

Families of varieties of general type

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with the collaboration of
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Preface

The aim of this book is to generalize the moduli theory of algebraic curves—developed by Riemann, Klein, Teichmüller, Mumford and Deligne—to higher dimensional algebraic varieties.

Starting with the theory of algebraic surfaces worked out by Castelnuovo, Enriques, Severi, Kodaira, and ending with Mori’s program, it became clear that the correct analog of a smooth projective curve of genus ≥ 2 is a smooth projective variety with ample canonical class. We establish a moduli theory for these objects, their limits and generalizations.

The first attempt to write a book on higher dimensional moduli theory was the 1993 Summer School at Salt Lake City. Some notes were written, but it soon became evident that, while the general aims of a theory were clear, most of the theorems were open and even some of the basic definitions unsettled.

The project was taken up again at an AIM conference in 2004, which eventually resulted in solving the moduli-theoretic problems related to singularities; these were written up in [Kol13b].

After almost 30 years, we now have a complete theory, the result of the work of many people.

While much of the early theory emphasized the construction of moduli spaces, later developments in the theory of stacks emphasized families. We also follow this approach, and spend most of the time understanding families. Once this is done at the right level, the existence of moduli spaces becomes a natural consequence.

Two major approaches to moduli—the geometric invariant theory of Mumford and the Hodge theory of Griffiths—are mostly absent from this book. Both of these are very powerful and give a lot of information in the cases when they are known to apply. They each deserve a full, updated treatment of their own. However, so far neither gave a full description of the moduli of surfaces, much less of higher dimensional varieties. It would be very interesting to develop a synthesis of the 3 methods and gain better understanding in the future.

ACKNOWLEDGMENTS . Throughout the years, I learned a lot from my teachers, colleagues and students.

My interest in moduli theory was kindled by my thesis advisor T. Matsusaka, and the early influences of S. Mori and N.I. Shepherd-Barron have been crucial to my understanding of the subject.

The original 1993 group included S. Abramovich, V. Alexeev, A. Corti, A. Grassi, B. Hassett, S. Keel, S. Kovács, T. Luo, K. Matsuki, J. M^cKernan, G. Megyesi and D. Morrison; many of them have been active in this area since. My students A. Corti, S. Kovács, T. Kuwata, E. Szabó, N. Tziolas worked on several aspects of the early theory.

I gave several lecture series about moduli at Princeton, the many comments and corrections of colleagues K. Ascher, G. Farkas, M. Fulger, S. Grushevsky, M. Lieblich, J. Moraga, T. Murayama, A. Okounkov, R. Pandharipande, Zs. Patakfalvi, C. Raicu, J. Waldron, and students C. Araujo, G. Di Cerbo, A. Hogadi, L. Ji, D. Kim, Y. Liu, A. Sengupta, C. Stibitz, Y-C. Tu, D. Villalobos-Paz, C. Xu, Z. Zhuang, R. H. Zong have been very helpful.

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Moduli theory has been developed and shaped by the works on many people. Advances in minimal model theory—especially the series of papers by C. Hacon, J. McKernan, and C. Xu—made it possible to extend the theory from surfaces to all dimensions. The projectivity of moduli spaces was gradually proved by E. Viehweg, O. Fujino, S. Kovács and Zs. Patakfalvi. After early works of V. Alexeev and P. Hacking, many examples have been worked out by V. Alexeev and his co-authors, A. Brunsyate, P. Engel, A. Knudson, R. Pardini and A. Thompson. Recent works of K. Ascher, D. Bejleri, K. DeVleming, G. Inchiostro and Y. Liu give very detailed information on especially important examples.

The influences of V. Alexeev, A. Corti, S. Kovács and C. Xu have been especially important for me.

Notation

We follow the notation and conventions of [Har77, KM98, Koll13b].

Notation in standard usage. The following are standard in the birational geometry literature, but may not be generally known.

A pair (X, D) consist of a scheme X and a divisor D on it, the coefficients can be in \mathbb{Z}, \mathbb{Q} or \mathbb{R} . The divisor is frequently called the *boundary* of the pair. When we start with a scheme X and a compactification $X^* \supset X$, frequently $X^* \setminus X$ is also called a *boundary*. (When pairs were first used systematically, D was constructed as $X^* \setminus X$, and the name stuck. Neither of these uses corresponds to the standard notion of boundary in topology.)

The *rounding down* of a real number d is denoted by $\lfloor d \rfloor$. For a divisor $D = \sum d_i D_i$ we use $\lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i$, where the D_i are distinct, irreducible divisors.

A divisor D on a proper, irreducible variety is *big* if $\lfloor mD \rfloor$ defines a birational map for $m \gg 1$.

We are mostly interested in proper pairs (X, Δ) with log canonical singularities (11.4). Such a pair is of *general type* if $K_X + \Delta$ is big. In examples we encounter pairs with $K_X + \Delta \equiv 0$ (called (log)-Calabi-Yau pairs) or with $-(K_X + \Delta)$ is ample (called (log)-Fano pairs).

A *map* or *rational map* is defined on a dense set; it is denoted by $-\dashrightarrow$. A *morphism* is everywhere defined; it is denoted by \rightarrow .

The exceptional set of a morphism is denoted by $\text{Ex}(f)$.

Affine n -space with coordinates (x_1, \dots, x_n) is denoted by $\mathbb{A}^n(x_1, \dots, x_n)$ or $\mathbb{A}_{\mathbf{x}}^n$. Same conventions for \mathbb{P}^n .

$\mathbb{A}_{\mathbf{x}}^m / \frac{1}{n}(a_1, \dots, a_m)$ denotes the quotient of $\mathbb{A}_{\mathbf{x}}^m$ by the action $x_i \mapsto \epsilon^{a_i} x_i$ where ϵ is a primitive n th root of unity.

The *completion* of a pointed scheme $(x \in X)$ is denoted by \hat{X} , or \hat{X}_x is we want to emphasize the point. For $\hat{\mathbb{A}}^n$, the point is assumed the origin, unless otherwise noted.

We distinguish the Picard group Pic and the Picard scheme **Pic**.

Our conventions. The following is a list of conventions that we follow most of the time, but define them at each occurrence.

The normalization of a scheme X is usually denoted by \bar{X} . If D is a reduced divisor on X , then usually \bar{D} denotes its preimage in \bar{X} . Then \bar{D}^n denotes the normalization of \bar{D} . Unfortunately, $\bar{}$ is also frequently used to denote the compactification of a scheme or moduli space.

$S^\circ \subset S$ usually denotes an open, dense subset. Then sheaves or divisors on S° are usually indicated by F° or D° . If G is an algebraic group then G° denotes the identity component.

We write moduli functors in italic and moduli spaces in roman. Thus for stable varieties we have \mathcal{SV} (functor) and SV (moduli space).

We use $\text{Hom}_X(F, G)$ for the set of homomorphisms and $\mathcal{H}om_X(F, G)$ for the sheaf of homomorphisms.

Base change. Assume that we are given objects over a base scheme S and a morphism $q : T \rightarrow S$. The objects obtained by pull-back to T are usually denoted either by a subscript T or by q^* . The fiber over a point $s \in S$ is denoted by a subscript s .

The most common usage is when $X \rightarrow S$ is a morphism and $X_T := X \times_S T$. If F is a sheaf on X , its pull-back to X_T is denoted by F_T . However, we frequently encounter the situation that the fiber product is not the ‘right’ pull-back, it needs to be ‘corrected.’ Roughly speaking, this happens when the fiber product picks up some embedded subsheaf, and the ‘correct’ pull-back is the quotient by it. We call this the *divisorial pull-back*, see (4.1.7). The divisorial pull-back of a sheaf F is denoted by $q^{[*]}F$.

Brackets are used to denote something naturally associated to an object. We use it to denote the cycle associated to a subscheme (1.3), the point in the moduli space corresponding to a variety/pair or the ‘corrected’ fiber as in (2.73).

CHAPTER 1

Introduction

The moduli space of smooth projective curves of genus $g \geq 2$, and its compactification by the moduli space of stable projective curves of genus g , are, quite possibly, the most studied of all algebraic varieties.

The aim of this book is to generalize the moduli theory of curves to surfaces and to higher dimensional varieties. In the introduction we start to outline how this is done, and, more importantly, to explain why the answer for surfaces is much more complicated than for curves. On the positive side, once we get the moduli theory of surfaces right, the higher dimensional theory works the same.

Section 1.1 is a quick review of the history of moduli problems, culminating in an outline of the basic moduli theory of curves. The paper [AJP16] led me to look at some of the early works on moduli, including Riemann, Cayley, Klein, Hilbert, Siegel, Teichmüller, Weil, Grothendieck and Mumford. This gave a much better understanding of how the modern theory relates to the earlier works; see [Kol18d] for an account that emphasizes the historical connections.

In Section 1.2 we outline how the theory should unfold for higher dimensional varieties. Details of going from curves to higher dimensions are given in the next 2 sections. Section 1.3 introduces canonical models, which are the basic objects of moduli theory in higher dimensions. Starting from stable curves, Section 1.4 leads up to the definition of stable varieties, their higher dimensional analogs. Then we show, by a series of examples, why flat families of stable varieties are *not* the correct higher dimensional analogs of flat families of stable curves. Finding the correct replacement has been one of the main difficulties of the whole theory.

While the moduli theory of curves serves as our guideline, it also has many good properties that do not generalize. Sections 1.5–1.9 are devoted to examples that show what can go wrong with moduli theory in general, or with stable varieties in particular.

First in Section 1.5 we show that the simple combinatorial recipe of going from a nodal curve to stable curve has no analog for surfaces. Next we give a collection of examples showing how easy it is to end up with rather horrible moduli problems. Hypersurfaces and other interesting examples are discussed in Section 1.6, and alternate compactifications of the moduli of curves in Section 1.7. Section 1.8 illustrates the differences between fine and coarse moduli spaces.

Problems with our theory in positive characteristic are detailed in Section 1.9

1.1. Short history of moduli problems

Let \mathbf{V} be a ‘reasonable’ class of objects in algebraic geometry, for instance, \mathbf{V} could be all subvarieties of \mathbb{P}^n , all coherent sheaves on \mathbb{P}^n , all smooth curves or all projective varieties. The aim of the theory of moduli is to understand all ‘reasonable’ families of objects in \mathbf{V} , and to construct an algebraic variety (or

scheme, or algebraic space) whose points are in ‘natural’ one-to-one correspondence with the objects in \mathbf{V} . If such a variety exists, we call it the *moduli space* of \mathbf{V} , and denote it by $M_{\mathbf{V}}$. The simplest, classical examples are given by the theory of linear systems and families of linear systems.

1.1 (Linear systems). Let X be a smooth projective variety over an algebraically closed field k and L a line bundle on X . The corresponding linear system is

$$\mathcal{L}inSys(X, L) = \{\text{effective divisors } D \text{ such that } \mathcal{O}_X(D) \cong L\}.$$

The objects in $\mathcal{L}inSys(X, L)$ are in natural one-to-one correspondence with the points of the projective space $\mathbb{P}(H^0(X, L)^\vee)$ which is traditionally denoted by $|L|$. (We follow the Grothendieck convention for \mathbb{P} as in [Har77].) Thus, for every effective divisor D such that $\mathcal{O}_X(D) \cong L$, there is a unique point $[D] \in |L|$.

Moreover, this correspondence between divisors and points is given by a universal family of divisors over $|L|$. That is, there is an effective Cartier divisor $\text{Univ}_L \subset |L| \times X$ with projection $\pi: \text{Univ}_L \rightarrow |L|$ such that

$$\pi^{-1}[D] = D$$

for every effective divisor D linearly equivalent to L .

The classical literature never differentiates between the linear system as a set and the linear system as a projective space. There are, indeed, few reasons to distinguish them as long as we work over a fixed base field k . If, however, we pass to a field extension $K \supset k$, the advantages of viewing $|L|$ as a k -variety appear. For any $K \supset k$, the set of effective divisors D defined over K such that $\mathcal{O}_X(D) \cong L$ corresponds to the K -points of $|L|$. Thus the scheme theoretic version automatically gives the right answer over every field.

1.2 (Jacobians of curves). Let C be a smooth projective curve (or Riemann surface) of genus g . As discovered by Abel and Jacobi, there is a variety $\text{Jac}^\circ(C)$ of dimension g whose points are in natural one-to-one correspondence with degree 0 line bundles on C . As before, the correspondence is given by a universal line bundle $L^{\text{univ}} \rightarrow C \times \text{Jac}^\circ(C)$, called the Poincaré bundle. That is, for any point $p \in \text{Jac}^\circ(C)$, the restriction of L^{univ} to $C \times \{p\}$ is the degree 0 line bundle corresponding to p .

A somewhat subtle point is that, unlike in (1.1), the universal line bundle L^{univ} is not unique (and need not exist if the base field is not algebraically closed). This has to do with the fact that while a divisor $D \subset X$ has no automorphisms fixing X , any line bundle $L \rightarrow C$ has automorphisms that fix C : we can multiply every fiber of L by the same nonzero constant.

1.3 (Cayley forms and Chow varieties). Cayley developed a method to associate a hypersurface in the Grassmannian $\text{Gr}(\mathbb{P}^1, \mathbb{P}^3)$ to a curve in \mathbb{P}^3 , in two papers with the same title [Cay60, Cay62]. The resulting moduli spaces have been used, but did not seem to have acquired a name. It was Chow who understood how to deal with reducible and multiple varieties, and proved that one gets a projective moduli space [CvdW37]. The name *Chow variety* seems standard, but I use Cayley-Chow for the correspondence that was discovered by Cayley.

See Section 3.1 for an outline, [HP47] for a classical introduction and [Kol96, Secs.I.3–4] for a more recent treatment.

Let k be an algebraically closed field and X a normal, projective k -variety. Fix a natural number m . An m -cycle on X is a finite, formal linear combination $\sum a_i Z_i$ where the Z_i are irreducible, reduced subvarieties of dimension m and $a_i \in \mathbb{Z}$. We

usually assume tacitly that all the Z_i are distinct. An m -cycle is called *effective* if $a_i \geq 0$ for every i .

Let $Y \subset X$ be a closed subscheme of dimension m . Let $Y_i \subset Y$ be its m -dimensional irreducible components, $Z_i := \text{red } Y_i$ and $y_i \in Y_i$ the generic point. Let a_i be the length of the Artin ring \mathcal{O}_{y_i, Y_i} . We define the *fundamental cycle* of Y as $[Y] := \sum a_i Z_i$. Thus the fundamental cycle ignores lower dimensional associated components and from the m -dimensional components it keeps only the underlying reduced variety and the length at the generic points.

It turns out that there is a k -variety $\text{Chow}_m(X)$, called the *Chow variety* of X whose points are in ‘natural’ one-to-one correspondence with the set of effective m -cycles on X . (Since we did not fix the degree of the cycles, $\text{Chow}_m(X)$ is not actually a variety but a countable disjoint union of projective, reduced k -schemes.) The point of $\text{Chow}_m(X)$ corresponding to a cycle $Z = \sum a_i Z_i$ is also usually denoted by $[Z]$.

As for linear systems, it is best to describe the ‘natural correspondence’ by a universal family. The situation is, however, more complicated than before.

There is a family (or rather an effective cycle) $\text{Univ}_m(X)$ on $\text{Chow}_m(X) \times X$ with projection $\pi: \text{Univ}_m(X) \rightarrow \text{Chow}_m(X)$ such that for every effective m -cycle $Z = \sum a_i Z_i$,

(1.3.1) the support of $\pi^{-1}[Z]$ is $\sum Z_i$, and

(1.3.2) the fundamental cycle of $\pi^{-1}[Z]$ equals Z if $a_i = 1$ for every i .

If the characteristic of k is 0, then the only problem in (2) is a clash between the traditional cycle-theoretic definition of the Chow variety and the scheme-theoretic definition of the fiber. It is easy to define a cycle-theoretic notion of fiber that restores equality in (2) for every Z ; see [Kol96, I.3]. In positive characteristic the situation is more problematic; a possible solution is described in [Kol96, I.4].

The example of a ‘perfect’ moduli problem is the theory of *Hilbert schemes*, introduced in [Gro62, Lect.IV]. See [Mum66], [Kol96, I.1–2] or [Ser06, Sec.4.3] for detailed treatments and Section 3.1 for a summary.

1.4 (Hilbert schemes). Let k be an algebraically closed field and X a projective k -scheme. Set

$$\mathcal{Hilb}(X) = \{\text{closed subschemes of } X\}.$$

Then there is a k -scheme $\text{Hilb}(X)$, called the *Hilbert scheme* of X whose points are in a ‘natural’ one-to-one correspondence with closed subschemes of X . The point of $\text{Hilb}(X)$ corresponding to a subscheme $Y \subset X$ is frequently denoted by $[Y]$. There is a universal family $\text{Univ}(X) \subset \text{Hilb}(X) \times X$ such that

(1.4.1) the first projection $\pi: \text{Univ}(X) \rightarrow \text{Hilb}(X)$ is flat, and

(1.4.2) $\pi^{-1}[Y] = Y$ for every closed subscheme $Y \subset X$.

The beauty of the Hilbert scheme is that it describes not just subschemes but all flat families of subschemes as well. To see what this means, note that for any morphism $g: T \rightarrow \text{Hilb}(X)$, by pull-back we obtain a flat family of subschemes of X parametrized by T

$$T \times_{g, \text{Hilb}(X)} \text{Univ}(X) \subset T \times X.$$

It turns out that every family is obtained this way:

(1.4.3) For every T and for every closed subscheme $Z_T \subset T \times X$ that is flat and proper over T , there is a unique $g: T \rightarrow \text{Hilb}(X)$ such that

$$Z_T = T \times_{g, \text{Hilb}(X)} \text{Univ}(X).$$

This takes us to the next, functorial approach to moduli problems.

1.5 (Hilbert functor and Hilbert scheme). Let $X \rightarrow S$ be a morphism of schemes. Define the *Hilbert functor* of X/S as a functor that associates to a scheme $T \rightarrow S$ the set

$$\mathcal{Hilb}_{X/S}(T) = \{\text{subschemes } Z \subset T \times_S X \text{ that are flat and proper over } T\}.$$

The basic existence theorem of Hilbert schemes then says that, if $X \rightarrow S$ is quasi-projective, there is a scheme $\text{Hilb}_{X/S}$ such that for any S scheme T ,

$$\mathcal{Hilb}_{X/S}(T) = \text{Mor}_S(T, \text{Hilb}_{X/S}).$$

Moreover, there is a universal family $\pi: \text{Univ}_{X/S} \rightarrow \text{Hilb}_{X/S}$ such that the above isomorphism is given by pulling back the universal family.

We can summarize these results as follows

PRINCIPLE 1.6. $\pi: \text{Univ}_{X/S} \rightarrow \text{Hilb}_{X/S}$ contains all the information about proper, flat families of subschemes of X/S and does it in the most succinct way.

This example leads us to a general definition:

DEFINITION 1.7 (Fine moduli spaces). Let \mathbf{V} be a ‘reasonable’ class of projective varieties (or schemes, or sheaves, or ...). In practice ‘reasonable’ may mean several restrictions, but for the definition we only need the following weak assumption:

(1.7.1) Let $K \supset k$ be a field extension. Then a k -variety X_k is in \mathbf{V} iff $X_K := X_k \times_{\text{Spec } k} \text{Spec } K$ is in \mathbf{V} .

Following (1.5), define the corresponding moduli functor as

$$\mathcal{Varieties}_{\mathbf{V}}(T) := \left\{ \begin{array}{l} \text{Flat families } X \rightarrow T \text{ such that} \\ \text{every fiber is in } \mathbf{V}, \\ \text{modulo isomorphisms over } T. \end{array} \right\} \quad (1.7.2)$$

We say that a scheme $\text{Moduli}_{\mathbf{V}}$, or, more precisely, a flat morphism

$$u: \text{Univ}_{\mathbf{V}} \rightarrow \text{Moduli}_{\mathbf{V}}$$

is a *fine moduli space* for the functor $\mathcal{Varieties}_{\mathbf{V}}$ if the following holds:

(1.7.3) For every scheme T , pulling back gives an equality

$$\mathcal{Varieties}_{\mathbf{V}}(T) = \text{Mor}(T, \text{Moduli}_{\mathbf{V}}).$$

Applying the definition to $T = \text{Spec } K$, where K is a field, we see that every fiber of $u: \text{Univ}_{\mathbf{V}} \rightarrow \text{Moduli}_{\mathbf{V}}$ is in \mathbf{V} and the K -points of the fine moduli space $\text{Moduli}_{\mathbf{V}}$ are in one-to-one correspondence with the K -isomorphism classes of objects in \mathbf{V} .

We consider the existence of a fine moduli space as the ideal possibility. Unfortunately, it is rarely achieved.

Next we see what happens with the simplest case, for smooth curves of fixed genus.

1.8 (Moduli functor and moduli space of smooth curves). Following (1.7) we define the moduli functor of smooth curves of genus g as

$$\text{Curves}_g(T) := \left\{ \begin{array}{l} \text{Smooth, proper families } S \rightarrow T, \\ \text{every fiber is a curve of genus } g, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

It turns out that there is no fine moduli space for curves of genus g . Every curve C with nontrivial automorphisms causes problems; there cannot be any point $[C]$ corresponding to it in a fine moduli space. Actually, problems arise already when \mathbf{V} consist of a single curve! See Section 1.8 for such examples.

It has been, however, understood for a long time that there is some kind of an object, denoted by M_g , and called the *coarse moduli space* (or simply *moduli space*) of curves of genus g that comes close to being a fine moduli space:

(1.8.1) For any algebraically closed field k , the k -points of M_g are in a ‘natural’ one-to-one correspondence with isomorphism classes of smooth curves of genus g defined over k . Let us denote the correspondence by $C \mapsto [C] \in M_g$.

(1.8.2) For any smooth family of genus g curves $h: S \rightarrow T$ there is a ‘moduli map’ $m_{h,T}: T \rightarrow M_g$ such that for every geometric point $p \in T$, the image $m_{h,T}(p)$ is the point corresponding to the fiber $[h^{-1}(p)]$.

(1.8.3) However, we do *not* assume that every $h: S \rightarrow T$ produces a pull-back family, and different families of curves may induce the same $h: S \rightarrow T$.

For elliptic curves we get $M_1 = \mathbb{A}^1$ and the moduli map is given by the j -invariant, as was known to Dedekind and Klein; see [KF92]. They also knew that there is no universal family over M_1 . The theory of Abelian integrals due to Abel, Jacobi and Riemann does essentially the same for all curves, though in this case a clear moduli theoretic interpretation seems to have been done only later; see the historical sketch at the end of [Sha74], [Sie69, Chap.4] or [GH94, Chap.2] for modern treatments. For smooth plane curves, and more generally for smooth hypersurfaces in any dimension, the invariant theory of Hilbert produces coarse moduli spaces. Still, a precise definition and proof of existence of M_g appeared only in [Tei44] in the analytic case and in [Mum65] in the algebraic case. See [AJP16] or [Kol18d] for historical accounts.

1.9 (Coarse moduli spaces). [Mum65]

As in (1.7), let \mathbf{V} be a ‘reasonable’ class. When there is no fine moduli space, we still can ask for a scheme that best approximates its properties.

We look for schemes M for which there is a natural transformation of functors

$$T_M: \text{Varieties}_g(*) \longrightarrow \text{Mor}(*, M).$$

Such schemes certainly exist, for instance, if we work over a field k then we can take $M = \text{Spec } k$. All schemes M for which T_M exists form an inverse system which is closed under fiber products. Thus, as long as we are not unlucky, there is a universal (or largest) scheme with this property. Though it is not usually done, it should be called the *categorical moduli space*.

This object can be rather useless in general. For instance, fix n, d and let $\mathbf{H}_{n,d}$ be the class of all hypersurfaces of degree d in \mathbb{P}_k^{n+1} up to isomorphisms. We see in (1.56) that a categorical moduli space exists and it is $\text{Spec } k$.

To get something more like a fine moduli space, we require that it give a one-to-one parametrization, at least set theoretically. Thus we say that a scheme $\text{Moduli}_{\mathbf{V}}$ is a *coarse moduli space* for \mathbf{V} if the following hold.

(1.9.1) There is a natural transformation of functors

$$\text{ModMap}: \text{Varieties}_{\mathbf{V}}(*) \longrightarrow \text{Mor}(*, \text{Moduli}_{\mathbf{V}}),$$

(1.9.2) $\text{Moduli}_{\mathbf{V}}$ is universal satisfying (1), and

(1.9.3) for any algebraically closed field $K \supset k$,

$$\text{ModMap}: \text{Varieties}_{\mathbf{V}}(\text{Spec } K) \xrightarrow{\cong} \text{Mor}(\text{Spec } K, \text{Moduli}_{\mathbf{V}}) = \text{Moduli}_{\mathbf{V}}(K)$$

is an isomorphism (of sets).

1.10 (Moduli functors versus moduli spaces). While much of the early work on moduli, especially since [Mum65], put the emphasis on the construction of fine or coarse moduli spaces, recently the emphasis shifted towards the study of the families of varieties, that is towards moduli functors and moduli stacks. The main task is to understand what kind of objects form ‘nice’ families. Once a good concept of ‘nice families’ is established, the existence of a coarse moduli space should be nearly automatic. The coarse moduli space is not the fundamental object any longer, rather it is only a convenient way to keep track of certain information that is only latent in the moduli functor or moduli stack.

1.11 (Compactifying M_g). While the basic theory of algebraic geometry is local, that is, it concerns affine varieties, most really interesting and important objects in algebraic geometry and its applications are global, that is, projective or at least proper.

The moduli spaces M_g are not compact, in fact the moduli functor of smooth curves discussed so far has a definitely local flavor. Most naturally occurring smooth families of curves live over affine schemes, and it is not obvious how to write down any family of smooth curves over a projective base. For many reasons it is useful to find geometrically meaningful compactifications of M_g . The answer to this situation is to allow not just smooth curves but also singular curves in our families.

Concentrating on 1-parameter families, the main question is the following:

(1.11.1) *Let B be a smooth curve, $B^\circ \subset B$ an open subset and $\pi^\circ: S^\circ \rightarrow B^\circ$ a smooth family of genus g curves. Find a ‘natural’ extension*

$$\begin{array}{ccc} S^\circ & \subset & S \\ \pi^\circ \downarrow & & \downarrow \pi \\ B^\circ & \subset & B, \end{array}$$

where $\pi: S \rightarrow B$ is a flat family of (possibly singular) curves.

We would like the extension to be unique and behave well with respect to pulling back families over curves, and for families over higher dimensional bases.

The answer, proposed in [DM69] has been so successful that it is hard to imagine a time when it was not the ‘obvious’ solution. Let us first review the definition of [DM69]. In Section 1.6 we see, by examples, why this concept has not been so obvious.

DEFINITION 1.12 (Stable curve). A *stable curve* over an algebraically closed field k is a proper, geometrically connected k -curve C such that the following hold:

(Local property) The only singularities of C are ordinary nodes.

(Global property) The canonical (or dualizing) sheaf ω_C is ample.

A *stable curve* over a scheme T is a flat, proper morphism $\pi: S \rightarrow T$ such that every geometric fiber of π is a stable curve. (The arithmetic genus of the fibers is a locally constant function on T , but we usually also tacitly assume that it is constant.)

The moduli functor of stable curves of genus g is

$$\overline{\text{Curves}}_g(T) := \left\{ \begin{array}{l} \text{Stable curves of genus } g \text{ over } T, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

THEOREM 1.13. [DM69] *For every $g \geq 2$, the moduli functor of stable curves of genus g has a coarse moduli space \bar{M}_g . Moreover, \bar{M}_g is projective, normal, has only quotient singularities, and contains M_g as an open dense subset.*

\bar{M}_g has a rich and intriguing intrinsic geometry which is related to major questions in many branches of mathematics and theoretical physics; see [FM13] for a collection of surveys and [Pan18a, Pan18b] for overviews.

1.2. Moduli for varieties of general type

The aim of this book is to use the moduli of stable curves as guideline, and develop a moduli theory for varieties of general type. (See (1.22) for some comments on the non-general type cases.)

Here we outline the main steps of the plan with some comments. Most of the rest of the book is then devoted to accomplish these goals.

STEP 1.14 (Higher dimensional analogs of smooth curves). It has been understood since the beginnings of the theory of surfaces that, for surfaces of Kodaira dimension ≥ 0 , the correct moduli theory should be birational, not biregular. That is, the points of the moduli space should correspond not to *isomorphism* classes of surfaces but to *birational* equivalence classes of surfaces. There are two ways to deal with this problem.

First, one can work with smooth families, but consider two families $V \rightarrow B$ and $W \rightarrow B$ equivalent if there is a *fiberwise birational* map between them; that is, a rational map $V \dashrightarrow W$ that induces a birational equivalence of the fibers $V_b \dashrightarrow W_b$ for every $b \in B$. This seems rather complicated technically.

The second, much more useful method relies on the observation that every birational equivalence class of surfaces of Kodaira dimension ≥ 0 contains a unique *minimal model*, that is, a smooth projective surface S^m whose canonical class is nef. Therefore, one can work with families of minimal models, modulo isomorphisms. With the works of [Mum65, Art74] it became clear that, for surfaces of general type, it is even better to work with the *canonical model*, which is a mildly singular projective surface S^c whose canonical class is ample. The resulting class of singularities has been since established in all dimensions; they are called *canonical singularities* (1.33).

Principle 1.14.1. In moduli theory, the main objects of study are projective varieties with ample canonical class and with canonical singularities.

Implicit in this claim is that every flat family of varieties of general type produces a flat family of canonical models, we discuss this in (1.36).

See Section 1.3 for more details on this step.

STEP 1.15 (Higher dimensional analogs of stable curves). The correct definition of the higher dimensional analogs of stable curves was much less clear. An approach through geometric invariant theory was investigated [Mum77], but never fully developed. In essence, the GIT approach starts with a particular method of construction of moduli spaces, and then tries to see for which class of varieties does it work. The examples of [WX14] suggest that geometric invariant theory is unlikely to give a good compactification for the moduli of surfaces.

A different framework was proposed in [KSB88]. Instead of building on geometric invariant theory, it focuses on 1-parameter families and uses Mori’s program as its basic tool.

Before we give the definition, recall a key step of the proof of (1.13) that establishes separatedness and properness of \bar{M}_g . (The traditional name is stable ‘reduction,’ but ‘extension’ is much more descriptive.)

Stable extension for curves 1.15.1. Let B be a smooth curve, $B^\circ \subset B$ a dense, open subset and $\pi^\circ: S^\circ \rightarrow B^\circ$ a flat family of smooth, projective curves of genus ≥ 2 . Then there is a finite surjection $p: A \rightarrow B$ and a diagram

$$\begin{array}{ccccc} S^\circ \times_B A & \subset & S_A^{\text{ss}} & \xrightarrow{\tau} & S_A^{\text{stab}} \\ \downarrow & & \downarrow \pi_A^{\text{ss}} & & \downarrow \pi_A^{\text{st}} \\ B^\circ \times_B A & \subset & A & = & A, \end{array}$$

where

- (a) $\pi_A^{\text{ss}}: S_A^{\text{ss}} \rightarrow A$ is a flat family of reduced, nodal curves (called a semi-stable extension),
- (b) $\tau: S_A^{\text{ss}} \rightarrow S_A^{\text{st}}$ is the relative canonical model (11.26), and
- (c) $\pi_A^{\text{st}}: S_A^{\text{st}} \rightarrow A$ is a flat family of stable curves.

A detailed proof is given in (2.49), for now we build on this to state the main theses of [KSB88] about higher dimensional moduli problems.

Principle 1.15.2. In higher dimensions, we should follow the proof of the Stable reduction theorem (1.15.1). The resulting fibers give the right class of stable varieties.

Principle 1.15.3. As in (1.12), a connected k -scheme X is stable iff it satisfies the following two conditions:

(Local property) A restriction on the singularities of X (so-called ‘semi-log-canonical’ singularities).

(Global property) The canonical (or dualizing) sheaf ω_X is ample.

The definition of semi-log-canonical is not important for now (1.41), the key point is that the only global restriction is the ampleness of ω_X .

STEP 1.16 (Higher dimensional analogs of families of stable curves I). The definition (1.7) is very natural within our usual framework of algebraic geometry, but it hides a very strong supposition:

Unwarranted assumption 1.16.1. If \mathbf{V} is a ‘reasonable’ class of varieties then any flat family whose fibers are in \mathbf{V} is a ‘reasonable’ family.

In Grothendieck’s foundations of algebraic geometry flatness is one of the cornerstones and there are many ‘reasonable’ classes for which flat families are indeed the ‘reasonable’ families. Nonetheless, even when the base of the family is a smooth

curve, (1.16.1) needs arguing, but the assumption is especially surprising when applied to families over non-reduced schemes T . Consider, for instance, the case when T is the spectrum of an Artinian k -algebra. Then T has only one closed point $t \in T$. A flat family $p: X \rightarrow T$ has only one fiber X_t , and our only restriction is that X_t be in our class \mathbf{V} . Thus (1.16.1) declares that we care only about X_t . Once X_t is in \mathbf{V} , every flat deformation of X_t over T is automatically ‘reasonable.’

A crucial conceptual point in the moduli theory of higher dimensional varieties is the realization that, starting with families of surfaces, flatness of the map $X \rightarrow T$ is not enough: allowing all flat families whose fibers are in a ‘reasonable’ class leads to the wrong moduli problem.

The simple fact is that basic numerical invariants, like the self intersection of the canonical class or even the Kodaira dimension fail to be locally constant in flat families of stable varieties, even when the singularities are quite mild and the base is a smooth curve. We give a series of such examples in (1.42–1.47).

The difficulty of working out the correct concept has been one of the main stumbling blocks of the general theory.

Principle 1.16.2. Flat families of stable varieties $X \rightarrow T$ are the correct higher dimensional analogs of flat families of stable curves (1.12) if the canonical sheaves ω_{X_t} are locally free, but **not** in general.

For families over smooth curves, the Stable extension theorem (1.15.1) is again our guide to the correct definition.

Stable morphisms 1.16.3. Let $p: Y \rightarrow B$ be a proper morphism from a normal variety to a smooth curve. Then p is *stable* iff

- (a) all fibers Y_s are semi-log-canonical,
- (b) K_Y is relatively ample, and
- (c) K_Y is \mathbb{Q} -Cartier.

Note that this is a direct generalization of the notion of stable family of curves (1.12), except that here we have to add the \mathbb{Q} -Cartier condition (which was automatic for curves).

We discuss this in detail in Section 2.4.

STEP 1.17 (Higher dimensional analogs of families of stable curves II). Extending the definition (1.16.3) from smooth curves to other base schemes turned out to be very difficult. There were 2 main proposals in [KSB88] and [Vie95]. They turned out to be equivalent over reduced base schemes, we explain this in Section 3.4. However, the 2 versions differ already for surfaces with quotient singularities [AK19a]. We treat these topics in Sections 6.2–6.3 and 6.6.

The problem becomes even harder when we treat not just stable varieties but stable pairs. Finding the correct definition turned out to be the longest-standing open question of the theory. An answer was developed in [Kol19], we devote Chapter 7 to explaining it.

STEP 1.18 (Representability of moduli functors). The question is the following. Let $p: X \rightarrow S$ be an arbitrary projective morphism. Can we understand all morphism $q: T \rightarrow S$ such that $X \times_S T \rightarrow T$ is a family in our moduli theory?

Representability 1.18.1. We say that a moduli theory \mathbf{M} is *representable* if there is a monomorphism $j : S^{\mathbf{M}} \rightarrow S$ such that $X \times_S T \rightarrow T$ is a family in \mathbf{M} iff q factors as $q : T \rightarrow S^{\mathbf{M}} \rightarrow S$.

That is, $X \times_S S^{\mathbf{M}} \rightarrow S^{\mathbf{M}}$ is in \mathbf{M} and $S^{\mathbf{M}}$ is the universal scheme with this property.

Representability is rarely mentioned for the moduli of curves, since it easily follows from general principles. The Flattening decomposition theorem of [Mum66, Lect.8] says that flatness is representable (3.19), and for proper, flat morphisms, being a family of stable curves is represented by an open subscheme.

