FUNDAMENTAL GROUPS OF RATIONALLY CONNECTED VARIETIES

JÁNOS KOLLÁR

1. Introduction

Let X be a smooth, projective, unirational variety and $U \subset X$ an open set. The aim of this paper is to find a smooth rational curve $C \subset X$ such that the fundamental group of $C \cap U$ surjects onto the fundamental group of U. Following the methods of [Kollár99] and [Colliot-Thélène00], a positive answer over $\mathbb{C}$ translates to a positive answer over any p-adic field. This gives a rather geometric proof of the theorem of [Harbater87] about the existence of Galois covers of the line over p-adic fields (1.4). We also obtain a slight generalization of the results of [Colliot-Thélène00] about the existence of certain torsors over open subsets of the line over p-adic fields (1.6).

If $U = X$ then $\pi_1(X)$ is trivial (cf. (2.3)), thus any rational curve C will do. If $X \setminus U$ is a divisor with normal crossings and C intersects every irreducible component of $X \setminus U$ transversally, then the *normal* subgroup of $\pi_1(U)$ generated by the image of $\pi_1(C \cap U)$ equals $\pi_1(U)$ by a simple argument. (See, for instance, the beginning of (4.2).) It is also not hard to produce rational curves C such that the image of $\pi_1(C \cap U)$ has finite index in $\pi_1(U)$ (cf. (3.3)). These results suggest that we are very close to a complete answer, but surjectivity is not obvious. Differences between surjectivity and finiteness of the index appear in many similar situation; see, for instance, [Kollár95, Part I] or [Napier-Ramachandran98].

The present proof relies on the machinery of rationally connected varieties developed in the papers [Ko-Mi-Mo92a, Ko-Mi-Mo92b, Ko-Mi-Mo92c]. The relevant facts are recalled in Section 2.

The main geometric result is the following:

Theorem 1.1. *Let K be an algebraically closed field of characteristic zero and X a smooth, projective variety over K which is rationally connected (2.1). Let $U \subset X$ be an open subset and $x_0 \in U$ a point. Then there is an open subset $0 \in V \subset \mathbb{A}^1$ and a morphism $f : V \to U$*

such that $f(0) = x_0$ *and*

$$\pi_1(V, 0) \twoheadrightarrow \pi_1(U, x_0) \quad \text{is surjective.}$$

Moreover, we can assume that the following also hold:

1. $H^1(\mathbb{P}^1, \bar{f}^* T_X(-2)) = 0$ *where* $\bar{f} : \mathbb{P}^1 \to X$ *is the unique extension of* f.
2. $\bar{f}$ *is an embedding if* $\dim X \geq 3$ *and an immersion if* $\dim X = 2$.

Corollary 1.2. *Let* K *be a* p*-adic field and* X *a smooth, projective variety over* K *which is rationally connected over* $\bar{K}$. *Let* $U \subset X$ *be an open subset and* $x_0 \in U(K)$ *a point. Then there is an open subset* $0 \in V \subset \mathbb{A}^1$ *and a morphism* $f : V \to U$ *(all defined over* K*) such that* $f(0) = x_0$ *and*

$$\pi_1(V, 0) \twoheadrightarrow \pi_1(U, x_0) \quad \text{is surjective,}$$

where π_1 *here denotes the algebraic fundamental group.*

Remark 1.3. More generally, (1.2) holds for any field K of characteristic zero such that every curve with a smooth K-point contains a Zariski dense set of K-points. Characterizations of this property are given in [Pop96, 1.1]. The following are some interesting classes of such fields:

1. Fields complete with respect to a discrete valuation.
2. Quotient fields of local Henselian domains.
3. $\mathbb{R}$ and all real closed fields.
4. Pseudo algebraically closed fields, cf. [Fried-Jarden86, Chap. 10]

Corollary 1.4. [Harbater87] *Let* G *be a finite group and* K *a field of characteristic zero as in (1.3). Then there is a Galois cover* $g : C \to \mathbb{P}^1_K$ *with Galois group* G *such that* C *is geometrically irreducible and* $g^{-1}(0{:}1) \cong G$.

