September 20, 2010

# TOPICS IN ALGEBRAIC GEOMETRY

JÁNOS KOLLÁR

0.1. **Grassmannians.** Study the family of all linear subspaces of a vector space. Especially recommended if you are interested in representations of groups like $GL_n$ etc.

Ref. [Has07, Chap.11]

Extensions: [LB09].

0.2. **Riemann–Roch on curves.** Give a formula for the dimension of the space of functions with preassigned poles. Much of algebraic geometry starts here.

Ref. [Ful89, Chap.8]

Extensions: You should really learn some cohomology if you plan to go further. The standard text is [Har77, Chap.III]. Good luck!

0.3. **Groebner bases.** As you will see, algebraic geometry is good with existence theorems. Groebner bases give a method that allows one to do actual computations.

Ref: [CLO97, Chap.2]

Extensions: learn the program *Macaulay* and do some work with it. (With this you will be mostly on your own since I never used it.)

0.4. **Examples of algebraic group actions.** Through examples, study algebraic groups like $GL_n$, $PGL_n$ and their actions.

Ref. [Har95, Lect.10]

Extensions: This is a vast topic, one can start with [Hum75] or [Bor91].

0.5. **Curve singularities.** Local study of plane curves, that is, given a plane curve $C := (f(x, y) = 0)$, we try to understand $C$ in a small neighborhood of the origin. First step: near the origin we try to write $y$ as a function of $x$ and give some kind of infinite series expansion.

Applications: resolution of singularities and understanding the topology of $C \cap (|x|^2 + |y|^2 = \epsilon^2)$ as a subset of the 3-sphere $(|x|^2 + |y|^2 = \epsilon^2)$ of radius $\epsilon$.

Ref: [BK86, Sec.8]

Extensions: This is long enough, but one can do other resolution methods [Kol07b, Chap.1] or higher dimensional singularities [AGZV85].

0.6. **Rational varieties.** Mostly through examples like plane conics, quadrics and cubics in $\mathbb{P}^3$ study the geometry and arithmetic of rational varieties.

Ref: [KSC04, Chap.1]

Extensions: [Kol08, Secs.1–3] or [Kol02] or [KSC04, Chap.2].

0.7. **Elliptic curves.** Essentially the study of plane cubic curves and their other incarnations. The geometry is well understood; many deep open number theoretic questions remain.

Refs. The most elementary is [Rei88, §2]. You should also go through [Cas91, Secs.6–9].

Extensions: [Sil09] and much that lies beyond.

0.8. **Elliptic functions.** A beautiful treatment is in [Sie88, Chap.1]. Needs only the basics of 1-variable complex analytic functions.

Extensions. You can continue with [Sie88].

0.9. **Hasse principle.** First prove that a quadric over $\mathbb{Q}$ has a point iff it has a point over $\mathbb{Q}_p$ for every $p$. Then show that the analogous statement fails for cubic surfaces.

Ref. [Cas91, Secs.1–5], [Ser73, Chap.IV] and [SD62, Mor65].

Extensions: (I am still looking.)

0.10. **Tarski-Seidenberg Theorem.** A subset $X \subset \mathbb{R}^n$ is basic semialgebraic if it is given by conditions $p_i(x_1, \dots, x_n) \geq 0$ where the $p_i$ are polynomials. Taking finite unions, intersections, complements we get semialgebraic sets.

*Theorem.* The projection of a semialgebraic set is again a semialgebraic set.

Start with the complex case: Chevalley's theorem that images of algebraic varieties are constructible sets.

Ref: [BCR98, Chap.2]

Possible extension: What should be the right notion for subsets of $\mathbb{Q}_p^n$?

0.11. **Chevalley's theorem on invariants of finite groups.** Let $G \subset GL(n,k)$ be a finite group. We get an action on $k[x_1, \dots, x_n]$. The question is: what is the subring of invariants $k[x_1, \dots, x_n]^G \subset k[x_1, \dots, x_n]$.

