared with the discrepancy $a(G^w, D, \operatorname{Diff}_D \Delta)$. If $(D, \operatorname{Diff}_D \Delta)$ is canonical, then $a(G^w, D, \operatorname{Diff}_D \Delta) \ge 0$, hence (5.39.3) implies that the coefficient of G^w in $\operatorname{Diff}_{D^w} \Delta^w$ is ≤ 0. Thus it is in fact zero. This is, however, not enough to conclude that the Kodaira dimension or the plurigenera and unchanged.

An extra problem is that, as in (5.23), ϕ_D could be a contraction for one minimal model $\phi : X \dashrightarrow X^m$ but not for another minimal model. We usually think of various minimal models as being "essentially the same," but here we have to distinguish them carefully.

(5.39.7) Although the divisor F defined in (5.39.6) is g-exceptional, this does not imply that $F|_{D_Y}$ is $g|_{D_Y}$-exceptional. Thus (5.39.4) need not be an equality, not even if ϕ_D is an isomorphism.

There is one straightforward case where neither of these difficulties appears.

LEMMA 5.40. *Notation and assumptions as in* (5.39). *Assume in addition that* $D^w \cap \operatorname{Ex}(\phi^{-1})$ *has codimension* ≥ 2 *in* D^w.

Then $\bigl(D^w, \operatorname{Diff}_{D^w} \Delta^w\bigr)$ *is a weak canonical model of* $\bigl(D, \operatorname{Diff}_D \Delta\bigr)$.

Note that since ϕ is a contraction, $\operatorname{codim}_{X^w} \operatorname{Ex}(\phi^{-1}) \geq 2$ but we assume that $\operatorname{codim}_{D^w}\bigl(D^w \cap \operatorname{Ex}(\phi^{-1})\bigr) \geq 2$.

Proof. It is clear that

$$\bigl(X^w \setminus \operatorname{Ex}(\phi^{-1})\bigr) \cap (\phi_D)_* \operatorname{Diff}_D \Delta = \bigl(X^w \setminus \operatorname{Ex}(\phi^{-1})\bigr) \cap \operatorname{Diff}_{D^w} \Delta^w.$$

If $D^w \cap \operatorname{Ex}(\phi^{-1})$ has codimension ≥ 2 in D^w, then $D^w \setminus \operatorname{Ex}(\phi^{-1})$ has non-empty intersection with every divisor in D^w. Thus, in this case, ϕ_D is a contraction and $(\phi_D)_* \operatorname{Diff}_D \Delta = \operatorname{Diff}_{D^w} \Delta^w$. Together with (5.39.3) these imply that $\bigl(D^w, \operatorname{Diff}_{D^w} \Delta^w\bigr)$ is a weak canonical model of $\bigl(D, \operatorname{Diff}_D \Delta\bigr)$. $\square$

Next we show that, in some cases, it is enough to know that ϕ is a local isomorphism at all generic points of $D \cap \operatorname{Supp} \Delta$.

LEMMA 5.41. *Notation as in* (5.39). *Assume in addition that that*

(1) $\phi : X \dashrightarrow X^w$ *is obtained by an* $(X, D + \Delta)$*-MMP,*
(2) *none of the irreducible components of* Δ *are contracted by* ϕ*,*
(3) (X, Δ) *and* $\bigl(D, \operatorname{Diff}_D \Delta\bigr)$ *are canonical and*
(4) ϕ *is a local isomorphism at all generic points of* $D \cap \operatorname{Supp} \Delta$.

Then

(5) $D^w \cap \operatorname{Ex}(\phi^{-1})$ *has codimension* ≥ 2 *in* D^w
(6) ϕ_D *is a contraction and*
(7) $\bigl(D^w, \operatorname{Diff}_{D^w} \Delta^w\bigr)$ *is a weak canonical model of* $(D, \operatorname{Diff}_D \Delta)$.

Proof. Let $G^w \subset D^w$ be a divisor. Assume to the contrary that ϕ is not a local isomorphism over the generic point of G^w. Using (1) and that discrepancies increase as we go from X to a weak canonical model (see, for instance, [**KM98**, 3.50] or [**Kol13c**, 1.23]) there is a divisor E over X^w such that

$$a(E, X, D + \Delta) < a(E, X^w, D^w + \Delta^w) \leq 0.$$

By the definition of canonical (1.78) and (1.93.3) this implies that $\operatorname{center}_X E$ is either one of the irreducible components of Δ or one of the irreducible components of $D \cap \Delta$. The first is impossible by (2) and the second by (3). This contradiction with (4) proves (5), which in turn implies (6–7). $\square$

The following is a slight generalization of [**HMX13**, Sec.4].

COROLLARY 5.42. *Let $f : (X, \Delta = \sum_{i\in I} a_i D_i) \to B$ be locally stable such that it has a canonical model*

$$\begin{tikzcd}(X,\Delta) \arrow[rr, dashed, "\phi^c"] \arrow[dr, "f"'] & & (X^c,\Delta^c) \arrow[dl, "f^c"] \\ & B & \end{tikzcd}$$

Assume that

(1) *X is $\mathbb{Q}$-factorial,*
(2) *the fibers $\big(X_b, \Delta_b\big)$ are canonical for every $b \in B$ and*
(3) *the fibers of the Stein factorization $D_i \to B_i$ of $f|_{D_i} : D_i \to B$ are reduced and irreducible for every i.*

Then $X^c_b \cap \mathrm{Ex}\big((\phi^c)^{-1}\big)$ has codimension ≥ 2 in X^c_b for every $b \in B$ and therefore $\big(X^c_b, \Delta^c|_{X^c_b}\big)$ is the canonical model of $\big(X_b, \Delta_b\big)$ for every $b \in B$.

Proof. Taking the canonical model commutes with flat base changes $B' \to B$ (2.44). Thus we may assume without loss of generality that each $D_i \to B$ has reduced and irreducible fibers.

For $i \in I$ set $c_i := \max\big\{0, -a(D_i, X^c, \Delta^c)\big\}$ and $\Gamma := \sum_{i\in I} c_i D_i$. Note that $c_i \leq a_i$ for every i and equality holds if D_i is not ϕ^c exceptional. Thus $\phi^c_*\Delta = \phi^c_*\Gamma$ and $f^c : \big(X^c, \Delta^c\big) \to B$ is also the canonical model of $f : (X, \Gamma) \to B$ by [**Kol13c**, 1.27]. Let

$$\begin{tikzcd}(X,\Gamma) \arrow[rr, dashed, "\phi^m"] \arrow[dr, "f"'] & & (X^m,\Gamma^m) \arrow[dl, "f^m"] \\ & B & \end{tikzcd} \tag{5.42.4}$$

be a minimal model of $f : (X, \Gamma) \to B$ obtained by an (X, Γ)-MMP. Since $\psi : X^m \to X^c$ is a morphism (cf. [**Kol13c**, 1.26.6]), it is enough to prove that $X^m_b \cap \mathrm{Ex}\big((\phi^m)^{-1}\big)$ has codimension ≥ 2 in X^m_b for every $b \in B$. We check that $\phi^m : (X, \Gamma) \dashrightarrow (X^m, \Gamma^m)$ satisfies the assumptions of (5.41).

We assumed (5.41.1) and (5.41.3) holds since $\Gamma \leq \Delta$. In order to see (5.41.2) note that if $c_i > 0$ then $c_i = -a(D_i, X^c, \Delta^c) = -a(D_i, X^c, \Gamma^c)$ and the latter equals $-a(D_i, X^m, \Gamma^m)$ (cf. [**Kol13c**, 1.21]). This implies that D_i is not contracted by ϕ^m (cf. [**Kol13c**, 1.19.5^m]).

The subtle point is (5.41.4). By (3) each D_i contributes exactly 1 irreducible component with coefficient c_i to $\mathrm{Diff}_{X_b} \Gamma = \Gamma|_{X_b}$. Thus the sum of the coefficients of $\mathrm{Diff}_{X_b} \Gamma$ is exactly $\sum_{i\in I} c_i$.

Since none of the D_i gets contracted by ϕ^m, each D^m_i contributes at least 1 irreducible component with coefficient c_i to $\mathrm{Diff}_{X^m_b} \Gamma$. Hence the sum of the coefficients of $\mathrm{Diff}_{X^m_b} \Gamma^m$ is at least $\sum_{i\in I} c_i$. Combining this with (5.39.3) we conclude that

$$\big(\phi^m_b\big)_* \mathrm{Diff}_{X_b} \Gamma = \mathrm{Diff}_{X^m_b} \Gamma^m \quad \text{for every } b \in B. \tag{5.42.5}$$

By inversion of adjunction (1.93), the coefficient of a divisor G in the different is also the minimal discrepancy of a divisor whose center is G. Thus (5.42.5) and [**Kol13c**, 1.23] imply that ϕ^m is a local isomorphism at every generic point of $D \cap \Delta$.

Thus (5.42) applies and hence $X^c_b \cap \mathrm{Ex}\big((\phi^c)^{-1}\big)$ has codimension ≥ 2 in X^c_b for every $b \in B$. By (5.40), this implies that $\big(X^c_b, \Delta^c|_{X^c_b}\big)$ is the canonical model of $\big(X_b, \Delta_b\big)$ for every $b \in B$. $\square$

5.7. Simultaneous canonical modifications

If S is smooth then the simultaneous canonical modification of $f:(X,\Delta)\to S$ is also the canonical modification of (X,Δ). This suggests that one should consider the canonical modification of (X,Δ) and try to prove that it is a simultaneous canonical modification.

5.43 (Proof of (5.17) over curves). Let C be a smooth curve and $f:(X,\Delta)\to C$ a flat, projective morphism of pure relative dimension n that satisfies the assumptions of (5.17).

Each $c\mapsto \bigl(\pi_c^*H_c^{n-i}\cdot(K_{X_c^{\rm can}}+\Delta_0^{\rm can})^i\bigr)$ is a constructible function on C. Thus, in order to prove (5.17.1) we may assume that C is the spectrum of a DVR with closed point $0\in C$ and generic point $g\in C$. We may also assume that X is reduced, thus f is flat.

By (5.28), $\bigl(\pi_0^*H_0^n\bigr)\le\bigl(\pi_g^*H_g^n\bigr)$ and equality holds iff X_0 is generically reduced. It is thus enough to deal with the latter case. Then X is generically normal along X_0 and we can replace X by its normalization without changing any of the assumptions or conclusions. We may now also assume that X is irreducible.

Let $\pi:\bigl(Y,\Delta^Y=\pi_*^{-1}\Delta\bigr)\to(X,\Delta)$ denote the canonical modification.

Write $Y_0=\sum_i e_iE_i$ where $e_0=1$ and E_0 is the birational transform of X_0. (For now E_0 is allowed to be reducible.) Set $E:=\operatorname{red}Y_0=\sum E_i$. Let $\tau:\bar E_0\to E_0$ denote the normalization and write $\tau^*\bigl(K_Y+E+\Delta^Y\bigr)=K_{\bar E_0}+D_0$ where $D_0=\operatorname{Diff}_{\bar E_0}\bigl(E-E_0+\Delta^Y\bigr)$ as in (1.90). Choose $m\ge 0$ such that $K_Y+E+\Delta^Y+m\pi^*H$ is ample over C. We claim the following sequence of (in)equalities.

$$\begin{aligned}
&\bigl(K_{X_g^{\rm can}}+\Delta_g^{\rm can}+m\pi_g^*H\bigr)^n\\
&\quad=\bigl(K_{Y_g}+\Delta_g^Y+m\pi_g^*H\bigr)^n\\
&\quad=\bigl(K_Y+\Delta^Y+m\pi^*H\bigr)^n\cdot[Y_g]\\
&\quad=\bigl(K_Y+E+\Delta^Y+m\pi^*H\bigr)^n\cdot[Y_g]\\
&\quad=\bigl(K_Y+E+\Delta^Y+m\pi^*H\bigr)^n\cdot[Y_0]\\
&\quad=\textstyle\sum_i e_i\bigl((K_Y+E+\Delta^Y+m\pi^*H)|_{E_i}\bigr)^n\\
&\quad\ge\bigl(K_{\bar E_0}+D_0+m\pi_0^*H\bigr)^n\\
&\quad\ge\operatorname{vol}\bigl(K_{X_0^{\rm can}}+\Delta_0^{\rm can}+m\pi_0^*H\bigr)\\
&\quad=\bigl(K_{X_0^{\rm can}}+\Delta_0^{\rm can}+m\pi_0^*H\bigr)^n.
\end{aligned}\tag{5.43.1}$$

The first equality holds since (Y_g,Δ_g^Y) is the canonical model of (X_g,Δ_g), hence $\Delta_g^{\rm can}=\Delta_g^Y$. The second equality is clear. We are allowed to add E in the fourth row since it is disjoint from Y_g. We can then replace Y_g by Y_0 since they are algebraically equivalent and compute the latter one component at a time. $K_Y+E+\Delta^Y+m\pi^*H$ is ample, thus if we keep only the summands corresponding to E_0, we get the first inequality, which is an equality iff $Y_0=E_0$.

The second inequality follows from (10.34), once we check that $\sigma_*^{-1}\Delta_0\le D_0$ where $\sigma:=\pi_0\circ\tau:\bar E_0\to\bar X_0$ is the natural map. Since D_0 is effective, this is clear for σ-exceptional divisors. Otherwise, either π is an isomorphism over the generic point of a divisor D_0^i (hence D_0^i has the same coefficients in $\sigma_*^{-1}\Delta_0$ and D_0) or $\sigma_*^{-1}D_0^i$ is contained in another irreducible component of $\operatorname{red}Y_0$. In this case $\sigma_*^{-1}D_0^i$ appears in D_0 with coefficient 1 and in $\sigma_*^{-1}\Delta_0$ with coefficient ≤ 1 by assumption (5.16.5). This proves the second inequality and, by (10.34), if equality holds then $D_0=\sigma_*^{-1}\Delta_0$. The last equality is a general property of ample divisors.

As we noted in (5.13), the inequality proved in (5.43.1) is equivalent to

$$I\big(\pi_g^* H_g, K_{X_g^{\mathrm{can}}} + \Delta_g^{\mathrm{can}}\big) \succeq I\big(\pi_0^* H_0, K_{X_0^{\mathrm{can}}} + \Delta_0^{\mathrm{can}}\big)$$

which proves (5.17.1).

If equality holds everywhere in (5.43.1) then $Y_0 = E_0$, $D_0 = \sigma_*^{-1}\Delta_0$ and $\big(\bar{E}_0, D_0\big)$ is canonical. On the other hand, D_0 is the sum of $\sigma_*^{-1}\Delta_0$ and of the conductor of $\bar{E}_0 \to E_0 = Y_0$. Thus the conductor is 0, hence Y_0 is normal and irreducible, $D_0 = (\pi_0)_*^{-1}\Delta_0$ and $\big(Y_0, (\pi_0)_*^{-1}\Delta_0\big)$ is canonical. Since $K_{Y_0} + D_0$ is ample over X_0, these show that $\big(Y_0, (\pi_0)_*^{-1}\Delta_0\big)$ is the canonical modification of (X_0, Δ_0). Thus the canonical modification of (X, Δ) is also the simultaneous canonical modification, proving (5.17.2). □

In close analogy with (5.16), we can define simultaneous log canonical and semi-log-canonical modifications.

Definition 5.44. Let (X, Δ) be a pair over a field k. Its *log-canonical modification* is a proper, birational morphism $\pi : \big(X^{\mathrm{lc}}, \Delta^{\mathrm{lc}}\big) \to (X, \Delta)$ such that $\Delta^{\mathrm{lc}} = \pi_*^{-1}\Delta + E$ where E contains every π-exceptional divisor with coefficient 1, $K_{X^{\mathrm{lc}}} + \Delta^{\mathrm{lc}}$ is π-ample and $\big(X^{\mathrm{lc}}, \Delta^{\mathrm{lc}}\big)$ is log-canonical. (Note: If X is normal, this agrees with the usual definition (5.15). If X is not normal and $B \subset Y$ is a prime divisor such that X is singular along $\pi(B)$ then E contains B with coefficient 1. In particular, if (X, Δ) is slc then its log-canonical modification is the normalization $(\bar{X}, \bar{\Delta} + \bar{D})$ as in (5.2.7).)

Let $f : (X, \Delta) \to S$ be a morphism that satisfies the conditions (5.2.1–4). A *simultaneous lc modification* is a proper morphism $\pi : (Y, \Delta^Y) \to (X, \Delta)$ such that $f \circ \pi : (Y, \Delta^Y) \to S$ is locally stable and $\pi_s : (Y_s, \Delta_s^Y) \to (X_s, \Delta_s)$ is the lc modification for every $s \in S$.

If $f : (X, \Delta) \to S$ is locally stable then the lc modification of a fiber is its normalization; usually these do not form a flat family. We introduce the notion of simultaneous slc modification to remedy this problem.

Let (X, Δ) be a pair over a field k that is slc in codimension 1. Its *semi-log-canonical modification* is a proper, birational morphism $\pi : \big(X^{\mathrm{slc}}, \Delta^{\mathrm{slc}}\big) \to (X, \Delta)$ such that π is an isomorphism over codimension 1 points of X, $\Delta^{\mathrm{slc}} = \pi_*^{-1}\Delta + E$ where E contains every π-exceptional divisor with coefficient 1, $K_{X^{\mathrm{slc}}} + \Delta^{\mathrm{slc}}$ is π-ample and $\big(X^{\mathrm{slc}}, \Delta^{\mathrm{slc}}\big)$ is slc.

If X is normal, then the semi-log-canonical modification is automatically normal and it agrees with the log-canonical modification.

In general lc modifications are conjectured to exist but there are slc pairs without slc modification, see [**Kol13c**, 1.40]. In both cases existence is known when $K_X + \Delta$ is $\mathbb{Q}$-Cartier, see [**OX12**].

Let $f : (X, \Delta) \to S$ be a morphism that satisfies the conditions (5.2.1–4). A *simultaneous slc modification* is a proper morphism $\pi : (Y, \Delta^Y) \to (X, \Delta)$ such that $f \circ \pi : (Y, \Delta^Y) \to S$ is locally stable and $\pi_s : (Y_s, \Delta_s^Y) \to (X_s, \Delta_s)$ is the slc modification for every $s \in S$.

As we saw in the Examples 5.25–5.26, the existence question is more complicated for the simultaneous log canonical modification than for the simultaneous canonical modification.

THEOREM 5.45. *Let C be a smooth curve, $f : (X, \Delta) \to C$ a projective morphism of pure relative dimension n satisfying (5.2.1–4). Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and, for every $c \in C$, the semi-log-canonical modification $\pi_c : \big(X_c^{\mathrm{slc}}, \Delta_c^{\mathrm{slc}}\big) \to (X_c, \Delta_c)$ exists. Then*

(1) *$c \mapsto I\big(\pi_c^* H_c^{n-2}, K_{X_c^{\mathrm{slc}}} + \Delta_c^{\mathrm{slc}}\big)$ is a lexicographically lower semicontinuous function on C and*
(2) *$f : (X, \Delta) \to C$ has a simultaneous semi-log-canonical modification iff this function is locally constant.*

Proof. Using (2.56) we may assume that X is normal. Next we closely follow the proof of (5.43).

Let $\pi : (Y, \Delta^Y) \to (X, \Delta)$ denote the log-canonical modification; this exists by (5.15). Note that here $\Delta^Y = \pi_*^{-1}\Delta + F$ where F is the sum of all π-exceptional divisors that dominate C.

Write $Y_0 = \sum_i e_i E_i$ where $e_0 = 1$ and E_0 is the birational transform of X_0. Let $\tau : \bar{E}_0 \to E_0$ denote the normalization and write $\tau^*\big(K_Y + Y_0 + \Delta^Y\big) = K_{\bar{E}_0} + D_0$. Choose $m \geq 0$ such that $K_Y + Y_0 + \Delta^Y + m\pi^* H$ is ample over C. As in the proof of (5.43) we get that

$$\begin{array}{rcl} \big(K_{X_g^{\mathrm{lc}}} + \Delta_g^{\mathrm{lc}} + m\pi_g^* H\big)^n & \geq & \big(K_{\bar{E}_0} + D_0 + m\pi_0^* H\big)^n \quad \text{and} \\ \mathrm{vol}\big(K_{X_0^{\mathrm{lc}}} + \Delta_0^{\mathrm{lc}} + m\pi_0^* H\big) & = & \big(K_{X_0^{\mathrm{lc}}} + \Delta_0^{\mathrm{lc}} + m\pi_0^* H\big)^n. \end{array}$$

It remains to prove that $\big(K_{\bar{E}_0} + D_0 + m\pi_0^* H\big)^n \geq \mathrm{vol}\big(K_{X_0^{\mathrm{lc}}} + \Delta_0^{\mathrm{lc}} + m\pi_0^* H\big)$.

We have $\sigma : \bar{E}_0 \to X_0$ and we can apply (10.35) provided every q-exceptional divisor $\bar{F}_0 \subset \bar{E}_0$ appears in D_0 with coefficient 1.

By the definition of lc modifications, every divisor F_i that is exceptional for $Y \to X$ appears in Δ^Y with coefficient 1. If $K_X + \Delta$ is $\mathbb{Q}$-Cartier then the exceptional set of $Y \to X$ has pure codimension 1. In this case $\tau(\bar{F}_0)$ is contained in a divisor that is exceptional for $Y \to X$. Thus, by adjunction, $\bar{F}_0$ appears in D_0 with coefficient 1.

If (X_0, Δ_0) is slc at a point x_0 then (X, Δ) is also slc at x_0 by inversion of adjunction (1.93) hence π is a local isomorphism over x_0. Thus $\pi_0 : (Y_0, \Delta_0^Y) \to (X_0, \Delta_0)$ is an isomorphism over codimension 1 points of X_0.

The rest of the proof works as before. □

If $K_X + \Delta$ is not $\mathbb{Q}$-Cartier then it can happen that an exceptional divisor $\bar{F}_0 \subset \bar{E}_0$ is not contained in any exceptional divisor of $X^{\mathrm{lc}} \to X$. In such cases we lose control of the coefficient of $\bar{F}$ in D_0. This occurs in (5.25) over the 4 singular points that lie on D_0.

5.8. Mostly flat families of line bundles

So far we have studied morphisms that were known to be stable in codimension 1. Next we turn to investigating sheaves that are known to be invertible in codimension 1; a topic we already encountered in Section 2.6. This leads to the proofs of (5.7) and (5.4). Many of the results proved here are developed for arbitrary coherent sheaves in Chapter 9.

DEFINITION 5.46 (Mostly flat families of line bundles). Let $f : X \to S$ be a morphism and L a mostly flat family of divisorial sheaves (3.51). Thus there is a closed subscheme $Z \subset X$ with complement $j : U := X \setminus Z \hookrightarrow X$ such that $Z \cap X_s$

has codimension ≥ 2 in X_s for every $s \in S$, $f|_U : U \to S$ is flat over S with pure, S_2 fibers and $L|_U$ is a line bundle.

We say that L is a *mostly flat families of line bundles* if the S_2-hulls

$$L_s^H = (j_s)_*\big(G|_{U_s}\big) \tag{5.46.1}$$

are locally free over the S_2-hull of the fibers of f

$$X_s^H = \operatorname{Spec}_{X_s}(j_s)_*\big(\mathcal{O}_{U_s}\big). \tag{5.46.2}$$

(If U_s is normal, which is the main case, then X_s^H is the normalization of X_s.)

We may as well assume that $\mathcal{O}_X = j_*\mathcal{O}_U$ (equivalently, that $\operatorname{depth}_Z \mathcal{O}_X \geq 2$) and then (10.4) implies that there is a dense open subset $S^0 \subset S$ such that L is a line bundle on $X^0 := f^{-1}(S^0)$.

A mostly flat family of line bundles L on X is called *fiber-wise ample* if L_s^H is ample for every $s \in S$.

Example 5.47. Let $f' : X' \to \mathbb{A}^1$ be a family of degree 4 surfaces in $\mathbb{P}^3$ such that X'_0 contains a line ℓ but the Picard number of X'_t is 1 for some $t \neq 0$. Then $\ell \subset X'$ can be contracted and we get $\pi : X' \to X$ and $f : X \to \mathbb{A}^1$. (Usually X is only an analytic or algebraic space.) Here X_0 is a K3 surface with a node. Set $L := \pi_*\mathcal{O}_X(2)$.

Then L is a mostly flat family of fiber-wise ample line bundles yet f itself is not projective.

Our aim is to find conditions to ensure that a mostly flat family of line bundles is a flat family of line bundles. We start with 1-parameter families.

5.48 (Euler characteristic and specialization). Let $(0, T)$ be the spectrum of a DVR, $f : X \to T$ a proper morphism of pure relative dimension n. We can usually harmlessly assume that X is S_2. Thus the generic fiber X_g is S_2 and the special fiber X_0 is S_1. Let L be a mostly flat family of line bundles on X.

By the assumptions (3.51) L is S_2 and there is a subset $Z_0 \subset X_0$ of codimension ≥ 2, called the *degeneracy set* of L, such that L is locally free on $X \setminus Z_0$ and $X_0 \setminus Z_0$ is S_2.

L_0 is also S_1, hence $L_0 \to L_0^H$ is an injection. By semicontinuity we have $h^0\big(X_0, L_0^H\big) \geq h^0\big(X_0, L_0\big) \geq h^0\big(X_g, L_g\big)$. Applying this inequality to powers of L we obtain that

$$\operatorname{vol}(L_0^H) = \lim \frac{h^0\big(X_0, (L_0^H)^{\otimes m}\big)}{m^n/n!} \geq \lim \frac{h^0\big(X_g, L_g^{\otimes m}\big)}{m^n/n!} = \operatorname{vol}(L_g). \tag{5.48.1}$$

If L is fiber-wise ample then the volume equals the self-intersection number, thus

$$\big((L_0^H)^n\big) \geq \big((L_g^H)^n\big). \tag{5.48.2}$$

In order to get more precise information, note that we have an exact sequence

$$0 \to L_0 \xrightarrow{r_0^L} L_0^H \to Q \to 0 \tag{5.48.3}$$

which defines the sheaf Q whose support is contained in the degeneracy set Z_0. Thus

$$\begin{aligned} \chi\big(X_0, L_0^H\big) &= \chi(X_0, L_0) + \chi(X_0, Q) \\ &= \chi(X_g, L_g) + \chi(X_0, Q). \end{aligned} \tag{5.48.4}$$

Let $\mathcal{O}_X(1)$ be an ample line bundle. Twisting (5.48.3) by $\mathcal{O}_X(m)$ and taking Euler characteristic we obtain that

$$\chi\big(X_0, L_0^H(m)\big) \succeq \chi\big(X_g, L_g(m)\big) \tag{5.48.5}$$

and equality holds $\Leftrightarrow Q = 0 \Leftrightarrow r_0^L : L_0 \to L_0^H$ is an isomorphism. If X_0 is S_2 then this is further equivalent to L being locally free.

If the degeneracy set Z_0 is finite then $Q = 0$ iff $\chi(X_0, Q) = 0$, hence

$$L_0 = L_0^H \quad \Leftrightarrow \quad \chi\big(X_0, L_0^H\big) = \chi\big(X_g, L_g\big). \tag{5.48.6}$$

As before, if X_0 is S_2 then this is further equivalent to L being locally free.

Remark 5.49. Let $f : X \to S$ be a proper morphism of pure relative dimension n and L a line bundle on X. It is not well understood under what conditions is the function $s \mapsto \operatorname{vol}(L_s)$ constructible; see [**Les14, PS13**].

Lemma 5.50. *Let $f : X \to S$ be a proper morphism of pure relative dimension n, A a relatively ample line bundle on X and L a mostly flat family of fiber-wise ample line bundles. Then*

(1) *$s \mapsto \big(A_s^i \cdot (L_s^H)^{n-i}\big)$ is constructible and upper semicontinuous for every i and*
(2) *if $\big((L_s^H)^n\big)$ is constant (as a function of s) then so is every $\big(A_s^i \cdot (L_s^H)^{n-i}\big)$.*

Proof. As we noted in (5.46), after passing to a dense open subset $S^0 \subset \operatorname{red} S$ and normalizing $X^0 := f^{-1}(S^0)$, we may assume that L is a line bundle. Thus the functions $s \mapsto \big(A_s^i \cdot (L_s^H)^{n-i}\big)$ are locally constant on S^0 and so constructible on S by Noetherian induction. Together with (5.48.2) this implies semicontinuity for $i = 0$.

For $i > 0$ we prove (1) by induction on n. We may assume that S is local and A is relatively very ample.

Let $Y \subset X$ be a hypersurface cut out by a general section of A. By a Bertini-type theorem (10.11) the restriction $L|_Y$ is a mostly flat family of fiber-wise ample line bundles on $Y \to S$. Furthermore

$$\big(A_s^i \cdot (L_s^H)^{n-i}\big) = \big(Y_s \cdot A_s^{i-1} \cdot (L_s^H)^{n-i}\big) = \big((A|_Y)_s^{i-1} \cdot \big((L|_Y)_s^H\big)^{n-i}\big) \tag{5.50.3}$$

and the latter is constructible and upper semicontinuous by induction.

In order to see (2) note that $L^m \otimes A^{-1}$ is also a mostly flat family of fiber-wise ample line bundles for $m \gg 1$ and

$$m^n\big((L_s^H)^n\big) = \textstyle\sum_i \binom{n}{i} \big(A_s^i \cdot \big((L^m \otimes A^{-1})_s^H\big)^{n-i}\big). \tag{5.50.4}$$

By (1) all summands on the right are constructible and upper semicontinuous. Therefore, if the sum is constant as a function of s, then so is every summand. Finally note that

$$\big(\big((L^m \otimes A^{-1})_s^H\big)^n\big) = \textstyle\sum_i (-1)^i m^{n-i} \binom{n}{i} \big(A_s^i \cdot (L_s^H)^{n-i}\big). \tag{5.50.5}$$

If the left side is constant for $m \gg 1$, as a function of s, then every summand on the right is constant. $\square$

5.51 (Proof of (5.7)). The assertions (5.7.1) and (5.7.3) are proved in (5.50.1). Furthermore, (5.50.2) shows that (5.7.2) implies (5.7.4).

Thus it remains to prove (5.7.2). We start with the case when S is the spectrum of a DVR; this implies the general case by (4.36).

Our argument has 3 parts. The first step, when the relative dimension is 2, is done in (5.53).

The next step is induction on the dimension. We may assume that S is local and A is relatively very ample. Let $Y \subset X$ be a general hypersurface cut out by a general section of A. Then (10.11) ensures that $L^H|_Y = (L|_Y)^H$. The restriction $L|_Y$ is a mostly flat family of fiber-wise ample line bundles on $Y \to S$ and, as we noted in (5.50.3),

$$\left(A_s^{n-2} \cdot (L_s^H)^2\right) = \left((A|_Y)_s^{n-3} \cdot \left((L|_Y)_s^H\right)^2\right).$$

Thus, by induction, $L^H|_Y$ is a line bundle. This implies that L^H is a line bundle along Y. Therefore L^H is a line bundle, except possibly at finitely many points $Z \subset X$.

Finally we need to exclude this finite set Z when the fiber dimension is at least 3. This follows from (2.90), which we have not proved yet.

Alternatively, we can use (2.84) and conclude that $L^{[m]}$ is a line bundle for some $m > 0$. Then a short global argument given in (5.54) shows that L itself is locally free. □

5.52 (Start of the proof of (5.4)). Note that (5.4.4) follows from (5.50.1).

Next we consider (5.4.5) when S is the the spectrum of a DVR; the general setting is postponed to (5.63).

Thus assume that we have a smooth curve C and $f : (X, \Delta) \to C$ satisfying the assumptions (5.4.1–3) and such that

$$c \mapsto \left(\pi_c^* H^{n-2} \cdot (K_{\bar{X}_c} + \bar{D}_c + \bar{\Delta}_c)^2\right)$$

is a constant function on C. We aim to prove that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

As a first step, we replace (X, Δ) by its normalization. This leaves the assumptions and the numerical conclusion unchanged. By (2.56), a demi-normal pair $(X, \Delta) \to C$ with slc generic fibers is slc iff its normalization is lc. Thus the conclusion is also unchanged.

It would seem that we should use (5.7). However, a key assumption of (5.7) is that every fiber is S_2; this is true but not obvious in our case. Thus we consider two separate cases.

If $n = 2$ then the weak numerical criterion (5.34) implies (5.4). However, for $n \geq 3$ the weak numerical criterion also involves the terms $\left(\pi_c^* H^{n-i} \cdot (K_{\bar{X}_c} + \bar{D}_c + \bar{\Delta}_c)^i\right)$ for $i \geq 3$; these are unknown to us.

Instead, using the already established $n = 2$ case and the Bertini–type result (10.11) as in (5.51), we may assume that $f : (X, \Delta) \to C$ is locally stable outside a subset of codimension ≥ 3. We can now apply (5.5), or rather, the special case proved in (5.35), to complete the argument. □

PROPOSITION 5.53. *Let T be an irreducible, regular, 1-dimensional scheme and $f : X \to T$ a flat, proper morphism of relative dimension 2 with S_2 fibers. Let L be a mostly flat family of line bundles on X. Then*

(1) $t \mapsto \left(L_t^H \cdot L_t^H\right)$ *is upper semicontinuous and*
(2) L *is locally free on X iff the above function is constant.*

Proof. If L is locally free then $\left(L_t^H \cdot L_t^H\right) = \left(L \cdot L \cdot [X_t]\right)$ is independent of $t \in T$. To see the converse we may assume that T is local with closed point $0 \in T$

and generic point $g \in T$. Note that L is locally free, except possibly at a finite set $Z_0 \subset X_0$, and $L_g^H \cong L_g$.

For each $t \in T$, the Euler characteristic is a quadratic polynomial

$$\chi\big(X_t, (L_t^H)^{\otimes m}\big) = a_t m^2 + b_t m + c_t,$$

and we know from Riemann–Roch that $a_t = \frac{1}{2}\big(L_t^H \cdot L_t^H\big)$ and $c_t = \chi\big(X_t, \mathcal{O}_{X_t}\big)$. Furthermore, (5.48.4) implies that

$$a_0 m^2 + b_0 m + c_0 \geq a_g m^2 + b_g m + c_g \quad \text{for every } m \in \mathbb{Z}. \tag{5.53.3}$$

For $m \gg 1$ the quadratic terms dominate, which gives that

$$\big(L_0^H \cdot L_0^H\big) = 2a_0 \geq 2a_g = \big(L_g \cdot L_g\big). \tag{5.53.4}$$

Assume now that $(L_0^H \cdot L_0^H) = (L_g \cdot L_g)$. Then $a_0 = a_g$ thus (5.53.3) implies that

$$b_0 m + c_0 \geq b_g m + c_g \quad \text{for every } m \in \mathbb{Z}. \tag{5.53.5}$$

For $m \gg 1$ this implies that $b_0 \geq b_g$ and for $m \ll -1$ that $-b_0 \geq -b_g$. Thus $b_0 = b_g$ and $c_0 = c_g$ also holds since f is flat. Therefore we have equality in (5.53.3).

Thus L is a flat family of locally free sheaves by (5.48.6). □

Local extension problems.

We are now ready to complete the proofs of (2.90) and (5.7).

In both cases, the only remaining case is when some reflexive tensor power $L^{[m]}$ is locally free on X and in Section 2.9 we even settled the case when $\operatorname{char} k(x) \nmid m$.

Below we give first the global argument of [**Kol16a**] and then discuss how it was localized in [**dJ15**].

PROPOSITION 5.54. *Let T be the spectrum of a DVR with closed point $0 \in T$ and generic point $g \in T$. Let $f : X \to T$ be a projective morphism with S_2 fibers and L a mostly flat family of line bundles such that $L^{[m]}$ is locally free for some $m > 0$. Then L is locally free.*

Proof. We claim an equality of the Hilbert polynomials

$$\chi\big(X_0, (L_0^H)^{\otimes r}\big) = \chi\big(X_g, L_g^{\otimes r}\big). \tag{5.54.1}$$

Since both sides are polynomials in r, it is sufficient to prove that they are equal for all multiples of m.

Note that $L^{[m]}|_{X_0}$ and $(L_0^H)^{\otimes m}$ are both locally free sheaves that agree outside a codimension 2 subset, hence they are isomorphic. Thus

$$\begin{array}{rclcl} \chi\big(X_0, (L_0^H)^{\otimes rm}\big) & = & \chi\big(X_0, \big(L^{[m]}|_{X_0}\big)^{\otimes r}\big) & = & \\ & = & \chi\big(X_g, \big(L^{[m]}|_{X_g}\big)^{\otimes r}\big) & = & \chi\big(X_g, L_g^{\otimes rm}\big), \end{array} \tag{5.54.2}$$

where the last equality holds since L_g is a line bundle (5.46). In particular we conclude that

$$\chi\big(X_0, L_0^H\big) = \chi\big(X_g, L_g\big). \tag{5.54.3}$$

Let now $\mathcal{O}_{X/S}(1)$ be an f-ample invertible sheaf. We can apply the same argument to any $L(m)$ to obtain that

$$\chi\big(X_0, L_0^H(m)\big) = \chi\big(X_g, L_g(m)\big) \quad \forall\ m \in \mathbb{Z}. \tag{5.54.4}$$

By (5.48.5) this implies that L is locally free. □

5.55 (Extension problems). Let (x, X) be a local Noetherian scheme, $s \in m_x$ a non-zerodivisor and $D := (s = 0)$. Set $U := X \setminus \{x\}$ and $U_D := U \cap D$. Let F be a coherent sheaf on U such that s is a non-zerodivisor on F. There are 2 natural quotient maps associated to this set-up. First, $r_U : F \to F/sF$ gives $h_U : H^0(U, F) \to H^0(U, F/sF)$. Second, let $j : U \hookrightarrow X$ denote the injection, then r_U extends to a map $r_X := j_* r_U : j_*F \to j_*(F/sF)$. It is clear that h_U is surjective iff r_X is, thus we can formulate our question in 2 equaivalent forms.

Local extension problem 5.55.1. When are the above maps

$$h_U : H^0(U, F) \to H^0(U, F/sF) \quad \text{and} \quad r_X : j_*F \to j_*(F/sF) \quad \text{surjective?}$$

A key observation of [**dJ15**] is that while $j_*F \to j_*(F/sF)$ is the case we need, one can run induction on the dimension if j_*F and $j_*(F/sF)$ are replaced by suitable subsheaves $F_X \subset j_*F$ and $F_D \subset j_*(F/sF)$. This leads to the following definition.

A *extension problem* associated to $r_U : F \to F/sF$ consists of coherent sheaves F_X on X and F_D on D plus a map $r_X : F_X \to F_D$ such that

$$\big(r_X : F_X \to F_D\big)|_U = \big(r_U : F \to F/sF\big). \tag{5.55.2}$$

Observe that r_X induces a morphism

$$r_D : F_X|_D \to F_D \tag{5.55.3}$$

that is an isomorphism over U_D. Thus both its kernel and cokernel have finite length. Set

$$\delta(F_X, F_D) := \operatorname{length}(\operatorname{coker} r_D) - \operatorname{length}(\ker r_D). \tag{5.55.4}$$

Claim 5.55.5. Using the above notation, assume that $x \notin \operatorname{Ass}(F_X)$. Then $\delta(F_X, F_D)$ depends only on F_D and F, not on F_X.

This suggests that the role of F_X is not very important, hence from now on we will think of an extension problem as a pair (F, F_D) and we choose F_X later in a convenient way. Thus from now on we write

$$\delta(F, F_D) := \delta(F_X, F_D) \tag{5.55.6}$$

for any suitable choice of F_X.

Proof. The assumption $x \notin \operatorname{Ass}(F_X)$ implies that F_X can be identified with a subsheaf of j_*F and if F'_X, F''_X give such extensions problems then so does their intersection. Thus it is enough to check that $\delta(F'_X, F_D) = \delta(F''_X, F_D)$ where $m_x F'_X \subset F''_X \subset F'_X$. Then, as we go from $F'_X \to F_D$ to $F''_X \to F_D$, the length of the kernel and of the cokernel both increase by $\operatorname{length}(F'_X/F''_X)$. □

Next we show that the original question of surjectivity of the restriction map $H^0(U, F) \to H^0(U_D, F/sF)$ between infinite dimensional vector spaces is equivalent to the vanishing of δ.

Claim 5.55.7. Assume that j_*F and $j_*(F/sF)$ are both coherent. Then $H^0(U, F) \to H^0(U_D, F/sF)$ is surjective iff $\delta\big(F, j_*(F/sF)\big) = 0$.

Proof. We can choose $F_X := j_*F$. Then $\operatorname{depth}_x(F_X/sF_X) \geq 1$, hence $r_X : F_X/sF_X \to j_*(F/sF)$ is injective. Thus $\delta\big(F, j_*(F/sF)\big) = 0$ iff r_X is an isomorphism. Every global section of F/sF extends to a global section of $j_*(F/sF)$ and then lifts to a global section of j_*F since X is affine. □

5.56 (Maps of extension problems). A *map* of extension problems

$$\alpha : (F, F_D) \to (G, G_D) \tag{5.56.1}$$

is a pair of maps sitting in a commutative diagram

$$\begin{array}{ccc} F & \overset{\alpha_U}{\to} & G \\ r^F \downarrow & & \downarrow r^G \\ F_D & \overset{\alpha_D}{\to} & G_D. \end{array} \tag{5.56.2}$$

We do *not* assume that α_U extends to a map between F_X and G_X. However, we can always choose F_X and G_X so that such an extension exists. Indeed, α_U gives a map $\bar{\alpha}_U : G_X \to j_* F$. We can thus replace G_X by $\bar{\alpha}_U^{-1}(F_X)$ to get $\alpha_X : G_X \to F_X$.

Correspondingly, an *exact sequence* of extension problems

$$0 \to (F, F_D) \to (G, G_D) \to (H, H_D) \to 0 \tag{5.56.3}$$

is a commutative diagram of 2 exact sequences

$$\begin{array}{ccccccccc} 0 & \to & F & \to & G & \to & H & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F_D & \to & G_D & \to & H_D & \to & 0. \end{array} \tag{5.56.4}$$

As before, we do not assume exactness for F_X, G_X, H_X . However, we claim that one can always choose F_X, G_X, H_X such that the sequences

$$\begin{array}{cccccccccc} 0 & \to & F_X & \to & G_X & \to & H_X & \to & 0 & \text{and} \\ & & \downarrow & & \downarrow & & \downarrow & & & \\ 0 & \to & F_X/sF_X & \to & G_X/sG_X & \to & H_X/sH_X & \to & 0 & \end{array} \tag{5.56.5}$$

are also exact. Indeed, first we choose G_X, then set $F_X := \ker\bigl[G_X \to j_* H\bigr]$. A problem is that there may not be a map $F_X \to F_D$, but such a map exists if we first replace G_X by $m_x^r G_X$ for some $r \gg 1$. Finally set $H_X := G_X/F_X$.

Lemma 5.57. *Consider an exact sequence of extension problems as in (5.56.3). Then*

$$\delta(G, G_D) = \delta(F, F_D) + \delta(H, H_D).$$

Proof. Choose F_X, G_X, H_X such that the sequences in (5.56.5) are also exact. Then the claim follows from (5.55.5) and the snake lemma applied to

$$\begin{array}{ccccccccc} 0 & \to & F_X/sF_X & \to & G_X/sG_X & \to & H_X/sH_X & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F_D & \to & G_D & \to & H_D & \to & 0. \end{array} \qquad \square$$

Definition 5.58. We say that (L, L_D) is a *line bundle extension problem* if L is a line bundle on U and L_D is a line bundle on D. Since (x, D) is local, in fact $L_D \cong \mathcal{O}_D$.

If $\operatorname{depth}_x D \geq 2$ then L_D is uniquely determined by $L|_{U_D}$, thus, in this case, line bundle extension problems are in one-to-one correspondence with the kernel of $\operatorname{Pic}^{\mathrm{loc}}(x, X) \to \operatorname{Pic}^{\mathrm{loc}}(x, D)$.

If (F, F_D) is an extension problem then $\bigl(F \otimes L, F_D \otimes L_D\bigr)$ is also an extension problem. Note that $F_D \cong F_D \otimes L_D$, thus we are changing the sheaf on U but keeping the sheaf on D fixed.

Now we come to the key point: δ behaves like a Hilbert polynomial.

PROPOSITION 5.59. *Let (F, F_D) be an extension problem and (L, L_D) a line bundle extension problem. Then*

$$n \mapsto \delta\big(F \otimes L^n, F_D \otimes L_D^n\big)$$

is a polynomial of degree $\leq \dim(\operatorname{Supp} F_X) - 1$.

Proof. We use induction on $\dim(\operatorname{Supp} F_X)$. If $\dim(\operatorname{Supp} F_X) \leq 1$ then $U \cap \operatorname{Supp} F_X$ is affine and L is trivial on $\operatorname{Spec}_U(\mathcal{O}_U / \operatorname{Ann} F)$. Thus $F \otimes L^n \cong F$ and $F_D \otimes L_D^n \cong F_D$ shows that $\delta\big(F \otimes L^n, F_D \otimes L_D^n\big)$ is constant.

Next we claim that for every (F, F_D) there is an exact sequence of extension problems

$$0 \to (F, F_D) \to (F \otimes L, F_D \otimes L_D) \to (Q, Q_D) \to 0.$$

To see this note that U is quasi-affine, so we can choose a global section g of L that does not vanish at any of the associated points of F or of F/sF. Thus we get injections

$$1_U \otimes g : F \to F \otimes L \quad \text{and} \quad 1_D \otimes g : (F/sF) \to (F/sF) \otimes L.$$

Thus $1_D \otimes g$ gives a map $F_D \to j_*(F/sF) \otimes L_D$ whose image need not be contained in $F_D \otimes L_D$. However, this can be rectified if we replace g by hg for suitable $h \in m_x$. Thus we get an injection $(F, F_D) \to (F \otimes L, F_D \otimes L_D)$ and (Q, Q_D) is defined as its cokernel.

Tensoring with a line bundle extension problem is clearly exact, thus we also have

$$0 \to (F \otimes L^{n-1}, F_D \otimes L_D^{n-1}) \to (F \otimes L^n, F_D \otimes L_D^n) \to (Q \otimes L^{n-1}, Q_D \otimes L_D^{n-1}) \to 0.$$

By (5.55.9) and induction this shows that

$$\delta(F \otimes L^n, F_D \otimes L_D^n) - \delta(F \otimes L^{n-1}, F_D \otimes L_D^{n-1})$$

is a polynomial of degree $\leq \dim(\operatorname{Supp} F_X) - 2$. Thus $\delta(F \otimes L^n, F_D \otimes L_D^n)$ is a polynomial of degree $\leq \dim(\operatorname{Supp} F_X) - 1$. □

5.60 (Proof of (2.90)). Let L be a line bundle on U such that $L|_{U_D} \cong \mathcal{O}_{U_D}$ and $L^m \cong \mathcal{O}_U$ for some $m > 0$.