Proof. Let $G \subset GL(n, K)$ be a faithful representation. Set $U = GL(n)/G$ with quotient map $h : GL(n) \to U$ and let x_0 be the image of the identity matrix. Then U is unirational, thus by (1.2) there is a $0 \in V \subset \mathbb{A}^1$ and a morphism $f : V \to U$ such that $\pi_1(V) \twoheadrightarrow \pi_1(U)$ is onto. $h : GL(n) \to U$ is étale and proper, thus it corresponds to a quotient $\pi_1(U) \twoheadrightarrow G$. The fiber product $W := GL(n) \times_U V \to V$ corresponds to the surjective homomorphism

$$\pi_1(V) \twoheadrightarrow \pi_1(U) \twoheadrightarrow G.$$

Thus W is connected and $W \to V$ is a Galois cover with Galois group G. Since W has a K-point, it is also geometrically connected. The preimage of $0 \in V$ is isomorphic to G (the disjoint union of $|G|$ copies of Spec_K). $W \to V$ can be extended to a (ramified) Galois cover of the whole $\mathbb{P}^1_K$. $\square$

Remark 1.5. The above proof works in positive characteristic if we know that for every subgroup $H < G$ the quotient $GL(n)/H$ has a smooth compactification.

The following result was proved by [Colliot-Thélène00] for finite groups, which is probably the most important for applications.

Corollary 1.6. *Let K be a field of characteristic zero as in (1.3), G a linear algebraic group scheme over K and A a principal homogeneous G-space. Then there is an open set $0 \in V \subset \mathbb{A}^1_K$ and a geometrically irreducible G-torsor $g : W \to V$ such that $g^{-1}(0) \cong A$ (as a G-space).*

Proof. Assume that G acts on A from the left and choose an embedding $G \subset GL(n)$ over K. $A \times GL(n)$ has a diagonal left action by G and a right action by $GL(n)$ acting only on $GL(n)$.

The right G-action makes the morphism $h : G\backslash(A \times GL(n)) \to G\backslash(A \times GL(n))/G =: U$ into a G-torsor. Let $x_0 \in U$ be the image of $G\backslash(A \times G)$. The fiber of h over x_0 is isomorphic to A. Let G^0 be the connected component of G. Then $G\backslash(A \times GL(n)) \to G\backslash(A \times GL(n))/G^0$ is smooth with connected fibers and $G\backslash(A \times GL(n))/G^0 \to U$ is étale and proper. Let $0 \in V \subset \mathbb{A}^1$ and $f : V \to U$ be as in the proof of (1.4). Then $W := (G\backslash(A \times GL(n))) \times_U V$ works. $\square$

A geometric application is the following:

Corollary 1.7. *For every $2 \leq g \leq 13$ there is an open set $0 \in V_g \subset \mathbb{C}$ and a smooth proper morphism with genus g fibers $S_g \to V_g$ such that the image of the monodromy representation is the full Teichmüller group.*

Proof. The moduli of curves is unirational for $g \leq 13$. Apply (1.1) to the open subset of curves without automorphisms $U_g \subset M_g$. $\square$

2. Rationally connected varieties and morphisms of rational curves

Rationally connected varieties were introduced in [Ko-Mi-Mo92b] as a higher dimensional generalization of rational and unirational varieties. A surface is rationally connected iff it is rational. In higher dimensions rationality and unirationality are very hard to check. The notion of rational connectedness concentrates on rational curves on a variety. The following characterizations were developed in the papers [Ko-Mi-Mo92b], [Kollár96, IV.3], [Kollár98, 4.1.2].

Definition–Theorem 2.1. Let K be an algebraically closed field of characteristic zero. A smooth proper variety X over K is called *rationally connected* if it satisfies any of the following equivalent properties:

1. There is an open subset $\emptyset \neq X^0 \subset X$, such that for every $x_1, x_2 \in X^0$, there is a morphism $f : \mathbb{P}^1 \to X$ satisfying $x_1, x_2 \in f(\mathbb{P}^1)$.
2. For every $x_1, \dots, x_n \in X$, there is a morphism $f : \mathbb{P}^1 \to X$ satisfying $x_1, \dots, x_n \in f(\mathbb{P}^1)$.
3. There is a morphism $f : \mathbb{P}^1 \to X$ such that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$. (This is equivalent to f^*T_X being ample.)
4. There is a variety P and a dominant morphism $F : \mathbb{P}^1 \times P \to X$ such that $F((0{:}1) \times P)$ is a point. We can also assume that $H^1(\mathbb{P}^1, F_p^*T_X(-2)) = 0$ for every p where $F_p := F|_{\mathbb{P}^1 \times \{p\}}$.
5. Let $z_1, \dots, z_n \in \mathbb{P}^1$ be distinct points and $m_1, \dots, m_n$ natural numbers. For each $i = 1, \dots, n$ let $f_i : \operatorname{Spec} K[t]/(t^{m_i}) \to X$ be a morphism. Then there is a variety P and a dominant morphism $F : \mathbb{P}^1 \times P \to X$ such that
   (a) the Taylor series of F_p at z_i coincides with f_i up to order m_i for every i and $p \in P$,
   (b) $H^1(\mathbb{P}^1, F_p^*T_X(-\sum m_i)) = 0$ for every i and p. □

Another easy result that we need is the following.

Lemma 2.2. *(cf.* [Kollár96, II.3.5.4,II.3.10.1 and II.3.11]*) Let X, P be smooth varieties and $F : \mathbb{P}^1 \times P \to X$ a dominant morphism such that $F((0{:}1) \times P)$ is a point. Then there is a dense open set $P^0 \subset P$ such that $H^1(\mathbb{P}^1, F_p^*T_X(-2)) = 0$ for every $p \in P^0$ where $F_p := F|_{\mathbb{P}^1 \times \{p\}}$.*

Conversely, let $f : \mathbb{P}^1 \to X$ be a morphism such that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$. Then there is a pointed variety $p_0 \in P$ and a dominant morphism $F : \mathbb{P}^1 \times P \to X$ such that $F((0{:}1) \times P)$ is a point, F is smooth away from $(0{:}1) \times P$ and $F_{p_0} = f$. □

The following result was proved by [Serre59] for unirational varieties and by [Campana91, Ko-Mi-Mo92c] in general.

Proposition 2.3. *A smooth, proper, rationally connected variety is simply connected.* □

In the course of the proof we repeatedly encounter the following situation. We have morphisms $f^i : \mathbb{P}^1 \to X$ each passing through the same point $x_0 \in X$. We would like to have a family of morphisms $f_t : \mathbb{P}^1 \to X$ such that the union of the maps f^i can be considered as the limit of the maps f_t as $t \mapsto 0$. The following lemma is a technical formulation of this idea. Its statement is a bit complicated since we also want to keep track of the field over which the f_t are defined.

Lemma 2.4. *(cf.* [Kollár99, 3.2]*) Let K be a field and $x_0 \in X$ a smooth, proper, pointed K-scheme. Let S be a zero dimensional reduced K-scheme and $f_0 : \mathbb{P}^1_S \to X$ a morphism such that*

1. *$H^1(\mathbb{P}^1_S, f_0^* T_X(-2)) = 0$, and*
2. *$f_0(S \times \{(0\!:\!1)\}) = \{x_0\}$.*

Then there are

3. *a smooth pointed curve $0 \in D$ over K,*
4. *a smooth surface Y with a proper morphism $h : Y \to D$ and a section $B \subset Y$ of h, and*
5. *a morphism $F : Y \to X$,*

such that

6. *$h^{-1}(0)$ is the union of $\mathbb{P}^1_S$ with a copy B_0 of $\mathbb{P}^1_K$ such that $B_0 \cap \mathbb{P}^1_S = S \times \{(0\!:\!1)\}$ and $B_0 \cap B$ is a single point.*
7. *F restricted to $\mathbb{P}^1_S$ coincides with f_0 and $F(B_0 \cup B) = \{x_0\}$,*
8. *$h^{-1}(D^0) \cong \mathbb{P}^1_K \times D^0$, where $D^0 := D \setminus \{0\}$,*
9. *$H^1(\mathbb{P}^1_d, F_d^* T_X(-2)) = 0$ for every $d \in D^0$ where $F_d := F|_{h^{-1}(d)}$.*