Both of these become quite complicated in higher dimensions. Since flatness is only part of our assumptions, we need a different way of pulling back families. The theory of Hulls and husks [Kol08a] was developed for this reason, leading to the notion of divisorial pull-back, defined in Section 4.1. With these, representability is proved in Sections 3.5, 4.5 and 7.6 in increasing generality.

Representability also implies that being a stable family can be tested on 0-dimensional subschemes of T , that is, on spectra of Artin rings. This is the reason why formal deformation theory is such a powerful tool [Ill71, Art76, Ser06].

The previous steps form the basis of a good moduli theory. Once we have them, it is quite straightforward to construct the corresponding moduli space.

STEP 1.19 (Two moduli spaces). Let C be a stable curve of genus $g \geq 2$. Then ω_C^r is very ample for $r \geq 3$ and any basis of its global sections gives an embedding

$$C \hookrightarrow \mathbb{P}^{r(2g-2)-g}.$$

Thus all stable curves of genus g appear in the Chow variety or Hilbert scheme of $\mathbb{P}^{r(2g-2)-g}$. Representability (1.18) then implies that we get a moduli space of all r -canonically embedded stable curves

$$\text{EmbStab}_g \subset \text{Hilb}(\mathbb{P}^{r(2g-2)-g}). \quad (1.19.1)$$

For a fixed C , the embedding $C \hookrightarrow \mathbb{P}^{r(2g-2)-g}$ give an orbit of $\text{Aut}(\mathbb{P}^{r(2g-2)-g})$, thus we should get the moduli space as

$$\overline{\mathbf{M}}_g = \text{EmbStab}_g / \text{Aut}(\mathbb{P}^{r(2g-2)-g}). \quad (1.19.2)$$

Starting with [Mum65] and [Mat64], much effort was devoted to understanding quotient like (1.19.2). Already for curves the method of [Mum65] is quite subtle, generalizations to surfaces [Gie77] and to higher dimensions [Vie95] are quite hard. For surfaces and in higher dimensions these approaches handle only the interior of the moduli space (where we have only canonical singularities). In fact, [WX14] suggest that GIT methods do not work for the whole moduli space.

When GIT works, it automatically gives a quasi-projective moduli space. It turns out to be much easier to obtain quotients that are algebraic spaces. The general quotient theorems of [Kol97, KM97] take care of this question completely; see also [Ols16]. So we consider the quotient problems fully solved for our purposes.

The same approach works in all dimensions. We fix $r > 0$ such that $\omega_X^{[r]}$ is very ample, and the rest of the proof works without changes.

For curves and $r \geq 3$ works, but, starting with surfaces, a uniform choice of r is no longer possible. The strongest results say that if we fix the dimension b and the volume v , then there is an $r = r(n, v)$ such that $\omega_X^{[r]}$ is very ample. We discuss this in (1.21).

However, it turns out to be much easier to prove that if we stay within an irreducible component of the moduli space, then a uniform choice of r exists; see (4.69).

Once we have our moduli spaces, we start to investigate their properties. We should not expect to get moduli spaces that are as nice as for curves. For instance, even for smooth surfaces with ample canonical class, the moduli spaces can have arbitrarily complicated singularities and scheme structures [Vak06].

Nonetheless, we have 2 types of basic positive results.

STEP 1.20 (Separatedness and properness). The valuative criteria of separatedness and properness translate to functors as follows.

We start with a smooth curve B , an open subset $B^\circ \subset B$, and a stable family $\pi^\circ: X^\circ \rightarrow B^\circ$.

Separatedness 1.20.1. There is at most one extension to

$$\begin{array}{ccc} X^\circ & \subset & X \\ \pi^\circ \downarrow & & \downarrow \pi \\ B^\circ & \subset & B, \end{array}$$

where $\pi: X \rightarrow B$ is also stable.

We obtain a similar translation of the valuative criterion of properness, but here we have to pay attention to the difference between coarse and fine moduli spaces.

Valuative-properness 1.20.2. There is a finite surjection $p: A \rightarrow B$ such that there is a unique extension

$$\begin{array}{ccc} X^\circ \times_B A & \subset & X_A \\ \downarrow & & \downarrow \pi_A \\ B^\circ \times_B A & \subset & A, \end{array}$$

where $\pi_A: X_A \rightarrow A$ is also stable.

Thus the valuative criterion of properness is exactly the general version of the Stable reduction theorem (1.15.1).

STEP 1.21 (Discrete invariants, boundedness and projectivity). The most important discrete invariant of a smooth projective curve C is its genus. The genus is unchanged under smooth deformations and all smooth curves with the same genus form a single family M_g . Thus, in effect, the genus is the only discrete invariant of a smooth projective curve and it completely determines the other ones, like the Euler characteristic $\chi(C, \mathcal{O}_C)$ or the Hilbert polynomial $\chi(C, \omega_C^m)$.

In a similar manner, we would like to find discrete invariant of (locally) stable varieties that are unchanged by (locally) stable deformations.

The basic such invariant is the Hilbert ‘polynomial’ $\chi(X, \omega_X^{[m]})$. We have to keep in mind that ω_X need not be locally free, so instead of its tensor powers $\omega_X^{\otimes m}$ we need to use its reflexive hull $\omega_X^{[m]}$. Therefore $m \mapsto \chi(X, \omega_X^{[m]})$ is not actually a polynomial, rather a polynomial with periodic coefficients.

For stable varieties the most important invariant, called the *volume*, is $\text{vol}(X) := (K_X^n)$ (where $n = \dim X$). This is also the leading coefficient of the Hilbert polynomial (times $n!$). The volume is positive, but it is frequently a rational number since K_X is only \mathbb{Q} -Cartier.

For $m = 0$ we get the Euler characteristic $\chi(X, \mathcal{O}_X)$, but it turns out that the individual groups $h^i(X, \mathcal{O}_X)$ are also deformation invariants by [KK10]; see Section 2.5.

Our moduli spaces satisfy the valuative criterion of properness, but this implies properness only for schemes of finite type. Proving the latter turned out to be extremely difficult. For smooth varieties this was solved by [Mat72], for stable surfaces by [Ale93] and the general stable case is settled in [HMX18].

Once we have a moduli space which is a proper algebraic space, we would like to prove that it is projective. For surfaces this was done in [Kol90] and extended to higher dimensions in [Fuj18] and [KP17].

These last 2 topics each deserve a detailed treatment of their own; we make only a few more comments in (6.5).

1.22 (Moduli for varieties of non-general type).

In contrast with varieties of general type, the moduli theory for varieties of non-general type is very complicated.

A general problem, illustrated by Abelian, elliptic and K3 surfaces is that a typical deformation of such an algebraic surface over \mathbb{C} is a non-algebraic complex analytic surface. Thus any algebraic theory captures only a small part of the full analytic deformation theory.

The moduli question for analytic surfaces has been studied, especially for complex tori and K3 surfaces. In both cases it seems that one needs to add some extra structure (for instance, fixing a basis in some topological homology group) in order to get a sensible moduli space. (As an example of what could happen, note that the 3-dimensional space of Kummer surfaces is dense in the 20-dimensional space of all K3 surfaces, cf. [PŠŠ71].)

Even if one restricts to the algebraic case, compactifying the moduli space seems rather difficult. Detailed studies of Abelian varieties and K3 surfaces show that there are many different compactifications depending on additional choices, see [KKMSD73, AMRT75].

It is only with the works of [Ale02] that a geometrically meaningful compactification of the moduli of principally polarized Abelian varieties became available. This relies on the observation that a pair (A, Θ) consisting of a principally polarized Abelian variety A and its theta divisor Θ behaves as if it were a variety of general type.

A moduli theory for K-stable Fano varieties was developed quite recently, see [Xu20] for an overview.

1.3. From smooth curves to canonical models

Here we discuss the considerations that led to Principle 1.14.1.

In the theory of curves, the basic objects are smooth projective curves. We frequently study any other curve by relating it to smooth projective curves. As a close analog, in higher dimensions the moduli functor of smooth varieties is

$$\text{Smooth}(S) := \left\{ \begin{array}{l} \text{Smooth, proper families } X \rightarrow S, \\ \text{modulo isomorphisms over } S. \end{array} \right\}$$

This, however, gives a rather badly behaved and mostly useless moduli functor already for surfaces. First of all, it is very non-separated.

1.23 (Non-separatedness in the moduli of smooth surfaces of general type). We construct two smooth families of projective surfaces $f_i: X^i \rightarrow B$ over a pointed smooth curve $b \in B$ such that

(1.23.1) all the fibers are smooth, projective surfaces of general type,

(1.23.2) $X^1 \rightarrow B$ and $X^2 \rightarrow B$ are isomorphic over $B \setminus \{b\}$,

(1.23.3) the fibers X_b^1 and X_b^2 are *not* isomorphic.

As the construction shows, this type of behavior happens every time we look at deformations of a surface that contains at least three (-1) -curves.

Let $f: X \rightarrow B$ be a smooth family of projective surfaces over a smooth (affine) pointed curve $b \in B$. Let $C_1, C_2, C_3 \subset X$ be three sections of f , all passing through a point $x_b \in X_b$ with independent tangent directions and disjoint elsewhere.

Set $X^1 := B_{C_1}B_{C_2}B_{C_3}X$, where we first blow-up $C_3 \subset X$, then the birational transform of C_2 in $B_{C_3}X$ and finally the birational transform of C_1 in $B_{C_2}B_{C_3}X$. Similarly, set $X^2 := B_{C_1}B_{C_3}B_{C_2}X$. Since the C_i are sections, all these blow-ups give smooth families of projective surfaces over B .

Over $B \setminus \{b\}$ the curves C_i are disjoint, thus X^1 and X^2 are both isomorphic to $B_{C_1+C_2+C_3}X$, the blow-up of $C_1 + C_2 + C_3 \subset X$.

We claim that, by contrast, the fibers of X_b^1 and X_b^2 are not isomorphic to each other for a general choice of the C_i .

To see this, choose local analytic coordinates t at $b \in B$ and (x, y, t) at $x_b \in X$. The curves C_i are defined by equations

$$C_i = (x - a_it - (\text{higher terms}) = y - b_it - (\text{higher terms}) = 0).$$

The blow-up $B_{C_i}X$ is given by

$$B_{C_i}X = (u_i(x - a_it - (\text{higher terms})) = v_i(y - b_it - (\text{higher terms}))) \subset X \times \mathbb{P}^1_{u_i v_i}.$$

On the fiber over b these give the same blow-up

$$B_{x_b}(X_b) = (ux = vy) \subset X_b \times \mathbb{P}^1_{uv}$$

Thus we see that the birational transform of C_j intersects the central fiber $(B_{C_i}X)_b = B_{x_b}(X_b)$ at the point

$$\frac{u}{v} = \frac{a_j - a_i}{b_j - b_i} \in \{x_b\} \times \mathbb{P}^1_{uv}.$$

The fibers $(B_{C_2}B_{C_3}X)_b$ and $(B_{C_3}B_{C_2}X)_b$ are isomorphic to each other since they are obtained from $B_{x_b}(X_b)$ by blowing up the same point

$$\frac{u}{v} = \frac{a_2 - a_3}{b_2 - b_3} \quad \text{resp.} \quad \frac{u}{v} = \frac{a_3 - a_2}{b_3 - b_2}.$$

When we next blow up the birational transform of C_1 on $(B_{C_2}B_{C_3}X)_b$ (resp. on $(B_{C_3}B_{C_2}X)_b$) this gives the blow-up of the point

$$\frac{a_1 - a_3}{b_1 - b_3} \quad \text{resp.} \quad \frac{a_1 - a_2}{b_1 - b_2}, \quad (1.23.4)$$

and these are different, unless $C_1 + C_2 + C_3$ is locally planar at x_b .

So far we have seen that the identity $X_b = X_b$ does not extend to an isomorphism between the fibers X_b^1 and X_b^2 .

If X_b is of general type, then $\text{Aut } X_b$ is finite, hence, to ensure that X_b^1 and X_b^2 are not isomorphic, we need to avoid finitely many other possible coincidences in (1.23.4).

The main reason, however, why we do not study the moduli functor of smooth varieties up to isomorphism is that, in dimension two, smooth projective surfaces do not form the *smallest* basic class. Given any smooth projective surface S , one can blow up any set of points $Z \subset S$ to get another smooth projective surface $B_Z S$ which is very similar to S . Therefore, the basic object should be not a single smooth projective surface but a whole *birational equivalence class* of smooth projective surfaces. Thus it would be better to work with smooth, proper families $X \rightarrow S$ modulo birational equivalence over S . That is, with the moduli functor

$$\text{GenType}_{\text{bir}}(S) := \left\{ \begin{array}{l} \text{Smooth, proper families } X \rightarrow S, \\ \text{every fiber is of general type,} \\ \text{modulo birational equivalences over } S. \end{array} \right\} \quad (1.23.5)$$

In essence this is what we end up doing, see (1.36), but it is very cumbersome to deal with birational equivalence over a base scheme. Nonetheless, working with birational equivalence classes leads to a separated moduli functor.

PROPOSITION 1.24. *Let $f_i: X^i \rightarrow B$ be two smooth families of projective varieties over a smooth curve B . Assume that the generic fibers $X^1_{k(B)}$ and $X^2_{k(B)}$ are birational and the pluricanonical system $|mK_{X^1_{k(B)}}|$ is nonempty for some $m > 0$. Then, for every $b \in B$, the fibers X^1_b and X^2_b are birational.*

Remark. By [KT19] the conclusion holds even if the pluricanonical systems are empty. However, the proof is much simpler in our case, and shows the role of the canonical class clearly.

Proof. Pick a birational map $\phi: X^1_{k(B)} \dashrightarrow X^2_{k(B)}$ and let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of ϕ . Let $Y \rightarrow \Gamma$ be the normalization with projections $p_i: Y \rightarrow X^i$. Note that both of the p_i are open embeddings on $Y \setminus (\text{Ex } p_1 \cup \text{Ex } p_2)$. Thus if we prove that neither $p_1(\text{Ex } p_1 \cup \text{Ex } p_2)$ nor $p_2(\text{Ex } p_1 \cup \text{Ex } p_2)$ contains a fiber of f_1 or f_2 , then $p_2 \circ p_1^{-1}: X^1 \dashrightarrow X^2$ restricts to a birational map $X^1_b \dashrightarrow X^2_b$ for every $b \in B$. (Thus the fiber Y_b contains an irreducible component that is the graph of the birational map $X^1_b \dashrightarrow X^2_b$, but it may have other components too; see (1.26.9).)

We use the canonical class to compare $\text{Ex } p_1$ and $\text{Ex } p_2$. Since the X^i are smooth,

$$K_Y \sim p_i^* K_{X^i} + E_i, \quad \text{where } E_i \geq 0 \text{ and } \text{Supp } E_i = \text{Ex } p_i. \quad (1.24.1)$$

Assume for simplicity that B is affine and let $\text{Bs}|mK_{X^i}|$ denote the set-theoretic base locus. By assumption, $|mK_{X^i}|$ is not empty and since B is affine, $\text{Bs}|mK_{X^i}|$ does not contain any of the fibers of f_i .

Every section of $\mathcal{O}_Y(mK_Y)$ pulls back from X^i , thus

$$\text{Bs}|mK_Y| = p_i^{-1}(\text{Bs}|mK_{X^i}|) + \text{Supp } E_i.$$

Comparing these for $i = 1, 2$, we conclude that

$$p_1^{-1}(\text{Bs}|mK_{X^1}|) + \text{Supp } E_1 = p_2^{-1}(\text{Bs}|mK_{X^2}|) + \text{Supp } E_2.$$

Therefore,

$$p_1(\text{Supp } E_2) \subset p_1(\text{Supp } E_1) + \text{Bs}|mK_{X^1}|.$$

Since E_1 is p_1 -exceptional, $p_1(E_1)$ has codimension ≥ 2 in X^1 , hence it does not contain any of the fibers of f_1 . We saw that $\text{Bs}|mK_{X^1}|$ does not contain any of the fibers either. Thus $p_1(\text{Exp}_1 \cup \text{Exp}_2)$ does not contain any of the fibers and similarly for $p_2(\text{Exp}_1 \cup \text{Exp}_2)$. \square

REMARK 1.25. A result of [MM64] says that, more generally, (1.24) holds as long as the fibers X_b^i are not birationally ruled, that is, not birational to a variety of the form $Z \times \mathbb{P}^1$. The proof of [MM64], relies on the study of exceptional divisors over a smooth variety; see [KSC04, Sec.4.5] for an overview. Exceptional divisors over a singular variety are much less understood. By contrast, the above proof focuses on the role of the canonical class. It is worthwhile to go back and check that the proof works if the X^i are normal, as long as (1.24.1) holds; the latter is essentially the definition of terminal singularities.

It is precisely the property (1.24.1) and its closely related variants that lead us to the correct class of singular varieties for moduli purposes.

Since it is much harder to work with a whole equivalence class, it would be desirable to find a particularly nice surface in every birational equivalence class. This is achieved by the theory of minimal models of algebraic surfaces. By a result of Enriques (cf. [BPV84, III.4.5]), every birational equivalence class of surfaces \mathbf{S} contains a unique smooth projective surface whose canonical class is nef (that is, has nonnegative degree on every effective curve), except when \mathbf{S} contains a ruled surface $C \times \mathbb{P}^1$ for some curve C . This unique surface is called the *minimal model* of \mathbf{S} .

It would seem at first sight that (1.24) implies that the moduli functor of minimal models is separated. There is, however, a quite subtle problem.

1.26 (Non-separatedness in the moduli of minimal models). We construct two smooth families of projective surfaces $f_i: X^i \rightarrow B$ over a pointed smooth curve $b \in B$ such that

- (1.26.1) all the fibers are smooth, projective minimal models,
- (1.26.2) $X^1 \rightarrow B$ and $X^2 \rightarrow B$ are isomorphic over $B \setminus \{b\}$,
- (1.26.3) the fibers X_b^1 and X_b^2 are isomorphic, but
- (1.26.4) $X^1 \rightarrow B$ and $X^2 \rightarrow B$ are *not* isomorphic.

While it is not clear from our construction, similar problems happen for any smooth family of surfaces where the general fiber has ample canonical class and a special fiber has nef (but not ample) canonical class, see [Art74, Bri68b, Rei80].

Let $X_0 := (f(x_1, \dots, x_4) = 0) \subset \mathbb{P}^3$ be a surface of degree n that has an ordinary double point (10.44) at $p = (0:0:0:1)$ as its sole singularity and contains the pair of lines $(x_1x_2 = x_3 = 0)$. Let g be homogeneous of degree $n-1$ such that x_4^{n-1} appears in it with nonzero coefficient. Consider the family of surfaces

$$X := (f(x_1, \dots, x_4) + tx_3g(x_1, \dots, x_4) = 0) \subset \mathbb{P}_{\mathbf{x}}^3 \times \mathbb{A}_t^1.$$

Note that X_t is smooth for general $t \neq 0$ and X contains the pair of smooth surfaces $(x_1x_2 = x_3 = 0)$.

For $i = 1, 2$, let $X^i := B_{(x_i, x_3)}X$ denote the blow-up of $(x_i = x_3 = 0)$ with induced morphisms $\pi_i: X^i \rightarrow X$ and $f_i: X^i \rightarrow \mathbb{A}^1$. There is a natural birational map $\phi := \pi_2^{-1} \circ \pi_1: X^1 \dashrightarrow X^2$. Let B_pX denote the blow-up of $p = ((0:0:0:1), 0)$ with exceptional divisor $E \subset B_pX$.

We claim that the following hold.

- (1.26.5) The $f_i: X^i \rightarrow \mathbb{A}^1$ are projective families of surfaces which are smooth over a neighborhood of $(t = 0)$.
- (1.26.6) For $n \geq 5$, the fibers X_t^i have ample canonical class for $t \neq 0$ and nef canonical class for $t = 0$.
- (1.26.7) $X^1 \times_X X^2$ is isomorphic to $B_p X$, hence it is smooth and irreducible.
- (1.26.8) The map ϕ is an isomorphism over $\mathbb{A}^1 \setminus \{0\}$ but it is not an isomorphism over 0.
- (1.26.9) The fiber of $X^1 \times_X X^2$ over $(t = 0)$ has two irreducible components. One of these components is the graph of an isomorphism $X_0^1 \cong X_0^2$. The other component is $E \cong \mathbb{P}^1 \times \mathbb{P}^1$.
- (1.26.10) Thus $\phi: X^1 \dashrightarrow X^2$ is an isomorphism over $\mathbb{A}^1 \setminus \{0\}$, the $X^i \rightarrow \mathbb{A}^1$ have isomorphic fibers over $0 \in \mathbb{A}^1$, but ϕ is not an isomorphism over \mathbb{A}^1 .

(It is not hard to see that, for general choice of f and g , the X_t have no birational self-maps, thus the only possible isomorphism between X^1 and X^2 would be the identity on X . Thus, by (6), in this case, X^1 and X^2 are not isomorphic to each other.)

Note that $(x_i = x_3 = 0)$ defines a Weil divisor in X which is Cartier outside the point p . Thus all 3 blow-ups are isomorphisms over $X \setminus \{p\}$. This means that all the above claims are local near p .

We prove the claims in (10.45) after choosing better local coordinates near p that make all the assertions (5–10) transparent.

All such problems go away when the canonical class is ample.

PROPOSITION 1.27. *Let $f_i: X^i \rightarrow B$ be two smooth families of projective varieties over a smooth curve B . Assume that the canonical classes K_{X^i} are f_i -ample. Let $\phi: X_{k(B)}^1 \cong X_{k(B)}^2$ be an isomorphism of the generic fibers.*

Then ϕ extends to an isomorphism $\Phi: X^1 \cong X^2$.

Proof. Let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of ϕ . Let $Y \rightarrow \Gamma$ be the normalization, with projections $p_i: Y \rightarrow X^i$ and $f: Y \rightarrow B$. As in (1.24), we use the canonical class to compare the X^i . Since the X^i are smooth,

$$K_Y \sim p_i^* K_{X^i} + E_i \quad \text{where } E_i \text{ is effective and } p_i\text{-exceptional.} \quad (1.27.1)$$

Since $(p_i)_* \mathcal{O}_Y(mE_i) = \mathcal{O}_{X^i}$ for every $m \geq 0$, we get that

$$\begin{aligned} (f_i)_* \mathcal{O}_{X^i}(mK_{X^i}) &= (f_i)_*(p_i)_* \mathcal{O}_Y(m p_i^* K_{X^i}) &= \\ &= (f_i)_*(p_i)_* \mathcal{O}_Y(m p_i^* K_{X^i} + mE_i) &= \\ &= (f_i)_*(p_i)_* \mathcal{O}_Y(mK_Y) &= f_* \mathcal{O}_Y(mK_Y). \end{aligned}$$

Since the K_{X^i} are f_i -ample, $X^i = \text{Proj}_B \sum_{m \geq 0} (f_i)_* \mathcal{O}_{X^i}(mK_{X^i})$. Putting these together, we get the isomorphism

$$\begin{aligned} \Phi: X^1 &\cong \text{Proj}_B \sum_{m \geq 0} (f_1)_* \mathcal{O}_{X^1}(mK_{X^1}) \cong \\ &\cong \text{Proj}_B \sum_{m \geq 0} f_* \mathcal{O}_Y(mK_Y) \cong \\ &\cong \text{Proj}_B \sum_{m \geq 0} (f_2)_* \mathcal{O}_{X^2}(mK_{X^2}) \cong X^2. \quad \square \end{aligned}$$

REMARK 1.28. As in (1.25), it is again worthwhile to investigate the precise assumptions behind the proof. The smoothness of the X^i is used only through the pull-back formula (1.27.1), which is weaker than (1.24.1).

If (1.27.1) holds, then, even if the K_{X_i} are not f_i -ample, we obtain an isomorphism

$$\mathrm{Proj}_B \sum_{m \geq 0} (f_1)_* \mathcal{O}_{X^1}(mK_{X^1}) \cong \mathrm{Proj}_B \sum_{m \geq 0} (f_2)_* \mathcal{O}_{X^2}(mK_{X^2}). \quad (1.28.1)$$

Thus it is of interest to study objects as in (1.28.1) in general.

Let us start with the absolute case, when X is a smooth projective variety over a field k . Its *canonical ring* is the graded ring

$$R(X, K_X) := \sum_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X)).$$

In some cases the canonical ring tells us very little about X . For instance, if X is rational or Fano then $R(X, K_X)$ is the base field k and if X is Calabi-Yau then $R(X, K_X)$ is isomorphic to the polynomial ring $k[t]$. One should thus focus on the cases when the canonical ring is large. The following theorem and the resulting definition is due to [Lit71]. See [Laz04, Sec.2.1.C] for a detailed treatment.

THEOREM-DEFINITION 1.29. *For a smooth projective variety X of dimension n , the following are equivalent.*

(1.29.1) $h^0(X, \mathcal{O}_X(mK_X)) \geq \epsilon \cdot m^n$ for some $\epsilon > 0$ and $m \gg 1$.

(1.29.2) $\mathrm{Proj} R(X, K_X)$ has dimension n .

(1.29.3) The natural map $X \dashrightarrow \mathrm{Proj} R(X, K_X)$ is birational.

If these hold, then we say that X is of general type.

This enables us to find a distinguished variety in any birational equivalence class.

DEFINITION 1.30 (Canonical models). Let X be a smooth projective variety of general type over a field k such that its canonical ring $R(X, K_X)$ is finitely generated. We define its *canonical model* as

$$X^{\mathrm{can}} := \mathrm{Proj}_k R(X, K_X).$$

If Y is a smooth projective variety birational to X then Y^{can} is isomorphic to X^{can} . Thus X^{can} is also the canonical model of the whole birational equivalence class containing X . (Taking Proj of a non-finitely generated ring may result in a quite complicated scheme. It does not seem profitable to contemplate what would happen in our case.)

Now we know [BCHM10] that the canonical ring $R(X, K_X)$ is always finitely generated in characteristic 0, thus X^{can} is a projective variety. On the other hand, X^{can} can be singular. Originally this was viewed as a major obstacle but now it seems only as a technical problem.

DEFINITION 1.31 (Canonical class and canonical sheaf). Let X be a smooth variety over a field k . As in [Sha74, III.6.3] or [Har77, p.180], the *canonical sheaf* of X is $\omega_X := \wedge^{\dim X} \Omega_{X/k}$. Any divisor D such that $\mathcal{O}_X(D) \cong \omega_X$ is called a *canonical divisor*. Their linear equivalence class is called the *canonical class*, denoted by K_X . (Note that both books assume that X is nonsingular. However, they tacitly assume that k is algebraically closed, hence nonsingularity implies smoothness. The definition, however, works over any field k as long as X is smooth over k .)

Let X be a normal variety over a perfect field k . Let $j: X^{\text{sm}} \hookrightarrow X$ be the inclusion of the locus of smooth points. Then $X \setminus X^{\text{sm}}$ has codimension ≥ 2 , therefore, restriction from X to X^{sm} is a bijection on Weil divisors and on linear equivalence classes of Weil divisors. Thus there is a unique linear equivalence class K_X of Weil divisors on X such that $K_X|_{X^{\text{sm}}} = K_{X^{\text{sm}}}$. It is called the *canonical class* of X . In general, K_X does not contain any Cartier divisors.

The push-forward $\omega_X := j_*\omega_{X^{\text{sm}}}$ is a rank 1 coherent sheaf on X , called the *canonical sheaf* of X . The canonical sheaf ω_X agrees with the *dualizing sheaf* ω_X° as defined in [Har77, p.241]. (Note that [Har77] defines the dualizing sheaf only if X is proper. In general, take a normal compactification $\bar{X} \supset X$ and use $\omega_{\bar{X}}^\circ|_X$ instead. For more details, see [KM98, Sec.5.5], [Har66] or [Con00].)

More generally, as long as ω_X is locally free outside a codimension ≥ 2 subset of X , we can work with ω_X and K_X as in the normal case. We use this mostly when X has nodes at codimension 1 points (11.9).

With this definition in place, we can give the following abstract characterization of canonical models.

THEOREM 1.32. *A normal proper variety Y is a canonical model iff*

(1.32.1) m_0K_Y is Cartier and ample for some $m_0 > 0$, and

(1.32.2) there is a resolution $f: X \rightarrow Y$ (that is, a proper birational morphism where X is smooth) and an effective, f -exceptional divisor E such that

$$m_0K_X \sim f^*(m_0K_Y) + E.$$

Proof. For now we prove only the ‘if’ part since this is what we need for the examples. For the converse, see [Rei80] or [Kol13b, 1.15].

Note that for any $r > 0$, $f_*\mathcal{O}_X(rE) = \mathcal{O}_Y$ since E is effective and f -exceptional. Thus, by the projection formula,

$$\begin{aligned} H^0(X, \mathcal{O}_X(rm_0K_X)) &= H^0(Y, f_*\mathcal{O}_X(rm_0K_X)) \\ &= H^0(Y, \mathcal{O}_Y(rm_0K_Y) \otimes f_*\mathcal{O}_X(rE)) \\ &= H^0(Y, \mathcal{O}_Y(rm_0K_Y)). \end{aligned}$$

Therefore

$$\begin{aligned} \text{Proj} \sum_m H^0(X, \mathcal{O}_X(mK_X)) &= \text{Proj} \sum_r H^0(X, \mathcal{O}_X(rm_0K_X)) \\ &= \text{Proj} \sum_r H^0(Y, \mathcal{O}_Y(rm_0K_Y)) = Y. \quad \square \end{aligned}$$

This makes it possible to give a local definition of the singularities that occur on canonical models.

DEFINITION 1.33. A normal variety Y has *canonical singularities* if

(1.33.1) m_0K_Y is Cartier for some $m_0 > 0$ and

(1.33.2) there is a resolution $f: X \rightarrow Y$ and an effective, f -exceptional divisor E such that $m_0K_X \sim f^*(m_0K_Y) + E$.

It is easy to show that this is independent of the resolution $f: X \rightarrow Y$; see [Kol13b, 2.12]. (It is not hard to define canonical singularities without assuming the existence of a resolution as in [Kol13b, Sec.2.1] or [Luo87].)

Equivalently, Y has canonical singularities iff every point $y \in Y$ has an étale neighborhood which is an open subset on some canonical model.

A complete list of canonical singularities is known in dimension 2 and almost known in dimension 3 [Rei80]. The following examples are useful to keep in mind.

- (1.33.3) Smooth points are canonical.
- (1.33.4) The hypersurface singularity $(x_1x_2 + f(x_3, \dots, x_n) = 0)$ is canonical iff f is not identically 0.
- (1.33.5) The quotient singularity $\mathbb{A}^d/\frac{1}{n}(1, n-1, a_3, \dots, a_d)$ is canonical for $d \geq 3$ if $(n, a_i) = 1$. Its canonical class is Cartier iff $n \mid a_3 + \dots + a_d$. (For arbitrary quotient singularities the Reid-Tai criterion [Rei80] determines canonicity, but it is not easy to write it down in closed form.)
- (1.33.6) The cone $C_d(\mathbb{P}^n)$ over the Veronese embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d)))$ has a canonical singularity iff $d \leq n+1$. Its canonical class is Cartier iff $d \mid n+1$. (See (2.35) or [Kol13b, 3.1] for the case of general cones.)

WARNING 1.34 (\mathbb{Q} -Cartier condition). While (1.33.1) may seem like a small technical condition, in many cases it turns out to be extremely important.

First of all, we cannot pull back arbitrary divisors, so (1.33.2) does not even make sense if K_Y is not \mathbb{Q} -Cartier. This is a real problem starting with dimension 3. (For normal surfaces one can define the pull-back of all divisors in a sensible way (11.47); see also (11.48).)

The issue becomes more serious for families of varieties. Unexpected jumps of the Kodaira dimension happen precisely when the canonical class of the total space is not \mathbb{Q} -Cartier, see (1.43–1.46).

The most difficult aspects appear for non-normal varieties. The gluing theory of [Kol13b, Chap.5] is almost entirely devoted to proving that in some cases the canonical divisor is \mathbb{Q} -Cartier; see (11.21) for a key consequence.

DEFINITION 1.35 (Moduli of canonical models). The moduli functor of canonical models is

$$\text{CanMod}(S) := \left\{ \begin{array}{l} \text{Flat, proper families } X \rightarrow S, \\ \text{every fiber is a canonical model,} \\ \text{modulo isomorphisms over } S. \end{array} \right\} \quad (1.35.1)$$

This is an improved version of the birational moduli functor $\text{GenType}_{\text{bir}}$ (1.23.5).

Warning. In retrospect, it seems only by luck that this definition gives the correct functor. See (1.16.2) and the examples after it.

1.36 (From GenType to CanMod). Let $p : Y \rightarrow S$ be a smooth, projective morphism of varieties over a field of characteristic 0. Assume that S is reduced and connected and the fibers Y_s are of general type. By (1.37), the plurigenera $h^0(Y_s, \omega_{Y_s}^m)$ are independent of $s \in S$. Thus, by Grauert's theorem [Har77, III.12.9], the sheaves $p_*\omega_{Y/S}^m$ are locally free for $m \geq 1$. We get the flat family of canonical models

$$Y_S^{\text{stab}} := \text{Proj}_S \oplus_m p_*\omega_{Y/S}^m. \quad (1.36.1)$$

Thus there is a natural transformation T_{CanMod} which, for any reduced scheme S gives a map of sets

$$T_{\text{CanMod}}(S) : \text{GenType}_{\text{bir}}(S) \rightarrow \text{CanMod}(S).$$

By definition, if $X_i \rightarrow S$ are two smooth, proper families of varieties of general type such that $T_{\text{CanMod}}(S)(X_1) = T_{\text{CanMod}}(S)(X_2)$ then X_1 and X_2 are birational, thus $T_{\text{CanMod}}(S)$ is injective. It is not surjective, but we have the following partial surjectivity statement.

Claim 1.36.2. Let $Y \rightarrow S$ be a flat family of canonical models. Then there is a dense open subset $S^\circ \subset S$ and a smooth, proper family of varieties of general type $Y^\circ \rightarrow S^\circ$ such that $T_{\text{CanMod}}(S^\circ)(Y^\circ) = [Y|_{S^\circ}]$.

For surfaces the following result goes back to [KS58], and the 3-fold case is proved in [KM92, 12.5.1]. The higher dimensional case for smooth projective morphisms over a smooth base is proved by [Siu98], and by [Nak04, Chap.VI] when the fibers have canonical singularities. The complex analytic case is settled in [Kol21].

THEOREM 1.37. *Let $\pi : V \rightarrow B$ be a flat, projective morphism whose fibers are of general type and have canonical singularities. Then the canonical models of the fibers form a flat, proper morphism $\pi^c : V^c \rightarrow B$, and the natural map $V \dashrightarrow V^c$ is fiberwise birational.*

1.4. From stable curves to stable varieties

Next we discuss the reasoning behind Step 1.15.

Let C be a nodal curve with normalized irreducible components C_i . We frequently view C as an object assembled from the pieces C_i . Note that the restriction of ω_C to C_i is not ω_{C_i} , rather $\omega_{C_i}(P_i)$, where $P_i \subset C_i$ are the preimages of the nodes of C .

Similarly, if X is a scheme with simple normal crossing singularities (11.2) and normalized irreducible components X_i , then the restriction of ω_X to X_i is not ω_{X_i} , rather $\omega_{X_i}(D_i)$, where $D_i \subset X_i$ is the preimage of $\text{Sing } X$ on X_i .

This suggests that we should develop a theory of ‘canonical models’ where the role of the canonical class is played by a divisor of the form $K_X + D$, where D is a simple normal crossing divisor (11.2).

DEFINITION 1.38 (Canonical models of pairs). Let (X, D) be a projective snc pair (11.2). We define the *canonical ring* of the pair (X, D) as

$$R(X, K_X + D) := \sum_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X + mD)).$$

It is conjectured (but known only for $\dim X \leq 4$ in characteristic 0) that the canonical ring of a pair (X, D) is finitely generated. If this holds then $X^{\text{can}} := \text{Proj}_k R(X, K_X + D)$ is a normal projective variety. We say that (X, D) is of *general type* if the natural map $\pi : X \dashrightarrow X^{\text{can}}$ is birational and then $(X^{\text{can}}, D^{\text{can}} := \pi_* D)$ is called the *canonical model* of (X, D) .