We apply (5.59) to the trivial extension problem $(\mathcal{O}_U, \mathcal{O}_D)$. Since $\operatorname{depth}_x D \geq 2$, every isomorphism $\mathcal{O}_{U_D} \cong \mathcal{O}_{U_D}$ is multiplication by a unit, so the actual choice of the isomorphism $\mathcal{O}_U|_{U_D} \cong \mathcal{O}_{U_D}$ does not matter.

Thus we obtain that $n \mapsto \delta\big(L^n, L_D^n\big)$ is a polynomial function of n.

If $L^n \cong \mathcal{O}_U$ then $\big(L^n, L_D^n\big) \cong (\mathcal{O}_U, \mathcal{O}_D)$, thus $\delta\big(L^n, L_D^n\big) = 0$ whenever m divides n. A polynomial with infinitely many roots is identically zero, thus $\delta\big(L^n, L_D^n\big) = 0$ for every n. In particular $\delta\big(L, L_D\big) = 0$. Thus the constant 1 section of $L_D \cong \mathcal{O}_D$ lifts to a global section of L by (5.55.5) and so $L \cong \mathcal{O}_U$ by (2.86). □

5.9. Families over higher dimensional bases

Here we complete the proofs of Theorems 5.3–5.17. In all cases the first part asserts that a certain constructible function on the base scheme S is upper or lower semicontinuous. For constructible functions semicontinuity can be checked along spectra of DVR's and this was already done in all cases.

The remaining part is to show that if our functions are locally constant on S then certain constructions produce a flat family of varieties or sheaves. Again, in all cases we have already checked that this holds when the base is a smooth curve.

Going from curves to arbitrary reduced base schemes is easiest in the following example.

5.61 (Proof of a special case of Theorem 5.11). We make the extra assumption that $f : (X, \Delta) \to S$ is locally stable. This is rather special but, for applications, this is one of the main cases.

We may assume that S is connected. Since $f : (X, \Delta) \to S$ is locally stable, there is an $r \geq 1$ such that $\omega_{X/S}^{[r]}(r\Delta)$ is locally free. We claim that $f_*\omega_{X/S}^{[mr]}(mr\Delta)$ is locally free for all $mr \geq 2$ and

$$\operatorname{Proj}_S \textstyle\sum_{m\geq 2} f_*\omega_{X/S}^{[mr]}(mr\Delta) \tag{5.61.1}$$

is the simultaneous canonical model. By Grauert's theorem [**Gra60**] it is enough to show that

$$h^0\bigl(X_s, \omega_{X_s}^{[mr]}(mr\Delta_s)\bigr) \quad \text{is independent of } s \in S. \tag{5.61.2}$$

(Note that [**Har77**, III.12.9] states Grauert's theorem for S integral, but the proof works for reduced bases as well.)

By assumption, each fiber X_s has a canonical model X_s^c and

$$H^0\bigl(X_s, \omega_{X_s}^{[mr]}(mr\Delta_s)\bigr) = H^0\bigl(X_s^c, \omega_{X_s^c}^{[mr]}(mr\Delta_s^c)\bigr), \tag{5.61.3}$$

by definition (5.8). Thus it is enough to prove that

$$h^0\bigl(X_s^c, \omega_{X_s^c}^{[mr]}(mr\Delta_s^c)\bigr) \quad \text{is independent of } s \in S. \tag{5.61.4}$$

Since $K_{X_s^c} + \Delta_s^c$ is ample, a general form of Kodaira's vanishing theorem [**Fuj14**, 1.9] implies that

$$H^0\bigl(X_s^c, \omega_{X_s^c}^{[mr]}(mr\Delta_s^c)\bigr) = \chi\bigl(X_s^c, \omega_{X_s^c}^{[mr]}(mr\Delta_s^c)\bigr) \tag{5.61.5}$$

holds whenever $mr \geq 2$. Thus it remains to show that

$$\chi\bigl(X_s^c, \omega_{X_s^c}^{[mr]}(mr\Delta_s^c)\bigr) \quad \text{is independent of } s \in S. \tag{5.61.6}$$

This can be checked after base change to every $T \to S$ where T is the spectrum of a DVR.

We have already proved in (5.37) that $f_T : \bigl(X_T, \Delta_T\bigr) \to T$ has a simultaneous canonical model $f_T^c : \bigl(X_T^c, \Delta_T^c\bigr) \to T$ that is flat over T, hence the Euler characteristic over the special fiber equals Euler characteristic over the generic fiber. $\square$

Note that, instead of using Kodaira's vanishing theorem, we could have used Serre vanishing to obtain (5.61.5) for $m \geq m(s)$. One can use Noetherian induction to show that a fixed lower bound m_0 works for all $s \in S$. As in (5.61.1) we get that

$$\operatorname{Proj}_S \textstyle\sum_{m\geq m_0} f_*\omega_{X/S}[m]\bigl(\lfloor m\Delta\rfloor\bigr)$$

is the simultaneous canonical model.

We are trying to use similar arguments for the other theorems, but there is no sheaf on X to which Grauert's theorem could be applied to. We go around this problem by constructing certain universal flat sheaves and comparing their base space with S and X. In the next proof this step is hidden in the reference to (4.45) which ultimately relies on (4.33) where the Hilbert scheme and its universal family appear explicitly. For some of the other theorems the Hilbert scheme is not sufficient. We need the theory of hulls and husks, to be defined in Chapter 9, which was developed with exactly such situations in mind.

5.62 (Proof of Theorem 5.1). We already proved the case when S is the spectrum of a DVR in (5.33). As we noted above, this implies (5.1.1) in general. Thus it remains to prove that if $s \mapsto \big(K^n_{X_s}\big)$ is constant then $f: X \to S$ is stable.

In view of (5.33) we know that $f_T : X_T \to T$ is stable for every $T \to S$ where T is the spectrum of a DVR. Thus $f : X \to S$ is stable by (4.45). □

Another typical example is the following proof of Theorems 5.3, 5.4 and 5.5. Note that in these cases we can not apply (5.7) since f is not assumed to be flat and its fibers are not assumed to be S_2.

5.63 (Proof of Theorems 5.3–5.5). By (4.48) there is an $m > 0$ such that $\omega^{[m]}_{\bar X_s}(m\bar\Delta_s)$ is locally free for every $s \in S$. Let next $\pi : H \to S$ denote the fiber product (over S) of the hulls (9.58)

$$\pi_m : \operatorname{Hull}\big(\omega^{[m]}_{X/S}(m\Delta)\big) \to S \quad \text{and of} \quad \pi_0 : \operatorname{Hull}(\mathcal{O}_X) \to S.$$

We aim to show that $\pi : H \to S$ is an isomorphism. By (9.59), both π_m and π_0 are locally closed decompositions, hence so is their fiber product.

Let T be the spectrum of a DVR and $g : T \to S$ a morphism that maps the generic point of T to a generic point of S. We apply (5.33), (5.52) or (5.35) to the divisorial pull-back $f_T : (X_T, \Delta_T) \to T$. We conclude that $f_T : (X_T, \Delta_T) \to T$ is stable (resp. locally stable) and hence $\omega^{[m]}_{X/T}(m\Delta_T)$ is locally free by (3.62).

Thus $g : T \to S$ factors through $\pi : H \to S$ hence $\pi : H \to S$ is an isomorphism by (3.49). In particular, $f : X \to S$ is flat with S_2 fibers and $\omega^{[m]}_{X/S}(m\Delta)$ is locally free. Therefore all fibers are slc by (1.85) hence $f : (X, \Delta) \to S$ is stable (resp. locally stable). □

5.64 (Proof of Theorems 5.10–5.11). Both claims were already established over the spectrum of a DVR, see (5.36) and (5.37). This implies the semicontinuity assertions in both cases.

It remains to show that if the volume is constant then $f : X \to S$ (resp. $f : (X, \Delta) \to S$) has a simultaneous canonical model.

Consider the moduli space of marked stable pairs $\pi : \mathrm{MSP}^{\mathrm{sn}} \to S$ and set

$$W := \big\{(X^{\mathrm{c}}_s, \Delta^{\mathrm{c}}_s) : s \in S\big\} \subset \mathrm{MSP}^{\mathrm{sn}}.$$

In order to prove that W is a closed subset, first we claim that it is constructible. This is clear since the canonical model over a generic point of S extends to a canonical model over an open subset of S and we can finish by Noetherian induction. Thus closedness needs to be checked over spectra of DVR's, and the latter follows from (5.36) and (5.37).

Thus W is a scheme and the projection π induces a geometric bijection $W \to S$ which is finite by (5.36) and (5.37). Thus $W \to S$ is an isomorphism since S is seminormal.

If each $(X^{\mathrm{c}}_s, \Delta^{\mathrm{c}}_s)$ is rigid, then $W \subset \mathrm{MSP}^{\mathrm{sn}}_{\mathrm{rigid}}$ and there is a universal family

$$\mathrm{Univ}^{\mathrm{sn}}_{\mathrm{rigid}} \to \mathrm{MSP}^{\mathrm{sn}}_{\mathrm{rigid}}.$$

Therefore the pull-back of the universal family $\mathrm{Univ}^{\mathrm{sn}}_{\mathrm{rigid}}$ to W gives the simultaneous canonical model over $S \cong W$.

In general we make the same proof work by rigidifying $f : (X, \Delta) \to S$. Note that it is enough to construct the simultaneous canonical model étale locally.

After replacing S by an étale neighborhood $(s', S') \to (s, S)$, we may assume that there are r sections $\sigma_i : S' \to X'$ such that $\bigl(X'_{s'}, \Delta'_{s'}, \sigma_1(s'), \dots, \sigma_r(s')\bigr)$ is rigid and the $\sigma_i(s')$ are smooth points of $X_{s'} \setminus \operatorname{Supp} \Delta_{s'}$ such that $(X'_{s'}, \Delta'_{s'}) \dashrightarrow (X^{\rm c}_s, \Delta^{\rm c}_s)$ is a local isomorphism at these points.

By (4.78), after further shrinking S' we may assume that the same holds at every $t \in S'$. We can now run the previous argument over S' using

$$W' := \bigl\{(X^{\rm c}_t, \Delta^{\rm c}_t, \sigma_1(t), \dots, \sigma_r(t)) : t \in S'\bigr\} \subset \mathrm{MpSP}^{\rm sn},$$

as in (4.82), to prove that the simultaneous canonical model exists over S'. □

5.65 (Proof of Theorem 5.17). The proof follows very closely the arguments in (5.64).

Both claims were already established over the spectrum of a DVR, see (5.43). This implies the semicontinuity assertion in general.

In order to complete the proof of (5.17) it remains to show that if $s \mapsto I\bigl(\pi_s^* H_s, K_{X_s^{\rm can}}\bigr)$ is constant then $f : (X, \Delta) \to S$ has a simultaneous canonical modification. Since the simultaneous canonical modification is unique, it is sufficient to construct it étale locally over S. So pick a point $s_0 \in S$, in the sequel we are free to replace S by smaller neighborhoods of s_0.

Choose $m > 0$ such that $K_{X_s^{\rm can}} + m\pi_s^* H_s$ is ample for every $s \in S$. Next choose a general $D \in |mH|$ such that $\bigl(X^{\rm can}_{s_0}, \Delta^{\rm can}_{s_0} + \pi^*_{s_0} D_{s_0}\bigr)$ is log canonical. We claim that, possibly after shrinking S, $\bigl(X^{\rm can}_s, \Delta^{\rm can}_s + \pi^*_s D_s\bigr)$ is log canonical for every $s \in S$. By (4.48) this condition defines a constructible subset of S and, by (5.43), it contains every generalization of s_0. Thus it contains an open neighborhood of s_0. Thus $\bigl(X^{\rm can}_s, \Delta^{\rm can}_s + \pi^*_s D_s\bigr)$ is a stable pair for every $s \in S$.

Consider the moduli space of marked stable pairs $\pi : \mathrm{MSP}^{\rm sn} \to S$ and set

$$W := \bigl\{(X^{\rm can}_s, \Delta^{\rm can}_s + \pi^*_s D_s) : s \in S\bigr\} \subset \mathrm{MSP}^{\rm sn}.$$

In order to prove that W is a closed subset, first we claim that it is constructible. This is clear since the canonical modification over a generic point of S extends to a canonical modification over an open subset of S and we can finish by Noetherian induction. Thus closedness needs to be checked over spectra of DVR's, and the latter follows from (5.43).

Thus W is a scheme and the projection π induces a geometric bijection $W \to S$ which is finite by (5.43). Thus $W \to S$ is an isomorphism since S is seminormal.

If each $(X^{\rm can}_s, \Delta^{\rm can}_s + \pi^*_s D_s)$ is rigid, then $W \subset \mathrm{MSP}^{\rm sn}_{\rm rigid}$ and there is a universal family

$$\mathrm{Univ}^{\rm sn}_{\rm rigid} \to \mathrm{MSP}^{\rm sn}_{\rm rigid}.$$

Therefore the pull-back of the universal family $\mathrm{Univ}^{\rm sn}_{\rm rigid}$ to W gives the simultaneous canonical modification over $S \cong W$.

As in (5.64), we can make the same proof work in general by rigidifying $f : (X, \Delta) \to S$ using étale-local sections. □

CHAPTER 6

Infinitesimal deformations

By Principle 1.43, not every flat deformation of a stable variety should be allowed in their moduli theory. The "good" deformations should be are compatible with powers of the dualizing sheaf. Three variants of this compatibility have been studied in the past.

NOTATION 6.1. We are ultimately interested in schemes with semi-log-canonical singularities, but for the basic definitions we need to assume only that X is a pure dimensional, S_2 scheme over a field k such that

(1.a) there is a closed subset $Z \subset X$ of codimension ≥ 2 such that $\omega_{X\setminus Z}$ is locally free and

(1.b) there is an $m > 0$ such that $\omega_X^{[m]}$ is locally free,

where $\omega_X^{[m]}$ denotes the reflexive hull of $\omega_X^{\otimes m}$. The smallest such $m > 0$ is called the *index* of ω_X. Both of these conditions are satisfied by schemes with semi-log-canonical singularities.

Let $(0, T)$ be a local scheme such that $k(0) \cong k$ and $p : X_T \to T$ a flat deformation of $X \cong X_0$. As in (2.6), for every $r \in \mathbb{Z}$ we have natural restriction maps

$$\mathcal{R}^{[r]} : \omega^{[r]}_{X_T/T}|_{X_0} \to \omega^{[r]}_{X_0}. \tag{6.1.2}$$

These maps are isomorphisms over $X \setminus Z$ and we are interested in understanding those cases when they are isomorphisms over X. By (9.26) if T is Artinian then the following conditions are equivalent for any fixed $r \in \mathbb{Z}$:

(3.a) $\mathcal{R}^{[r]}$ is an isomorphism,

(3.b) $\mathcal{R}^{[r]}$ is surjective,

(3.c) $\omega^{[r]}_{X_T/T}$ is flat over T.

DEFINITION 6.2. Let $p : X_T \to T$ be a flat deformation as in Notation 6.1.

(6.2.1) We call $p : X_T \to T$ a qG-*deformation* if the conditions (6.1.3.a–c) hold for every r. It is enough to check these for $r = 1, \dots, \mathrm{index}(\omega_X)$. (qG is short for "Quotient of Gorenstein," but this is misleading if $\dim X \geq 3$.)

These deformations were introduced and studied by Kollár and Shepherd-Barron [**KSB88**] as the class most suitable for compactifying the moduli of varieties of general type. A list of log canonical surface singularities with qG-smoothings is given in [**KSB88**]. In the key case of cyclic quotient singularities the list was earlier established by Wahl [**Wah80**, 2.7], though he viewed them as examples of W-deformations (see below).

(6.2.2) We call $p : X_T \to T$ a *Viehweg-type deformation* (or V-deformation) if the conditions (6.1.3.a–c) hold for every r divisible by $\mathrm{index}(\omega_X)$. It is enough to check this for $r = \mathrm{index}(\omega_X)$. These deformations are used in the monograph

[**Vie95**]. Actually, [**Vie95**] considers the—a priori weaker—condition: $\mathcal{R}^{[r]}$ is an isomorphism for some $r > 0$ divisible by $\operatorname{index}(\omega_X)$. One can see that in this case (6.1.3.a–c) hold for every r divisible by $\operatorname{index}(\omega_X)$, at least in characteristic 0; see (2.92). The two notions are different in positive characteristic by (4.40).

(6.2.3) We call $p : X_T \to T$ a *Wahl-type deformation* (or W-deformation) if the conditions (6.1.3.a–b) hold for $r = -1$. These deformations were considered in [**Wah80, Wah81**] and called ω^*-constant deformations there.

(6.2.4) We call $p : X_T \to T$ a VW-deformation if it is both a V-deformation and a W-deformation.

It is clear that every qG-deformation is also a VW-deformation. Understanding the precise relationship between these 4 classes has been a long standing open problem. For reduced base spaces we have the following, which is a combination of (2.76) and (3.68).

Theorem 6.3. *A flat deformation of a log canonical scheme over a reduced, local scheme of characteristic 0 is a V-deformation iff it is a* qG*-deformation.*

This raised the possibility that every V-deformation of a log-canonical singularity is also a qG-deformation over arbitrary base schemes. It would be enough to check this for Artinian bases. Here we focus on first order deformations and prove that these 2 classes are quite different from each other.

Definition 6.4. Let X be a scheme satisfying the conditions (6.1.1–2). Let $T^1(X)$ denote the set of isomorphism classes of deformations of X over $\operatorname{Spec}_k k[\epsilon]$. This is a (possibly infinite dimensional) k-vector space. Let $T^1_{\mathrm{qG}}(X) \subset T^1(X)$ denote the space of first order qG-deformations, $T^1_V(X)$ the space of first order V-deformations, $T^1_W(X)$ the space of first order W-deformations and $T^1_{VW}(X)$ the space of first order VW-deformations. We have obvious inclusions

$$T^1_{\mathrm{qG}}(X) \subset T^1_{VW}(X) \subset T^1_V(X), T^1_W(X) \subset T^1(X),$$

but the relationship between $T^1_V(X)$ and $T^1_W(X)$ is not clear.

These $T^1_*(X)$ are the tangent spaces to the corresponding miniversal deformation spaces; we denote these by $\operatorname{Def}_{\mathrm{qG}}(X), \operatorname{Def}_V(X)$ and so on. See [**Art76**] or [**Loo84**] for precise definitions and introductions or (2.24–2.28) for details on surface quotient singularities.

We completely describe first order qG-, V- and W-deformations of cyclic quotient singularities. The precise answers are stated in Section 6.2. The main conclusion is that qG-deformations and V-deformations are quite different over Artinian bases; its proof is given in (6.42).

Theorem 6.5. *Let $S_{n,q} := \mathbb{A}^2/\frac{1}{n}(1,q)$ denote the quotient of $\mathbb{A}^2$ by the cyclic group action generated by $(x,y) \mapsto (\eta x, \eta^q y)$, where η is a primitive nth root of unity. Then*

$$\dim T^1_V\big(S_{n,q}\big) - \dim T^1_{VW}\big(S_{n,q}\big) = \operatorname{embdim}\big(S_{n,q}\big) - 4 \quad \text{or} \quad \operatorname{embdim}\big(S_{n,q}\big) - 5.$$

In particular, if $\operatorname{embdim}\big(S_{n,q}\big) \geq 5$ *then $S_{n,q}$ has V-deformations that are not VW-deformations, hence also not* qG*-deformations.*

By contrast, qG-deformations and VW-deformations are quite close to each other, as shown by the next result, proved in (6.44).

THEOREM 6.6. *Let* $S_{n,q} := \mathbb{A}^2/\frac{1}{n}(1,q)$ *denote the quotient of* $\mathbb{A}^2$ *by the cyclic group action generated by* $(x,y) \mapsto (\eta x, \eta^q y)$, *where* η *is a primitive nth root of unity.*

(1) *If* $(n, q+1) = 1$ *then* $\operatorname{Def}_{\mathrm{qG}}\bigl(S_{n,q}\bigr) = \operatorname{Def}_{VW}\bigl(S_{n,q}\bigr) = \{0\}$.
(2) *If* $S_{n,q}$ *admits a* qG*-smoothing then* $\operatorname{Def}_{\mathrm{qG}}\bigl(S_{n,q}\bigr) = \operatorname{Def}_{VW}\bigl(S_{n,q}\bigr)$.
(3) *In general* $\dim T^1_{\mathrm{qG}}\bigl(S_{n,q}\bigr) \le \dim T^1_{VW}\bigl(S_{n,q}\bigr) \le \dim T^1_{\mathrm{qG}}\bigl(S_{n,q}\bigr) + 1$.

COROLLARY 6.7. *The cyclic quotient singularities for which every V-deformation is a* qG*-deformation are the following.*

(1) *Double points:* $\mathbb{A}^2/\frac{1}{n}(1, n-1)$ *for* $n \ge 1$.
(2) *Triple points:* $\mathbb{A}^2/\frac{1}{ab-1}(1, ab-b-1)$ *for* $a, b \ge 2$.
(3) *Quadruple points:* $\mathbb{A}^2/\frac{1}{a(ab-2)}\bigl(1, (ab-2)(a-1)-1\bigr)$ *for* $a, b \ge 2$.

The list includes all triple points but only some of the quadruple points.

6.1. First order deformations—with Klaus Altmann

In this section we study first order infinitesimal deformations of normal varieties. We describe the deformations of the smooth locus and then try to understand when a deformation of the smooth locus extends to a deformation of the whole variety. The final aim is to get an explicit obstruction theory for lifting sections of powers of the dualizing sheaf. This turns out to be given by the classical notion of divergence.

6.8 (First order thickening). Let k be a field and R a k-algebra. Consider the algebra $R[\epsilon]$ where ϵ is a new variable satisfying $\epsilon^2 = 0$. It is flat over $k[\epsilon]$ and $R[\epsilon] \otimes_{k[\epsilon]} k \cong R$. Thus we can think of $R[\epsilon]$ as the trivial first order deformation of R.

Let $v : R \to R$ be a k-linear derivation. Then

$$\alpha_v : r_1 + \epsilon r_2 \mapsto r_1 + \epsilon\bigl(v(r_1) + r_2\bigr) \tag{6.8.1}$$

defines an automorphism of $R[\epsilon]$ that is trivial modulo (ϵ). Conversely, every automorphism of $R[\epsilon]$ that is trivial modulo (ϵ) arises this way. (The product (or Leibnitz) rule for v is equivalent to the multiplicativity of α_v.)

Let X be a k-scheme. The trivial first order deformation of X is

$$X[\epsilon] := X \times_k \operatorname{Spec}_k k[\epsilon]. \tag{6.8.2}$$

As in (6.8.1), every derivation $v : \mathcal{O}_X \to \mathcal{O}_X$ defines an automorphism α_v of $X[\epsilon]$ that is trivial modulo (ϵ). This gives an exact sequence

$$0 \to \operatorname{Hom}(\Omega^1_X, \mathcal{O}_X) \to \operatorname{Aut}(X[\epsilon]) \to \operatorname{Aut}(X) \to 1. \tag{6.8.3}$$

If X is smooth, or at least normal, then $\mathcal{H}om(\Omega^1_X, \mathcal{O}_X)$ is the tangent sheaf T_X of X, hence we can rewrite the sequence as

$$0 \to H^0(X, T_X) \xrightarrow{\alpha} \operatorname{Aut}(X[\epsilon]) \to \operatorname{Aut}(X) \to 1. \tag{6.8.4}$$

Aside. On a differentiable manifold M one can identify the Lie algebra of all vector fields with the Lie algebra of the automorphism group. If X is a smooth variety, then this identification works if X is proper but not otherwise. For instance, an affine curve C of genus ≥ 1 has only finitely many automorphisms but $H^0(C, T_C)$ is infinite dimensional. Infinitesimal thickenings restore the connection between vector fields and automorphisms.

6.9 (Locally trivial first order deformations). Let k be a field and X a k-scheme. A *deformation* of X over $A := \operatorname{Spec}_k k[\epsilon]$ is a flat A-scheme X' together with an isomorphism $X' \times_A \operatorname{Spec} k \cong X$. The set of isomorphism classes of first order deformations is denoted by $T^1(X)$. It is easy to see that $T^1(X)$ is naturally a k-vector space whose zero is the trivial deformation $X[\epsilon]$, but this is not very important for us now. See [**Art76**] or [**Har10**] for detailed discussions.

We say that X' is *locally trivial* if there is an affine cover $X = \cup_i X_i$ such that each X'_i is a trivial deformation of X_i.

We aim to classify all locally trivial first order deformations of arbitrary k-schemes X, but our main interest is in cases when X is smooth and quasi-projective.

Let $X = \cup_i X_i$ be an affine cover. This gives an affine cover $X' = \cup_i X'_i$ and we assume that each X'_i is a trivial deformation of X_i. Fix trivializations $\phi_i : X'_i \cong X_i[\epsilon]$. Over $X'_{ij} := X'_i \cap X'_j$ we have 2 trivializations, these differ by an automorphism

$$\alpha_{ij} := \phi_j^{-1} \circ \phi_i : X'_{ij} \to X'_{ij}, \tag{6.9.1}$$

which is the identity on X_{ij}. By (6.8.1) the automorphisms α_{ij} correspond to $v_{ij} \in \operatorname{Hom}\bigl(\Omega^1_{X_{ij}}, \mathcal{O}_{X_{ij}}\bigr)$ and these form a 1-cocycle $D := \{v_{ij}\}$. Changing the trivializations changes the cocyle by a coboundary. Thus we get a well defined element

$$D = D(X') \in H^1\bigl(X, \mathcal{H}om(\Omega^1_X, \mathcal{O}_X)\bigr). \tag{6.9.2}$$

The construction can be reversed. It is left to the reader to check that $D(X')$ is independent of the choices we made. The final outcome is the following.

Claim 6.9.3. Let X be a k-scheme. There is a one-to-one correspondence, denoted by $D \mapsto X_D$, between

(a) elements of $H^1\bigl(X, \mathcal{H}om(\Omega^1_X, \mathcal{O}_X)\bigr)$ and
(b) locally trivial deformations of X over $\operatorname{Spec}_k k[\epsilon]$, up-to isomorphism.

Furthermore, if X is normal then $H^1\bigl(X, \mathcal{H}om(\Omega^1_X, \mathcal{O}_X)\bigr) = H^1(X, T_X)$. □

Next we check that every first order deformation of a smooth variety Y is locally trivial. To see this we may assume that Y is affine. Then Y' is also affine and we can fix a vector space isomorphism $k[Y'] \cong k[Y] \otimes k[\epsilon]$. Pick a point $p \in Y$, local coordinates $y_1, \dots, y_n$ and their trivial lifts $y'_1, \dots, y'_n \in k[Y']$. Any other $z \in k[Y]$ satisfies a monic, separable equation $F(z, \mathbf{y}) = 0$. We claim that z has a unique lift $z' \in k[Y']$ such that $F(z', \mathbf{y}') = 0$. To see this pick any lift z^*. Then $F\bigl(z^*, \mathbf{y}'\bigr) = \epsilon G(z)$ for some $G(z) \in k[Y]$. We are looking for z' in the form $z' = z^* + \epsilon g$ where $g \in k[Y]$. Since $F\bigl(z^* + \epsilon g, \mathbf{y}'\bigr) = \epsilon G(z) + \epsilon g \cdot \partial F(z, \mathbf{y})/\partial z$, we see that $g = -G(z)\bigl(\partial F(z, \mathbf{y})/\partial z\bigr)^{-1}$ is the unique solution. We do this for a finite set of generators $\{z_i\}$ of $k[Y]$ to get a trivialization in a neighborhood where all the $\partial F_i(z, \mathbf{y})/\partial z$ are invertible.

Combining with (6.9.3), this proves the following. (See [**Har77**, Exrc.II.8.6] for a slightly different proof.)

Claim 6.9.4. Every deformation of a smooth, affine variety over $k[\epsilon]$ is trivial. □

6.10 (Arbitrary first order deformations). Let k be a field and X a normal k-variety. Let $U \subset X$ be the smooth locus, $Z \subset X$ the singular locus and $j : U \hookrightarrow X$ the natural injection.

Let $X' \to \operatorname{Spec}_k k[\epsilon]$ be a flat deformation of X. By restriction it induces a flat deformation U' of U. Note that U' uniquely determines X'. Indeed, $\operatorname{depth}_Z \mathcal{O}_X \geq 2$ since X is normal, hence $\operatorname{depth}_Z \mathcal{O}_{X'} \geq 2$ since $\mathcal{O}_{X'}$ is an extension of 2 copies of $\mathcal{O}_X$. Therefore $\mathcal{O}_{X'} = j_*\mathcal{O}_{U'}$ by (9.7). Thus we have an injection

$$T^1(X) \hookrightarrow T^1(U) = H^1(U, T_U).$$

Following [**Sch71**], our plan is to study $T^1(X)$ by first describing $T^1(U)$ and then understanding which $D \in H^1(U, T_U)$ correspond to a deformation of X; see also [**vE90**]. The second step is accomplished in (6.13).

Definition 6.11. Let X be a k-scheme. Given $v \in \operatorname{Hom}(\Omega^1_X, \mathcal{O}_X)$, *differentiation* by v is defined as the composite

$$v(\ ) : \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{v} \mathcal{O}_X. \tag{6.11.1}$$

Let $x_1, \dots, x_n$ be (analytic or étale) local coordinates at a smooth point of X and write $v = \sum_i v_i \frac{\partial}{\partial x_i}$. Then the above maps are

$$v : f \mapsto \textstyle\sum_i \frac{\partial f}{\partial x_i} dx_i \mapsto \sum_i v_i \frac{\partial f}{\partial x_i}.$$

Thus if X is smooth and v is identified with a section of T_X, then (6.11.1) agrees with the usual definition.

Next let $D \in H^1\big(X, \mathcal{H}om(\Omega^1_X, \mathcal{O}_X)\big)$ and choose a representative 1-cocyle $D = \{v_{ij}\}$ using an affine cover $X = \cup X_i$. For any $s \in H^0(X, \mathcal{O}_X)$ the derivatives $\{v_{ij}\big(s|_{X_{ij}}\big)\}$ form a 1-cocycle with values in $\mathcal{O}_X$. This defines $D(s) \in H^1(X, \mathcal{O}_X)$. We think of it either as a *cohomological differentiation* map

$$D : H^0(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \tag{6.11.2}$$

or as a k-bilinear map

$$H^1\big(X, \mathcal{H}om(\Omega^1_X, \mathcal{O}_X)\big) \times H^0(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X). \tag{6.11.3}$$

If X is normal then we can rewrite this as

$$H^1(X, T_X) \times H^0(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X). \tag{6.11.4}$$

Let X_D be the deformation of X corresponding to D. Its structure sheaf sits in an exact sequence

$$0 \to \epsilon\mathcal{O}_X \to \mathcal{O}_{X_D} \to \mathcal{O}_X \to 0. \tag{6.11.5}$$

Taking cohomology we see that D in (6.11.2) is the connecting map

$$H^0\big(X_D, \mathcal{O}_{X_D}\big) \to H^0(X, \mathcal{O}_X) \xrightarrow{D} H^1(X, \mathcal{O}_X). \tag{6.11.6}$$

Warning 6.11.7. Note that although $H^0(X, \mathcal{O}_X)$ and $H^1(X, \mathcal{O}_X)$ are both $H^0(X, \mathcal{O}_X)$-modules, the map D is usually not an $H^0(X, \mathcal{O}_X)$-module homomorphism. Indeed, the constant section $1_X \in H^0(X, \mathcal{O}_X)$ always lifts, hence $D(1_X) = 0$. Thus D is an $H^0(X, \mathcal{O}_X)$-module homomorphism iff it is identically 0.

We can summarize the above considerations as follows.

Lemma 6.12. *Let X be a k-scheme, $D \in H^1\big(X, \mathcal{H}om(\Omega^1_X, \mathcal{O}_X)\big)$ and X_D the corresponding deformation of X. Then a global section $s \in H^0(X, \mathcal{O}_X)$ lifts to $s_D \in H^0(X_D, \mathcal{O}_{X_D})$ iff $D(s) \in H^1(X, \mathcal{O}_X)$ is zero.* □

Corollary 6.13. *Let X be a normal, affine variety and $U \subset X$ its smooth locus. Let U_D be the deformation of U corresponding to $D \in H^1(U, T_U)$. Then*

(1) *U_D extends to a flat deformation X_D of X iff $D : H^0(U, \mathcal{O}_U) \to H^1(U, \mathcal{O}_U)$ (as in (6.11.2)) is identically 0.*
(2) *$T^1(X)$ is the left kernel of $H^1(U, T_U) \times H^0(U, \mathcal{O}_U) \to H^1(U, \mathcal{O}_U)$.*

Proof. Assume that U_D extends to a flat deformation X_D of X. Since X is affine, so is X_D and so $H^0(X_D, \mathcal{O}_{X_D}) \to H^0(X, \mathcal{O}_X)$ is surjective. Thus $D : H^0(U, \mathcal{O}_U) \to H^1(U, \mathcal{O}_U)$ is identically 0 by (6.12).

Conversely, if $D : H^0(U, \mathcal{O}_U) \to H^1(U, \mathcal{O}_U)$ is identically 0 then $H^0(U_D, \mathcal{O}_{U_D}) \to H^0(U, \mathcal{O}_U)$ is surjective and $H^0(U, \mathcal{O}_U) = H^0(X, \mathcal{O}_X)$ since X is normal. We can then take $X_D := \operatorname{Spec}_k H^0(U_D, \mathcal{O}_{U_D})$. This proves the first claim and the second is a reformulation of it. □

REMARK 6.14. If X is not affine, one can restate (6.13) as follows. $D \in H^1(U, T_U)$ gives a k-linear map $D : \mathcal{O}_X = j_*\mathcal{O}_U \to R^1 j_*\mathcal{O}_U = \mathcal{H}^2_Z(\mathcal{O}_X)$ where $Z := X \setminus U$ is the singular locus. Then U_D extends to a flat deformation X_D of X iff $D : \mathcal{O}_X \to \mathcal{H}^2_Z(\mathcal{O}_X)$ is identically 0.

6.15 (Lie derivative). Let M be a smooth, real manifold and v a vector field on M. By integrating v we get a 1-parameter family of diffeomorphisms ϕ_t of M. The *Lie derivative* of a covariant tensor field S is defined as

$$L_v S := \tfrac{d}{dt}\big(\phi_t^* S\big)_{t=0}. \tag{6.15.1}$$

In local coordinates $\{y_i\}$ write $v = \sum_i v_i \frac{\partial}{\partial y_i}$. The Lie derivatives of a function s and of a 1-form dy_j are given by the formulas

$$L_v s = v(s) = \textstyle\sum_i v_i \frac{\partial s}{\partial y_i} \quad \text{and} \quad L_v(dy_j) = dv_j. \tag{6.15.2}$$

Since functions and 1-forms generate the algebra of covariant tensors, the Lie derivative is uniquely determined by the formulas (6.15.2). One can extend the definition to all tensors by duality.

We can transplant this definition to algebraic geometry as follows.

Let Y be a smooth variety over a field k and $v \in H^0(Y, T_Y)$ a vector field. By (6.8.4) v can be identified with an automorphism α_v of $Y[\epsilon]$. We write Ω_Y for the module of derivations (frequently denoted by Ω^1_Y). The covariant tensors are sections of the algebra $\sum_{m\ge 0} \Omega_Y^{\otimes m}$.

Let $S \in H^0\big(Y, \sum_{m\ge 0} \Omega_Y^{\otimes m}\big)$ be a covariant tensor on Y. It has a trivial extension to $Y[\epsilon]$; denote it by $S[\epsilon]$. Thus $\alpha_v^*\big(S[\epsilon]\big)$ is a global section of $\sum_{m\ge 0} \Omega_{Y[\epsilon]}^{\otimes m}$. Since α_v is the identity on X, $\alpha_v^*\big(S[\epsilon]\big) - S[\epsilon]$ is divisible by ϵ and we can define the *Lie derivative* of S by the formula

$$\alpha_v^*\big(S[\epsilon]\big) = S[\epsilon] + \epsilon L_v S. \tag{6.15.3}$$

Expanding the identity $\alpha_v^*\big(S_1[\epsilon] \otimes S_2[\epsilon]\big) = \alpha_v^*\big(S_1[\epsilon]\big) \otimes \alpha_v^*\big(S_2[\epsilon]\big)$ shows that the Lie derivative is a k-linear derivation of the tensor algebra

$$L_v : \textstyle\sum_{m\ge 0} \Omega_Y^{\otimes m} \to \sum_{m\ge 0} \Omega_Y^{\otimes m}. \tag{6.15.4}$$

The Lie derivative preserves natural quotient bundles of $\Omega_X^{\otimes m}$. Thus we get similar maps L_v for symmetric and skew-symmetric tensors. Our main interest is in powers of ω_X. The corresponding map

$$L_v : \omega_Y^m \to \omega_Y^m \tag{6.15.5}$$

is obtained using the identification $\Omega_Y^{\otimes n} \twoheadrightarrow \Omega_Y^n = \omega_Y$ where $n = \dim Y$.

From (6.8.1) we see that

$$\alpha_v^*\big(s[\epsilon]\big) = s[\epsilon] + \epsilon v(s) \quad \text{and} \quad \alpha_v^*(dy_j) = d\big(\alpha_v^*(y_j)\big) = dy_j + \epsilon dv_j. \tag{6.15.6}$$

Comparing with (6.15.2) we see that the algebraic definition coincides with the differential geometry definition.

6.16 (Cartan formula). This is an identity which holds for exterior forms S

$$L_v(S) = d\big(v\lrcorner S\big) + v\lrcorner dS, \tag{6.16.1}$$

where $\lrcorner$ denotes *contraction* or *inner product* by a vector field $v \in H^0(Y, T_Y)$ obtained as follows. We have the contraction map $T_Y \otimes \Omega_Y^m \to \Omega_Y^{m-1}$, thus every $v \in H^0(Y, T_Y)$ gives the $\mathcal{O}_Y$-linear map

$$v\lrcorner : \Omega_Y^m \to \Omega_Y^{m-1}. \tag{6.16.2}$$

In (analytic or étale) local coordinates $y_1, \dots, y_n$ write $v = \sum_i v_i \frac{\partial}{\partial y_i}$. Then

$$v\lrcorner\big(dy_1 \wedge \dots \wedge dy_m\big) = \sum_r (-1)^{r-1} v_r \cdot dy_1 \wedge \dots \wedge \widehat{dy_r} \wedge \dots \wedge dy_m, \tag{6.16.3}$$

where the hat indicates that we omit that term.

The prove (6.16.1), one first checks that $S \mapsto d\big(v\lrcorner S\big) + v\lrcorner dS$ is also a derivation. Thus it is sufficient to verify (6.16.1) for a generating set of exterior forms. For functions and for dy_j we recover the identities (6.15.2).

6.17. As in (6.11), let Y be a smooth k-variety. Pick $D \in H^1(Y, T_Y)$ and choose a representative 1-cocycle $D = \{v_{ij}\}$ using an affine cover $Y = \cup Y_i$. For any $S \in H^0\big(Y, \Omega_Y^{\otimes m}\big)$ the Lie derivatives $\{L_{v_{ij}}\big(S|_{Y_{ij}}\big)\}$ form a 1-cocycle with values in $\Omega_Y^{\otimes m}$. This defines

$$L_D(S) \in H^1\big(Y, \Omega_Y^{\otimes m}\big), \tag{6.17.1}$$

which we view as a *cohomological differentiation* map

$$L_D : \sum H^0\big(Y, \Omega_Y^{\otimes m}\big) \to \sum H^1\big(Y, \Omega_Y^{\otimes m}\big). \tag{6.17.2}$$

As we noted in (6.15), the map L_D respects natural quotient bundles of $\Omega_Y^{\otimes m}$. Thus we get similar maps for symmetric and skew-symmetric tensors and for powers of ω_Y

$$L_D : \sum H^0\big(Y, \omega_Y^m\big) \to \sum H^1\big(Y, \omega_Y^m\big). \tag{6.17.3}$$

For $m = 0$ the map $L_D : H^0\big(Y, \omega_Y^0\big) \to H^1\big(Y, \omega_Y^0\big)$ agrees with the map $D : H^0(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$ defined in (6.11.2).

As in (6.11.7), L_D is a k-linear differentiation which is usually not $H^0(Y, \mathcal{O}_Y)$-linear. However, if the map $D : H^0(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y)$ is zero then L_D is $H^0(Y, \mathcal{O}_Y)$-linear; this holds both for the general case (6.17.2) and the special one (6.17.3).

Arguing as in (6.12) we obtain the following lifting criterion.

LEMMA 6.18. *Let Y be a smooth k-variety and Y_D a first order deformation of Y. Then $S \in H^0\big(Y, \Omega_Y^{\otimes m}\big)$ lifts to $S_D \in H^0\big(X_D, \Omega_{Y_D}^{\otimes m}\big)$ iff $L_D(S) \in H^1\big(Y, \Omega_Y^{\otimes m}\big)$ is zero.* □

Divergence.

Next we consider what the previous method gives for ω_Y and its powers using (6.17.3).

6.19 (Divergence). Let Y be a smooth k-variety, $\sigma \in H^0(Y, \omega_Y^m)$ and $v \in H^0(Y, T_Y)$. Then σ and $L_v\sigma$ are both sections of the line bundle ω_Y^m, hence their quotient is a rational function, called the *divergence* of v with respect to σ,

$$\nabla_\sigma v := \frac{L_v \sigma}{\sigma}. \tag{6.19.1}$$

(Most books seem to use this terminology only when σ is a nowhere 0 section of ω_Y and σ is frequently suppressed in the notation.)

In order to compute this, start with a section σ of ω_Y. Since $d\sigma = 0$, Cartan's formula (6.16) shows that $L_v : \omega_Y \to \omega_Y$ is the composite map

$$L_v : \omega_Y = \Omega_Y^n \xrightarrow{v \lrcorner} \Omega_Y^{n-1} \xrightarrow{d} \Omega_Y^n = \omega_Y. \tag{6.19.2}$$

In local coordinates $y_1, \dots, y_n$ assume that $\sigma = dy_1 \wedge \dots \wedge dy_n$ and $v = \sum_i v_i \frac{\partial}{\partial y_i}$. Contraction by v sends σ to

$$\textstyle\sum_i (-1)^{i-1} v_i \; dy_1 \wedge \dots \wedge \widehat{dy_i} \wedge \dots \wedge dy_n. \tag{6.19.3}$$

Exterior differentiation now gives that

$$L_v \sigma = d(v \lrcorner \sigma) = \textstyle\sum_i \frac{\partial v_i}{\partial y_i} \cdot \sigma. \tag{6.19.4}$$

That is, the usual formula holds for the divergence:

$$\nabla_\sigma v = \nabla_{\mathbf{y}} v := \textstyle\sum_i \frac{\partial v_i}{\partial y_i}. \tag{6.19.5}$$

For powers of ω_Y this gives the next formula.

LEMMA 6.20. *Let Y be a smooth k-variety of dimension n. Let $v \in H^0(Y, T_Y)$ be a vector field, $s \in H^0(Y, \mathcal{O}_Y)$ a function and $\sigma \in H^0(Y, \omega_Y)$ an n-form. Then*

$$\nabla_{(s\sigma^m)} v = \tfrac{v(s)}{s} + m \nabla_\sigma v. \tag{6.20.1}$$

Proof. This is really just the assertion that the Lie derivative is a derivation, but it is instructive to do the local computations.

The claimed identities are local, so we may work with local coordinates $y_1, \dots, y_n$ and assume that $\sigma = dy_1 \wedge \dots \wedge dy_n$. Write $v = \sum_i v_i \frac{\partial}{\partial y_i}$. We need to compute how the isomorphism α_v acts on $s\sigma^m$. It sends y_i to $y_i + \epsilon v(y_i) = y_i + \epsilon v_i$, thus

$$\alpha_v^*(dy_i) = \bigl(1 + \epsilon \tfrac{\partial v_i}{\partial y_i}\bigr) dy_i + \epsilon \bigl(\textstyle\sum_{j \neq i} \frac{\partial v_i}{\partial y_j} dy_j\bigr). \tag{6.20.2}$$

Next we wedge these together. Any two epsilon terms wedge to 0 since $\epsilon^2 = 0$. Thus $\epsilon\bigl(\sum_{j \neq i} \frac{\partial v_i}{\partial y_j} dy_j\bigr)$ gets killed unless it is wedged with all the other dy_j, but the result is then zero in the exterior algebra. Hence the only term that survives is

$$\begin{aligned} \textstyle\prod_i \bigl(1 + \epsilon \frac{\partial v_i}{\partial y_i}\bigr) \cdot dy_1 \wedge \dots \wedge dy_n &= \bigl(1 + \epsilon \textstyle\sum_i \frac{\partial v_i}{\partial y_i}\bigr) \cdot dy_1 \wedge \dots \wedge dy_n \\ &= \bigl(1 + \epsilon \nabla_{\mathbf{y}} v\bigr) \cdot dy_1 \wedge \dots \wedge dy_n. \end{aligned} \tag{6.20.3}$$

Thus we get that $s\sigma^m$ is mapped to

$$\begin{aligned} &\bigl(s + \epsilon v(s)\bigr)\bigl(1 + m\epsilon \nabla_{\mathbf{y}} v\bigr) \cdot \sigma^m \\ &\quad = \bigl(s + \epsilon v(s) + m\epsilon s \nabla_{\mathbf{y}} v\bigr) \cdot \sigma^m \\ &\quad = s\sigma^m + \epsilon \cdot \bigl(\tfrac{v(s)}{s} + m \nabla_{\mathbf{y}} v\bigr) \cdot s\sigma^m. \quad \square \end{aligned} \tag{6.20.4}$$

NOTATION 6.21. Let X be a normal, affine k-variety and X_D a flat deformation of X over $k[\epsilon]$ corresponding to $D \in T^1(X)$. Let $U \subset X$ be the smooth locus. By (6.10) we can think of D as a cohomology class $D \in H^1(U, T_U)$. By (6.11.2) D induces a map

$$D : H^0(U, \mathcal{O}_U) \to H^1(U, \mathcal{O}_U) \tag{6.21.1}$$

which is identically zero by (6.13.2). There is a natural exact sequence

$$0 \to \omega_U^m \to \omega_{U_D}^m \to \omega_U^m \to 0. \tag{6.21.2}$$

Taking cohomologies gives an exact sequence

$$H^0\big(U_D, \omega_{U_D}^m\big) \to H^0\big(U, \omega_U^m\big) \stackrel{\delta_m}{\to} H^1\big(U, \omega_U^m\big). \tag{6.21.3}$$

As we noted at the end of (6.17), δ_m is $H^0(U, \mathcal{O}_U)$-linear since D in (6.21.1) is 0.

It was observed in [**Ste88**] that, for cyclic quotients, the deformation obstruction computed in [**EV85**] equals the divergence. The next result shows that this is a general phenomenon.