Proof. Let us start with any curve $0 \in D'$ and $Y' := \mathbb{P}^1 \times D'$. Let $S' \subset \mathbb{P}^1 \times \{0\}$ be a subscheme isomorphic to S and Y'' the blow up of $S' \subset D'$ with projection $h' : Y'' \to D'$. We can define a morphism $f' : (h')^{-1}(0) \to X$ by setting f' to be f_0 on the exceptional divisor of $Y'' \to Y'$ and the constant morphism to $\{x_0\}$ on the birational transform of $\mathbb{P}^1 \times \{0\}$. Fix any section $B' \subset Y'$ which does not pass through S'. We are done if f' can be extended to $F' : Y'' \to X$ as required. In general this is not possible, but such an extension exists after a suitable étale base change $(0 \in D) \to (0 \in D')$. This is proved in [Ko-Mi-Mo92b, 1.2] and [Kollár99, 2.2]. $\square$

3. Fundamental groups of fibers of morphisms

We need some easy results about the variation of fundamental groups for fibers of nonproper morphisms.

Lemma 3.1. *Let K be an algebraically closed field of characteristic zero, $\bar{Z}, D$ irreducible K-varieties and $f : \bar{Z} \to D$ a smooth and proper morphism with connected fibers. Let $Z \subset \bar{Z}$ be an open subset such that $(\bar{Z} \setminus Z) \to D$ is smooth. Let $z_0 \in Z(K)$ be a K-point, $d_0 = f(z_0)$ and Z_0 the fiber of $Z \to D$ through z_0. Then there is an exact sequence*

$$\pi_1(Z_0, z_0) \to \pi_1(Z, z_0) \to \pi_1(D, d_0) \to 1.$$

Proof. Over $\mathbb{C}$ the fibration $Z(\mathbb{C}) \to D(\mathbb{C})$ is a topological fiber bundle, thus we have the above exact sequence. To settle the algebraic case, let $Y \to Z$ be any connected finite degree étale cover and extend it to a finite morphism $\bar{Y} \to \bar{Z}$ where $\bar{Y}$ is normal. Since $\bar{Z} \setminus Z$ is smooth over D, the same holds for $\bar{Y} \to D$. (This is a special case of Abhyankar's lemma, cf. [SGA1, XIII.5.2].) The generic fiber of $\bar{Y} \to D$ is irreducible. Since $\bar{Y} \to D$ is smooth and proper, every fiber is irreducible. This is equivalent to the exactness of the above sequence. □

The following technical lemma is an upper semi-continuity statement for the fundamental groups of fibers of nonproper morphisms.

Lemma 3.2. *Let K be an algebraically closed field of characteristic zero, W a normal surface over K, $f : W \to D$ a (not necessarily proper) morphism to a curve with connected fibers. Let $B \subset W$ be a connected subset, one of whose irreducible components is a section of f. Let $d_0 \in D$ be a K-point and C_0 an irreducible component of $f^{-1}(d_0)$ with a K-point $b_0 \in C_0 \cap B$. Let $x_0 \in U$ be a pointed K-scheme and $h : W \to U$ a morphism such that $h(B) = \{x_0\}$. Then there is an open subset $D^0 \subset D$ such that for every $d \in D^0$ and for every K-point $b_d \in C_d \cap B$,*

$$\operatorname{im}[\pi_1(C_d, b_d) \to \pi_1(U, x_0)] \supset \operatorname{im}[\pi_1(C_0, b_0) \to \pi_1(U, x_0)].$$

Proof. Choose a normal compactification $\bar{f} : \bar{W} \to D$. Let $D^0 \subset D$ be any open subset such that $\bar{f}$ is smooth with irreducible fibers over D^0 and $\bar{W} \setminus W \to D$ is unramified over D^0. Set $W^0 := f^{-1}(D^0)$. $\pi_1(C_0, b_0) \to \pi_1(U, x_0)$ factors through $\pi_1(W, b_0) \to \pi_1(U, x_0)$ and it has the same image as $\pi_1(W, b) \to \pi_1(U, x_0)$ for any $b \in B(K)$. $\pi_1(W^0, b) \to \pi_1(W, b)$ is surjective by (3.3). By (3.1) there is an exact sequence