The proof of the ‘if’ part of the following characterization goes exactly as in (1.32).

THEOREM 1.39. *A pair (Y, B) , consisting of a proper normal variety Y and an effective, reduced Weil divisor B , is the canonical model of a simple normal crossing pair iff*

(1.39.1) $m_0(K_Y + B)$ is Cartier and ample for some $m_0 > 0$, and

(1.39.2) there is a resolution $f : X \rightarrow Y$, an effective, reduced simple normal crossing divisor $D \subset X$ such that $f(D) = B$, and an effective, f -exceptional divisor E such that

$$m_0(K_X + D) \sim f^*(m_0(K_Y + B)) + E.$$

Warning 1.39.3. If $B = 0$, it can happen that $(X, 0)$ is the canonical model of a pair, but X is not a canonical model (1.32). To see this, choose a resolution $f: X \rightarrow Y$, and let $F_i \subset X$ be the f -exceptional divisors. Although $B = 0$, in (1.39.2) we can still take $D = \sum F_i$. Thus (1.39.2) can be rewritten as

$$m_0 K_X \sim f^*(m_0 K_Y) + E - m_0 \sum F_i.$$

This looks like (1.32.2), but $E - m_0 \sum F_i$ need not be effective; it can contain divisors with coefficients $\geq -m_0$.

This is the source of some terminological problems. Originally $R(X, K_X + D)$ was called the ‘log canonical ring’ and $\text{Proj}_k R(X, K_X + D)$ the ‘log canonical model.’ Since the canonical ring is just the $D = 0$ special case of the ‘log canonical ring,’ it seems more convenient to drop the prefix ‘log.’ However, log canonical singularities are quite different from canonical singularities, so ‘log’ cannot be omitted there. See also (5.8) for other inconsistencies in the standard usage of ‘canonical model.’

As in (1.33), this can be reformulated as a definition. (For now we assume that every irreducible component of B appears in B with coefficient 1; later we also consider cases when the coefficients are rational or real.)

DEFINITION 1.40. Let (Y, B) be a pair consisting of a normal variety Y and a reduced Weil divisor B . Then (Y, B) is *log canonical*, or has *log canonical singularities* iff the conditions (1.39.1–2) are satisfied. We say that Y is log canonical if (Y, \emptyset) is.

If (Y, B) is log canonical and B is \mathbb{Q} -Cartier then Y is also log canonical (11.4.1). However, if B is not \mathbb{Q} -Cartier then K_Y is also not \mathbb{Q} -Cartier, so Y is not log canonical.

A complete list of log canonical singularities is known in dimension 2, see [Kol13b, Sec.2.2]. The following examples are useful to keep in mind.

- (1.40.1) $(\mathbb{A}^n, (x_1 \cdots x_r = 0))$ is log canonical for any $r \leq n$.
- (1.40.2) Let g be a homogeneous polynomial of degree d that defines a smooth hypersurface $(g = 0) \subset \mathbb{P}^{r-1}$. If $d \leq r$ then the hypersurface singularity $(g(x_1, \dots, x_r) + f(x_{r+1}, \dots, x_n) = 0)$ is log canonical for every f .
- (1.40.3) All quotient singularities $\mathbb{A}^d / \frac{1}{n}(a_1, \dots, a_d)$ are log canonical. These are the simplest ones (they are even log terminal), but they already show many of the complicated behavior of arbitrary log canonical singularities.
- (1.40.4) A cone $C(X)$ over a Calabi-Yau variety is log canonical, see (2.35) or [Kol13b, 3.1].

We are now ready to define the higher dimensional analogs of stable curves.

DEFINITION 1.41 (Stable varieties). Let k be a field and Y a reduced, proper scheme over k . Let $Y_i \rightarrow Y$ be the irreducible components of the normalization of Y and $D_i \subset Y_i$ the reduced preimage of the non-normal locus of Y . Then Y is *semi-log-canonical*—usually abbreviated as *slc*—or *locally stable* iff

- (1.41.1) at codimension 1 points, Y is either smooth or has a node (11.9),
- (1.41.2) each (Y_i, D_i) is log canonical, and
- (1.41.3) K_Y is \mathbb{Q} -Cartier, that is, mK_Y is Cartier for some $m > 0$. Equivalently, $\omega_Y^{[m]}$, the double dual of $\omega_Y^{\otimes m}$, is locally free for some $m > 0$.

Y is a *stable variety* iff, in addition

(1.41.4) Y is projective and K_Y is ample.

As we noted in (1.34), the \mathbb{Q} -Cartier condition for K_Y is quite hard to interpret in terms of the (Y_i, D_i) . See (11.21) or the more detailed [Kol13b, Chap.5].

For now we only deal with examples where K_Y is obviously Cartier or \mathbb{Q} -Cartier.

Jump of K^2 and of the Kodaira dimension

We give examples of flat families of projective surfaces $\{S_t : t \in \mathbb{D}\}$ such that S_0 has quotient singularities and the S_t are smooth for $t \neq 0$, but the self intersection of the canonical class $K_{S_t}^2$ varies with t . We also give examples where K_{S_t} is ample for $t = 0$ but not even big for $t \neq 0$. As we noted in (1.40.3), among log canonical singularities, the quotient singularities are the mildest.

As we already noted in (1.34), such jumps happen when the canonical class of the total space is not \mathbb{Q} -Cartier.

EXAMPLE 1.42 (Degree 4 surfaces in \mathbb{P}^5). There are 2 families of nondegenerate degree 4 smooth surfaces in \mathbb{P}^5 . These were classified by Del Pezzo, see [EH87] for a modern treatment.

One family consists of Veronese surfaces $\mathbb{P}^2 \subset \mathbb{P}^5$ embedded by $\mathcal{O}(2)$. The general member of the other family is $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^5$ embedded by $\mathcal{O}(2, 1)$, special members are embeddings of the ruled surface \mathbb{F}_2 . The two families are distinct since

$$K_{\mathbb{P}^2}^2 = 9 \quad \text{and} \quad K_{\mathbb{P}^1 \times \mathbb{P}^1}^2 = 8.$$

For both of these surfaces, a smooth hyperplane section is a degree 4 rational normal curve in \mathbb{P}^4 .

For us the most interesting degree 4 singular surface in \mathbb{P}^5 is the cone over the degree 4 rational normal curve in \mathbb{P}^4 ; denote it by $T_0 \subset \mathbb{P}^5$. The minimal resolution of T_0 is the ruled surface $p: \mathbb{F}_4 \rightarrow T_0$. Let $E, F \subset \mathbb{F}_4$ be the exceptional curve and the fiber of the ruling. Then $K_{\mathbb{F}_4} = -2E - 6F$ and $p^*(2K_{T_0}) = -3E - 12F$. Thus

$$2(K_{\mathbb{F}_4} + E) = p^*(2K_{T_0}) + E$$

shows that T_0 has log canonical singularities. We also get that $K_{T_0}^2 = 9$.

A key feature is that one can write T_0 as a limit of smooth surfaces in two distinct ways, corresponding to the two ways of writing the degree 4 rational normal curve in \mathbb{P}^4 as a hyperplane section of a surface. (See [Kol13b, 3.9] for a concrete description of these deformations.)

From the first family, we get T_0 as the special fiber of a flat family whose general fiber is \mathbb{P}^2 . This family is denoted by $\{T_t : t \in \mathbb{C}\}$. From the second family, we get T_0 as the special fiber of a flat family whose general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$. This family is denoted by $\{T'_t : t \in \mathbb{C}\}$. (In general, one needs to worry about the possibility of getting embedded points at the vertex. However, by [Kol13b, 3.10], in both cases the special fiber is indeed T_0 .)

Note that K^2 is constant in the family $\{T_t : t \in \mathbb{C}\}$ but jumps at $t = 0$ in the family $\{T'_t : t \in \mathbb{C}\}$.

These are, however, families of rational surfaces with negative canonical class, and we are interested in stable varieties.

Next we take a suitable cyclic cover (11.14) of the two families to get similar examples with ample canonical class.

EXAMPLE 1.43 (Jump of Kodaira dimension I).

We give examples of two flat families of projective surfaces S_t and S'_t such that

- (1.43.1) $S_0 \cong S'_0$ has log canonical singularities and ample canonical class,
- (1.43.2) S_t is a smooth surface with ample canonical class for $t \neq 0$, and
- (1.43.3) S'_t is a smooth, elliptic surface with $K_{S'_t}^2 = 0$ for $t \neq 0$.

With T_0 as in (1.42), let $\pi_0: S_0 \rightarrow T_0$ be a double cover, ramified along a smooth quartic hypersurface section. Note that $K_{T_0} \sim_{\mathbb{Q}} -\frac{3}{2}H$ where H is the hyperplane class. Thus, by the Hurwitz formula,

$$K_{S_0} \sim_{\mathbb{Q}} \pi_0^*(K_{T_0} + 2H) \sim_{\mathbb{Q}} \frac{1}{2}\pi_0^*H.$$

So S_0 has ample canonical class and $K_{S_0}^2 = 2$. Since π_0 is étale over the vertex of T_0 , S_0 has 2 singular points, locally (in the analytic or étale topology) isomorphic to the singularity on T_0 . Thus S_0 is a stable surface.

Both of the smoothings in (1.42) lift to smoothings of S_0 .

From T_t we get a smoothing S_t where $\pi_t: S_t \rightarrow \mathbb{P}^2$ is a double cover, ramified along a smooth octic. Thus S_t is smooth, $K_{S_t} \sim_{\mathbb{Q}} \pi_t^*\mathcal{O}_{\mathbb{P}^2}(1)$ is ample and $K_{S_t}^2 = 2$.

From T'_t we get a smoothing S'_t where $\pi'_t: S'_t \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is a double cover, ramified along a smooth curve of bidegree $(8, 4)$. One of the families of lines on $\mathbb{P}^1 \times \mathbb{P}^1$ pulls back to an elliptic pencil on S'_t and $K_{S'_t}^2 = 0$. Thus S'_t is not of general type for $t \neq 0$.

EXAMPLE 1.44 (Jump of Kodaira dimension II). A similar pair of examples is obtained by working with triple covers ramified along a cubic hypersurface section of the surface families in (1.42). The family over T_t has ample canonical class and $K^2 = 3$. As before, the family over T'_t is elliptic and has $K^2 = 0$.

EXAMPLE 1.45 (Jump of Kodaira dimension III).

We construct a flat family of surfaces whose central fiber is the quotient of the square of the Fermat cubic curve by $\mathbb{Z}/3$:

$$S_F^* \cong (u_1^3 = v_1^3 + w_1^3) \times (u_2^3 = v_2^3 + w_2^3) / \frac{1}{3}(1, 0, 0; 1, 0, 0), \quad (1.45.1)$$

thus it has Kodaira dimension 0. The general fiber is \mathbb{P}^2 blown up at 12 points.

In \mathbb{P}^3 consider two lines $L_1 = (x_0 = x_1 = 0)$ and $L_2 = (x_2 = x_3 = 0)$. The linear system $|\mathcal{O}_{\mathbb{P}^2}(2)(-L_1 - L_2)|$ is spanned by the 4 reducible quadrics $x_i x_j$ for $i \in \{0, 1\}$ and $j \in \{2, 3\}$. They satisfy a relation $(x_0 x_2)(x_1 x_3) = (x_0 x_3)(x_1 x_2)$. Thus we get a morphism

$$\pi: B_{L_1+L_2}\mathbb{P}^3 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

which is a \mathbb{P}^1 -bundle whose fibers are the birational transforms of lines that intersect both of the L_i .

Let $S \subset \mathbb{P}^3$ be a cubic surface such that $\mathbf{p} := S \cap (L_1 + L_2)$ is 6 distinct points. Then we get $\pi_S: B_{\mathbf{p}}S \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

In general, none of the lines connecting 2 points of \mathbf{p} is contained in S . Thus in this case π_S is a finite triple cover.

At the other extreme we have the Fermat-type surface

$$S_F := (x_0^3 + x_1^3 = x_2^3 + x_3^3) \subset \mathbb{P}^3.$$

We can factor both sides and write its equation as $m_1 m_2 m_3 = n_1 n_2 n_3$. The 9 lines $L_{ij} := (m_i = n_j = 0)$ are all contained in S_F . Let $L'_{ij} \subset B_{\mathbf{p}}S_F$ denote

their birational transforms. Then the self-intersections $(L'_{ij} \cdot L'_{ij})$ equal -3 and π_{S_F} contracts these 9 curves L'_{ij} . Thus the Stein factorization of π_{S_F} gives a triple cover $S_F^* \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and S_F^* has 9 singular points of type $\mathbb{A}^2/\frac{1}{3}(1,1)$. We see furthermore that

$$-3K_{S_F} \sim \sum_{ij} L_{ij} \quad \text{and} \quad -3K_{B_{\mathbb{P}}S_F} \sim \sum_{ij} L'_{ij}.$$

Thus $-3K_{S_F^*} \sim 0$.

To see that the two surfaces denoted by S_F^* are isomorphic, use the map of the surface (1.45.1) to $\mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$(u_1:v_1:w_1) \times (u_2:v_2:w_2) \mapsto (v_1:w_1) \times (v_2:w_2),$$

and the rational map to the cubic surface given by

$$(u_1:v_1:w_1) \times (u_2:v_2:w_2) \mapsto (v_2u_1u_2^2:u_1u_2^2:v_1u_2^3:u_2^3).$$

EXAMPLE 1.46 (Jump of Kodaira dimension IV). The previous examples are quite typical in some sense. If S_0 is any projective rational surface with quotient singularities, then there is a flat family of surfaces $\{S_t\}$ such that S_t is a smooth rational surface for $t \neq 0$.

To see this, take a minimal resolution $S'_0 \rightarrow S_0$. Let H'_0 be the pull-back of an ample Cartier divisor from S_0 . Since S'_0 is a smooth rational surface, it can be obtained from a minimal smooth rational surface by blowing up points. We can deform S'_0 by moving these points into general position (and also deforming the minimal smooth rational surface if necessary). Thus we see that if S_0 is singular then a general deformation S'_t of S'_0 is obtained by blowing up points in \mathbb{P}^2 in general position. One can see, (cf. [dF05, 2.4]) that every smooth rational curve on S'_t with negative self-intersection is a (-1) -curve. In particular, none of the exceptional curves of $S'_0 \rightarrow S_0$ lift to S'_t hence H'_t is ample for general t . As before, we get a flat deformation $\{S_t\}$ such that $S_t \cong S'_t$ for $t \neq 0$.

Many recent constructions of surfaces of general type start with a particular rational surface S_0 with quotient singularities and show that it has a flat deformation to a smooth surface with ample canonical class; see [LP07, PPS09a, PPS09b]. Thus such an S_0 has flat deformations of general type and also flat deformations that are rational.

Even more surprisingly, a surface with ample canonical class can have non-algebraic deformations.

EXAMPLE 1.47 (Non-algebraic deformations). We construct a projective surface X_0 with a quotient singularity, ample canonical class and two deformations

$$(1.47.1) \text{ an algebraic one } g^{\text{alg}} : X^{\text{alg}} \rightarrow D, \text{ where } g \text{ is flat, projective, and}$$

$$(1.47.2) \text{ a complex analytic one } g^{\text{an}} : X^{\text{an}} \rightarrow \mathbb{D}, \text{ where } g^{\text{an}} \text{ is flat, proper,}$$

such that

$$(1.47.3) X_s^{\text{alg}} \text{ is a smooth, algebraic, K3 surface blown up at 3 points for } s \neq 0, \text{ and}$$

$$(1.47.4) X_s^{\text{an}} \text{ is a smooth, non-algebraic, K3 surface blown up at 3 points for very general } s \in \mathbb{D}.$$

Let us start with a K3 surface $Y \subset \mathbb{P}^3$ with a hyperplane section $C \subset Y$ and 3 points $p_i \in C$. Blow up these points to get $\pi : Z \rightarrow Y$ with exceptional curves $E = E_1 + E_2 + E_3$. Let $C_Z \subset Z$ be the birational transform of C and $H = \pi^*C - \frac{2}{3}E$.

If the p_i are smooth points on C , then $\pi^*C = C_Z + E$, hence $H = C_Z + \frac{1}{3}E$. Since $(H \cdot C_Z) = 2$, $(H \cdot E_i) = \frac{2}{3}$ and $Z \setminus (C_Z + E) \cong Y \setminus C$ is affine, we see that H is ample by the Nakai-Moishezon criterion.

If the p_i are double points on C , then $\pi^*C = C_Z + 2E$, hence $H = C_Z + \frac{4}{3}E$. Then $(C_Z \cdot E_i) = 2$, $(H \cdot C_Z) = 0$ and $(H \cdot E_i) = \frac{2}{3}$. We see that $3H$ is semiample and it contracts C_Z . Let the resulting surface be X_0 , and $F_i \subset X_0$ the images of the E_i .

Note that in this case C_Z is a smooth, rational curve and $(C_Z^2) = -8$. Thus X_0 has a single quotient singularity of type $\mathbb{C}^2/\frac{1}{8}(1,1)$. We also get that $(F_i^2) = -\frac{1}{2}$ and $(F_i \cdot F_j) = \frac{1}{2}$ for $i \neq j$. Furthermore, $K_{X_0} \sim F_1 + F_2 + F_3$ is ample.

In order to construct the algebraic family, start with $C \subset Y$ where C is a rational curve with 3 nodes. The deformation is obtained by moving the points into general position. Blowing up the points we get H that is ample on the general fibers and contracts the birational transform of C in the special fiber. Thus we get $g^{\text{alg}} : X^{\text{alg}} \rightarrow D$.

For the analytic case, we choose a deformation $Y \rightarrow \mathbb{D}$ of Y_0 whose very general fibers are non-algebraic K3 surfaces. Take 3 sections $B_i \subset Y$ that pass through the 3 nodes of C . Blow them up and then contract the birational transform of C . The contraction extends to the total space by [MR71]. We get $g^{\text{an}} : X^{\text{an}} \rightarrow \mathbb{D}$ whose central fiber is X_0 and the other fibers are non-algebraic, K3 surfaces blown up at 3 points.

EXAMPLE 1.48 (More rational surfaces with ample canonical class). [Kol08b, Sec.5] Given natural numbers a_1, a_2, a_3, a_4 , consider the surface

$$S = S(a_1, a_2, a_3, a_4) := (x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_4 + x_4^{a_4}x_1 = 0) \subset \mathbb{P}(w_1, w_2, w_3, w_4),$$

where $w'_i = a_{i+1}a_{i+2}a_{i+3} - a_{i+2}a_{i+3} + a_{i+3} - 1$ (with indices modulo 4), and $w_i = w'_i / \gcd(w'_1, w'_2, w'_3, w'_4)$.

It is easy to see that S has only quotient singularities (at the 4 coordinate vertices). It is proved in [Kol08b, Thm.39] that S is rational if $\gcd(w'_1, w'_2, w'_3, w'_4) = 1$. (By [Kol08b, 38], this happens with probability ≥ 0.75 .)

$\mathbb{P}(w_1, w_2, w_3, w_4)$ has isolated singularities iff the $\{w_i\}$ are pairwise relatively prime. (It is easy to see that for $1 \leq a_i \leq N$, this happens for at least $c \cdot N^{4-\epsilon}$ of the 4-tuples.) In this case the canonical class of S is

$$K_S = \mathcal{O}_{\mathbb{P}}(\prod a_i - 1 - \sum w_i)|_S.$$

From this it is easy to see that if $a_1, a_2, a_3, a_4 \geq 4$ then K_S is ample and K_S^2 converges to 1 as $a_1, a_2, a_3, a_4 \rightarrow \infty$.

1.5. From nodal curves to stable curves and surfaces

As we saw in (1.15.1) stable reduction for families of curves $C \rightarrow S$ involves 2 main steps.

- First we transform a given proper family of curves $C \rightarrow S$ into a proper, flat family $C_1 \rightarrow S_1$ whose fibers are reduced, nodal curves. This needs a base change $S_1 \rightarrow S$ that involves choices and then a sequence of blow-ups that again involves choices. It does not seem possible to do either of these in a functorial way. It is sometimes convenient to choose S_1 to be smooth.

- Once we have a proper, flat family $C_1 \rightarrow S_1$ whose fibers are reduced, nodal curves and whose base is smooth, we take the relative canonical model (11.28) to get

the stable family $C_1^{\text{stab}} \rightarrow S_1$. Note that every form of the MMP over a base scheme assumes that the singularities of the total space (as opposed to the singularities of the fibers) are mild. Hence this approach can possibly work only if S_1 is at least log canonical.

Nonetheless, the next result says that one can go from flat families of nodal curves to flat families of stable curves in a functorial way over an arbitrary base.

THEOREM 1.49. *For every $g \geq 2$ there is a natural transformation $C \mapsto C^{\text{stab}}$*

$$\left\{ \begin{array}{l} \text{proper, flat families of} \\ \text{reduced, nodal,} \\ \text{genus } g \text{ curves} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{stable families of} \\ \text{genus } g \text{ curves} \end{array} \right\}.$$

In order to get something reasonable, we require that if C is a smooth, projective curve then $C^{\text{stab}} = C$. We assume that the curves in question are geometrically connected and by the genus of a proper nodal curve C we mean the arithmetic genus $h^1(C, \mathcal{O}_C)$.

Proof. We outline the main steps, leaving some details to the reader.

First let C be a proper, reduced, nodal curve over an algebraically closed field. We start with 2 recipes to construct C^{stab} . With both approaches, we first obtain the largest semistable subcurve $C^{\text{ss}} \subset C$.

- (1.a) (Using MMP) Find an irreducible component $C' \subset C$ on which ω_C has negative degree. Equivalently, $C' \cong \mathbb{P}^1$ and it meets the rest of C in 1 point only. Contract (or discard) this component. Repeat if possible.
- (1.b) (Using ω_C) C^{ss} is the support of the global sections of ω_C .

Once we have C^{ss} we continue to get C^{stab} as follows.

- (2.a) Find an irreducible component $C' \subset C^{\text{ss}}$ on which $\omega_{C^{\text{ss}}}$ has degree 0. Equivalently, $C' \cong \mathbb{P}^1$ and it meets the rest of C^{ss} in 2 points only; call them p, q . Contract this component. Equivalently, discard C' and identify the points p, q to get a node. Repeat if possible.
- (2.b) (Using the canonical ring) $C^{\text{stab}} = \text{Proj} \sum_m H^0(C^{\text{ss}}, \omega_{C^{\text{ss}}}^m)$.

We call an irreducible component $C' \subset C$ *stable* if it is kept in the above process.

Warning 1.49.3. Note that (2.b) works only for semistable curves. As an example, let $C = C_1 \cup C_2$ be a curve with a single node p with $g(C_1) \geq 2$ and $C_2 \cong \mathbb{P}^1$. Show that

$$\sum_{m \geq 0} H^0(C, \omega_C^m) = \sum_{m \geq 0} H^0(C_1, \omega_{C_1}^m((m-1)[p]))$$

is **not** finitely generated.

For an arbitrary field k we need to show that if C_k is defined over k then $(C_k)^{\text{stab}}$ is also defined over k , giving us $(C_k)^{\text{stab}}$.

Next let T be the spectrum of a DVR and $C_T \rightarrow T$ a proper, flat family of reduced, nodal curves. We extend the above procedure from the central fiber C to C_T as follows.

- (4.a) Let $C' \subset C$ be an irreducible component that we need to contract. If there is an irreducible component $C'_T \subset C_T$ whose central fiber is C' , then contract C'_T .
- (4.b) Otherwise we contract C' to a point and leave the generic fiber unchanged.

Once we know the stable irreducible components $C^i \subset C$, we can also recover the corresponding stable curve in the following way.

(1.49.5) Pick non-nodal points $p^i \in C^i$ on each stable, irreducible component and set $L := \mathcal{O}_C(\sum p^i)$. Then, for $m \gg 1$, $H^1(C, L^m) = 0$, L^m is globally generated and maps C onto C^{stab} .

Let now $g : C_S \rightarrow S$ be a proper, flat family of reduced, nodal curves over an arbitrary base. It is enough to construct the stable family étale-locally. Thus we may assume that $(0, S)$ is local. Let $C_0^i \subset C_0$ be the stable irreducible components. Pick non-nodal points $p^i \in C_0^i$ and let $D^i \subset C_S$ be sections that meet C_0 only at p^i . (Usually this needs an étale base change of S .) Set $L_S := \mathcal{O}_{C_S}(\sum D^i)$. The construction in (4.a–b) shows that the D^i intersect the other fibers at their stable components. Then (5) shows the following.

(1.49.6) For $m \gg 1$, $R^1 g_* L^m = 0$, $g_* L^m$ is locally free and maps C_S onto C_S^{stab} . \square

DEFINITION 1.50 (Stabilization functor). Attempting to generalize (1.49) to higher dimensions, the best would be to have a functor

$$\left\{ \begin{array}{l} \text{proper, flat} \\ \text{locally stable families} \end{array} \right\} \longrightarrow \{ \text{stable families} \},$$

that agrees with $X \rightarrow X^{\text{can}}$ on log canonical varieties of general type.

One can further restrict the singularities of the fibers and talk about stabilization functors for families of smooth varieties, simple normal crossing varieties (11.2), and so on.

We see below that such a stabilization functor does exist for smooth families, but not for more complicated singularities. We discuss this phenomenon in detail in Section 5.2. This is another reason why the moduli theory of higher dimensional varieties is much more complicated.

THEOREM 1.51 (Stabilization functor for surfaces).

(1.51.1) For smooth, projective surfaces of general type, $S \mapsto S^{\text{can}}$ is a stabilization functor.

(1.51.2) For projective surfaces of general type with quotient singularities, $S \mapsto S^{\text{can}}$ is **not** a stabilization functor.

(1.51.3) For projective surfaces with (simple) normal crossing singularities, $S \mapsto S^{\text{can}}$ is **not** a stabilization functor.

(1.51.4) For projective surfaces with (simple) normal crossing singularities, S^{can} does not even make sense in general.

Proof. As we already noted in (1.36), the first part has been long known as the deformation invariance of the plurigenera (if the base scheme is reduced). As in (1.49), more work is needed for nonreduced bases.

We give a series of concrete examples for (2) in (1.52), but it may be worthwhile to give a conceptual explanation of what goes wrong.

Assume for simplicity that we have a flat, projective morphism $p : X \rightarrow \mathbb{A}^1$. Assume furthermore that K_X is \mathbb{Q} -Cartier and the fibers are surfaces with quotient singularities only. Then we get the stable model $p^{\text{stab}} : X^{\text{stab}} \rightarrow \mathbb{A}^1$ as the relative canonical model (11.28).

To see how difficulties appear, we look at the MMP step by step. Let $\pi_0 : X_0 \rightarrow X'_0$ denote the contraction of an extremal ray. It always extends to a contraction

$\pi : X \rightarrow X'$. If π is a divisorial contraction then we can continue the MMP with X' . (This is always the case for families of curves.) However, if π is a flipping contraction, then we need to construct the flip $\pi^+ : X^+ \rightarrow X'$ and continue with X^+ instead.

Under the flip the central fiber X_0 is replaced by a new central fiber X_0^+ . This involves removing the K -negative extremal curves $C_0 \subset X_0$ and then adding in some K -positive curves $C_0^+ \subset X_0^+$. In the MMP for surfaces we only contract curves, never add new curves, thus we get

Problem 1.51.5. $X_0 \mapsto X_0^+$ is **not** a step of the MMP for X_0 .

In most cases this implies that the central fiber of $X^{\text{can}} \rightarrow \mathbb{A}^1$ is **not** isomorphic to $(X_0)^{\text{can}}$. In (1.52) we also give similar examples to show that this can also happen with snc fibers. These examples show that in the settings of (2) or (3), there is no stabilization functor, not even if we work over 1-dimensional smooth bases. In general, the formation of the stable model commutes with étale base changes, but not with taking closed fibers.

Next consider such a family $X \rightarrow \mathbb{A}_u^1$ and glue it to the trivial family $q : Y := X_0 \times \mathbb{A}_v^1 \rightarrow \mathbb{A}_v^1$ along the central fibers to get a locally stable family

$$r : X \amalg_{X_0} Y \rightarrow (uv = 0).$$

For this family over the pair of lines $(uv = 0)$

- (6.a) $p^{\text{stab}} : X^{\text{stab}} \rightarrow \mathbb{A}_u^1$ and $q^{\text{stab}} : Y^{\text{stab}} \rightarrow \mathbb{A}_v^1$ both exists, yet
- (6.b) their central fibers $(X^{\text{can}})_0$ and $(Y^{\text{can}})_0$ are **not** isomorphic, so
- (6.c) $r : X \amalg_{X_0} Y \rightarrow (uv = 0)$ does **not** have a stable model.

Claim (4) is not a precise assertion, but we expect that, even over algebraically closed fields, there is no ‘sensible’ way to associate a stable surface to every projective, normal crossing surface. For example, [Koll11c] constructs irreducible, projective surfaces S with normal crossing singularities for which the canonical ring $\sum_{m \geq 0} H^0(S, \omega_S^m)$ is **not** finitely generated. We present a similar example in (1.53). However, we also do not know any definition of ‘sensible.’ \square

EXAMPLE 1.52. We start with the simplest flip $\phi : X \dashrightarrow X^+$ sitting in the following diagram.

$$\begin{array}{ccc} Y \dashrightarrow Y^+ & & \\ q \swarrow & & \searrow q^+ \\ X & \xrightarrow{\phi} & X^+ \\ p \searrow & & \swarrow p^+ \\ & Z & \end{array} \quad (1.52.1)$$

To construct this, let e denote the negative section and f a fiber of the ruled surface \mathbb{F}_1 . The linear system $|e + 2f|$ is very ample and gives an embedding $\mathbb{F}_1 \hookrightarrow \mathbb{P}^4$. Let $Z \subset \mathbb{P}^5$ denote the cone over \mathbb{F}_1 with vertex $v \in Z$.

Let $\pi : Y^+ \rightarrow Z$ denote the blow-up of the vertex, with exceptional surface $E^+ \cong \mathbb{F}_1$. We can contract E^+ along the \mathbb{P}^1 -bundle direction to get

$$p^+ : (C^+ \subset X^+) \rightarrow (v \in Z).$$

Flopping $e \subset E^+ \subset Y^+$ we get $Y^+ \dashrightarrow Y$. The birational transform of E^+ is $E \cong \mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$. Contracting $E \subset Y$ we get

$$p : (C \subset X) \rightarrow (v \in Z).$$

It is not hard to compute that

$$K_X \cdot C = -\frac{1}{2} \quad \text{and} \quad K_{X^+} \cdot C^+ = 1. \quad (1.52.2)$$

We have constructed the diagram (1.52.1), next we turn the 3-folds in it into families of surfaces. To this end let $|H|$ be a general pencil of degree d hypersurface sections of Z . For the computation at the end we will need $d \geq 4$. By pull-back we get pencils of divisors on all the 3-folds in (1.52.1). Blowing up the base loci we get flat families of surfaces

$$\tilde{X} \rightarrow \mathbb{P}^1, \quad \tilde{X}^+ \rightarrow \mathbb{P}^1 \quad \text{and} \quad \tilde{Y} \rightarrow \mathbb{P}^1. \quad (1.52.3)$$

Let $0 \in \mathbb{P}^1$ denote the point corresponding to the unique surface $H_0 \in |H|$ passing through v . Over $\mathbb{P}^1 \setminus \{0\}$, all 3 families in (1.52.3) are isomorphic to each other and the fibers are hypersurface sections of $Z \setminus \{v\}$. The interesting question is what happens over 0, more precisely, over the vertex $v \in Z$. Let the special fibers be

$$S_0 \subset \tilde{X}, \quad S_0^+ \subset \tilde{X}^+ \quad \text{and} \quad S_0^Y \subset \tilde{Y}. \quad (1.52.4)$$

Note that H_0 is a normal, projective surface with a single singular point $v \in H_0$. The singularity is formally isomorphic to the quotient $\mathbb{A}^2/\frac{1}{3}(1, 1)$.

We see that S_0^+ is smooth, it is the blow up of $v \in H_0$. The exceptional curve of $S_0^+ \rightarrow H_0$ is $C^+ \cong \mathbb{P}^1$. It has self-intersection -3 and $K_{S_0^+} \cdot C^+ = 1$.

The surface S_0 is singular with a unique singularity formally isomorphic to the quotient $\mathbb{A}^2/\frac{1}{4}(1, 1)$. Blowing up the singular point gives S'_0 , the birational transform of S_0 on Y . S'_0 is obtained from S^+ by blowing up a point on C^+ . The exceptional curve of $S_0 \rightarrow H_0$ is $C \cong \mathbb{P}^1$. It has self-intersection $-\frac{3}{4}$ and $K_{S_0} \cdot C = -\frac{1}{2}$.

The pull-back of H_0 on Y is S_0^Y , which is the union of S'_0 and of E . The 2 components intersect transversally along a curve, which is the exceptional curve of the blow-up $S'_0 \rightarrow S_0$ on S'_0 and a smooth conic on $E \cong \mathbb{P}^2$.

By (1.52.2) $K_{S_0^+}$ is positive on the exceptional curve C^+ . If the degree d of $|H|$ is large then $|K_{H_0}|$ is ample and so is $K_{S_0^+}$. (It turns out the $d \geq 4$ is enough.) Thus we conclude that

Claim 1.52.5. $\tilde{X}^+ \rightarrow \mathbb{P}^1$ is a stable family of surfaces. Its central fiber is $S_0^+ \not\cong H_0$. \square

By contrast K_{S_0} is negative on the exceptional curve C , so the contraction $S_0 \rightarrow H_0$ can be the 1st step of the MMP for S_0 . Since $|K_{H_0}|$ is ample we see that

Claim 1.52.6. H_0 is the canonical model of S_0 . \square

The surface S_0^Y is reducible. The restriction of its dualizing sheaf to $E \cong \mathbb{P}^2$ is $\omega_{\mathbb{P}^2}(2)$, which is negative. Thus following (1.49.1–2) we should contract E . This gives $S_0^Y \rightarrow S_0$ and then (1.52.6) gives H_0 after contracting C . Thus

Claim 1.52.7. H_0 is the stable model of S_0^Y . \square

Since $\tilde{X}^+ \rightarrow \mathbb{P}^1$ is the stable model for both $\tilde{X} \rightarrow \mathbb{P}^1$ and $\tilde{Y} \rightarrow \mathbb{P}^1$ in (1.52.3), we see that the formation of the stable model does not commute with taking fibers over the point $0 \in \mathbb{P}^1$.

Following [Koll11c], next we give an example of a projective, normal crossing surface that does not have a canonical model.

EXAMPLE 1.53. We start with a smooth plane curve $C \subset \mathbb{P}^2$ of degree d and a line L intersecting C transversally. Let $c \in C \cap L$ be one of the intersection points. Fix distinct points $p, q \in \mathbb{P}^1$. In $\mathbb{P}_p^2 := \{p\} \times \mathbb{P}^2$ we get C_p, L_p and similarly for $C_q, L_q \subset \mathbb{P}_q^2$. We have the ‘identity’ $\tau : C_p \cong C_q$.

Let $\bar{S} \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a surface of bidegree $(e, d+1)$ such that $\bar{S} \cap \mathbb{P}_p^2 = C_p \cup L_p$ and $\bar{S} \cap \mathbb{P}_q^2 = C_q \cup L_q$. We can further arrange that \bar{S} is smooth, except for an ordinary node at $c_p \in C_p \cap L_p$.

Let $\bar{S}' \rightarrow \bar{S}$ be obtained by blowing up c_p and c_q . We get exceptional curves E'_p, E'_q and birational transforms C'_p and C'_q . Note that \bar{S}' is smooth and $E'_p + E'_q + C'_p + C'_q$ is an snc divisor. We can now glue C'_p to C'_q using the ‘identity’ $\tau' : C'_p \cong C'_q$ to obtain the non-normal surface $S' := \bar{S}'/(\tau')$. It has normal crossing self-intersection along a curve $C \cong C'_p \cong C'_q$. Note that $K_{S'} + E_p + E_q$ is a Cartier divisor.

Claim 1.53.1. The projective, normal crossing pair $(S', E_p + E_q)$ does not have a canonical model.