THEOREM 6.22. *Let X, $U \subset X$, $D = \{v_{ij}\} \in H^1(U, T_U)$ and X_D be as above (6.21). Assume that $\omega_U^{[m]}$ has a nowhere 0 section σ_m for some $m > 0$ such that* $\operatorname{char} k \nmid m$. *Set $\nabla_{\sigma_m} D := \big\{\nabla_{\sigma_m}(v_{ij})\big\} \in H^1(U, \mathcal{O}_U)$. Then*

(1) *$\nabla D := \frac{1}{m}\nabla_{\sigma_m} D \in H^1(U, \mathcal{O}_U)$ is independent of the choice of m and σ_m.*
(2) *The boundary map $\delta_m : H^0\big(U, \omega_U^m\big) \to H^1\big(U, \omega_U^m\big)$ defined in (6.21.3) is multiplication by $m\nabla D$.*
(3) *$\omega_{U_D}^{[m]}$ is free $\Leftrightarrow$ it is locally free $\Leftrightarrow$ $\nabla D = 0$ in $H^1(U, \mathcal{O}_U)$.*

Proof. Choose affine charts $\{U_i\}$ on U such that $D = \{v_{ij}\}$ and $\sigma_m|_{U_{ij}} = s_{ij}\sigma_{ij}^m$ for some $\sigma_{ij} \in H^0(U_{ij}, \omega_{U_{ij}})$. Any other section of ω_U^m can be written as $g\sigma_m$ where $g \in H^0(U, \mathcal{O}_U)$. Using (6.20) we obtain that

$$\nabla_{\sigma_m} D = \big\{\nabla_{\sigma_m}(v_{ij})\big\} = \Big\{\frac{v_{ij}(s_{ij})}{s_{ij}} + m\nabla_{\sigma_{ij}}(v_{ij})\Big\}. \tag{6.22.4}$$

Similarly, we get that

$$\nabla_{g\sigma_m} D = \Big\{\frac{v_{ij}(gs_{ij})}{gs_{ij}} + m\nabla_{\sigma_{ij}}(v_{ij})\Big\}. \tag{6.22.5}$$

Since

$$\frac{v_{ij}(gs_{ij})}{gs_{ij}} = \frac{v_{ij}(g)}{g} + \frac{v_{ij}(s_{ij})}{s_{ij}}, \tag{6.22.6}$$

subtracting (6.22.4) from (6.22.5) yields

$$\nabla_{g\sigma_m} D - \nabla_{\sigma_m} D = \tfrac{1}{g} D(g) \in H^1(U, \mathcal{O}_U). \tag{6.22.7}$$

As we noted in (6.21), $D(g) = 0$ in $H^1(U, \mathcal{O}_U)$. Thus $\nabla_{g\sigma_m} D = \nabla_{\sigma_m} D$ (as classes in $H^1(U, \mathcal{O}_U)$). Independence of the choice of m is shown by the formula

$$\nabla_{(\sigma_m^r)} D = \Big\{\frac{v_{ij}(s_{ij}^r)}{s_{ij}^r} + rm\nabla_{\sigma_{ij}}(v_{ij})\Big\} = r \cdot \Big\{\frac{v_{ij}(s_{ij})}{s_{ij}} + m\nabla_{\sigma_{ij}}(v_{ij})\Big\}. \tag{6.22.8}$$

Thus ∇D is well defined and this proves (1–2).

Finally, $\omega_{U_D}^{[m]}$ is free iff σ_m lifts to a section of $\omega_{X_D}^{[m]}$ and $\nabla D \cdot \sigma_m$ is the lifting obstruction. This implies (3). $\square$

REMARK 6.23. Let $x \in X$ be an isolated normal singularity and $U := X \setminus \{x\}$. Then $H^1(U, \mathcal{O}_U) = H^2_x(X, \mathcal{O}_X)$ and $H^1(U, T_U) = H^2_x(X, T_X)$. Thus if $\omega_U^m \cong \mathcal{O}_U$ for some $m > 0$ then the divergence can be thought of as a map

$$\nabla : T^1(X) \to H^2_x(X, \mathcal{O}_X).$$

If $\operatorname{depth}_x \mathcal{O}_X \geq 3$ then $H^2_x(X, \mathcal{O}_X) = 0$ by Grothendieck's vanishing theorem (10.18.5), thus in this case the divergence vanishes and sections of ω_U^m lift to all first order deformations. This, however, already follows from (6.21.3) since $H^1(U, \omega_U^m) = H^1(U, \mathcal{O}_U) = H^2_x(X, \mathcal{O}_X) = 0$.

If X is log canonical and ω_X is locally free, then sections of ω_X lift to any deformation by [**KK17**], see also (2.69). By (6.22) this implies that $\nabla : T^1(X) \to H^1(U, \mathcal{O}_U)$ is the zero map.

This should either have a direct proof or some interesting consequences.

Next we give explicit forms of the maps in the general theory for $X := \mathbb{A}^2$ and $U := \mathbb{A}^2 \setminus \{(0,0)\}$. At first this seems quite foolish to do since we already know that a smooth affine variety has only trivial infinitesimal deformations. However, we will be able to use these computations to get very detailed information about deformations of 2-dimensional cyclic quotient singularities; a very interesting subject.

NOTATION 6.24. Let k be a field, $X = \mathbb{A}^2_{xy}$ and $U := X \setminus \{(0,0)\}$. Using the affine charts $U_0 := U \setminus (x = 0)$, $U_1 := U \setminus (y = 0)$ and $U_{01} := U \setminus (xy = 0)$ we compute that

$$H^1(U, \mathcal{O}_U) = \Big\langle \frac{1}{x^i y^j} : i, j \geq 1 \Big\rangle \tag{6.24.1}$$

and also that

$$H^1(U, T_U) = \Big\langle \frac{1}{x^i y^j} \cdot \frac{\partial}{\partial x},\ \frac{1}{x^i y^j} \cdot \frac{\partial}{\partial y} : i, j \geq 1 \Big\rangle.$$

Note that $H^1(U, \mathcal{O}_U)$ is naturally a quotient of

$$H^0\big(U_{01}, \mathcal{O}_{U_{01}}\big) = k\big[x^i y^j : i, j \in \mathbb{Z}\big];$$

the basis in (6.24.1) depends on the choice of coordinates x, y. Similarly, $H^1(U, T_U)$ is naturally a quotient of $H^0\big(U_{01}, T_{U_{01}}\big)$.

It is very convenient computationally that the diagonal subgroup $\mathbb{G}_m^2 \subset \mathrm{GL}_2$ acts on these cohomology groups and subsequent constructions are $\mathbb{G}_m^2$-equivariant. In order to keep track of this action it is better to use the $\mathbb{G}_m^2$-invariant differential operators

$$\partial_x := x \frac{\partial}{\partial x} \quad \text{and} \quad \partial_y := y \frac{\partial}{\partial y}. \tag{6.24.2}$$

Thus $\partial_x(x^r y^s) = r x^r y^s$, $\partial_y(x^r y^s) = s x^r y^s$ and

$$H^1(U, T_U) = \Big\langle \frac{\partial_x}{x^i y^j} : i \geq 2, j \geq 1 \Big\rangle \bigoplus \Big\langle \frac{\partial_y}{x^i y^j} : i \geq 1, j \geq 2 \Big\rangle. \tag{6.24.3}$$

The $\mathbb{G}_m^2$-eigenspaces in $H^1(U, T_U)$ are usually 2-dimensional

$$\Big\langle \frac{\partial_x}{x^i y^j}, \frac{\partial_y}{x^i y^j} \Big\rangle \quad \text{for} \quad i, j \geq 2. \tag{6.24.4.a}$$

The 1-dimensional eigenspaces are

$$\Big\langle \frac{\partial_x}{x^i y} \Big\rangle \quad \text{and} \quad \Big\langle \frac{\partial_y}{x y^j} \Big\rangle \quad \text{for} \quad i, j \geq 2. \tag{6.24.4.b}$$

The pairing $H^1(U,T_U)\times H^0(U,\mathcal{O}_U)\to H^1(U,\mathcal{O}_U)$ defined in (6.11.3) is especially transparent using the bases (6.24.1–4) since

$$\frac{a\partial_x - b\partial_y}{x^iy^j}\bigl(x^ry^s\bigr) = (ar-bs)\cdot x^{r-i}y^{s-j}. \tag{6.24.5}$$

This is identically 0 as an element of $H^0(U_{01},\mathcal{O}_{U_{01}})$ iff $ar-bs=0$. It is more important to know when this is 0 as an element of $H^1(U,\mathcal{O}_U)$. The latter holds iff

(6.a) either $ar-bs=0$ or
(6.b) $r\geq i$ or $s\geq j$.

This easily implies that the left kernel of $H^1(U,T_U)\times H^0(U,\mathcal{O}_U)\to H^1(U,\mathcal{O}_U)$ is trivial, hence $T^1(\mathbb{A}^2)=0$ by (6.13.2); but this we already knew.

Combining (6.18) and (6.20) gives the following.

LEMMA 6.25. *Using the above notation, let $D\in H^1(U,T_U)$ and U_D the corresponding deformation. Then $f(dx\wedge dy)^m$ lifts to a section of $\omega^m_{U_D}$ iff*

$$D(f)+mf\nabla D\in H^1(U,\mathcal{O}_U)\quad \textit{vanishes.}\qquad \square \tag{6.25.1}$$

We are thus interested in computing the kernels of the operators

$$(D,f)\mapsto D(f)+mf\nabla D.$$

We start by describing the kernel of ∇.

6.26 (Computing the divergence). Set $D:=(a\partial_x-b\partial_y)x^{-i}y^{-j}$. By explicit computation,

$$\nabla\Bigl(\frac{a\partial_x-b\partial_y}{x^iy^j}\Bigr) = -\frac{a(i-1)-b(j-1)}{x^iy^j}. \tag{6.26.1}$$

Thus ∇D is identically zero iff $a(i-1)-b(j-1)=0$. If D is a nonzero element of $H^1(U,T_U)$ then $i,j>0$ and then ∇D is 0 as an element of $H^1(U,\mathcal{O}_U)$ iff it is identically zero.

If $(i,j)=(1,1)$ then $\nabla D=0$ but then D vanishes in $H^1(U,T_U)$. If $\nabla D=0$ and $i=1,j>1$ then $b=0$ and again D vanishes in $H^1(U,T_U)$. Thus we conclude that

$$\ker\bigl[H^1(U,T_U)\stackrel{\nabla}{\to}H^1(U,\mathcal{O}_U)\bigr] = \Bigl\langle\frac{(j-1)\partial_x-(i-1)\partial_y}{x^iy^j} : i,j\geq 2\Bigr\rangle. \tag{6.26.2}$$

COROLLARY 6.27. *Let $D\in H^1(U,T_U)$. Then $D(xy),\nabla D\in H^1(U,\mathcal{O}_U)$ are both 0 iff D is contained in the subspace*

$$K_{VW} := \Bigl\langle\frac{\partial_x-\partial_y}{(xy)^i} : i\geq 2\Bigr\rangle\subset H^1(U,T_U).$$

Proof. Corresponding to the 2 cases in (6.24.6.a–b), the kernel of the map $D\mapsto D(xy)\in H^1(U,\mathcal{O}_U)$ is a direct sum of 2 subspaces

$$K_1 := \Bigl\langle\frac{\partial_x-\partial_y}{x^iy^j} : i,j\geq 2\Bigr\rangle\quad\text{and}\quad K_2 := \Bigl\langle\frac{\partial_y}{xy^j},\frac{\partial_x}{x^iy} : i,j\geq 2\Bigr\rangle. \tag{6.27.1}$$

Combining this with (6.26.2) gives the claim. $\square$

6.2. Deformations of cyclic quotient singularities—with Klaus Altmann

In this section we discuss what the general theory of the previous section says about deformations of 2-dimensional quotient singularities. The results are very explicit for cyclic quotient singularities.

6.28 (Deformation of quotients). Let k be a field, X an affine k-scheme that is S_2, $x \in X$ a closed point and $U := X \setminus \{x\}$. Let G be a finite group acting on X such that x is a G-fixed point and the action is free on U. The quotient map $\pi_U : U \to U/G$ is finite and étale. This extends to a finite map $\pi_X : X \to X/G$ which is ramified at x.

$\mathcal{O}_{U/G}$ is identified with the G-invariant subsheaf $(\pi_* \mathcal{O}_U)^G$ and similarly $\omega_{U/G}$ is identified with $(\pi_* \omega_U)^G$. (For the latter we need that the action is free). Thus we get that

$$\begin{aligned} &H^0(U/G, \mathcal{O}_{U/G}) = H^0(U, \mathcal{O}_U)^G = H^0(X, \mathcal{O}_X)^G \quad \text{and} \\ &H^0(U/G, \omega^{[m]}_{U/G}) = H^0(U, \omega^{[m]}_U)^G = H^0(X, \omega^{[m]}_X)^G. \end{aligned} \tag{6.28.1}$$

If $\operatorname{char} k \nmid |G|$ then the G-invariant subsheaf is a direct summand, hence by taking cohomologies we similarly see that

$$H^1(U/G, \mathcal{O}_{U/G}) = H^1(U, \mathcal{O}_U)^G \quad \text{and} \quad H^1(U/G, T_{U/G}) = H^1(U, T_U)^G. \tag{6.28.2}$$

If $D \in H^1(U, T_U)$ is G-invariant then the deformation U_D descends to a deformation $(U/G)_D$ of U/G and these give all first order deformations of U/G. If $H^0(U/G, \mathcal{O}_{U/G})$ is flat over $k[\epsilon]$ then its spectrum gives a flat deformation of X/G and every flat deformation of X/G that is locally trivial on U/G arises this way.

Thus, using (6.13) we get the following fundamental observation.

THEOREM 6.29. [**Sch71**] *Let k be a field, X a smooth, affine k-variety, $x \in X$ a closed point and $U := X \setminus \{x\}$. Let G be a finite group acting on X such that x is a G-fixed point, the action is free on U and* $\operatorname{char} k \nmid |G|$. *Then $T^1(X/G)$ is the left kernel of the pairing*

$$H^1(U, T_U)^G \times H^0(U, \mathcal{O}_U)^G \to H^1(U, \mathcal{O}_U)^G. \tag{6.29.1}$$

More generally, if X is normal, the left kernel corresponds to those flat deformations of X/G that are locally trivial on U/G. □

Next we compute the terms in (6.29.1) for cyclic quotient singularities.

NOTATION 6.30. For the rest of the section we use the following notation.

Set $X := \mathbb{A}^2$ and $U := \mathbb{A}^2 \setminus \{(0,0)\}$. G denotes a cyclic group of order n with generator $g \in G$. The G-action, denoted by $\frac{1}{n}(1,q)$, is given by

$$g : (x, y) \mapsto (\eta x, \eta^q y),$$

where η is a primitive nth root of unity. Thus $\operatorname{char} k \nmid n$ and $(n,q) = 1$ since the action is free outside the origin. The corresponding ring of invariants is

$$R_{nq} := k[x,y]^G = k\big[x^i y^j : i, j \geq 0,\ i + qj \equiv 0 \mod n\big], \tag{6.30.1}$$

and the corresponding quotient singularity is

$$S_{n,q} := \mathbb{A}^2 / \tfrac{1}{n}(1,q) = \operatorname{Spec}_k R_{nq}. \tag{6.30.2}$$

While we work with this affine model, all the results apply to its localization, Henselisation or completion at the origin.

We can also choose $\eta' = \eta^q$ as our primitive nth root of unity. This shows the isomorphism

$$S_{n,q} \cong S_{n,q'} \quad \text{where} \quad qq' \equiv 1 \mod n. \tag{6.30.3}$$

Note that $q \equiv q' \mod n$ iff $n \mid q^2 - 1$.

Various ways of studying such singularities go back a long time. The first relevant work might be [**Jun08**] followed by [**Hir53**]. Most of the following formulas can be found in [**Rie74**]; see [**Ste13**] for an introduction and many examples.

The G-action preserves the monomials, hence R_{nq} has a generating set consisting of monomials. A non-minimal generating set can be constructed as follows. For any $0 < j < n$ let $0 < \gamma_j < n$ be the unique integer such that $\gamma_j + qj \equiv 0 \mod n$. Then

$$x^n, x^{\gamma_1}y, x^{\gamma_2}y^2, \dots, x^{\gamma_{n-1}}y^{n-1}, y^n$$

is a generating set of R_{nq}. We know that $\gamma_1 = n-q$ and $\gamma_{n-1} = q$. This is a minimal generating set of R_{nq} as a $k[x^n, y^n]$-module, but usually not as a k-algebra. Indeed, $x^{\gamma_i}y^i$ divides $x^{\gamma_j}y^j$ if $\gamma_i < \gamma_j$ and $i < j$. In any concrete case one can use this observation to get a minimal set of algebra generators.

We label the monomials of the minimal generating set as $M_i = x^{a_i}y^{b_i}$, ordered by increasing y-powers

$$M_0 = x^n, M_1 = x^{n-q}y = x^{a_1}y^{b_1}, M_2 = x^{a_2}y^{b_2}, \dots, M_r = y^n. \tag{6.30.4}$$

At the same time the a_i form a decreasing sequence. Indeed, if $b_i < b_j$ and $a_i \leq a_j$ then M_i divides M_j so the sequence would not be minimal.

From (6.31.2) we obtain that there are relations of the form

$$M_i^{c_i} = M_{i-1}M_{i+1} \quad \text{for} \quad i = 1, \dots, r-1. \tag{6.30.5}$$

This tells us that the a_i and the c_i are recursively defined by

$$a_0 = n, a_1 = n-q, c_i = \lceil a_{i-1}/a_i \rceil, a_{i+1} = c_i a_i - a_{i-1}. \tag{6.30.6}$$

Similarly, $b_0 = 0, b_1 = 1$ and $b_{i+1} = c_i b_i - b_{i-1}$. These imply that $(a_i, a_{i+1}) = (b_i, b_{i+1}) = 1$ for every i and that the c_i are computed by the modified continued fraction expansion

$$\frac{n}{n-q} = c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \cfrac{1}{c_4 - \cfrac{1}{\dots}}}} \tag{6.30.7}$$

The following observations about the a_i, b_i, c_i are quite useful. The first 2 follow from the original construction of the M_i, the 3rd from (6.30.5) and the last one is equivalent to (6.31.3).

(8.a) $a_{i-1} = \min\{\alpha > 0 : \exists x^\alpha y^\beta \in R_{nq} \text{ such that } \beta < b_i\}$ for $i > 0$.
(8.b) $b_{i+1} = \min\{\beta > 0 : \exists x^\alpha y^\beta \in R_{nq} \text{ such that } \alpha < a_i\}$ for $i < r$.
(8.c) $c_i - 1 = \lfloor \frac{a_{i-1}}{a_i} \rfloor = \lfloor \frac{b_{i+1}}{b_i} \rfloor$ for $0 < i < r$.
(8.d) $a_i b_{i+1} - a_{i+1} b_i = n$ for $0 \leq i < r$.

Note that $r+1$ is the embedding dimension of S_{nq} and r is its multiplicity. Thus $r = 2$ iff $M_1 = M_{r-1} = xy$ and hence we have the A_{n-1}-singularity $\mathbb{A}^2/\frac{1}{n}(1,-1)$. These are exceptional for many of the subsequent formulas, so we assume from now on that $r \geq 3$.

6.31 (Cones and semigroups). Let $v_0, v_1 \in \mathbb{Z}^2$ be primitive vectors and $C := \mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_1 \subset \mathbb{R}^2$ the closed cone spanned by them. Let $\bar{C}(\mathbb{Z})$ be the closed, convex hull of $\left(\mathbb{Z}^2 \cap C\right) \setminus \{(0,0)\}$ and $N(C)$ the part of the boundary of $\bar{C}(\mathbb{Z})$ that connects v_0 and v_1. Let $m_0 = v_0, m_1, \dots, m_{r-1}, m_r = v_1$ be the integral points in $N(C)$ as we move from v_0 to v_1. We leave it to the reader to prove that

(1) the m_i generate the semigroup $\mathbb{Z}^2 \cap C$,
(2) there are natural numbers $c_1, \dots, c_{r-1} \geq 2$ such that $c_i m_i = m_{i-1} + m_{i+1}$ holds for every i and
(3) the triangles with vertices $\{(0,0), m_i, m_{i+1}\}$ all have the same area.

Thus $R(C)$, the semigroup algebra of $\mathbb{Z}^2 \cap C$, is generated by $m_0, \dots, m_s$.

For $1 \leq q < n$ and $(n,q) = 1$ consider the cone C_{nq} spanned by $v_0 = (1,0)$ and $v_1 = (q,n)$. Then

$$\mathbb{Z}^2 \cap C_{nq} = \left\langle \left(\tfrac{i}{n}, \tfrac{j}{n}\right) : i,j \geq 0, i + qj \equiv 0 \mod n \right\rangle.$$

Thus we see that the semigroup algebra $R(C_{nq})$ is isomorphic to the algebra of invariants R_{nq} defined in (6.30.1). (It is not hard to see that, up-to the action of $\mathrm{SL}(2,\mathbb{Z})$, every rational cone in $\mathbb{R}^2$ is of the form C_{nq}.)

6.32 (Computing $T^1(S_{nq})$). Continuing with the notation of (6.28–6.30) we see that $D \in H^1(U, T_U)^G$ is in $T^1(S_{nq})$ iff $D(M_i) = 0 \in H^1(U, \mathcal{O}_U)$ for every i.

Since the pairing (6.29.1) is $\mathbb{G}_m^2$-equivariant, it is sufficient to consider one eigenspace at a time. As in (6.24.4.a–b), the eigenspaces in $H^1(U, T_U)^G$ are usually 2-dimensional and of the form

$$\left\langle \frac{\partial_x}{M}, \frac{\partial_y}{M} \right\rangle \tag{6.32.1}$$

where M is a monomial in the M_i-s involving both x, y. The exceptions are 1-dimensional subspaces. For every $s \geq 0$ we have two of them

$$\left\langle \frac{\partial_x}{M_0^s M_1} \right\rangle \quad \text{and} \quad \left\langle \frac{\partial_y}{M_{r-1} M_r^s} \right\rangle. \tag{6.32.2}$$

Thus we can write $D = (\alpha\partial_x - \beta\partial_y)/M$. Note that

$$D(x^a y^b) = (\alpha a - \beta b)\frac{x^a y^b}{M}, \tag{6.32.3}$$

thus if $a < \operatorname{ord}_x M$ and $b < \operatorname{ord}_y M$ then this is zero in $H^1(U, \mathcal{O}_U)$ iff $\beta/\alpha = a/b$. Thus if M is divisible by at least 2 different monomials M_i, M_j for $0 < i, j < r$ then $D(M_i) = 0$ and $D(M_j) = 0$ imply that we need to satisfy both of the equations $\beta/\alpha = a_i/b_i$ and $\beta/\alpha = a_j/b_j$, a contradiction. We get a similar contradiction for the eigenspaces (6.32.2) if $s > 0$. We are left with the cases when $M = M_i^s$ for some $0 < i < r$. If $s \geq 2$ then $D(M_i) = 0$ implies that $D = (b_i\partial_x - a_i\partial_y)/M_i^s$. Then $b_i a_j - a_i b_j \neq 0$ for $j \neq i$ hence $D(M_j) = (b_i a_j - a_i b_j)(M_j/M_i^s)$ vanishes in $H^1(U, \mathcal{O}_U)$ iff $sa_i \leq a_j$ or $sb_i \leq b_j$. If $j < i$ then $b_j < b_i$, hence $sa_i \leq a_j$ must hold. Since the a_j form a decreasing sequence, we need $sa_i \leq a_{i-1}$. Similarly, $sb_j \leq b_{j+1}$. By (6.30.8.c) these are equivalent to $s \leq c_i - 1$.

We have thus proved the following.

PROPOSITION 6.33. [**Rie74, Pin77**] *Let $M_i = x^{a_i} y^{b_i}$ for $i = 0, \dots, r$ be the generators of R_{nq} as in (6.30.3). Then $T^1(S_{nq}) \subset H^1(U, T_U)$ has a basis consisting of*

$$\left\{ \frac{\partial_x}{M_1}, \frac{\partial_y}{M_{r-1}} \right\} \quad \textit{and} \quad \left\{ \frac{\partial_x}{M_i}, \frac{\partial_y}{M_i} : 2 \leq i \leq r-2 \right\} \tag{6.33.1}$$

plus the possibly empty set

$$\left\{\frac{b_i\partial_x - a_i\partial_y}{M_i^s} : 1 \le i \le r-1,\ 2 \le s \le c_i - 1\right\} \tag{6.33.2}$$

where $c_i = \lceil \frac{a_{i-1}}{a_i} \rceil = \lceil \frac{b_{i+1}}{b_i} \rceil$ *is defined in (6.30.5).*

6.34 (Powers of ω). Fix any $m \in \mathbb{Z}$. Then $H^0(U, \omega_U^m)$ has a basis consisting of $M(dx \wedge dy)^m$ where M is any monomial. Thus $H^0\big(S_{nq}, \omega_{S_{nq}}^{[m]}\big) = H^0\big(U/G, \omega_{U/G}^m\big)$ has a basis consisting of

$$\big\{x^a y^b (dx \wedge dy)^m : a + qb \equiv -m(1+q) \mod n\big\}. \tag{6.34.1}$$

For $D \in T^1(S_{nq})$ let S_D denote the corresponding deformation. By (6.25) $x^a y^b (dx \wedge dy)^m f$ lifts to a section of $\omega_{S_D}^{[m]}$ iff

$$D(x^a y^b) + m x^a y^b \nabla D = 0 \in H^1(U, \mathcal{O}_U). \tag{6.34.2}$$

It is enough to check (6.34.2) for a minimal generating set of $H^0\big(S_{nq}, \omega_{S_{nq}}^{[m]}\big)$ as an R_{nq}-module. In any given case this can be worked out by hand, but there are 2 instances where the answer is simple.

(6.34.3) If $n \mid (q+1)m$ then $H^0\big(S_{nq}, \omega_{S_{nq}}^m\big)$ is cyclic with generator $1 \cdot (dx \wedge dy)^m$.

(6.34.4) If $m = -1$ then $xy(dx \wedge dy)^{-1}$ is G-invariant. Thus every other $x^a y^b (dx \wedge dy)^{-1}$ is a multiple of it, save for powers of x or y. Thus $\omega_{S_{nq}}^{-1}$ has 3 generating sections:

$$\frac{xy}{dx \wedge dy},\ \frac{x^{q+1}}{dx \wedge dy},\ \frac{y^{q'+1}}{dx \wedge dy}.$$

6.35 (V-deformations). If $n \mid (q+1)m$ then $1 \cdot (dx \wedge dy)^m$ is a generator by (6.34.3) thus the condition (6.34.2) is equivalent to $\nabla D = 0$.

Therefore $T_V^1(S_{nq})$ equals the intersection of $T^1(S_{nq})$ with the kernel of ∇. The former was computed in (6.33) the latter in (6.26.2). Thus we see that a basis of $T_V^1(S_{nq})$ is

$$\left\{\frac{(b_i - 1)\partial_x - (a_i - 1)\partial_y}{M_i} : 2 \le i \le r-2\right\} \tag{6.35.1.a}$$

and, if M_i is a power of xy for some i, then we have to add

$$\left\{\frac{\partial_x - \partial_y}{M_i^s} : 2 \le s \le c_i - 1\right\}. \tag{6.35.1.b}$$

6.36 (W-deformations). By (6.34.4), $\omega_{X/G}^{-1}$ has 3 generating sections. Thus, by (6.34.2), D corresponds to a W-deformation iff

(1.a) $D(xy) - xy\nabla D = 0$,

(1.b) $D(x^{q+1}) - x^{q+1}\nabla D = 0$ and $D(y^{q'+1}) - y^{q'+1}\nabla D = 0$.

The first of these conditions is especially strong. We do not compute it here, rather go directly to the next case where the answer is simpler.

6.37 (VW-deformations). Combining (6.35) and (6.36) we get the description of VW-deformations. These satisfy the conditions

(1.a) $\nabla D = 0$,

(1.b) $D(xy) = 0$,

(1.c) $D(x^{q+1}) = 0$ and $D(y^{q'+1}) = 0$.

We computed the subspace K_{VW} where (1.a) and (1.b) both hold in (6.27). It is spanned by the derivations $(\partial_x - \partial_y)(xy)^{-i}$ for $i \geq 2$. Comparing this with (6.33) we get the following.

Claim 6.37.2. If $T^1_{VW}(S_{nq}) \neq 0$ then R_{nq} has a minimal generator of the form $M_i = (xy)^a$. $\square$

In order to put this into a cleaner form, assume that $(xy)^s$ is the smallest G-invariant power of xy. Note that $(xy)^n = M_0 M_r$ is G-invariant but it is not one of the M_i. We have $s(q+1) \equiv 0 \mod n$, thus if $s < n$ then $b := (n, q+1) > 1$. We have thus shown the following.

Claim 6.37.3. If $(n, q{+}1) = 1$ then $T^1_{\mathrm{qG}}(S_{nq}) = T^1_{VW}(S_{nq}) = 0$ and $\dim T^1_V(S_{nq}) = r - 3$. $\square$

Claim 6.37.4. Assume that $M_i = (xy)^a$ for some i (so $a_i = b_i = a$). Then the space of VW-deformations is spanned by

$$\left\{ \frac{\partial_x - \partial_y}{M_i^s} :\ 1 \leq s \leq \min\bigl\{c_i - 1, \tfrac{q+1}{a}, \tfrac{q'+1}{a}\bigr\} \right\}.$$

Proof. The first restriction on s we get from (6.33.2). The condition $D(x^{q+1}) = 0$ is equivalent to $sa \leq q+1$ and $D(y^{q'+1}) = 0$ is equivalent to $sa \leq q'+1$. These give the last 2 restrictions. $\square$

We thus need to compare the 2 upper bounds occurring in (6.35.1.b) and (6.37.4). The key is the following general estimate.

LEMMA 6.38. *Using the notation of (6.30) we have*

$$\tfrac{n}{a_i b_i} \leq \tfrac{a_{i-1}}{a_i}, \tfrac{b_{i+1}}{b_i} < \tfrac{n}{a_i b_i} + 1.$$

Proof. Note that $n = a_i b_{i+1} - a_{i+1} b_i$ by (6.30.8.d). Dividing by $a_i b_i$ we get that

$$\tfrac{n}{a_i b_i} = \tfrac{b_{i+1}}{b_i} - \tfrac{a_{i+1}}{a_i}.$$

Since the a_i form a decreasing sequence, $\frac{a_{i+1}}{a_i} < 1$. $\square$

The final estimate connecting (6.35.1.b) and (6.37.4) is easier to state using a different system of indexing the singularities.

NOTATION 6.39. Set $b = (n, q{+}1)$ and write $n = ab,\ q{+}1 = bc$ where $(a, c) = 1$. The inverse (modulo ab) of $bc - 1$ is written as $bc' - 1$. We thus have the singularity

$$S_{abc} := S_{nq} = \mathbb{A}^2/\tfrac{1}{ab}(1, bc-1) \cong \mathbb{A}^2/\tfrac{1}{ab}(1, bc'-1) \tag{6.39.1}$$

Note that $(xy)^a$ is the smallest G-invariant power of xy but it need not be among the generators M_i; see (6.41).

COROLLARY 6.40. *Assume in addition that $M_i = (xy)^a$ for some i. Then*

$$\lfloor \tfrac{b}{a} \rfloor \leq \min\bigl\{c_i - 1, \tfrac{q+1}{a}, \tfrac{q'+1}{a}\bigr\} \leq c_i - 1 \leq \lfloor \tfrac{b}{a} \rfloor + 1. \tag{6.40.1}$$

Proof. First we claim that

$$\tfrac{b}{a} \leq \min\bigl\{\tfrac{a_{i-1}}{a_i}, \tfrac{b_{i+1}}{b_i}, \tfrac{q+1}{a}, \tfrac{q'+1}{a}\bigr\} \leq \min\bigl\{\tfrac{a_{i-1}}{a_i}, \tfrac{b_{i+1}}{b_i}\bigr\} < \tfrac{b}{a} + 1. \tag{6.40.2}$$

To see this note that $q = bc - 1, q' = bc' - 1$. Thus $b \leq q+1, q'+1$, so it is enough to show that

$$\tfrac{b}{a} \leq \min\bigl\{\tfrac{a_{i-1}}{a_i}, \tfrac{b_{i+1}}{b_i}\bigr\} < \tfrac{b}{a} + 1.$$

Since $n = ab$ and $a = a_i = b_i$, the latter is equivalent to (6.38). Taking the round-down gives (1) using (6.30.8.c). $\square$

EXAMPLE 6.41. Assume that $x^\alpha y^\beta$ is G-invariant. From $\alpha + \beta(bc-1) \equiv 0 \mod ab$ we see that $\alpha \equiv \beta \mod b$. Thus if $0 < \alpha, \beta \le 2b$ then either $\alpha = \beta$ or $\alpha = \beta \pm b$.

It turns out that if $a \le b$ then we can write down these invariants explicitly. Corresponding to the first case we have $(xy)^a$ (and its square). In order to get the other cases, let $0 < e < a$ (resp. $0 < e' < a$) be the unique solution of $ec \equiv -1 \mod a$ (resp. $e'c' \equiv -1 \mod a$). Then $(b+e) + e(bc-1) = b(ec+1) \equiv 0 \mod ab$ and $e'(bc'-1) + (b+e') = b(e'c'+1) \equiv 0 \mod ab$. Thus we get the minimal generators

$$M_{i-1} = x^{b+e}y^e,\ M_i = x^a y^a,\ M_{i+1} = x^{e'}y^{b+e'}.$$

This gives that

$$c_i - 1 = \lfloor \tfrac{b+e}{a} \rfloor = \lfloor \tfrac{b+e'}{a} \rfloor.$$

Fixing a, b we can choose any $0 < e < a$ such that $(a, e) = 1$ and then solve for c. Thus we see that if $b \equiv 0 \mod a$ then $\lfloor \frac{b}{a} \rfloor = c_i - 1$ for every e and if $b \equiv -1 \mod a$ then $\lfloor \frac{b}{a} \rfloor = c_i - 2$ for every e but otherwise both are possible for suitable choice of e.

We see in (6.43) that the condition $a \le b$ holds iff S_{abc} has a nontrivial qG-deformation, so this is a natural class to consider.

6.42 (Proof of (6.5)). Comparing (6.35) and (6.37) we see that the derivations listed in (6.35.1) give V-deformations but not W-deformations. The only possible exception occurs if $M_i = (xy)^a$ for some i. Thus we have 2 cases.

If $M_i = (xy)^a$ does not occur then $\dim T^1_V(S_{nq}) = \dim T^1_{VW}(S_{nq}) + r - 3$.

If $M_i = (xy)^a$ for some i then (6.35.1) gives $r - 4$ basis vectors that give V-deformations but not W-deformations. By (6.40), there is at most 1 derivation as in (6.35.2) that gives a V-deformation that is not a W-deformation. $\square$

6.43 (qG-deformations). From (6.25) and (6.30.4)) we see that D corresponds to a qG-deformation iff $D(x^i y^j) + m x^i y^j \nabla D = 0$ whenever $i + j(bc-1) \equiv -mbc \mod ab$.

First we use this for $1 \cdot (dx \wedge dy)^{ab}$ to conclude that $\nabla D = 0$. Second, we note that since $(a, c) = 1$, the congruence $i + j(bc-1) \equiv -mbc \mod ab$ holds for some m iff $i \equiv j \mod b$. The ring of such monomials is generated by x^b, xy, y^b. Thus D gives a first order qG-deformation iff

(1.a) $\nabla D = 0$,
(1.b) $D(xy) = 0$,
(1.c) $D(x^b) = 0$ and $D(y^b) = 0$.

We thus get that $T^1_{\mathrm{qG}}(S_{abc})$ is spanned by the derivations

$$\left\{ \frac{\partial_x - \partial_y}{(xy)^{as}} :\ 1 \le s \le \lfloor b/a \rfloor \right\}. \tag{6.43.2}$$

The corresponding deformations were written down in [**Wah80**, 2.7]. The canonical cover of S_{abc} is

$$(uv - w^b = 0) \cong \mathbb{A}^2/\tfrac{1}{b}(1, bc-1) = \mathbb{A}^2/\tfrac{1}{b}(1, -1), \tag{6.43.3}$$

hence $u = x^b, v = y^b, w = xy$ and

$$S_{abc} \cong (uv - w^b = 0)/\tfrac{1}{a}(1, bc-1, c). \tag{6.43.4}$$

Thus we get explicit qG-deformations of S_{abc}:

$$(uv - w^b - t_1 w^{b-a} - \cdots - t_r w^{b-ra} = 0)/\tfrac{1}{a}(1, bc-1, c). \tag{6.43.5}$$

To make this $\mathbb{G}_m^2$-equivariant, the $\mathbb{G}_m^2$-action on t_i should be the same as on $(xy)^{ai}$. Thus (6.43.5) describes a smooth subscheme T of $\mathrm{Def}_{\mathrm{qG}}(S_{abc}) \subset \mathrm{Def}(S_{abc})$ and $\dim T = \lfloor b/a \rfloor$. By (6.43.2), the tangent space of $\mathrm{Def}_{\mathrm{qG}}(S_{abc})$ has dimension $\lfloor b/a \rfloor$, so $T = \mathrm{Def}_{\mathrm{qG}}(S_{abc})$ and $\mathrm{Def}_{\mathrm{qG}}(S_{abc})$ is smooth.

In particular, there is a nontrivial 1-parameter qG-deformation iff $a \le b$ and there is a qG-smoothing iff $a|b$. Note that $a \le b$ is equivalent to $ab \le b^2$ and we have proved the following.

Claim 6.43.6. The singularity S_{nq} has

(a) a qG-smoothing iff $n|(q+1)^2$ and
(b) a nontrivial qG-deformation iff $n \le (n, q+1)^2$. Furthermore,
(c) $\dim T^1_{\mathrm{qG}}(S_{nq}) = \lfloor b/a \rfloor = \lfloor (n, q+1)^2/n \rfloor$. □

If $a|b$ then write $b = ad$. We get the singularities

$$W_{adc} := \tfrac{1}{a^2 d}(1, adc-1) \cong (uv - w^{ad} = 0)/\tfrac{1}{a}(1, -1, c). \tag{6.43.7}$$

In this case $b/a = c_i - 1$ hence the above arguments give the following.

Claim 6.43.8. For the singularities $W_{adc} = \mathbb{A}^2/\frac{1}{a^2 d}(1, adc-1)$ every VW-deformation is a qG-deformation. □

6.44 (Proof of (6.6)). Note that (6.6.1) follows from (6.37.3) and (6.6.2) from (6.43.8) for first order deformations. Since $\mathrm{Def}_{\mathrm{qG}}\big(S_{n,q}\big)$ is smooth by (2.28) or by the explicit description (6.43.5), equality of the tangent spaces $T^1_{\mathrm{qG}}\big(S_{n,q}\big) = T^1_{VW}\big(S_{n,q}\big)$ implies that $\mathrm{Def}_{\mathrm{qG}}\big(S_{n,q}\big) = \mathrm{Def}_{VW}\big(S_{n,q}\big)$.

In order to prove (6.6.3) we consider 2 cases. If R_{nq} does not have a minimal generator of the form $M_i = (xy)^a$ then $T^1_{VW}(S_{nq}) = T^1_{\mathrm{qG}}(S_{nq}) = \{0\}$ by (6.37.4).

Otherwise, we have proved in (6.43) that

$$\dim T^1_{\mathrm{qG}}\big(\mathbb{A}^2/\tfrac{1}{ab}(1, bc-1)\big) = \lfloor \tfrac{b}{a} \rfloor$$

and (6.40) shows that

$$\dim T^1_{VW}\big(\mathbb{A}^2/\tfrac{1}{ab}(1, bc-1)\big) = \min\big\{c_i - 1, \tfrac{q+1}{a}, \tfrac{q'+1}{a}\big\} \le \lfloor \tfrac{b}{a} \rfloor + 1. \quad \square$$

Next we list those cyclic quotients singularities for which every V-deformation is a qG-deformation.

Example 6.45 (Double points). These are the A_n singularities; every deformation is a qG-deformation.

Example 6.46 (Triple points). For cyclic quotient triple points the minimal generators of its coordinate ring are $x^n, x^{n-q}y, xy^{n-q'}, y^n$. Thus $\frac{n}{n-q}$ has a 2-step continued fraction expansion involving c_1, c_2. Setting $c_1 = e, c_2 = d$ we have the singularities

$$\mathbb{A}^2/\tfrac{1}{ed-1}(1, ed-d-1). \tag{6.46.1}$$

with invariants

$$x^{ed-1}, x^d y, xy^e, y^{ed-1}. \tag{6.46.2}$$

By (6.35) we have $T^1_V = T^1_{\mathrm{qG}} = 0$.

Example 6.47 (Quadruple points). By (6.6), every cyclic quotient singularity of multiplicity 4 has a V-deformation that is not a qG-deformation, unless M_2 is a power of xy. Thus in this case the minimal generators of its coordinate ring are

$$x^n, x^{n-q}y, x^a y^a, xy^{n-q'}, y^n. \tag{6.47.1}$$

The equation $M_2^{c_2} = M_1 M_3$ now implies that $q = q'$. Thus $\frac{n}{n-q}$ has a 3-step continued fraction expansion involving $c_1, c_2, c_3 = c_1$. By expanding it we see that $c_1 = a$. Setting $c_2 = d$ the singularity is

$$\mathbb{A}^2/\tfrac{1}{a(ad-2)}\bigl(1, (ad-2)(a-1)-1\bigr) \tag{6.47.2}$$

and the minimal generators of the ring of invariants are

$$x^{a(ad-2)}, x^{ad-1}y, x^a y^a, xy^{ad-1}, y^{a(ad-2)}. \tag{6.47.3}$$

Thus $\lfloor (ad-2)/a \rfloor = d-1 = c_2 - 1$ and hence, by (6.35) and (6.43), $T^1_V = T^1_{\mathrm{qG}}$ is spanned by

$$\left\{ \frac{\partial_x - \partial_y}{(xy)^{as}} : \ 1 \le s \le d-1 \right\}. \tag{6.47.4}$$

These singularities admit a qG-smoothing iff $a = 2$. Then, after replacing $d-1$ by d, the normal form becomes

$$\mathbb{A}^2/\tfrac{1}{4d}(1, 2d-1). \tag{6.47.5}$$

Together with the A_n-series, these are the only cyclic quotient singularities with a qG-smoothing for which every V-deformation is a qG-deformation.

Example 6.48 (Higher multiplicity points). By (6.6), every cyclic quotient singularity of multiplicity ≥ 5 has V-deformations that are not qG-deformations.

CHAPTER 9

Hulls and Husks

Given a coherent sheaf F over a proper scheme, the quot-scheme—introduced by Grothendieck—parametrizes all quotients $F \twoheadrightarrow Q$ of F. In many applications it is necessary to understand not only surjections $F \twoheadrightarrow Q$ but also “almost surjections” $F \to G$. The precise notion should depend on the application; for us the most important is to study morphisms $F \to G$ from F to a pure sheaf G that are surjective at all generic points of $\operatorname{Supp} G$. Such objects are called *quotient husks.* Special cases appeared in [**Kol08a, PT09, AK10, Kol11b**]. The aim of this chapter is to study quotient husks, prove that they have a fine moduli space $\operatorname{QHusk}(F)$ and then apply this to families of hulls.

In Section 9.1 we recall basic results on S_2 sheaves; the proofs are based on [**Gro60, Gro62a**].

Then we turn to the study of hulls of coherent sheaves. The notion of S_2*-hull* (or hull for short) is the proper generalization of the concept of reflexive hull of a module over a normal integral domain. In Section 9.2 we discuss the absolute case and in Section 9.3 the relative case. For many applications the key is the following.

Question 9.1. Let $f : X \to S$ be a proper morphism and F a coherent sheaf on X. Do the hulls $F_s^{[**]}$ of the fibers F_s form a coherent sheaf that is flat over S?

If the answer is yes, the resulting sheaf is called the *universal hull* of F over S. Local criteria for its existence are studied in Section 9.4.

In order to get global criteria, husks and quotient husks are defined in Section 9.5. In Section 9.6, the first main result of the Chapter proves that if $X \to S$ is projective and F is a coherent sheaf on X then the functor of all quotient husks with a given Hilbert polynomial has a fine moduli space $\operatorname{QHusk}_p(X)$ which is a proper algebraic space over S. The proof closely follows the arguments given in [**Kol08a**].

This is used in a global study if hulls in Section 9.7. A third answer to our question is given in Section 9.8 in terms of a decomposition of S into locally closed subschemes. This can be viewed as a generalization of the Flattening Decomposition Theorem [**Mum66**, Lect.8].

Moduli spaces of hulls of powers of the relative dualizing sheaf $\omega_{X/S}^{\otimes m}$ were used to define moduli spaces of stable varieties and pairs.

These results are partially extended to algebraic spaces in Section 9.9.

9.1. S_2 sheaves

In this section we collect some well known results about pushing forward and S_2 sheaves.

Assumptions. In this section we work with arbitrary Noetherian schemes.

DEFINITION 9.2. Let F be a quasi-coherent sheaf on a scheme X. Its *annihilator,* denoted by $\operatorname{Ann}(F)$, is the largest ideal sheaf $I \subset \mathcal{O}_X$ such that $I \cdot F = 0$. The *support* of F is the zero set $Z(I) \subset X$, denoted by $\operatorname{Supp} F$.

The *dimension* of F at a point x, denoted by $\dim_x F$, is the dimension of its support at x. The dimension of F is $\dim F := \dim \operatorname{Supp} F$.

The set of all associated points (or primes) of a quasi-coherent sheaf F is denoted by $\operatorname{Ass}(F)$. An associated point of F is called *embedded* if it is contained in the closure of another associated point of F. Let $\operatorname{emb}(F) \subset F$ denote the largest subsheaf whose associated points are all embedded points of F. Thus $F/\operatorname{emb}(F)$ has no embedded points hence it is S_1 (9.6). Informally speaking, $F \mapsto F/\operatorname{emb} F$ is the best way to associate an S_1 sheaf to an arbitrary coherent sheaf.

If F is coherent then it has only finitely many associated points and $\operatorname{Supp} F$ is the union of their closures.

Let $Z \subset X$ be a closed subscheme. Then $\operatorname{tors}_Z F \subset F$ denotes the subsheaf of all local sections whose support is contained in Z. There is a natural isomorphism $\operatorname{tors}_Z F = \mathcal{H}^0_Z(X, F)$.

Assume that X has a dimension function. Then we use $\operatorname{tors}(F) \subset F$ to denote the subsheaf of all local sections whose support has dimension $< \dim \operatorname{Supp} F$. A coherent sheaf F is called *pure* (of dimension n) if (the closure of) every associated point of F has dimension n. Thus $\operatorname{pure}(F) := F/\operatorname{tors}(F)$ is the maximal *pure quotient* of F.

If $\operatorname{Supp} F$ is pure dimensional then $\operatorname{emb}(F) = \operatorname{tors}(F)$. If X is pure dimensional, F is coherent and $\dim F = \dim X$, then our terminology agrees with every usage of torsion that we know of.

9.3 (Regular sequences and depth). Let A be a ring and M a nonzero A-module. Recall that $x \in A$ is *M-regular* if it is not a zero divisor on M, that is, if $m \in M$ and $xm = 0$ implies that $m = 0$. Equivalently, if x is not contained in any of the associated primes of M.