$$\pi_1(C_d, b_d) \to \pi_1(W_0, b_d) \to \pi_1(D^0, d) \to 1.$$

Let $B^0 \subset B \cap W^0$ be a section of f. Then $\pi_1(B^0, b_d)$ maps onto $\pi_1(D^0, d)$ and the image of $\pi_1(B^0, b_d)$ in $\pi_1(U, x_0)$ is trivial. Thus

$$\operatorname{im}[\pi_1(C_d, b_d) \to \pi_1(U, x_0)] = \operatorname{im}[\pi_1(W^0, b_d) \to \pi_1(U, x_0)]$$

and we are done. □

Lemma 3.3. *(cf.* [SGA1, IX.5.6], [Campana91, 1.3]*). Let X, Y be normal varieties, $x_0 \in X$ a closed point and $f : X \to Y$ a dominant morphism. Then $\operatorname{im}[\pi_1(X, x_0) \to \pi_1(Y, f(x_0))]$ has finite index in $\pi_1(Y, f(x_0))$. If f is an open immersion then $\pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is surjective.* □

4. Proof of the main results

The theory of free morphisms of curves (cf. [Kollár96, II.3]) suggests that morphisms $f : \mathbb{P}^1 \to X$ such that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$ behave rather predictably. Therefore we concentrate on such morphisms. First we establish that there is a unique maximal subgroup of $\pi_1(U)$ obtainable from such a morphism.

Lemma 4.1. *Let X be a smooth, proper, rationally connected variety over an algebraically closed field of characteristic zero. Let $U \subset X$ be an open set and $x_0 \in U$ a point. Then there is a unique finite index subgroup $H < \pi_1(U, x_0)$ with the following properties:*

1. *There is a morphism $f : \mathbb{P}^1 \to X$ such that $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$, $f(0{:}1) = x_0$ and $H = \operatorname{im}[\pi_1(f^{-1}(U), (0{:}1)) \to \pi_1(U, x_0)]$.*
2. *Let g be any morphism $g : \mathbb{P}^1 \to X$ such that $g(0{:}1) = x_0$ and $H^1(\mathbb{P}^1, g^*T_X(-2)) = 0$. Then $H \supset \operatorname{im}[\pi_1(g^{-1}(U), (0{:}1)) \to \pi_1(U, x_0)]$.*

Proof. First we find one morphism as in (4.1.2) such that

$$\operatorname{im}[\pi_1(g^{-1}(U), (0{:}1)) \to \pi_1(U, x_0)]$$

has finite index in $\pi_1(U, x_0)$.

Let $F : \mathbb{P}^1 \times P \to X$ be as in (2.1.4). Let $P^0 \subset P$ be an open subset such that $F^{-1}(X \setminus U) \to P$ is étale over P^0. Pick any poit $p \in P^0$. By (3.1) there is an exact sequence

$$\begin{aligned} \pi_1(F_p^{-1}(U), (0{:}1) \times \{p\}) &\to \pi_1((\mathbb{P}^1 \times P^0) \cap F^{-1}(U), (0{:}1) \times \{p\}) \\ &\to \pi_1(P^0, p) \to 1. \end{aligned}$$

The section $(0{:}1) \times P^0$ is mapped to a point by F, thus

$$\pi_1(F_p^{-1}(U), (0{:}1) \times \{p\}) \quad \text{and} \quad \pi_1((\mathbb{P}^1 \times P^0) \cap F^{-1}(U), (0{:}1) \times \{p\})$$

have the same image in $\pi_1(U, x_0)$. The latter image has finite index by (3.3), thus $g := F_p : \mathbb{P}^1 \to X$ is as desired.

In order to finish, it is sufficient to prove that if $g_1, g_2 : \mathbb{P}^1 \to X$ are as in (4.1.2) then there is a third morphism $g : \mathbb{P}^1 \to X$ such that

$$\operatorname{im}[\pi_1(g^{-1}(U), (0{:}1)) \to \pi_1(U, x_0)] \supset \operatorname{im}[\pi_1(g_i^{-1}(U), (0{:}1)) \to \pi_1(U, x_0)]$$

for $i = 1, 2$. To do this, let $S = \{1, 2\}$ be a 2 point scheme and take $f_0 : \mathbb{P}^1_S \to X$ to be g_i on $\{i\} \times \mathbb{P}^1$. Construct $h : Y \to D$ and $F : Y \to X$ as in (2.4). Set $W := F^{-1}(U)$ and apply (3.2) twice for $i = 1, 2$ with $C_0 := f_i^{-1}(U)$. For any $d \in D^0$, $g := F_d$ has the required property. $\square$

4.2 (Proof of (1.1)).