Proof. The normalization of $(S', E_p + E_q)$ is $(\bar{S}', E'_p + E'_q + C'_p + C'_q)$, thus the only ‘sensible’ thing to do is to construct the canonical model of $(\bar{S}', E'_p + E'_q + C'_p + C'_q)$ and then glue the images of C'_p and C'_q together. We compute that

$$(K_{\bar{S}'} + E'_p + E'_q + C'_p + C'_q) \cdot E'_p = -1 \quad \text{and} \quad (K_{\bar{S}'} + E'_p + E'_q + C'_p + C'_q) \cdot E'_q = -1.$$

Thus we need to contract E'_p and E'_q . Then we get the surface $(\bar{S}, C_p + C_q)$. Note that

$$\omega_{\bar{S}} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(e-2, d+1-3)|_{\bar{S}},$$

which is ample for $e \geq 3, d \geq 3$. We compute below in (1.53.2.a–b) that if $d \geq 4$ then $K_{\bar{S}} + C_p + C_q$ is ample. We assume this from now on. Therefore the only possible choice for the canonical model of $(S', E'_p + E'_q)$ is $\bar{S}/(\tau)$.

What is wrong with $\bar{S}/(\tau)$? So far it did not matter that S is smooth at c_q and has a node at c_p , but now this becomes crucial.

Claim 1.53.2. The canonical class of $\bar{S}/(\tau)$ is not \mathbb{Q} -Cartier. Thus its canonical ring is not finitely generated.

Proof. \bar{S} is smooth along C_q , hence the usual adjunction gives that

$$(K_{\bar{S}} + C_p + C_q)|_{C_q} = K_{C_q}. \quad (1.53.2.a)$$

By contrast, \bar{S} has a node along C_p , and this modifies the adjunction formula to

$$(K_{\bar{S}} + C_p + C_q)|_{C_p} = K_{C_p} + \frac{1}{2}[c_p]; \quad (1.53.2.b)$$

see [Kol13b, 4.3] for this computation. This means that we cannot match up local generating sections of the sheaf $\omega_{\bar{S}}^m(mC'_p + mC'_q)$ at the points c_p and at c_q ; see [Kol13b, 5.12] for the precise statement and proof.

This easily implies that finite generation fails, see [Kol10, Exrc.97].

1.6. Examples of bad moduli problems

Now we turn to a more general overview of moduli problems. The aim of this section is to present examples of moduli problems that seem quite reasonable at first sight but turn out to have rather bad properties.

Moduli of hypersurfaces.

The Chow and Hilbert varieties describe families of hypersurfaces in a fixed projective space \mathbb{P}^n . For many purposes it is more natural to consider the moduli functor of hypersurfaces modulo isomorphisms. We consider what kind of ‘moduli spaces’ one can obtain in various cases.

DEFINITION 1.54 (Hypersurfaces modulo linear isomorphisms).

Over an algebraically closed field k , we consider hypersurfaces $X \subset \mathbb{P}_k^n$ where $X_1, X_2 \subset \mathbb{P}_k^n$ are considered isomorphic if there is an automorphism $\phi \in \text{Aut}(\mathbb{P}_k^n)$ such that $\phi(X_1) = X_2$. (One could also consider hypersurfaces modulo isomorphisms which do not necessarily extend to an isomorphism of the ambient projective space. It is easy to see that smooth hypersurfaces can have such nonlinear isomorphisms only for $(d, n) \in \{(3, 2), (4, 3)\}$. A smooth cubic curve in \mathbb{P}^2 has an infinite automorphism group, but only finitely many extend to an automorphism of \mathbb{P}^2 . Similarly, a smooth quartic surface in \mathbb{P}^3 can have an infinite automorphism group as in (1.67), but only finitely many extend to an automorphism of \mathbb{P}^3 . See also [SS17, Ogu17] for further interesting examples of isomorphisms of smooth quartic surfaces in \mathbb{P}^3 .)

Over an arbitrary base scheme S , we consider pairs $(X \subset P)$ where P/S is a \mathbb{P}^n -bundle for some n and $X \subset P$ is a closed subscheme, flat over S such that every fiber is a hypersurface. There are two natural invariants, the dimension of P and the degree of X . Thus for any given n, d we get a functor

$$\mathcal{HypSur}_{n,d}(S) := \left\{ \begin{array}{l} \text{Flat families } X \subset P \\ \text{such that } \dim_S P = n, \deg X = d, \\ \text{modulo isomorphisms over } S. \end{array} \right\}$$

One can also consider various subfunctors, for instance $\mathcal{HypSur}_{n,d}^{\text{red}}$, $\mathcal{HypSur}_{n,d}^{\text{norm}}$, $\mathcal{HypSur}_{n,d}^{\text{can}}$, $\mathcal{HypSur}_{n,d}^{\text{lc}}$, or $\mathcal{HypSur}_{n,d}^{\text{sm}}$ where we allow only reduced (resp. normal, canonical, log canonical or smooth) hypersurfaces.

Our aim is to investigate what the ‘coarse moduli spaces’ of these functors look like. Our conclusion is that in many cases there cannot be any scheme or algebraic space that is a coarse moduli space: any ‘coarse moduli space’ would have to have very strange topology. Assume for simplicity that we work over an infinite field.

Let $\mathcal{HypSur}_{n,d}^*$ be any subfunctor of $\mathcal{HypSur}_{n,d}$ and assume that it has a coarse moduli space $\text{HypSur}_{n,d}^*$. By definition, the set of k -points of $\text{HypSur}_{n,d}^*$ is $\mathcal{HypSur}_{n,d}^*(\text{Spec } k)$. We can also get some idea about the Zariski topology of $\text{HypSur}_{n,d}^*$ using various families of hypersurfaces.

For instance, we can study the closure \bar{U} of a subset $U \subset \text{HypSur}_{n,d}^*(\text{Spec } k)$ using the following observation:

- Assume that there is a flat family of hypersurfaces $\pi: X \rightarrow S$ and a dense open subset $S^\circ \subset S$ such that $[X_s] \in U$ for every $s \in S^\circ(k)$. Then $[X_s] \in \bar{U}$ for every $s \in S(k)$.

Next we write down flat families of hypersurfaces $\pi: X \rightarrow \mathbb{A}^1$ in $\mathcal{HypSur}_{n,d}^*$ such that for $t \neq 0$ the fibers X_t are isomorphic to each other but X_0 is not isomorphic to them. Such a family corresponds to a morphism $\tau: \mathbb{A}^1 \rightarrow \text{HypSur}_{n,d}^*$ such that $\tau(\mathbb{A}^1 \setminus \{0\}) = [X_1]$ but $\tau(\{0\}) = [X_0]$. This implies that the point $[X_1]$ is not closed and its closure contains $[X_0]$.

This is not very surprising in a scheme, but note that X_1 itself is defined over our base field k , so $[X_1]$ is supposed to be a k -point. On a k -scheme, k -points are closed. Thus we conclude that if there is any family as above, the moduli space $\text{HypSur}_{n,d}^*$ cannot be a k -scheme or a quasi-separated algebraic space [Sta15, Tag 08AL].

The simplest way to get such families is by the following construction.

EXAMPLE 1.55 (Deformation to cones). Let $f(x_0, \dots, x_n)$ be a homogeneous polynomial of degree d and $X := (f = 0)$ the corresponding hypersurface. For some $0 \leq i < n$ consider the family of hypersurfaces

$$\mathbf{X} := (f(x_0, \dots, x_i, tx_{i+1}, \dots, tx_n) = 0) \subset \mathbb{P}^n \times \mathbb{A}_t^1 \quad (1.55.1)$$

with projection $\pi: \mathbf{X} \rightarrow \mathbb{A}_t^1$. If $t \neq 0$ then the substitution

$$x_j \mapsto x_j \quad \text{for } j \leq i, \quad \text{and} \quad x_j \mapsto t^{-1}x_j \quad \text{for } j > i$$

shows that the fiber X_t is isomorphic to X . If $t = 0$ then we get the cone over $X \cap (x_{i+1} = \dots = x_n = 0)$:

$$X_0 = (f(x_0, \dots, x_i, 0, \dots, 0) = 0) \subset \mathbb{P}^n.$$

This is a hypersurface iff $f(x_0, \dots, x_i, 0, \dots, 0)$ is not identically 0.

Already these simple deformations show that various moduli spaces of hypersurfaces have very few closed points.

COROLLARY 1.56. *The sole closed point of $\text{HypSur}_{d,n}$ is $[(x_0^d = 0)]$.*

Proof. Take any $X = (f = 0) \subset \mathbb{P}^n$. After a general change of coordinates, we can assume that x_0^d appears in f with nonzero coefficient. For $i = 0$ consider the family (1.55.1).

Then $X_0 = (x_0^d = 0)$, hence $[X]$ cannot be closed point unless $X \cong X_0$. It is quite easy to see that if $X \rightarrow S$ is a flat family of hypersurfaces whose generic fiber is a d -fold plane, then every fiber is a d -fold plane. This shows that $[(x_0^d = 0)]$ is a closed point. \square

COROLLARY 1.57. *The only closed points of $\text{HypSur}_{d,n}^{\text{red}}$ are $[(f(x_0, x_1) = 0)]$ where f has no multiple roots.*

Proof. If X is a reduced hypersurface of degree d , there is a line that intersects it in d distinct points. We can assume that this is the line $(x_2 = \dots = x_n = 0)$. For $i = 1$ consider the family (1.55.1).

Then $X_0 = (f(x_0, x_1, 0, \dots, 0) = 0)$ where $f(x_0, x_1)$ has d distinct roots. Since X_0 is reduced, we see that none of the other hypersurfaces correspond to closed points.

It is not obvious that the points corresponding to $(f(x_0, x_1, 0, \dots, 0) = 0)$ are closed, but this can be easily established by studying the moduli of d points in \mathbb{P}^1 ; see [Mum65, Chap.3] or [Dol03, Sec.10.2]. \square

A similar argument establishes the normal case:

COROLLARY 1.58. *The only closed points of $\text{HypSur}_{d,n}^{\text{norm}}$ are $[(f(x_0, x_1, x_2) = 0)]$ where $(f(x_0, x_1, x_2) = 0) \subset \mathbb{P}^2$ is a nonsingular curve.* \square

In the above examples the trouble comes from cones. Cones can be normal, but they are very singular by other measures; they have a singular point whose multiplicity equals the degree of the variety. So one could hope that high multiplicity points cause the problems. This is true to some extent as the next theorems and examples show. For proofs see [Mum65, Sec.4.2] and [Dol03, Sec.10.1].

THEOREM 1.59. *Each of the following functors has a coarse moduli space which is a quasi-projective variety.*

- (1.59.1) *The functor of smooth hypersurfaces $\mathcal{HypSur}_{d,n}^{\text{sm}}$.*
(1.59.2) *For $d \geq n + 1$, the functor $\mathcal{HypSur}_{d,n}^{\text{can}}$ of hypersurfaces with canonical singularities.*
(1.59.3) *For $d > n + 1$, the functor $\mathcal{HypSur}_{d,n}^{\text{lc}}$ of hypersurfaces with log canonical singularities.*
(1.59.4) *For $d > n + 1$, the functor $\mathcal{HypSur}_{d,n}^{\text{low-mult}}$ of those hypersurfaces that have only points of multiplicity $< \frac{d}{n+1}$.*

EXAMPLE 1.60. Consider the family of even degree d hypersurfaces

$$((x_0^{d/2} + t^d x_1^{d/2})x_1^{d/2} + x_2^d + \cdots + x_n^d = 0) \subset \mathbb{P}^n \times \mathbb{A}_t^1.$$

For $t \neq 0$ the substitution

$$(x_0 : x_1 : x_2 : \cdots : x_n) \mapsto (tx_0 : t^{-1}x_1 : x_2 : \cdots : x_n).$$

transforms the equation of X_t to

$$X := ((x_0^{d/2} + x_1^{d/2})x_1^{d/2} + x_2^d + \cdots + x_n^d = 0) \subset \mathbb{P}^n.$$

X has a single singular point which is at $(1:0:\cdots:0)$ and has multiplicity $d/2$.

For $t = 0$ we obtain the hypersurface

$$X_0 := (x_0^{d/2} x_1^{d/2} + x_2^d + \cdots + x_n^d = 0).$$

X_0 has 2 singular points of multiplicity $d/2$, hence it is not isomorphic to X .

Thus we conclude that $[X]$ is not a closed point of the ‘moduli space’ of those hypersurfaces of degree d that have only points of multiplicity $\leq d/2$.

This is especially interesting when $d \leq n$ since in this case X_0 has canonical singularities (1.33).

Thus we see that for $d \leq n$, the functor $\mathcal{HypSur}_{d,n}^{\text{can}}$ parametrizing hypersurfaces with canonical singularities does not have a coarse moduli space. By contrast, for $d > n$ the coarse moduli scheme $\mathcal{HypSur}_{d,n}^{\text{can}}$ exists and is quasi-projective by (1.59).

Other non-separated examples.

The nonseparated examples produced so far all involved ruled or at least uniruled varieties. Next we consider some examples of nonseparatedness where the varieties are not uniruled. The bad behavior is due to the singularities and not to the global structure.

EXAMPLE 1.61 (Double covers of \mathbb{P}^1). Let $f(x, y)$ and $g(x, y)$ be two cubic forms without multiple roots, neither divisible by x or y . Consider 2 families of curves

$$\begin{aligned} S_1 &:= (f(x_1, y_1)g(t^2x_1, y_1) = z_1^2) \subset \mathbb{P}(1, 1, 3) \times \mathbb{A}^1 \quad \text{and} \\ S_2 &:= (f(x_2, t^2y_2)g(x_2, y_2) = z_2^2) \subset \mathbb{P}(1, 1, 3) \times \mathbb{A}^1. \end{aligned}$$

Note that $\omega_{S_i/\mathbb{A}^1}$ is relatively ample and the general fiber of $\pi_1: S_i \rightarrow \mathbb{A}^1$ is a smooth curve of genus 2.

The central fibers are $(f(x_1, y_1)g(0, y_1) = z_1^2)$ resp. $(f(x_2, 0)g(x_2, y_2) = z_2^2)$. By assumption $g(0, y_1) = a_1 y_1^3$ and $f(x_2, 0) = a_2 x_2^3$ where the $a_i \neq 0$. Setting $z_1 = a_1^{1/2} w_1 y_1$ and $z_2 = a_2^{1/2} w_2 x_2$ gives the normalizations. Hence the central fibers are elliptic curves with a cusp. Their normalization is isomorphic to $(f(x_1, y_1)y_1 = w_1^2)$ resp. $(x_2 g(x_2, y_2) = w_2^2)$, and these are, in general, not isomorphic to each other.

This also shows that along the central fibers, the only singularities are at $(1:0:0; 0)$ and at $(0:1:0; 0)$. Up to canceling units, the local equations are $g(t^2, y_1) = z_1^2$ resp. $f(x_2, t^2) = z_2^2$. (These are simple elliptic with minimal resolution a single smooth elliptic curve of self intersection -1 .) Hence the S_i are normal surfaces, each having 1 simple elliptic singular point.

Finally, the substitution

$$(x_1 : y_1 : z_1; t) = (x_2 : t^2 y_2 : t^3 z_2; t)$$

transforms $f(x_1, y_1)g(t^2 x_1, y_1) - z_1^2$ into

$$f(x_2, t^2 y_2)g(t^2 x_2, t^2 y_2) - t^6 z_2^2 = t^6 (f(x_2, t^2 y_2)g(x_2, y_2) - z_2^2),$$

thus the two families are isomorphic over $\mathbb{A}^1 \setminus \{0\}$

EXAMPLE 1.62 (Limits of double covers of \mathbb{P}^3). Let $a_i(x, y)$ and $b_i(u, v)$ be homogeneous forms of degree n . Consider 2 families of threefolds

$$\begin{aligned} X_1 &:= (a_1(x, y) + t^{2n} b_1(u, v))(a_2(x, y) + b_2(u, v)) = w^2 \subset \mathbb{P}(1^4, n) \times \mathbb{A}^1, \quad \text{and} \\ X_2 &:= (a_1(x, y) + b_1(u, v))(t^{2n} a_2(x, y) + b_2(u, v)) = w^2 \subset \mathbb{P}(1^4, n) \times \mathbb{A}^1. \end{aligned}$$

Claim.

- (1.62.1) For general a_i, b_i , the central fibers of the $X_i \rightarrow \mathbb{A}^1$ are normal. Their singularities are canonical iff $n \leq 3$, and log-canonical iff $n \leq 4$.
- (1.62.2) The resolutions of the central fibers are of general type if $n \geq 7$, have Kodaira dimension 1 if $n = 5, 6$ and are rationally connected if $n \leq 4$.
- (1.62.3) The general fibers of $X_i \rightarrow \mathbb{A}^1$ have only cA_1 -singularities and their canonical class is trivial if $n = 4$ and ample if $n \geq 5$.
- (1.62.4) The two families are isomorphic over $\mathbb{A}^1 \setminus \{0\}$ but not isomorphic over \mathbb{A}^1 .

Proof. For general a_i, b_i , the surface $S := (a_2(x, y) + b_2(u, v) = 0) \subset \mathbb{P}^3$ is smooth and $T := (a_1(x, y) = 0)$ has only transverse intersection with it away from the line $L := (x = y = 0)$. The central fiber of $X_1 \rightarrow \mathbb{A}^1$ is the double cover $\pi: X_{10} \rightarrow \mathbb{P}^3$ ramified along $S \cup T$. At a general point of L the function $b_2(u, v)$ is nonzero and the local equation of the double cover can be made into $p^2 = a_1(x, y)$. At special points b_2 can have simple zeros. Here the equation is $p^2 = s \cdot a_1(x, y)$.

Let $g: P' := B_L \mathbb{P}^3 \rightarrow \mathbb{P}^3$ denote the blow up with exceptional divisor E . Let $S' \subset P'$ denote the birational transform of S and $T' \subset P'$ the birational transform of T . Note that T' is the union of n disjoint planes from the linear system $M = |g^* \mathcal{O}_{\mathbb{P}^3}(1)(-E)|$ and $S' + T' + E$ is an snc divisor if the a_i, b_i are general. The fiber product $P' \times_{\mathbb{P}^3} X_{10}$ can be realized as a double cover $X_{10}^* \rightarrow P'$ ramified along $S' + T' + nE$. This is not normal along E . Its normalization $\pi': X' \rightarrow X_{10}^* \rightarrow P'$ is again a double cover that ramifies along $S' + T' + E$ if n is odd and along $S' + T'$ if n is even. Since $S' + T' + E$ is an snc divisor, X_{10}^* has only canonical singularities (1.33). Let $g_X: X'_{10} \rightarrow X_{10}$ denote the induced morphism.

The canonical classes of X_{10} and of X'_{10} are computed by the Hurwitz formulas

$$K_{X_{10}} \sim \pi^* \mathcal{O}_{\mathbb{P}^3}(n-4) \quad \text{and} \quad K_{X'_{10}} \sim \pi'^*(g^* \mathcal{O}_{\mathbb{P}^3}(n-4)(-\lfloor \frac{n-2}{2} \rfloor E)).$$

Thus we obtain that

$$K_{X'_{10}} \sim g_X^* K_{X_{10}} - \lfloor \frac{n-2}{2} \rfloor \pi'^* E.$$

This shows that X_{10} has canonical singularities if $n \leq 3$ and log canonical singularities if $n = 4$, proving (2). (Note that for $n = 5$ the formula gives $K_{X'_{10}} \sim g_X^* K_{X_{10}} - \pi'^* E$, but π' ramifies along E so $\pi'^* E$ is a divisor with multiplicity 2.)

Furthermore, if $n \geq 7$ then $n-5 \geq \lfloor \frac{n-2}{2} \rfloor$, thus

$$g^* \mathcal{O}_{\mathbb{P}^3}(n-4)(-\lfloor \frac{n-2}{2} \rfloor E) \supset g^* \mathcal{O}_{\mathbb{P}^3}(n-4)(-(n-5)E) = g^* \mathcal{O}_{\mathbb{P}^3}(1)((n-5)M),$$

which shows that X'_{10} is of general type.

If $n = 5, 6$ then X'_{10} has Kodaira dimension 1 and $\pi'^* M$ is a pencil of K3 surfaces. For a general plane M in this pencil, we get a double cover ramified along the quintic curve $M \cap S$ plus the line L when $n = 5$. The ramification is along the sextic curve $M \cap S$ when $n = 6$.

The computations for the central fiber of $X_2 \rightarrow \mathbb{A}^1$ are the same.

The general fibers of $X_i \rightarrow \mathbb{A}^1$ are double covers of \mathbb{P}^3 ramified along two smooth surfaces which intersect transversally. This gives the singularities ($p^2 = qr$). The Hurwitz formula computes the canonical class.

Finally, the substitution

$$(x : y : u : v : w : t) \mapsto (t^2 x : t^2 y : u : v : t^n w : t)$$

transforms $(a_1(x, y) + t^{2n} b_1(u, v))(a_2(x, y) + b_2(u, v)) - w^2$ into

$$\begin{aligned} & (a_1(t^2 x, t^2 y) + t^{2n} b_1(u, v))(a_2(t^2 x, t^2 y) + b_2(u, v)) - t^{2n} w^2 \\ & = t^{2n} \left((a_1(x, y) + b_1(u, v))(t^{2n} a_2(x, y) + b_2(u, v)) - w^2 \right). \end{aligned}$$

Let us end our study of hypersurfaces with a different type of example. This shows that the moduli problem for hypersurfaces usually includes smooth limits that are not hypersurfaces. These pose no problem for the general theory, but they show that it is not always easy to see what schemes one needs to include in a moduli space.

EXAMPLE 1.63 (Smooth limits of hypersurfaces). [[Mor75](#)]

Fix integers $a, b > 1$ and $n \geq 2$. We construct a family of smooth n -folds X_t such that X_t is a smooth hypersurface of degree ab for $t \neq 0$ and X_0 is not isomorphic to a smooth hypersurface.

It is not known if similar examples exist for $n \geq 3$ and $\deg X$ a prime number; see [[OS20](#)] for the cases $\deg X \leq 7$.

Fix $\mathbb{P}(1^{n+1}, a)$ with coordinates x_0, \dots, x_n, z . Let f_a, g_{ab} be general homogeneous forms of degree a (resp. ab) in x_0, \dots, x_n . Consider the family of complete intersections

$$X_t := (tz - f_a(x_0, \dots, x_n) = z^b - g_{ab}(x_0, \dots, x_n) = 0) \subset \mathbb{P}(1^{n+1}, a).$$

For $t \neq 0$ we can eliminate z to obtain a degree ab smooth hypersurface

$$X_t \cong (f_a^b(x_0, \dots, x_n) = t^b g_{ab}(x_0, \dots, x_n)) \subset \mathbb{P}^{n+1}.$$

For $t = 0$ we see that $\mathcal{O}_{X_0}(1)$ is not very ample but realizes X_0 as a b -fold cyclic cover ([11.14](#))

$$X_0 \rightarrow (f_a(x_0, \dots, x_n) = 0) \subset \mathbb{P}^{n+1}$$

of a degree a smooth hypersurface. In particular, X_0 is not isomorphic to a smooth hypersurface.

More unexpected examples.

We start with an example showing that seemingly equivalent moduli problems may lead to different moduli spaces.

EXAMPLE 1.64. We start with the moduli space P_{n+1} of $n + 1$ points in \mathbb{C} up to translations. We can view such a point set as the zeros of a unique polynomial of degree $n + 1$ whose leading term is x^{n+1} . We can use a translation to kill the coefficient of x^n and the universal polynomial is then given by

$$x^{n+1} + a_2x^{n-1} + \cdots + a_{n+1}.$$

Thus $P_{n+1} \cong \mathbb{C}^n$ with coordinates a_2, \dots, a_{n+1} .

Let us now look at those point sets where n of the points coincide. There are 2 ways to formulate this as a moduli problem:

(1.64.1) unordered point sets $p_0, \dots, p_n \in \mathbb{C}$ where at least n of the points coincide, up to translations, or

(1.64.2) unordered point sets $p_0, \dots, p_n \in \mathbb{C}$ plus a point $q \in \mathbb{C}$ such that $p_i = q$ at least n -times, up to translations.

If $n \geq 2$ then q is uniquely determined by the points p_0, \dots, p_n , so it would seem that the two formulations are equivalent. We claim, however, that the two versions have non-isomorphic moduli spaces.

If the n -fold point is at t then the corresponding polynomial is $(x - t)^n(x + nt)$. By expanding it we get that

$$a_i = t^i \left[(-1)^i \binom{n}{i} + (-1)^{i-1} n \binom{n}{i-1} \right] \quad \text{for } i=2, \dots, n+1.$$

This shows that the space $R_{n+1} \subset P_{n+1}$ of polynomials with an n -fold root is a cuspidal rational curve given as the image of the map

$$t \mapsto \left(a_i = t^i \left[(-1)^i \binom{n}{i} + (-1)^{i-1} n \binom{n}{i-1} \right] : i = 2, \dots, n+1 \right).$$

So the moduli space R_{n+1} of the first variant (1) is a cuspidal rational curve.

By contrast, the space \bar{R}_{n+1} of the second variant (2) is a smooth rational curve, the isomorphism is given by

$$(p_0, \dots, p_n; q) \mapsto \sum_i (p_i - q) \in \mathbb{C}.$$

Not surprisingly, the map that forgets the n -fold root gives $\pi: \bar{R}_{n+1} \rightarrow R_{n+1}$ which is the normalization map.

Next we have 2 examples of moduli functors that are not representable (1.18.1).

They suggest that varieties whose canonical class is not ample present special challenges.

EXAMPLE 1.65. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree 4 with infinite automorphism group (1.67).

Let $\mathbf{S} \rightarrow W$ be the universal family of smooth degree 4 surfaces in \mathbb{P}^3 . The isomorphism classes of the pairs $(S, \mathcal{O}_S(1))$ correspond to the $\text{Aut}(\mathbb{P}^3)$ -orbits in W . We see below that the fibers isomorphic to S form countably many $\text{Aut}(\mathbb{P}^3)$ -orbits.

For any $g \in \text{Aut } S$, $g^* \mathcal{O}_S(1)$ gives another embedding of S into \mathbb{P}^3 . Two such embeddings are projectively equivalent iff $g^* \mathcal{O}_S(1) \cong \mathcal{O}_S(1)$, that is, when

$g \in \text{Aut}(S, \mathcal{O}_S(1))$. The latter can be viewed as the group of automorphisms of \mathbb{P}^3 that map S to itself. Thus $\text{Aut}(S, \mathcal{O}_S(1))$ is a closed subscheme of the algebraic variety $\text{Aut}(\mathbb{P}^3) \cong \text{PGL}_4$. Since $\text{Aut} S$ is discrete, this implies that $\text{Aut}(S, \mathcal{O}_S(1))$ is finite. Hence the fibers of $\mathbf{S} \rightarrow W$ that are isomorphic to S lie over countably many $\text{Aut}(\mathbb{P}^3)$ -orbits, corresponding to $\text{Aut} S / \text{Aut}(S, \mathcal{O}_S(1))$.

EXAMPLE 1.66. We construct a smooth, proper family of surfaces $X \rightarrow C$ over a smooth curve such that

- (1.66.1) every fiber has nef canonical class,
- (1.66.2) the generic fiber has ample canonical class,
- (1.66.3) $X \rightarrow C$ is locally projective but
- (1.66.4) $X \rightarrow C$ is not projective.

Start with hypersurfaces of degree $d \geq 5$ in \mathbb{P}^4 that contain a fixed 2-plane L . These hypersurfaces form a very ample linear system on the blow-up $B_L \mathbb{P}^4$, hence, by the Lefschetz theorem, the class group of a general $Y \subset \mathbb{P}^4$ is generated by L and the hyperplane class H .

It is easy to see that a general Y has $(d-1)^2$ ordinary double points as its singularities and a general hyperplane containing L intersects Y in $L + S$ where S is also smooth.

The singularities of Y can be resolved either by blowing up L or by blowing up S (10.45). Either of these results in a projective variety, but next we mix these up.

Partition the set of ordinary double points into two nonempty subsets D_1, D_2 . Let $Y_1 := B_L(Y \setminus D_2)$ and $Y_2 := B_S(Y \setminus D_1)$. Both of these contain $Y \setminus (D_1 + D_2)$ as an open subset. By gluing them together, we get a proper variety Y^* . We claim that Y^* is not projective.

Indeed, let $E_i \subset Y^*$ be an exceptional curve mapping to a node in D_i . Let $L^* \subset Y^*$ (resp. $H^* \subset Y^*$) denote the birational transforms of L (resp. H). Then, as in (10.45), $L^* \cdot E_1 = +1$, $L^* \cdot E_2 = -1$ and $H^* \cdot E_i = 0$. Thus no linear combination $aL^* + bH^*$ has positive degree on both E_1 and E_2 . Since $\text{Pic} Y^*$ is generated by L^* and H^* , this implies that there is no ample divisor on Y^* . Moreover, this also shows that if $X^* \rightarrow Y^*$ is a proper birational morphism that is an isomorphism near $E_1 + E_2$ and $X \subset X^*$ is an open set that contains $E_1 + E_2$, then X is not quasi-projective.

It is now easy to construct a family of surfaces as required. Let $H_1, H_2 \subset \mathbb{P}^4$ be general hyperplanes and $Y' := B_{H_1 \cap H_2} Y$ the blow up. The pencil $|H_1, H_2|$ defines a morphism $f': Y' \rightarrow \mathbb{P}^1$. Since the H_i are general, we may assume that there are finite sets $B_0, B_1, B_2 \subset \mathbb{P}^1$ such that the following holds

- (1.66.5) for $b \notin \cup B_i$, the fiber Y'_b is smooth,
- (1.66.6) for $b \in B_1$ (resp. $b \in B_2$), the fiber Y'_b has a single node which is at one of the points of D_1 (resp. D_2).

Set $X^* := B_{H_1 \cap H_2} Y^*$ and $f^*: X^* \rightarrow Y^* \rightarrow \mathbb{P}^1$. Finally let $C := \mathbb{P}^1 \setminus B_0$ and $X := (f^*)^{-1}(C) \subset X^*$ with $f := f^*|_X$.

By the computations of (10.45), $f: X \rightarrow C$ is smooth. By construction, f is projective over $C \setminus B_i$ for $i = 1, 2$ but X itself is not quasi-projective.

The following examples are useful in various constructions.

EXAMPLE 1.67 (Surfaces with infinite discrete automorphism group). Let us start with a smooth genus 1 curve E defined over a field K . Any point $q \in E(K)$

defines an involution τ_q where $\tau_q(p)$ is the unique point such that $p + \tau_q(p) \sim 2q$. (Equivalently, we can set q as the origin, then $\tau_q(p) = -p$.) The first formulation shows that if L/K is a quadratic extension, then any $Q \in E(L)$ also defines an involution τ_Q where $\tau_Q(p)$ is the unique point such that $p + \tau_Q(p) \sim Q$.

Given points $q_1, q_2 \in E(K)$, we see that $p \mapsto \tau_{q_2} \circ \tau_{q_1}(p)$ is translation by $2q_1 - 2q_2$. Similarly, given $Q_i \in E(L_i)$, $p \mapsto \tau_{Q_2} \circ \tau_{Q_1}(p)$ is translation by $Q_1 - Q_2$. Usually these translations have infinite order.

Let now $g: S \rightarrow C$ be a smooth, minimal, elliptic surface with generic fiber E over $k(C)$. By the above, any section or double section of g gives an involution of S and two involutions usually generate an infinite group of automorphisms of S .

As a concrete example, let $S \subset \mathbb{P}^3$ be a smooth quartic that contains 3 lines L_i . The pencil of planes through L_1 gives an elliptic fibration and L_2, L_3 are sections. Thus these K3 surfaces usually have an infinite automorphism group.

As another example, let $S \subset \mathbb{P}^3$ be a quartic with a double point $p \in S$. Projecting S from p exhibits the blow-up $B_p S$ as a double cover of \mathbb{P}^2 , hence we get a Galois involution τ_p . If S has 2 nodes, the two involutions usually generate an infinite group of automorphisms of the minimal resolution of S .

1.7. Compactifications of M_g

Here we consider what happens if we try define other compactifications of M_g . First we give a complete study of a compactified moduli functor of genus 2 curves that uses only irreducible curves.

Moduli of genus 2 curves.

DEFINITION 1.68. Let $\mathcal{M}_2^{\text{irr}}$ be the moduli functor of flat families of irreducible curves of arithmetic genus 2 which are either

(1.68.1) smooth,

(1.68.2) nodal,

(1.68.3) rational with 2 cusps or

(1.68.4) rational with a triple point whose complete local ring is isomorphic to $\mathbb{C}[[x, y, z]]/(xy, yz, zx)$.

The aim of this subsection is to prove the following. (See [Mum65, Chap.3] or [Dol03, Sec.10.2] for the relevant background on GIT quotients.)

PROPOSITION 1.69. *Let $\mathcal{M}_2^{\text{irr}}$ be the moduli functor defined above. Then*

(1.69.1) *the coarse moduli space M_2^{irr} exists and equals the geometric invariant theory quotient (8.41) of the symmetric power $\text{Sym}^6 \mathbb{P}^1 // \text{Aut}(\mathbb{P}^1)$, but*

(1.69.2) *$\mathcal{M}_2^{\text{irr}}$ is a very bad moduli functor.*

Proof. A smooth curve of genus 2 can be uniquely written as a double cover $\tau: C \rightarrow \mathbb{P}^1$, ramified at 6 distinct points $p_1, \dots, p_6 \in \mathbb{P}^1$, up to automorphisms of \mathbb{P}^1 . Thus, M_2 is isomorphic to the space of 6 distinct points in \mathbb{P}^1 , modulo the action of $\text{Aut}(\mathbb{P}^1)$. If some of the 6 points coincide, we get singular curves as double covers.

It is easy to see the following (cf. [Mum65, Chap.3], [Dol03, Sec.10.2]).

(1.69.3) A point set is semi-stable iff it does not contain any point with multiplicity ≥ 4 . Equivalently, if the corresponding genus 2 cover has only nodes and cusps.

- (1.69.4) The properly semistable point sets are of the form $3p_1 + p_2 + p_3 + p_4$ where the p_2, p_3, p_4 are different from p_1 but may coincide with each other. Equivalently, the corresponding genus 2 cover has at least one cusp.
- (1.69.5) Point sets of the form $2p_1 + 2p_2 + 2p_3$ where the p_1, p_2, p_3 are different from each other give the only semistable case when the double cover is reducible. It has two smooth rational components meeting each other at 3 points.

In the properly semistable case, generically the double cover is an elliptic curve with a cusp over p_1 . As a special case we can have $3p_1 + 3p_2$, giving as double cover a rational curve with 2 cusps. Note that the curves of this type have a 1-dimensional moduli (the cross ratio of the points p_1, p_2, p_3, p_4 or the j -invariant of the elliptic curve), but they all correspond to the same point in $\text{Sym}^6 \mathbb{P}^1 // \text{Aut}(\mathbb{P}^1)$. (See (1.61) for an explicit construction.) Our definition (1.68) aims to remedy this non-uniqueness by always taking the most degenerate case; a rational curve with 2 cusps (1.68.3).

In case (5), write the reducible double cover as $C = C_1 + C_2$. The only obvious candidate to get an irreducible curve is to contract one of the two components C_i . We get an irreducible rational curve; denote it by C'_j where $j = 3 - i$. Note that C'_j has one singular point which is analytically isomorphic to the 3 coordinate axes in \mathbb{A}^3 . The resulting singular rational curves C'_j are isomorphic to each other. These are listed in (1.68.4).

Let $p: X \rightarrow S$ be any flat family of irreducible, reduced curves of arithmetic genus 2. The trace map (cf. [BPV84, III.12.2]) shows that $R^1 p_* \omega_{X/S} \cong \mathcal{O}_S$. Thus, by cohomology and base change (cf. [Har77, III.12.11]), $p_* \omega_{X/S}$ is locally free of rank 2. Set $P := \mathbb{P}_S(p_* \omega_{X/S})$. Then P is a \mathbb{P}^1 -bundle over S and we have a rational map $\pi: X \rightarrow P$. If X_s has only nodes and cusps, then ω_{X_s} is locally free and generated by global sections, thus π is a morphism along X_s .