A sequence $x_1, \dots, x_r \in A$ is an *M-regular sequence* if x_1 is not a zero divisor on M and x_i is not a zero divisor on $M/(x_1, \dots, x_{i-1})M$ for all $i = 2, \dots, r$.

Let $\operatorname{rad} A$ denote the *radical* of A, that is, the intersection of all maximal ideals. Let $I \subset \operatorname{rad} A$ be an ideal. The *depth* of M along I is the maximum length of an M-regular sequence $x_1, \dots, x_r \in I$. It is denoted by $\operatorname{depth}_I M$.

It turns out that if A is Noetherian and M is finite over A then all maximal M-regular sequences $x_1, \dots, x_r \in I$ have the same length; see for instance [**Mat86**, p.127] or [**Eis95**, Sec.17]. (Note that in [**Eis95**, 17.4] the necessary assumption $I \subset \operatorname{rad} A$ is left out.) This is a quite subtle result which makes the depth computable in practice: Pick any $x_1 \in I$ that is not contained in any of the associated primes of M, then $\operatorname{depth}_I M = 1 + \operatorname{depth}_I(M/x_1M)$.

Among other useful consequences we see that $\operatorname{depth}_I M$ depends only on $\sqrt{I}$ and it is the minimum of the depths computed for the localizations A_m. Furthermore, if A is finite over B and $J \subset B$ is an ideal then $\operatorname{depth}_J M = \operatorname{depth}_{JA} M$. (Here the first depth is computed over the ring B and the second over the ring A.)

Warning. While the above definition of depth makes sense for arbitrary rings and ideals, it can give wrong results. For instance, take $A = k[x, y]$, $I = (x)$ and $M = A/(x-1)$. Then both x and xy, x are maximal M-regular sequences.

DEFINITION 9.4. Let F be a coherent sheaf on X. The *depth of F at x*, denoted by $\operatorname{depth}_x F$, is defined as the depth of its localization F_x along $m_{x,X}$ (as an $\mathcal{O}_{x,X}$-module). For a closed subscheme $Z \subset X$ we set

$$\operatorname{depth}_Z F := \inf\{\operatorname{depth}_z F : z \in Z\}. \tag{9.4.1}$$

If $X = \operatorname{Spec} A$ is affine, $Z = V(I)$ for some ideal $I \subset \operatorname{rad} A$ and $M = H^0(X, F)$ then $\operatorname{depth}_Z F = \operatorname{depth}_I M$.

Warning. This definition is for coherent sheaves only. See [**Gro68**, Exp.III] for the correct definition of depth for quasi-coherent sheaves.

A coherent sheaf F is called S_m (or it is said to satisfy *Serre's condition S_m*) if

$$\operatorname{depth}_x F \geq \min\{m, \operatorname{codim}(x, \operatorname{Supp} F)\} \quad \text{for all } x \in \operatorname{Supp} F. \tag{9.4.2}$$

We say that F is S_m at x if the localization F_x is S_m (as an $\mathcal{O}_{x,X}$-module).

Comments. Frequently condition (9.4.2) is stated for all $x \in X$. For the purposes of the latter version, one should say that the zero module has infinite depth. This, however, messes up other conventions, so we just ignore this problem.

In practice there are two cases that are especially interesting and useful. If $m \geq \dim F$ then S_m is equivalent to CM; see, for instance, [**Kol13c**, 2.58]. This is pretty much the ideal situation, but if it does not hold, usually one can not do anything about it. The other very useful case is condition S_2. Not every sheaf is S_2, but, as we see in (9.14), to any coherent sheaf one can usually associate a coherent S_2 sheaf in a natural way, and this is very helpful in many proofs.

Warning. It is important to note that being S_m at x is *not* the same as $\operatorname{depth}_x F \geq m$; neither implies the other. The difference is clear already for $m = 1$:

- $\operatorname{depth}_x F \geq 1$ iff x is not an associated point of F (cf. (9.6)) and
- F is S_1 at x iff x is not contained in the closure of an embedded associated point of F.

As another example, let $(x \in X)$ be the localization of $k[x_1, \dots, x_4]$ at the origin and $M = k[x_1, \dots, x_4] + k[x_1, \dots, x_4]/(x_3, x_4)$. Then $\operatorname{depth}_{(x_1,\dots,x_4)} M = 2$ but $\operatorname{depth}_{(x_3,x_4)} M = 0$. Thus M is not even S_1.

By contrast, if F has maximal depth at x, that is, if $\operatorname{depth}_x F = \dim_x F$, then $\operatorname{depth}_z F = \operatorname{codim}(z, \operatorname{Supp} F)$ holds for every point $z \in X$. This is one reason why being CM is much better behaved. (See (10.15) for a discussion of the general case.)

9.5 (Depth and flatness). Let $p : Y \to X$ be a morphism and G a coherent sheaf on Y that is flat over X. It is easy to see that for any point $y \in Y$ we have

$$\operatorname{depth}_y G = \operatorname{depth}_{p(y)} X + \operatorname{depth}_y G_{p(y)}. \tag{9.5.1}$$

Similarly, assume that $p : Y \to X$ is flat and let F be a coherent sheaf on X. Then

$$\operatorname{depth}_y p^*F = \operatorname{depth}_{p(y)} F + \operatorname{depth}_y Y_{p(y)}. \tag{9.5.2}$$

In particular, if $p : Y \to X$ is flat with S_m fibers and F is a quasi-coherent S_m sheaf on X then p^*F is also S_m. The converse also holds if p is faithfully flat.

The assumption on the fibers is necessary and a flat pull-back of an S_m sheaf need not be S_m; not even for products. Let X_1, X_2 be k-schemes. Then $X_1 \times X_2$ is S_m iff both of the X_i are S_m.

Condition S_1 can be described in terms of embedded points.

Lemma 9.6. *Let F be a coherent sheaf on a scheme X and $Z \subset X$ a closed subscheme. Then* $\operatorname{depth}_Z F \geq 1$ *iff none of the associated points of F is contained in Z. In particular, F is S_1 iff it has no embedded associated points.*

Proof. An element $x_1 \in m \subset A$ is not a zero divisor on a module M iff it is not contained in any of the associated primes of M. If M has only finitely many associated primes, then there exists such an x_1 iff m is not an associated prime of M. □

We will repeatedly use the following lemma which gives several characterizations of S_2 sheaves.

Lemma 9.7. *Let F be a coherent, S_1 sheaf and $Z \subset \operatorname{Supp} F$ a nowhere dense subscheme. The following are equivalent.*

(1) $\operatorname{depth}_Z F \geq 2$.
(2) *Let Q be a quasi-coherent sheaf such that* $\operatorname{Supp} Q \subset Z$. *Then every exact sequence $0 \to F \to F' \to Q \to 0$ splits.*
(3) *Let $0 \to F \to F' \to Q \to 0$ be an exact sequence such that* $\operatorname{Supp} Q \subset Z$ *and $Q \neq 0$. Then F' has an associated prime contained in* $\operatorname{Supp} Q$.
(4) $F = j_*\big(F|_{X\setminus Z}\big)$ *where $j : X \setminus Z \hookrightarrow X$ is the natural injection.*

Proof. By assumption $\operatorname{tors}_Z F = 0$. Thus, if there is a splitting locally then the unique splitting is given by $\operatorname{tors}_Z F' \subset F'$. We can thus work with a module M over a local ring (A, m).

Let $x_1 \in m$ be M-regular. If $\operatorname{depth}_m M < 2$ then $\operatorname{depth}_m M/x_1M < 1$, hence there is a submodule $x_1M \subsetneq N \subset M$ such that N/x_1M has codimension ≥ 2. Then $M' := x_1^{-1}M' \supset M$ gives an extension violating (2). Clearly (2) $\Rightarrow$ (3).

Let $0 \to M \to M' \to Q \to 0$ be an extension and set $N' \operatorname{tors}_Z M'$. If $N' \twoheadrightarrow Q$ then the extension splits. Thus we can replace M' by a suitable $M'' \subset M'/N'$ to get an extension $0 \to M \to M'' \to Q'' \to 0$ where $Q'' \neq 0$ is cyclic and $\operatorname{Ass} M'' = \operatorname{Ass} M$. By localizing at a generic point of $\operatorname{Supp} Q''$, we may even assume that $mQ'' = 0$.

Now take any $x_1 \in m$ that is not a zero divisor on M and M''. By assumption $x_1M'' \subset M$ and thus M/x_1M has a nonzero submodule supported on $\operatorname{Supp} Q''$. Thus $\operatorname{depth}_m M < 2$.

Finally, since F has no associated primes contained in Z, there is an injection $F \hookrightarrow j_*\big(F|_{X\setminus Z}\big)$. The quotient is supported on Z, thus (4) is equivalent to (3). □

Corollary 9.8. *Let F be a coherent, S_2 sheaf and G any coherent sheaf. Then $\mathcal{H}om_X(G, F)$ is also S_2.*

Proof. It is clear that every irreducible component of $\operatorname{Supp} \mathcal{H}om_X(G, F)$ is also an irreducible component of $\operatorname{Supp} F$.

Let $Z \subset \operatorname{Supp} F$ be a closed subset of codimension ≥ 2 and $j : X \setminus Z \hookrightarrow X$ the injection. Any homomorphism $\phi : G|_{X\setminus Z} \to F|_{X\setminus Z}$ uniquely extends to

$$j_*\phi : j_*\big(G|_{X\setminus Z}\big) \to j_*\big(F|_{X\setminus Z}\big).$$

Since F is S_2, the target equals F. We have a natural map $G \to j_*\big(G|_{X\setminus Z}\big)$ (whose kernel is the subsheaf of suctions whose support is in Z). Thus

$$\mathcal{H}om_X(G, F) = j_*\big(\mathcal{H}om_X(G, F)|_{X\setminus Z}\big),$$

hence $\mathcal{H}om_X(G, F)$ is S_2. □

An important property of S_2 sheaves is the following, which can be obtained by combining [**Har77**, III.7.3] and [**Har77**, III.12.11].

PROPOSITION 9.9 (Enriques–Severi–Zariski lemma). *Let $f : X \to S$ be a projective morphism and F a coherent sheaf on X that is flat over S with S_2 fibers of pure dimension ≥ 2. Then $f_*F(-m) = R^1 f_*F(-m) = 0$ for $m \gg 1$.*

Therefore, if $H \in |\mathcal{O}_X(m)|$ does not contain any of the associated points of F then the restriction map $f_*F \to (f|_H)_*(F|_H)$ is an isomorphism. □

9.2. Hulls of coherent sheaves

Let X be an integral, normal scheme and F a coherent sheaf on X. The reflexive hull of F is the double dual $F^{**} := \mathcal{H}om_X\big(\mathcal{H}om_X(F, \mathcal{O}_X), \mathcal{O}_X\big)$. We would like to extend this notion to arbitrary schemes and arbitrary coherent sheaves. For this the key properties of the reflexive hull are the following.

- F^{**} is S_2 and
- F^{**} is the smallest S_2 sheaf containing $F/(\text{torsion})$.

These are the properties that we use to define the hull of a sheaf. Note, however, that for this we need to agree what the torsion subsheaf of a sheaf should be. Two natural candidates are discussed in (9.2):

- $\operatorname{emb} F$, the subsheaf corresponding to embedded points and
- $\operatorname{tors} F$, the largest subsheaf whose support has dimension $< \dim F$.

These are the same if $\operatorname{Supp} F$ is irreducible but quite different in general. For example, if X is a disjoint union of normal schemes of different dimensions, then $\operatorname{emb} \mathcal{O}_X = 0$ while $\operatorname{tors} \mathcal{O}_X$ is the structure sheaf of the lower dimensional components. A theory of hulls using $\operatorname{emb} F$ is developed in [**Kol15**].

Here we work with $\operatorname{tors} F$. An advantage is that $F/\operatorname{tors} F$, and hence the hull, are pure dimensional; this is quite important in our applications. A disadvantage is that for this to make sense, one needs the dimension function to be quite well behaved. In all final applications we work with schemes of finite type, so this will not be a problem.

ASSUMPTION 9.10. In this section we consider schemes X such that

(1) $\dim W$ is finite for every irreducible subscheme $W \subset X$ and
(2) if $W_1 \subsetneq W_2$ is a maximal (with respect to inclusion) irreducible subscheme of the irreducible $W_2 \subset X$, then $\dim W_1 = \dim W_2 - 1$.

These hold for schemes of finite type over a field (by standard dimension theory) and also for schemes of finite type over a local CM scheme; see [**Sta15**, Tags 00NM and 02JT]. However, these conditions are not satisfied by many naturally occurring schemes; a typical example is the localization of $k[x, y]$ at $(x, y) \cup (x - 1)$.

(More generally, one could work instead with any scheme that admits a dimension function, see [**Kol15**].)

A useful property of pure sheaves is the following.

LEMMA 9.11. *Let $p : X \to Y$ be a finite morphism and F a coherent sheaf on X. Then F is pure and S_m iff p_*F is pure and S_m.*

Proof. The last remark of (9.3) implies that the depth is preserved by push-forward. Thus the only question is whether (co)dimension is preserved or not; this is where (9.10) is used. □

Definition 9.12 (Hull of a sheaf). Let X be a scheme as in (9.10) and F a coherent sheaf on X. Set $n := \dim F$. The S_2 *hull,* or *pure hull,* or simply *hull,* of F is a coherent sheaf $F^{[**]}$ together with a map $q : F \to F^{[**]}$ such that

(1) $\operatorname{Supp}(\ker q)$ has dimension $\le n-1$,
(2) $\operatorname{Supp}(\operatorname{coker} q)$ has dimension $\le n-2$ and
(3) $F^{[**]}$ is pure and S_2.

We see below that a pure hull is unique and it exists if X is excellent.

By definition, $F^{[**]} = (F/\operatorname{tors} F)^{[**]}$, hence it is enough to construct pure hulls of pure, coherent sheaves.

The notation $F^{[**]}$ is chosen to emphasize the close connection between the pure hull and the reflexive hull F^{**}; see (9.13) for the precise statement. We introduce a relative version, denoted by F^H in (9.17). If $X \to \operatorname{Spec} k$ is a k-scheme then $F^{[**]} = F^H$.

The following property is clear from the definition.

(4) Let G be a pure, coherent, S_2 sheaf and $F \subset G$ a subsheaf. Then $G = F^{[**]}$ iff $\dim(G/F) \le \dim G - 2$.

From (9.11) and (9.5) we obtain the following base change properties of hulls.

(5) Let $p : X \to Y$ be a finite morphism. Then $p_*\big(F^{[**]}\big) = \big(p_*F\big)^{[**]}$.
(6) $g : Z \to X$ be flat and pure dimensional with S_2 fibers. Then $g^*\big(F^{[**]}\big) = \big(g^*F\big)^{[**]}$.

In many cases the pure hull coincides with the usual reflexive hull.

Proposition 9.13. *Let X be an irreducible, normal scheme and F a torsion free coherent sheaf on X such that $F^{**} := \mathcal{H}om_X\big(\mathcal{H}om_X(F, \mathcal{O}_X), \mathcal{O}_X\big)$ is coherent. Then $F^{[**]} = F^{**}$.*

Proof. F is locally free outside a codimension ≥ 2 subset $Z \subset X$. Thus the natural map $F \to F^{**}$ is an isomorphism over $X \setminus Z$. Since F^{**} is S_2 by (9.8), it satisfies the assumptions of (9.12). $\square$

This can be used to construct the pure hull over finite type schemes. Indeed, we may assume that X is affine and $X = \operatorname{Supp} F$. By Noether normalization there is a finite surjection $p : X \to \mathbb{A}^n$. Thus, by (9.12.5) and (9.13), $F^{[**]}$ can be identified with $\big(p_*F\big)^{**}$. Next we prove that pure hulls exist over excellent schemes; see [**Kol15**] for a more general result.

Proposition 9.14. *Let X be an excellent scheme and F a pure, coherent sheaf on X.*

(1) *There is a closed subset $Z \subset \operatorname{Supp} F$ of dimension $\le \dim F - 2$ such that F is S_2 over $X \setminus Z$.*
(2) *Let $Z \subset \operatorname{Supp} F$ be any closed subset of dimension $\le \dim F - 2$ such that F is S_2 over $X \setminus Z$. Then the following hold.*
    (a) $F^{[**]} = j_*\big(F|_{X\setminus Z}\big)$.
    (b) *Let $G \supset F$ be any coherent sheaf such that $G|_{X\setminus Z} = F|_{X\setminus Z}$. Then $F \to F^{[**]}$ uniquely extends to $G \to F^{[**]}$.*

Proof. The first claim follows from (10.17). To see (2.a), note that $j_*\big(F|_{X\setminus Z}\big)$ is coherent by (10.16), S_2 over $X \setminus Z$ by assumption and $\operatorname{depth}_Z j_*\big(F|_{X\setminus Z}\big) \ge 2$ by (9.7). Thus $j_*\big(F|_{X\setminus Z}\big)$ is a hull of F.

If G as in (2.b), then we get $G \to j_*(G|_{X\setminus Z}) = j_*(F|_{X\setminus Z})$.

Let $F^{[**]}$ be any hull of F. Then $F^{[**]}|_{X\setminus Z}$ is a hull of $F|_{X\setminus Z}$; let $W \subset X \setminus Z$ be the support of their quotient. Then $\operatorname{codim}_X W \geq 2$ hence $F^{[**]}|_{X\setminus Z} = F|_{X\setminus Z}$ by (9.7.2). Thus we get a map $F^{[**]} \to j_*(F|_{X\setminus Z})$. Applying (9.7) again gives that $F^{[**]} = j_*(F|_{X\setminus Z})$. □

9.15 (Quasi-coherent hulls). The formula (9.14.2.a) suggests that one should define the hull of a quasi-coherent sheaf F as

$$F^{[**]} := \varinjlim\, (j_Z)_*(F|_{X\setminus Z}),$$

where Z runs through all codimension ≥ 2 closed subsets of $\operatorname{Supp} F$. It is easy to see that $F^{[**]}$ is always S_2 (as defined in [**Gro68**, Exp.III]) and it agrees with our definition whenever both are defined.

Since j_* is left exact, we obtain that the formation of the hull is also left exact.

COROLLARY 9.16. *Let $0 \to F_1 \to F_2 \to F_3$ be an exact sequence of coherent sheaves of the same dimension. Then the hulls also form an exact sequence $0 \to F_1^{[**]} \to F_2^{[**]} \to F_3^{[**]}$.* □

9.3. Relative hulls

Next we develop a relative version of the notion of hull for coherent sheaves on a scheme X over a base scheme S.

In the absolute case, the hull is an S_2 sheaf that we can associate to any coherent sheaf on X, in particular, the hull does not have embedded points.

In the relative case, assume for simplicity that $f : X \to S$ is smooth; then $\mathcal{O}_X$ should be its own "relative hull." Note, however, that the structure sheaf $\mathcal{O}_X$ has no embedded points if and only if the base scheme S has no embedded points. Thus if we want to say that $\mathcal{O}_X$ is its own "relative hull" then we have to distinguish embedded points that are caused by S (these are allowed) from other embedded points (these are forbidden).

The distinction between these two types of embedded points seems to be meaningful only if F is flat over a sufficiently large open subset of $\operatorname{Supp} F$. Further restrictions need to be imposed if we want to allow base changes.

DEFINITION 9.17 (Hull over a base scheme). Let $f : X \to S$ be a morphism of finite type and F a coherent sheaf on X. Let n denote the maximum fiber dimension of $\operatorname{Supp} F \to S$.

A *hull* (or *relative hull*) of F over S is a coherent sheaf F^H together with a morphism $q : F \to F^H$ such that

(1) $\operatorname{Supp}(\ker q) \to S$ has fiber dimension $\leq n-1$,
(2) $\operatorname{Supp}(\operatorname{coker} q) \to S$ has fiber dimension $\leq n-2$,
(3) there is a closed subset $Z \subset X$ with complement $U := X \setminus Z$ such that
  (a) $Z \to S$ has fiber dimension $\leq n-2$,
  (b) $(F/\ker q) \to F^H$ is an isomorphism over U.
  (c) $F^H|_U$ is flat over S with pure, S_2 fibers and
  (d) $\operatorname{depth}_Z F^H \geq 2$.

Note that $\operatorname{Supp}(\operatorname{coker} q) \subset Z$ by (3.b) hence in fact (3.a) implies (2). We state the latter separately to emphasize the parallels with the definition of the absolute hull

(9.12). If S is the spectrum of a field then clearly $F^H = F^{[**]}$. Note, however, that while the hull is always defined, the relative hull frequently does not exist.

For instance, let $f : X := \mathbb{A}^2_{st} \to S := \mathbb{A}^1_t$ be the projection and $F \subset \mathcal{O}_X$ the ideal sheaf of the point $(0,0)$. Then $F^{[**]} = \mathcal{O}_X$ but $F \to \mathcal{O}_X$ is not a relative hull since $\operatorname{coker}(F \to \mathcal{O}_X)$ has codimension 1 on the fiber X_0.

(It would have been more consistent to denote the hull by F^{h}, but a superscript h is frequently used to denote the Henselization.)

Relative hulls are easy to understand if the base scheme is 1-dimensional and regular.

Lemma 9.18. *Let $(0,T)$ be the spectrum of a DVR, $f : X \to T$ a morphism of finite type and $q : F \to G$ a map between pure, coherent sheaves on X that are flat over T. Then G is a relative hull of F iff the following hold.*

(1) *G_g is the hull of F_g,*
(2) *G_0 is S_1,*
(3) *$q_0 : F_0 \to G_0$ is an isomorphism outside a subset $Z_0 \subset \operatorname{Supp} G_0$ of codimension ≥ 2.*

Proof. Assume that $G = F^H$ and let $Z \subset X$ be as in (9.17). By assumption $G|_{X \setminus Z}$ has S_2 fibers thus $G|_{X\setminus Z}$ is S_2. Hence G is S_2 since $\operatorname{depth}_Z G \geq 2$ and so G_0 is S_1 and $q_0 : F_0 \to G_0$ is an isomorphism outside $X_0 \cap Z$.

Conversely, if (1–3) hold then G is S_2 by (1–2). By (9.14) there is a closed subset $Z_1 \subset X_0$ of codimension ≥ 2 such that F_0 is S_2 over $X_0 \setminus Z_0$. Thus $q : F \to G$ satisfies the conditions (9.17.1–3) where Z is the union of 3 closed sets: Z_0, Z_1 and the closure of $\operatorname{Supp}(\operatorname{coker} q_g)$. □

As a special case, we get the following characterization of relative hulls.

Corollary 9.19. *Let $(0,T)$ be the spectrum of a DVR, $f : X \to T$ a morphism of finite type and F a pure, coherent sheaf on X that is flat over T. Then $F = F^H$ $\Leftrightarrow$ F is S_2 $\Leftrightarrow$ F_g is S_2 and F_0 is S_1.* □

Corollary 9.20 (Bertini theorem for relative hulls). *Let $(0,T)$ be the spectrum of a DVR, $X \subset \mathbb{P}^n_T$ a quasi-projective scheme and F a coherent sheaf on X with relative hull $q : F \to F^H$. Then $q|_L : F|_L \to F^H|_L$ is the relative hull of $F|_L$ for a general hyperplane $L \subset \mathbb{P}^n_T$.*

Proof. We use (10.9) both for the special fiber X_0 and the generic fiber X_g. We get open subsets $U_0 \subset \check{\mathbb{P}}^n_0$ and $U_g \subset \check{\mathbb{P}}^n_g$ such that

(1) $F^H|_{L_0}$ is S_1 for $L_0 \in U_0$,
(2) $(F/\operatorname{tors} F)|_{L_0} = (F|_{L_0})/\operatorname{tors}(F|_{L_0})$ for $L_0 \in U_0$,
(3) $(F|_{L_0})/\operatorname{tors}(F|_{L_0}) \to G_{L_0}$ is an isomorphism outside a subset of codimension ≥ 2 for $L_0 \in U_0$ and
(4) $F^H|_{L_g}$ is the hull of $F|_{L_g}$ for $L_g \in U_g$.

Let $W_T \subset \check{\mathbb{P}}^n_T$ denote the closure of $\check{\mathbb{P}}^n_g \setminus U_g$. For dimension reasons, W_T does not contain $\check{\mathbb{P}}^n_0$. Thus any hyperplane corresponding to a section through a point of $U_0 \setminus W_T$ works. (See (9.29) for some examples with non-general hyperplanes.) □

Next we state the precise conditions needed for the existence of relative hulls. Then we show that a relative hull is unique and does not depend on the choice of $Z \subset X$, generalizing (9.14).

LEMMA 9.21. *Let $f : X \to S$ be a morphism of finite type and F a coherent sheaf on X. Then F has a relative hull iff*

(1) *there is a coherent subsheaf $\operatorname{tors}_S F \subset F$ such that $\operatorname{Supp}(\operatorname{tors}_S F) \to S$ has fiber dimension $\leq n-1$ and*
(2) *a closed subset $Z \subset X$ such that $f|_Z : Z \to S$ has fiber dimension $\leq n-2$ such that*
(3) *$\bigl(F/\operatorname{tors}_S F\bigr)|_{X \setminus Z}$ is flat over S with pure, S_2 fibers.*

Furthermore, if these conditions are satisfied then

(4) *$F^H = j_*\bigl(\bigl(F/\operatorname{tors}_S F\bigr)|_{X\setminus Z}\bigr)$ is the unique relative hull of F over S.*
(5) *Let $G \supset F$ be any coherent sheaf such that $G|_{X\setminus Z} = F|_{X\setminus Z}$. Then $F \to F^H$ uniquely extends to $G \to F^H$.*

Proof. If $q : F \to F^H$ is a relative hull then the conditions (9.21.1–3) are satisfied and $\operatorname{tors}_S(F) = \ker(q)$.

Conversely, assume that the conditions (9.21.1–3) are satisfied. We can harmlessly replace F by $F/\operatorname{tors}_S(F)$. Write $U := X \setminus Z$. Then $j_*\bigl(F|_U\bigr)$ is coherent by (10.16), $F \to j_*\bigl(F|_U\bigr)$ is an isomorphism over U by construction and $\operatorname{depth}_Z j_*\bigl(F|_U\bigr) \geq 2$ by (9.7).

The last claim follows from the universal property of the push-forward and it implies that F^H is independent of the choice of Z. $\square$

We see from the above construction that the torsion subsheaf $\operatorname{tors}_S(F)$ plays essentially no role and one can always work with the quotient $F/\operatorname{tors}_S(F)$ instead of F. This leads to the following generalization of (5.46).

DEFINITION 9.22 (Mostly flat families of S_2 sheaves). Let $f : X \to S$ be a morphism and F a coherent sheaf on X. We say that F is a *mostly flat family of S_2 sheaves* if there is a closed subscheme $Z \subset X$ with complement $U := X \setminus Z$ such that

(1) $Z \cap X_s$ has codimension ≥ 2 in $\operatorname{Supp} F_s$ for every $s \in S$ and
(2) $F|_U$ is flat over S with pure, S_2 fibers.

By (9.21), if these hold then F has a hull F^H. Conversely, if F has a hull then $F/\operatorname{tors}_S F$ is a mostly flat family of S_2 sheaves.

The following is a direct analog of (9.12.4).

COROLLARY 9.23. *Let $f : X \to S$ be a morphism of finite type and G a coherent sheaf on X that is flat over S with pure, S_2 fibers of dimension n. Let $F \subset G$ be a subsheaf. Then $G = F^H$ iff the fiber dimension of $\operatorname{Supp}(G/F) \to S$ is $\leq n-2$.* $\square$

9.4. Universal hulls

For many applications a key question is to understand the behavior of relative hulls under a base change.

NOTATION 9.24. Let $f : X \to S$ be a morphism of finite type and F a coherent sheaf satisfying the conditions (9.21.1–3). For any $g : T \to S$ we have a base-change diagram

$$\begin{array}{ccc} X_T & \xrightarrow{g_X} & X \\ f_T \downarrow & & \downarrow f \\ T & \xrightarrow{g} & S. \end{array} \tag{9.24.1}$$

By pull-back we obtain $Z_T := g_X^{-1}(Z)$, $U_T := g_X^{-1}(U)$ and $F_T := g_X^* F$. Note that $f_T : X_T \to T, Z_T$ and F_T again satisfies (9.21.1–3).

In general $g_X^*\bigl(F^H\bigr)$ is not the relative hull of F_T. Thus we need to distinguish

$$\bigl(F^H\bigr)_T := g_X^*\bigl(F^H\bigr) \quad \text{and} \quad \bigl(F_T\bigr)^H := \bigl(g_X^* F\bigr)^H. \tag{9.24.2}$$

Since the two sheaves agree over U_T, (9.21.5) implies that there is a natural map

$$r_T^F : \bigl(F^H\bigr)_T \to \bigl(F_T\bigr)^H \tag{9.24.3}$$

We call r_T^F the *restriction map,* especially when T is a subscheme of S. We see in (9.26) that r_T^F is an isomorphism along $g_X^{-1}(x)$ if F is flat with pure, S_2-fiber at x.

Definition 9.25. Let $f : X \to S$ be a morphism of finite type and F a coherent sheaf on X satisfying (9.21.1–3).

We say that F^H is a *universal hull* of F at $x \in X$ if the restriction map r_T^F (9.24.3) is an isomorphism along $g_X^{-1}(x)$ for every $g : T \to S$. We say that F^H is a *universal hull* of F if it is a universal hull at every $x \in X$. That is, F^H is a universal hull iff $g_X^*(F^H)$ is the hull of $g_X^* F$ for every $g : T \to S$. Equivalently, iff the functor $F \mapsto F^H$ *commutes with base change.*

We say that $F \mapsto F^H$ is *universally flat* if $(F_T)^H$ is flat over T for every $g : T \to S$.

The following theorem gives several characterizations of universal hulls.

Theorem 9.26. *Let $f : X \to S$ be a morphism of finite type, F a mostly flat family of S_2 sheaves (9.22) and $F \to F^H$ the relative hull of F over S. The following are equivalent.*

(1) *F^H is a universal hull of F.*
(2) *$F \mapsto F^H$ is universally flat.*
(3) *F^H is flat over S and has pure, S_2 fibers.*
(4) *F^H is flat over S and has pure, S_2 fibers over closed points of S.*
(5) *For every closed point $s \in S$ the restriction map $r_s^F : F^H \to (F_s)^H$ is surjective.*
(6) *$(F_A)^H$ is a universal hull of F_A for every Artin scheme $A \to S$.*

Proof. The only obvious implications are (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) but (4) $\Rightarrow$ (3) directly follows from the openness of the S_2-condition (10.2).

Note that the properties in (3) are preserved by base change, thus $\bigl(F^H\bigr)_T$ is flat over T and $\bigl((F^H)_T\bigr)_t$ is S_2 for every point $t \in T$. By (9.23) this implies that $(F^H)_T$ is the relative hull of F_T. Therefore $(F^H)_T = (F_T)^H$, so $F \mapsto F^H$ is universally flat and commutes with base change. That is, (3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) both hold.

If (4) holds then $\bigl(F^H\bigr)_s = (F_s)^H$ by (9.12.4), thus (4) $\Rightarrow$ (5). Applying (10.68) to every localization of S at closed points shows that (5) $\Rightarrow$ (4).

Next we show that (2) $\Rightarrow$ (6). We may assume that $S = \operatorname{Spec} A$ where (A, m) is local, Artinian. Choose the smallest $r \geq 0$ such that $m^{r+1} = 0$; so $m^r \cong \oplus_i A/m$, the sum of certain number of copies of A/m. This gives an injection $j_r : \oplus_i F_s \hookrightarrow F$ which then extends to $j_r^H : \oplus_i (F_s)^H \hookrightarrow F^H$.

Since F^H is flat over A, the image $j_r^H\bigl(\oplus_i (F_s)^H\bigr)$ is also isomorphic to $(m^r) \otimes_A F^H$ which is the same as $\oplus_i (F^H)_s$. Thus $(F_s)^H = (F^H)_s$ and, by the above arguments, (2) implies the properties (1–5) for local, Artinian base schemes.

In order to see (6) $\Rightarrow$ (5) we may replace S by its completion at s. For $r \in \mathbb{N}$ set $A_r := \operatorname{Spec}_S \mathcal{O}_S/m_s^r$. By base change we get $f_r : X_r \to A_r$ and $F_r := F|_{X_r}$. By assumption $(F_r)^H$ is flat over A_r and we have proved that $F \mapsto F^H$ commutes with base change over Artin schemes. Set

$$\tilde{F} := \varprojlim (F_r)^H.$$

Then $\tilde{F}$ is flat over S, coherent (cf. [**Har77**, II.9.3.A]), agrees with F over U and $\tilde{F} \to F_s^H$ is surjective. Thus $\tilde{F} = F^H$ by (9.23), giving (5). $\square$

We can restate the characterization (9.26.3) as follows.

Corollary 9.27. *Let $f : X \to S$ be a morphism of finite type, $q : F \to G$ a map of coherent sheaves on X. Let n denote the maximum fiber dimension of* $\operatorname{Supp}(F) \to S$. *Then G is the universal hull of F over S iff the following hold.*

1. *$q_s : F_s \to G_s$ is an isomorphism at all n-dimensional points of X_s for every $s \in S$.*
2. *G is flat with purely n-dimensional, S_2 fibers over S and*
3. $\operatorname{Supp}(\operatorname{coker}(q)) \to S$ *has fiber dimension $\le n-2$.* $\square$

Combining (9.27) and (10.3) shows that a relative hull is a universal hull over a dense open subset of the base. Thus Noetherian induction gives the following. A much more precise form will be proved in (9.59).

Corollary 9.28 (Universal hull decomposition, weak form). *Let $f : X \to S$ be a proper morphism and F a coherent sheaf on X. Then there is a locally closed decomposition $j : S' \to S$ such that $j_X^* F$ has a universal hull.* $\square$

The following example illustrates several aspects of (9.26).

Example 9.29. Let X be a projective variety with ample line bundle $\mathcal{O}_X(1)$ and $C(X)$ the corresponding affine cone over X with vertex v. Set $C^*(X) := C(X) \setminus \{v\}$ with natural injection $j : C^*(X) \hookrightarrow C(X)$. If F is a coherent sheaf on X then by pull-back we get a coherent sheaf $C(F)$ on $C^*(X)$ and

$$H^0\big(C^*(X), C(F)\big) = \textstyle\sum_{m\in\mathbb{Z}} H^0\big(X, F(m)\big). \tag{9.29.1}$$

Next let $g : X \to S$ be a flat family of projective varieties and F a coherent sheaf on X that is flat over S and of pure relative dimension ≥ 1. We get the relative cones

$$C_S^*(X) := \operatorname{Spec}_S \textstyle\sum_{m\in\mathbb{Z}} g_*\mathcal{O}_X(m) \quad \text{and} \quad C_S(X) := \operatorname{Spec}_S \sum_{m\in\mathbb{N}} g_*\mathcal{O}_X(m)$$

with natural injection $j : C_S^*(X) \hookrightarrow C_S(X)$. Note that $C_S^*(X)$ is a $\mathbb{G}_m$-bundle over X, thus $C_S^*(X)$ is flat over S but $C_S(X)$ need not be flat over S.

Let $C_S(F)$ denote the coherent sheaf on $C_S^*(X)$ corresponding to $\sum_{m\in\mathbb{Z}} g_*F(m)$; then $C_S(F)^H = j_*C_S(F)$ by (9.21).

By (9.26), $C_S(F)^H$ is a universal hull iff the restriction map $r_s : j_*C_S(F) \to (j_s)_*C(F_s)$ is surjective for every point $s \in S$. Using the grading given by the cone structure this holds iff the restriction map $r_s(m) : g_*F(m) \to H^0\big(X_s, F_s(m)\big)$ is surjective for every point $s \in S$ and every $m \in \mathbb{Z}$. Using the Cohomology and base change theorem we conclude that

$$C_S(F)^H \text{ is a universal hull} \quad \Leftrightarrow \quad g_*F(m) \text{ is locally free } \forall\, m \in \mathbb{Z}. \tag{9.29.2}$$

In order to get some concrete examples, let E_1, E_2 be elliptic curves and $A = E_1 \times E_2$ with projections $\pi_i : A \to E_i$. Let L_i be line bundles of degree $d \ge 3$ on E_i; these give a very ample line bundle $\mathcal{O}_A(1) := \pi_1^* L_1 \otimes \pi_2^* L_2$. Let $C \in |\mathcal{O}_A(1)|$ be a smooth curve. Note that $(C^2) = 2g(C) - 2 = 2d^2$.

Set $S := \operatorname{Pic}^{2d}(E_1) \times \operatorname{Pic}^0(E_2)$ and $X := A \times S$ with projection $g : X \to S$. $H := C \times S$ gives a hyperplane section $C_S(H) \subset C_S(X)$. The universal bundles give $F := \pi_1^* M \otimes \pi_2^* T$. We check that $C_S^*(F)^H$ is not universal along $\operatorname{Pic}^{2d}(E_1) \times \{\mathcal{O}_{E_2}\}$ but if we choose $T = \mathcal{O}_{E_2}$ and M general then the restriction of $C_S(F)$ to the hyperplane section $C_S(H) \subset C_X(X)$ has a universal hull.

By (9.29.2) these claims are equivalent to computing some cohomologies on A and on C. So let M be a line bundle of degree $2d$ on E_1 and T a line bundle of degree 0 on E_2. One easily computes that $h^0\big(A, (\pi_1^* M \otimes \pi_2^* T)(m)\big)$ is independent of M, T if $m \neq 0$ but

$$h^0\big(A, \pi_1^* M \otimes \pi_2^* T\big) = 2d \cdot h^0(E_2, T)$$

jumps at $T = 0$. Similarly we obtain that $h^0\big(C, (\pi_1^* M \otimes \pi_2^* T)(m)|_C\big)$ is independent of M, T if $m \neq 0, -1$ but $h^0\big(C, (\pi_1^* M \otimes \pi_2^* T)_C\big)$ jumps iff $h^0\big(C, (\pi_1^* M \otimes \pi_2^* T)(-1)|_C\big)$ jumps iff $(\pi_1^* M \otimes \pi_2^* T)|_C \cong \omega_C$.

This shows that the hull $C_S^*(F)^H$ is not universal along $\operatorname{Pic}^{2d}(E_1) \times \{\mathcal{O}_{E_2}\}$ but if we choose $T = \mathcal{O}_{E_2}$ and M general then then the restriction of $C_S(F)$ to the hyperplane section $C_S(H) \subset C_X(X)$ has a universal hull.

The following result is a restatement of (10.70), see also [**Kol95a**, Thm.12].

THEOREM 9.30. *Let $f : X \to S$ be a morphism and F a mostly flat family of S_2 sheaves (9.22). Assume that, for every $s \in S$, the hull $(F_s)^H$ is coherent and* $\operatorname{depth}_{Z_s}(F_s)^H \geq 3$. *Then F^H is the universal hull of F over S.* □

9.5. Husks of coherent sheaves

ASSUMPTION 9.31. In this section we continue with the assumptions (9.10).

DEFINITION 9.32. Let X be a scheme and F a coherent sheaf on X. An n-dimensional *quotient husk* of F is a quasi-coherent sheaf G together with a homomorphism $q : F \to G$ such that

(1) G is pure of dimension n and
(2) $q : F \to G$ is surjective at all generic points of $\operatorname{Supp} G$.

A quotient husk is called a *husk* if $n = \dim F$ and

(3) $q : F \to G$ is an isomorphism at all n-dimensional points of X.

If $h \in \operatorname{Ann}(F)$ then $h \cdot F = 0$, hence $h \cdot G \subset G$ is supported in dimension $< n$, thus it is 0. Therefore G is also an $\mathcal{O}_X / \operatorname{Ann}(F)$ sheaf and so the particular choice of X matters very little.

Any coherent sheaf F has a maximal husk

$$M(F) := \varinjlim (j_Z)_* \big(F|_{X \setminus Z}\big),$$

where Z runs through all closed subsets of $\operatorname{Supp} F$ such that $\dim Z < \dim F$. If $\dim F \geq 1$ then $M(F)$ is never coherent, but it is the union of coherent husks. Thus a coherent sheaf has many different coherent husks and there is no maximal coherent husk.

LEMMA 9.33. *Let F be a coherent sheaf on X and $q : F \to G$ an n-dimensional (quotient) husk of F.*

(1) *Let $g : X \to Z$ be a finite S-morphism. Then g_*G is an n-dimensional (quotient) husk of g_*F.*
(2) *Let $h : Y \to X$ be a flat morphism of pure relative dimension r with S_1 fibers. Then h^*G is an $(n+r)$-dimensional (quotient) husk of h^*F.*

Proof. If g is a finite morphism and M is a sheaf then the associated primes of g_*M are the images of the associated primes of M. This implies (1). Similarly, if h is flat then the associated primes of h^*M are the preimages of the associated primes of M. Since h^*G is S_1 by (9.5), we get (2). $\square$

9.34 (Bertini theorem for (quotient) husks). Let F be a coherent sheaf on a quasi-projective variety $X \subset \mathbb{P}^n$ and $q : F \to G$ a coherent (quotient) husk. Let $H \subset \mathbb{P}^n$ be a general hyperplane. Then $G|_H$ is pure by (10.9). If, in addition, H does not contain any of the associated primes of G/F then $q|_H : F|_H \to G|_H$ as also a (quotient) husk.

Definition 9.35. Let X be a scheme and F a coherent sheaf on X. Set $n := \dim F$. A husk $q : F \to G$ is called *tight* if $q : F/\operatorname{tors} F \hookrightarrow G$ is an isomorphism at all $(n-1)$-dimensional points of X.

Thus the hull $q : F \to F^{[**]}$ defined in (9.12) is a tight husk of F. We see below that the hull is the maximal tight husk.

Lemma 9.36. *Let X be a scheme and F a coherent sheaf on X with hull $q : F \to F^{[**]}$.*
(1) *Let $r : F \to G$ be any tight husk. Then q extends uniquely to an injection $q_G : G \hookrightarrow F^{[**]}$.*
(2) *$F^{[**]}$ is the unique tight husk that is S_2.*

Proof. After replacing F with $F/\operatorname{tors} F$ we may assume that F is pure. Set $Z := \operatorname{Supp}(G/F) \cup \operatorname{Supp}(F^{[**]}/F)$. Then Z has codimension ≥ 2 and F is S_2 on $X \setminus Z$. Thus, by using (9.14.2) for F we get that

$$G \subset j_*\big(G|_{X\setminus Z}\big) = j_*\big(F|_{X\setminus Z}\big) = F^{[**]},$$

proving (1). If G is also S_2, then, (9.14.2) gives that $G = F^{[**]}$. $\square$

Lemma 9.37. *Let X be a projective scheme, F a coherent sheaf of pure dimension n and $F \to G$ a quotient husk. The following are equivalent.*
(1) *$G = F^{[**]}$.*
(2) *G is S_2 and $\chi\big(X, F(t)\big) - \chi\big(X, G(t)\big)$ has degree $\leq n-2$.*
(3) *$\chi\big(X, F^{[**]}(t)\big) \equiv \chi\big(X, G(t)\big)$.*

Proof. The exact sequence $0 \to K \to F \to G \to Q \to 0$ defines the sheaves K, Q and

$$\chi\big(X, F(t)\big) - \chi\big(X, G(t)\big) = \chi\big(X, K(t)\big) - \chi\big(X, Q(t)\big).$$

Note that K has pure dimension n and Q has dimension $\leq n-1$.

If $G = F^{[**]}$ then $K = 0$ and $\dim Q \leq n-2$ which implies (2) and (1) $\Rightarrow$ (3) is obvious.

Conversely, assume that $\chi\big(X, F(t)\big) - \chi\big(X, G(t)\big)$ has degree $\leq n-2$. Since $\deg \chi\big(X, Q(t)\big) \leq n-1$, we see that $\deg \chi\big(X, K(t)\big) \leq n-1$. However, K has pure dimension n thus in fact $K = 0$ and so G is a tight husk of F. If G is S_2 then (9.36) implies that $G = F^{[**]}$, hence (2) $\Rightarrow$ (1).

Finally, if (3) holds then $\chi\big(X, F(t)\big) - \chi\big(X, G(t)\big)$ has degree $\leq n-2$, hence, as we proved, G is a tight husk of F. By (9.36.1) G is a subsheaf of $F^{[**]}$. Thus $G = F^{[**]}$ since they have the same Hilbert polynomials. □

Definition 9.38 (Husks over a base scheme). Let $f : X \to S$ be a morphism and F a coherent sheaf on X. A *quotient husk* of F over S is a quasi-coherent sheaf G together with a homomorphism $q : F \to G$ such that

(1) G is flat and pure over S and
(2) $q_s : F_s \to G_s$ is a quotient husk for every $s \in S$.

A quotient husk is called a *husk* if

(3) $q_s : F_s \to G_s$ is a husk for every $s \in S$.

As before, G is also an $\mathcal{O}_X/\operatorname{Ann}(F)$ sheaf and so X matters very little.

Warning. The notion of a (quotient) husk over S does depend on $f : X \to S$. If S is pure, S_2 and $f : F \to G$ is a (quotient) husk over S then it is a (quotient) husk as defined in (9.32), but the converse does not hold. The two notions are even more different if the base scheme is not S_2.

Our main interest is in the relative case; we sometimes omit "over S" if our choice of S is clear from the context.

Husks and quotient husks are preserved by base change. That is, let $q : F \to G$ be a (quotient) husk over S amnd $g : T \to S$ a morphism. Set $X_T := X \times_S T$ and let $g_X : X_T \to X$ be the first projection. Then $g_X^* q : g_X^* F \to g_X^* G$ is a (quotient) husk over T.

9.39 (Openness of husks). Let $\pi : X \to S$ be a morphism and $q : F \to G$ a map of coherent sheaves on X. Assume that G is flat and pure over S. By the Nakayama lemma, for a map between sheaves it is an open condition to be surjective and for a surjective map with flat target it is an open condition to be fiber-wise injective (cf. [**Mat86**, 22.5]). Thus the set of points

$$\big\{x \in X : q_{\pi(x)} : F_{\pi(x)} \to G_{\pi(x)} \text{ is a local isomorphism at } x\big\}$$

is open in X. In particular, if π is proper then

$$\big\{s \in S : q_s : F_s \to G_s \text{ is a (quotient) husk}\big\}$$

is open in S.

9.6. Moduli space of quotient husks

Definition 9.40. Let $f : X \to S$ be a proper morphism and F a coherent sheaf on X. Let $\mathcal{QH}usk(F)(*)$ (resp. $\mathcal{H}usk(F)(*)$) be the functor that to a scheme $g : T \to S$ associates the set of all coherent quotient husks (resp. husks) of $g_X^* F$, where $g_X : T \times_S X \to X$ is the projection.

By (9.39) $\mathcal{H}usk(F)(*)$ is an open subfunctor of $\mathcal{QH}usk(F)(*)$.