*Theorem.* If $k = \mathbb{R}$ and $G$ is generated by reflections then $k[x_1, \dots, x_n]^G$ is again a polynomial ring.

Ref: [Che55]

Extensions: You should work out the corresponding result for $k = \mathbb{C}$ and other fields. Other directions: [Ben93, Chaps.1–3].

0.12. **Simple singularities.** We work with power series $f(x_1, \dots, x_n)$. Two power series $f, g$ are considered equivalent if there is a coordinate change given by power series $x_i \mapsto \phi_i(\mathbf{x})$ such that $f(\phi_1(\mathbf{x}), \dots, \phi_n(\mathbf{x})) = g(\mathbf{x})$.

Given two power series $f, g$, we can view $f + \epsilon g$ as perturbations of $f$. A very fruitful question of singularity theory asks: what can we say about the perturbations of a polynomial or power series $f$?

The aim is to classify those power series $f(x_1, \dots, x_n)$ that have only finitely many inequivalent perturbations.

Ref. Probably the best is to think about this and then get the proof from [KM98, 4.24–25] by replacing Steps 6 and 9. Or you can look at the general case in [AGZV85, Secs.11–].

Extensions: More than you want is in [AGZV85, Secs.11–15].

0.13. **Chow's theorem.** This is the following.

*Theorem.* Let $Z \subset \mathbb{CP}^n$ be a Euclidean closed subset that is locally definable as a common zero set of analytic functions. Then $Z$ is algebraic, that is, globally the common zero set of polynomials.

It is helpful if you are somewhat familiar with several variable complex analytic functions.

Ref: [Mum95, Chap.4].

Extensions. If you are up to it, read [Ser56].

0.14. **Minimal degree varieties.** The aim is to classify irreducible subvarieties of $\mathbb{P}^n$ that are not contained in any linear subspace and whose degree is as small as possible. Nice concrete geometry.

Ref: [EH87]

Extensions: minimal multiplicity local rings [Sal79]; Castelnuovo bound for space curves [ACGH85, Sec.III.2]; other extremal examples [Rus00, CMR04].

0.15. **Pointless varieties and large fields.** The aim is to show that if $k$ is a field such that there is a geometrically irreducible $k$-variety without $k$-points (for instance if $k = \mathbb{R}$ then the conic $(x^2 + y^2 + z^2 = 0) \subset \mathbb{P}^2$ is such) then there is also such a plane projective curve.

Ref. [Kol07a, Sec.1]

Extensions. Best is to study the Weil estimates for points over finite fields (I am still looking for an elementary introduction.).

## References


[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 267, Springer-Verlag, New York, 1985. MR 770932 (86h:14019)

[AGZV85] V. I. Arnold, S. M. Guseĭn-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Vol. I*, Monographs in Mathematics, vol. 82, Birkhäuser Boston Inc., Boston, MA, 1985, The classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous and Mark Reynolds. MR 777682 (86f:58018)

[BCR98] Jacek Bochnak, Michel Coste, and Marie-Francoise Roy, *Real algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 36, Springer-Verlag, Berlin, 1998, Translated from the 1987 French original, Revised by the authors. MR 1659509 (2000a:14067)

[Ben93] D. J. Benson, *Polynomial invariants of finite groups*, London Mathematical Society Lecture Note Series, vol. 190, Cambridge University Press, Cambridge, 1993. MR 1249931 (94j:13003)

[BK86] Egbert Brieskorn and Horst Knörrer, *Plane algebraic curves*, Birkhäuser Verlag, Basel, 1986, Translated from the German by John Stillwell. MR 886476 (88a:14001)

[Bor91] Armand Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR 1102012 (92d:20001)

[Cas91] J. W. S. Cassels, *Lectures on elliptic curves*, London Mathematical Society Student Texts, vol. 24, Cambridge University Press, Cambridge, 1991. MR 1144763 (92k:11058)

[Che55] Claude Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. **77** (1955), 778–782. MR 0072877 (17,345d)