If X_s is as in (1.68.4), then ω_{X_s} is not locally free and π is not defined at the singular point. $\pi|_{X_s}$ is birational and the 3 local branches of X_s at the singular point correspond to 3 points on $\mathbb{P}(H^0(X_s, \omega_{X_s}))$.

The branch divisor of π is a degree 6 multisection of $P \rightarrow S$, all of whose fibers are stable point sets. Thus we have a natural transformation

$$\mathcal{M}_2^{\text{irr}}(*) \rightarrow \text{Mor}(*, \text{Sym}^6 \mathbb{P}^1 // \text{Aut}(\mathbb{P}^1)).$$

We have already seen that we get a bijection

$$\mathcal{M}_2^{\text{irr}}(\mathbb{C}) \cong (\text{Sym}^6 \mathbb{P}^1 // \text{Aut}(\mathbb{P}^1))(\mathbb{C}).$$

Since $\text{Sym}^6 \mathbb{P}^1 // \text{Aut}(\mathbb{P}^1)$ is normal, we conclude that it is the coarse moduli space. This completes the proof of (1.69.1).

The assertion (1.69.2) is more a personal opinion. There are 3 main things ‘wrong’ with the functor $\mathcal{M}_2^{\text{irr}}(*)$. Let us consider them one at a time.

1.69.6 (Stable reduction questions).

At the set-theoretic level, we have our moduli space $M_2^{\text{irr}} = \text{Sym}^6 \mathbb{P}^1 // \text{Aut}(\mathbb{P}^1)$, but what about at the level of families?

The first indications are good. Let $\pi_B: S_B \rightarrow B$ be a stable family of genus 2 curves. Assume that no fiber is of type (1.69.5). Let $b_i \in B$ be the points corresponding to fibers with 2 components of arithmetic genus 1. Let $p: A \rightarrow B$ be a double cover ramified at the points b_i . Consider the pull-back family $\pi_A: S_A \rightarrow A$. Set $a_i = p^{-1}(b_i)$ and let $s_i \in \pi_A^{-1}(a_i)$ be the point where the 2 components meet.

Since we took a ramified double cover, each $s_i \in S_A$ is a double point. Thus if we blow up every s_i , the exceptional curve appears in the fiber with multiplicity 1. We can now contract the birational transforms of the elliptic curves to get a family where all these reducible fibers are replaced by a rational curve with 2 cusps. We have proved the following analog of (1.15.1):

Lemma 1.69.6.1. Let $\pi: S \rightarrow B$ be a stable family of genus 2 curves such that no fiber has 2 smooth rational components. Then, after a suitable double cover $A \rightarrow B$, the pull-back $S \times_B A$ is birational to another family where each reducible fiber is replaced by a rational curve with 2 cusps. \square

This solved our problem for 1-parameter families, but, as it turns out, we have problems over higher dimensional bases. In particular, there is no universal family over any base scheme Y that finitely dominates $\text{Sym}^6 \mathbb{P}^1 // \text{Aut}(\mathbb{P}^1)$, not even locally in any neighborhood of the properly semistable point. Indeed, this would give a proper, flat family of curves of arithmetic genus 2 over a 3-dimensional base $\pi: X \rightarrow Y$ where only finitely many of the fibers (the ones over the unique properly semistable point) have cusps. However, the next result shows that there is no such family.

Proposition 1.69.6.2. Let $\pi: X \rightarrow Y$ be a proper, flat family of curves of arithmetic genus 2. Assume that X_0 is a rational curve with a cusp for some $0 \in Y$ and that $\dim_0 Y \geq 3$. Then there is a curve $0 \in C \subset Y$ such that X_y has a cusp for every $y \in C$.

Proof. This follows from the deformation theory of the cusp which says that every flat deformation of a cusp is induced by pull-back from the 2-parameter family

$$\begin{array}{ccc} (y^2 = x^3 + ux + v) & \subset & \mathbb{A}_{xy}^2 \times \mathbb{A}_{uv}^2 \\ p \downarrow & & \downarrow \\ \mathbb{A}_{uv}^2 & = & \mathbb{A}_{uv}^2. \end{array}$$

(See Section 10.5 or [Art76, AGZV85, Har10] for introductions.)

Thus our family π gives an analytic morphism $\tau: Y \rightarrow \mathbb{A}_{uv}^2$ (defined in some neighborhood of $0 \in Y$) and $C = \tau^{-1}(0, 0) \subset Y$ is the required curve along which the fiber has a cusp. \square

1.69.7 (Failure of representability).

Following (1.69.6.2), consider the universal deformation of the rational curve with 2 cusps. This is given as

$$\begin{array}{ccc} (z^2 = (x^3 + uxy^2 + vy^3)(y^3 + syx^2 + tx^3)) & \subset & \mathbb{P}^2(1, 1, 3) \times \mathbb{A}_{uvst}^4 \\ p \downarrow & & \downarrow \\ \mathbb{A}_{uvst}^4 & = & \mathbb{A}_{uvst}^4. \end{array}$$

Let us work in a neighborhood of $(0, 0, 0, 0) \in \mathbb{A}^4$ where the 2 factors $x^3 + uxy^2 + vy^3$ and $y^3 + syx^2 + tx^3$ have no common roots. There are 3 types of fibers of p .

- i) $p^{-1}(0, 0, 0, 0)$ is a rational curve with 2 cusps.
- ii) $p^{-1}(a, b, 0, 0)$ and $p^{-1}(0, 0, a, b)$ are irreducible with exactly 1 cusp if $(a, b) \neq (0, 0)$.
- iii) $p^{-1}(a, b, c, d)$ is irreducible with at worst nodes otherwise.

Thus the curves that we allow in our moduli functor $\mathcal{M}_2^{\text{irr}}$ do not form a representable family. Even worse, the subfamily

$$\begin{array}{ccc} (z^2 = (x^3 + uxy^2 + vy^3)y^3) & \subset & \mathbb{P}^2(1, 1, 3) \times \text{Spec } k[[u, v]] \\ p \downarrow & & \downarrow \\ \text{Spec } k[[u, v]] & = & \text{Spec } k[[u, v]] \end{array}$$

is not allowed in our moduli functor $\mathcal{M}_2^{\text{irr}}$, but the family

$$\begin{array}{ccc} (z^2 = (x^3 + uxy^2 + vy^3)(y^3 + u^nyx^2 + v^nx^3)) & \subset & \mathbb{P}^2(1, 1, 3) \times \text{Spec } k[[u, v]] \\ p \downarrow & & \downarrow \\ \text{Spec } k[[u, v]] & = & \text{Spec } k[[u, v]] \end{array}$$

is allowed. Over $\text{Spec } k[u, v]/(u^n, v^n)$ the two families are isomorphic. Since deformation theory is essentially a study of families over Artin rings, this means that the usual methods cannot be applied to understand the functor $\mathcal{M}_2^{\text{irr}}$.

1.69.8 (Unusual non-separatedness).

A quite different type of problem arises at the curve corresponding to $2p_1 + 2p_2 + 2p_3$.

Write the double cover as $C = C_1 + C_2$. As before, if we contract one of the two components C_i , we get an irreducible rational curve C'_j where $j = 3 - i$ as in (1.68.4).

Since the curves C'_1 and C'_2 are isomorphic, from the set-theoretic point of view this is a good solution. However, as in (1.26), something strange happens with families. Let $p: S \rightarrow \mathbb{A}^1$ be a family of stable curves whose central fiber $S_0 := p^{-1}(0)$ is isomorphic to $C = C_1 + C_2$. We have two ways to construct a family with an irreducible central fiber: contract either of the two irreducible components C_i . Thus we get two families

$$S \xrightarrow{\pi_i} S_i \xrightarrow{p_i} \mathbb{A}^1 \quad \text{with } p_i^{-1}(0) \cong C'_{3-i}.$$

Over $\mathbb{A}^1 \setminus \{0\}$ the two families are naturally isomorphic to $S \rightarrow \mathbb{A}^1$, hence to each other, yet this isomorphism does not extend to an isomorphism of S_1 and S_2 . Indeed, the closure of the graph of the resulting birational map is given by the image $(\pi_1, \pi_2): S \rightarrow S_1 \times_{\mathbb{A}^1} S_2$. Thus the corresponding moduli functor is not separated.

We claimed above that, by contrast, the coarse moduli space is \bar{M}_2 , hence separated. A closer study reveals the source of this discrepancy: we have been thinking of schemes instead of algebraic spaces. The occurrence of such problems in moduli theory was first observed by [Art74]. The aim of the next paragraph is to show how such examples arise.

1.69.9 (Bug-eyed covers). [Art74, Kol92a]

A non-separated scheme always has ‘extra’ points. The typical example is when we take two copies of a scheme $X \times \{i\}$ for $i = 0, 1$, an open dense subscheme $U \subsetneq X$ and glue $U \times \{0\}$ to $U \times \{1\}$ to get $X \amalg_U X$. The non-separatedness arises from having 2 points in $X \amalg_U X$ for each point in $X \setminus U$.

By contrast, an algebraic space can be non-separated by having no extra points, only extra tangent directions. The simplest example is the following.

On \mathbb{A}_t^1 consider two equivalence relations. The first is $R_1 \rightrightarrows \mathbb{A}^1$ given by

$$(t_1 = t_2) \cup (t_1 = -t_2) \subset \mathbb{A}_{t_1}^1 \times \mathbb{A}_{t_2}^1.$$

Then $\mathbb{A}_t^1/R_1 \cong \mathbb{A}_u^1$ where $u = t^2$.

The second is the étale equivalence relation $R_2 \rightrightarrows \mathbb{A}^1$ given by

$$\mathbb{A}^1 \xrightarrow{(1,1)} \mathbb{A}^1 \times \mathbb{A}^1 \quad \text{and} \quad \mathbb{A}^1 \setminus \{0\} \xrightarrow{(1,-1)} \mathbb{A}^1 \times \mathbb{A}^1.$$

(Note that we take the disconnected union of the two components, instead of their union as 2 lines in $\mathbb{A}^1 \times \mathbb{A}^1$ intersecting at the origin.)

One can also obtain \mathbb{A}_t^1/R_2 by taking the quotient of the nonseparated scheme $\mathbb{A}^1 \amalg_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1$ by the (fixed point free) involution that interchanges $(t, 0)$ and $(-t, 1)$.

The morphism $\mathbb{A}_t^1 \rightarrow \mathbb{A}_t^1/R_2$ is étale, thus $\mathbb{A}_t^1/R_2 \neq \mathbb{A}_t^1/R_1$. Nonetheless, there is a natural morphism

$$\mathbb{A}_t^1/R_2 \rightarrow \mathbb{A}_t^1/R_1$$

which is one-to-one and onto on closed points. The difference between the 2 spaces is seen by the tangent vectors at the origin. The tangent space of \mathbb{A}_t^1/R_2 at the origin is spanned by $\partial/\partial t$ while the tangent space of \mathbb{A}_t^1/R_1 at the origin is spanned by

$$\frac{\partial}{\partial u} = \frac{1}{2t} \cdot \frac{\partial}{\partial t}.$$

Other compactifications of M_g .

While M_g has many compactifications besides \bar{M}_g , it is only recently that a systematic search begun for other geometrically meaningful examples. The papers [Sch91, HH13, Smy13] contain many examples.

Our attempt to replace the moduli functor of stable curves of genus 2 with another one that parametrizes only irreducible curves was not successful, but some of the problems seemed to have arisen from the symmetry that forced us to make artificial choices.

We can avoid such choices for other values of the genus using the following observation.

Let $\pi: S \rightarrow B$ be a flat family of curves with smooth general fiber and reduced special fibers. If $C_b := \pi^{-1}(b)$ is a singular fiber and C_{bi} are the irreducible components of its normalization then

$$\begin{aligned} \sum_i h^1(C_{bi}, \mathcal{O}_{C_{bi}}) &\leq h^1(C_b, \mathcal{O}_{C_b}) = 1 - \chi(C_b, \mathcal{O}_{C_b}) = 1 - \chi(C_{gen}, \mathcal{O}_{C_{gen}}) \\ &= h^1(C_{gen}, \mathcal{O}_{C_{gen}}), \end{aligned}$$

where C_{gen} is the general smooth fiber. In particular, there can be at most 1 irreducible component with geometric genus $> \frac{1}{2}g(C_{gen})$.

From this it is easy to prove the following:

Let B be a smooth curve and $S^\circ \rightarrow B^\circ$ a smooth family of genus g curves over an open subset of B . Then there is at most one normal surface $S \rightarrow B$ extending S° such that every fiber of $S \rightarrow B$ is irreducible and of geometric genus $> \frac{1}{2}g(C_{gen})$.

Moreover, if $S_{stab} \rightarrow B$ is a stable family extending S° and every fiber of $S_{stab} \rightarrow B$ contains an irreducible curve of geometric genus $> \frac{1}{2}g(C_{gen})$, then we obtain S from S_{stab} by contracting all connected components of curves of geometric genus $< \frac{1}{2}g(C_{gen})$ that are contained in the fibers. (It is not hard to show that $S \rightarrow B$ exists, at least as an algebraic space.)

In fact, this way we obtain a partial compactification $M_g \subset M'_g$ such that

(1.69.1) M'_g parametrizes smoothable irreducible curves of arithmetic genus g and geometric genus $> \frac{1}{2}g$.

(1.69.2) Let $M_g \subset M_g'' \subset \bar{M}_g$ be the largest open subset parametrizing curves that contain an irreducible component of geometric genus $> \frac{1}{2}g$. Then there is a natural morphism $M_g'' \rightarrow M_g'$.

So far so good, but, as we see next, we cannot extend M_g' to a compactification in a geometrically meaningful way. This happens for every $g \geq 3$; the following example with $g = 13$ is given by simple equations.

This illustrates a general pattern: one can easily propose partial compactifications that work well for some families but lead to contradictions for some others.

EXAMPLE 1.70. Consider the surface $F := (x^8 + y^8 + z^8 = u^2) \subset \mathbb{P}^3(1, 1, 1, 4)$ and on it the curve $C := F \cap (xyz = 0)$. C has 3 irreducible components $C_x = (x = 0)$, $C_y = (y = 0)$, $C_z = (z = 0)$ which are smooth curves of genus 3. C itself has arithmetic genus 13.

We work with a 3-parameter family of deformations

$$T := (xyz - ux^3 - vy^3 - wz^3 = 0) \subset F \times \mathbb{A}_{uvw}^3. \quad (1.70.1)$$

For general $uvw \neq 0$ the fiber of the projection $\pi: T \rightarrow \mathbb{A}^3$ is a smooth curve of genus 13. If one of the u, v, w is zero, then generically we get a curve with 2 nodes hence with geometric genus 11.

If two of the coordinates are zero, say $v = w = 0$, then we have a family

$$T_x := (x(yz - ux^2) = 0) \subset F \times \mathbb{A}_u^1.$$

For $u \neq 0$, the fiber $C_{u,0,0}$ has 2 irreducible components. One is $C_x = (x = 0)$, the other is $(yz - tx^2 = 0)$ which is a smooth genus 7 curve.

Thus the proposed rule says that we should contract $C_x \subset C_{u,0,0}$.

Similarly, by working over the v and the w -axes, the rule tells us to contract $C_y \subset C_{0,v,0}$ for $v \neq 0$ and $C_z \subset C_{0,0,w}$ for $w \neq 0$.

It is easy to see that over $\mathbb{A}^3 \setminus \{(0, 0, 0)\}$ these contractions can be performed (at least among algebraic spaces). Thus we obtain

$$\begin{array}{ccc} T \setminus \{\pi^{-1}(0, 0, 0)\} & \xrightarrow{p^\circ} & S^\circ \\ \pi \downarrow & & \tau^\circ \downarrow \\ \mathbb{A}^3 \setminus \{(0, 0, 0)\} & = & \mathbb{A}^3 \setminus \{(0, 0, 0)\} \end{array} \quad (1.70.2)$$

where τ° is flat with irreducible fibers.

Claim 1.70.3. There is no proper family of curves $\tau: S \rightarrow \mathbb{A}^3$ that extends τ° . (We do not require τ to be flat.)

Proof. Assume to the contrary that $\tau: S \rightarrow \mathbb{A}^3$ exists and let

$$\Gamma \subset T \times_{\mathbb{A}^3} S$$

be the closure of the graph of p° . Since p° is a morphism on $T \setminus \{\pi^{-1}(0, 0, 0)\}$, we see that the first projection $\pi_1: \Gamma \rightarrow T$ is an isomorphism away from $\pi^{-1}(0, 0, 0)$. Since $T \times_{\mathbb{A}^3} S \rightarrow \mathbb{A}^3$ has 2-dimensional fibers, we conclude that $\dim \pi_1^{-1}(\pi^{-1}(0, 0, 0)) \leq 2$. T is, however, a smooth 4-fold, hence the exceptional set of any birational map to T has pure dimension 3. Thus $\Gamma \cong T$ and so p° extends to a morphism $p: T \rightarrow S$.

Now we see that the rule lands us in a contradiction over the origin $(0, 0, 0)$. Here all 3 components $C_x, C_y, C_z \subset C_{0,0,0} = C$ should be contracted. This is impossible to do since this would give that the central fiber of $S \rightarrow \mathbb{A}^3$ is a point.

1.8. Coarse and fine moduli spaces

As in (1.7), let \mathbf{V} be a ‘reasonable’ class of projective varieties (or schemes, or ...) and $\mathcal{V}arieties_{\mathbf{V}}$ the corresponding functor. The aim of this section is to study the difference between coarse and fine moduli spaces, mostly through a few examples. We are guided by the following:

PRINCIPLE 1.71. *Let \mathbf{V} be a ‘reasonable’ class as above and assume that it has a coarse moduli space $\text{Moduli}_{\mathbf{V}}$. Then $\text{Moduli}_{\mathbf{V}}$ is a fine moduli space iff $\text{Aut}(V)$ is trivial for every $V \in \mathbf{V}$.*

From the point of view of algebraic stacks, a precise version is given in [LMB00, 8.1.1]. Our construction of the moduli spaces in Chapter 8 also shows that this principle is true for various moduli spaces of polarized varieties, see Section 8.6.

The rest of the section is devoted to some simple examples illustrating (1.71). The direction \Rightarrow is rather easy to see if $\text{Aut}(V)$ is finite for every $V \in \mathbf{V}$, see (1.74.2). However, (1.71) fails in some cases, as shown by (1.74.3). The direction \Leftarrow is subtler. It again holds for polarized varieties but a precise version needs careful attention to descent theory and the difference between schemes and algebraic spaces.

1.72 (Moduli of varieties without automorphisms). As above, let \mathbf{V} be a ‘reasonable’ class of varieties with a coarse moduli space $\text{Moduli}_{\mathbf{V}}$. Let us make the following

Assumption 1.72.1. $\text{Aut}(V) = \{1\}$ is an open condition in flat families with fibers in \mathbf{V} .

If this holds then there is an open subscheme $\text{Moduli}_{\mathbf{V}}^{\text{rigid}} \subset \text{Moduli}_{\mathbf{V}}$ that is a coarse moduli space for varieties in \mathbf{V} without automorphisms. By (1.71) $\text{Moduli}_{\mathbf{V}}^{\text{rigid}}$ should be a fine moduli space. In many important cases $\text{Moduli}_{\mathbf{V}}^{\text{rigid}}$ is dense in $\text{Moduli}_{\mathbf{V}}$, thus one can understand much about the coarse moduli space $\text{Moduli}_{\mathbf{V}}$ by studying the fine moduli space $\text{Moduli}_{\mathbf{V}}^{\text{rigid}}$.

Let $X \rightarrow S$ be a flat family with fibers in \mathbf{V} and $\pi: \text{Aut}(X/S) \rightarrow S$ the scheme representing automorphisms of the fibers; cf. [Kol96, I.1.10]. If \mathbf{V} satisfies the valuative criterion of separatedness (1.20) and all automorphisms are finite, then π is proper. More careful attention to the scheme structure of the automorphism groups shows that in fact $\text{Aut}(V) = \{1\}$ is an open condition.

The following example, however, shows that (1.72.1) does not hold for all smooth projective surfaces.

Example 1.72.2. Let S be a smooth projective surface such that $G := \text{Aut}(S) = \langle \tau \rangle \cong \mathbb{Z}/p$ has prime order ≥ 3 and there is a τ -fixed point $s \in S$ such that the G action on $\mathbb{P}(T_s S)$ is faithful.

For instance, if $f(x, y, z)$ is a general homogeneous form of degree pd then we can take S to be the degree p cyclic cover $(w^p = f(x, y, z)) \subset \mathbb{P}^3(1, 1, 1, d)$ and s to be any branch point.

Take now a smooth (affine) curve $s \in C \subset S$ such that the stabilizer of $T_s C \subset T_s S$ is trivial. For $0 \leq i < p$ let $C_i \subset S \times C$ be the image of $(\tau^i, 1): C \rightarrow S \times C$. By shrinking C we may assume that the C_i intersect only at (s, s) .

Let $X_0 \rightarrow S \times C$ denote the blow up of C_0 . The birational transforms C'_i are disjoint for $0 < i < p$. We can now blow up the C'_i for $0 < i < p$ simultaneously to obtain

$$\pi: X \rightarrow S \times C \rightarrow C.$$

If $c \neq s$ then the fiber X_c is obtained from S by blowing up the G -orbit of the point $c \in C \subset S$. Thus the G -action on S lifts to a G -action on X_c .

For $c = s$ we get a fiber X_s which is obtained from S in two steps.

First we blow up s to get $B_s S$ with exceptional curve $E \subset B_s S$. The G -action on S lifts to a G -action on $B_s S$. Second, we blow up the $(p - 1)$ intersection points $E \cap C'_i$ for $0 < i < p$ but we do not blow up the point $E \cap C'_0$. There is no G -orbit with $p - 1$ elements, thus the G -action on $B_s S$ does not lift to X_s and $\text{Aut}(X_s) = \{1\}$.

Example 1.72.3. A similar jump of the automorphism group also happens for Enriques surfaces. By the works of [BP83, Dol84, Kon86], the automorphism group of a general Enriques surface is infinite, but there are special Enriques surfaces with finite automorphism group.

Next we see what goes wrong in the presence of automorphisms. We start with a concrete example.

EXAMPLE 1.73 (Moduli theory of the curve $(z^2 = x^{2n} - 1)$, I.).

A seemingly trivial, but actually quite subtle and revealing, example is the moduli theory of the hyperelliptic curve C , given by a projective equation as

$$C = (z^2 = x^{2n} - y^{2n}) \subset \mathbb{P}^2(1, 1, n).$$

Let k be an algebraically closed field. Following the pattern of (1.8), as a first approximation, our moduli functor should be

$$\text{Curves}_C(T) := \left\{ \begin{array}{l} \text{Smooth families } S \rightarrow T \text{ such that} \\ \text{every fiber is isomorphic to } C, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

This is the right definition if T is reduced, but not otherwise, so for now we restrict ourselves to reduced base schemes.

Since the k -points of the coarse moduli space are in one-to-one correspondence with the k -isomorphism classes of objects, a coarse moduli space for Curves_C has a unique k -point.

The only possible choice for the universal family is now

$$u: C \rightarrow \text{Spec } k.$$

Any k -scheme T has a unique morphism $g: T \rightarrow \text{Spec } k$ and by pull-back we obtain the trivial family

$$g^*u: C \times T \rightarrow T.$$

It is easy to see, however, that for many schemes T , there are other families in $\text{Curves}_C(T)$. Take, for instance, $T = \mathbb{A}^* := \mathbb{A}^1 \setminus \{0\}$ and consider the surface

$$S_1^* := (z^2 = x^{2n} - ty^{2n}) \subset \mathbb{P}^2(1, 1, n)_{xyz} \times \mathbb{A}_t^*.$$

S_1^* is smooth and the fibers of the projection $\pi_1: S_1^* \rightarrow \mathbb{A}^*$ are smooth hyperelliptic curves of genus $n - 1$. The substitution $y' := \sqrt[n]{t} \cdot y$ shows that each geometric fiber is isomorphic to the curve $C := (z^2 = x^{2n} - y^{2n}) \subset \mathbb{P}^2(1, 1, n)$. We claim, however, that, for $n \geq 3$, the family $\pi_1: S_1^* \rightarrow \mathbb{A}^*$ is different from the trivial family $\pi_2: S_2^* := (C \times \mathbb{A}^*) \rightarrow \mathbb{A}^*$. We can write the latter as

$$S_2^* := (z^2 = x^{2n} - y^{2n}) \subset \mathbb{P}^2(1, 1, n)_{xyz} \times \mathbb{A}_t^*.$$

To see the difference note that a hyperelliptic curve (of genus ≥ 2) has a unique degree 2 map to \mathbb{P}^1 . In our two families the corresponding maps are the coordinate projection

$$\mathbb{P}^2(1, 1, n)_{xyz} \times \mathbb{A}_t^* \rightarrow \mathbb{P}_{xy}^1 \times \mathbb{A}_t^*$$

restricted to S_1^* (resp. S_2^*).

The branch curve of $S_1^* \rightarrow \mathbb{P}_{xy}^1 \times \mathbb{A}_t^*$ is the irreducible curve

$$B_1^* := (x^{2n} - ty^{2n} = 0) \subset \mathbb{P}_{xy}^2 \times \mathbb{A}_t^*,$$

whereas the branch curve of $S_2^* \rightarrow \mathbb{P}_{xy}^1 \times \mathbb{A}_t^*$ is the reducible curve

$$B_2^* := (x^{2n} - y^{2n} = 0) \subset \mathbb{P}_{xy}^2 \times \mathbb{A}_t^*.$$

Thus the two families are not isomorphic.

We also see that the two families become isomorphic after a finite and surjective base change. Consider the substitution $t = u^{2n}$. By pulling back S_1^* , we get the family

$$T_1^* := (z^2 = x^{2n} - u^{2n}y^{2n}) \subset \mathbb{P}^2(1, 1, n)_{xyz} \times \mathbb{A}_u^*.$$

By setting $y_1 := uy$, T_1^* becomes isomorphic to the trivial family

$$T_2^* := (z^2 = x^{2n} - y_1^{2n}) \subset \mathbb{P}^2(1, 1, n)_{xy_1z} \times \mathbb{A}_u^*,$$

which is also obtained by pulling back the trivial family S_2^* to \mathbb{A}_u^* .

We can put these considerations in a somewhat more general setting as follows.

1.74 (Isotrivial families). Let X be a smooth projective variety over \mathbb{C} and assume for simplicity that $\text{Aut}(X)$ is a discrete group. We are interested in the functor, which to a reduced scheme T associates the set

$$\mathcal{I}so\text{triv}_X(T) := \left\{ \begin{array}{l} \text{Smooth families } \mathbf{X} \rightarrow T \text{ such that} \\ \text{every fiber is isomorphic to } X, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

More precisely, we should distinguish between the algebraic and the complex analytic versions $\mathcal{I}so\text{triv}_X^{\text{alg}}(*)$ and $\mathcal{I}so\text{triv}_X^{\text{an}}(*)$. It turns out that allowing T to be a complex analytic space is a minor difference, but allowing \mathbf{X} to be complex analytic creates a substantial change. Let us start complex analytically.

Lemma 1.74.1. Assume that $\text{Aut}(X)$ is a discrete group. Then families in $\mathcal{I}so\text{triv}_X^{\text{an}}(T)$ are in one-to-one correspondence with the $\text{Aut}(X)$ -conjugacy classes of group homomorphisms $\text{Hom}(\pi_1(T, t), \text{Aut}(X))$.

Proof. Since $\text{Aut}(X)$ is a discrete group, over any contractible subset of T the family has a unique trivialization. Thus, if we fix a point $t \in T$ and an isomorphism $\mathbf{X}_t \cong X$, then the various families are classified by the monodromy representation

$$\rho: \pi_1(T, t) \rightarrow \text{Aut}(X).$$

If we do not fix an isomorphism $\mathbf{X}_t \cong X$, then we have to work with conjugacy classes of such homomorphisms. \square

It is not hard to go from an analytic classification to an algebraic one.

Lemma 1.74.2. Notation and assumptions as above.

(1.74.1) Two such algebraic families $\mathbf{X}_i \rightarrow T$ are algebraically isomorphic iff they are analytically isomorphic.

(1.74.2) $\mathbf{X} \rightarrow T$ is projective iff the image of ρ is finite.

(1.74.3) $\mathbf{X} \rightarrow T$ is an algebraic space iff $\mathbf{X} \rightarrow T$ is projective.

Proof. Assume that $\mathbf{X}_i \rightarrow T$ are algebraic and consider the scheme parametrizing relative isomorphisms $\text{Isom}_T(\mathbf{X}_1, \mathbf{X}_2)$ (cf. [Kol96, Sec.I.1]). By our assumptions $\text{Isom}_T(\mathbf{X}_1, \mathbf{X}_2) \rightarrow T$ is étale, thus it has an algebraic section iff it has an analytic section. This proves (1).

Assume that $\mathbf{X} \rightarrow T$, corresponding to $\rho: \pi_1(T, t) \rightarrow \text{Aut}(X)$, is projective and let L be a relatively ample divisor on \mathbf{X} . Then $c_1(L|_X) \in H^2(X, \mathbb{Z})$ is invariant under $\text{im } \rho$. For some $d > 0$, the Néron-Severi group $\text{NS}(X)$ is generated by effective divisors of degree $\leq d$ (with respect to $c_1(L|_X)$). There are only finitely many such divisor classes, hence a finite index subgroup of the image of ρ acts trivially on $\text{NS}(X)$. For any projective variety X , the subgroup $\text{Aut}^\tau(X)$ of $\text{Aut}(X)$ that acts trivially on $\text{NS}(X)$ is an algebraic group (cf. [Kol96, I.1.10.2]). Since $\text{Aut}(X)$ is assumed discrete, $\text{Aut}^\tau(X)$ is finite. Thus $\text{im } \rho$ is finite, proving one direction of (2).

Conversely, assume that $G := \text{im } \rho$ is finite and let $T' \rightarrow T$ be the étale cover corresponding to G . On the trivial family $X \times T'$ consider the action of G where we act on T' by deck transformations and on X by ρ . The quotient $\mathbf{X} := (X \times T')/G$ exists and is projective (cf. [Kol13b, 9.29]).

The proof of (3) is left to the reader; we will not use it. \square

Corollary 1.74.3. Let X be a smooth projective variety over \mathbb{C} such that $\text{Aut}(X)$ is a discrete group. Then $X \rightarrow \text{Spec } \mathbb{C}$ is a fine moduli space for $\text{Isotriv}_X^{\text{an}}(*)$ iff $\text{Aut}(X) = \{1\}$.

Proof. If $\text{Aut}(X) \neq \{1\}$ then there is a nontrivial homomorphism $\mathbb{Z} \rightarrow \text{Aut}(X)$. This gives a locally trivial but globally nontrivial complex analytic family over \mathbb{C}^* (or over any elliptic curve) that cannot be the pull-back of $X \rightarrow \text{Spec } \mathbb{C}$. Conversely, if $\text{Aut}(X) = \{1\}$ then $\text{Isotriv}_X^{\text{an}}(T)$ consists of the trivial family for any T .

Corollary 1.74.4. Let X be a smooth projective variety over \mathbb{C} such that $\text{Aut}(X)$ is discrete and torsion-free. Then for any T , the trivial family $X \times T$ gives the only algebraic family in $\text{Isotriv}_X^{\text{alg}}(T)$. In particular, $X \rightarrow \text{Spec } \mathbb{C}$ is a fine moduli space for $\text{Isotriv}_X^{\text{alg}}(*)$.

Proof. By our assumption, the only homomorphism $\rho: \pi_1(T, t) \rightarrow \text{Aut}(X)$ with finite image is the trivial one. It corresponds to the trivial family $X \times T \rightarrow T$. \square

The next construction gives such an example that is birational to an Abelian surface.

Example 1.74.5. Let $0 \in E$ be an elliptic curve such that $\text{End}(0 \in E) \cong \mathbb{Z}$, (that is, without complex multiplication). Then the automorphism group of its square is

$$\text{Aut}((0, 0) \in E \times E) \cong GL(2, \mathbb{Z})$$

and the isomorphism is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [(x, y) \mapsto (ax + by, cx + dy)].$$

Take 3 points $P_1 = (0, 0)$, $P_2 = (x_2, 0)$ and $P_3 = (0, x_3)$ where $x_3 \in E$ is 3-torsion and $x_2 \in E$ is non-torsion. It is easy to see that $\{0\} \times E$ (resp. $E \times \{0\}$) are the

only elliptic curves in $E \times E$ that contain 2 of the points whose difference is torsion (resp. non-torsion). Thus we conclude that

$$\text{Aut}(P_1 + P_2 + P_3, E \times E) = \left\{ \begin{pmatrix} 1 & 3m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}.$$

Let now X be the surface obtained from $E \times E$ by blowing up the 3 points P_i . Since the only rational curves on X are the 3 exceptional curves, we conclude that

$$\text{Aut}(X) = \text{Aut}(P_1 + P_2 + P_3, E \times E) \cong \mathbb{Z}.$$

EXAMPLE 1.75 (Moduli theory of the curve $(z^2 = x^{2n} - 1)$, II.).

Another reincarnation of the phenomenon observed in (1.73) occurs if we notice that C is already defined over \mathbb{Q} and we try to construct the moduli space as $\text{Spec } \mathbb{Q}$.

Over an algebraically closed field, C is isomorphic to any of the curves

$$C_{ab} = (z^2 = ax^{2n} - by^{2n}) \subset \mathbb{P}^2(1, 1, n) \quad \text{for } a, b \neq 0.$$

Over other fields, however, the curves C_{ab} need not be isomorphic. For instance, over \mathbb{R} , we can obtain $(z^2 = x^{2n} + y^{2n})$ whose set of real points consists of 2 circles, $(z^2 = x^{2n} - y^{2n})$ whose set of real points consists of 1 circle and $(z^2 = -x^{2n} - y^{2n})$ whose set of real points is empty.

The situation is even worse over \mathbb{Q} . For instance, as p runs through all prime numbers, the curves $C_{1p} = (z^2 = x^{2n} - py^{2n})$ are pairwise non-isomorphic for $n \geq 4$.

A simple way to see this is to note that the ramification locus of the projection $C_{1p} \rightarrow \mathbb{P}_{xy}^1$ is an isomorphism invariant of C_{1p} . In our case, the ramification locus is the scheme $\text{Spec}_{\mathbb{Q}} \mathbb{Q}(\sqrt[n]{p})$, and these fields are different from each other for different values of p . For instance, the only ramified primes in $\mathbb{Q}(\sqrt[n]{p})/\mathbb{Q}$ are p and possibly some divisors of $2n$. Thus as p runs through the set of primes not dividing $2n$, we get pairwise non-isomorphic fields and hence non-isomorphic curves C_{1p} .

1.76 (Field of moduli). Let $X \subset \mathbb{P}^n$ be a projective variety defined over some large field, for example \mathbb{C} . Any set of defining equations involves only finitely many elements of \mathbb{C} , thus X can be defined over a finitely generated subextension of \mathbb{C} . It is a natural question to ask: Is there a smallest subfield $K \subset \mathbb{C}$ such that X can be defined by equations over K .

There are three variants for this question.

- (1.76.1) Fix coordinates on \mathbb{P}^n and view X as a specific subvariety. In this case a smallest subfield exists; see [Wei46, Sec.I.7] or [KSC04, Sec.3.4]. This is a special case of the existence of Hilbert schemes (1.5).
- (1.76.2) No embedding of X is fixed. Thus we are looking for a field $K \subset \mathbb{C}$ and a K -variety X_K such that $X \cong (X_K)_{\mathbb{C}}$. We see in (1.79) that this may lead to rather complicated behavior.
- (1.76.3) As an intermediate choice, fix an embedding $X \hookrightarrow \mathbb{P}^n$ but do not fix the coordinates on \mathbb{P}^n . Equivalently, we work with a pair (X, L) where L is a very ample line bundle on X . This is the question that we consider next. Note that, if the canonical line bundle on X is ample or anti-ample, we can harmlessly identify X with the pair $(X, \mathcal{O}_X(mK_X))$ if mK_X is very ample. (There are two further natural variants of this approach. We may decide not to distinguish between the pairs (X, L) and (X, L^m) for $m > 0$ or we may identify (X, L) and

(X, L') if L is numerically equivalent to L' . Both of these lead to minor technical differences only.)