If f is projective, H is an f-ample divisor class and $p(t)$ is a polynomial, then $\mathcal{QH}usk_p(F)(*) \subset \mathcal{QH}usk(F)(*)$ (resp. $\mathcal{H}usk_p(F)(*) \subset \mathcal{H}usk(F)(*)$) denotes the subfunctor of all coherent quotient husks (resp. husks) of $g_X^* F$ with Hilbert polynomial $p(t)$. That is, quotient husks $F \to G$ such that $f_*\big(G \otimes H^m\big)$ is locally free of rank $p(m)$ for all $m \gg 1$.

The main existence theorem of this section is the following.

THEOREM 9.41. *Let $f : X \to S$ be a projective morphism and F a coherent sheaf on X. Let H be an f-ample divisor class and $p(t)$ a polynomial. Then* $\mathcal{QH}usk_p(F)$ *has a fine moduli space* $\mathrm{QHusk}_p(F) \to S$ *which is a proper algebraic space over S.*

Our construction establishes $\mathrm{QHusk}_p(F)$ as an algebraic space. When S is a point, the projectivity of $\mathrm{QHusk}_p(F)$ is proved in [**Lin15**], see also [**Wan15**] for earlier results.

As we noted, $\mathcal{H}usk(F)$ is an open subfunctor of $\mathcal{QH}usk(F)$, thus $\mathcal{H}usk_p(F)$ is represented by an open subspace $\mathrm{Husk}_p(F) \subset \mathrm{QHusk}_p(F)$, which is usually not closed. There are, however, many important cases when $\mathrm{Husk}_p(F)$ is also proper over S.

DEFINITION 9.42. Let $f : X \to S$ be a morphism and F a coherent sheaf on X. Let $n = \max_{s\in S}\dim(F_s)$ We say that F is *generically flat on every fiber* (or, more precisely, on every fiber of $\operatorname{Supp} F \to S$) if F is flat at every n-dimensional point of every fiber X_s. If F is coherent, then this is equivalent to the following.

There is a subscheme $Z \subset X$ such that

(1) $F|_{X\setminus Z}$ is flat over S, and
(2) $\dim(X_s \cap Z) < n$ for every $s \in S$.

If f is proper and F is coherent, then $\operatorname{Supp} F \to S$ is also proper. If, in addition, F is generically flat on every fiber then $s \mapsto \dim(X_s \cap \operatorname{Supp} F)$ is locally constant. To simplify notation we always assume that it is actually constant.

COROLLARY 9.43. *Let $f : X \to S$ be a projective morphism and F a coherent sheaf that is generically flat on every fiber. Let H be an f-ample divisor class and $p(t)$ a polynomial. Then* $\mathcal{H}usk_p(F)$ *has a fine moduli space* $\mathrm{Husk}_p(F) \to S$ *which is a proper algebraic space over S.*

We start the proof of (9.41) by establishing the valuative criteria of properness and separatedness. Then we define certain open subfunctors $\mathcal{QH}usk_p^m(F) \subset \mathcal{QH}usk_p(F)$ and construct their moduli spaces $\mathrm{QHusk}_p^m(F)$ using quot-schemes (9.48). At the end we check that $\mathcal{QH}usk_p(F) = \mathcal{QH}usk_p^m(F)$ for $m \gg 1$. The rest of the section is devoted to the details of these arguments. The implication (9.41) $\Rightarrow$ (9.43) is proved at the end of (9.44).

As a preliminary step, note that the problem is local on S, thus we may assume that S is affine. Then f, X, F are defined over a finitely generated subalgebra of $\mathcal{O}_S$, hence we may assume in the sequel that S is of finite type.

9.44 (The valuative criteria of separatedness and properness). More generally, we show that $\mathcal{QH}usk(F)$ satisfies the valuative criteria of separatedness and properness whenever f is proper.

Let T be the spectrum of an excellent DVR with closed point $0 \in T$ and generic point $t \in T$. Given $g : T \to S$ let $g_X : T \times_S X \to X$ denote the projection. We have the coherent sheaf g_X^*F and, over the generic point, a quotient husk $q_t : g_X^*F_t \to g_X^*G_t$. We aim to extend it to a quotient husk $\tilde q : g_X^*F \to \tilde G$.

Let $K \subset g_X^*F$ be the largest subsheaf that agrees with $\ker q_t$ over the generic fiber. Then g_X^*F/K is a coherent sheaf on X_T and none of its associated primes is contained in X_0. Thus g_X^*F/K is flat over T. Let $Z_0 \subset X_0$ be the union of the embedded primes of $(g_X^*F/K)_0$.

By construction q_t descends to a morphism $q_t' : (g_X^*F/K)_t \hookrightarrow g_X^*G_t$. Let $Z_t \subset \operatorname{Supp}(g_X^*F/K)_t$ be the closed subset where q_t' is not an isomorphism and $Z_T \subset X_T$ its closure. Finally set $Z = Z_0 \cup (Z_T \cap X_0)$.

g_X^*F/K restricted to $X_T \setminus \big(Z_0 \cup Z_T\big)$ is flat and pure over T and $g_X^*G_t$ is pure on $X_t = X_T \setminus X_0$. Furthermore, when restricted to $X_T \setminus (X_0 \cup Z_T)$, both of these sheaves are naturally isomorphic to g_X^*F/K. Thus we can glue them to get a single sheaf G' defined on $X_T \setminus Z$ that is is flat and pure over T.

Let $j : X_T \setminus Z \hookrightarrow X_T$ be the injection. By (9.45) $\tilde{G} := j_*G'$ is the unique extension that is flat and pure over T hence

$$\tilde{q} : g_X^*F \to g_X^*F/K \to \tilde{G}$$

is the unique quotient husk extending $q_t : F_t \to G_t$. Thus $\mathcal{QH}usk(F)$ satisfies the valuative criteria of separatedness and properness.

Furthermore, if f is projective then $\tilde{G}_0$ has the same Hilbert polynomial as G_t.

Finally note that if F is generically flat over S and $q_t : g_X^*F_t \to g_X^*G_t$ is a husk then $K \subset g_X^*F$ is zero at the generic points of $X_0 \cap \operatorname{Supp} g_X^*F$, thus $\tilde{q} : g_X^*F \to g_X^*F/K \to \tilde{G}$ is a husk.

This shows that if F is generically flat over S then $\operatorname{Husk}(F)$ is closed in $\operatorname{QHusk}(F)$ hence (9.43) follows from (9.41).

The following extension result is the key to understanding flat families of coherent sheaves over spectra of discrete valuation rings.

Corollary 9.45. *Let T be the spectrum of an excellent DVR with closed point $0 \in T$, generic point $t \in T$ and $g : X \to T$ a morphism. Let $Z \subset X_0$ be a closed subscheme and $j : X \setminus Z \hookrightarrow X$ the natural injection. Let G be a coherent sheaf on $X \setminus Z$. Assume that*

(1) *G is flat over T,*
(2) *Z does not contain any of the generic points of $\big(\overline{W}\big)_0$ for any associated prime W of G.*

Then $F = j_*G$ is the unique extension of G to a coherent sheaf on X such that

(3) *F is flat over T and*
(4) *F_0 has no associated primes contained in Z.*

Note that assumption (2) is automatic if $\dim Z < \dim W_t$ for every associated prime W_t of G_t.

Proof. If F_0 has no associated primes contained in Z then $\operatorname{depth}_{Z_0} F_0 \geq 1$ hence $\operatorname{depth}_{Z_0} F \geq 2$ and so $F = j_*G$ by (9.7)

Conversely, j_*G is coherent by (10.16) and $\operatorname{depth}_{Z_0} F \geq 2$ by (9.7) which implies that $\operatorname{depth}_{Z_0} F_0 \geq 1$. □

9.46 (Construction of bounded open subfunctors). For a given $m \in \mathbb{N}$ let $\mathcal{QH}usk_p^m(F) \subset \mathcal{QH}usk_p(F)$ be the open subfunctor of all quotient husks $F_s \to G_s$ such that $G_s(m)$ is generated by global sections and its higher cohomologies vanish.

Let E be any coherent sheaf on X that is flat over S with proper support. If $H^i(X_s, E_s) = 0$ for some $s \in S$ and all $i > 0$, then this vanishing holds in an open neighborhood of $s \in S$. Thus all sections of E_s lift to nearby fibers, hence, if E_s is globally generated then so are the nearby $E_{s'}$. Thus we expect that, for every m, the moduli space $\operatorname{QHusk}_p^m(F)$ is an open subscheme of $\operatorname{QHusk}_p(F)$.

We prove later that $\mathcal{QH}usk_p^m(F) = \mathcal{QH}usk_p(F)$ for $m \gg 1$.

9.47 (Construction of $\mathrm{QHusk}_p^m(F)$). We use the existence and basic properties of quot-schemes (9.48) and hom-schemes (9.49).

By assumption, each $G_s(m)$ can be written as a quotient of $\mathcal{O}_{X_s}^{\oplus p(m)}$. Let

$$Q_{p(t)} := \mathrm{Quot}^0_{p(t)}(\mathcal{O}_X^{\oplus p(m)}) \subset \mathrm{Quot}(\mathcal{O}_X^{\oplus p(m)})$$

be the universal family of quotients $q_s : \mathcal{O}_{X_s}^{\oplus p(m)} \twoheadrightarrow M_s$ that have Hilbert polynomial $p(t)$, are pure, have no higher cohomologies and the induced map

$$q_s : H^0\big(X_s, \mathcal{O}_{X_s}^{\oplus p(m)}\big) \to H^0\big(X_s, M_s\big)$$

is an isomorphism. Openness of purity is the $m = 1$ case of (10.3), the other two properties were discussed in (9.46).

Let $\pi : Q_{p(t)} \to S$ be the structure map, $\pi_X : Q_{p(t)} \times_S X \to X$ the second projection and M the universal sheaf on $Q_{p(t)} \times_S X$.

By (9.39), the hom-scheme $\underline{\mathrm{Hom}}(\pi_X^* F, M)$ (9.49) has an open subscheme $W_{p(t)}$ parametrizing maps from F to M that are surjective outside a subset of dimension $\leq n-1$. Let $\sigma : W_{p(t)} \to Q_{p(t)}$ be the structure map, and $\sigma_X : W_{p(t)} \times_S X \to Q_{p(t)} \times_S X$ the fiber product.

Note that $W_{p(t)}$ parametrizes triples

$$w := \Big[F_w \stackrel{r_w}{\to} G_w \stackrel{q_w}{\twoheadleftarrow} \mathcal{O}_{X_w}(-m)^{\oplus p(m)}\big)\Big]$$

where $r_w : F_w \to G_w$ is a quotient husk with Hilbert polynomial $p(t)$ and $q_w(m) : \mathcal{O}_{X_w}^{\oplus p(m)} \to G_w(m)$ is a surjection that induces an isomorphism on the spaces of global sections.

Let $w' \in W_{p(t)}$ be another point corresponding to the triple

$$w' := \Big[F_{w'} \stackrel{r_{w'}}{\to} G_{w'} \stackrel{q_{w'}}{\twoheadleftarrow} \mathcal{O}_{X_{w'}}(-m)^{\oplus p(m)}\big)\Big].$$

such that

$$\big[F_w \stackrel{r_w}{\to} G_w\big] \cong \big[F_{w'} \stackrel{r_{w'}}{\to} G_{w'}\big].$$

The difference between w and w' comes from the different ways that we can write $G_w \cong G_{w'}$ as quotients of $\mathcal{O}_{X_w}(-m)^{\oplus p(t)}$. Since we assume that the induced maps

$$q_w(m), q_{w'}(m) : H^0\big(X_w, \mathcal{O}_{X_w}^{\oplus p(m)}\big) \rightrightarrows H^0\big(X_w, G_w(m)\big) = H^0\big(X_w, G_{w'}(m)\big)$$

are isomorphisms, the different choices of q_w and $q_{w'}$ correspond to different bases in $H^0\big(X_w, G_w(mH)\big)$. Thus the fiber of $\mathrm{Mor}(*, W_{p(t)}) \to \mathcal{QH}usk_p(F)(*)$ over $\pi \circ \sigma(w) = \pi \circ \sigma(w') =: s \in S$ is a principal homogeneous space under the algebraic group $\mathrm{GL}\big(p(m), k(s)\big) = \mathrm{Aut}\big(H^0\big(X_s, G_s(m)\big)\big)$.

Thus the group scheme $\mathrm{GL}\big(p(m), S\big)$ acts on $W_{p(t)}$ and

$$\mathrm{QHusk}_p^m(F) = W_{p(t)}/\mathrm{GL}\big(p(m), S\big)$$

by (**??**).

9.48 (Quot-schemes). Let $f : X \to S$ be a morphism and F a coherent sheaf on X. $\mathcal{Q}uot(F)(*)$ denotes the functor that to a scheme $g : T \to S$ associates the set of all quotients of $g_X^* F$ that are flat over T with proper support, where $g_X : T \times_S X \to X$ is the projection.

If $F = \mathcal{O}_X$, then a quotient can be identified with a subscheme of X, thus $\mathcal{Q}uot(\mathcal{O}_X) = \mathcal{H}ilb(X)$, the Hilbert functor.

If H is an f-ample divisor class and $p(t)$ a polynomial, then $\mathcal{Q}uot_p(F)(*)$ denotes those flat quotients that have Hilbert polynomial $p(t)$.

By [**Gro62b**], $\mathcal{Q}uot_p(F)$ is bounded, proper, separated and it has a fine moduli space $\operatorname{Quot}_p(F)$. See [**Ser06**, Sec.4.4] for a detailed proof. If $F = \mathcal{O}_X$, then $\operatorname{Quot}(\mathcal{O}_X) = \operatorname{Hilb}(X)$, the Hilbert scheme of X.

Note that one can write F as a quotient of $\mathcal{O}_{\mathbb{P}^n}(-m)^r$ for some m, r, thus $\mathcal{Q}uot_p(F)$ can be viewed as a subfunctor of $\mathcal{Q}uot(\mathcal{O}^r_{\mathbb{P}^n})$. The theory of $\mathcal{Q}uot(\mathcal{O}^r_{\mathbb{P}^n})$ is essentially the same as the study of the Hilbert functor, discussed in [**Mum66**] and [**Kol96**, Sec.I.1].

9.49 (Hom-schemes). Let $f : X \to S$ be proper morphism and F, G coherent sheaves on X. $\underline{\mathcal{H}om}(F, G)(*)$ denotes the functor that to a scheme $g : T \to S$ associates the set of all homomorphisms of $g_X^* F$ to $g_X^* G$ where $g_X : T \times_S X \to X$ is the projection.

As a special case of [**Gro60**, III.7.7.8–9], if G is flat over S then $\underline{\mathcal{H}om}(F, G)(*)$ is represented by an S-scheme $\underline{\operatorname{Hom}}(F, G)$. That is, for any $g : T \to S$, there is a natural isomorphism

$$\operatorname{Hom}_T(g_X^* F, g_X^* G) \cong \operatorname{Mor}_S\bigl(T, \underline{\operatorname{Hom}}(F, G)\bigr).$$

To see this, note that there is a natural identification between

(1) homomorphisms $\phi : F \to G$, and
(2) quotients $\Phi : (F + G) \twoheadrightarrow Q$ that induce an isomorphism $\Phi|_G : G \cong Q$.

Let $\pi : \operatorname{Quot}(F + G) \to S$ denote the quot-scheme parametrizing quotients of $F + G$ with universal quotient $u : \pi_X^*(F + G) \to Q$, where π_X denotes the induced map $\pi_X : \operatorname{Quot}(F + G) \times_S X \to X$.

Consider now the restriction of u to $u_G : \pi_X^* G \to Q$. By (9.39) there is an open subset

$$\operatorname{Quot}^0(F + G) \subset \operatorname{Quot}(F + G)$$

that parametrizes those quotients $v : F + G \to Q$ that induce an isomorphism $v_G : G \cong Q$. Thus $\underline{\operatorname{Hom}}(F, G) = \operatorname{Quot}^0(F + G)$. □

9.50 (A boundedness condition). Let $X \subset \mathbb{P}^N$ be a projective scheme over a field. As a temporary convenience, let us say that a coherent sheaf G on X satisfies condition $\mathcal{B}(m)$ if

(1) $H^i(X, G(r)) = 0$ for $i > 0$ and $r \geq m$,
(2) $H^0(X, G(r)) \otimes H^0(X, \mathcal{O}_X(1)) \twoheadrightarrow H^0(X, G(r + 1))$ for $r \geq m$.

Note that (2) implies that $G(m)$ is generated by global sections. Thus it can be written as a quotient of $\mathcal{O}_X^{p(m)}$ where $p(m) = h^0(X, G(m))$ (and $p(t)$ is the Hilbert polynomial of G). In particular, all sheaves G that satisfy $\mathcal{B}(m)$ and have Hilbert polynomial $p(t)$ form a bounded family by (9.48).

While we care about the latter conclusion, the assumptions (1–2) are better suited for inductive arguments.

Condition $\mathcal{B}(m)$ should be thought of as a crude version of *Castelnuovo-Mumford regularity*; see [**Laz04**, Sec.I.1.8] for a detailed treatment.

PROPOSITION 9.51. *Let $X \subset \mathbb{P}^N$ be a projective scheme over a field and G a pure, coherent sheaf of dimension ≥ 2 on X with Hilbert polynomial $p(t)$. Let $H \subset X$ be a hyperplane section such that $G_H := G \otimes \mathcal{O}_H$ is also pure. Assume that G_H satisfies condition $\mathcal{B}(m_H)$.*

Then there is an m_X, depending only on m_H and on $p(t)$, such that G satisfies condition $\mathcal{B}(m_X)$.

Proof. Using the cohomology sequence of

$$0 \to G(r-1) \to G(r) \to G_H(r) \to 0$$

we conclude that $H^i(X, G(r-1)) \cong H^i(X, G(r))$ for $i \geq 2$ and $r \geq m_H + 1$. Thus, by Serre's vanishing, $H^i(X, G(r)) = 0$ for $i \geq 2$ and $r \geq m_H + 1$.

For $i = 1$ we have only an exact sequence

$$H^0(X, G(r)) \stackrel{b(r)}{\to} H^0(X, G_H(r)) \to H^1(X, G(r-1)) \stackrel{c(r)}{\to} H^1(X, G(r)) \to 0,$$

which shows that $b(r)$ is onto iff $c(r)$ is an isomorphism.

If $b(r)$ is onto for some $r \geq m_H$ then $b(r+1)$ is also onto by (9.52). Thus $c(r)$ is an isomorphism for every $r \geq m_H$. By Serre's vanishing, this again gives that $H^1(X, G(r)) = 0$ for every $r \geq m_H$.

Otherwise $h^1(X, G(r-1)) > h^1(X, G(r))$. In either case we get that

$$H^1(X, G(r)) = 0 \quad \text{for } r \geq m_H + h^1(X, G(m_H)).$$

Since $h^1(X, G(m_H)) = h^0(X, G(m_H)) - p(m_H)$, we are done if we can bound $h^0(X, G(m_H))$ from above. Since G and G_H are pure,

$$h^0(X, G(r)) \leq \textstyle\sum_{i \geq 0} h^0(X, G_H(r-i)),$$

thus it is enough to bound the sum on the right. Note that $h^0(X, G_H(m_H)) = \chi(X, G_H(m_H)) = p(m_H) - p(m_H - 1)$ and

$$h^0(X, G_H(r)) \neq 0 \Rightarrow h^0(X, G_H(r-1)) < h^0(X, G_H(r)).$$

This bounds $h^0(X, G(m_H))$ and $h^1(X, G(m_H))$ from above. $\square$

Lemma 9.52. *Let $X \subset \mathbb{P}^N$ be a projective scheme, $H := (s = 0) \subset X$ a hyperplane section. Let G be a coherent sheaf on X such that s is not a zero divisor on G and set $G_H := G \otimes \mathcal{O}_H$. Assume that*

(1) *$H^0(X, G) \twoheadrightarrow H^0(H, G_H)$ and*
(2) *$H^0(H, G_H) \otimes H^0(X, \mathcal{O}_X(1)) \twoheadrightarrow H^0(H, G_H(1))$.*

Then, for every $m \geq 1$ we have

(3) *$H^0(X, G(m)) \twoheadrightarrow H^0(H, G_H(m))$ and*
(4) *$H^0(X, G) \otimes H^0(X, \mathcal{O}_X(m)) \twoheadrightarrow H^0(X, G(m))$.*

Proof. By induction it is enough to show this for $m = 1$. Consider the following diagram.

$$\begin{array}{ccc}
H^0(X, G) \otimes H^0(X, \mathcal{O}_X) & = & H^0(X, G) \\
\downarrow & & \downarrow \\
H^0(X, G) \otimes H^0(X, \mathcal{O}_X(1)) & \to & H^0(X, G(1)) \\
\downarrow & & \downarrow \\
H^0(H, G_H) \otimes H^0(X, \mathcal{O}_X(1)) & \to & H^0(H, G_H(1)).
\end{array}$$

Here the right vertical sequence is exact and the assumptions say that the lower left vertical and bottom horizontal arrows are surjective. The conclusion follows by an easy diagram chasing. $\square$

We are now ready to prove the boundedness of $\mathrm{QHusk}^m_p(F)$.

Proposition 9.53. *Let $f : X \to S$ be a projective morphism, H an f-ample divisor class , $p(t)$ a polynomial and F a coherent sheaf on X. Then there is an m such that for every point $s \to S$, every quotient husk of F_s with Hilbert polynomial $p(t)$ satisfies the condition $\mathcal{B}(m)$.*

Proof. The proof is by induction on $n := \deg p(t)$ which is also the dimension of the husks. If $n = 0$ then $\dim G_s = 0$ and $\mathcal{B}(m)$ holds for every m.

Next consider the case $n = 1$. Let d be the leading coefficient of $p(t)$. We may assume that $X \subset \mathbb{P}^N_S$, H is the hyperplane class and F is a quotient of a direct sum $\oplus_i \mathcal{O}_{\mathbb{P}^N}(-q)$ for some q. Set $C_s := \operatorname{Spec} \mathcal{O}_{\mathbb{P}^N} / \operatorname{Ann}(G_s)$ and note that $\deg C_s \le d$. We have a morphism $\oplus_i \mathcal{O}_{C_s}(-q) \to G_s$ that is surjective at all 1-dimensional points. This gives surjections

$$\oplus_i H^1\bigl(C_s, \mathcal{O}_{C_s}(r-q)\bigr) \twoheadrightarrow H^1\bigl(X_s, G_s(r)\bigr).$$

Thus $H^1\bigl(X_s, G_s(r)\bigr) = 0$ for $r \ge q + d - m - 1$ by (9.54).

Finally assume that $n \ge 2$. Let $F_s \to G_s$ be a quotient husk and $H_s \subset X_s$ a hyperplane section. As long as H_s does not contain any of the associated primes of $F_s, G_s, G_s/F_s$ and $G_s|_{H_s}$ is pure (10.9) we see that $G_s|_{H_s}$ is a quotient husk of $F_s|_{H_s}$ with Hilbert polynomial $p(t) - p(t-1)$.

If $X \subset \mathbb{P}^N_S$ then the restrictions $F_s|_{H_s}$ are fibers of a coherent sheaf on

$$X \times_S \check{\mathbb{P}}^N_S \to \check{\mathbb{P}^N}_S$$

where $\check{\mathbb{P}}^N_S$ is the dual projective space bundle parametrizing all hyperplanes in $\mathbb{P}^N_S$. Therefore, by induction, the $G_s|_{H_s}$ satisfy $\mathcal{B}(m_1)$ for some m_1 by induction. Thus, by (9.51), the G_s satisfy $\mathcal{B}(m)$ for some fixed m. □

Lemma 9.54. *Let $X \subset \mathbb{P}^n$ be a subscheme of dimension m and degree d. Then $H^m(X, \mathcal{O}_X(r)) = 0$ for $r \ge d - m - 1$.*

Proof. Choose coordinates on $\mathbb{P}^n$ such that $(x_0 = \cdots = x_m = 0)$ is disjoint from X and set $L := (x_{m+1} = \cdots = x_n = 0)$ with ideal sheaf I_L. Consider the $\mathbb{G}_m$-action

$$\rho_t : (x_0 : \cdots : x_n) \mapsto (x_0 : \cdots : x_m : tx_{m+1} : \cdots : tx_n).$$

As $t \to 0$, the flat limit of the schemes $\rho_t(X)$ is a subscheme X_0 whose support is L. By semicontinuity it is enough to prove that $H^m(X_0, \mathcal{O}_{X_0}(r)) = 0$ for $r \ge d-m-1$. The top cohomology is unchanged by removing embedded points, thus we may assume that X_0 is pure, in particular $\mathcal{O}_{X_0}$ is a quotient of $\mathcal{O}_{\mathbb{P}^n}/I^d_L$.

Note that $\mathcal{O}_{\mathbb{P}^n}/I^d_L$ is a successive extension of line bundles of the form $\mathcal{O}_L(b)$ where $0 \ge b \ge 1 - d$. Thus $H^m\bigl(\mathbb{P}^n, (\mathcal{O}_{\mathbb{P}^n}/I^d_L)(r)\bigr) = 0$ for $r + 1 - d \ge -m$. □

9.7. Hulls and Hilbert polynomials

Let $f : X \to S$ be a projective morphism with relatively ample line bundle $\mathcal{O}_X(1)$. For a coherent sheaf F on X we aim to understand flatness of F and of its hull F^H in terms of the Hilbert polynomials $\chi\bigl(X_s, F_s(t)\bigr)$ of the fibers F_s. Note that the $\chi\bigl(X_s, F_s(t)\bigr)$ carry no information about the nilpotents in $\mathcal{O}_S$.

Theorem 9.55. *Using the above notation, assume that S is reduced. Then $s \mapsto \chi\bigl(X_s, F_s(*)\bigr)$ is an upper semicontinuous function on S and F is flat over S iff this function is locally constant.*

Proof. By generic flatness [**Eis95**, 14.4], there is a dense open subset $S^0 \subset S$ such that F is flat over S^0. Thus the function $s \mapsto \chi\bigl(X_s, F_s(t)\bigr)$ is locally constant on S^0, hence constructible on S by Noetherian induction.

It is thus enough to prove upper semicontinuity when $(0, S)$ is the spectrum of a DVR with generic point g. Let $t_0(F) \subset F$ denote the largest subsheaf supported on X_0. Then $F/t_0(F)$ is flat over S hence

$$\chi\bigl(X_g, F_g(t)\bigr) = \chi\bigl(X_g, \bigl(F/t_0(F)\bigr)_g(t)\bigr) = \chi\bigl(X_0, \bigl(F/t_0(F)\bigr)_0(t)\bigr).$$

Furthermore, a moments thought shows that there is an exact sequence

$$0 \to t_0(F)_0 \to F_0 \to \bigl(F/t_0(F)\bigr)_0 \to 0,$$

hence $\chi\bigl(X_0, F_0(*)\bigr) \succeq \chi\bigl(X_0, \bigl(F/t_0(F)\bigr)_0(*)\bigr)$ and equality holds iff $t_0(F) = 0$.

The last claim is proved (although not stated) in [**Har77**, III.9.9]. □

We have similar results for the Hilbert polynomials of hulls.

THEOREM 9.56. *Let $f : X \to S$ be a projective morphism with relatively ample line bundle $\mathcal{O}_X(1)$ and F a mostly flat family of coherent, S_2 sheaves. Assume that S is reduced. Then $s \mapsto \chi\bigl(X_s, F_s^{[**]}(*)\bigr)$ is an upper semicontinuous function and F^H is a universal hull iff this function is locally constant on S.*

Proof. As in the proof of (9.55) we obtain that $s \mapsto \chi\bigl(X_s, F_s^{[**]}(t)\bigr)$ is constructible and it is enough to prove upper semicontinuity when $(0, S)$ is the spectrum of a DVR with generic point g. The argument closely parallels (5.48.3–5)).

We may replace F by its hull, hence we may assume that F is S_2 and flat over S. In particular, $\chi\bigl(X_0, F_0(t)\bigr) = \chi\bigl(X_g, F_g(t)\bigr)$.

Furthermore, F_0 is S_1, hence the restriction map (9.24) $r_0^F : F_0 \to F_0^H$ is an injection. The exact sequence

$$0 \to F_0 \to F_0^H \to Q_0 \to 0$$

defines Q_0 and

$$\chi\bigl(X_0, F_0^H(t)\bigr) = \chi\bigl(X_0, F_0(t)\bigr) + \chi\bigl(X_0, Q_0(t)\bigr).$$

This gives that

$$\chi\bigl(X_0, F_0^H(t)\bigr) \succeq \chi\bigl(X_0, F_0(t)\bigr) \equiv \chi\bigl(X_g, F_g(t)\bigr)$$

and equality holds iff $r_0^F : F_0 \to F_0^H$ is an isomorphism. By (9.26), in this case F^H is a universal hull.

We have thus proved that if $s \mapsto \chi\bigl(X_s, F_s^{[**]}(t)\bigr)$ is locally constant and S is regular and 1-dimensional then F^H is a universal hull of F. We show in (9.60) that this implies the general case. □

PROPOSITION 9.57. *Let $f : X \to S$ be a projective morphism with relatively ample line bundle $\mathcal{O}_X(1)$ and F a mostly flat family of coherent, S_2 sheaves. Then F^H is a universal hull iff for every local, Artinian ring (A, m_A) with residue field $k = A/m_A$ and every morphism* $\operatorname{Spec} A \to S$ *we have*

$$\chi\bigl(X_A, (F_A)^H(t)\bigr) \equiv \chi\bigl(X_k, (F_k)^H(t)\bigr) \cdot \operatorname{length} A.$$

Proof. We show that the condition holds iff $(F_A)^H$ is flat over A and then conclude using (9.26.6).

Pick a maximum length filtration of A and lift it to a filtration

$$0 = G_0^U \subset G_1^U \subset \cdots \subset G_r^U = F_A|_{U_A}$$

such that $G_{i+1}^U/G_i^U \cong F_k|_{U_k}$ and $r = \operatorname{length} A$. By pushing it forward to X_A we get a filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_r = (F_A)^H$$

such that $G_{i+1}/G_i \subset (F_k)^H$. Therefore

$$\chi\bigl(X_A, (F_A)^H(t)\bigr) \preceq \chi\bigl(X_k, (F_k)^H(t)\bigr) \cdot \operatorname{length} A$$

and equality holds iff $G_{i+1}/G_i = (F_k)^H$ for every i, that is, iff F_A^H is flat over A. □

9.8. Moduli space of universal hulls

DEFINITION 9.58. Let $f : X \to S$ be a morphism and F a coherent sheaf on X. For a scheme $g : T \to S$ set $\mathcal{H}ull(F)(T) = 1$ if $g_X^* F$ has a universal hull and $\mathcal{H}ull(F)(T) = \emptyset$ if $g_X^* F$ does not have a universal hull, where $g_X : T \times_S X \to X$ is the projection.

The following result is the key to many applications of the theory.

THEOREM 9.59 (Flattening decomposition for universal hulls). *Let $f : X \to S$ be a projective morphism and F a coherent sheaf on X. Then*

(1) *$\mathcal{H}ull(F)$ is bounded, separated and it has a fine moduli space* $\operatorname{Hull}(F)$.
(2) *The structure map* $\operatorname{Hull}(F) \to S$ *is a locally closed decomposition (3.48).*

Proof. Let n be the maximal fiber dimension of $\operatorname{Supp} F \to S$ and $S_n \subset S$ the closed subscheme parametrizing n-dimensional fibers. We construct $\operatorname{Hull}_n(F)$, the fine moduli space of n-dimensional universal hulls and then repeat the argument for $S \setminus S_n$.

Let $\pi : \operatorname{Husk}(F) \to S$ be the structure map, $\pi_X : \operatorname{Husk}(F) \times_S X \to X$ the second projection and $q_{\mathrm{univ}} : \pi_X^* F \to G_{\mathrm{univ}}$ the universal husk. The set of points $y \in \operatorname{Husk}(F)$ such that $(G_{\mathrm{univ}})_y$ is S_2 and has pure dimension n is open by (10.3). The fiber dimension of

$$\operatorname{Supp} \operatorname{coker}[\pi_X^* F \to G_{\mathrm{univ}}] \to \operatorname{Husk}(F)$$

is upper semicontinuous. Thus there is a largest open set $W_n \subset \operatorname{Husk}(F)$ parametrizing husks $F_s \to G_s$ such that G_s is S_2, has pure dimension n and $\dim \operatorname{Supp} G_s/F_s \leq n-2$. By (9.27), $\operatorname{Hull}_n(F) = W_n$.

Since hulls are unique (9.21), $\operatorname{Hull}(F) \to S$ is a monomorphism (3.47). In order to prove that each $\operatorname{Hull}_p(F) \to S$ is a locally closed embedding, we check the valuative criterion (3.49).

Let $(0, T)$ be the spectrum of a DVR with generic point g and $p : T \to S$ a morphism such that the hulls of F_g and of F_0 have the same Hilbert polynomials. Let G_g denote the hull of F_g and extend G_g to a husk $F_T \to G_T$. By assumption and by flatness

$$\chi\bigl(X_0, (G_T)_0(t)\bigr) = \chi\bigl(X_g, (G_T)_g(t)\bigr) = \chi\bigl(X_g, (F_g)^H(t)\bigr) = \chi\bigl(X_0, (F_0)^H(t)\bigr).$$

Hence $(G_T)_0 = (F_0)^H$ by (9.37) and so G_T is the relative hull of F_T. Thus G_T defines the lifting $T \to \operatorname{Hull}_p(F)$. □

9.60 (End of the proof of (9.56)). Let T be the spectrum of a DVR and $p : T \to S$ a morphism. We have already proved in (9.56) that if $s \mapsto \chi\bigl(X_s, F_s^{[**]}(t)\bigr)$ is locally constant then p^*F has a universal hull. Thus $p : T \to S$ lifts to $\tilde{p} : T \to \operatorname{Hull}(F)$ and so $\operatorname{Hull}(F) \to S$ is proper. As we show in (3.47), a proper monomorphism is a closed embedding. Since $\operatorname{Hull}(F) \to S$ is also surjective, it is an isomorphism if S is reduced. $\square$

For non-projective morphisms we have the following variant of (9.59).

THEOREM 9.61. *Let (S, m_S) be a complete local ring, R a finite type S-algebra and F a finite R-module that is mostly flat with S_2 fibers over S (9.22). Then there is a quotient $S \twoheadrightarrow S^u$ which represents $Hull(F)$ for local S-algebras.*

Equivalently, we claim that for every local morphism $h : (S, m_S) \to (T, m_T)$ the hull $(F_T)^H$ is universal iff there is a factorization $h : S \twoheadrightarrow S^u \to T$. Compared with (9.59), we only identify the stratum containing the closed point of $\operatorname{Spec} S$.

Proof. By (9.26), $(F_T)^H$ is a universal hull iff $(F_A)^H$ is a universal hull for every Artin quotient $q_A : T \twoheadrightarrow A$, and h factors through S^u iff $q_A \circ h$ factors through S^u for every A. Thus it is enough to construct $S \twoheadrightarrow S^u$ that has the required property for every Artin algebra $h : S \to A$.

We follow the usual method of deformation theory [**Art76, Ses75, Har10**]. As a first step we construct S^u.

For an ideal $I \subset S$ set $F_I := F \otimes (R/IR)$. First we claim that if $(F_I)^H$ and $(F_J)^H$ are universal hulls then so is $(F_{I\cap J})^H$. To see this, start with the exact sequence

$$0 \to S/(I \cap J) \to S/I + S/J \to S/(I + J) \to 0. \tag{9.61.1}$$

F is mostly flat over S, thus (9.61.1) stays left exact after tensoring by F and taking the hull. Thus we obtain the exact sequence

$$0 \to (F_{I\cap J})^H \to (F_I)^H + (F_J)^H \to (F_{I+J})^H. \tag{9.61.2}$$

$(F_J)^H \to (F_{I+J})^H$ is surjective since $(F_J)^H$ is a universal hull, hence (9.61.2) is also right exact.

Set $k := S/m_S$. Since $(F_I)^H$ is a universal hull, $(F_I)^H \otimes k \cong (F_m)^H$, and the same holds for J and $I+J$. Thus tensoring (9.61.2) with k yields an exact sequence

$$(F_{I\cap J})^H \otimes k \to (F_m)^H + (F_m)^H \xrightarrow{p} (F_m)^H \to 0. \tag{9.61.3}$$

Since $\ker p \cong (F_m)^H$ we see that $(F_{I\cap J})^H \otimes k \to (F_m)^H$ is surjective. By (9.26) this implies that $(F_{I\cap J})^H$ is a universal hull.

Let $I^u \subset S$ be the intersection of all those ideals I such that $(F_I)^H$ is a universal hull and $S^u := S/I^u$. By (9.62) we obtain that $(F_{S^u})^H$ is a universal hull.

By construction, if $h : S \twoheadrightarrow W := S/I_W$ is a quotient such that $(F_W)^H$ is a universal hull then $I^u \subset I_W$. We still need to prove that if (A, m_A) is a local Artin S-algebra such that $(F_A)^H$ is a universal hull then $h : S \to A$ factors through S^u.

Let $K := A/m_A$ denote the residue field. Working inductively we may assume that there is an ideal $J \subset A$ such that $J \cong K$ and $h' : S \to A/J$ factors through S^u. Therefore $h : S \to A$ factors through $S \to S/m_S I^u$, thus we may replace S by $S/m_S I^u$ and assume that in fact $m_S I^u = 0$. In this holds then I^u is a finite dimensional k-vector space and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & I^u & \to & S & \to & S^u & \to & 0 \\ & & \downarrow\lambda & & \downarrow h & & \downarrow h' & & \\ 0 & \to & K & \to & A & \to & A' & \to & 0 \end{array} \tag{9.61.4}$$

for some k-linear map $\lambda : I^u \to K$. If $\lambda = 0$ then h factors through S^u, thus we need to get a contradiction if $\lambda \neq 0$.

Set $X := \operatorname{Spec}_S R$ and let $i : U \hookrightarrow X$ be the largest open set over which F is flat over S. For any $S \to T$ by base change we get $i : U_T \hookrightarrow X_T$. Let $\mathcal{G}_T$ denote the restriction of the sheaf $\tilde{F}_T$ to U_T. Then $i_*\mathcal{G}_T$ is the sheaf associated to $(F_T)^H$ and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & I^u \otimes_k i_*\mathcal{G}_k & \to & i_*\mathcal{G}_S & \to & i_*\mathcal{G}_{S^u} & \stackrel{\delta}{\to} & I^u \otimes_k R^1 i_*\mathcal{G}_k \\ & & \downarrow\lambda & & \downarrow h & & \downarrow h' & & \downarrow\lambda \\ 0 & \to & i_*\mathcal{G}_K & \to & i_*\mathcal{G}_A & \to & i_*\mathcal{G}_{A'} & \stackrel{\Delta}{\to} & R^1 i_*\mathcal{G}_K. \end{array} \tag{9.61.5}$$

Note that $\Delta = 0$ since $i_*\mathcal{G}_A$ is a universal hull. The right hand square can be factored as

$$\begin{array}{ccccc} \delta : i_*\mathcal{G}_{S^u} & \to & i_*\mathcal{G}_k & \stackrel{\delta_k}{\to} & I^u \otimes_k R^1 i_*\mathcal{G}_k \\ \downarrow h' & & \downarrow h_k & & \downarrow \lambda\otimes\mathbf{1} \\ \Delta : i_*\mathcal{G}_{A'} & \to & i_*\mathcal{G}_K & \stackrel{\Delta_K}{\to} & K \otimes_k R^1 i_*\mathcal{G}_k. \end{array} \tag{9.61.6}$$

By assumption $\Delta = 0$. Thus by (9.63) there is a nonzero $\mu : I^u \to k$ such that $\mu \circ \delta_k = 0$. Set $S' := S/\ker\mu$ and $J := I^u/\ker\mu$. The extension $J \to S' \to S^u$ gives the exact sequence

$$0 \to J \otimes_k i_*\mathcal{G}_k \to i_*\mathcal{G}_{S'} \to i_*\mathcal{G}_{S^u} \stackrel{\mu\circ\delta}{\longrightarrow} J \otimes_k R^1 i_*\mathcal{G}_k. \tag{9.61.7}$$

Since $\mu \circ \delta = 0$ the map $i_*\mathcal{G}_{S'} \to i_*\mathcal{G}_{S^u}$ is surjective, and so is the composite

$$i_*\mathcal{G}_{S'} \to i_*\mathcal{G}_{S^u} \to i_*\mathcal{G}_k.$$

Thus $i_*\mathcal{G}_{S'}$ is a universal hull by (9.26). This contradicts the maximal choice of S^u. $\square$

The next lemma says that on a complete local ring, all topologies given by m-primary ideals are equivalent. Note that this does not hold for non-complete rings. For example, the intersection of the ideals

$$I_r := \big(y - \sin x, (x,y)^r\big) \subset k[x,y]_{(x,y)}$$

is trivial, yet none of them is contained in $(x,y)^2$.

Lemma 9.62. *Let (S,m) be a complete local ring and $I_1 \supset I_2 \supset \cdots$ m-primary ideals such that $\cap_i I_i = \{0\}$. Let $J \subset S$ be an m-primary ideal. Then there is a $k = k(J)$ such that $J \supset I_k$.*

Proof. It is enough to prove this for the ideals $J = m^j$. For any j, the $(I_k + m^j)/m^j$ form a descending chain of ideal in the Artin ring S/m^j. Thus the chain stabilizes at some $F_j \subset S/m^j$ and the natural maps $F_{j+1} \to F_j$ are surjective. If $F_j \neq 0$ for some j then we get an inverse system of elements $r_i \in F_i$ for $i \geq j$. Since S is complete, they have a limit $s \in S$ and $s \in I_k + m^j$ for every k, j. By Krull's intersection theorem $I_k = \cap_s (I_k + m^s)$, thus $s \in I_k$ for every k. This is impossible since $\cap_k I_k = \{0\}$ by assumption. $\square$

The following is a simple linear algebra lemma.

LEMMA 9.63. *Let $k \subset K$ be a field extension, M, N (possibly infinite dimensional) k-vector spaces and $\delta_i : M \to N$ linear maps. The following are equivalent:*

(1) *There are $\lambda_i \in K$ (not all 0) such that $\sum_i \lambda_i \delta_i : K \otimes M \to K \otimes N$ is zero.*
(2) *There are $\mu_i \in k$ (not all 0) such that $\sum_i \mu_i \delta_i : M \to N$ is zero.* □

One can see that (9.61) does not hold for arbitrary local schemes S, but the following consequence was pointed out by E. Szabó.

COROLLARY 9.64. *The conclusion of (9.61) remains true if S is a Henselian local ring which is the localization of an algebra of finite type over a field or over an excellent DVR.*

Proof. There is a general theorem [**Art69**, 1.6] about representing functors over Henselian local rings, we check that its conditions are satisfied.

Let $\hat{S}$ denote the completion of S. Let $\mathcal{F}$ be the functor from local S-algebras to sets defined as follows:

$$\mathcal{F}(h : S \to T) = \begin{cases} 1 & \text{if } (F_T)^H \text{ is a universal hull,} \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to see that if $\mathcal{F}(h : S \to T) = 1$ then there is a factorization $S \to T' \to T$ such that T' is of finite type over S and $\mathcal{F}(h' : S \to T') = 1$. By definition this means that $\mathcal{F}$ is locally of finite presentation over S (see e.g. [**Art69**, 1.5]). The universal family over $(\hat{S})^u$ gives an effective versal deformation of the fiber over m_S. The existence of S^u now follows from [**Art69**, 1.6]. □

9.9. Hulls and husks over algebraic spaces

The previous proofs used in an essential way the projectivity of $X \to S$. Here we present an alternate approach that does not use projectivity, works for algebraic spaces but leaves properness unresolved. The proofs were worked out jointly with M. Lieblich.

THEOREM 9.65. *Let S be a Noetherian algebraic space and $p : X \to S$ a proper morphism of algebraic spaces. Let F be a coherent sheaf on X. Then $\mathcal{QH}usk(F)$ is separated and it has a fine moduli space* QHusk(F).

Proof. Let $f : X \to S$ be a proper morphism. The functor of flat families of coherent sheaves $\mathcal{F}lat(X/S)$ is represented by an algebraic stack $\mathrm{Flat}(X/S)$ which is locally of finite type but very non-separated; see [**LMB00**, 4.6.2.1].

Let $\sigma : \mathrm{Flat}(X/S) \to S$ be the structure morphism and $U_{X/S}$ the universal family over $\mathrm{Flat}(X/S)$. By (10.3), there is an open substack

$$\mathrm{Flat}^n(X/S) \subset \mathrm{Flat}(X/S)$$

parametrizing pure sheaves of dimension n. Let $U^n_{X/S}$ be the corresponding universal family.

Consider $X \times_S \mathrm{Flat}^n(X/S)$ with coordinate projections π_1, π_2. The stack

$$\underline{\mathrm{Hom}}\big(\pi_1^* F, \pi_2^* U^n_{X/S}\big)$$

parametrizes all maps from the sheaves F_s to pure, n-dimensional sheaves N_s.

We claim that $\mathcal{H}usk(F)$ is an open substack of $\underline{\mathrm{Hom}}\big(\pi_1^* F, \pi_2^* U^n_{X/S}\big)$. Indeed, as in the proof of (9.49), for a map of sheaves $M \to N$ with N flat over S, it is an open condition to be an isomorphism at the generic points of the support.

As we discussed in (9.44), $\mathcal{H}usk(F)$ satisfies the valuative criteria of separatedness and properness. Thus the diagonal of $\mathcal{H}usk(F)$ is a monomorphism. Every algebraic stack with this property is an algebraic space; see [**LMB00**, Sec.8]. $\square$

In the projective case, the Hilbert polynomial was used to write $\mathrm{QHusk}(F)$ as a disjoint union of subschemes $\mathrm{QHusk}_p(F)$ that are proper over S. In the proper but non-projective case we do not have Hilbert polynomials, but one could still hope that the connected components of $\mathrm{QHusk}(F)$ are proper over S. This fails even for the quot-scheme but the following weaker variant should be true. (The proof claimed in [**Kol08a**] is incorrect.)

CONJECTURE 9.66. *Every irreducible component of* $\mathrm{QHusk}(F)$ *is proper.*

The construction of $\mathrm{Hull}(F)$ given in (9.59) applies to algebraic spaces as well but it does not give boundedness. Nonetheless, we claim that $\mathrm{Hull}(F)$ is of finite type. First, it is locally of finite type since $\mathrm{QHusk}(F)$ is. Second, we claim that $\operatorname{red}\mathrm{Hull}(F)$ is dominated by an algebraic space of finite type. In order to see this, consider the (reduced) structure map $\operatorname{red}\mathrm{Hull}(F) \to \operatorname{red} S$. It is an isomorphism at the generic points, hence there is an open dense $S^0 \subset \operatorname{red} S$ such that S^0 is isomorphic to an open subspace of $\operatorname{red}\mathrm{Hull}(F)$. Repeating this for $\operatorname{red} S \setminus S^0$, by Noetherian induction we eventually write $\operatorname{red}\mathrm{Hull}(F)$ as a disjoint union of finitely many locally closed subspaces of $\operatorname{red} S$. (We do not claim, however, that every irreducible component of $\operatorname{red}\mathrm{Hull}(F)$ is a locally closed subspace of $\operatorname{red} S$.)