Let $H < \pi_1(U, x_0)$ be the subgroup obtained in (4.1). We are done if $H = \pi_1(U, x_0)$. Otherwise there is a corresponding irreducible étale cover $(x_0' \in U') \to (x_0 \in U)$. Let $w : X' \to X$ be the normalization of X in the function field of U'. X is simply connected by (2.3), hence by the purity of branch loci (cf. [SGA1, X.3.1]) there is a divisor $D' \subset X'$ such that w ramifies along D'. Let $D \subset X$ be the image of D'. By construction $D \subset X \setminus U$. We derive a contradiction as follows.

Let $f : \mathbb{P}^1 \to X$ be a morphism such that $f(0{:}1) = x_0$ and $f(\mathbb{P}^1)$ intersects D transversally at a point $x_1 := f(1{:}1)$. This implies that the local fundamental group of $\mathbb{P}^1 \setminus \{(1{:}1)\}$ at $(1{:}1)$ surjects onto the local fundamental group of $X \setminus D$ at x_1. (See [Grothendieck-Murre71] for the local fundamental group of a divisor in a variety. There it is called the fundamental group of the formal neighborhood of a divisor.) Therefore, if f lifts to $f' : \mathbb{P}^1 \to X'$ then $X' \to X$ is étale at $f'(1{:}1)$. If $X' \to X$ is a Galois extension the we already have a contradiction since $X' \to X$ ramifies everywhere above D. In the non–Galois case p may be unramified along some of the irreducible components of $p^{-1}(D)$, and we only get a contradiction if the image of f' intersects D'.

If $X' \to X$ is not Galois, we need to proceed in a somewhat roundabout way. First I give the outlines; a precise version is given afterwards.

It is clear from the definition that there are many maps $\mathbb{P}^1 \to X'$, thus X' (or rather any desingularization of X') should be rationally connected. This indeed follows from (4.3) applied to any $F : \mathbb{P}^1 \times P \to X$ as in (2.1.4). Thus by (2.1.5) there is a morphism $f' : \mathbb{P}^1 \to X'$ which passes through x_0' and intersects D' at a smooth point x_1'. We obtain a contradiction if $w \circ f'$ is the limit of a sequence of maps $f_t : \mathbb{P}^1 \to X$ such that

1. f_t lifts to $f_t' : \mathbb{P}^1 \to X'$ and f' is the limit of the maps f_t',
2. the image of f_t intersects D transversally.

First we prove a general lifting property for families of morphisms and then we proceed to construct the morphism f'.

Lemma 4.3. *Let V be a normal variety and $G : \mathbb{P}^1 \times V \to X$ a morphism such that $G((0{:}1) \times V) = \{x_0\}$. Assume that $H^1(\mathbb{P}^1, G_v^* T_X(-2)) = 0$ for some $v \in V$. Then G can be lifted to $G' : \mathbb{P}^1 \times V \to X'$.*

Proof. First we show that such a lifting exists over an open subset of $\mathbb{P}^1 \times V$. Choose $V^0 \subset V$ such that $H^1(\mathbb{P}^1, G_v^* T_X(-2)) = 0$ for every $v \in V^0$ (this is possible by the upper semi continuity of cohomology groups) and such that we have an exact sequence

$$\pi_1(G_v^{-1}(U), (0{:}1) \times v) \to \pi_1((G^0)^{-1}(U), (0{:}1) \times v) \to \pi_1(V^0, v) \to 1$$

for every $v \in V^0$ where $G^0 := G|_{\mathbb{P}^1 \times V^0}$ (this is possible by (3.1)). Then

$$\operatorname{im}[\pi_1((G^0)^{-1}(U), (0{:}1) \times v) \to \pi_1(U, x_0)] \subset H,$$

thus $G^0|_{(G^0)^{-1}(U)}$ can be lifted to $G' : (G^0)^{-1}(U) \to U'$. This extends to a rational map $G' : \mathbb{P}^1 \times V^0 \dashrightarrow X'$.