How is this connected with moduli theory?

Let \mathbf{V} be a class of varieties with a coarse moduli space $\text{Moduli}_{\mathbf{V}}$. Assume that $X \in \mathbf{V}$ can be defined by equations over a field K ; that is, there is a K -scheme $X_K \rightarrow \text{Spec } K$ whose geometric fiber is isomorphic to X . By the definition of a coarse moduli space, this corresponds to a morphism $\text{Spec } K \rightarrow \text{Moduli}_{\mathbf{V}}$. In particular, we get an injection of the residue field of $\text{Moduli}_{\mathbf{V}}$ at $[X]$ into K . Conversely, if $\text{Moduli}_{\mathbf{V}}$ is a fine moduli space, then X can be defined over the residue field of $[X] \in \text{Moduli}_{\mathbf{V}}$. Thus we have proved the following:

Lemma 1.76.4. If $\text{Moduli}_{\mathbf{V}}$ is a fine moduli space then the residue field of $\text{Moduli}_{\mathbf{V}}$ at $[X]$ is the smallest field K such that X can be defined over K as in (1.76.2). \square

A consequence is that, for fine moduli spaces, the residue field of $\text{Moduli}_{\mathbf{V}}$ at $[X]$ depends only on X and not on the choice of \mathbf{V} .

In general, let us define the *field of moduli* of X as the (function field of) the coarse moduli space of the functor $\mathcal{I}so\text{triv}_X(*)$, where, generalizing the concept in (1.74) from \mathbb{C} to arbitrary fields, for any reduced scheme T we set

$$\mathcal{I}so\text{triv}_X(T) := \left\{ \begin{array}{l} \text{Smooth families } \mathbf{X} \rightarrow T \text{ such that} \\ \text{every geometric fiber is isomorphic to } X, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

As we see in (1.79), $\mathcal{I}so\text{triv}_X(*)$ need not have a coarse moduli space. We thus introduce the following variant. For a pair (X, L) , where L is an ample line bundle on X , set

$$\mathcal{I}so\text{triv}_{(X,L)}(T) := \left\{ \begin{array}{l} \text{Smooth families } \mathbf{X} \rightarrow T \text{ plus a} \\ \text{relatively ample line bundle } \mathbf{L} \text{ such that} \\ \text{every geometric fiber is isomorphic to } (X, L), \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

Then $\mathcal{I}so\text{triv}_{(X,L)}(*)$ has a coarse moduli space, the spectrum of the field of moduli of (X, L) (1.76).

In order to avoid some problems with infinite Galois groups (1.79), the following lemma is stated for number fields only.

Lemma 1.76.5. Let X be a smooth projective variety defined over a number field L . For a field K the following are equivalent.

- (1.76.1) The field of moduli of X is contained in K .
- (1.76.2) There is a K -scheme T such that $\mathcal{I}so\text{triv}_X(T) \neq \emptyset$.
- (1.76.3) For any $\sigma \in \text{Gal}(\bar{K}/K)$, the variety X^σ is isomorphic to X over \bar{K} . (Here X^σ is obtained by applying σ to a set of defining equations of X .)

Proof. The interesting part is (3) \Rightarrow (2). Choose a finite extension $K(\alpha)/K$ such that $L \subset K(\alpha)$, where α is a root of a polynomial $p(t) \in K[t]$ of degree d . Let

$$f_i(x_0, \dots, x_m) \in K(\alpha)[x_0, \dots, x_m] : i = 1, \dots, r$$

be defining equations of X (in some projective embedding) over $K(\alpha)$. Since $K(\alpha) = K + \alpha K + \dots + \alpha^{d-1}K$, we can also think of the f_i as

$$f_i(\alpha, x_0, \dots, x_m) \in K[\alpha, x_0, \dots, x_m],$$

where $\deg_\alpha f_i < d$. Consider now the K -scheme

$$Y_K := (f_1(t, x_0, \dots, x_m) = \dots = f_r(t, x_0, \dots, x_m) = p(t) = 0) \subset \mathbb{P}_K^m \times \mathbb{A}_t^1.$$

The second projection gives $\pi: Y_K \rightarrow \text{Spec}_K K[t]/(p(t))$. One of the geometric fibers of π is $X_{\bar{L}}$, the others are its conjugates $X_{\bar{L}}^\sigma$. If (3) holds then $\pi: Y_K \rightarrow \text{Spec}_K K(\alpha)$ is an isotrivial family over the K -scheme $\text{Spec}_K K(\alpha)$, which shows (2). \square

In (1.78) we construct a hyperelliptic curve whose field of moduli is \mathbb{Q} yet it cannot be defined over \mathbb{R} . The first such examples are in [Ear71, Shi72].

1.77 (Field of moduli for hyperelliptic curves). Let A be a smooth hyperelliptic curve of genus ≥ 2 . Over an algebraically closed field, A has a unique degree 2 map to \mathbb{P}^1 . Let $B \subset \mathbb{P}^1$ be the branch locus, that is, a collection of $2g + 2$ points in \mathbb{P}^1 . If the base field k is not closed, then A has a unique degree 2 map to a smooth genus 0 curve Q . (One can always think of Q as a conic in \mathbb{P}^2 .) Thus A is defined over a field k iff the pair $(B \subset \mathbb{P}^1)$ can be defined over k .

The latter problem is especially transparent if A is defined over \mathbb{C} and we want to know if it is defined over \mathbb{R} or if its field of moduli is contained in \mathbb{R} .

Up to isomorphism, there are 2 real forms of \mathbb{P}^1 . One is \mathbb{P}^1 , corresponding to the anti-holomorphic involution $(x:y) \mapsto (\bar{x}:\bar{y})$, which, after a coordinate change, can also be written as $\sigma_1: (x:y) \mapsto (\bar{y}:\bar{x})$. (In the latter form the real points form the unit circle.) The other is the ‘empty’ conic, corresponding to the anti-holomorphic involution $\sigma_2: (x:y) \mapsto (-\bar{y}:\bar{x})$. Thus (1.76.5) gives the following.

Lemma 1.77.1. Let $A \rightarrow \mathbb{P}^1$ be a smooth hyperelliptic curve of genus ≥ 2 over \mathbb{C} and $B \subset \mathbb{C}\mathbb{P}^1$ the branch locus. Then

(1.77.1) A can be defined over \mathbb{R} iff there is a $g \in \text{Aut}(\mathbb{C}\mathbb{P}^1)$ such that gB is invariant under σ_1 or σ_2 .

(1.77.2) The field of moduli of A is contained in \mathbb{R} iff there is $h \in \text{Aut}(\mathbb{C}\mathbb{P}^1)$ such that hB equals B^{σ_1} or B^{σ_2} .

Note that if $(gB)^\sigma = gB$ then $B^\sigma = (g^\sigma)^{-1}gB$ shows that (1) \Rightarrow (2). Conversely, if $B^\sigma = hB$ and we can write $h = (g^\sigma)^{-1}g$ then $(gB)^\sigma = gB$.

EXAMPLE 1.78. Here is an example of a hyperelliptic curve C whose field of moduli is \mathbb{Q} but C cannot be defined over \mathbb{R} .

Pick $\alpha = a + ib$ where a, b are rational. Consider the hyperelliptic curve

$$C(\alpha) := \left(z^2 - (x^8 - y^8)(x^2 - \alpha y^2)(\bar{\alpha}x^2 + y^2) = 0 \right) \subset \mathbb{P}^3(1, 1, 6).$$

Its complex conjugate is

$$C(\bar{\alpha}) := \left(z^2 - (x^8 - y^8)(x^2 - \bar{\alpha}y^2)(\alpha x^2 + y^2) = 0 \right) \subset \mathbb{P}^3(1, 1, 6).$$

Note that $C(\alpha)$ and $C(\bar{\alpha})$ are isomorphic, as shown by the substitution

$$(x, y, z) \mapsto (iy, x, z).$$

In particular, over the \mathbb{Q} -scheme $\text{Spec}_\mathbb{Q} \mathbb{Q}[t]/(t^2 + 1)$ we have a curve

$$C(a, b) := \left(z^2 - (x^8 - y^8)(x^2 - (a + tb)y^2)((a - tb)x^2 + y^2) = 0 \right) \subset \mathbb{P}^3(1, 1, 6)$$

whose geometric fibers are isomorphic to $C(\alpha)$. Thus the field of moduli of $C(\alpha)$ is \mathbb{Q} by (1.76.5).

We claim that, for sufficiently general a, b , the curve $C(\alpha)$ cannot be defined over \mathbb{Q} , not even over \mathbb{R} . By (1.77) we need to show that there is no anti-holomorphic involution that maps the branch locus to itself. In the affine chart $y \neq 0$, the ramification points of $C(\alpha) \rightarrow \mathbb{P}^1$ are:

- (1.78.1) the 8th roots of unity corresponding to $x^8 - y^8$, and
 (1.78.2) the 4 points $\pm\beta, \pm i/\bar{\beta}$ where $\beta^2 = \alpha$.

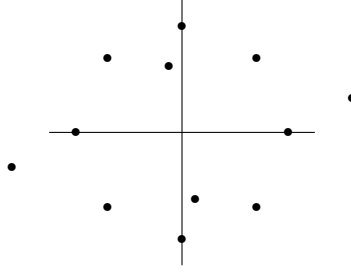
The anti-holomorphic automorphisms of the Riemann sphere map circles to circles. Out of our 12 points, the 8 roots of unity lie on the circle $|z| = 1$, but no other 8 can lie on a circle. Thus any anti-holomorphic automorphism that maps our configuration to itself, must fix the unit circle $|z| = 1$ and map the 8th roots of unity to each other.

The only such anti-holomorphic involutions are

- (1.78.3) Reflection on the line $\mathbb{R} \cdot \epsilon$ where ϵ is a 16th root of unity, and
 (1.78.4) $z \mapsto 1/\bar{z}$ or $z \mapsto -1/\bar{z}$.

A short case analysis shows that $C(\alpha)$ is not isomorphic (over \mathbb{C}) to a real curve, as long as β^{16} is not a positive real number.

The configuration depicted below shows 12 points p_1, \dots, p_{12} on \mathbb{C} that are invariant under $z \mapsto i/\bar{z}$ but not invariant under any anti-holomorphic involution.



EXAMPLE 1.79. We give an example of a smooth projective surface S such that if S is defined over a field extension K/\mathbb{C} then $\text{trdeg } K \geq 2$ but the intersection of all such fields of definition is \mathbb{C} .

Let X be a smooth projective variety such that

- (1.79.1) $\text{Aut}(X)$ is an infinite discrete group whose general orbit is Zariski dense in X and
 (1.79.2) $\text{Aut}(X)$ is generated by 2 finite subgroups G_1, G_2 .

By (1.74.5), one such example is $B_0(E \times E)$, the blow up of the square of an elliptic curve at a point, as shown by the subgroups generated by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

There are also K3 surfaces with infinite automorphism group generated by 2 involutions (1.67).

Let $\Delta \subset X \times X$ be the diagonal and, using one of the projections, consider the family of smooth varieties

$$f: Y := B_\Delta(X \times X) \rightarrow X.$$

Note that $Y \rightarrow X$ is the universal family of the varieties of the form $B_x X$ for $x \in X$. This shows that $f: Y \rightarrow X$ cannot be obtained by pull-back from any family over a lower dimensional base.

In particular, if $x \in X$ is general, then $\text{Aut}(B_x X) = \mathbb{Z}/2$ if $X = B_0(E \times E)$ and $\text{Aut}(B_x X) = 1$ if X is a K3 surface. The action of $\text{Aut}(X)$ lifts to the diagonal action on Y .

Let $G \subset \text{Aut}(X)$ be a finite subgroup. There is an open subset $U_G \subset X$ such that G operates on U_G without fixed points. Thus $f/G: Y/G \rightarrow X/G$ is a family of smooth varieties over U_G/G and $Y|_{U_G} \cong Y/G \times_{X/G} U_G$.

Let $K = \mathbb{C}(X)$ denote the function field of X . The variety we are interested in is Y_K , the generic fiber of $Y \rightarrow X$. The above considerations show that Y_K can be defined over $\mathbb{C}(X/G) = K^G$ for every finite subgroup $G \subset \text{Aut}(X)$.

Note that $K = \mathbb{C}(X)$ is a function field of transcendence degree $\dim X$ over \mathbb{C} and so are the subfields K^G . On the other hand, the intersection $K^{G_1} \cap K^{G_2}$ is \mathbb{C} . Indeed, any function in $K^{G_1} \cap K^{G_2}$ is constant on every G_1 -orbit and also on every G_2 -orbit. By assumption (2), it is also constant along every $\text{Aut}(X)$ -orbit, hence constant by assumption (1).

This phenomenon is also connected with the behavior of ample line bundles on $\pi_i: Y \rightarrow Y/G_i$. Although both of the Y/G_i are projective, there are no ample line bundles L_i on Y/G_i such that $\pi_1^* L_1 \cong \pi_2^* L_2$.

1.9. Problems in positive characteristic

The following examples illustrate several problems that appear for the moduli problem of pairs in positive characteristic. In fact, such problems appear already for the moduli of \mathbb{P}^1 with 4 weighted points. Much of the work in Chapters 6–8 is devoted to showing that similar problems do not arise in characteristic 0.

EXAMPLE 1.80 (Inseparable base change). Let k be a field of characteristic $p > 0$, X a smooth, projective k -variety and S a smooth k -variety. Let D be an effective, relative Cartier divisor on $X \times S \rightarrow S$.

Let $\tau_S: S \rightarrow S'$ be a finite, purely inseparable map of degree q to another smooth k -variety S' . (For example, we can take τ to be a power of the Frobenius.) Taking product by X we get a finite, purely inseparable map $\tau: X \times S \rightarrow X \times S'$ of degree q , and hence a divisor $D' := \tau_* D$ on $X \times S'$. Then D' is also an effective, relative Cartier divisor on $X \times S' \rightarrow S'$ and $\tau^* D' = qD$. Thus

$$(X \times S', \frac{c}{q} D') \rightarrow S' \quad (1.80.1)$$

is a well defined family whose pull-back to S is

$$(X \times S, \frac{c}{q} (qD)) \rightarrow S. \quad (1.80.2)$$

If we follow our usual custom and identify $\frac{c}{q} (qD) = cD$, then (1.80.1) is also a family in our moduli problem. In particular, if \mathbf{M} is the categorical moduli space (1.9) then the moduli map $\phi: S \rightarrow \mathbf{M}$ can be factored as

$$\phi: S \xrightarrow{\tau_S} S' \xrightarrow{\phi'} \mathbf{M}. \quad (1.80.3)$$

If this happens for every $S \rightarrow S'$ then $\phi(S)$ is a closed point of \mathbf{M} . This means that the points of the categorical moduli space depend only on X , not on the choice of the boundary divisor D .

That is, if we assume that $D = \frac{1}{p}(pD)$: the divisor part becomes invisible in the categorical moduli space (1.9).

The concept of marking, to be introduced in Section 8.1, aims to solve the above issues. The solution seems satisfactory in characteristic 0, but further problems, similar to (1.80), appear in positive characteristic. We illustrate these next.

1.81 (Cartier versus \mathbb{Q} -Cartier). One of the early key conceptual steps of the Minimal Model Program was the realizations that, starting with dimension 3, minimal models can be singular. Moreover, their canonical class need not be Cartier. It was gradually understood that the more general \mathbb{Q} -Cartier condition is the important one. Typical theorems start with the assumption that a divisor or \mathbb{Q} -divisor D be \mathbb{Q} -Cartier, without worrying about which multiple of D is Cartier.

The smallest $m \in \mathbb{N}$ such that mD is Cartier is called the *Cartier index* or simply *index* of D .

In moduli theory we frequently start with pairs (X, B) where X is smooth and B is Cartier, but in compactifying their moduli space we encounter pairs (X', B') where X' is singular and $K_{X'}$ and B' are only \mathbb{Q} -Cartier. Even worse, it can happen that only $K_{X'} + B'$ is \mathbb{Q} -Cartier. It is usually quite hard to bound the Cartier index of $K_{X'} + B'$ in terms of numerical invariants of (X, B) . We know that these bounds exist, but, with rare exceptions, these are pure existence results. Thus the usual approach is to work with pairs (X, B) where $K_X + B$ is \mathbb{Q} -Cartier.

This seems to be quite satisfactory in characteristic 0, more generally, when the index is relatively prime to the characteristic.

1.82 (Moduli of points on \mathbb{P}^1). We consider the moduli problem of $n = 2r + 1 \geq 3$ unordered points in \mathbb{P}^1 . Fix an index set I of n elements. There is, I believe, only one way of defining the objects of this theory.

(1.82.1) (Geometric objects) $(\mathbb{P}^1, \sum_{i \in I} [p_i])$ where the p_i are distinct points.

(1.82.2) (Objects over a field) (\mathbb{P}^1, Z) where $Z \subset \mathbb{P}^1$ is a geometrically reduced, 0-dimensional subscheme of degree n .

The question becomes more subtle when families are considered.

(1.82.3) (Families) $(P_S \rightarrow S, D)$ where $P_S \rightarrow S$ is locally trivial \mathbb{P}^1 -bundle, and $D \subset P_S$ is a divisor over S of degree n . (One could consider either Zariski or étale locally trivial \mathbb{P}^1 -bundles, this does not matter for the current purposes.)

The key question is, what kind of divisor is D ? For ordered points the traditional choice is to take D to be a union of sections of $P_S \rightarrow S$, but for unordered points we have 2 natural choices.

(1.82.3.a) (Cartier) D is a relative Cartier divisor over S ; equivalently, $D \rightarrow S$ is flat. This is closest to the traditional choice of union of sections.

(1.82.3.b) (\mathbb{Q} -Cartier) D is a relative \mathbb{Q} -Cartier divisor over S . This is more in the spirit of the higher dimensional theory, where we inevitably encounter \mathbb{Q} -Cartier divisors.

(1.82.4) (Base spaces) Ideally we should work over arbitrary base schemes, but it turns out that unexpected things happen even when the base is quite nice. We consider 3 classes of base schemes.

(1.82.4.a) (Reduced)

(1.82.4.b) (Seminormal)

(1.82.4.c) (Weakly normal)

The cases (3.a–b) and (4.a–c) are in principle independent, thus we have 6 different settings for the moduli problem.

Expectation 1.82.5. We expect that there is a fine moduli space and it is

$$M_{0,n}/S_n \cong (\mathrm{Sym}^n \mathbb{P}^1 \setminus (\mathrm{diagonal}))/\mathrm{PGL}_2.$$

We prove that this holds in characteristic 0 but fails for 2 of the 6 cases in positive characteristic.

THEOREM 1.83. *Consider the above 6 settings of the moduli problem of $n \geq 3$ unordered points in \mathbb{P}^1 over a field k .*

- (1.83.1) *If $\mathrm{char} k = 0$ then $M_{0,n}/S_n$ is a fine moduli space in all 6 settings.*
- (1.83.2) *If $\mathrm{char} k > 0$ then $M_{0,n}/S_n$ is a fine moduli space, provided either (1.82.3.a) or (1.82.4.c) holds.*
- (1.83.3) *If $\mathrm{char} k > 0$, and we work in the settings (1.82.3.b+4.a) or (1.82.3.b+4.b), then $M_{0,n}/S_n$ is not even a coarse moduli space. In fact the categorical moduli space (1.9) is $\mathrm{Spec} k$.*

Proof. If D is flat over S then choosing an open cover $S = \cup_j U_j$ and isomorphisms $P_{U_j} \cong \mathbb{P}^1 \times U_j$ gives morphisms $\phi_j : U_j \rightarrow \mathrm{Hilb}_n(\mathbb{P}^1) = M_{0,n}$. Changing the local trivialization changes the ϕ_j by an element of $\mathrm{Aut}(\mathbb{P}^1)$. Thus the ϕ_j glue to give a global morphism $\phi : S \rightarrow M_{0,n}/S_n$.

If $\mathrm{char} k = 0$ then a relatively \mathbb{Q} -Cartier divisor is Cartier by (4.39). The same holds in any characteristic if the base is weakly normal by (4.42). These show (1) and (2).

The proof of (3) relies on the following construction.

Let k be a field of characteristic $p > 0$, B a smooth projective curve over k and S a k -variety, for example a smooth curve. Let Δ be an effective, relative Cartier divisor on $B \times S \rightarrow S$. Factor a power of the Frobenius as

$$F_{p^m} : S \xrightarrow{\tau_S} T \rightarrow S,$$

such that τ_S is birational. Taking product with B we get $\tau : B \times S \rightarrow B \times T$ and hence a divisor $\Delta_T := \tau_* \Delta$ on $B \times T$.

Since τ is birational, the coefficients of Δ_T are the same as the coefficients of Δ , thus we do not have the problem encountered in (1.80). However, the Cartier index gets multiplied by p^m . Indeed, $(1_B \times F_{p^m})^* \Delta = p^m \Delta$, we see that $(\tau')^* \Delta = p^m \Delta_T$. In particular, if $r\Delta$ is Cartier then $p^m r \Delta_T$ is Cartier.

Example 1.83.4. A typical example with concrete equations is the following. Take $S = \mathbb{A}_x^1$ and let T be the spectrum of $k[x^2, x^s] \cong k[u, v]/(u^s - v^2)$. Set $D := (z - xt = 0) \subset \mathbb{A}_x^1 \times \mathbb{P}_{zt}^1$ and let $D_s \subset \mathrm{Spec} k[u, v]/(u^s - v^2) \times \mathbb{P}_{zt}^1$ denote its image. We claim that D_s is \mathbb{Q} -Cartier if $\mathrm{char} k > 0$. Assume that p is odd.

To see this choose $m > 0$ such that $p^m \geq s$ and set $c := \frac{1}{2}(p^m - s)$. We claim that $p^m D_s$ is a Cartier divisor with equation $z^{p^m} = u^c v \cdot t^{p^m}$. Indeed, pulling back to the normalization we get

$$z^{p^m} - (x^2)^c x^s \cdot t^{p^m} = (z - xt)^{p^m}.$$

For us the key point is that Δ_T is still \mathbb{Q} -Cartier. We have thus proved the following.

Claim 1.83.5. If $(B \times S, \Delta) \rightarrow S$ is in our moduli problem using (1.82.3.b) then so is $(B \times T, \Delta_T) \rightarrow T$. \square

Assume now that we work in the settings (1.82.3.b+4.a) or (1.82.3.b+4.b) and let \mathbf{M}_n be the categorical moduli space. If $(\mathbb{P}^1 \times S, D)$ is a family of n points on \mathbb{P}^1 then we get a moduli map $\phi : S \rightarrow \mathbf{M}_n$. By the above construction, for any $\tau_T : S \rightarrow T$ we get a factorization

$$\phi : S \xrightarrow{\tau_T} T \xrightarrow{\phi_T} \mathbf{M}_n.$$

Corollary 1.83.6. If the universal push-out of all the birational, universal homeomorphisms $\tau_T : S \rightarrow T$ is $S \rightarrow \text{Spec } k$, then the moduli map $\phi : S \rightarrow \mathbf{M}_n$ is constant.

Instead of proving this in general, we work out some typical examples.

Example 1.83.7. The map $\text{Spec } k[x] \rightarrow \text{Spec } k[(x-c)^r, (x-c)^s]$ is a birational, universal homeomorphism for any $(r, s) = 1$ and $c \in k$. The universal push-out of all of them is $\text{Spec } k[x] \rightarrow \text{Spec } k$; cf. (10.88).

Indeed, if $f(x) \in k[(x-c)^r, (x-c)^s]$ vanishes at c then it has a zero of multiplicity $\geq \min\{r, s\}$. Thus only the constants are contained in the intersection of all of them.

These examples settle the case (1.82.3.b+4.a) but the curves $\text{Spec } k[(x-c)^r, (x-c)^s]$ are not seminormal if $r, s > 1$. It is, however, not hard to get similar seminormal examples.

Example 1.83.8. Let $K := k(t)$ be a function field of characteristic $p > 0$. Set $q := p^m$. We check in (10.74) that $R_q := K + (x^q - t)K[x] \subset K[x]$ is seminormal and $\text{Spec } K[x] \rightarrow \text{Spec } R_q$ is birational. It is again easy to check that the universal push-out of all of them is $\text{Spec } K[x] \rightarrow \text{Spec } K$. \square

Over an algebraically closed field, this yields 2-dimensional examples.

Example 1.83.9. Set $R_q := k[x] + (y^q - x)k[x, y] \subset k[x, y]$. R_q is seminormal but not weakly normal and its normalization is $k[x, y]$. The conductor ideal is $(y^q - x)k[x, y]$. It is a principal ideal in $k[x, y]$ but not in R_q .

The map $\text{Spec } k[x, y] \rightarrow \text{Spec } R_q$ is a birational. It is again easy to check that the universal push-out of all of them is $\text{Spec } k[x, y] \rightarrow \text{Spec } k[x]$. Thus if we combine the maps $\text{Spec } k[x, y] \rightarrow \text{Spec } R_q$ with all linear coordinate changes, then the universal push-out is $\text{Spec } k[x, y] \rightarrow \text{Spec } k$. \square

One-parameter families

In [Kol13b] we studied in detail canonical and semi-log-canonical varieties, especially their singularities; a summary of the main results is given in Section 11.1. These are the objects that correspond to the points in a moduli functor/stack of canonical and semi-log-canonical varieties. We start the study of the general moduli problem with 1-parameter families.

In traditional moduli theory, for instance for curves, smooth varieties or sheaves, the description of all families over 1-dimensional regular schemes pretty much completes the story: the definitions and theorems have obvious generalizations to families over an arbitrary base. The best examples are the valuative criteria of separatedness and properness; we discussed these in (1.20). In our case, however, much remains to be done in order to work over arbitrary base schemes.

Two notions of locally stable or semi-log-canonical families are introduced in Section 2.1. Their equivalence is proved in characteristic 0, but remains open in general. For surfaces, one can give a rather complete étale-local description of all locally stable families; this is worked out in Section 2.2.

A series of higher dimensional examples is presented in Section 2.3. These show that stable degenerations of smooth projective varieties can get rather complicated.

Next we turn to global questions and define our main objects, stable families in Section 2.4. The main result says that stable families satisfy the valuative criteria of separatedness and properness.

Cohomological properties of stable families are studied in Section 2.5. In particular, we show that in a proper locally stable family $f: X \rightarrow C$, the basic numerical invariants $h^i(X_c, \mathcal{O}_{X_c})$ and $h^i(X_c, \omega_{X_c})$ are independent of $c \in C$. We also show that being CM is deformation invariant.

In the next two sections we turn to a key problem of the theory: understanding the difference between the divisor-theoretic and the scheme-theoretic restriction of divisors, equivalently, the role of embedded points. The general theory is outlined in Section 2.6. Then in Section 2.7 we show that if all the coefficients of the boundary divisor are $> \frac{1}{2}$, then we need not worry about embedded points in moduli questions.

In order to get the stronger form of the local stability criterion, we prove several Grothendieck–Lefschetz-type theorems in Sections 2.8–2.9, building on the techniques of Section 2.6.

2.1. Locally stable families

Following the pattern established in Section 1.4, we expect that the definition of a stable family $f: (X, \Delta) \rightarrow S$ consists of some local conditions describing the singularities of f and a global condition, that $K_X + \Delta$ be f -ample. We are now ready to formulate the correct local condition, at least for 1-parameter families.

Assumptions. While the basic definitions (2.2–2.3) are formulated for arbitrary schemes, starting with (2.4) we work over a field of characteristic 0. However, the results of this Section are conjectured to hold in general, except (2.12).

We already defined stable varieties in (1.41). The basic objects of our moduli theory are their generalizations.

DEFINITION 2.1 (Stable pairs). A *locally stable pair* (X, Δ) over a field k consists of

(2.1.1) a proper, pure dimensional, geometrically reduced k -scheme X , and

(2.1.2) an effective \mathbb{R} -divisor Δ , such that

(2.1.3) (X, Δ) has semi-log-canonical (abbreviated as slc) singularities (11.11).

(X, Δ) is a *stable pair* if, in addition

(2.1.4) X is proper, and

(2.1.5) $K_X + \Delta$ is an ample \mathbb{R} -Cartier divisor.

Thus a locally stable pair is the same as an slc pair; we usually use the former terminology for fibers of families.

If $\Delta = 0$, we have a *stable variety* as in (1.41).

DEFINITION 2.2. Let C be a regular 1-dimensional scheme. A *family of varieties* over C is a flat morphism of finite type $f: X \rightarrow C$, whose fibers are pure dimensional and geometrically reduced. For $c \in C$, let $X_c := f^{-1}(c)$ denote the fiber of f over c .

A *family of pairs* over C is a family of varieties $f: X \rightarrow C$ plus an effective \mathbb{R} -divisor Δ on X such that, for every $c \in C$, the support of Δ does not contain any irreducible component of X_c , and none of the irreducible components of $X_c \cap \text{Supp } \Delta$ is contained in $\text{Sing } X_c$. The latter condition holds if the fibers are slc pairs and it turns out to be technically crucial, so it is much easier to assume it from the beginning.

The assumptions imply that X is regular at the generic points of $X_c \cap \text{Supp } \Delta$, thus Δ is an \mathbb{R} -Cartier divisor at the generic points of $X_c \cap \text{Supp } \Delta$. In particular, $\Delta_c := \Delta|_{X_c}$ is a well-defined \mathbb{R} -divisor on X_c . Thus the pair-fibers (X_c, Δ_c) make sense.

DEFINITION 2.3. Let $f: (X, \Delta) \rightarrow C$ be a family of pairs over a regular 1-dimensional scheme. We say that $f: (X, \Delta) \rightarrow C$ is *locally stable* or *semi-log-canonical* (usually abbreviated as *slc*) if $(X, X_c + \Delta)$ is semi-log-canonical for every closed point $c \in C$.

Since X_c is a Cartier divisor, this implies that (X, Δ) is semi-log-canonical, hence X is demi-normal.

Warning. While the definition is made for arbitrary regular 1-dimensional schemes C , not much is known in positive and mixed characteristics, see (2.15).

As we noted in Section 1.4, usually (2.3) can not be reformulated as a condition about the fibers of f only. (Significant exceptions are discussed in (2.5) and (2.7).) The following result, however, comes close to achieving this.

THEOREM 2.4. Let C be a smooth curve over a field of characteristic zero and $f: (X, \Delta) \rightarrow C$ a family of pairs over C with Δ effective. For any $c \in C$ and $p \in X_c := f^{-1}(c)$ the following are equivalent.

(2.4.1) $f: (X, \Delta) \rightarrow C$ is locally stable in an open neighborhood of p in X .

(2.4.2) $K_{X/C} + \Delta$ is \mathbb{R} -Cartier at p and (X_c, Δ_c) is semi-log-canonical in an open neighborhood of p in X_c .

(2.4.3) $K_{X/C} + \Delta$ is \mathbb{R} -Cartier at p and $(\bar{X}_c, \text{Diff}_{\bar{X}_c}(\Delta))$ is log canonical in an open neighborhood of $\pi^{-1}(p)$ in \bar{X}_c , where $\pi: \bar{X}_c \rightarrow X_c$ denotes the normalization.

While it is hard to see how (2.3) could be generalized to families over higher dimensional bases, the variants (2.4.2–3) make sense in general. This observation leads to the general definition of our moduli functor in Chapters 6–8.

Proof. If (3) holds then inversion of adjunction (11.20) shows that $(X, X_c + \Delta)$ is semi-log-canonical in a neighborhood of p and, by [Kol13b, 4.10] this continues to hold for nearby fibers $X_{c'}$. Thus (3) \Rightarrow (1) and the converse also holds since (11.20) works both ways.

Since X_c is a Cartier divisor in X , the restriction $\Delta|_{X_c}$ equals the different $\text{Diff}_{X_c}(\Delta)$ by (11.18). Furthermore, by (11.17.5)

$$K_{\bar{X}_c} + \text{Diff}_{\bar{X}_c}(\Delta) = \pi^*(K_{X_c} + \text{Diff}_{X_c}(\Delta)).$$

Thus it would seem that (11.11) says that (2) \Leftrightarrow (3).

This is almost the case, except that in order to apply (11.11) we need to know that X_c is demi-normal.

By assumption X_c is geometrically reduced and an easy local computation shows that X_c is either smooth or has nodes at codimension 1 points; see [Kol13b, 2.33]. Thus it remains to prove that X_c is S_2 .

This is actually quite subtle. We outline three different proofs, all of which provide valuable insight.

First, if the generic fiber is klt, then, by (2.14), (X, Δ) is klt. Thus X is CM by (11.5) and so is every fiber X_c . In general, however, (X, Δ) is not klt and X is not CM. However, CM is much more than we need.

The second method looks carefully at what weaker versions of CM would still imply that the fibers are S_2 . Since the X_c are Cartier divisors in X , it is enough to prove that X is S_3 . As noted in [Kol13b, 3.6], X is not S_3 in general; fortunately this is not a problem for us. If $g \in C$ is the generic point, then a local ring of X_g is also a local ring of X , hence X_g is S_2 if X is S_2 . Therefore (X_g, Δ_g) is semi-log-canonical. If $c \in C$ is a closed point and $p \in X_c$ has codimension ≥ 2 , then $p \in X$ has codimension ≥ 3 , thus $\text{depth}_p \mathcal{O}_X \geq 3$ by (11.8), hence $\text{depth}_p \mathcal{O}_{X_c} \geq 2$. Thus again X_c is S_2 .

Third, we know that X_c is a Cartier divisor on a demi-normal scheme. A local version of the Enriques-Severi-Zariski lemma, proved in [Gro68, XIII.2.1], says that if D is a Cartier divisor on an S_2 scheme and $p \in D$ has codimension ≥ 2 then $\hat{D}_p \setminus \{p\}$ is connected, where \hat{D}_p denotes the completion of D at p . Thus X_c has this local connectedness property.

Furthermore, X_c is the union of log canonical centers of $(X, X_c + \Delta)$. Therefore, X_c is seminormal by (11.25.2). These two observations together imply that X_c is S_2 , hence demi-normal. \square

2.5 (When is $K_{X/C} + \Delta$ automatically Cartier or \mathbb{R} -Cartier?). In (2.4.2–3) we make a fiber-wise assumption (that (X_c, Δ_c) be slc) and a total space assumption (that $K_{X/C} + \Delta$ be \mathbb{R} -Cartier).

If the latter condition is automatic, then we have a fiber-wise stability criterion. Section 1.4 contains examples of families of surfaces with quotient singularities where $K_{X/C}$ is not \mathbb{R} -Cartier but the situation gets better in dimension ≥ 3 .

We prove in Section 2.8 that if (X_c, Δ_c) is slc and there is a subset $Z \subset X_c$ such that $K_{X/C} + \Delta$ is \mathbb{R} -Cartier on $X \setminus Z$ and $\dim Z \leq \dim X_c - 3$, then $K_{X/C} + \Delta$ is \mathbb{R} -Cartier everywhere.

Also, prove in (2.93) that if $K_{X/S} + \Delta$ is \mathbb{Q} -Cartier, then $m(K_{X/S} + \Delta)$ is Cartier at a point $p \in X_s$ iff $m(K_s + \Delta_s)$ is Cartier at p .

2.6 (The relative dualizing sheaf I). Let $f: (X, \Delta) \rightarrow C$ be locally stable. The relative dualizing sheaf $\omega_{X/C} \cong \mathcal{O}_C(K_{X/C})$ exists. (Since ω_C is locally free, we can define it as $\omega_{X/C} := \omega_X \otimes f^*\omega_C^{-1}$. A more conceptual construction will be given in (2.69).)

By (11.16), for $c \in C$ there is a Poincaré residue (or adjunction) map (11.16)

$$\mathcal{R}: \omega_{X/C}|_{X_c} \rightarrow \omega_{X_c}. \quad (2.6.1)$$

The map exists for any flat morphism $f: X \rightarrow C$ and general duality theory implies that it is an isomorphism if the fibers are CM. It is, however, not an isomorphism in general but we prove in (2.68) that, for locally stable morphisms, the adjunction map is an isomorphism. Thus $\omega_{X/C}$ can be thought of as a flat family of the dualizing sheaves of the fibers.