These together imply that $\mathrm{Hull}(F)$ is of finite type. (Indeed, if $U \to V$ is a surjection, U is of finite type and V is locally of finite type then V is of finite type.)

As in (9.21.5), the structure map $\mathrm{Hull}(F) \to S$ is a monomorphism. However, in the non-projective case, it need not be a locally closed decomposition (9.68). We can summarize these considerations in the following theorem.

THEOREM 9.67 (Flattening decomposition for hulls). *Let $f : X \to S$ be a proper morphism of algebraic spaces and F a coherent sheaf on X. Then*

(1) $\mathcal{H}ull(F)$ *is separated and it has a fine moduli space* $\mathrm{Hull}(F)$*,*
(2) $\mathrm{Hull}(F)$ *is an algebraic space of finite type over S and*
(3) *the structure map* $\mathrm{Hull}(F) \to S$ *is a surjective monomorphism.* $\square$

EXAMPLE 9.68. Let C, D be two smooth projective curves. Pick points $p, q \in C$ and $r \in D$. Let X be the surface obtained from the blow-up $B_{(p,r)}(C \times D)$ by identifying $\{q\} \times D$ with the birational transform of $\{p\} \times D$. Note that X is a proper but non-projective scheme and there is a natural proper morphism $\pi : X \to C'$ where C' is the nodal curve obtained from C by identifying the points p, q.

Then $\mathrm{Hull}(\mathcal{O}_X) = C \setminus \{q\}$ and the natural map $C \setminus \{q\} \to C'$ is a surjective monomorphism but not a locally closed embedding.

CHAPTER 10

Ancillary results

10.1. Flat families of S_m sheaves

Here we consider how the S_m property varies in flat families of coherent sheaves.

DEFINITION 10.1. Recall that a coherent sheaf F on a scheme X satisfies *Serre's condition* S_m if

$$\operatorname{depth}_x F \geq \min\{m, \dim_x F\} \quad \text{for every } x \in X.$$

F is called *Cohen-Macaulay* or *CM* if

$$\operatorname{depth}_x F = \dim_x F \quad \text{for every } x \in X.$$

It is easy to see that if X is CM then $\operatorname{Supp} F$ is locally pure dimensional. We will usually assume that $\operatorname{Supp} F$ is pure dimensional.

THEOREM 10.2. [**Gro60**, IV.12.1.6] *Let $\pi : X \to S$ be a morphism of finite type and F a coherent sheaf on X that is flat over S. Fix $m \in \mathbb{N}$. Then the set of points*

$$\{x \in X : F_{\pi(x)} \text{ is pure and } S_m \text{ at } x\}$$

is open in X

This immediately implies the following variant for proper morphisms.

COROLLARY 10.3. *Let $\pi : X \to S$ be a proper morphism and F a coherent sheaf on X that is flat over S. Fix $m \in \mathbb{N}$. Then the set of points*

$$\{s \in S : F_s \text{ is pure and } S_m\}$$

is open in S. $\square$

For non-proper morphisms we get the following.

COROLLARY 10.4. *Let S be an integral scheme, $\pi : X \to S$ a morphism of finite type and F a coherent sheaf on X. Assume that F is pure and S_m. Then there is a dense open subset $S^0 \subset S$ such that F_s is pure and S_m for every $s \in S^0$.*

Proof. Let $Z \subset X$ denote the set of points $x \in X$ such that either F is not flat at x or $F_{\pi(x)}$ is not pure and S_m at x. Note that Z is closed in X by (10.2) and by generic flatness [**Eis95**, 14.4].

The local rings of the generic fiber of π are also local rings of X, hence the restriction of F to the generic fiber is pure and S_m. Thus Z is disjoint from the generic fiber of π. Therefore $\pi(Z) \subset S$ is a constructible subset that does not contain the generic point, hence $S \setminus \pi(Z)$ contains a dense open subset $S^0 \subset S$. $\square$

10.5 (Nagata's openness criterion). In many cases one can check openness of a subset of a scheme using the following easy to prove test, which is sometimes called the Nagata openness criterion.

Let X be a Noetherian topological space and $U \subset X$ an arbitrary subset. Then U is open iff the following conditions are satisfied.

(1) If $x_1 \in \bar{x}_2$ and $x_1 \in U$ then $x_2 \in U$.
(2) If $x \in U$ then there is a nonempty open $V \subset \bar{x}$ such that $V \subset U$.

Assume now that we want to use this to check openness of a fiber-wise property $\mathcal{P}$ for a morphism $\pi : X \to S$.

We start with condition (10.5.1). Pick points $x_1, x_2 \in X$ such that $x_1 \in \bar{x}_2$.

Let T be the spectrum of a DVR with closed point $0 \in T$, generic point $t_g \in T$ and $q : T \to X$ a morphism such that $q(0) = x_1$ and $q(t_g) = x_2$. After base change using $\pi \circ q$ we get $Y \to T$. Usually one can not guarantee that the residue fields are unchanged under q. However, if property $\mathcal{P}$ is invariant under field extensions, then it is enough to check (10.5.1) for $Y \to T$. Thus we may assume that S is the spectrum of a DVR.

As for (10.5.2), we can replace S by the closure of $\pi(x)$. Then $\pi(x)$ is the generic point of S and, by passing to an open subset, we may assume that S is regular.

We can summarize these considerations in the following form.

Proposition 10.6 (Openness criterion). *Let $\mathcal{P}$ be a property of coherent sheaves over local rings over fields that is invariant under field extensions. The following are equivalent.*

(1) *Let $\pi : X \to S$ be a morphism of finite type and F a coherent sheaf on X that is flat over S. Then the set of points*

$$\{x \in X : F_{\pi(x)} \textit{satisfies property } \mathcal{P} \textit{ at } x\} \quad \textit{is open in } X.$$

(2) *The following 2 special cases of (1) hold, where $\sigma : S \to X$ denotes a section.*
   (a) *S is the spectrum of a DVR with closed point 0, generic point g and $\mathcal{P}$ holds for $\sigma(0) \in X_0$ then $\mathcal{P}$ holds for $\sigma(g) \in X_g$.*
   (b) *S is the spectrum of a regular ring with generic point g and $\mathcal{P}$ holds for $\sigma(g) \in X_g$ then $\mathcal{P}$ holds for all points in a nonempty open subset $U \subset \sigma(S)$.* □

10.7 (Proof of (10.2)). By (10.6) we may assume that S is affine and regular. We may also assume that π is affine and $X = \operatorname{Supp} F$.

First we check (10.6.2.a) for $m = 1$. (Note that pure and S_1 is equivalent to pure.) Let $W \subset X$ be an associated prime of F. Then $W \cap X_0$ is an associated prime of F_0. Since F_0 is pure, $W \cap X_0$ is an irreducible component of $\operatorname{Supp} F_0$ hence W is an irreducible component of $\operatorname{Supp} F$. Thus F_g is also pure.

Next we check (10.6.2.a) for $m > 1$. We already know that every fiber of F is pure. By (10.8) there is a subset $Z \subset X$ of relative codimension ≥ 2 such that F is CM over $X \setminus Z$. Let $Z \subset H \subset X$ be a Cartier divisor that does not contain any of the associated primes of F_0. Then $F|_H$ is flat over S and $\big(F|_H\big)_0 = F_0|_H$ is pure and S_{m-1}. Thus, by induction, $F|_H$ is pure and S_{m-1} on the generic fiber, hence F_{s_g} is pure and S_m along H. It is even CM on $X \setminus H$, hence F_{s_g} is pure and S_m.

The proof of (10.6.2.b) follows a similar pattern. We start with $m = 1$. We may assume that F_{s_g} is pure. By Noether normalization, there is a finite surjection $p : X \to \mathbb{A}^n_S$ for some n. Note that p_*F is flat over S and it is pure on the generic fiber by (9.11), hence torsion free. Using (9.11) in the reverse direction for the other fibers, we are reduced to the case when $X = \mathbb{A}^n_S$ and F is torsion free at $x := \sigma(g)$ on the generic fiber. Thus there is an injection of the localizations $F_x \hookrightarrow \mathcal{O}^m_{x,X}$. By generic flatness [**Eis95**, 14.4], the quotient $\mathcal{O}^m_{x,X}/F_x$ is flat over an open, dense subset $S^0 \subset S$. Thus if $s \in S^0$ then we have an injection $F|_U \hookrightarrow \mathcal{O}^m_U$. Thus every fiber F_s is torsion free over $U \cap \pi^{-1}(S^0)$.

For $m > 1$ we follow the same argument as above using $Z \subset H \subset X$ and induction. □

LEMMA 10.8. *Let $\pi : X \to S$ be a morphism of finite type and F a coherent sheaf on X that is flat over S. Assume that* $\operatorname{Supp} F$ *is pure-dimensional over S. Set*

$$\text{cm-locus}(F) := \{x \in \operatorname{Supp} F : F_{\pi(x)} \text{ is CM at } x\}.$$

Then, for every $s \in S$,

(1) *$X_s \cap \text{cm-locus}(F)$ is dense in* $\operatorname{Supp} F_s$ *and*
(2) *if F_s is pure then $X_s \setminus \text{cm-locus}(F)$ has codimension ≥ 2 in* $\operatorname{Supp} F_s$.

Proof. We may assume that π is affine and $X = \operatorname{Supp} F$. By Noether normalization, there is a finite surjection $p : X \to Y := \mathbb{A}^n_S$ for some n.

Since p_*F is flat over S, it is locally free at a point $y \in Y$ iff the restriction of p_*F to the fiber $Y_{p(y)}$ is locally free at y. The latter holds outside a codimension ≥ 1 subset of each fiber Y_s. If F is pure then p_*F is torsion free on each fiber, and then local freeness holds outside a subset of codimension ≥ 2. □

Let F be a coherent, S_m sheaf on $\mathbb{P}^n$. If a hyperplane $H \subset \mathbb{P}^n$ does not contain any of the irreducible components of $\operatorname{Supp} F$ then $F|_H$ is S_{m-1}, essentialy by definition. The following result says that $F|_H$ is even S_m for general hyperplanes, though we can not be very explicit about the meaning of "general."

COROLLARY 10.9 (Bertini theorem for S_m). *Let F be a coherent, pure, S_m sheaf on a finite type k-scheme and $|V|$ a base point free linear system on X. Then there is a dense open subset $U \subset |V|$ such that $F|_H$ is also pure and S_m for $H \in U$.*

Proof. Let $Y \subset X \times |V|$ be the incidence correspondence (that is, the set of pairs (point $\in H$) with projections π and $\check{\pi}$. Note that π is a $\mathbb{P}^{n-1}$-bundle for $n = \dim |V|$, thus π^*F is also pure and S_m by (9.5).

By (10.4) there is a dense open subset $U \subset |V|$ such that $F|_H$ is also pure and S_m for $H \in U$. For a divisor H, the restriction $F|_H$ is isomorphic to the restriction of π^*F to the fiber of $\check{\pi}$ over $H \in |V|$. □

COROLLARY 10.10 (Bertini theorem for hulls). *Let $|V|$ be a base point free linear system on a finite type k-scheme X. Let F be a coherent sheaf on X with hull $q : F \to F^{[**]}$. Then there is a dense open subset $U \subset |V|$ such that*

$$\big(F^{[**]}\big)|_H = \big(F|_H\big)^{[**]} \quad \text{for } H \in U.$$

Proof. If $H \in |V|$ is general then $\dim(\operatorname{tors} F)|_H = \dim \operatorname{tors} F - 1$ and $(F/\operatorname{tors} F)|_H$ is S_1 by (10.9). Similarly, $\big(F^{[**]}\big)|_H$ is S_2 and $(F/\operatorname{tors} F)|_H \to \big(F^{[**]}\big)|_H$ is an isomorphism outside $H \cap \operatorname{coker} q$. □

Corollary 10.11 (Bertini theorem for S_m in families). *Let T be the spectrum of a local ring, $X \subset \mathbb{P}^n_T$ a quasi-projective scheme and F a coherent sheaf on X that is flat over T with pure, S_m fibers.*

Assume that either X is projective over T or $\dim T \leq 1$. Then $F|_{H\cap X}$ is also flat over T with pure and S_m fibers for a general hyperplane $H \subset \mathbb{P}^n_T$.

Proof. The hyperplanes correspond to sections of $\check{\mathbb{P}}^n_T \to T$. If X is projective over T then we use (10.9) for the special fiber X_0 and conclude using (10.3).

If $\dim T = 1$ then we use (10.9) both for the special fiber X_0 and the generic fibers X_{g_i}. We get open subsets $U_0 \subset \check{\mathbb{P}}^n_0$ and $U_{g_i} \subset \check{\mathbb{P}}^n_{g_i}$. Let $W_i \subset \check{\mathbb{P}}^n_T$ denote the closure of $\check{\mathbb{P}}^n_{g_i} \setminus U_{g_i}$. For dimension reasons, W_i does not contain $\check{\mathbb{P}}^n_0$. Thus any hyperplane corresponding to a section through a point of $U_0 \setminus \big(\cup_i W_i\big)$ works. $\square$

10.12 (Associated points of restrictions). Let X be a scheme, $D \subset X$ a Cartier divisor and F a coherent sheaf on X. We aim to compare $\operatorname{Ass}(F)$ and $\operatorname{Ass}(F|_D)$. If D does not contain any of the associated points of a sheaf G then $\operatorname{Tor}^1(G, \mathcal{O}_D) = 0$. Thus if $0 = F_1 \subset \cdots \subset F_r = F$ is a filtration of F by subsheaves and D does not contain any of the associated points of F_i/F_{i-1} then $0 = F_1|_D \subset \cdots \subset F_r|_D = F|_D$ is a filtration of $F|_D$ and $F_i|_D/F_{i-1}|_D \cong (F_i/F_{i-1})|_D$. In particular

$$\operatorname{Ass}(F|_D) \subset \cup_i \operatorname{Ass}\big((F_i/F_{i-1})|_D\big).$$

By (10.22) we can choose the F_i such that $\operatorname{Ass}(F_i/F_{i-1})$ is a single associated point of F for every i. Thus it remains to understand $\operatorname{Ass}(G|_D)$ when G is pure. Let $G^H \supset G$ denote the hull of G and set $Q := G^H/G$. As we noted above, if D does not contain any of the associated points of Q then $G^H|_D \supset G|_D$, thus $\operatorname{Ass}(G^H|_D) = \operatorname{Ass}(G|_D)$. Finally, since G^H is S_2, the restriction $G^H|_D$ is S_1 hence its associated points are exactly the generic points of $D \cap \operatorname{Supp} G$. We have thus proved the following.

Claim 10.12.1. Let X be an excellent scheme and F a coherent sheaf on X. Then there are finitely many points $x_i \in X$ such that the following holds.

Let $D \subset X$ be a Cartier divisor that does not contain any of the x_i. Then

(a) the associated points of $F|_D$ are exactly the generic points of $D \cap \bar{x}$ for all $x \in \operatorname{Ass}(F)$ and

(b) $\big(F/\operatorname{emb}(F)\big)|_D \cong (F|_D)/\big(\operatorname{emb}(F|_D)\big)$. $\square$

Example 10.13. If $\dim T \geq 2$ then (10.11) does not hold for non-proper maps. Here is a similar example for the classical Bertini theorem on smoothness. Set

$$X := (x^2 + y^2 + z^2 = s) \setminus (x = y = z = s = 0) \subset \mathbb{A}^3_{xyz} \times \mathbb{A}^2_{st}$$

with smooth second projection $f : X \to \mathbb{A}^2_{st}$. Over the origin we start with the hyperplane $H_{00} := (x = 0)$, it is a typical member of the base point free linear system $|ax + by + cz = 0|$.

A general deformation of it is given by $H_{st} := x + b(s,t)y + c(s,t)z = d(s,t)$. It is easy to compute that the intersection $H_{st} \cap X_{st}$ is singular iff $s(1 + b^2 + c^2) = d^2$. This equation describes a curve in $\mathbb{A}^2_{st}$ that passes through the origin.

The next result describes how the associated points of fibers of a flat sheaf fit together. The proof is a refinement of the arguments used in (10.7).

Theorem 10.14. *Let $f : X \to S$ be a morphism of finite type and F a coherent sheaf on X. Then the following hold.*

(1) *There are finitely many locally closed subschemes* $W_i \subset X$ *such that for every* $s \in S$ *the associated points of* F_s *are exactly the generic points of the* $(W_i)_s$.
(2) *If* F *is flat over* S *then we can choose the* W_i *to be closed and such that each* $f|_{W_i} : W_i \to f(W_i)$ *is equidimensional.*

Proof. Using Noetherian induction it is enough to prove that (1) holds over a non-empty open subset of $\operatorname{red} S$. We may thus assume that S is integral with generic point $g \in S$.

Assume first that X is integral and F is torsion free. By Noether normalization, after again passing to some non-empty open subset of S there is a finite surjection $p : X \to \mathbb{A}^m_S$. Then p_*F is torsion free of generic rank say r, hence there is an injection $j : p_*F \hookrightarrow \mathcal{O}^r_{\mathbb{A}^m_S}$. After again passing to some non-empty open subset we may assume that $\operatorname{coker}(j)$ is flat over S, thus

$$j_s : p_*(F_s) = (p_*F)_s \hookrightarrow \mathcal{O}^r_{\mathbb{A}^m_s}$$

is an injection for every $s \in S$. Thus each F_s is torsion free and its associated points are exactly the generic points of the fiber X_s.

In general, we use (10.21) for the generic fiber and then extend the resulting filtration to X. Thus, after replacing S by a non-empty open subset if necessary, we may assume that there is a filtration $0 = F^0 \subset \cdots \subset F^n = F$ such that each F^{m+1}/F^m is a coherent, torsion free sheaf over some integral subscheme $W_m \subset X$. As we proved, we may assume that the associated points of each $\left(F^{m+1}/F^m\right)_s$ are exactly the generic points of the fiber $(W_m)_s$. Using generic flatness we may also assume that each F^{m+1}/F^m is flat over S. Then the associated points of each F_s are exactly the generic points of the fibers $(W_m)_s$ for every m. This proves (1).

In order to see (2), consider first the case when the base $(0 \in T)$ is the spectrum of a DVR. The filtration given by (10.21) for the generic fiber extends to a filtration $0 = F^0 \subset \cdots \subset F^n = F$ over X giving closed integral subschemes $W_m \subset X$. Since T is the spectrum of a DVR, the F^{m+1}/F^m are flat over T, hence the associated points of F_0 are exactly the generic points of the fibers $(W_m)_0$ for every m.

To prove (2) in general, we take the $W_i \subset X$ obtained in (1) and replace them by their closures. A possible problem arises if $f|_{W_i} : W_i \to f(W_i)$ is not equidimensional. Assume that $W_i \to f(W_i)$ has generic fiber dimension d and let $(W_i)_s$ be a special fiber. Pick any closed point $x \in (W_i)_s$ and the spectrum of a DVR $(0 \in T)$ mapping to W_i such that the special point of T maps to x and the generic point of T to the generic point of W_i. After base change to T we see that F_s has a d-dimensional associated subscheme containing x. Thus $(W_i)_s$ is covered by d-dimensional associated subschemes of F_s. Since F_s is coherent, this is only possible if $\dim(W_i)_s = d$ and every generic point of the $(W_i)_s$ is an associated point of F_s. $\square$

10.15 (Semicontinuity and depth). Let X be a scheme and F a coherent sheaf on X. As we noted in (9.4), the function $x \mapsto \operatorname{depth}_x F$ is not lower semi continuous. This is, however, caused by the non-closed points. A quick way to see this is the following.

Assume that X is regular and let $0 \in X$ be a closed point. By the Auslander–Buchsbaum formula (cf. [**Eis95**, 19.9]) F_0 has a projective resolution of length $\dim X - \operatorname{depth}_0 F$. Thus there is an open subset $0 \in U \subset X$ such that $F|_U$ has a projective resolution of length $\dim X - \operatorname{depth}_0 F$. This shows that

$$\operatorname{depth}_x F \geq \operatorname{depth}_0 F - \dim \bar{x} \quad \forall x \in U. \tag{10.15.1}$$

That is, $x \mapsto \operatorname{depth}_x F$ is lower semi continuous for closed points. In general, we have the following analog of (10.2).

Proposition 10.15.2. Let $\pi : X \to S$ be a morphism of finite type and F a coherent sheaf on X that is flat over S with pure fibers. Let $0 \in X$ be a closed point. Then there is an open subset $0 \in U \subset X$ such that

$$\operatorname{depth}_x F_{\pi(x)} \geq \operatorname{depth}_0 F_{\pi(0)} - \text{tr-deg}_{k(\pi(x))} k(x) \quad \forall x \in U,$$

where $F_{\pi(x)}$ is the restriction of F to the fiber $X_{\pi(x)}$ and tr-deg denotes the transcendence degree. Hence $x \mapsto \operatorname{depth}_x F_{\pi(x)}$ is lower semi continuous on closed points.

In order to see this, using Noether normalization and (10.8.1) as in (10.7), we can reduce to the case when $X = \mathbb{A}^n_S$ for some n. Next we take a projective resolution of the fiber $F_{\pi(0)}$ and lift it to a suitable neighborhood $0 \in U \subset X$ using the flatness of F. □

10.2. Cohomology over non-proper schemes

The cohomology theory of coherent sheaves is trivial over affine schemes and well understood over proper schemes. If X is a scheme and $j : U \hookrightarrow X$ is an open subscheme then one can study the cohomology theory of coherent sheaves on U by understanding the cohomology theory of coherent sheaves on X and the higher direct image functors $R^i j_*$. The key results are (10.16) and (10.19).

We start with the basic coherence result for push-forwards.

Proposition 10.16. [**Gro60**, IV.5.11.1] *Let X be an excellent scheme, $Z \subset X$ a closed subscheme and $U := X \setminus Z$ with injection $j : U \hookrightarrow X$. Let G be a coherent sheaf on U. Then j_*G is coherent iff* $\operatorname{codim}_{\bar{W}}(Z \cap \bar{W}) \geq 2$ *for every associated point W of G.*

The case of arbitrary Noetherian schemes is discussed in [**Kol15**].

Proof. This is a local question, hence we may assume that X is affine. By (10.22) and (10.24), G has a filtration $0 = G_0 \subset \cdots \subset G_r = G$ such that each G_{m+1}/G_m is isomorphic to a subsheaf of some $\mathcal{O}_W$ where W is an associated prime of G. Since j_* is left exact, it is enough to show that each $j_*\mathcal{O}_W$ is coherent.

Let $\bar{W} \subset X$ denote the closure of W and $p : V \to W$, $\bar{p} : \bar{V} \to \bar{W}$ the normalizations. Since X is excellent, p and $\bar{p}$ are finite. $\mathcal{O}_{\bar{V}}$ is S_2 (by Serre's criterion) and so is $\bar{p}_*\mathcal{O}_{\bar{V}}$ by (9.11). Thus

$$j_*\mathcal{O}_W \subset j_*\big(p_*\mathcal{O}_V\big) = j_*\big(\bar{p}_*\mathcal{O}_{\bar{V}}\big),$$

where the equality follows from (9.7) using $\operatorname{codim}_{\bar{W}}(Z \cap \bar{W}) \geq 2$. Thus $j_*\mathcal{O}_W$ is coherent. □

It is frequently quite useful to know that coherent sheaves are "nice" over large open subsets. For finite type schemes this was established in (10.8).

Proposition 10.17. *Let X be a Noetherian scheme. Assume that every integral subscheme $W \subset X$ has an open dense subscheme $W^0 \subset W$ that is regular (or at lest CM). Let F be a coherent sheaf on X.*

(1) *There is a closed subset* $Z_1 \subset \operatorname{Supp} F$ *of codimension* ≥ 1 *such that* F *is CM on* $X \setminus Z_1$.
(2) *If* F *is* S_1 *then there is a closed subset* $Z_2 \subset \operatorname{Supp} F$ *of codimension* ≥ 2 *such that* F *is CM on* $X \setminus Z_2$.

Proof. The question is local, hence after removing the intersections of different irreducible components of $\operatorname{Supp} F$ we may assume that $\operatorname{Supp} F$ is irreducible. Since an extension of CM sheaves of the same dimensional support is CM (cf. [**Kol13c**, 2.60–62]), using (10.22) we may assume that F is torsion free over an integral subscheme $W \subset X$. Then F is locally free over a dense open subset $W^0 \subset W$ and we can take $Z_1 := W \setminus W^*$, where W^* is the regular locus of W^0.

In order to prove (2), we may assume that X is affine. Let $s = 0$ be a local equation of Z_1. We apply the first part to F/sF to obtain a closed subset $Z_2 \subset \operatorname{Supp}(F/sF)$ of codimension ≥ 1 such that F/sF is CM on $X \setminus Z_2$. Thus F is CM on $X \setminus Z_2$. □

10.18 (Cohomology over quasi-affine schemes). (See [**Gro67**] for details.)

Let X be an affine scheme, $Z \subset X$ a closed subscheme and $U := X \setminus Z$. Here our primary interest is in the case when $Z = \{x\}$ is a closed point.

For a quasi-coherent sheaf F on X, let $H^0_Z(X, F)$ denote the space of global sections whose support is in Z. There is a natural exact sequence

$$0 \to H^0_Z(X, F) \to H^0(X, F) \to H^0(U, F|_U).$$

This induces a long exact sequence of the corresponding higher cohomology groups. Since X is affine, $H^i(X, F) = 0$ for $i > 0$, hence the long exact sequence breaks up into a shorter exact sequence

$$0 \to H^0_Z(X, F) \to H^0(X, F) \to H^0(U, F|_U) \to H^1_Z(X, F) \to 0 \tag{10.18.1}$$

and a collection of isomorphisms

$$H^i(U, F|_U) \cong H^{i+1}_Z(X, F) \quad \text{for } i \geq 1. \tag{10.18.2}$$

The vanishing of the local cohomology groups is closely related to the depth of the sheaf F. Two instances of this follow from already established results. First, for coherent sheaves (9.6) can be restated as

$$H^0_Z(X, F) = 0 \iff \operatorname{depth}_Z F \geq 1. \tag{10.18.3}$$

Second, (9.7) tells us when the map $H^0(X, F) \to H^0(U, F|_U)$ in (10.18.1) is an isomorphism. This implies that, for coherent sheaves,

$$H^0_Z(X, F) = H^1_Z(X, F) = 0 \iff \operatorname{depth}_Z F \geq 2. \tag{10.18.4}$$

More generally, Grothendieck's vanishing theorem (see [**Gro67**, Sec.3] or [**BH93**, 3.5.7]) says that

$$H^i_Z(X, F) = 0 \quad \text{for } i < \operatorname{depth}_Z F. \tag{10.18.5}$$

Combined with (10.18.2–3) this shows that

$$H^i\big(U, F|_U\big) = 0 \quad \text{for } 1 \leq i \leq \operatorname{depth}_Z F - 2. \tag{10.18.6}$$

All the above groups are naturally modules over $H^0(X, \mathcal{O}_X)$ and we need to understand when they are finitely generated.

More generally, let G be a coherent sheaf on U. When is the group $H^i(U, G)$ a finite $H^0(X, \mathcal{O}_X)$-module? Since X is affine,

$$H^i(U, G) = H^0\big(X, R^i j_* G\big),$$

where $j : U \hookrightarrow X$ denotes the natural open embedding. Thus $H^i(U, G)$ is a finite $H^0(X, \mathcal{O}_X)$-module iff $R^i j_* G$ is a coherent sheaf. For $i \geq 1$, the sheaves $R^i j_* G$ are supported on Z, which implies the following.

Lemma 10.18.7. Notation as above. Assume that $i \geq 1$.

(a) Every associated prime of $H^i(U, G)$ (viewed as an $H^0(X, \mathcal{O}_X)$-module) is contained in Z.
(b) If $Z = \{x\}$ then $H^i(U, G)$ is a finite $H^0(X, \mathcal{O}_X)$-module iff $H^i(U, G)$ has finite length. □

The general finiteness condition is stated in (10.19); but first we work out the special cases that we use. We start with $H^0(U, G)$; here we have the following restatement of (10.16).

Lemma 10.18.8. Let X be an excellent, affine scheme, $Z \subset X$ a closed subscheme, $U := X \setminus Z$ and G a coherent sheaf on U. Assume in addition that $Z \cap \bar{W}_i$ has codimension ≥ 2 in $\bar{W}_i$ for every associated prime $W_i \subset U$ of G. Then $H^0(U, G)$ is a finite $H^0(X, \mathcal{O}_X)$-module. □

It is considerably harder to understand finiteness for $H^1(U, G)$. The following special case is used in Section 5.8.

Lemma 10.18.9. Let X be an excellent scheme, $Z \subset X$ a closed subscheme, $U := X \setminus Z$ and G a coherent sheaf on U. Assume in addition that

(a) G is S_2,
(b) there is a coherent CM sheaf F on X and an injection $G \hookrightarrow F|_U$,
(c) Z has codimension ≥ 3 in $\operatorname{Supp} F$.

Then $R^1 j_* G$ is coherent.

Proof. Set $Q = F|_U / G$. Since G is S_2, it has no extensions with a sheaf whose support has codimension ≥ 2 by (9.7), thus every associated prime of Q has codimension ≤ 1 in $\operatorname{Supp} F$. Thus Q satisfies the assumptions of (10.16) and so $j_* Q$ is coherent. By (10.18.4) $R^1 j_* \big(F|_U\big) = 0$, hence the exact sequence

$$0 \to j_* G \to j_* \big(F|_U\big) \to j_* Q \to R^1 j_* G \to R^1 j_* \big(F|_U\big) = 0$$

shows that $R^1 j_* G$ is coherent. □

Not every S_2-sheaf can be realized as a subsheaf of a CM sheaf, but this can be arranged in some important cases.

Lemma 10.18.10. Notation as above. Assume in addition that

(a) X is embeddable into a regular, affine scheme R as a closed subscheme.
(b) $\operatorname{Supp} G$ has pure dimension $n \geq 3$ and $Z = \{x\}$ is a closed point.
(c) G is S_2.

Then $H^1(U, G)$ has finite length. Thus, if X is of finite type over a field k, then $H^1(U, G)$ is a finite dimensional k-vector space.

Outline of proof. X plays essentially no role. Let $Y \subset R$ be a complete intersection subscheme defined by $\dim R - n$ elements of $\operatorname{Ann} G$. Then Y is Gorenstein, we can view G as a coherent sheaf on $Y \setminus \{x\}$ and $H^i(X \setminus \{x\}, G) = H^i(Y \setminus \{x\}, G)$. Thus it is enough to prove vanishing of the latter for $i = 1$.

By (10.18.11) there is an embedding $G \hookrightarrow \mathcal{O}^m_{Y \setminus \{x\}}$, hence (10.18.9) applies. □

Lemma 10.18.11. Let U be a quasi-affine scheme of pure dimension n and G a pure coherent sheaf on U of dimension n. Assume that

(1) either U is reduced
(2) or U is Gorenstein at its generic points.

Then G is isomorphic to a subsheaf of $\mathcal{O}_U^m$ for some m. □

Outline of proof. Assume that such an embedding exists at the generic points. Then we have an embedding $G \hookrightarrow \mathcal{O}_U^m$ over some dense open set $U^0 \subset U$. Pick $s \in \mathcal{O}_U$ invertible at the generic points and vanishing along $U \setminus U^0$. Multiplying by s^r for $r \gg 1$ gives the embedding $G \hookrightarrow \mathcal{O}_U^m$.

The remaining question is, what happens at the generic point. The existence of the embedding is clear if U is reduced.

In general, we are reduced to the following algebra question: given an Artin ring A, when is every finite A-module M a submodule of A^m for some m? Usually the answer is no. However, local duality theory (see, for instance, [**Eis95**, Secs.21.1–2]) shows that every finite A-module is a submodule of ω_A^m for some m. Finally A is Gorenstein iff $A \cong \omega_A$. □

Much of the following result can be established by the above methods, but it is easier to prove it using local duality theory; see [**Gro68**, VIII.2.3] for details.

THEOREM 10.19. *Let X be an excellent scheme, $Z \subset X$ a closed subscheme, $U := X \setminus Z$ and $j : U \hookrightarrow X$ the open embedding. Assume in addition that X is locally embeddable into a regular scheme. For a coherent sheaf G on U and $n \in \mathbb{N}$ the following are equivalent.*

(1) *$R^i j_* G$ is coherent for $i < n$.*
(2) $\operatorname{depth}_u G \geq n$ *for every point $u \in U$ such that* $\operatorname{codim}_{\bar{u}}(Z \cap \bar{u}) = 1$. □

10.3. Dévissage

Dévissage is a method that writes a coherent sheaf as an extension of simpler coherent sheaves and uses these to prove various theorems. There are many ways to do this, different ones are useful in different contexts.

NOTATION 10.20. Let X be a Noetherian scheme and F a coherent sheaf on X. Let $\{w_i : i = 1, \dots, m\}$ be the associated points of F in some fixed order.

Let $Z \subset X$ be a closed subscheme. As in (9.2) we write $\operatorname{tors}_Z F \subset F$ for the largest subsheaf whose support is contained in Z.

F is called *torsion free* on X if every associated point of F is a generic point of X.

LEMMA 10.21. *Using the notation of (10.20), assume in addition that $w_j \notin W_i$ for $i < j$. Then F admits a unique filtration $0 = G_0 \subset G_1 \cdots \subset G_m = F$ such that G_i/G_{i-1} is (isomorphic to) a coherent sheaf that is supported on W_i and is torsion free (as a sheaf on W_i) for $i = 1, \dots, m$. Moreover, the natural map* $\operatorname{tors}_{W_i} F \to G_i/G_{i-1}$ *is an isomorphism at w_i for $i = 1, \dots, m$.*

Proof. It is easy to see that we must set $G_1 = \operatorname{tors}_{W_1} F$. Then pass to F/G_1 and use induction on the number of associated points. □

For an arbitrary ordering of the w_i the filtration still exists but it is not canonical and not even the generic rank of the graded pieces is unique, see (10.23).

LEMMA 10.22. *Using the notation of (10.20), F admits a finite filtration $0 = G_0 \subset G_1 \cdots \subset G_m = F$ such that G_i/G_{i-1} is (isomorphic to) a coherent sheaf that is supported and torsion free on W_i for $i = 1, \dots, m$.*

Proof. First set $G' := \operatorname{tors}_{W_1} F$. Then the associated points of F/G' are those w_j that are not contained in W_1. Let $G'' \subset G'$ be the largest subsheaf whose support is not dense in W_1. Set $Z = \operatorname{Supp} G''$. A power of I_Z kills G'', hence, by the Artin–Rees lemma, $G'' \cap I_Z^r G' = 0$ for some r. By Noetherian induction, there is a coherent subsheaf $H \subset \big(F/I_Z^r G'\big)$ such that none of the w_i are associated points of H and every associated point of $\big(F/I_Z^r G'\big)/H$ is among the w_i. Finally let $G_1 \subset F$ be the preimage of H. Every associated point of F/G_1 is among the $\{w_i : i \geq 2\}$ by construction. The associated points of G_1 are w_1 and possibly a few others that are distinct from the $\{w_i : i \geq 2\}$. However, since $G_1 \subset F$, its associated points are a subset of $\{w_i : i = 1, \dots, m\}$. Thus w_1 is the only associated point of G_1.

Next we pass to F/G_1 and finish by induction as before. □

EXAMPLE 10.23. The graded pieces G_i/G_{i-1} depend on the ordering of the w_i in (10.22), even their generic rank can change. For example, let $X = \operatorname{Spec} k[x, y]$ and F the sheaf corresponding to $k[x, y]/(xy, y^2)$. The associated points are (y) and (x, y). If we take (x, y) first, we get the most natural filtration

$$0 \to k \xrightarrow{y} k[x, y]/(xy, y^2) \to k[x, y]/(y) \to 0.$$

If we take (y) first then for every $n \geq 1$ we get different possibilities

$$0 \to k[x, y]/(y) \xrightarrow{x^n} k[x, y]/(xy, y^2) \to k[x, y]/(xy, x^n, y^2) \to 0.$$

The above filtration can be further refined.

LEMMA 10.24. *Let W be an irreducible, Noetherian scheme and F a torsion free, coherent sheaf on W. Then F admits a finite filtration $0 = G_0 \subset G_1 \cdots \subset G_m = F$ such that G_i/G_{i-1} is a torsion free, coherent sheaf on $\operatorname{red} W$ of generic rank 1 for $i = 1, \dots, m$.*

Proof. Let $J \subset \mathcal{O}_W$ be the nil-radical. Let $U \subset W$ be a dense, open, affine subset and $s \in H^0(U, F|_U)$ a nonzero section such that $J \cdot s = 0$. We can take G_1 to be the subsheaf of local sections ϕ such that $\phi|_V \subset \mathcal{O}_V \cdot (s|_V)$ for some dense, open subset $V \subset W$. We then pass to F/G_1 and repeat the process. □

Over a quasi-affine scheme, any global section of G_1 shows the following consequence, which is also easy to prove directly.

COROLLARY 10.25. *Let X be a Noetherian, quasi-affine scheme and F a coherent sheaf on X with associated points $\{w_i : i = 1, \dots, m\}$. For every i there are injections $\mathcal{O}_{W_i} \hookrightarrow F$ where $W_i := \bar{w}_i$.* □

The following is probably the best known variant of dévissage and it is sufficient for most applications.

COROLLARY 10.26. *The K-group of coherent sheaves on a Noetherian scheme is generated by the structure sheaves of closed, integral subvarieties.*

Proof. Using (10.21) and (10.24) we need to deal with the case when F is a torsion free sheaf of generic rank 1 on an integral scheme W. There is a nowhere dense subscheme $Z \subset W$ and an injection $\mathcal{O}_W(-Z) \hookrightarrow F$. Thus

$$[F] = [\mathcal{O}_W(-Z)] + [F/\mathcal{O}_W(-Z)] = [\mathcal{O}_W] - [\mathcal{O}_Z] + [F/\mathcal{O}_W(-Z)],$$

where the bracket denotes the class of a sheaf in the K-group of coherent sheaves. By Noetherian induction the claim holds for $F/\mathcal{O}_W(-Z)$. □

This is probably as far as one can go on a general Noetherian scheme. On an integral quasi projective scheme every torsion free, coherent sheaf of generic rank 1 has a subsheaf that is a line bundle. If X is quasi affine, we can choose the line bundle to be trivial. The quotient has smaller dimensional support but we do not know its associated points. Thus we get the following, where the Z_i need not be closures of associated points of F.

Lemma 10.27. *Let X be a quasi-projective scheme and F a coherent sheaf on X. Then F admits a finite filtration $0 = G_0 \subset G_1 \cdots \subset G_m = F$ such that G_i/G_{i-1} is isomorphic to a line bundle on a closed, integral subvariety $Z_i \subset X$ for $i = 1, \dots, m$.* □

Lemma 10.28. *Let X be a Noetherian, quasi affine scheme and F a coherent sheaf on X. Then F admits a finite filtration $0 = G_0 \subset G_1 \cdots \subset G_m = F$ such that G_i/G_{i-1} is isomorphic to the structure sheaf of a closed, integral subvariety $Z_i \subset X$ for $i = 1, \dots, m$.* □

10.4. Volumes and intersection numbers

We have used several general results that compare intersection numbers and volumes under birational morphisms.

Definition 10.29. [**Laz04**, Sec.2.2.C] Let X be a proper scheme of dimension n over a field k and L a divisorial sheaf on X (3.50). Its *volume* is defined as

$$\operatorname{vol}(L) := \lim \frac{h^0(X, L^{[m]})}{m^n/n!}.$$

This defines the volume of any divisor that is Cartier at its generic points, and the notion extends to $\mathbb{Q}$-Cartier divisors. If L is nef then $\operatorname{vol}(L) = (L^n)$.

Proposition 10.30. *Let $p : Y \to X$ be a birational morphism of normal, proper varieties of dimension n. Let D_Y be a p-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $D_X := p_*(D_Y)$ is also $\mathbb{Q}$-Cartier. Then*

(1) $\operatorname{vol}(D_X) \geq \operatorname{vol}(D_Y)$ *and*
(2) *if D_X is ample then equality holds in (1) iff $D_Y \sim_{\mathbb{Q}} p^*D_X$.*

Furthermore, let H is an ample divisor on X. Then

(3) $I(H, D_X) \succeq I(p^*H, D_Y)$ *where and*
(4) *equality holds in (3) iff $D_Y \sim_{\mathbb{Q}} p^*D_X$.*

The general case when equality holds in (1) is considered in (10.37).

Proof. Write $D_Y = p^*D_X - E$ where E is p-exceptional. By assumption $-E$ is p-nef, hence E is effective by [**KM98**, 3.39]. Thus $\operatorname{vol}(D_X) = \operatorname{vol}(p^*D_X) \geq \operatorname{vol}(D_Y)$, proving (1).

Set $r = \dim\bigl(p(\operatorname{Supp} E)\bigr)$. For any $\mathbb{Q}$-Cartier divisors A_i on X the intersection number $\bigl(p^*A_1 \cdots p^*A_j \cdot E\bigr)$ vanishes whenever $j > r$. Thus, if $j > r$ then

$$\bigl(p^*H^j \cdot D_Y^{n-j}\bigr) = \bigl(p^*H^j \cdot (p^*D_X - E)^{n-j}\bigr) = \bigl(p^*H^j \cdot p^*D_X^{n-j}\bigr) = \bigl(H^j \cdot D_X^{n-j}\bigr)$$

and for $j = r$ we get that

$$\bigl(p^*H^r \cdot D_Y^{n-r}\bigr) = \bigl(H^r \cdot D_X^{n-r}\bigr) + \bigl(p^*H^r \cdot (-E)^{n-r}\bigr).$$

Thus we need to understand $\bigl(p^*H^r \cdot (-E)^{n-r}\bigr)$. We may assume that H is very ample. Intersecting with p^*H is then equivalent to restricting to the preimage of a general member of $|H|$. Using this r-times, we get a birational morphism $p' : Y' \to X'$ between varieties of dimension $n - r$ and an effective, nonzero, p-exceptional $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor E' such that $-E'$ is p'-nef and $p'(E'_i)$ is 0-dimensional. Thus, by (10.31), $\bigl(p^*H^r \cdot (-E)^{n-r}\bigr) = (-E')^{n-r} < 0$ which proves (3–4).

If D_X is ample then we can use this for $H := D_X$. Then $\bigl(H^r \cdot D_X^{n-r}\bigr) = \bigl(D_X^n\bigr)$ and we get (2). □

Lemma 10.31. *Let $p : Y \to X$ be a proper, birational morphism of normal schemes. Let E be an effective, nonzero, p-exceptional $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $p(E)$ is 0-dimensional and $-E$ is p-nef. Set $n = \dim E$.*

Then $-(-E)^{n+1} = \bigl(-E|_E\bigr)^n > 0$.

Proof. Assume that there is an effective, nonzero, p-exceptional $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor F such that $p(F) = p(E)$, $-F$ is p-nef and $-(-F)^{n+1} > 0$. Note that E, F have the same support, namely $p^{-1}\bigl(p(E)\bigr)$, thus $E - \epsilon F$ is effective for $0 < \epsilon \ll 1$. Thus $0 > (-\epsilon F)^n \geq (-E)^n$ by (10.32).

Such a divisor F exists on the normalization of the blow-up $B_{p(E)}X$. Let now $Z \to X$ be a proper, birational morphism that dominates both Y and $B_{p(E)}X$. We can apply the above observation to the pull-backs of E and F to Z. □

Lemma 10.32. *Let N_1, N_2 be $\mathbb{Q}$-Cartier divisors with proper support on an n-dimensional scheme. Assume that there exists an effective divisor with proper support E such that $E \sim_{\mathbb{Q}} N_1 - N_2$ and the $N_i|_E$ are both nef. Then $(N_1^n) \geq (N_2^n)$.*

Proof. $(N_1^n) - (N_2^n) = E \cdot \sum_{i=0}^{n-1} N_i^i N_2^{n-1-i}$. □

The next results compare the volumes of different perturbations of the canonical divisor.

Lemma 10.33. *Let X be a normal, proper variety of dimension n and D an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier, nef and big. Let Y be a smooth, proper variety birational to X. Then*

(1) $\operatorname{vol}(K_Y) \leq \bigl(K_X + D\bigr)^n$ *and*
(2) *equality holds iff $D = 0$ and X has canonical singularities.*

Proof. Let Z be a normal, proper variety birational to X such that there are morphisms $q : Z \to Y$ and $p : Z \to X$. Write

$$K_Z \sim_{\mathbb{Q}} q^*K_Y + E \quad \text{and} \quad K_Z \sim_{\mathbb{Q}} p^*(K_X + D) - p_*^{-1}D + F \tag{10.33.3}$$

where E is effective, q-exceptional and F is p-exceptional (not necessarily effective). Thus

$$q^*K_Y \sim_{\mathbb{Q}} p^*(K_X + D) - p_*^{-1}D + F - E. \tag{10.33.4}$$

Write $F-E=G^+-G^-$ where G^+,G^- are effective and without common irreducible components. Note that G^+ is p-exceptional. If $m>0$ is sufficiently divisible then

$$H^0\big(Z,\mathcal{O}_Z\big(mp^*(K_X+D)+mG^+\big)\big)=H^0\big(Z,\mathcal{O}_Z\big(mp^*(K_X+D)\big)\big)$$

and hence also

$$\begin{aligned}&H^0\big(Z,\mathcal{O}_Z\big(mp^*(K_X+D)-p_*^{-1}(mD)+mG^+-mG^-\big)\big)\\&\quad=H^0\big(Z,\mathcal{O}_Z\big(mp^*(K_X+D)-p_*^{-1}(mD)-mG^-\big)\big).\end{aligned}$$

Thus

$$\begin{array}{rcl}\operatorname{vol}(K_Y) &=& \operatorname{vol}\big(p^*(K_X+D)-p_*^{-1}D+G^+-G^-\big)\\ &=& \operatorname{vol}\big(p^*(K_X+D)-p_*^{-1}D-G^-\big)\\ &\leq& \operatorname{vol}\big(p^*(K_X+D)\big)\\ &=& \big(p^*(K_X+D)\big)^n=(K_X+D)^n.\end{array}$$

Furthermore, by (10.37) equality holds iff $p_*^{-1}D+G^-=0$, that is, when $D=0$ and $G^-=0$. In such a case (10.33.4) becomes

$$q^*K_Y\sim_{\mathbb{Q}}p^*K_X+G^+\quad\text{and } G^+\text{ is effective.}$$

Thus $a(E,X)\geq a(E,Y)$ for every divisor E (cf. [**Kol13c**, 2.5]), hence X has canonical singularities. □

A similar birational statement does not hold for pairs in general, but a variant holds if Y is a resolution of X. We can also add some other auxiliary divisors; these are needed in our applications.