Next let $\Gamma \subset (\mathbb{P}^1 \times V) \times X$ be the graph of G and $\Gamma^* \subset (\mathbb{P}^1 \times V) \times X'$ its preimage. The rational lifting G' corresponds to an irreducible component $\Gamma' \subset \Gamma^*$ such that the projection $\Gamma' \to (\mathbb{P}^1 \times V)$ is birational. Since $X' \to X$ is finite, so is $\Gamma^* \to \Gamma$. Thus $\Gamma' \to \Gamma \to \mathbb{P}^1 \times V$ is finite and birational, hence an isomorphism. $\square$

Let $\phi : X'' \to X'$ be any desingularization and $x_1' \in D'$ a point such that D' and X' are smooth at x_1' and ϕ^{-1} is a local isomorphism near x_1'. By (2.1.5) there is a dominant morphism $F : \mathbb{P}^1 \times P \to X'$ such that

1. $F((0{:}1) \times P) = \{x_0'\}$,
2. $F((1{:}1) \times P) = \{x_1'\}$, and
3. the image of $F_p : \mathbb{P}^1 \to X'$ is transversal to D' at x_1' for every $p \in P$.

Let us now consider the dominant morphism $w \circ F : \mathbb{P}^1 \times P \to X$. By the first part of (2.2), there is a $p_0 \in P$ such that $H^1(\mathbb{P}^1, (w \circ F_{p_0})^* T_X(-2)) = 0$. Thus by the second part of (2.2) there is a pointed variety $q_0 \in Q$ and a morphism $G : \mathbb{P}^1 \times Q \to X$ such that $G_{q_0} = w \circ F_{p_0}$ and G is smooth away from $(0{:}1) \times Q$. In particular, $G^{-1}(D) \subset \mathbb{P}^1 \times Q$ is a generically smooth divisor, hence there is a dense open set $Q^0 \subset Q$ such that the projection $G^{-1}(D) \to Q$ is smooth over Q^0. This means that the image of $G_q : \mathbb{P}^1 \to X$ intersects D transversally for every $q \in Q^0$. (Note that $G^{-1}(D)$ denotes the inverse image scheme.)

By (4.3), G can be lifted to $G' : \mathbb{P}^1 \times Q \to X'$. On $\mathbb{P}^1 \times \{q_0\}$ the lifting agrees with F_{p_0}, hence $G'(\mathbb{P}^1 \times \{q_0\})$ intersects D' at the point x_1'. This implies that $(G')^{-1}(D')$ is a divisor and so by shrinking Q we may assume that the image of G_q' intersects D' for every $q \in Q$. Thus $G_q = w \circ G_q'$ is never transversal to D, a contradiction.

Condition (1.1.1) holds by construction and a general choice of f satisfies (1.1.2) by [Kollár96, II.3.14]. $\square$

4.4 (Proof of (1.2))**.**

Pick $f_1 : V_1 \to U_{\bar{K}}$ defined over $\bar{K}$ such that $\pi_1(V_1, 0) \to \pi_1(U_{\bar{K}}, x_0)$ is surjective and $H^1(\mathbb{P}^1, \bar{f}_1^* T_X(-2)) = 0$. f_1 is defined over a finite Galois extension $L \supset K$; let $f_i : V_i \to U_{\bar{K}}$ be its conjugates. Each of these extends to a morphism $\bar{f}_i : \mathbb{P}^1 \to X_{\bar{K}}$. Let $S = \operatorname{Spec}_K L$. We can view the morphisms f_i as one morphism $f_0 : \mathbb{P}^1_S \to X$ defined over K. By (2.4) we obtain $h : Y \to D$ and $F : Y \to X$, all defined over K. Let $d \in D^0(\bar{K})$ be any point. Then by (3.2) we see that

$$\operatorname{im}[\pi_1(Y_d, 0) \xrightarrow{F_d} \pi_1(U_{\bar{K}}, x_0)] \supset \operatorname{im}[\pi_1(V_1, 0) \xrightarrow{f_1} \pi_1(U_{\bar{K}}, x_0)],$$