[CLO97] David Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, second ed., Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1997, An introduction to computational algebraic geometry and commutative algebra. MR 1417938 (97h:13024)

[CMR04] Ciro Ciliberto, Massimiliano Mella, and Francesco Russo, *Varieties with one apparent double point*, J. Algebraic Geom. **13** (2004), no. 3, 475–512. MR 2047678 (2005b:14078)

[EH87] David Eisenbud and Joe Harris, *On varieties of minimal degree (a centennial account)*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 3–13. MR 927946 (89f:14042)

[Ful89] William Fulton, *Algebraic curves*, Advanced Book Classics, Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1989, An introduction to algebraic geometry, Notes written with the collaboration of Richard Weiss, Reprint of 1969 original. MR 1042981 (90k:14023)

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

[Har95] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995, A first course, Corrected reprint of the 1992 original. MR 1416564 (97e:14001)

[Has07] Brendan Hassett, *Introduction to algebraic geometry*, Cambridge University Press, Cambridge, 2007. MR 2324354 (2008d:14001)

[Hum75] James E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York, 1975, Graduate Texts in Mathematics, No. 21. MR 0396773 (53 #633)

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR 1658959 (2000b:14018)

[Kol02] János Kollár, *Unirationality of cubic hypersurfaces*, J. Inst. Math. Jussieu **1** (2002), no. 3, 467–476. MR 1956057 (2003m:14082)

[Kol07a] ———, *Algebraic varieties over PAC fields*, Israel J. Math. **161** (2007), 89–101. MR 2350157 (2009a:12003)

[Kol07b] ———, *Lectures on resolution of singularities*, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007. MR 2289519 (2008f:14026)

[Kol08] ———, *Looking for rational curves on cubic hypersurfaces*, Higher-dimensional geometry over finite fields, NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur., vol. 16, IOS, Amsterdam, 2008, Notes by Ulrich Derenthal, pp. 92–122. MR 2484078 (2009k:14047)

[KSC04] János Kollár, Karen E. Smith, and Alessio Corti, *Rational and nearly rational varieties*, Cambridge Studies in Advanced Mathematics, vol. 92, Cambridge University Press, Cambridge, 2004. MR 2062787 (2005i:14063)

[LB09] V. Lakshmibai and Justin Brown, *Flag varieties*, Texts and Readings in Mathematics, vol. 53, Hindustan Book Agency, New Delhi, 2009, An interplay of geometry, combinatorics, and representation theory. MR 2474907 (2010f:14053)

[Mor65] L. J. Mordell, *On the conjecture for the rational points on a cubic surface*, J. London Math. Soc. **40** (1965), 149–158. MR 0169815 (30 #58)

[Mum95] David Mumford, *Algebraic geometry. I*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Complex projective varieties, Reprint of the 1976 edition. MR 1344216 (96d:14001)

[Rei88] Miles Reid, *Undergraduate algebraic geometry*, London Mathematical Society Student Texts, vol. 12, Cambridge University Press, Cambridge, 1988. MR 982494 (90a:14001)

[Rus00] Francesco Russo, *On a theorem of Severi*, Math. Ann. **316** (2000), no. 1, 1–17. MR 1735076 (2001h:14070)

[Sal79] Judith D. Sally, *Cohen-Macaulay local rings of maximal embedding dimension*, J. Algebra **56** (1979), no. 1, 168–183. MR 527163 (80e:14022)

[SD62] H. P. F. Swinnerton-Dyer, *Two special cubic surfaces*, Mathematika **9** (1962), 54–56. MR 0139989 (25 #3413)

[Ser56] Jean-Pierre Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 1–42. MR 0082175 (18,511a)

[Ser73] J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York, 1973, Translated from the French, Graduate Texts in Mathematics, No. 7. MR 0344216 (49 #8956)

[Sie88] C. L. Siegel, *Topics in complex function theory. Vols. I–III*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1988. MR 1008930 (90h:30002)

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009. MR 2514094 (2010i:11005)