The isomorphism in (2.6.1) is easy to prove if the fibers are dlt or if $K_{X/C}$ is \mathbb{Q} -Cartier (2.77.2). A proof for slc fibers, following [Kol11a] and [Kol13b, 7.22], follows directly from (11.8).

The general case, when C is replaced by an arbitrary base scheme, is quite subtle. The known proofs use the Du Bois property of X_c . The projective case was proved in [KK10] and the quasi-projective one in [KK20]. We discuss these in Section 2.5.

It is also worth noting that the powers of the Poincaré residue map

$$\mathcal{R}^m: \omega_{X/C}^{[m]}|_{X_c} \rightarrow \omega_{X_c}^{[m]} \quad (2.6.2)$$

are isomorphisms for locally stable maps if $\Delta = 0$, but not in general; see (2.77.2) and (2.42).

Note that if ω_{X_c} is locally free then (2.6.1) implies that $\omega_{X/C}$ is also locally free along X_c . Thus (2.68) and (2.4) imply the following.

COROLLARY 2.7 (Deformations if K_{X_c} is Cartier). *Let $f: X \rightarrow C$ be a flat morphism of finite type over a field of characteristic 0 such that X_c is slc and ω_{X_c} is locally free for some $c \in C$. Then $\omega_{X/C}$ is locally free near X_c and f is locally stable near X_c . \square*

Note that (2.7) is a special property of slc varieties. Analogous claims fail both for normal varieties (2.43) and for pairs (X, D) . To see the latter, consider a flat family X_c of smooth quadrics in \mathbb{P}^3 becoming a quadric cone for $c = 0$. Let $D_c \subset X_c$ be two disjoint lines that degenerate to a pair of distinct lines on X_0 . Then K_{X_c}, D_c are both Cartier divisors for every c , but on the total space X they give a divisor $K_X + D$ that is not even \mathbb{Q} -Cartier.

If X_c is canonical then K_{X_c} is Cartier in codimension 2. We can thus use (2.7) in codimension 2 and then (2.5) in higher codimensions obtain the next result.

COROLLARY 2.8 (Deformations if X_c is canonical). *Let $f: X \rightarrow C$ be a flat morphism of finite type over a field of characteristic 0 such that X_c is canonical for some $c \in C$. Then K_X is \mathbb{Q} -Cartier and hence f is locally stable near X_c . \square*

Permanence properties.

We start with the invariance of local stability under quasi-étale morphisms.

LEMMA 2.9. *Let C be a smooth curve over a field of characteristic zero and $f: (X, \Delta) \rightarrow C$ a family of pairs over C . Let $\pi: Y \rightarrow X$ be quasi-étale where Y is demi-normal.*

If f is locally stable then so is $\pi \circ f$, and the converse also holds if π is surjective.

Proof. This follows directly from (2.3) and (11.13.3).

Note that $\pi_c: Y_c \rightarrow X_c$ need not be quasi-étale. However, codimension 1 ramification can occur only along the singular locus of X_c . A typical example is

$$\mathbb{A}_{xy}^2 \xrightarrow{\pi} \mathbb{A}^2/\frac{1}{n}(1, -1) \xrightarrow{f} \mathbb{A}^1,$$

where $\pi \circ f(x, y) = xy$. \square

Next we consider base changes $C' \rightarrow C$. (See (2.15) for some results in positive characteristic.)

PROPOSITION 2.10. *Let C be a smooth curve over a field of characteristic zero and $g: C' \rightarrow C$ a quasi-finite morphism. If $f: (X, \Delta) \rightarrow C$ is locally stable then so is the pull-back*

$$g^*f: (X', \Delta') := (X \times_C C', \Delta \times_C C') \rightarrow C'.$$

Proof. We may assume that $g: (c', C') \rightarrow (c, C)$ is a finite, local morphism, étale away from c' . Set $D := X_c$ and $D' := X'_{c'}$. By (11.13.5), $(X, D + \Delta)$ is lc iff $(X', D' + \Delta')$ is. The rest follows from (2.4). \square

The following result shows that one can usually reduce questions about locally stable families to the special case when X is normal.

PROPOSITION 2.11. *Let C be a smooth curve over a field of characteristic zero and $f: (X, \Delta) \rightarrow C$ a family of pairs over C . Assume that X is demi-normal and let $\pi: \bar{X} \rightarrow X$ denote the normalization with conductor $\bar{D} \subset \bar{X}$ (11.10).*

(2.11.1) *If $f: (X, \Delta) \rightarrow C$ is locally stable then so is $f \circ \pi: (\bar{X}, \bar{D} + \bar{\Delta}) \rightarrow C$.*

(2.11.2) *If $K_X + \Delta$ is \mathbb{R} -Cartier and $f \circ \pi: (\bar{X}, \bar{D} + \bar{\Delta}) \rightarrow C$ is locally stable then so is $f: (X, \Delta) \rightarrow C$.*

Proof. Fix a closed point $c \in C$. By (11.21) or [Kol13b, 5.38], if $K_X + \Delta$ is \mathbb{R} -Cartier, then $(X, X_c + \Delta)$ is slc iff $(\bar{X}, \bar{X}_c + \bar{D} + \bar{\Delta})$ is lc. \square

The next result allows us to pass to hyperplane sections. This is quite useful in proofs that use induction on the dimension. (As with many Bertini-type theorems, the characteristic 0 assumption is essential.)

PROPOSITION 2.12 (Bertini theorem for local stability). *Let $f: (X, \Delta) \rightarrow C$ be locally stable and $H \in |H|$ a general divisor in a basepoint-free linear system on X . Then the following morphisms are also locally stable.*

(2.12.1) $f: (X, H + \Delta) \rightarrow C$,

(2.12.2) $f|_H: (H, \Delta|_H) \rightarrow C$ and

(2.12.3) the composite $f \circ \pi: (Y, \pi^{-1}(\Delta)) \rightarrow C$ where $\pi: Y \rightarrow X$ is a μ_m -cover ramified along H ; see (11.14).

Proof. As we noted in (2.3), we can assume that X is normal. Let $p: Y \rightarrow X$ be a log resolution of (X, Δ) such that

$$p^{-1}(\text{Supp } \Delta) + \text{Ex}(p) + (\text{any fiber of } f \circ p)$$

is an snc divisor [Kol13b, 10.46]. Pick $H \in |H|$ such that $p^{-1}(H) = p_*^{-1}(H)$ and

$$p^{-1}(H) + p^{-1}(\text{Supp } \Delta) + \text{Ex}(p) + (\text{any fiber of } f \circ p)$$

is an snc divisor. Then every exceptional divisor of p has the same discrepancy with respect to $(X, X_c + \Delta)$ and $(X, X_c + H + \Delta)$. Therefore, $(X, X_c + H + \Delta)$ is slc for every $c \in C$. Thus $f: (X, H + \Delta) \rightarrow C$ is locally stable, proving (1). By adjunction, this implies that $(H, H_c + \Delta|_H)$ is slc for every $c \in C$, proving (2). By (11.13),

$$(Y, Y_c + \pi^{-1}(\Delta)) \text{ is slc} \iff (X, X_c + (1 - \frac{1}{m})H + \Delta) \text{ is slc.}$$

The latter holds since even $(X, X_c + H + \Delta)$ is slc for every $c \in C$. \square

2.13 (Inverse Bertini theorem, weak form). Inversion of adjunction (11.20) implies that if $f|_H: (H, \Delta|_H) \rightarrow C$ is locally stable then $f: (X, H + \Delta) \rightarrow S$, and hence also $f: (X, \Delta) \rightarrow S$, are locally stable in a neighborhood of H . A much stronger result will be proved in (5.6).

The following simple result shows that if $f: (X, \Delta) \rightarrow C$ is locally stable, then (X, Δ) behaves as if it were *canonical*, as far as divisors over closed fibers are concerned. In some situations, for instance in (2.48), this is a very useful observation, but at other times the technical problems caused by log canonical centers in the generic fiber are hard to overcome.

PROPOSITION 2.14. *Let $f: (X, \Delta) \rightarrow C$ be a locally stable morphism. Let E be a divisor over X such that $\text{center}_X E \subset X_c$ for some closed point $c \in C$. Then $a(E, X, \Delta) \geq 0$. Therefore every log center of (X, Δ) dominates C . In particular, if the generic fiber is klt (resp. canonical) then (X, Δ) is also klt (resp. canonical).*

Proof. Since $(X, X_c + \Delta)$ is semi-log-canonical, $a(E, X, X_c + \Delta) \geq -1$. Let $\pi: Y \rightarrow X$ be a proper birational morphism such that E is a divisor on Y and let b_E denote the coefficient of E in $\pi^*(X_c)$. Then b_E is an integer and it is positive since $\text{center}_X E \subset X_c$. Thus,

$$a(E, X, \Delta) = a(E, X, X_c + \Delta) + b_E \geq -1 + b_E \geq 0.$$

In particular, none of the log centers of (X, Δ) are contained in X_c . \square

2.15 (Some results in positive characteristic). As we already noted, very few of the previous theorems are known in positive characteristic, but the following partial results are sometimes helpful.

(2.15.1) Let (X, Δ) be a pair and $g: Y \rightarrow X$ a smooth morphism. By [Kol13b, 2.14.2], if (X, Δ) is slc, lc, klt, ... then so is $(Y, g^*\Delta)$.

(2.15.2) As a special case of [Kol13b, 2.14.4] we see that if (X, Δ) is slc then, for every smooth curve C , the trivial family $(X, \Delta) \times C \rightarrow C$ is locally stable.

(2.15.3) The proof of (2.14) works in any characteristic. Applying this to a trivial family will have interesting consequences in (8.47).

(2.15.4) Let (X_i, Δ_i) be two pairs that are slc, lc, klt, \dots . Then their product $(X_1 \times X_2, X_1 \times \Delta_2 + \Delta_1 \times X_2)$ is also slc, lc, klt, \dots . This is a generalization of (2.15.2) and can be proved by the same method as in [Kol13b, 2.14.2], using [Kol13b, 2.22].

(2.15.5) Assume that $f: (X, \Delta) \rightarrow C$ is locally stable and let $g: C' \rightarrow C$ be a tamely ramified morphism. Then the pull-back

$$g^*f: (X \times_C C', \Delta \times_C C') \rightarrow C'$$

is also locally stable. This follows from (11.13.3) as in (2.10); see [Kol13b, 2.42] for details.

(2.15.6) Neither the wildly ramified nor the inseparable case of (2.15.5) is known. By [HZ20], the inseparable case would imply the wildly ramified one. The case when all fibers are snc divisors is treated in (2.56).

Other deformations of ω .

The dualizing sheaf plays a very special role in algebraic geometry, thus it is natural to focus on understanding the powers of the relative dualizing sheaf. [LN18] studies other deformations of ω that behave as well as one would expect from locally stable families. The next result, closely related to [LN18, 7.18], says that the relative dualizing sheaf is the ‘best’ deformation of the dualizing sheaf of a fiber.

PROPOSITION 2.16. *Let C be a smooth curve over a field of characteristic 0 and $f: X \rightarrow C$ a flat morphism. Assume that X_0 is slc and there is a rank 1, reflexive sheaf L on X and a restriction morphism $\mathfrak{R}_L: L|_{X_0} \rightarrow \omega_{X_0}$ such that its reflexive powers*

$$\mathfrak{R}_L^{[m]}: L^{[m]}|_{X_0} \rightarrow \omega_{X_0}^{[m]} \quad (2.16.1)$$

are isomorphisms for every m . Then

$$\mathfrak{R}^{[m]}: \omega_{X/C}^{[m]}|_{X_0} \rightarrow \omega_{X_0}^{[m]} \quad (2.16.2)$$

is an isomorphism for every m .

Proof. We prove that $\omega_{X/C} \hat{\otimes} L^{-1}$ is a line bundle, where $\hat{\otimes}$ denotes the double dual of the usual tensor product. If this holds then (2.16.2) is obtained from (2.16.1) by tensoring with $\omega_{X/C} \hat{\otimes} L^{-1}$ and we are done.

Let n be the smallest positive integer such that $\omega_{X_0}^{[n]}$ is locally free. By assumption, then $L^{[n]}$ is also locally free. Being a line bundle is a local property, we may thus assume that X is local, hence $L^{[n]}$ is free. By (11.14) we can take a cyclic cover $\pi: Y \rightarrow X$ such that $\pi_*\mathcal{O}_Y = \sum_{i=0}^{n-1} L^{[-i]}$ and

$$\pi_*\omega_{Y/C} \cong \mathcal{H}om_X(\pi_*\mathcal{O}_Y, \omega_{X/C}) = \sum_{i=0}^{n-1} L^{[i]} \hat{\otimes} \omega_{X/C}.$$

The resulting $g: Y \rightarrow C$ is flat, Y_0 is slc by (11.13.3) and ω_{Y_0} is locally free. Therefore $\omega_{Y/C}$ is locally free by (2.7), hence free since Y is semilocal. Thus $\pi_*\omega_{Y/C} \cong \pi_*\mathcal{O}_Y$ and so one of the summands $L^{[i]} \hat{\otimes} \omega_{X/C}$ is free. Restriction to X_0 tells us that in fact $i = n - 1$. Next note that

$$\begin{aligned} \omega_{X/C} &\cong \omega_{X/C} \hat{\otimes} L^{[n-1]} \hat{\otimes} L \hat{\otimes} L^{[-n]} \\ &\cong L \hat{\otimes} (\omega_{X/C} \hat{\otimes} L^{[n-1]}) \hat{\otimes} L^{[-n]} \\ &\cong L \otimes \mathcal{O}_X \otimes L^{[-n]} \cong L, \end{aligned}$$

where in the last line we changed to the usual tensor product since the tensor product of a reflexive sheaf and of a line bundle is reflexive. Thus (2.16.2) is obtained from (2.16.1) by tensoring with a line bundle. \square

2.2. Locally stable families of surfaces

In this section we develop a rather complete local picture of slc families of surfaces. That is, we start with a pointed, local slc pair $(x \in X_0, \Delta_0)$ and aim to describe all locally stable deformations over local schemes $0 \in S$

$$\begin{array}{ccc} (X_0, \Delta_0) & \subset & (X_S, \Delta_S) \\ \downarrow & & \downarrow \\ 0 & \in & S. \end{array}$$

In the study of singularities it is natural to work étale-locally. That is, two pointed schemes $(x_1 \in X_1)$ and $(x_2 \in X_2)$ are considered the same if there is a third pointed scheme $(x_3 \in X_3)$ and an elementary étale morphisms of pointed schemes

$$(x_1 \in X_1) \xleftarrow{\pi_1} (x_3 \in X_3) \xrightarrow{\pi_2} (x_2 \in X_2),$$

where an étale morphism is called *elementary étale* if the induced maps on the residue fields $\pi_i^*: k(x_i) \rightarrow k(x_3)$ are isomorphisms; see [Sta15, Tag 02LD]. (This notion is also called strictly étale and strongly étale in the literature.) We will mostly work over algebraically closed fields and then being elementary is automatic.

Since we have not yet defined the notion of a locally stable family in general, we concentrate on the case when S is the spectrum of a DVR.

We start by recalling the classification of lc surface singularities. This has a long history, starting with [DV34]. For simplicity we work over an algebraically closed field. It turns out that lc surface singularities have a very clear description using their dual graphs and this is independent of the characteristic. (By contrast, the equations of the singularities depend on the characteristic.)

DEFINITION 2.17 (Dual graph). Let $(0 \in S)$ be a normal surface singularity over an algebraically closed field and $f: S' \rightarrow S$ the minimal resolution with irreducible exceptional curves $\{C_i\}$. We associate to this a *dual graph* $\Gamma = \Gamma(0 \in S)$ whose vertices correspond to the C_i . We use the negative of the self-intersection number $(C_i \cdot C_i)$ to represent a vertex and connect two vertices C_i, C_j by r edges iff $(C_i \cdot C_j) = r$. In the lc cases, the C_i are almost always smooth rational curves and $(C_i \cdot C_j) \leq 1$, so we get a very transparent picture.

The *intersection matrix* of the resolution is $(-(C_i \cdot C_j))$. This matrix is positive definite (essentially by the Hodge index theorem). Its determinant is denoted by

$$\det(\Gamma) := \det(-(C_i \cdot C_j)).$$

For example, if $\Gamma = \{ 2 \text{ --- } 2 \text{ --- } 2 \}$ then

$$\det(\Gamma) = \det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = 4.$$

Let B be a curve on S and B_i the local analytic branches of B that pass through $0 \in S$. The *extended dual graph* (Γ, B) has an additional vertex for each B_i , represented by \bullet , and it is connected to C_j by r edges if $(f_*^{-1} B_i \cdot C_j) = r$.

Next we list the dual graphs of all lc pairs $(0 \in S, B)$, starting with the terminal and canonical ones. For proofs see [Ale93] or [Koll13b, Sec.3.3].

2.18 (List of canonical surface singularities I). Here $(0 \in S)$ is a normal surface singularity over an algebraically closed field and $B \subset S$ a curve (with coefficient 1).

CASE 1 (Terminal). $(0 \in S, B)$ is terminal iff $B = \emptyset$ and S is smooth at 0.

CASE 2 (Canonical). $(0 \in S, B)$ is canonical iff either B and S are both smooth at 0 or $B = \emptyset$ and Γ is one of the following. The corresponding singularities are called *Du Val* singularities or *rational double points* or *simple* surface singularities. See [Dur79] for more information. The equations below are correct only in characteristic zero; see [Art77] for the general case.

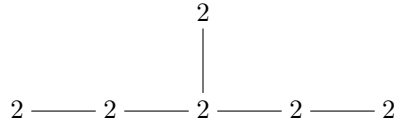
A_n : $x^2 + y^2 + z^{n+1} = 0$, with $n \geq 1$ curves in the dual graph:



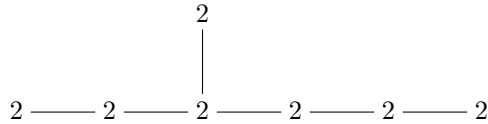
D_n : $x^2 + y^2z + z^{n-1} = 0$, with $n \geq 4$ curves in the dual graph:



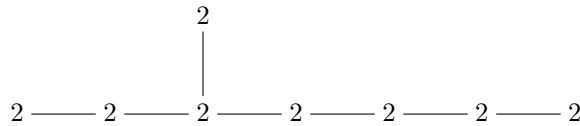
E_6 : $x^2 + y^3 + z^4 = 0$, with 6 curves in the dual graph:



E_7 : $x^2 + y^3 + yz^3 = 0$, with 7 curves in the dual graph:



E_8 : $x^2 + y^3 + z^5 = 0$, with 8 curves in the dual graph:



Before moving to the plt cases, we need more terminology.

DEFINITION 2.19. A connected graph is a *twig* if all vertices have ≤ 2 edges. Thus such a graph is of the form



A connected graph is a *tree with 1 fork* if there is a vertex (called the root) with 3 edges and all other vertices have ≤ 2 edges. Thus such a dual graph is of the form

$$\begin{array}{c} \Gamma_1 \text{ --- } c_0 \text{ --- } \Gamma_2 \\ | \\ \Gamma_3 \end{array}$$

where each Γ_i is a twig joined to c_0 at an end vertex. We will mainly be interested in the cases when $\det(\Gamma) \in \{2, 3, 4, 5, 6\}$. These are

$$\begin{aligned} \det(\Gamma) = 2 &\Leftrightarrow \Gamma \text{ is } 2 \\ \det(\Gamma) = 3 &\Leftrightarrow \Gamma \text{ is } 3 \text{ or } 2 - 2, \\ \det(\Gamma) = 4 &\Leftrightarrow \Gamma \text{ is } 4 \text{ or } 2 - 2 - 2, \\ \det(\Gamma) = 5 &\Leftrightarrow \Gamma \text{ is } 5 \text{ or } 2 - 2 - 2 - 2 \text{ or } 2 - 3 \text{ or } 3 - 2, \\ \det(\Gamma) = 6 &\Leftrightarrow \Gamma \text{ is } 6 \text{ or } 2 - 2 - 2 - 2 - 2. \end{aligned}$$

2.20 (List of log canonical surface singularities II).

CASE 3 (Purely log terminal). The names below reflect that, at least in characteristic 0, these singularities are obtained as the quotient of \mathbb{C}^2 by the indicated type of group. See [Bri68a] for details.

Subcase 3.1 (Cyclic quotient). B is smooth at 0 (or empty) and (Γ, B) is

$$\bullet \text{ --- } c_1 \text{ --- } \cdots \text{ --- } c_n \text{ or } c_1 \text{ --- } \cdots \text{ --- } c_n$$

Subcase 3.2 (Dihedral quotient).

$$\begin{array}{c} \\ \\ \\ \\ c_1 \text{ --- } \cdots \text{ --- } c_n \begin{array}{l} / \\ \backslash \end{array} \\ \\ \end{array}$$

Subcase 3.3 (Other quotient). The dual graph is a tree with 1 fork (2.19) with 3 possibilities for $(\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3))$:

(Tetrahedral) (2,3,3)

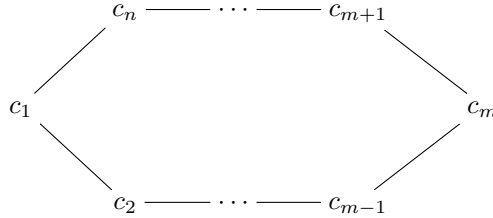
(Octahedral) (2,3,4)

(Icosahedral) (2,3,5).

CASE 4 (Log canonical with $B = 0$).

Subcase 4.1 (Simple elliptic). There is a unique exceptional curve E , it is smooth and of genus 1. If the self-intersection $r := -(E^2)$ is ≥ 3 then the singularity is isomorphic to the cone over the elliptic normal curve $E \subset \mathbb{P}^{r-1}$ of degree r .

Subcase 4.2 (Cusp). The dual graph is a circle of smooth rational curves



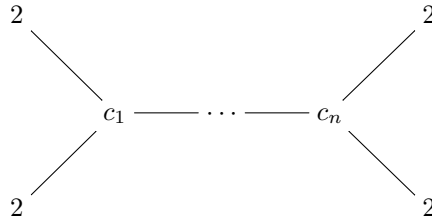
The cases $n = 1, 2$ are exceptional. For $n = 2$ we have 2 smooth rational curves meeting at 2 points and for $n = 1$ the unique exceptional curve is a rational curve with a single node. We can draw the dual graphs as

$$c_1 \equiv c_2 \quad \text{and} \quad \bigcirc c_1.$$

For example the dual graphs of the three singularities $(z(xy - z^2) = x^4 + y^4)$, $(z^2 = x^2(x + y^2) + y^7)$ and $(z^2 = x^2(x^2 + y^2) + y^5)$ are

$$3 \equiv 4, \quad \bigcirc 1 \quad \text{and} \quad \bigcirc 2.$$

Subcase 4.3 ($\mathbb{Z}/2$ -quotient of a cusp).



(For $n = 1$ it is a $\mathbb{Z}/2$ -quotient of a simple elliptic singularity.)

Subcase 4.4 (Simple elliptic quotient). The dual graph is a tree with 1 fork (2.19) with 3 possibilities for $(\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3))$:

$$(\mathbb{Z}/3\text{-quotient}) (3,3,3)$$

$$(\mathbb{Z}/4\text{-quotient}) (2,4,4)$$

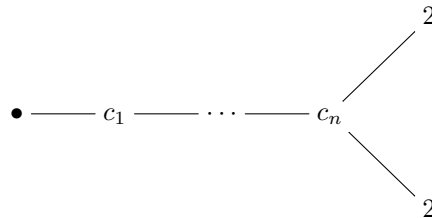
$$(\mathbb{Z}/6\text{-quotient}) (2,3,6).$$

CASE 5 (Log canonical with $B \neq 0$).

Subcase 5.1 (Cyclic). B has 2 smooth branches meeting transversally at 0 and (Γ, B) is



Subcase 5.2 (Dihedral).



2.21 (List of semi-log-canonical surface singularities III). The dual graphs are very similar to the previous ones but there are two possible changes due to the double curve of the surface S passing through the chosen point $0 \in S$.

In the normal case, the local picture represented by an edge is

$$(xy = 0) \subset \mathbb{A}^2, \quad \text{denoted by } \circ \text{ --- } \circ \quad \text{or} \quad \bullet \text{ --- } \circ,$$

where $(y = 0)$ is an exceptional curve and $(x = 0)$ is either an exceptional curve or a component of B . We can now have a non-normal variant

$$(xy = z = 0) \subset (xy = 0) \subset \mathbb{A}^3, \quad \text{denoted by } \circ \xrightarrow{\mathbf{d}} \circ \quad \text{or} \quad \bullet \xrightarrow{\mathbf{d}} \circ,$$

where $(x = z = 0)$ and $(y = z = 0)$ are the curves \circ or \bullet , and $(x = y = 0)$ the double curve of the surface.

The local picture represented by $\bullet \text{ --- } \circ$ also has a non-normal variant where (as long as $\text{char} \neq 2$) we create a pinch point by identifying the points $(0, y) \leftrightarrow (0, -y)$. The local equation is

$$(xy = z = 0) \subset (z^2 = xy^2) \subset \mathbb{A}^3, \quad \text{denoted by } \mathbf{p} \text{ --- } \circ,$$

where $(y = z = 0)$ is the double curve of the surface and $(x = z = 0)$ an exceptional curve.

CASE 6 (Semi-plt).

Subcase 6.1 (Higher pinch points). These are obtained from the cyclic dual graph of (2.20.Case.3.1) by replacing $\bullet \text{ --- } \circ$ by $\mathbf{p} \text{ --- } \circ$.

The simplest one is the pinch point, whose dual graph is $\mathbf{p} \text{ --- } 1$. The equation of the pinch point is $(x^2 = zy^2)$; it is its own semi-resolution.

As another example, start with the A_n singularity $(xy = z^{n+1})$ and pinch it along the line $(x = z = 0)$. The dual graph is

$$\mathbf{p} \text{ --- } 2 \text{ --- } \cdots \text{ --- } 2$$

with 2 occurring n -times. As a subring of $k[x, y, z]/(xy - z^{n+1})$ the coordinate ring is generated by (x, z, y^2, xy, yz) but $xy = z^{n+1}$. Thus $u_1 = x, u_2 = z, u_3 = y^2, u_4 = yz$ gives an embedding into \mathbb{A}^4 . The image is a triple point whose equations can be written as

$$\text{rank} \begin{pmatrix} u_2^n & u_4 & u_3 \\ u_1 & u_2^2 & u_4 \end{pmatrix} \leq 1.$$

Subcase 6.2. The dual graph is

$$\Gamma_1 \xrightarrow{\mathbf{d}} \Gamma_2$$

where the Γ_i are twigs such that $\det(\Gamma_1) = \det(\Gamma_2)$. Note that here we allow $\Gamma_i = \{1\}$ and $1 \xrightarrow{\mathbf{d}} 1$ corresponds to $(xy = 0) \subset \mathbb{A}^3$. Similarly $2 \xrightarrow{\mathbf{d}} 2$ corresponds to

$$(x_1y - z_1^2 = x_2 = z_2 = 0) \cup (x_2y - z_2^2 = x_1 = z_1 = 0) \subset \mathbb{A}^5.$$

(It is a good exercise to check that if $\det(\Gamma_1) \neq \det(\Gamma_2)$ then the canonical class of the resulting surface is not \mathbb{Q} -Cartier. The case $2 \xrightarrow{\mathbf{d}} 1$ is easy to compute by hand. The key in general is to compute the different on the double curve; see [Koll13b, 5.18] for details. This is one of the special cases of (11.21).

CASE 7 (Slc and $K_S + B$ Cartier).

Subcase 7.1 (Degenerate cusp). Here $B = 0$ and these are obtained from the dual graph of a cusp (2.20.Case.4.2) by replacing some of the edges $\circ - \circ$ with $\circ \xrightarrow{\mathbf{d}} \circ$.

The cases $n = 1, 2$ are again exceptional. For $n = 2$ we can replace either of the edges $\circ - \circ$ with $\circ \xrightarrow{\mathbf{d}} \circ$. For example, $(z^2 = x^2y^2)$ and $(z^2 = x^2y^2 + y^5)$ correspond to the dual graphs

$$1 \xrightarrow{\mathbf{d}} 1 \quad \text{and} \quad 2 \xrightarrow{\mathbf{d}} 2.$$

For $n = 1$ the unique exceptional curve is a rational curve with a single node. We can think of the dual graph as

$$\mathbf{d} \bigcirc c_1.$$

For example the singularities $(z^2 = x^2(x + y^2))$ and $(z^2 = x^2(x^2 + y^2))$ give the dual graphs

$$\mathbf{d} \bigcirc 1 \quad \text{and} \quad \mathbf{d} \bigcirc 2.$$

Subcase 7.2. These are obtained from the cyclic dual graph of (2.20.Case.5.1) by replacing some of the edges $\circ - \circ$ with $\circ \xrightarrow{\mathbf{d}} \circ$.

CASE 8 (Slc and $2(K_S + B)$ Cartier).

Subcase 8.1. Here $B = 0$ and these are obtained from the dual graph of a $\mathbb{Z}/2$ -quotient of a cusp (2.20.Case.4.3) by replacing some of the horizontal edges $\circ - \circ$ with $\circ \xrightarrow{\mathbf{d}} \circ$.

Subcase 8.2. These are obtained from the cyclic dual graph of (2.20.Case.5.1) by replacing at least one of $\bullet - \circ$ by $\mathbf{p} - \circ$ and replacing some of the edges $\circ - \circ$ with $\circ \xrightarrow{\mathbf{d}} \circ$.

Subcase 8.3. These are obtained from the dihedral dual graph of (2.20.Case.5.2) by replacing $\bullet - \circ$ by $\mathbf{p} - \circ$ and replacing some of the horizontal edges $\circ - \circ$ with $\circ \xrightarrow{\mathbf{d}} \circ$.

This completes the list of all slc surface singularities and now we turn to describing their locally stable deformations. An slc surface can be singular along a curve and the transversal hyperplane sections are nodes. Deformations of nodes are described in (11.9).

The situation is much more complicated for surfaces, so we start with the case $\Delta_0 = 0$. It would be natural to first try to understand all flat deformations of $(x \in X_0)$ and then decide which of these are locally stable. However, in many interesting cases, flat deformations are rather complicated, but a good description of all locally stable deformations can be obtained by relating them to locally stable deformations of certain cyclic covers of X (11.14).

PROPOSITION 2.22. *Let k be a field and (X, D) a local, slc scheme over k with D reduced. Assume that $\omega_X^{[m]}(mD) \cong \mathcal{O}_X$ for some $m \geq 1$ that is not divisible by $\text{char } k$ and let $\pi: (\tilde{X}, \tilde{D}) \rightarrow (X, D)$ be a corresponding μ_m -cover (11.14). Let R be a complete DVR with residue field k and set $S = \text{Spec } R$.*

Taking μ_m -invariants establishes a bijection between

- (2.22.1) *flat, local, slc morphisms $\tilde{f}: (\tilde{X}_S, \tilde{D}_S) \rightarrow S$ such that $(\tilde{X}_0, \tilde{D}_0) \cong (\tilde{X}, \tilde{D})$ plus a μ_m -action on $(\tilde{X}_S, \tilde{D}_S)$ extending the μ_m -action on (\tilde{X}, \tilde{D}) and*
- (2.22.2) *flat, local, slc morphisms $f: (X_S, D_S) \rightarrow S$ such that $(X_0, D_0) \cong (X, D)$.*

Note that $\omega_{\tilde{X}}(\tilde{D})$ is locally free, and, in many cases, this makes (\tilde{X}, \tilde{D}) much simpler than (X, D) . This reduction step is especially useful when $D = 0$, in which case $\omega_{\tilde{X}}$ is locally free. As we saw in (2.7), then all flat deformations of \tilde{X} are slc. For surfaces, this leads to an almost complete description of all slc deformations.

ASIDE 2.23 (Deformations of quotients). Let \tilde{X} be a scheme and G a finite group acting on it. The proof of (2.22) shows that G -equivariant deformations of \tilde{X} always induce flat deformations of $X := \tilde{X}/G$ provided the characteristic does not divide $|G|$.

The converse is, however, quite subtle and usually deformations of X are not related to any deformation of \tilde{X} . As an example, consider the family $(xy - z^n - tz^m = 0)$ for $m < n$. For $t = 0$ the fiber is isomorphic to $\mathbb{C}^2/\mathbb{Z}_n$ and for $t \neq 0$ the fiber has a singularity (analytically) isomorphic to $\mathbb{C}^2/\mathbb{Z}_m$. There is no relation between the corresponding degree n cover of the central fiber and the (local analytic) degree m cover of a general fiber.

However, if G acts freely outside a subset of codimension ≥ 3 and \tilde{X} is S_3 , then every deformation of X arises from a deformation of \tilde{X} [Kol95a, 12.7].

The following two examples show that the codimension ≥ 3 condition is not enough, not even for μ_m -covers.

1. Let E be an elliptic curve and S a K3 surface with a fixed point free involution τ . Set $Y = E \times S$ and $X = Y/\sigma$ where σ is the involution $(-1, \tau)$. Note that $p: Y \rightarrow X$ is an étale double cover, $h^1(Y, \mathcal{O}_Y) = 1$ and $h^1(X, \mathcal{O}_X) = 0$. Let H_X be a smooth ample divisor on X and H_Y its pull-back to Y . Consider the cones and general projections

$$\begin{array}{ccc} C_a(Y, H_Y) & \xrightarrow{p_c} & C_a(X, H_X) \\ \pi_Y \downarrow & & \downarrow \pi_X \\ \mathbb{A}^1 & = & \mathbb{A}^1 \end{array}$$

Since $h^1(X, \mathcal{O}_X) = 0$, the central fiber of π_X is the cone over H_X by [Kol13b, 3.10]. By contrast, the central fiber F_0 of π_Y is not S_2 since $h^1(Y, \mathcal{O}_Y) \neq 0$, again by [Kol13b, 3.10]. Thus, although the normalization of F_0 is the cone over H_Y , it is not isomorphic to it.

2. Let $g: X \rightarrow B$ be a smooth projective morphism to a smooth curve and H an ample line bundle on X . For large enough m and for every $r \in \mathbb{N}$, the direct images $g_*\mathcal{O}_X(rH)$ commute with base change, hence the cones $C_a(X_b, \mathcal{O}_{X_b}(mH|_{X_b}))$ form a flat family.

The cones $C_a(X_b, \mathcal{O}_{X_b}(H|_{X_b}))$ are μ_m -covers of the cones $C_a(X_b, \mathcal{O}_{X_b}(mH|_{X_b}))$, but they form a flat family only if $g_*\mathcal{O}_X(rH)$ commutes with base change for every r . That is, we get the required examples whenever $H^0(X_b, \mathcal{O}_{X_b}(H|_{X_b}))$ jumps for special values of b . The latter is easy to arrange, even on a family of smooth curves, as long as $\deg H|_{X_b} < 2g - 2$.

2.24 (Proof of (2.22)). Let us start with $f: (X_S, D_S) \rightarrow S$. Since $\omega_{X_S}^{[m]}(mD_S)$ is locally free, the restriction map

$$\omega_{X_S}^{[m]}(mD_S) \rightarrow \omega_{X_0}^{[m]}(mD_0) \cong \mathcal{O}_{X_0}$$

is surjective. Since X_S is affine, the constant 1 section lifts back to a nowhere zero section $s: \mathcal{O}_{X_S} \cong \omega_{X_S}^{[m]}(mD_S)$. Let $\tilde{f}: (\tilde{X}_S, \tilde{D}_S) \rightarrow S$ be the corresponding μ_m -cover (11.14).

\tilde{f} is also locally stable by (2.9). By (2.4), this implies that \tilde{X}_0 is S_2 , hence it agrees with the μ_m -cover of (X_0, D_0) .

To see the converse, let $g: Y \rightarrow S$ be any flat, affine morphism and G a reductive group (or group scheme) acting on Y with quotient $g/G: Y/G \rightarrow S$. Then $(g/G)_* \mathcal{O}_{Y/G} = (g_* \mathcal{O}_Y)^G$ is a direct summand of $g_* \mathcal{O}_Y$, hence g/G is also flat. Taking invariants commutes with base change since G is reductive. This shows that (1) \Rightarrow (2). \square

Assumptions. For the rest of this Section, we work in characteristic 0, though almost everything works in general as long as the characteristic does not divide m in (2.25), but very little has been proved otherwise.