and the latter image is $\pi_1(U_{\bar K}, x_0)$ by assumption. D^0 is a Zariski open set in a curve D with a smooth K-point 0. By (1.3) this implies that $D(K)$ is dense in D, hence $D^0(K) \neq \emptyset$. By choosing $d \in D^0(K)$ we obtain an open subset $0 \in V \subset \mathbb{A}^1$ and a morphism $f : V \to U$ (all defined over K) such that $f(0) = x_0$ and

$$\pi_1(V_{\bar K}, 0) \twoheadrightarrow \pi_1(U_{\bar K}, x_0) \quad \text{is surjective.}$$

The fundamental group of a K-scheme W is related to the fundamental group of $W_{\bar K}$ by the exact sequence (cf. [SGA1, IX.6.1])

$$1 \to \pi_1(W_{\bar K}, 0) \to \pi_1(W, 0) \to \mathrm{Gal}(\bar K/K) \to 1.$$

This implies that

$$\pi_1(V, 0) \twoheadrightarrow \pi_1(U, x_0) \quad \text{is also surjective.}$$

□

Example 4.5. For every $n \geq 4$ here I give an example of a rational threefold X, a normal crossing divisor $F \subset X$ and a smooth rational curve $B \subset X$ such that B intersects F everywhere transversally, B intersects every irreducible component of F and the image of $\pi_1(B \setminus F) \to \pi_1(X \setminus F)$ has index n in $\pi_1(X \setminus F)$.

Let us start with a similar surface example. Let $g_1 : \mathbb{P}^1 \to \mathbb{P}^1$ be a degree n morphism with critical points $x_1, \dots, x_{2n-2} \in \mathbb{P}^1$ and different critical values $y_1, \dots, y_{2n-2}$, Choose 3 other points $x_{2n-1}, x_{2n}, x_{2n+1} \in \mathbb{P}^1$ such that $g_1(x_{2n-1})$ and $g_1(x_{2n})$ are different critical values of g_1 and $g_1(x_{2n+1})$ is not a critical value of g_1. Let $g_2 : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism with $g_2(x_i) = x_i$ for $i \leq 2n+1$. Consider the morphism $h : (g_2, g_1) : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$

Set $D = \mathbb{P}^1 \times \{y_1, \dots, y_{2n-2}\}$ and $U = \mathbb{P}^1 \times \mathbb{P}^1 \setminus D$.

$\mathrm{im}\, h$ intersects every irreducible componnet of D transversally at $n-2$ points and the image of $\pi_1(h^{-1}(U)) \to \pi_1(U)$ has index n in $\pi_1(U)$.

Here $\mathrm{im}\, h$ is also tangent to every irreducible component of D. The tangencies can be resolved by 2n blow ups, but then the birational transform of $\mathrm{im}\, h$ does not intersect every boundary component. To remedy this situation, take another morphism $g_3 : \mathbb{P}^1 \to \mathbb{P}^1$ with $g_3(x_i) = x_i$ for $i \leq 2n+1$. Set $Y := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $F = D \times \mathbb{P}^1$ and $H := (g_3, g_2, g_1) : \mathbb{P}^1 \to Y$. Again the only problem is that $\mathrm{im}\, H$ is tangent to every irreducible component of F. These can be resolved by blowing up two suitable smooth curves. First we take a smooth curve which passes through every point of tangency and also through the point $H(x_{2n-1})$. After blow up, the birational transform of $\mathrm{im}\, H$ intersects every boundary component transversally but above each point there is a point common to two boundary components and to the birational transform of $\operatorname{im} H$. Next take a smooth curve which passes through all these points with a general tangent direction there and also through $H(x_{2n})$. We can also assume that neither of the two curves passes through $H(x_{2n+1})$. Doing two such blow ups creates two new boundary components and the birational transform of $\operatorname{im} H$ intersects both of them. The fundamental group computation is unchanged.

Varying g_2, g_3 we obtain many morphisms all of which pass through the point $H(x_{2n+1})$.

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Princeton University, Princeton NJ 08544-1000

kollar@math.princeton.edu