LEMMA 10.34. *Let X be a normal, proper variety of dimension n and Δ a reduced, effective $\mathbb{Q}$-divisor on X. Let A be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and D an effective $\mathbb{Q}$-divisor such that $K_X+\Delta+A+D$ is $\mathbb{Q}$-Cartier, nef and big. Let $p:Y\to X$ be any log resolution of (X,Δ). Then*

(1) $\operatorname{vol}(K_Y+p_*^{-1}\Delta+p^*A)\leq\big(K_X+\Delta+A+D\big)^n$ *and*
(2) *equality holds iff $D=0$ and (X,Δ) is canonical.*

Proof. As usual, write

$$K_Y+p_*^{-1}\Delta\sim_{\mathbb{Q}}p^*(K_X+\Delta+D)-p_*^{-1}D-F_1+F_2 \tag{10.34.3}$$

where the F_i are p-exceptional, effective and without common irreducible components. If $m>0$ is sufficiently divisible then

$$H^0\big(Y,\mathcal{O}_Y\big(mp^*(K_X+\Delta+A+D)+mF_2\big)\big)=H^0\big(Y,\mathcal{O}_Y\big(mp^*(K_X+\Delta+A+D)\big)\big)$$

and hence also

$$\begin{aligned}&H^0\big(Y,\mathcal{O}_Y\big(mp^*(K_X+\Delta+A+D)-p_*^{-1}(mD)-mF_1+mF_2\big)\big)\\&\quad=H^0\big(Y,\mathcal{O}_Y\big(mp^*(K_X+\Delta+A+D)-p_*^{-1}(mD)-mF_1\big)\big).\end{aligned}$$

Thus

$$\begin{array}{rcl}\operatorname{vol}(K_Z+p_*^{-1}\Delta+p^*A) &=& \operatorname{vol}\big(p^*(K_X+\Delta+A+D)-p_*^{-1}D+F_2-F_1\big)\\ &=& \operatorname{vol}\big(p^*(K_X+\Delta+A+D)-p_*^{-1}D-F_1\big)\\ &\leq& \operatorname{vol}\big(p^*(K_X+\Delta+A+D)\big)\\ &=& \big(p^*(K_X+\Delta+A+D)\big)^n=(K_X+\Delta+A+D)^n.\end{array}$$

Furthermore, by (10.37) equality holds iff $p_*^{-1}D+F_1=0$, that is, when $D=0$ and $F_1=0$. Thus (10.34.3) becomes

$$K_Z+p_*^{-1}\Delta\sim_{\mathbb{Q}}p^*(K_X+\Delta)+F_2$$

where F_2 is effective. This says that (X,Δ) is canonical. □

Essentially the same argument gives the following log canonical version.

LEMMA 10.35. *Let X be a normal, proper variety of dimension n, Δ a reduced, effective $\mathbb{Q}$-divisor on X and A a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Let $q : \bar{X} \to X$ be a proper birational morphism, $\bar{E}$ the reduced g-exceptional divisor, $\bar{\Delta} := q_*^{-1}\Delta$ and $\bar{D}$ an effective $\mathbb{Q}$-divisor on $\bar{X}$ such that $K_{\bar{X}} + \bar{\Delta} + \bar{E} + D + q^*A$ is $\mathbb{Q}$-Cartier, nef and big. Let $p : Y \to X$ be any log resolution of singularities with reduced exceptional divisor E. Then*

(1) $\operatorname{vol}(K_Y + p_*^{-1}\Delta + E + p^*A) \leq \big(K_{\bar{X}} + \bar{\Delta} + \bar{E} + \bar{D} + q^*A\big)^n$ *and*
(2) *equality holds iff $\bar{D} = 0$ and $(\bar{X}, \bar{\Delta} + \bar{E})$ is log canonical.* □

We have also used the following elementary estimate.

LEMMA 10.36. *Let $p : Y \to X$ be a separable, generically finite morphism between smooth, proper varieties. Then* $\operatorname{vol}(K_Y) \geq \deg(Y/X) \cdot \operatorname{vol}(K_X)$.

Proof. This is obvious if $\operatorname{vol}(K_X) = 0$, hence we may assume that K_X is big. Pulling back differential forms gives a natural map $p^*\omega_X \to \omega_Y$. This gives an injection

$$\omega_X^r \otimes p_*\omega_Y \hookrightarrow p_*\big(\omega_Y^{r+1}\big).$$

Since $p_*\omega_Y$ has rank $\deg(Y/X)$ and K_X is big, $H^0\big(X, \omega_X^r \otimes p_*\omega_Y\big)$ grows at least as fast as $\deg(Y/X) \cdot H^0\big(X, \omega_X^r\big)$. □

The following result describes the variation of the volume near a nef and big divisor. The assertions are special cases of [**FKL16**, Thms.A–B].

THEOREM 10.37. *Let X be a proper variety, L a big $\mathbb{Q}$-Cartier divisor and E a nonzero effective divisor. The following are equivalent.*

(1) $\operatorname{vol}(L - E) = \operatorname{vol}(L)$ *and*
(2) $H^0\big(\mathcal{O}_X(\lfloor mL - mE \rfloor)\big) = H^0\big(\mathcal{O}_X(\lfloor mL \rfloor)\big)$ *for every $m \geq 0$.*

If L is nef then these are further equivalent to

(3) $E = 0$.

Note that the implications (3) ⇒ (2) ⇒ (1) are clear but the converse is somewhat surprising. It says that although the volume measures only the asymptotic growth of the Hilbert function, one can not change the Hilbert function without changing the volume. For proofs see the original paper.

10.5. Double points

We used a variety of results about hypersurface double points. For the rest of the section we work with rings R that contain $\frac{1}{2}$. In this case, all the definitions that we have seen are equivalent to the ones given below. If $\frac{1}{2} \notin R$, there are differing conventions, especially if $\operatorname{char} R = 2$.

The following results on normal forms, deformations and resolutions of double points are well known, but not easy to find in one place.

DEFINITION 10.38. A *quadratic form* over a field k is a degree 2 homogeneous polynomial $q(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$. The *rank* of q is defined either as the dimension of the space spanned by the derivatives

$$\Big\langle \tfrac{\partial q}{\partial x_1}, \dots, \tfrac{\partial q}{\partial x_n} \Big\rangle, \tag{10.38.1}$$

or as the rank of the Hessian matrix

$$\operatorname{Hess}(q) := \left(\tfrac{\partial^2 q}{\partial x_i \partial x_j}\right), \tag{10.38.2}$$

or as the number of variables in any diagonalized form

$$q = a_1 y_1^2 + \cdots + a_r y_r^2 \quad \text{where} \quad a_i \in k^*. \tag{10.38.3}$$

More abstractly, if V is a k-vector space, we can think of q as an element of its symmetric square $S^2 V$. (With this convention, $V = \langle x_1, \dots, x_n\rangle$. It is also natural to think of $W := \operatorname{Spec}_k k[x_1, \dots, x_n]$ to be the basic object, then quadratic forms are elements of $S^2(W^*)$.)

Definition 10.39. Let (S, m) be a regular local ring with residue field k such that $\operatorname{char} k \neq 2$. We can identify m^2/m^3 with $S^2(m/m^2)$. Thus, for any $g \in m^2$, we can view $g_2 := g + m^3 \in m^2/m^3$ as a quadratic form.

We say that $S/(g)$ is a *double point* if $g_2 \neq 0$, a *cA point* if $\operatorname{rank} g_2 \geq 2$ and an *ordinary double point* if $\operatorname{rank} g_2 = \dim_k m/m^2$.

An ordinary double point is also called a *node,* especially if $\dim S/(g) = 1$.

10.40 (cA-singularities of hypersurfaces). Let Y be a smooth variety over a field of characteristic $\neq 2$ and $X = (g = 0) \subset Y$ a hypersurface. If $y_1, \dots, y_n$ are étale coordinates on Y then we can compute the Hessian as

$$\operatorname{Hess}_{\mathbf{y}}(g) = \left(\tfrac{\partial^2 g}{\partial y_i \partial y_j}\right). \tag{10.40.1}$$

Since the rank is lower semicontinuous, we see that, for every r,

$$\{p \in \operatorname{Sing} X : \operatorname{rank}_p g_2 \geq r\} \quad \text{is open in} \quad \operatorname{Sing} X. \tag{10.40.2}$$

For us the most interesting case is cA-singularities, that is, $r = 2$. The relative version of (10.40.2) is then the following.

Let $f : Y \to S$ be smooth and $X \subset Y$ a relative Cartier divisor. Then

$$\{p \in X : p \text{ is } cA \text{ (or smooth) on } X_{f(p)}\} \subset X \quad \text{is open.} \tag{10.40.3}$$

This implies that if $X \to S$ is proper and X_s has only cA-singularities (and smooth points) outside a closed subset $Z_s \subset X_s$ of codimension $\geq m$ for some $s \in S$ then then same holds in an open neighborhood $s \in S^0 \subset S$.

Corollary 10.41. *Let $\pi : X \to S$ be a flat and pure dimensional morphism. Then the set of points $\{x : X_{\pi(x)}$ is demi-normal at $x\}$ is open in X.*

Proof. Being S_2 is an open condition by (10.3). A proper S_1 scheme is geometrically reduced iff it is generically smooth and smoothness is an open condition. Thus being S_2 and geometrically reduced is an open condition.

It remains to show that having only nodes in codimension 1 is also an open condition. If all residue characteristics are $\neq 2$, this follows from (10.40.3) since having only cA-singularities in codimension 1 is an open condition.

See [**Kol13c**, 1.41] for the right definition and the universal deformation space of a node in characteristic 2. $\square$

Morse lemma.

Let f be a function on $\mathbb{R}^n$ that has an ordinary critical point at the origin. The Morse lemma says that in suitable local coordinates $y_1, \dots, y_n$ we can write f as $\pm y_1^2 \pm \cdots \pm y_n^2$; see [**Mil63**, p.6] and [**AGZV85b**, ??] for the precise differentiable and analytic versions. Algebraically the best is to work with formal power series.

LEMMA 10.42 (Formal Morse lemma). *Let k be a field of characteristic $\neq 2$ and $g \in k[[x_1, \dots, x_n]]$ a power series of multiplicity ≥ 2 such that $\operatorname{rank}_{\mathbf{0}} \operatorname{Hess}(g) = r$. Then there are local coordinates $y_1, \dots, y_n$ such that*

$$g = a_1 y_1^2 + \cdots + a_r y_r^2 + h(y_{r+1}, \dots, y_n),$$

where $a_i \in k^*$ and $\operatorname{mult} h \geq 3$.

We state and prove a more general version of this next.

Let (R, m) be a local ring and $g \in R[[x_1, \dots, x_n]]$ a power series. Reduction modulo m is denoted by $\bar{g}$. Thus $\bar{g}(x_1, \dots, x_n) \in (R/m)[[x_1, \dots, x_n]]$. We aim to understand those cases when $\operatorname{mult} \bar{g} = 2$. The next result is stated in a form that also works if $\operatorname{char}(R/m) = 2$.

LEMMA 10.43 (Formal Morse lemma with parameters). *Let (R, m) be a complete local ring and $G \in R[[x_1, \dots, x_N]]$ a power series of multiplicity ≥ 2. Assume that there is a quadratic form $q(x_1, \dots, x_n)$ such that*

(1) $\dim\langle \partial\bar{q}/\partial x_1, \dots, \partial\bar{q}/\partial x_n \rangle = n$ *and*
(2) $\bar{G}(x_1, \dots, x_n, 0, \dots, 0) - \bar{q}(x_1, \dots, x_n, 0, \dots, 0) \in (x_1, \dots, x_n)^3$.

Then there are local coordinates $y_1, \dots, y_n, x_{n+1}, \dots, x_N$ such that

(1) $y_i \equiv x_i \mod (x_1, \dots, x_N)^2 + m[[x_1, \dots, x_N]]$ *and*
(2) $G = q(y_1, \dots, y_n) + b$ *for some* $b \in (x_{n+1}, \dots, x_N)^3 + m[[x_{n+1}, \dots, x_N]]$.

Proof. Replacing R by $R[[x_{n+1}, \dots, x_N]]$ reduces everything to the case when $N = n$; we assume this from now on.

Let us start with the case when $R = k$ is a field. Set $x_{2,i} := x_i$. Assume inductively (starting with $r = 2$) that there are local coordinate systems $(x_{s,1}, \dots, x_{s,n})$ for $2 \leq s \leq r$ such that

$$\begin{array}{rcll} x_{s,i} & \equiv & x_{s-1,i} & \mod (x_1, \dots, x_n)^{s-1} \quad \text{and} \\ G & \equiv & q(x_{r,1}, \dots, x_{r,n}) & \mod (x_1, \dots, x_n)^{r+1}. \end{array}$$

Next we choose $x_{r+1,i} := x_{r,i} + h_{r,i}$ for suitable $h_{r,i} \in (x_1, \dots, x_n)^r$. Note that

$$q(x_{r+1,1}, \dots, x_{r+1,n}) = q(x_{r,1}, \dots, x_{r,n}) + \textstyle\sum_i h_{r,i} \frac{\partial q}{\partial x_i} \mod (x_1, \dots, x_n)^{2r}.$$

(We use this only modulo $(x_1, \dots, x_n)^{r+2}$.) Since q is nondegenerate,

$$\textstyle\sum_i \frac{\partial q}{x_i} (x_1, \dots, x_n)^r = (x_1, \dots, x_n)^{r+1}.$$

Thus we can choose the $h_{r,i}$ such that

$$G - q(x_{r+1,1}, \dots, x_{r+1,n}) \in (x_1, \dots, x_n)^{r+2}.$$

In the limit we get $(x_{\infty,1}, \dots, x_{\infty,n})$ as required.

Applying this $k = R/m$ we can assume from now on that

$$G - q(x_1, \dots, x_n) \in mR[[x_1, \dots, x_n]].$$

Working inductively (starting with $r = 1$) that there are local coordinate systems $(y_{s,1}, \dots, y_{s,n})$ for $2 \le s \le r$ such that

$$\begin{array}{lcll} y_{s,i} & \equiv & y_{s-1,i} \mod m^{s-1}R[[x_1,\dots,x_n]] & \text{and} \\ G & \equiv & q(y_{r,1},\dots,y_{r,n}) \mod m + m^r R[[x_1,\dots,x_n]]. \end{array}$$

Next we choose $y_{r+1,i} := y_{r,i} + c_{r,i}$ for suitable $c_{r,i} \in m^r R[[x_1,\dots,x_n]]$. Note that

$$q(y_{r+1,1},\dots,y_{r+1,n}) = q(y_{r,1},\dots,y_{r,n}) + \textstyle\sum_i c_{r,i}\frac{\partial q}{\partial x_i} \mod m^{2r}R[[x_1,\dots,x_n]].$$

(We use this only modulo $m^{r+1}R[[x_1,\dots,x_n]]$.) Since q is nondegenerate,

$$\textstyle\sum_i \frac{\partial q}{\partial x_i} m^r R[[x_1,\dots,x_n]] = (x_1,\dots,x_n)m^r R[[x_1,\dots,x_n]].$$

Thus we can choose the $c_{r,i}$ such that

$$G - q(y_{r+1,1},\dots,y_{r+1,n}) \in m + m^{r+1}R[[x_1,\dots,x_n]].$$

In the limit we get $(y_{\infty,1},\dots,y_{\infty,n})$ as required. □

In (1.28) we used various results on resolutions of double points of surfaces that contain a pair of lines and double points of 3–folds that contain a pair of planes. The normal forms can be obtained using the method of (10.43) but we did not follow how linear subvarieties transform under the (non-linear) coordinate changes used there. However, in the next examples one can be quite explicit about the coordinate changes and the resolutions.

10.44 (Ordinary double points of surfaces). Let $S := \bigl(h(x_1,x_2,x_3) = 0\bigr) \subset \mathbb{A}^3$ be a surface with an ordinary double point at the origin that contains the pair of lines $(x_1x_2 = x_3 = 0)$. Then h can be written as

$$h = f(x_1,x_2,x_3)x_1x_2 - g(x_1,x_2,x_3)x_3.$$

If the quadratic part has rank 3 then $f(0,0,0) \ne 0$ and we can write $g = x_1g_1 + x_2g_2 + x_3g_3$ for some polynomials g_i. Thus

$$h = f\bigl(x_1 - f^{-1}g_1x_3\bigr)\bigl(x_2 - f^{-1}g_2x_3\bigr) - \bigl(g_3 + f^{-1}g_1g_2\bigr)x_3^2.$$

Here $g_3 + f^{-1}g_1g_2$ is nonzero at $(0,0,0)$ and we can set

$$y_1 := x_1 - f^{-1}g_1x_3,\ y_2 := f\bigl(x_2 - f^{-1}g_2x_3\bigr)\bigl(g_3 + f^{-1}g_1g_2\bigr)^{-1} \quad \text{and} \quad y_3 := x_3$$

to bring the equation to the normal form $S = (y_1y_2 - y_3^2 = 0)$. The pair of lines is still $(y_1y_2 = y_3 = 0)$.

Now we consider 3 ways of resolving the singularity of X. First, one can blow up the origin $0 \in \mathbb{A}^3$. We get

$$B_0\mathbb{A}^3 \subset \mathbb{A}^3_{\mathbf{y}} \times \mathbb{P}^2_{\mathbf{s}}$$

defined by the equations $\{y_is_j = y_js_i : 1 \le i,j \le 3\}$. Besides these equations, B_0S is defined by $y_1y_2 - y_3^2 = s_1s_2 - s_3^2 = y_1s_2 - y_3s_3 = s_1y_2 - y_3s_3 = 0$.

One can also blow up (y_1,y_3). We get

$$B_{(y_1,y_3)}\mathbb{A}^3 \subset \mathbb{A}^3_{\mathbf{y}} \times \mathbb{P}^1_{u_1u_3}$$

defined by the equation $y_1u_3 = y_3u_1$. Besides this equation, $B_{(y_1,y_3)}S$ is defined by $y_1y_2 - y_3^2 = u_1y_2 - u_3y_3 = 0$.

These two blow-ups are actually isomorphic, as shown by the embedding

$$\mathbb{A}^3_{\mathbf{y}} \times \mathbb{P}^1_{u_1u_3} \hookrightarrow \mathbb{A}^3_{\mathbf{y}} \times \mathbb{P}^2_{\mathbf{s}} \quad : \quad \bigl((y_1,y_2,y_3),(u_1{:}u_3)\bigr) \mapsto \bigl((y_1,y_2,y_3),(u_1^2{:}u_3^2{:}u_1u_3)\bigr)$$

restricted to $B_{(y_1,y_3)}S$.

The same things happen if we blow up (y_2, y_3).

10.45 (Ordinary double points of 3-folds). Let $X := \bigl(h(x_1,\dots,x_4)=0\bigr) \subset \mathbb{C}^4$ be a hypersurface with an ordinary double point at the origin that contains the pair of planes $(x_1x_2 = x_3 = 0)$. Then h can be written as

$$h = f(x_1,\dots,x_4)x_1x_2 - g(x_1,\dots,x_4)x_3.$$

The quadratic part has rank 4 iff $f(0,\dots,0) \neq 0$ and x_4 appears in g with nonzero coefficient. In this case we can set

$$y_i := x_i \quad \text{for } i = 1,2,3, \text{ and} \quad y_4 := f^{-1}g$$

to bring the equation to the normal form $X = (y_1y_2 - y_3y_4 = 0)$. The original pair of planes is still $(y_1y_2 = y_3 = 0)$.

Now we consider 3 ways of resolving the singularity of X. First, one can blow up the origin $0 \in \mathbb{A}^4$. We get

$$B_0\mathbb{A}^4 \subset \mathbb{A}^4_{\mathbf{y}} \times \mathbb{P}^3_{\mathbf{s}}$$

defined by the equations $\{y_is_j = y_js_i : 1 \leq i,j \leq 4\}$ and $p : B_0X \to X$ by the additional equations

$$y_1y_2 - y_3y_4 = s_1s_2 - s_3s_4 = y_is_{3-i} - y_js_{7-j} = 0 \ : i \in \{1,2\}, j \in \{3,4\}.$$

The exceptional set is the smooth quadric $(s_1s_2 = s_3s_4) \subset \mathbb{P}^3$ lying over the origin $0 \in \mathbb{A}^4$.

One can also blow up (y_1, y_3). We get

$$B_{(y_1,y_3)}\mathbb{A}^4 \subset \mathbb{A}^4_{\mathbf{y}} \times \mathbb{P}^1_{u_1u_3}$$

defined by the equation $y_1u_3 = y_3u_1$. Besides this equation, $B_{(y_1,y_3)}X$ is defined by $y_1y_2 - y_3y_4 = u_1y_2 - u_3y_4 = 0$. The exceptional set is the smooth rational curve $E \cong \mathbb{P}^1_{u_1u_3}$ lying over the origin $0 \in \mathbb{A}^4$.

Note furthermore that the birational transform P^*_{24} of the plane $P_{24} := (y_2 = y_4 = 0)$ is the blown-up plane B_0P_{24}, but the birational transform P^*_{14} of the plane $P_{14} := (y_1 = y_4 = 0)$ is the plane $(y_1 = u_1 = 0)$. The latter intersects E at the point $(u_1 = 0) \in E$, thus $\bigl(P^*_{14} \cdot E\bigr) = 1$. Since $P^*_{14} + P^*_{24}$ is the pull-back of the Cartier divisor $(y_4 = 0)$, it has 0 intersection number with E. Thus $\bigl(P^*_{24} \cdot E\bigr) = -1$.

We claim that the rational map $p : \mathbb{A}^4_{\mathbf{y}} \times \mathbb{P}^3_{\mathbf{s}} \dashrightarrow \mathbb{A}^4_{\mathbf{y}} \times \mathbb{P}^1_{\mathbf{u}}$ given by

$$p_1 : (y_1,\dots,y_4, s_1{:}\cdots{:}s_4) \mapsto (y_1,\dots,y_4, s_1{:}s_3)$$

gives a morphism $p_1 : B_0X \to B_{(y_1,y_3)}X$.

To see this note that the quadric $Q := (s_1s_2 - s_3s_4 = 0)$ is isomorphic to $\mathbb{P}^1_{\mathbf{u}} \times \mathbb{P}^1_{\mathbf{v}}$, with the isomorphism given as

$$j : \bigl((u_0{:}u_1), (v_0{:}v_1)\bigr) \mapsto \bigl(u_0v_0{:}u_0v_1{:}u_1v_0{:}u_1v_1\bigr).$$

Thus the map $(s_1{:}\cdots{:}s_4) \mapsto (s_1{:}s_3)$ is the inverse of j followed by the 1st coordinate projection. Thus p_1 restricts to a morphism on $\mathbb{A}^4_{\mathbf{y}} \times Q$ and $B_0X \subset \mathbb{A}^4_{\mathbf{y}} \times Q$.

Similarly, we obtain $p_2 : B_0X \to B_{(y_2,y_3)}X$. Putting these together, we get an isomorphism

$$p_1 \times p_2 : B_0X \cong B_{(y_1,y_3)}X \times_X B_{(y_2,y_3)}X.$$

(The above considerations show that this is an isomorphism of reduced schemes, and this is all we need. However, by explicit computation, the right hand side is reduced, so we have a scheme theoretic isomorphism.) In particular, this shows that the two maps $p_i : B_{(y_i,y_3)}X \to X$ are not isomorphic to each other.

Finally, set $S := (y_3 = y_4) \subset X$. By the computations of (10.44), the p_i restrict to isomorphisms $p_i : B_0S \cong B_{(y_i,y_3)}S$. Thus $p^{-1}S = B_0S \cup E$ and B_0S is the graph of the isomorphism $p_2 \circ p_1^{-1} : B_{(y_1,y_3)}S \cong B_{(y_2,y_3)}S$.

10.6. Flatness criteria

Let $f : X \to S$ be a morphism that we would like to prove to be flat. If f is of finite type then flatness is an open property. Let $U \subset X$ denote the largest open set over which f is flat and set $Z := X \setminus U$. The situation is technically simpler if Z is a single closed point. To achieve this, one can use a Bertini-type theorem (10.46) to pass to a general hyperplane section of X and repeat if necessary. At the end we arrive at a finite type morphism $g : X' \to S$ that is flat except possibly at a finite set of points. Localizing at any one of them we have a local morphism of local schemes

$$f' : (x', X') \to (s', S')$$

that is flat over $X' \setminus \{x'\}$ and $k(x')/k(s')$ is finite field extension.

Alternatively, we can localize at a generic point of Z and then use (10.47) to reach the same situation.

If f is not of finite type then we have to be more careful since flatness is not an open property for morphisms of arbitrary Noetherian schemes. A morphism is flat iff it is flat at all points and the latter can be checked after localization. A local, Noetherian scheme is finite dimensional, there is thus a point of largest dimension where f is not flat. Localizing at that point we again get $f' : (x', X') \to (s', S')$ that is flat over $X' \setminus \{x'\}$.

If $k(x')/k(s')$ is finitely generated then we can again use (10.47) but the situation is more complicated in general. We wrangle with this issue in (10.73).

We can also complete X' and S', thus we are reduced to the case when we have a local morphism of complete, local, Noetherian schemes. Note, however, that some of our results hold only over base schemes that are normal, seminormal or reduced. These conditions are preserved by completion if S' is excellent but not in general.

PROPOSITION 10.46. *Let $(x, X) \to (s, S)$ be a local morphism of local schemes and F a coherent sheaf on X. Assume that $r \in m_{x,X}$ is a non-zerodivisor both on F and on F_s. Then F is flat over S iff F/rF is flat over S.*

Proof. By assumption we have an exact sequence

$$0 \to F \xrightarrow{r} F \to F/rF \to 0.$$

Tensoring with $k = k(s)$ gives

$$\operatorname{Tor}^1(k, F) \xrightarrow{r} \operatorname{Tor}^1(k, F) \to \operatorname{Tor}^1(k, F/rF) \to F_s \xrightarrow{r} F_s \to (F/rF)_s \to 0.$$

By the second assumption $r : F_s \to F_s$ is injective, hence we get a shorter exact sequence

$$\operatorname{Tor}^1(k, F) \xrightarrow{r} \operatorname{Tor}^1(k, F) \to \operatorname{Tor}^1(k, F/rF) \to 0.$$

F is flat over S iff $\operatorname{Tor}^1(k, F) = 0$ by the local criterion of flatness (see [**Mat86**, Sec.22] or [**Eis95**, Sec.6.4]) hence $\operatorname{Tor}^1(k, F/rF) = 0$ and so F/rF is flat over S. Conversely, if F/rF is flat over S then $\operatorname{Tor}^1(k, F/rF) = 0$ hence $r : \operatorname{Tor}^1(k, F) \to$

$\mathrm{Tor}^1(k,F)$ is surjective. Thus $\mathrm{Tor}^1(k,F)=0$ by the Nakayama lemma and so F is flat over S. □

10.47. Let $f : X \to S$ be a morphism, $x \in X$ a point and $s := f(x)$ its image. Assume that $c_1, \dots, c_n \in k(x)$ generate the field extension $k(x)/k(s)$. These define a point $s' \in \mathbb{A}^n_{k(s)}$ such that $k(x) \cong k(s')$.

Consider next the trivial lifting $f \times 1_n : X \times \mathbb{A}^n \to S \times \mathbb{A}^n$. We also have points $(s,s') \in S \times \mathbb{A}^n$ projecting to s and $(x,s') \in X \times \mathbb{A}^n$ projecting to x Thus we have a commutative diagram of pointed schemes

$$\begin{array}{ccc} (x,s') \in X \times \mathbb{A}^n & \xrightarrow{\pi_X} & x \in X \\ f \times 1_n \downarrow & & \downarrow f \\ (s,s') \in S \times \mathbb{A}^n & \xrightarrow{\pi_S} & s \in S \end{array} \tag{10.47.1}$$

such that $\pi_X^* : k(x) \to k(x,s')$ and $(f \times 1_n)^* : k(s,s') \to k(x,s')$ are isomorphisms.

The projections π_X, π_S are both smooth, hence flat. In particular

Claim 10.47.2. f is flat at x iff $f \times 1_n$ is flat at (x,s'). □

This reduces most flatness questions for local morphisms $f : (x,X) \to (s,S)$ with finitely generated residue field extension $k(x)/k(s)$ to the special case when the residue fields are isomorphic.

I do not know a similar simple trick that works for non-finitely generated residue field extensions. A different method, discussed in (10.73), applies whenever $k(x)/k(s)$ is separable, but non-finitely generated purely inseparable extensions cause numerous problems.

Flatness is usually easy to check if we know all the fibers of a morphism. For projective morphisms there are criteria using the Hilbert function; see [**Har77**, III.9.9] or (3.44). In the local case we have the following.

LEMMA 10.48. *Let S be a reduced scheme and $f : X \to S$ a morphism that is essentially of finite type, pure dimensional and its fibers are geometrically reduced. Then f is flat.*

Proof. By (3.46) it is enough to show this when (s,S) is the spectrum of a DVR. In this case f is flat iff none of the associated points of X is contained in X_s. By assumption X_s is reduced, so only generic points of X_s could occur. Then the corresponding irreducible component of X_s is also an irreducible component of X, but we also assumed that f has pure relative dimension. □

In many cases we have some information about the fibers, but we do not fully understand them. Thus we are looking for flatness criteria that do not require complete knowledge of the fibers.

10.49 (Format of flatness criteria). Let $\pi : X \to S$ be a morphism of Noetherian schemes, (s,S) local. Let F be a coherent sheaf on X and $Z \subset \operatorname{Supp} F_s$ a nowhere dense closed subset such that $F|_{X \setminus Z}$ is flat over S. We aim to prove various flatness theorems with assumptions

(1) on $\mathrm{pure}_Z(F_s)$,
(2) on $\mathrm{depth}_Z F$,
(3) on $F|_{X \setminus Z}$ and
(4) on S.

Our main focus is on $\operatorname{pure}_Z(F_s)$, the assumptions (2–3) are then chosen as needed.

Let G be another coherent sheaf on X such that $G|_{X\setminus Z}$ is flat over S and $X_s \cap \operatorname{Supp} G \subset Z$. Then $\operatorname{pure}_Z F_s = \operatorname{pure}_Z(F_s + G_s)$, so assumptions of type (1–2) do not give control over G. Thus we have to make sure that the assumptions of type (2) exclude G.

There are two ways of achieving this. Let $x_G \in \operatorname{Supp} G$ be a generic point and set $s_G := \pi(x_G)$. Then x_G is an associated point of the fiber $(F+G)_{s_G}$ and $X_s \cap \overline{x_G} \subset Z$. Thus the presence of G is excluded by the assumption: $\operatorname{depth}_x\bigl(F_{\pi(x)}\bigr) \geq 1$ whenever $X_s \cap \bar{x} \subset Z$. A similar argument with extensions suggests that we frequently need the stronger variant

(3') $\operatorname{depth}_x\bigl(F_{\pi(x)}\bigr) \geq 2$ whenever $X_s \cap \bar{x} \subset Z$.

This is a quite mild restriction and probably the best one can do for Noetherian schemes. It has a geometrically transparent reformulation if there is a "good" dimension function for $\pi : X \to S$. A precise definition is not important, we mean by this a function $x \mapsto \dim_S x$ that is upper semicontinuous on X, strictly decreasing under specialization and if a coherent sheaf G is flat over $S^0 \subset S$ then $s \mapsto \operatorname{Supp}\{\dim_S x : x \in \operatorname{Supp}(G_s)\}$ is locally constant on S^0.

The prime example for this is the usual dimension function if $X \to S$ is of finite type. If we have a "good" dimension function and $\operatorname{codim}(Z, X_s) \geq 2$ then we can replace (3') by the more convenient assumption

(3") the fibers of F over $S \setminus \{s\}$ are pure and S_2.

The key step in the following proofs is to exclude $\{x\}$ as an associated point of various sheaves associated to $X \to S$. We start with the case when $\dim X_s = 0$. With each increase of $\dim X_s$ the results become more general.

Flatness in relative codimension 0.

The basic result is the following, proved in [**Gro71**, II.2.3].

PROPOSITION 10.50. *Let $f : (x, X) \to (s, S)$ be a local morphism of local, Noetherian schemes of the same dimension such that $f^{-1}(s) = x$ holds scheme theoretically, that is, $m_{x,X} = m_{s,S}\mathcal{O}_X$. Assume that*

(1) *$k(x) \supset k(s)$ is separable and*
(2) *$\hat{S}$, the completion of S, is normal.*

Then f is flat at x.

Note that if S is normal and excellent then $\hat{S}$ is normal.

Proof. We may replace S and X by their completions. As we discuss in (10.73), we can factor f as

$$f : (x, X) \xrightarrow{p} (y, Y) \xrightarrow{q} (x, S)$$

where (y, Y) is also complete, local, Noetherian, $k(x) = k(y)$, $m_{x,X} = m_{s,S}\mathcal{O}_X$ and q is flat.

Thus $f^* : m_{y,Y}/m_{y,Y}^2 \to m_{x,X}/m_{x,X}^2$ is surjective, hence $p^* : \mathcal{O}_Y \to \mathcal{O}_X$ is surjective by the Nakayama lemma. Equivalently, $p : X \to Y$ is a closed embedding. It is thus an isomorphism, provided $\dim X = \dim Y$ and Y is integral.

In order to ensure these properties of Y we need to know more about q. If $k(x)/k(s)$ is finitely generated then q is the localization of a smooth morphism (10.73.3). Thus Y is normal and $\dim Y = \dim S$, as required. The general case is technically harder. We use that q is formally smooth and geometrically regular (10.73.4) to reach the same conclusions as before.

Thus p is an isomorphism, so $f = q$ and f is flat. $\square$

The next examples show that the assumptions in (10.50), and later in (10.53), are necessary.

EXAMPLE 10.51. 10.51.1. Assume that $\operatorname{char} k \neq 2$ and set $C := (y^2 = ax^2 + x^3)$ where $a \in k$ is not a square. Let $f : \bar{C} \to C$ denote the normalization. Then the fiber over the origin is the spectrum of $k(\sqrt{a})$, which is a separable extension of k. Here C is not normal and f is not flat.

10.51.2. The extension $\mathbb{C}[x, y] \subset \mathbb{C}[\frac{x}{y}, y]_{(y)}$ is not flat yet $(x, y) \cdot \mathbb{C}[\frac{x}{y}, y]_{(y)}$ is the maximal ideal and the residue field extension is purely transcendental. However, the dimension of the larger ring is 1.

A similar thing happens with the injection $\mathbb{C}[x, y] \hookrightarrow \mathbb{C}[[t]]$ given by $(x, y) \mapsto (t, \sin t)$. The fiber over the origin is the origin with reduced scheme structure.

10.51.3. On $\mathbb{C}[x, y]$ consider the involution $\tau(x) = -x, \tau(y) = -y$. The invariant ring is $\mathbb{C}[x^2, xy, y^2] \subset \mathbb{C}[x, y]$. The fiber over the origin is $\mathbb{C}[x, y]/(x^2, xy, y^2)$; it has length 3 and embedding dimension 2. The fiber over any other point has length 2. Thus the extension is not flat.

10.51.4. As in [**Kol95a**, 15.2], on $S := k[x_1, x_2, y_1, y_2]$ consider the involution $\tau(x_1, x_2, y_1, y_2) = (x_2, x_1, y_2, y_1)$. The ring of invariants is

$$R := k[x_1 + x_2, x_1x_2, y_1 + y_2, y_1y_2, x_1y_1 + x_2y_2].$$

The resulting extension is not flat along the invariant locus $(x_1 - x_2 = y_1 - y_2 = 0)$.

If $\operatorname{char} k = 2$ then $x_1 - x_2, y_1 - y_2$ are invariants. Set $P := (x_1 - x_2, y_1 - y_2)R$. Then $S/PS = S/(x_1 - x_2, y_1 - y_2)S \cong k[x_1, y_1]$ and $R/P \cong k[x_1^2, y_1^2]$.

Thus $S_P \supset R_p$ is a finite extension whose fiber over P is $k(x_1, y_1) \supset k(x_1^2, y_1^2)$. This is an inseparable field extension, generated by 2 elements.

These examples leave open only one question: what happens with curvilinear fibers.

DEFINITION 10.52 (Curvilinear schemes). Let k be a field and (A, m) a local, artinian k-algebra. We say that $\operatorname{Spec}_k A$ is *curvilinear* if A is cyclic as a $k[t]$-module for some t. That is, if A can be written as a quotient of $k[t]$. It is easy to see that this holds if

(1) either A/m is a finite, separable extension of k and m is a principal ideal,
(2) or A is a field extension of k of degree $= \operatorname{char} k$.

Let B be a semi-local artinian k-algebra. Then $\operatorname{Spec}_k B$ is called *curvilinear* if all of its irreducible components are curvilinear. If k is an infinite field, this holds iff B can be written as a quotient of $k[t]$. If K/k is a separable field extension and $\operatorname{Spec}_k B$ is curvilinear then so is $\operatorname{Spec}_K(B \otimes_k K)$.

Let $\pi : X \to S$ be a finite type morphism. The embedding dimension of fibers is upper semicontinuous, thus the set $\{x \in X : X_{\pi(x)} \text{ is curvilinear at } x\}$ is open.

THEOREM 10.53. *Let $f : X \to S$ be a finite type morphism with curvilinear fibers. Assume that S is normal and every associated point of X dominates S. Then f is flat.*

Proof. We start with the classical case when X, S are complex analytic, f is finite and $X \subset S \times \mathbb{C}$. Let $s \in S^{\text{ns}}$ be a smooth point. Then $S \times \mathbb{C}$ is smooth along

$\{s\} \times \mathbb{C}$ thus X is a Cartier divisor near X_s. In particular, f is flat over S^{ns}. Set $d := \deg f$. For each $s \in S^{\mathrm{ns}}$ there is a unique monic polynomial $t^d + a_{d-1}(s)t^{d-1} + \cdots + a_0(s)$ of degree d whose zero set is precisely $X_s \subset \mathbb{C}$. As in the proof of the analytic form of the Weierstrass preparation theorem (see, for instance, [**GH94**, p.8] or [**GR65**, Sec.II.B]) we see that the $a_i(s)$ are analytic functions on S^{ns}. By Hartogs's theorem they extend to analytic functions on the whole of S; we denote these still by $a_i(s)$. Thus

$$X = \bigl(t^d + a_{d-1}(s)t^{d-1} + \cdots + a_0(s) = 0\bigr) \subset S \times \mathbb{C}$$

is a Cartier divisor and f is flat. This completes the complex analytic case.

In general we argue similarly but replace the polynomial $t^d + a_{d-1}(s)t^{d-1} + \cdots + a_0(s)$ by the point in the Hilbert scheme corresponding to X_s.

Assume first that f is finite. Again set $d := \deg f$ and let $S^0 \subset S$ denote a dense open subset over which f is flat. Since f is finite, it is (locally) projective, thus we have

$$\begin{array}{ccc} \operatorname{Univ}_d(X/S) & \xrightarrow{p} & X \\ \downarrow & & \downarrow f \\ \operatorname{Hilb}_d(X/S) & \xrightarrow{\pi} & S \end{array}$$

parametrizing length d quotients of the fibers of f. If $s \in S^0$ then $\mathcal{O}_{X_s}$ has length d, hence its sole length d quotient is itself. Thus π is an isomorphism over S^0.

Let now $s \to S$ be an arbitrary geometric point. Then

$$X_s \cong \operatorname{Spec} k(s)[t]/\bigl(\textstyle\prod_i (t - a_i)^{m_i}\bigr)$$

for some $a_i \in k(s)$ and $m_i \in \mathbb{N}$. Thus the fiber of p over s is a finite set corresponding to length d quotients of $k(s)[t]/\bigl(\prod_i (t - a_i)^{m_i}\bigr)$, equivalently, to solutions of the equation $\sum_i m_i' = d$ where $0 \le m_i' \le m_i$. We have not yet proved that $\operatorname{Hilb}_d(X/S)$ has no embedded points over $\operatorname{Sing} S$, but we obtain that $\operatorname{pure}\bigl(\operatorname{Hilb}_d(X/S)\bigr) \to S$ is finite and birational, hence an isomorphism since S is normal. Furthermore, the natural map

$$\operatorname{pure}(p) : \operatorname{Univ}_d(X/S) \times_{\operatorname{Hilb}_d(X/S)} \operatorname{pure}\bigl(\operatorname{Hilb}_d(X/S)\bigr) \to X$$

is a closed immersion whose image is isomorphic to X over S^0. Thus $\operatorname{pure}(p)$ is an isomorphism and so f is flat and $\operatorname{Hilb}_d(X/S) \cong S$.

Finally we reduce the general case to the finite one. (Note that any finite type, quasi-finite morphism can be extended to a finite morphism, but there is no reason to believe that the extension still has curvilinear fibers.)

Flatness is a local question on X, thus pick $x \in X$. Set $s := f(x)$ and use (10.73.6) to get a diagram

$$\begin{array}{ccc} (x', X') & \xrightarrow{\pi} & (x, X) \\ g \downarrow & & \downarrow f \\ (y, Y) & \xrightarrow{h} & (s, S), \end{array}$$

where π is étale, g is finite, $g^{-1}(y) = \{x'\}$ and h is also étale. The latter implies that (y, Y) is also normal. As we noted in (10.52), $h \circ g = f \circ \pi$ has curvilinear fibers since π is étale. Thus g has curvilinear fibers, hence g is flat by the already established finite morphism case. Therefore f is also flat at x. $\square$

REMARK 10.54. Let $f : X \to S$ be a finite morphism with curvilinear fibers such that every associated point of X dominates a generic point of S. Assume that it has the same degree d over all generic points of S. The above proof shows that $\pi : \operatorname{pure}\bigl(\operatorname{Hilb}_d(X/S)\bigr) \to S$ is

- finite and birational over each irreducible component of S and
- an isomorphism over the locus where f is flat.

There are 3 further cases where one can conclude that π is an isomorphism. The first follows from (9.7).

Claim 10.54.1. Assume in addition that there is a closed subset $W \subset S$ such that f is flat over $S \setminus W$ and $\operatorname{depth}_W S \geq 2$. Then f is flat. □

Claim 10.54.2. Assume in addition that S is weakly normal and f is a well defined family of 0-cycles (3.19). Then f is flat.

Proof. By (4.16.8) the assumptions implies that there is a unique subscheme $[X_s]$ of X_s—possibly defined over a purely inseparable extension of $k(s)$—whose class is the Cayley-Chow fiber of $f : X \to S$. Thus $\pi : \operatorname{pure}\bigl(\operatorname{Hilb}_d(X/S)\bigr) \to S$ is geometrically injective and an isomorphism over a dense, open set of S. Since S is weakly normal, these imply that π is an isomorphism (3.29). □

Claim 10.54.3. Assume in addition that S is seminormal and f is a well defined family of 0-cycles that satisfies the field of definition condition (3.19). Then f is flat.

Proof. As before we get that $\pi : \operatorname{pure}\bigl(\operatorname{Hilb}_d(X/S)\bigr) \to S$ is geometrically injective and an isomorphism over a dense, open set of S. The field of definition condition says that the fiber over s is defined over $k(s)$. Since S is seminormal, π is an isomorphism (3.24). □

This result has a quite interesting consequence for unique factorization in power series rings. The first example of a unique factorization domain A such that $A[[t]]$ is not a UFD was constructed in [**Sam61**] and the situation was further clarified in [**Sto69, Dan70**]. An example of a complete, local UFD A such that $A[[t]]$ is not a UFD is given in (10.58). By contrast, the following result shows that many height 1 ideals in $A[[t]]$ are principal.

THEOREM 10.55 (Principal ideals in power series rings). *Let (R, m) be a normal, complete, local ring and $P \subset R[[x_1, \dots, x_n]]$ a height 1 prime ideal that is not contained in $mR[[x_1, \dots, x_n]]$. Then P is principal.*

Proof. We start with the case $n = 1$. Then $R[[x]]/P$ is finite over R and its fibers are curvilinear. Thus $R[[x]]/P$ is flat over R by (10.53). Since $(R/m)[[x]]$ is a DVR, the reduction of P modulo m is a principal ideal and its generator lifts back to a generator of P as in (4.20).

Next consider the case when $n \geq 2$ and R/m is infinite. Pick $g \in P \setminus mR[[x_1, \dots, x_n]]$ and let $\bar g \in (R/m)[[x_1, \dots, x_n]]$ denote its reduction by m. By assumption $\bar g$ is not identically zero, thus it contains a nonzero monomial $\bar a_I x^I$. Let $z_2, \dots, z_n$ be indeterminates and consider

$$\bar g(x_1, x_2 + z_2 x_1, \dots, x_n + z_n x_n).$$

The coefficient of $x_1^{|I|}$ is a nonzero polynomial of degree $\leq |I|$ in the variables z_i. If $|R/m| > |I|$ then we can choose $a_i \in R$ such that after the linear change of coordinates $y_1 = x_1,\ y_i = x_i + a_i x_1$, the power series $\bar g$ contains the monomial $y_1^{|I|}$. This means that

$$P \not\subset (m, y_2, \dots, y_n)R[[y_1, \dots, y_n]].$$

Set $R^* := R[[y_2, \dots, y_n]]$, its maximal ideal is $m^* := (m, y_2, \dots, y_n)$. Thus P is not contained in $m^* R^*[[y_1]]$. We have already proved that in this case P is principal.

If R/m is finite then (10.74.2) guarantees that there is a normal, local ring (R', m') faithfully flat over R such that $m' = mR'$ and $|R'/m'| > |I|$. We already proved that PR' is principal. Finally note that if PR' is principal then so is P by (4.22). □

Remark 10.56. The property proved in (10.55) almost characterizes normal rings. Indeed, let (R, m) be a non-normal, complete, local ring with algebraically closed residue field k. Then its normalization $(\bar R, \bar m)$ is also a complete, local ring with the same residue field k, so $\bar m \neq m$. Pick any $\bar r \in \bar m \setminus m$. Then $R[\bar r] \subset \bar R$ is a quotient of $R[[x]]$. The kernel of $R[[x]] \twoheadrightarrow R[\bar r]$ is a height 1, non-principal, prime ideal $P \subset R[[x]]$ which is not contained in $mR[[x]]$.

There is a strong similarity between (10.55) and Hensel's lemma. It is, however, not clear to us how to derive either one from the other.

Corollary 10.57 (Unique factorization in power series rings). *Let (R, m) be a normal, complete, local ring and $g \in R[[x_1, \dots, x_n]]$ a power series not contained in $mR[[x_1, \dots, x_n]]$. Then g has a unique factorization as $g = \prod_i p_i$ where each (p_i) is a prime ideal.*

Proof. Let P_i be a height 1 associated prime ideal of (g). Then (P_i) is not contained in $mR[[x_1, \dots, x_n]]$ thus it is principal by (10.55). □

Remark 10.58. A lemma of Gauss says that if R is a UFD then $R[t]$ is also a UFD. More generally, if Y is a normal scheme then $\mathrm{Cl}(Y \times \mathbb{A}^n) \cong \mathrm{Cl}(Y)$. If $\mathbb{A}^n$ is replaced by a smooth variety X then there is an obvious inclusion

$$\mathrm{Cl}(Y) \times \mathrm{Cl}(X) \hookrightarrow \mathrm{Cl}(Y \times X),$$

but, as the example below shows, this map is not surjective, not even if $\mathrm{Cl}(Y) = \mathrm{Cl}(X) = 0$.

Let $E \subset \mathbb{P}^2$ be a cubic defined over $\mathbb{Q}$ such that $\mathrm{Pic}(E)$ is generated by a degree 3 point $P := E \cap L$ for some line $L \subset \mathbb{P}^2$. Let $S \subset \mathbb{A}^3$ be the affine cone over E and $E^0 := E \setminus P$. Then $\mathrm{Cl}(S) = 0$ and $\mathrm{Cl}(E^0) = 0$. However, we claim that $\mathrm{Cl}(S \times E^0)$ is infinite.

To see this pick any $\phi \in \mathrm{End}(E)$. (For any m we have multiplication by $3m+1$ which sends $p \in E(\bar{\mathbb{Q}})$ to the unique point $\phi(p) \sim (3m+1)p - mP$.) The lines $\{\ell_p \times \{\phi(p)\} : p \in E\}$ sweep out a divisor in $S \times E$, where $\ell_p \subset S$ denotes the line over $p \in E$. It is not hard to see that this gives an isomorphism $\mathrm{End}(E) \cong \mathrm{Cl}(S \times E^0)$.

As another application, let R denote the complete local ring of S at its vertex. The above considerations also show that R is a UFD but $R[[t]]$ is not.

Flatness in relative codimension 1.

The following result is stated in all dimensions, but we will have even stronger theorems when the codimension is ≥ 2. The proof works well if $\mathrm{char}\, k(s) = 0$ or if the morphism is essentially of finite type, so we state the 2 versions separately.

THEOREM 10.59. *Let $f : X \to S$ be a morphism of Noetherian schemes, $s \in S$ a closed point and $Z \subset X_s$ a nowhere dense closed subset such that f is flat on $X \setminus Z$. Assume that*

(1) $\operatorname{pure}_Z(X_s)$ *is regular,*
(2) *if $X_s \cap \bar{x} \subset Z$ then $x \notin \operatorname{Ass}\bigl(X_{\pi(x)}\bigr)$,*
(3) $\operatorname{depth}_Z X \geq 1$,
(4) $\{s\}$ *is not an associated point of S and*
(5) $\operatorname{char} k(s) = 0$.

Then f is flat and X_s is regular.

THEOREM 10.60. *Let $f : X \to S$ be a morphism of Noetherian schemes that is essentially of finite type, $s \in S$ a closed point and $Z \subset X_s$ a nowhere dense closed subset such that f is flat on $X \setminus Z$. Assume that*

(1) $\operatorname{pure}_Z(X_s)$ *is geometrically regular,*
(2) *the fibers of f over $S \setminus \{s\}$ are pure and S_1,*
(3) $\operatorname{depth}_Z X \geq 1$ *and*
(4) $\{s\}$ *is not an associated point of S.*

Then f is flat and X_s is geometrically regular.

Proof. We start with (10.59). After localizing at a generic point $x \in Z$, we may assume that $Z = \{x\}$. The completion of a local ring preserves depth and the completion of a regular local ring is again regular, hence we may replace S and X by their completions at s and x. In particular, $\operatorname{pure}_x(X_s) \cong \operatorname{Spec}_{k(s)} k(x)[[t_1, \dots, t_n]]$ for some $n \geq 1$; here we use that $\operatorname{char} k(s) = 0$.

By (10.73.4) we can factor f as

$$f : (x, X) \xrightarrow{f'} (y, Y) \xrightarrow{q} (s, S),$$

where (y, Y) is complete, $k(x) = k(y)$ and q is faithfully flat, formally smooth and regular. We need to prove that f' is flat.

Lifting the t_i back to sections of $\mathcal{O}_X$ gives a finite morphism

$$\pi : X \to \hat{\mathbb{A}}^n_Y,$$

see (10.63) for the notation. We aim to prove that π is an isomorphism. For now we know that it is a local isomorphism along $X_s \setminus \{x\}$. Thus there is a closed subscheme $W \subset \hat{\mathbb{A}}^n_Y$ such that π is a local isomorphism outside W and $W \cap \hat{\mathbb{A}}^n_y = (0, y)$. In particular, $\pi|_W : W \to Y$ is finite.

We show next that π is a local isomorphism outside $(0, y) \in \hat{\mathbb{A}}^n_Y$. To see this pick any point $p \in S \setminus \{s\}$. Then π restricts to a finite morphism $\pi_p : X_p \to (\hat{\mathbb{A}}^n_Y)_p$ whose target is regular (since q is regular) and which is a local isomorphism outside the nowhere dense set $W \cap (\hat{\mathbb{A}}^n_Y)_p$. Thus π_p has a unique irreducible component that maps isomorphically onto $(\hat{\mathbb{A}}^n_Y)_p$ and all other associated points of X_p are contained in $\pi_p^{-1}(W)$. Such associated points are excluded by (2). Thus π_p is an isomorphism. Since $X \to S$ is flat over p, we see that π is a local isomorphism along X_p by (10.62). Finally,

$$\operatorname{depth}_{(0,y)} \hat{\mathbb{A}}^n_Y = n + \operatorname{depth}_y Y = n + \operatorname{depth}_s S \geq 1 + 1 = 2,$$

and X has no associated points supported on Z by (3). Hence π is an isomorphism by (9.7). This proves (10.59).

The proof of (10.60) is quite similar. We use (10.73.3) to factor f as

$$f : (x, X) \xrightarrow{f'} (y_1, Y_1) \xrightarrow{q} (s, S),$$

where (y_1, Y_1) is complete, $k(x)/k(y_1)$ is finite, purely inseparable and q is obtained from a smooth morphism by localization and completion. By (10.74) there is a finite, flat morphism $(y, Y) \to (y_1, Y_1)$ such that $k(y) = k(x)$. By base change we reduce everything to the fiber product $X \times_{Y_1} Y \to Y$. Geometric regularity of $\operatorname{pure}(X_s)$ is preserved by finite field extensions. The rest of the proof goes as before. □

REMARK 10.61. There should be a common generalization of the above theorems to morphism of Noetherian schemes assuming that $\operatorname{pure}_Z(X_s)$ is geometrically regular and (10.59.2–4) hold. Trying to follow the above proof, at the end we need to deal with an infinite purely inseparable extension $k(x)/k(y_1)$. Then $k(x) \otimes_{k(y_1)} k(x)$ and $X \times_{Y_1} Y \to Y$ are not Noetherian, hence the rest of the argument completely breaks down. However, I do not have any counter examples.

We have used the following easy lemma, cf. [**Mat86**, 22.5].

LEMMA 10.62. *Let (s, S) be a local Noetherian scheme and $\pi : X \to Y$ a finite morphism of Noetherian S-schemes. Assume that X is flat over S. Then π is an isomorphism iff $\pi_s : X_s \to Y_s$ is an isomorphism.* □

NOTATION 10.63. Let R be a ring and $Y = \operatorname{Spec} R$. We denote $\operatorname{Spec} R[[x_1, \dots, x_n]]$ by $\hat{\mathbb{A}}^n_Y$. If $X \to Y$ is a finite morphism then $\hat{\mathbb{A}}^n_X \cong X \times_Y \hat{\mathbb{A}}^n_Y$. However, $\hat{\mathbb{A}}^n_Y$ is *not* the product of $\hat{\mathbb{A}}^n$ with Y in any sense.

For example, if R is an integral domain with quotient field K then the generic fiber of $\hat{\mathbb{A}}^n_Y \to Y$ is the spectrum of the ring of power series over K that have bounded denominators. That is, power series of the form

$$\left\{\textstyle\sum_I a_I x^I : a_I \in K \quad \text{and} \quad \exists r \in R \quad \text{such that} \quad r a_I \in R \ \forall I.\right\}$$

This ring is regular (see [**Gro60**, 0.19.3.5, 0.19.7.1, 0.22.5.8] or [**Sta15**, Tag 07PM]) but in general more complicated than $K[[x_1, \dots, x_n]]$. For example, it can happen that the generic fiber is not a UFD, see (10.58).

EXAMPLE 10.64. We construct an example $f : X \to S$ to show that geometric regularity is needed in (10.60), even if X and S are affine varieties.

Let (s, S) be a 1-dimensional scheme such that $S \setminus \{s\}$ is regular and $g : Y \to (s, S)$ a flat morphism such that Y_s is regular. Let $W \subset Y$ be a closed subset such that $Z := Y_s \cap W$ is not empty and nowhere dense in W_s. Let $\pi : X \to Y$ be a finite, birational morphism such that $\operatorname{Supp}\bigl(\pi_* \mathcal{O}_X / \mathcal{O}_Y\bigr) = W$.

Then $\pi_s : X_s \to Y_s$ is a finite morphism that is an isomorphism over $Y_s \setminus W$. Since Y_s is regular, we get an isomorphism $\operatorname{pure}_Z X_s \cong Y_s$. Thus the composite $X \to S$ satisfies (10.59.1–4) but X_s is not regular.

In order to get such an $X \to S$, let k be a field of characteristic $p > 0$, $c \in k \setminus k^p$ and set $\gamma = c^{1/p}$. Consider the ring extensions

$$k[u, v]/(u^p - cv^p) \subset k[u, v, z, t]/(u^p - cv^p, z^p - c - vt^p).$$

Taking their spectra will give $Y \to S$.

Over the origin (u,v) the fiber is $k[z,t]/(z^p-c)\cong k(\gamma)[t]$ hence regular. If $v\neq 0$ then we can invert v and the rings become

$$k[u,v,v^{-1}]/(u^p-cv^p)\cong k[u,v,v^{-1}]/\big((u/v)^p-c\big)\cong k(\gamma)[v,v^{-1}] \quad\text{and}$$
$$k[u,v,v^{-1},z,t]/(u^p-cv^p,z^p-c-vt^p)\cong k(\gamma)[v,v^{-1},z,t]/\big((z-\gamma)^p-vt^p\big).$$

The former is regular while the latter is not normal along $(z-\gamma,t)$. Its normalization is obtained by setting $w:=(z-\gamma)/t$; it is the ring

$$k(\gamma)[v,v^{-1},z,t,w]/\big(w^p-v,z-\gamma-wt\big)\cong k(\gamma)[w,w^{-1},t].$$

Let F be the largest coherent, torsion free sheaf over Y that agrees with $\mathcal{O}_Y$ on $Y\setminus(z-\gamma=t=0)$ and with the normalization of Y over $Y\setminus(u=v=0)$. As we noted above, $X\to S$ has the required properties.

In codimension 1, an slc pair is either smooth or has nodes. Next we show that a close analog of (10.60) holds for nodal fibers if the base scheme is normal; the latter assumption is necessary by (10.67.1).

THEOREM 10.65. *Let (s,S) be a normal, local, excellent scheme, (x,X) a local, S_2 scheme and $f:X\to S$ a finite type morphism of pure relative dimension 1. Assume that the purified central fiber* $\operatorname{pure}(X_s)$ *has only nodes.*

Then f is flat with reduced fibers that have only nodes.

Proof. It is sufficient to prove flatness after a faithfully flat base change. By assumption $k(x)/k(s)$ is a finite extension, generated by n elements. (We can choose $n=1$ if $k(x)/k(s)$ is separable.) Choosing generators gives $s'\in\mathbb{A}^n_{k(s)}$ such that $k(x)\cong k(s')$.

Consider next the trivial lifting $f^{(n)}:\mathbb{A}^n_X\to\mathbb{A}^n_S$ and the points $(s',s)\in\mathbb{A}^n_S$ projecting to s and $(s',x)\in\mathbb{A}^n_X$ projecting to x. After localization and completion we get

$$f':\big((s',x),X'\big)\to\big((s',s),S'\big),$$

where all the assumptions hold and also $k(s',x)=k(s',s)$. Thus it is sufficient to prove the original claim when, in addition, $k(x)=k(s)$.

By (10.75.4) and (10.43) there is a finite, birational morphism onto a hypersurface

$$\pi:X\to H:=\big(q(x_1,x_2)+c=0\big)\subset\hat{\mathbb{A}}^2_S$$

where $c\in m_S$. If $c\neq 0$ then H is normal by (10.76), hence π is an isomorphism and we are done. If $c=0$ then the normalization of H is

$$H^n:=\hat{\mathbb{A}}^1_{S'}\quad\text{where}\quad S':=\big(q(t,1)=0\big)\subset\hat{\mathbb{A}}^1_S.$$

Note that $\tau:S'\to S$ is an étale double cover, and we have a natural exact sequence

$$0\to\mathcal{O}_S\to\tau_*\mathcal{O}_{S'}\to L\to 0,$$

where L is a line bundle on S. Thus we have an exact sequence

$$0\to\mathcal{O}_H\to\tau_*\mathcal{O}_{H^n}\to L\to 0,$$

so there is a coherent subsheaf $L'\subset L$ such that $\tau_*\mathcal{O}_{H^n}/\pi_*\mathcal{O}_X\cong L/L'$.

If $L'=0$ then $X\cong H^n$ and if $L'=L$ then $X\cong H$; the projection to S is flat in both cases. Otherwise $\operatorname{Supp}\big(\tau_*\mathcal{O}_{H^n}/\pi_*\mathcal{O}_X\big)=\operatorname{Supp}(L/L')$ has codimension ≥ 2 in H which is impossible since X is assumed to be S_2. □

With different methods, the following generalization of (10.65) is proved in [**Kol11b**]. The projectivity assumption should not be necessary.

THEOREM 10.66. *Let (s, S) be a normal, local scheme and $f : X \to S$ a projective morphism of pure relative dimension 1. Assume that X is S_2 and the purified central fiber* $\operatorname{pure}(X_s)$ *is*

(1) *either seminormal*
(2) *or has only simple, planar singularities.*

Then f is flat with reduced fibers that are seminormal in case (1) and have only simple, planar singularities in case (2). □

See [**AGZV85b**, I.p.245] for the conceptual definition of simple, planar singularities. For us it is quickest to note that a plane curve singularity $\big(f(x, y) = 0\big)$ is simple iff $\big(z^2 + f(x, y) = 0\big)$ is a Du Val surface singularity.

EXAMPLE 10.67. The next examples show that (10.66) does not generalize to most other curve singularities.

10.67.1 (Deformations of ordinary double points) Let $C \subset \mathbb{P}^2$ be a nodal cubic with normalization $p : \mathbb{P}^1 \to C$. Over the coordinate axes $S := (xy = 0) \subset \mathbb{A}^2$ consider the family X that is obtained as follows.

Over the x-axis take a smoothing of C, over the y-axis take $\mathbb{P}^1 \times \mathbb{A}^1_y$ and glue them over the origin using $p : \mathbb{P}^1 \to C$ to get $f : X \to S$.

Then X is seminormal and S_2, the central fiber is C with an embedded point yet f is not flat.

10.67.2 (Deformations of ordinary triple points) Consider the family of plane cubic curves

$$\mathbf{C} := \big((x^2 - y^2)(x + t) + t(x^3 + y^3) = 0\big) \subset \mathbb{A}^2_{xy} \times \mathbb{A}^1_t.$$

For every t the origin is a singular point, but it has multiplicity 3 for $t = 0$ and multiplicity 2 for $t \neq 0$. Thus blowing up the line $(x = y = 0)$ gives the normalization for $t \neq 0$ but it introduces an extra exceptional curve over $t = 0$. The normalization of $\mathbf{C}$ is obtained by contracting this extra curve. The fiber over $t = 0$ is then isomorphic to 3 lines though the origin in $\mathbb{A}^3$.

10.67.3 (Deformations of ordinary quadruple points) Let $\mathbf{C}_4 \to \mathbb{P}^{14}$ be the universal family of degree 4 plane curves and $\mathbf{C}_{4,1} \to S^{12}$ the 12-dimensional subfamily whose general members are elliptic curves with 2 nodes. We normalize both the base and the total space to get

$$\bar{\pi} : \bar{\mathbf{C}}_{4,1} \to \bar{S}^{12}.$$

We claim that the fiber of $\bar{\pi}$ over the plane quartic with an ordinary quadruple point $C_0 := (x^3y - xy^3 = 0)$ is C_0 with at least 2 embedded points. Most likely, the family is not even flat, but I have not checked this.

We prove this by showing that in different families of curves through $[C_0] \in S^{12}$ we get different flat limits.

To see this, note that the seminormalization C_0^{sn} of C_0 can be thought of as 4 general lines through a point in $\mathbb{P}^4$. In suitable affine coordinates we can write it as

$$k[x, y]/(x^3y - xy^3) \hookrightarrow k[u_1, \dots, u_4]/(u_iu_j : i \neq j)$$

using the map $(x, y) \mapsto (u_1 + u_3 + u_4, u_2 + u_3 - u_4)$. Any 3-dimensional linear subspace

$$\langle u_1, \dots, u_4\rangle \supset W_\lambda \supset \langle u_1 + u_3 + u_4, u_2 + u_3 - u_4\rangle.$$

corresponds to a projection of C_0^{sn} to $\mathbb{P}^3$; call the image $C_\lambda \subset \mathbb{P}^3$. Then C_λ is 4 general lines through a point in $\mathbb{P}^3$; thus it is a $(2,2)$-complete intersection curve of arithmetic genus 1. (Note that the C_λ are isomorphic to each other, but the isomorphism will not commute with the map to C_0 in general.) Every C_λ can be realized as the special fiber in a family $S_\lambda \to B_\lambda$ of $(2,2)$-complete intersection curves in $\mathbb{P}^3$ whose general fiber is a smooth elliptic curve.

By projecting these families to $\mathbb{P}^2$, we get a 1-parameter family $S'_\lambda \to B_\lambda$ of curves in S^{12} whose special fiber is C_0 .

Let now $\bar{S}'_\lambda \subset \bar{\mathbf{C}}_{4,1}$ be the preimage of this family in the normalization. Then $\bar{S}'_\lambda$ is dominated by the surface S_λ. There are two possibilities. First, if $\bar{S}'_\lambda$ is isomorphic to S_λ, then the fiber of $\bar{\mathbf{C}}_{4,1} \to \bar{S}^{12}$ over $[C_0]$ is C_λ. This, however, depends on λ, a contradiction. Second, if $\bar{S}'_\lambda$ is not isomorphic to S_λ, then the fiber of $\bar{S}'_\lambda \to B_\lambda$ over the origin is C_0 with some embedded points. Since C_0 has arithmetic genus 3, we must have at least 2 embedded points.

10.67.4. The above example also shows that the proposed inequality (3.57.3) does not always hold. We take L to be the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$. The generic fiber C_g is a curve in $\mathbb{P}^3$, thus $h^0(C_g, L_g) = 4$. The pure special fiber is the plane curve C_0, thus $h^0(C_0, L_0) = 3$.

Flatness in relative codimension ≥ 2.

Once we know flatness at codimension 1 points of the fibers, the following general result, valid for coherent sheaves, can be used to prove flatness everywhere. We no longer need any restrictions on the base scheme S.

THEOREM 10.68. *Let $f : X \to S$ be a morphism of Noetherian schemes, (s, S) local, $Z \subset \operatorname{Supp} F_s$ a nowhere dense closed subset and F a coherent sheaf on X. Assume that*

(1) $\operatorname{depth}_Z\big(F_s/\operatorname{tors}_Z(F_s)\big) \geq 2$,
(2) *F is flat over S along $X \setminus Z$ and*
(3) *$X_s \cap \bar{x} \not\subset Z$ for every $x \in \operatorname{Ass}(F_{f(x)})$.*

Then F is flat over S and $\operatorname{tors}_Z(F_s) = 0$.

Moreover, if f is of finite type then we can replace (3) by

(3') *the fibers of F over $S \setminus \{s\}$ are pure and* $\operatorname{depth}_Z F \geq 1$.

Proof. Set $X_n := \operatorname{Spec}_X(\mathcal{O}_X/f^*m^n_{s,S}\mathcal{O}_X)$, $F_n := F|_{X_n}$ and $m := m_{s,S}$. We may assume that S is m-adically complete. There are natural complexes

$$0 \to (m^n/m^{n+1}) \cdot F_0 \to F_{n+1} \xrightarrow{r_n} F_n \to 0, \tag{10.68.4}$$

which are exact on $X \setminus Z$ but not (yet) known to be exact along Z, except that r_n is surjective. We also know that

$$(m^n/m^{n+1}) \cdot \big(F_0/\operatorname{tors}_Z F_0\big) \to \ker r_n/\operatorname{tors}_Z(\ker r_n) \tag{10.68.5}$$

is an isomorphism on $X \setminus Z$. Since $\operatorname{depth}_Z\big(F_0/\operatorname{tors}_Z F_0\big) \geq 2$, this implies that (10.68.5) is an isomorphism on X by (9.7). Next we show that the induced map

$$r_n : \operatorname{tors}_Z F_{n+1} \to \operatorname{tors}_Z F_n \quad \text{is surjective.} \tag{10.68.6}$$

Set $K_{n+1} := r_n^{-1}\big(\operatorname{tors}_Z F_n\big)$. We have an exact sequence

$$0 \to \ker r_n/\operatorname{tors}_Z(\ker r_n) \to K_{n+1}/\operatorname{tors}_Z(\ker r_n) \to \operatorname{tors}_Z F_n \to 0. \tag{10.68.7}$$

Using that (10.68.5) is an isomorphism, we have $\operatorname{depth}_Z\bigl(\ker r_n/\operatorname{tors}_Z(\ker r_n)\bigr)\geq 2$, hence the sequence (10.68.7) splits by (9.7). We know that $\varprojlim(\operatorname{tors}_Z F_n)$ is a subsheaf of F, let w be a generic point of its support. Then $w\in \operatorname{Ass}(F_{f(w)})$ by (10.72.3) and $\operatorname{Supp}\bigl(\bar w\cap X_s\bigr)\subset Z$ by construction. Thus $\varprojlim(\operatorname{tors}_Z F_n)=0$ by (3) hence $\operatorname{tors}_Z F_n=0$ for every n by (10.68.6).

Thus (10.68.5) now says that $(m^n/m^{n+1})\cdot F_0\cong \ker r_n$. Therefore the sequences (10.68.4) are exact, F is flat and $\operatorname{tors}_Z F_0=0$. □

Putting together the above flatness criteria (10.50), (10.65), (10.66.1) and (10.68) gives the following strengthening of [**Hir58**].

THEOREM 10.69. *Let (s,S) be a normal, local, excellent scheme, X an S_2 scheme and $f:X\to S$ a finite type morphism of pure relative dimension n. Assume that* $\operatorname{pure}(X_s)$ *is*

(1) *either normal*
(2) *or seminormal and S_2.*

Then f is flat with reduced fibers that are normal in case (1) and seminormal and S_2 in case (2). □

Flatness in relative codimension ≥ 3.

We get an even stronger result in codimension ≥ 3; see [**Kol95a**, Thm.12]. [**LN16**] pointed out that the purity assumption in (3') is also necessary.

THEOREM 10.70. *Let $f:X\to S$ be a morphism of Noetherian schemes, (s,S) local, $Z\subset \operatorname{Supp} F_s$ a nowhere dense closed subset and $j:X_s\setminus Z\hookrightarrow X_s$ the natural injection. Assume that*

(1) *$j_*(F_s|_{X_s\setminus Z})$ is coherent and $\operatorname{depth}_Z\bigl(j_*(F_s|_{X_s\setminus Z})\bigr)\geq 3$,*
(2) *$F|_{X\setminus Z}$ is flat over S and $\operatorname{depth}_Z F\geq 2$,*
(3) *$\operatorname{depth}_x\bigl(F_{\pi(x)}\bigr)\geq 2$ whenever $X_s\cap\bar x\subset Z$ and $x\notin X_s$.*

Then F is flat over S and $F_s=j_*(F_s|_{X_s\setminus Z})$.

Moreover, if f is of finite type then we can replace (3) by

(3') *the fibers of F over $S\setminus\{s\}$ are pure and S_2.*

Proof. Set $m:=m_{s,S}$, $X_n:=\operatorname{Spec}_X(\mathcal{O}_X/f^*m^n\mathcal{O}_X)$ and $F_n:=F|_{X_n}$. We may assume that $\mathcal{O}_S$ and $\mathcal{O}_X$ are m-adically complete. Set $G_n:=F_n|_{X_n\setminus Z}$ and let j denote any of the injections $X_n\setminus Z\hookrightarrow X_n$. By assumption (2) we have exact sequences

$$0\to(m^n/m^{n+1})\cdot G_0\to G_{n+1}\longrightarrow G_n\to 0. \tag{10.70.4}$$

Pushing it forward we get the exact sequences

$$\begin{aligned}0\to(m^n/m^{n+1})\otimes j_*G_0\to j_*G_{n+1}\stackrel{r_n}{\to} j_*G_n\to\\ \to(m^n/m^{n+1})\otimes R^1j_*G_0.\end{aligned} \tag{10.70.5}$$

The first part of assumption (1) says that j_*G_0 is coherent and the second part implies (in fact is equivalent to) $R^1j_*G_0=0$ by [**Gro68**, III.3.3, II.6 and I.2.9] or (10.18).

Thus the r_n are surjective. This shows that $G:=\varprojlim j_*G_n$ is a coherent sheaf on X that is flat over S and $\operatorname{depth}_x\bigl(G_{\pi(x)}\bigr)\geq 1$ whenever $X_s\cap\bar x\subset Z$ and $x\notin X_s$. Furthermore, the natural map $\rho:F\to G$ is an isomorphism along $X_s\setminus Z$. Thus (10.71) implies that it is an isomorphism. So $F\cong G$ is flat with central fiber $j_*G_0=j_*(F_s|_{X_s\setminus Z})$. □

LEMMA 10.71. *Let $f : X \to S$ be a morphism of Noetherian schemes, (s, S) local and $Z \subset X_s$ a nowhere dense closed subset. Let F, G be coherent sheaves on X and $\phi : F \to G$ a morphism. Assume that*

(1) *G and F are flat over S along $X \setminus Z$,*
(2) *ϕ is an isomorphism along $X_s \setminus Z$ and*
(3) *for every point $x \in X \setminus X_s$ such that $X_s \cap \bar{x} \subset Z$, we have $\operatorname{depth}_x G_{f(x)} \geq 1$ and $\operatorname{depth}_x F_{f(x)} \geq 2$.*

Then ϕ is an isomorphism.

Moreover, if f is of finite type then we can replace (3) by

(3') *the fibers of G over $S \setminus \{s\}$ are pure and the fibers of F over $S \setminus \{s\}$ are pure and S_2.*

Proof. Set $W := \operatorname{Supp}(\ker \phi)$ and let $w \in W$ be a generic point. It is also an associated point of $F_{f(w)}$ by (10.72.3). Furthermore, $X_s \cap \bar{w} \subset Z$ hence $\operatorname{depth}_w F_{f(w)} \geq 1$ by (3), a contradiction. Next set $V := \operatorname{Supp}(\operatorname{coker} \phi)$ and let $v \in V$ be a generic point. As before, $X_s \cap \bar{v} \subset Z$, hence $\operatorname{depth}_v F_{f(v)} \geq 2$ by (3). Thus

$$0 \to F_v \to G_v \to \operatorname{coker} \phi_v \to 0$$

splits by (9.7), so $\operatorname{coker} \phi_v$ is a subsheaf of $G_{f(v)}$ but this contradicts $\operatorname{depth}_v G_{f(v)} \geq 1$ as before. $\square$

10.72 (Flatness and associated points). Let $f : X \to S$ be a morphism of Noetherian schemes and F a coherent sheaf on X.

Claim 10.72.1. If F is flat over S then $f(\operatorname{Ass}(F)) \subset \operatorname{Ass}(S)$.

Proof. Let $x \in X$ be an associated point of F and $s := f(x)$. Assume that s is not an associated point of S. Then there is an $r \in m_{s,S}$ such that $r : \mathcal{O}_S \to \mathcal{O}_S$ is injective near s. Tensoring with F shows that $r : F \to F$ is injective near X_s. Thus none of the points of X_s is in $\operatorname{Ass}(F)$. $\square$

Claim 10.72.2. Assume that F is flat with pure fibers over S. Then every $x \in \operatorname{Ass}(F)$ is a generic point of $\operatorname{Supp}(F_{f(x)})$.

Proof. Set $s := f(x)$. By (1) the annihilator of $m_{s,S}$ is a nonzero ideal $J \subset m_{s,S}$ which is also a $k(s)$-vector space. By assumption JF has nonzero sections supported on $\bar{x}$. On the other hand, since F is flat, $JF \cong J \otimes_{k(s)} F_s$ and the latter is assumed pure. Thus x is a generic point of $\operatorname{Supp}(F_s)$. $\square$

Claim 10.72.3. Assume that F is flat over S and $x \in \operatorname{Ass}(F)$. Then every generic point of $\operatorname{Supp}(\bar{x} \cap X_s)$ is an associated point of F_s.

Proof. Let $G \subset F$ be the largest subsheaf supported on $\bar{x}$. After localizing at a generic point of $\operatorname{Supp}(\bar{x} \cap X_s)$ we may assume that $\operatorname{Supp}(\bar{x} \cap X_s) = \{w\}$, a single closed point. There is a smallest $n \geq 0$ such that $G \subset m_{s,S}^n F$ but $G \not\subset m_{s,S}^{n+1} F$. Thus $m_{s,S}^n F / m_{s,S}^{n+1} F \cong (m_{s,S}^n / m_{s,S}^{n+1}) \otimes F_s$ has a nonzero subsheaf supported on w. $\square$

Note that flatness is needed for (10.72.3) as illustrated by the restriction of either of the coordinate projections to the union of the axes $(xy = 0)$.

10.7. Noether normalization

Noether's normalization theorem says that if X is an affine k-variety of dimension m then it admits a finite morphism onto $\mathbb{A}^m_k$. Equivalently, the structure morphism $X \to \operatorname{Spec} k$ can be factored as

$$X \stackrel{\text{finite}}{\longrightarrow} \mathbb{A}^m_k \to \operatorname{Spec} k.$$

We aim to generalize this to arbitrary morphisms. That is, we would like to factor an arbitrary morphism $p : X \to S$ as

$$X \xrightarrow{p_1} Y \xrightarrow{p_2} S,$$

where p_1 has "finiteness" properties and p_2 has "smoothness" properties.

In (10.73.7) we give an example of a morphism of pure relative dimension one $p : X \to S$ from an affine 3-fold X to a smooth, pointed surface $s \in S$ that can not be factored as

$$p : X \stackrel{\text{finite}}{\longrightarrow} \mathbb{A}^1 \times S \to S,$$

not even over a formal neighborhood of s. Such examples are quite typical and, although the projective version of Noether's normalization theorem is easy to generalize to the relative setting, there does not seem to be any sensible global affine analog over base schemes of dimension ≥ 2. There are, however, very useful local versions.

10.73 (Noether normalization, local version). Let $f : (x, X) \to (s, S)$ be a morphism of local, Noetherian schemes. We would like to factors f as

$$f : (x, X) \stackrel{p}{\to} (s', S') \stackrel{q}{\to} (s, S) \tag{10.73.1}$$

where p has "finiteness" properties and q has "smoothness" properties. The most useful version is (10.73.5), but let us start with the case when $k(x) \supset k(s)$ is a finitely generated field extension. Pick any transcendence basis $\bar{y}_1, \dots, \bar{y}_n$ of $k(x)/k(s)$ and lift these back to $y_1, \dots, y_n \in \mathcal{O}_X$. We can then take S' to be the localization of $\mathbb{A}^n_S$ at the generic point of the fiber over $s \in S$. Thus we have proved the following.

Claim 10.73.2. Let $f : (x, X) \to (s, S)$ be a local morphism of local, Noetherian schemes such that $k(x) \supset k(s)$ is a finitely generated field extension. Then we can factors f as

$$f : (x, X) \stackrel{p}{\to} (s', S') \stackrel{q}{\to} (s, S) \tag{10.73.2.a}$$

where $k(x)/k(s')$ is a finite field extension, q has relative dimension 0 and it is the localization of a smooth morphism. □

Combining this and (10.74) shows that if $k(x) \supset k(s)$ is arbitrary then we get a factorization (10.73.2.a) where $k(x)/k(s')$ is an algebraic field extension and q is formally smooth.

For flatness questions we can freely replace (x, X) and (s, S) by their completions and in the complete case we can do better.

Pick $\bar{y} \in k(x)$ that is separable over $k(s')$ with separable, monic equation $\bar{g}(\bar{y}) = 0$. If $\mathcal{O}_X$ is complete, Hensel's lemma tells us that we can lift $\bar{y}$ to $y \in \mathcal{O}_X$ such that y satisfies a separable, monic equation $g(y) = 0$. We can now replace S' with the completion of $\mathcal{O}_{S'}[y]/(g(y))$ along the central fiber and obtain the following.

Claim 10.73.3. Let $f:(x,X)\to(s,S)$ be a local morphism of local, complete, Noetherian schemes such that $k(x)\supset k(s)$ is a finitely generated field extension. Then we can factors f as

$$f:(x,X)\stackrel{p}{\to}(s',S')\stackrel{q}{\to}(s,S) \tag{10.73.3.a}$$

where p is finite, $k(x)/k(s')$ is a purely inseparable field extension, q has relative dimension 0 and it is the localization of a smooth morphism. □

Combining this with (10.74.3) gives the following.

Claim 10.73.4. Let $f:(x,X)\to(s,S)$ be a local morphism of local, complete, Noetherian schemes. Then we can factors f as

$$f:(x,X)\stackrel{p}{\to}(s',S')\stackrel{q}{\to}(s,S) \tag{10.73.4.a}$$

where $k(x)/k(s')$ is a purely inseparable field extension and q is formally smooth, faithfully flat, regular and of relative dimension 0. □

For flatness criteria the following form is the most useful.

Claim 10.73.5. Let $f:(x,X)\to(s,S)$ be a local morphism of local, complete, Noetherian schemes such that $k(x)/k(s)$ is separable. Set $n:=\dim X_s$

Then we can factors f as

$$f:(x,X)\stackrel{p}{\to}\big((s',0),\hat{\mathbb{A}}^n_{S'}\big)\stackrel{\pi}{\to}(s',S')\stackrel{q}{\to}(s,S) \tag{10.73.5.a}$$

such that

(b) p is finite, $k(x)=k(s',0)=k(s')$,
(c) π is the coordinate projection,
(d) q has relative dimension 0 and
(e) q is the localization of a smooth morphism if $k(x)/k(s)$ is finitely generated and formally smooth, faithfully flat and regular in general.

Proof. By (10.73.4) we have $q:(s',S')\to(s,S)$ such that $k(x)=k(s')$. Since $\mathcal{O}_{X_s}$ has dimension n, there are $\bar t_1,\dots,\bar t_n\in\mathcal{O}_{X_s}$ that generate an ideal that is primary to the maximal ideal. Lift these back to $t_1,\dots,t_n\in\mathcal{O}_X$. These define $p:(x,X)\to\big((s',0),\hat{\mathbb{A}}^n_{S'}\big)$. By construction

$$\mathcal{O}_X/\big(m_S,t_1,\dots,t_n\big)\cong\mathcal{O}_{X_s}/\big(\bar t_1,\dots,\bar t_n\big)$$

is finite over $k(s')$. Thus p is finite. □

The following variant is due to [**RG71**]; see also [**Sta15**, Tag 052D]. A related factorization theorem is proved in [**AFH94**].

Claim 10.73.6. Let $f:X\to S$ be a finite type morphism. Pick $s\in S$, $x\in X_s$ and set $n=\dim_x X_s$. Then there is a commutative diagram

$$\begin{array}{ccc}(x',X') & \stackrel{\pi}{\to} & (x,X)\\ g\downarrow & & \downarrow f\\ (y,Y) & \stackrel{h}{\to} & (s,S),\end{array}$$

where π is étale, g is finite, $g^{-1}(y)=\{x'\}$ and h is smooth of relative dimension n. □

Example 10.73.7. Let S denote the localization (or completion) of $\mathbb{A}^2_{st}$ at the origin and consider the affine scheme

$$X:=\big((x^3+y^3+1)(1+tx)+sy=0\big)\subset\mathbb{A}^2_{xy}\times S.$$

Then $\pi : X \to S$ is a family of curves. We claim that there is no quasi-finite morphism of it onto $\mathbb{A}^1 \times S$.

Assume to the contrary that such a map $g : X \to \mathbb{A}^1 \times S$ exists. Then g can be extended to a finite morphism $\bar{g} : \bar{X} \to \mathbb{P}^1 \times S$.

Here $\bar{X}_{(0,0)}$ is a compactification of $X_{(0,0)}$, hence a curve of geometric genus 1.

For $t \neq 0$ the line $(1 + tx = 0)$ gives an irreducible component of $\bar{X}_{(0,t)}$ that is a rational curve. As $t \to 0$, the limit of these rational curves is a union of rational, irreducible, geometric components of $\bar{X}_{(0,0)}$, a contradiction.

10.74 (Residue field extensions). Let (s, S) be a Noetherian, local scheme and $K/k(s)$ a field extension. We would like to find a Noetherian, local scheme (x, X) and a flat morphism $g : (x, X) \to (s, S)$ such that $g^* m_{s,S} = m_{x,X}$ (that is, the scheme theoretic fiber $g^{-1}(s)$ is the reduced point $\{x\}$) and $k(x) \cong K$. The answer is given in [**Gro60**, 0_{III}.10.3.1].

Claim 10.74.1. Such a $g : (x, X) \to (s, S)$ always exists.

Outline of proof. Let us start with extensions with 1 generator $K = k(s)(t)$. If t is transcendental over $k(s)$, we can take X to be the localization of $\mathbb{A}^1_S$ at the generic point of the fiber over $s \in S$. If t is algebraic, let $\bar{g}(z) \in k(s)[z]$ be a monic minimal polynomial of t. Lift it back to a monic polynomial $g(z) \in \mathcal{O}_S[z]$ and take X to be the localization of $(g(z) = 0) \subset \mathbb{A}^1_S$ at the central fiber. Combining these steps gives a solution for any finitely generated field extension $K/k(s)$. In the separable case the proof gives the following stronger form.

Claim 10.74.2. If $K/k(s)$ is a finitely generated separable extension then we can choose $g : (x, X) \to (s, S)$ to be the localization of a smooth morphism. In particular, if S is normal then so is X. □

The general case is proved by iterating the same steps but one also needs a somewhat tricky limit argument to show that the resulting scheme is Noetherian. See [**Gro60**, 0_{III}.10.3.1] for details. Combining it with [**And74**] we get the following.

Claim 10.74.3. If $K/k(s)$ is an arbitrary separable extension then we can choose $g : (x, X) \to (s, S)$ to be formally smooth. If S is complete then g is also regular. In particular, if S is normal then so is X. □

The following example illustrates some of the subtle aspects.

Example 10.74.4. Let k be a field. Fix a prime p and $a \in k \setminus k^p$. Set $S = k[x]_{(x-a)}$ with maximal ideal $m = (x - a)S$. Then $S/m \cong k$. Set $K = k\big(a^{1/p^n} : n = 1, 2, \dots\big)$. Then $K[t]_{(t-u)}$ is a solution, but here is a more interesting one.

Start with $k\big[x^{1/p^n} : n = 1, 2, \dots\big]$. This is not Noetherian since the ideal $\big(x^{1/p^n} : n = 1, 2, \dots\big)$ is not finitely generated. However, $x - a$ is irreducible in $k\big[x^{1/p^n}\big]$ for every n (equivalently, $x^{p^n} - a$ is irreducible in $k[x]$) hence also in $k\big[x^{1/p^n} : n = 1, 2, \dots\big]$. Thus $k\big[x^{1/p^n} : n = 1, 2, \dots\big]_{(x-a)}$ is a DVR with maximal ideal $(x - a)$.

As a concrete example, take $k = \mathbb{Q}$. Then $\mathbb{Q}[t^{1/n} : n = 1, 2, \dots]$ is not Noetherian and the polynomials $t, t - 1, t + 1$ all have infinitely many divisors. However, it seems that these are the only ones and $\mathbb{Q}\big[t^{1/n} : n = 1, 2, \dots, \frac{1}{t^3 - t}\big]$ is Noetherian.

Note also that if char $k = p$ then $K\big[x^{1/p^n} : n = 1, 2, \dots\big]_{(x-a)}$ is not Noetherian, as shown by the ideal $\big(x^{1/p^n} - a^{1/p^n} : n = 1, 2, \dots\big)$.

Remark on inseparable extensions 10.74.5. Infinite inseparable extensions do cause problems in the above arguments, leading to finite type assumptions in (10.60) and its consequences in positive characteristic. However, I do not know whether these restrictions are actually necessary or not.

From the technical point of view, one difficulty is that infinite inseparable extensions can lead to non-excellent schemes. For example, let k be a perfect field of characteristic $p > 0$ and $K := k(x_1, x_2, \dots)$, a purely transcendental extension in infinitely many variables. Then $K^p = k(x_1^p, x_2^p, \dots)$ is abstractly isomorphic to K. Now start with $K^p[[t]]$ and try to get a residue field extension K/K^p. The above method gives $R := \cup_L L[[t]] \subset K[[t]]$, where L runs through all finite degree subextensions of K/K^p. Note that $R \neq K[[t]]$ since $\sum_i x_i t^i$ is not in R.

It is easy to see that R is a DVR and its completion is $K[[t]]$, which is purely inseparable over R. Thus R is not excellent. It is the source of many counter examples in [**Nag62**].

10.75 (Noether normalization, birational version). An affine form of Noether's normalization theorem says that every geometrically reduced affine variety admits a finite, birational morphism onto a hypersurface. Over an infinite base field this can be obtained by a general linear projection.

More generally, we have the following relative version.

Claim 10.75.1. Let S be an affine, integral, positive dimensional scheme and $X \to S$ a finite, dominant morphism. The following are equivalent.

(a) The generic fiber X_{gen} is curvilinear (10.52) over $k(S)$.

(b) There is a Cartier divisor $H \subset \mathbb{A}^1_S$ that is finite over S and a morphism $\pi : X \to H$ such that π is an isomorphism outside a nowhere dense closed subset of H. (If X is integral then this says that π is birational.)

Proof. It is clear that (b) implies (a). To see the converse, let K denote the function field of S and A the semi-local ring of the generic points of X. If (a) holds then $A \cong K[t]/(g(t))$ for some monic polynomial $g(t)$ of degree m. Thinking of t as an element of A, there is a non-zero $c \in \mathcal{O}_S$ such that $x := ct \in \mathcal{O}_X$ and $cg(t) \in \mathcal{O}_S[t]$. Thus we can take $H := \big(c^m g(x/c) = 0\big) \subset \mathbb{A}^1_S$. $\square$

A construction as in (10.73.7) shows that the relative version of (10.75.1) fails for morphisms of finite type, but, as in (10.73.3–5), we can generalize (10.75.1) to morphisms of complete local schemes.

Thus let $f : (x, X) \to (s, S)$ be a local morphism of complete, Noetherian schemes. Assume that there is a finite, birational morphism onto a hypersurface

$$\pi_s : X_s \to H_s \subset \hat{\mathbb{A}}^{n+1}_{k(s)}. \tag{10.75.2}$$

By lifting the coordinate functions $\pi_s^*(x_i)$ arbitrarily to $\mathcal{O}_X$, π_s extends to a morphism

$$\pi : X \to \hat{\mathbb{A}}^{n+1}_S. \tag{10.75.3}$$

Note that π is finite since $\pi^{-1}(s, \mathbf{0}) = \pi_s^{-1}(\mathbf{0})$. Let us denote its image by $H \subset \hat{\mathbb{A}}^{n+1}_S$. The intersection of H with the central fiber $\hat{\mathbb{A}}^{n+1}_s$ is H_s. Thus, if f is flat at the generic points of X_s, then $\pi : X \to H$ is a local isomorphism at the generic points of H_s and so $H \to S$ is also flat at the generic points of H_s.

Furthermore, if S is normal then H is a relative Cartier divisor in $\hat{\mathbb{A}}^{n+1}_S$ by (10.55) hence we have proved the following.

Claim 10.75.4. Let $f : (x, X) \to (s, S)$ be a local morphism of complete, Noetherian schemes. Assume that S is normal and f is flat at the generic points of X_s. Let $\pi_s : X_s \to H_s \subset \hat{\mathbb{A}}^{n+1}_{k(s)}$ be a finite, birational morphism onto a hypersurface.

Then there is a relative Cartier divisor $H \subset \hat{\mathbb{A}}^{n+1}_S$ such that π_s extends to a finite, birational morphism of S-schemes $\pi : X \to H$. □

Informally speaking, normalizations of hypersurfaces describe all deformations over normal base schemes. (Normality of S is necessary by (10.67.1).)

This is a seemingly very useful observation, but in most cases it turns out to be extremely hard to understand the central fiber of the normalization. Next we discuss some examples where this approach leads to a complete answer. More delicate applications of this method are in [**dJvS91**].

PROPOSITION 10.76. *Let (R, m) be a complete, normal, local ring, $q(x_1, \dots, x_n)$ a nondegenerate quadratic form over R and $c \in m$. Then $R[[x_1, \dots, x_n]]/(q + c)$ is normal if $n \geq 3$ or $n = 2$ and $c \neq 0$.*

Proof. Set $S := \operatorname{Spec} R$, $X := \operatorname{Spec} R[[x_1, \dots, x_n]]/(q + c)$ with projection $\pi : X \to S$. Let $W \subset X$ denote the locus where π is not regular. Thus $X \setminus W$ is normal and, by Serre's criterion, X is normal if $\operatorname{depth}_W X \geq 2$.

The fiber of W over $\operatorname{Spec}(R/m)$ has codimension $n - 1$. Thus if $w \in W$ then

$$\operatorname{depth}_w X \geq \operatorname{depth}_{p(w)} S + \operatorname{codim}(w, X_{p(w)}) \geq \operatorname{depth}_{p(w)} S + n - 1.$$

We are done if $n \geq 3$ or if $n = 2$ and $p(w)$ is not a generic point of S. We finish by noting that if $c \neq 0$ then the generic fiber is regular by the Jacobian criterion. □

LEMMA 10.77. *Let (R, m) be a normal, complete, local ring such that the characteristic of R/m is $\neq 2$. Let $f \in R[[x_1, \dots, x_n]]$ be a power series such that $\bar{f}$ is not identically zero. By (10.57) we can write $f = g^2 h$ where h is square-free. Then $y \mapsto gz$ gives the normalization map*

$$R[[y, x_1, \dots, x_n]]/(y^2 - f) \hookrightarrow R[[z, x_1, \dots, x_n]]/(z^2 - h).$$

Proof. It is clear that the ring extension is finite and birational. Thus we need to show that $R[[z, x_1, \dots, x_n]]/(z^2 - h)$ is normal. Projection to $R[[x_1, \dots, x_n]]$ is étale away from $(h = 0)$, hence $R[[x_1, \dots, x_n]]$ is normal away from $(h = 0)$.

To check normality along $(h = 0)$ we localize at a generic point of $(h = 0)$. Then we have a DVR A with maximal ideal $(h)A$, otherwise we would have a multiple factor of h. The unique maximal ideal of $A[z]/(z^2 - h)$ is (z, h) and it is generated by z, thus $A[z]/(z^2 - h)$ is a DVR, hence normal. □

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