2.25 (Classification plan). We establish an étale-local description of all slc deformations of surface singularities in four steps.

- (2.25.1) Classify all slc surface singularities $(0, \tilde{S})$ with $\omega_{\tilde{S}}$ locally free.
- (2.25.2) Classify all flat deformations of these $(0, \tilde{S})$.
- (2.25.3) Classify all μ_m -actions on these surfaces and decide which ones correspond to our μ_m -covers.
- (2.25.4) Describe the μ_m -actions on the miniversal deformation spaces of these $(0, \tilde{S})$.

The first task was already accomplished in (2.18–2.21); we have Du Val singularities (2.18.Case.2), simple elliptic singularities and cusps (2.20.Cases.4.1–2) and degenerate cusps (2.21.Case.6). We can thus proceed to the next step (2.25.2).

2.26 (Deformations of slc surface singularities with K_S Cartier).

(Du Val singularities.) It is easy to work out the miniversal deformation space from the equations and (2.27). For each of the A_n, D_n, E_n cases the dimension of the miniversal deformation space is exactly n . For instance, for A_n we get

$$\begin{array}{ccccc} (xy + z^{n+1} = 0) & \subset & (xy + z^{n+1} + \sum_{i=0}^{n-1} t_i z^i = 0) & \subset & \mathbb{A}_{xyz}^3 \times \mathbb{A}_t^n \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \in & \mathbb{A}_t^n & = & \mathbb{A}_t^n. \end{array}$$

(Elliptic/cusp/degenerate cusp.) Let $(0 \in S)$ be one of these singularities and C_i the exceptional curves of the minimal (semi)resolution. Set $m = -(\sum C_i)^2$ and write $(0 \in S_m)$ to indicate such a singularity.

- (2.26.1) If $m = 1, 2, 3$ then $(0 \in S_m)$ is (isomorphic to) a singular point on a surface in \mathbb{A}^3 [Sai74, Lau77]. Their deformations are completely described by (2.27).
- (2.26.2) If $m = 4$ then $(0 \in S_4)$ is (isomorphic to) a singular point on a surface in \mathbb{A}^4 that is a complete intersection of 2 hypersurfaces. The miniversal deformation space of a complete intersection can be described in a manner similar to (2.27); see [Art76, Loo84, Har10].
- (2.26.3) If $m = 5$ then the deformations are completely described by the method of [BE77]; see [Har10, Sec.9].
- (2.26.4) If $m \geq 3$ and $(0 \in S_m)$ is simple elliptic, then it is (isomorphic to) the singular point of a projective cone $\tilde{S}_m \subset \mathbb{P}^m$ over an elliptic normal curve $E_m \subset$

\mathbb{P}^{m-1} . By [Pin74, Sec.9], every deformation of $(0 \in S_m)$ is the restriction of a deformation of $\bar{S}_m \subset \mathbb{P}^m$. In particular, any smoothing corresponds to a smooth surface of degree m in \mathbb{P}^m . The latter have been fully understood classically: these are the del Pezzo surfaces embedded by $|-K|$. In particular, a simple elliptic singularity $(0 \in S_m)$ is smoothable only for $m \leq 9$ [Pin74, Sec.9].

(2.26.5) The $m = 9$ case is especially interesting. Given an elliptic curve E , a degree 9 embedding $E_9 \hookrightarrow \mathbb{P}^8$ is given by global sections of a line bundle L_9 of degree 9 on E . Embeddings of E into \mathbb{P}^2 are given by line bundles L_3 of degree 3. If we take $(E \hookrightarrow \mathbb{P}^2)$ given by L_3 and then embed \mathbb{P}^2 into \mathbb{P}^9 by $\mathcal{O}_{\mathbb{P}^2}(3)$, then E is mapped to E_9 iff $L_3^{\otimes 3} \cong L_9$. For a fixed L_9 this gives 9 choices of L_3 . Thus a given $E_9 \hookrightarrow \mathbb{P}^8$ is a hyperplane section of a $\mathbb{P}^2 \hookrightarrow \mathbb{P}^9$ in 9 different ways. Correspondingly, the deformation space $(0 \in S_9)$ has 9 smoothing components. (This was overlooked in [Pin74, Sec.9].) The automorphism group of $(0 \in S_9)$ permutes these 9 components. See [LW86, Sec.6] for another description.

(2.26.6) For $m \geq 6$ the deformation theory of cusps is much harder. A full description is given in [GHK15].

(2.26.7) Degenerate cusps are all smoothable [Ste98].

2.27 (Deformations of hypersurface singularities). For general references, see [Art76, Loo84].

Let $0 \in X \subset \mathbb{A}_{\mathbf{x}}^n$ be a hypersurface singularity defined by an equation $(f(\mathbf{x}) = 0)$. Choose polynomials p_i that give a basis of

$$k[[x_1, \dots, x_n]] / \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right). \quad (2.27.1)$$

If $(0 \in X)$ is an isolated singularity, then the quotient has finite length, say N . In this case, the miniversal deformation of $(0 \in X)$ is given by

$$\begin{array}{ccccc} X & \subset & (f(\mathbf{x}) + \sum_i t_i p_i(\mathbf{x}) = 0) & \subset & \mathbb{A}_{\mathbf{x}}^n \times \mathbb{A}_{\mathbf{t}}^N \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \in & \mathbb{A}_{\mathbf{t}}^N & = & \mathbb{A}_{\mathbf{t}}^N. \end{array}$$

In particular, the miniversal deformation space $\text{Def}(X)$ is smooth.

If the quotient in (2.27.1) has infinite length, then it is best to think of the resulting infinite dimensional deformation space as an inverse system of deformations over Artin rings whose embedding dimension goes to infinity.

The next step (2.25.3) in the classification is to describe all μ_m -actions, but it is more transparent to consider reductive commutative groups. These are of the form $G \times \mathbb{G}_m^r$ where G is a finite, commutative group and $\mathbb{G}_m = \text{GL}(1)$ the multiplicative group of scalars, cf. [Hum75, Sec.16].

2.28 (Commutative groups acting on Du Val singularities).

The action of a reductive commutative group on \mathbb{A}^n can be diagonalized. Thus let $S \subset \mathbb{A}^3$ be a Du Val singularity which is invariant under a diagonal group action on \mathbb{A}^3 . It is easy to work through any one of the standard classification methods (for instance, the one in [KM98, 4.24]) to obtain the following normal forms. In each case we describe first the maximal connected group actions and then the maximal non-connected group actions.

(Main series: \mathbb{G}_m -actions)

A_n : $(xy + z^{n+1} = 0)$ and \mathbb{G}_m^2 acts with character $(1, -1, 0), (0, n+1, 1)$.

D_n : $(x^2 + y^2z + z^{n-1} = 0)$ and \mathbb{G}_m acts with character $(n-1, n-2, 2)$.

E_6 : $(x^2 + y^3 + z^4 = 0)$ and \mathbb{G}_m acts with character $(6, 4, 3)$.

E_7 : $(x^2 + y^3 + yz^3 = 0)$ and \mathbb{G}_m acts with character $(9, 6, 4)$.

E_8 : $(x^2 + y^3 + z^5 = 0)$ and \mathbb{G}_m acts with character $(15, 10, 6)$.

(Twisted versions: $\mu_r \times \mathbb{G}_m$ -actions)

A_n : $(x^2 + y^2 + z^{n+1} = 0)$. If $n+1$ is odd then \mathbb{G}_m acts with character $(n+1, n+1, 2)$ and μ_2 acts with character $(0, 1, 0)$. If $n+1$ is even then \mathbb{G}_m acts with character $(\frac{n+1}{2}, \frac{n+1}{2}, 1)$ and μ_2 acts with character $(0, 1, 0)$.

D_n : $(x^2 + y^2z + z^{n-1} = 0)$, \mathbb{G}_m acts with character $(n-1, n-2, 2)$ and μ_2 acts with character $(1, 1, 0)$.

D_4 : $(x^2 + y^3 + z^3 = 0)$, \mathbb{G}_m acts with character $(3, 2, 2)$ and μ_3 acts with character $(0, 1, 0)$.

E_6 : $(x^2 + y^3 + z^4 = 0)$ and \mathbb{G}_m acts with character $(6, 4, 3)$ and μ_2 acts with character $(1, 0, 0)$.

EXAMPLE 2.29 (Locally stable deformations of surface quotient singularities).

Let $(0 \in S)$ be a surface quotient singularity with Du Val cover $(0 \in \tilde{S}) \rightarrow (0 \in S)$. By (2.22), the classification of locally stable deformations of all such $(0 \in S)$ is equivalent to classifying all cyclic group actions on Du Val singularities $(0 \in \tilde{S})$ that are free outside the origin and whose action on $\omega_{\tilde{S}} \otimes k(0)$ is faithful. This is straightforward, though somewhat tedious, using (2.28). Alternatively, one can use the classification of finite subgroups of $\mathrm{GL}(2)$ as in [Bri68a].

Thus the miniversal locally stable deformation space, which we denote by $\mathrm{Def}_{\mathrm{qG}}(S)$ (6.64), is the fixed point set of the corresponding cyclic group action on $\mathrm{Def}(\tilde{S})$, hence it is also smooth.

A_n -series: $(xy + z^{n+1} = 0)/\frac{1}{m}(1, (n+1)c-1, c)$ for any m where $((n+1)c-1, m) = 1$. These are equivariantly smoothable only if $m|(n+1)c$.

D_n -series: $(x^2 + y^2z + z^{n-1} = 0)/\frac{1}{2k+1}(n-1, n-2, 2)$ where $(2k+1, n-2) = 1$. These are not equivariantly smoothable, but, for instance, if $2k+1|n-1$, they deform to the quotient singularity $\mathbb{A}^2/\frac{1}{2k+1}(-1, 2)$.

E_6 -series: $(x^2 + y^3 + z^4 = 0)/\frac{1}{m}(6, 4, 3)$ for $(m, 6) = 1$. For $m > 1$ all equivariant deformations are trivial, save for $m = 5$, when there is a 1-parameter family $(x^2 + y^3 + z^4 + \lambda yz = 0)/\frac{1}{5}(1, 4, 3)$.

E_7 -series: $(x^2 + y^3 + yz^3 = 0)/\frac{1}{m}(9, 6, 4)$ for $(m, 6) = 1$. For $m > 1$ all equivariant deformations are trivial, save for $m = 5$ and $m = 7$, when there are 1-parameter families $(x^2 + y^3 + yz^3 + \lambda xz = 0)/\frac{1}{5}(4, 1, 4)$ and $(x^2 + y^3 + yz^3 + \lambda z = 0)/\frac{1}{7}(2, 6, 4)$.

E_8 -series: $(x^2 + y^3 + z^5 = 0)/\frac{1}{m}(15, 10, 6)$ for $(m, 30) = 1$. For $m > 1$ all equivariant deformations are trivial, save for $m = 7$, when there is a 1-parameter family $(x^2 + y^3 + z^5 + \lambda yz = 0)/\frac{1}{7}(1, 3, 6)$.

A_n -twisted: $(x^2 + y^2 + z^{n+1} = 0)/\frac{1}{4m}(n+1, n+1+2m, 2)$ for any $(2m, n+1) = 1$. These are never equivariantly smoothable.

D_4 -twisted: $(x^2 + y^3 + z^3 = 0)/\frac{1}{18k+9}(9k+6, 1, 6k+4)$. All equivariant deformations are trivial.

EXAMPLE 2.30 (Quotients of simple elliptic and cusp singularities).

Since $(xy)^{-1}dx \wedge dy$ is invariant under the group action, it descends to a 2-form on H_n with poles along the curve $D_n := (xy = 0)/\frac{1}{n}(1, 1)$. Thus $K_{H_n} + D_n \sim 0$ and the pair (H_n, D_n) is lc. Our aim is to understand which deformations of H_n extend to a deformation of the pair (H_n, D_n) .

Claim 2.32.1. Fix $n \geq 7$ and let $\pi: X \rightarrow \mathbb{A}^1$ be a general smoothing of H_n . Then the divisor D_n can not be extended to a divisor D_X such that $\pi: (X, D_X) \rightarrow \mathbb{A}^1$ is locally stable.

However, there are special smoothings $\pi: X' \rightarrow \mathbb{A}^1$ for which such a divisor D'_X exists.

Proof. For $m \in \mathbb{N}$, let \mathbb{F}_m denote the ruled surface $\text{Proj}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(-m))$. Let $E_m \subset \mathbb{F}_m$ denote the section with self intersection $-m$ and $F \subset \mathbb{F}_m$ denote a fiber. Note that $K_{\mathbb{F}_m} \sim -(2E_m + (m+2)F)$.

For $a \geq 1$ set $A_{ma} := E + (m+a)F$. Then A_{ma} is very ample with self intersection $n := m + 2a$ and it embeds \mathbb{F}_m into \mathbb{P}^{n+1} as a surface of degree n . Denote the image by S_{ma} . A general hyperplane section of S_{ma} is a rational normal curve $C_n \subset \mathbb{P}^n$. Consider the affine cones $X_{ma} := C_a(S_{ma})$ and $H_n := C_a(C_n)$. We can choose coordinates such that

$$X_{ma} \subset \mathbb{A}_{x_1, \dots, x_{n+2}}^{n+2} \quad \text{and} \quad H_n = (x_{n+2} = 0).$$

The last coordinate projection gives $\pi: X_{ma} \rightarrow \mathbb{A}^1$ which is a flat deformation (in fact a smoothing) of H_n . By [Kol13b, 3.14.5]

$$\begin{aligned} H^0(X_{ma}, \mathcal{O}_{X_{ma}}(-K_{X_{ma}})) &= \sum_{i \in \mathbb{Z}} x_0^i \cdot H^0(S_{ma}, \mathcal{O}_{S_{ma}}(-K_{S_{ma}} + iA_{ma})) \\ &= \sum_{i \in \mathbb{Z}} x_0^i \cdot H^0(S_{ma}, \mathcal{O}_{S_{ma}}((2+i)E_m + (m+2+im+ia)F)). \end{aligned}$$

The lowest degree terms in the sum depend on m and a . For $i < -2$, we get 0. For $i = -2$ we have

$$H^0(S_{ma}, \mathcal{O}_{S_{ma}}((2-m-2a)F)) = H^0(S_{ma}, \mathcal{O}_{S_{ma}}((2-n)F)).$$

This is 0, unless $n = 2$, that is, when X is the quadric cone in \mathbb{A}^3 . Then D_2 is a Cartier divisor H_2 and so every deformation of H_2 extends to a deformation of the pair (H_2, D_2) . Thus assume next that $n \geq 3$.

For $i = -1$ we have the summand

$$H^0(S_{ma}, \mathcal{O}_{S_{ma}}(E_m + (2-a)F)).$$

This is again zero if $a \geq 3$, but for $a = 1$ we get a pencil $|E_m + F|$ (whose members are pairs of intersecting lines) and for $a = 2$ we get a unique member E_m (which is a smooth conic in \mathbb{P}^{n+1}). This shows the following.

Claim 2.32.2. For $a = 1, 2$ and any $m \geq 0$, the anticanonical class of the 3-fold X_{ma} contains a (possibly reducible) quadric cone $D \subset X_{ma}$ and $\pi: (X_{ma}, D) \rightarrow \mathbb{A}^1$ is locally stable. \square

For $a \geq 3$, we have to look at the next term

$$H^0(S_{ma}, \mathcal{O}_{S_{ma}}(2E_m + (m+2)F))$$

for a nonzero section. The corresponding linear system consists of reducible curves of the form $E_m + G_m$ where $G_m \in |E_m + (m+2)F|$. These curves have 2 nodes and arithmetic genus 1. Let $B \subset X_{ma}$ denote the cone over any such curve. Then (X_{ma}, B) is log canonical but $\pi: (X_{ma}, B) \rightarrow \mathbb{A}^1$ is not locally stable since the

restriction of B to H_n consists of $n + 2$ lines through the vertex. Thus we have proved:

Claim 2.32.3. For $a \geq 3$ and any $m \geq 0$, the anticanonical class of X_{ma} does not contain any divisor D for which $\pi: (X_{ma}, D) \rightarrow \mathbb{A}^1$ is locally stable. \square

Note finally that the surfaces S_{ma} with $n = m + 2a$ form an irreducible family. General points correspond to the largest possible value $a = \lfloor (n - 1)/2 \rfloor$. The surfaces with $a \leq 2$ correspond to a closed subset, which is a 2-dimensional subspace of the versal deformation space of H_n .

2.3. Examples of locally stable families

The aim of this section is to investigate, mostly through examples, fibers of locally stable morphisms. If (S, Δ) is slc then, for any smooth curve C , the projection $\pi: (S \times C, \Delta \times C) \rightarrow C$ is locally stable with fiber (S, Δ) . Thus, in general we can only say that fibers of locally stable morphisms are exactly the slc pairs.

The question becomes, however, quite interesting, if we look at special fibers of locally stable morphisms whose general fibers are ‘nice,’ for instance smooth or canonical. The main point is thus to probe the difference between arbitrary slc singularities and those slc singularities that occur on locally stable degenerations of smooth varieties. We focus on two main questions.

QUESTION 2.33. Let $f: X \rightarrow T$ be a locally stable morphism over a pointed curve $(0 \in T)$ such that X_t is smooth for $t \neq 0$.

- (2.33.1) Is X_0 CM?
- (2.33.2) Are the irreducible components of X_0 CM?
- (2.33.3) Is the normalization of X_0 CM?

QUESTION 2.34. Let $f: (X, \Delta) \rightarrow T$ be a locally stable morphism over a pointed curve $(0 \in T)$ such that X_t is smooth and Δ_t is snc for $t \neq 0$.

- (2.34.1) Do the supports of $\{\Delta_t: t \in T\}$ form a flat family of divisors?
- (2.34.2) Are the sheaves $\mathcal{O}_{X_0}(mK_{X_0} + \lfloor m\Delta_0 \rfloor)$ CM?
- (2.34.3) Do the sheaves $\{\mathcal{O}_{X_t}(mK_{X_t} + \lfloor m\Delta_t \rfloor): t \in T\}$ form a flat family?

A normal surface is always CM, and the (local analytic) irreducible components of an slc surface are CM. The latter follows from the classification of slc surfaces given in [Kol13b, Sec.2.2]. Starting with dimension 3, there are lc singularities that are not CM. The simplest examples are cones over Abelian varieties; see (2.35). On the other hand, we noted in (11.5) that canonical and log terminal singularities are CM and rational in characteristic 0.

Let us note next that the answer to (2.33.1) is positive, that is, X_0 is CM. Indeed, X is canonical by (2.14) and hence CM by (11.5). Therefore X_0 is also CM. A more complete answer to (2.33.1), without assuming that X_t is smooth or canonical for $t \neq 0$, is given in (2.67).

For locally stable families of pairs, the boundary provides additional sheaves whose CM properties are important to understand; this motivates (2.34). Unlike for (2.33), the answers to all of these are negative already for surfaces. The first convincing examples were discovered by Hassett (2.40). As a consequence, we see that we can not think of the deformations of (S, Δ) as a flat deformation of S and a flat deformation of Δ that are compatible in certain ways. In general it is

imperative to view (S, Δ) as a single object. See, however, Section 2.7 for many cases where viewing (S, Δ) as a pair does work well.

Our examples will be either locally or globally cones and we need some basic information about them.

2.35 (Cones). Let X be a projective scheme with an ample line bundle L . The *affine cone* over X with conormal bundle L is

$$C_a(X, L) := \text{Spec}_k \sum_{m \geq 0} H^0(X, L^m).$$

Away from the vertex, the cone is locally isomorphic to $X \times \mathbb{A}^1$, but the vertex is usually more complicated. If X is normal then so is $C_a(X, L)$ and its canonical class is Cartier (resp. \mathbb{Q} -Cartier) iff $\mathcal{O}_X(K_X) \sim L^m$ for some $m \in \mathbb{Z}$ (resp. $\mathcal{O}_X(rK_X) \sim L^m$ for some $r, m \in \mathbb{Z}$ with $r \neq 0$).

The following results are quite straightforward, see [Kol13b, Sec.3.1] for details.

Let X be a projective variety with rational singularities over a field of characteristic 0 and L an ample line bundle on X .

- (2.35.1) If $-K_X$ is ample then $C_a(X, L)$ is CM and has rational singularities.
- (2.35.2) If $-K_X$ is nef (for instance, X is Calabi-Yau), then
 - (2.35.2.a) $C_a(X, L)$ is CM iff $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$, and
 - (2.35.2.b) $C_a(X, L)$ has rational singularities iff $H^i(X, \mathcal{O}_X) = 0$ for $0 < i \leq \dim X$.
- Next let (X, Δ) be a projective, slc pair and L an ample Cartier divisor on X . Let $\Delta_{C_a(X, L)}$ denote the \mathbb{R} -divisor corresponding to Δ on $C_a(X, L)$. Assume that $K_X + \Delta \sim_{\mathbb{Q}} r \cdot L$ for some $r \in \mathbb{R}$. Then $(C_a(X, L), \Delta_{C_a(X, L)})$ is
 - (2.35.3) terminal iff $r < -1$ and (X, Δ) is terminal,
 - (2.35.4) canonical iff $r \leq -1$ and (X, Δ) is canonical,
 - (2.35.5) klt iff $r < 0$ (that is, $-(K_X + \Delta)$ is ample) and (X, Δ) is klt,
 - (2.35.6) dlt iff either $r < 0$ and (X, Δ) is dlt or $(X, \Delta) \cong (\mathbb{P}^n, (\prod x_i = 0))$ and the cone is $(\mathbb{A}^{n+1}, (\prod x_i = 0))$.
 - (2.35.7) lc iff $r \leq 0$ (that is, $-(K_X + \Delta)$ is nef) and (X, Δ) is lc,
 - (2.35.8) semi-log-canonical iff $r \leq 0$ and X is semi-log-canonical.

EXAMPLE 2.36 (Counterexample to (2.33.2)). Let $Q_0 \subset \mathbb{P}^4$ be the singular quadric ($xy - uv = 0$). Let $|A|$ and $|B|$ be the two families of planes on Q_0 and H the hyperplane class. Let $S_1 \in |2A + H|$ be a general member. Note that S_1 is smooth away from the vertex of Q_0 and at the vertex it has 2 local analytic components intersecting at a single point. In particular, S_1 is non-normal and non-CM. (The easiest way to see these is to blow up a plane $B_1 \in |B|$. Then $B_{B_1}Q_0 \rightarrow Q_0$ is a small resolution whose exceptional set E is a smooth rational curve. The birational transform of $|2A + H|$ is a very ample linear system whose general member is a smooth surface that intersects E in 2 points. This is the normalization of the surface S_1 .)

Let B_1, B_2 be planes in the other family. Then $X_0 := S_1 + B_1 + B_2 \sim 3H$, thus X_0 is a $(2) \cap (3)$ complete intersection in \mathbb{P}^4 . We can thus write X_0 as the limit of a smooth family of $(2) \cap (3)$ complete intersections X_t . The general X_t is a smooth K3 surface.

On the other hand, X_0 can also be viewed as a general member of a flat family whose special fiber is $A_1 + A_2 + B_1 + B_2 + H$. The latter is slc by (2.35), thus X_0

is also slc. Hence $\{X_t : t \in T\}$ is a locally stable family such that X_t is a smooth K3 surface for $t \neq 0$. Moreover, the irreducible component $S_1 \subset X_0$ is not CM.

In this case, the source of the problem is easy to explain. At its singular point, S_1 is analytically reducible. The local analytic branches of S_1 and the normalization of S_1 are both smooth.

One can, however, modify this example to get analytically irreducible non-CM examples, albeit in dimension 3. To see this, let

$$Y_0 := C(X_0) = C(S_1) + C(B_1) + C(B_2) \subset \mathbb{P}^5$$

be the cone over X_0 . It is still a $(2) \cap (3)$ complete intersection, thus we can write Y_0 as the limit of a smooth family of $(2) \cap (3)$ complete intersections Y_t . The general Y_t is a smooth Fano 3-fold.

By (2.35), Y_0 is slc, thus $\{Y_t : t \in T\}$ is a stable family such that Y_t is a smooth 3-fold for $t \neq 0$. Since S_1 is irreducible, the cone $C(S_1)$ is analytically irreducible at its vertex. It is non-normal along a line and non-CM.

One can check that the normalization of $C(S_1)$ is CM.

EXAMPLE 2.37 (Counterexample to (2.33.3)). As in (2.36), let $Q_0 \subset \mathbb{P}^4$ be the singular quadric ($xy - uv = 0$). On it, take a divisor

$$D_0 := A_1 + A_2 + \frac{1}{2}(B_1 + \cdots + B_4) + \frac{1}{2}H_4$$

where the A_i are planes in one family, the B_i are planes in the other family and H_4 is a general quartic section.

Note that (Q_0, D_0) is lc (2.35) and $2D_0$ is an octic section of Q_0 . We can thus write (Q_0, D_0) as the limit of a family (Q_t, D_t) where Q_t is a smooth quadric and $2D_t$ a smooth octic hypersurface section of Q_t .

Let us now take the double covers of Q_t ramified along $2D_t$ (11.14). We get a family of $(2) \cap (8)$ complete intersections $X_t \subset \mathbb{P}(1^5, 4)$. The general X_t is smooth with ample canonical class. The special fiber is irreducible, slc, but not normal along $A_1 + A_2$, which is the union of 2 planes meeting at a point.

Let $\pi: \bar{X}_0 \rightarrow Q_0$ denote the projection of the normalization of X_0 . Then

$$\pi_* \mathcal{O}_{\bar{X}_0} = \mathcal{O}_{Q_0} + \mathcal{O}_{Q_0}(4H - A_1 - A_2).$$

It is easy to compute that $\mathcal{O}_{Q_0}(4H - A_1 - A_2)$ is not CM (see, for instance, [Kol13b, 3.15]), so we conclude that \bar{X}_0 is not CM.

It is also interesting to note that the preimage of $A_1 + A_2$ in \bar{X}_0 is the union of 2 elliptic cones meeting at their common vertex. These are quite complicated lc centers.

EXAMPLE 2.38 (Counterexample to (2.33.2–3)). Here is an example of a locally stable family of smooth projective varieties $\{Y_t : t \in T\}$ such that

(2.38.1) the canonical class K_{Y_t} is ample and Cartier for every t ,

(2.38.2) Y_0 is slc and CM,

(2.38.3) the irreducible components of Y_0 are normal, but

(2.38.4) one of the irreducible components of Y_0 is not CM.

Let Z be a smooth Fano variety of dimension $n \geq 2$ such that $-K_Z$ is very ample, for instance $Z = \mathbb{P}^2$. Set $X := \mathbb{P}^1 \times Z$ and view it as embedded by $|-K_X|$ into \mathbb{P}^N for suitable N . Let $C(X) \subset \mathbb{P}^{N+1}$ be the cone over X .

Let $M \in |-K_Z|$ be a smooth member and consider the following divisors in X :

$$D_0 := \{(0 : 1)\} \times Z, \quad D_1 := \{(1 : 0)\} \times Z \quad \text{and} \quad D_2 := \mathbb{P}^1 \times M.$$

Note that $D_0 + D_1 + D_2 \sim -K_X$. Let $E_i \subset C(X)$ denote the cone over D_i . Then $E_0 + E_1 + E_2$ is a hyperplane section of $C(X)$ and $(C(X), E_0 + E_1 + E_2)$ is lc by (2.35).

For some $m > 0$, let $H_m \subset C(X)$ be a general intersection with a degree m hypersurface. Then

$$(C(X), E_0 + E_1 + E_2 + H_m)$$

is snc outside the vertex and is lc at the vertex. Set $Y_0 := E_0 + E_1 + E_2 + H_m$. Since $\mathcal{O}_{C(X)}(Y_0) \sim \mathcal{O}_{C(X)}(m+1)$, we can view Y_0 as an slc limit of a family of smooth hypersurface sections $Y_t \subset C(X)$.

The cone over X is CM by (2.35), hence its hyperplane section $E_0 + E_1 + E_2 + H_m$ is also CM. However, E_2 is not CM. To see this, note that E_2 is the cone over $\mathbb{P}^1 \times M$ and, by the Künneth formula,

$$H^i(\mathbb{P}^1 \times M, \mathcal{O}_{\mathbb{P}^1 \times M}) = H^i(M, \mathcal{O}_M) = \begin{cases} k & \text{if } i = 0, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus E_2 is not CM by (2.35).

EXAMPLE 2.39 (Easy counterexamples to (2.34)). There are some obvious problems with all of the questions in (2.34) if the D_t contain divisors with different coefficients. For instance, let C be a smooth curve and $D', D'' \subset \mathbb{A}^1 \times C =: S$ two sections of the 1st projection π_1 . Set $D := \frac{1}{2}(D' + D'')$. Then

$$\pi_1: (S, D) \rightarrow \mathbb{A}^1$$

is a stable family of 1-dimensional pairs. For general t , the sections D', D'' intersect C_t at two different points and then $\mathcal{O}_{C_t}(K_{C_t} + \lfloor D_t \rfloor) \cong \mathcal{O}_C(K_C)$. If, however, D', D'' intersect C_t at the same point $p_t \in C_t$, then $\mathcal{O}_{C_t}(K_{C_t} + \lfloor D_t \rfloor) \cong \mathcal{O}_C(K_C)(p_t)$.

Similarly, the support of D_t is 2 points for general t but only 1 point for special values of t .

In 1-dimension one can correct for these problems by a more careful bookkeeping of the different parts of the divisor D_t . However, starting with dimension 2, no correction seems possible, except when all the coefficients are $> \frac{1}{2}$ (2.81).

The following example is due to Hassett (unpublished).

EXAMPLE 2.40 (Counterexample to (2.34.1-3)). We start with the already studied example of deformations of the cone $S \subset \mathbb{P}^5$ over the degree 4 rational normal curve (1.42), but here we add a boundary to it. Fix $r \geq 1$ and let D_S be the sum of $2r$ lines. Then $(S, \frac{1}{r}D_S)$ is lc and $(K_S + \frac{1}{r}D_S)^2 = 4$.

As in (1.42), there are two different deformations of the pair (S, D_S) .

(2.40.1) First, set $P := \mathbb{P}^2$ and let D_P be the sum of r general lines. Then $(P, \frac{1}{r}D_P)$ is lc (even canonical if $r \geq 2$) and $(K_P + \frac{1}{r}D_P)^2 = 4$. The usual smoothing of $S \subset \mathbb{P}^5$ to the Veronese surface gives a family $f: (X, D_X) \rightarrow \mathbb{P}^1$ with general fiber (P, D_P) and special fiber (S, D_S) . We can concretely realize this as deforming $(P, D_P) \subset \mathbb{P}^5$ to the cone over a general hyperplane section. Note that for any general D_S there is a choice of lines D_P such that the above limit is exactly D_S .

The total space (X, D_X) is the cone over (P, D_P) (blown up along curve) and X is \mathbb{Q} -factorial. Thus by (11.5) the structure sheaf of an effective divisor on X is CM.

In particular, D_S is a flat limit of D_P . Since the D_P is a plane curve of degree r , we conclude that

$$\chi(\mathcal{O}_{D_S}) = \chi(\mathcal{O}_{D_P}) = -\frac{r(r-3)}{2}.$$

(2.40.2) Second, set $Q := \mathbb{P}^1 \times \mathbb{P}^1$ and let A, B denote the classes of the 2 rulings. Let D_Q be the sum of r lines from the A -family. Then $(Q, \frac{1}{r}D_Q)$ is canonical and $(K_Q + \frac{1}{r}D_Q)^2 = 4$. The usual smoothing of $S \subset \mathbb{P}^5$ to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $H := A + 2B$ gives a family $g: (Y, D_Y) \rightarrow \mathbb{P}^1$ with general fiber (Q, D_Q) and special fiber (S, D_S) . We can concretely realize this as deforming $(Q, D_Q) \subset \mathbb{P}^5$ to the cone over a general hyperplane section.

The total space (Y, D_Y) is the cone over (Q, D_Q) (blown up along curve) and Y is not \mathbb{Q} -factorial. However, $K_Q + \frac{1}{r}D_Q \sim_{\mathbb{Q}} -H$, thus $K_Y + \frac{1}{r}D_Y$ is \mathbb{Q} -Cartier and $(Y, S + \frac{1}{r}D_Y)$ is lc by inversion of adjunction (11.20) and so is $(Y, \frac{1}{r}D_Y)$.

In this case, however, D_S is not a flat limit of D_Q for $r > 1$. This follows, for instance, from comparing their Euler characteristic:

$$\chi(\mathcal{O}_{D_S}) = -\frac{r(r-3)}{2} \quad \text{and} \quad \chi(\mathcal{O}_{D_Q}) = r.$$

(2.40.3) Because of their role in the canonical ring, we are also interested in the sheaves $\mathcal{O}(mK + \lfloor \frac{m}{r}D \rfloor)$.

Let H_P be the hyperplane class of $P \subset \mathbb{P}^5$ (that is, 2 times a line $L \subset P$) and write $m = br + a$ where $0 \leq a < r$. Then

$$mK_P + \lfloor \frac{m}{r}D_P \rfloor + nH_P \sim (2n - 2m - a)L,$$

and hence

$$\begin{aligned} \chi(P, \mathcal{O}_P(mK_P + \lfloor \frac{m}{r}D_P \rfloor + nH_P)) &= \binom{2n-2m-a+2}{2} \\ &= \binom{2n-2m+2}{2} - a(2n-2m+1) + \binom{a}{2}. \end{aligned}$$

Again by (11.5) all divisorial sheaves on X are CM. Thus the restriction of $\mathcal{O}_X(mK_X + \lfloor \frac{m}{r}D_X \rfloor)$ to the central fiber S is $\mathcal{O}_S(mK_S + \lfloor \frac{m}{r}D_S \rfloor)$. In particular,

$$\chi(S, \mathcal{O}_S(mK_S + \lfloor \frac{m}{r}D_S \rfloor + nH_S)) = \binom{2n-2m+2}{2} - a(2n-2m+1) + \binom{a}{2}.$$

The other deformation again behaves differently. Write $m = br + a$ where $0 \leq a < r$. Then, for $H_Q \sim A + 2B$, we see that

$$mK_Q + \lfloor \frac{m}{r}D_Q \rfloor + nH_Q \sim (n-m-a)A + (2n-2m)B,$$

and therefore

$$\chi(Q, \mathcal{O}(mK_Q + \lfloor \frac{m}{r}D_Q \rfloor + nH_Q)) = \binom{2n-2m+2}{2} - a(2n-2m+1).$$

From this we conclude that the restriction of $\mathcal{O}_Y(mK_Y + \lfloor mD_Y \rfloor)$ to the central fiber S agrees with $\mathcal{O}_S(mK_S + \lfloor mD_S \rfloor)$ only if $a \in \{0, 1\}$, that is when $m \equiv 0, 1 \pmod{r}$. The if part was clear from the beginning. Indeed, if $a = 0$ then $\mathcal{O}_Y(mK_Y + \lfloor mD_Y \rfloor) = \mathcal{O}_Y(mK_Y + mD_Y)$ is locally free and if $a = 1$ then

$$\mathcal{O}_Y(mK_Y + \lfloor mD_Y \rfloor) = \mathcal{O}_Y(K_Y) \otimes \mathcal{O}_Y((m-1)K_Y + (m-1)D_Y)$$

is $\mathcal{O}_Y(K_Y)$ tensored with a locally free sheaf. Both of these commute with restrictions.

In the other cases we only get an injection

$$\mathcal{O}_Y(mK_Y + [mD_Y])|_S \hookrightarrow \mathcal{O}_S(mK_S + [mD_S])$$

whose quotient is a torsion sheaf of length $\binom{a}{2}$ supported at the vertex.

EXAMPLE 2.41 (Counterexample to (2.34.1)). As in (2.38), let Z be a smooth Fano variety of dimension $n \geq 2$ such that $-K_Z$ is very ample. Set $X := \mathbb{P}^1 \times Z$ but now view it as embedded by global sections of $\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_Z(-K_Z)$ into \mathbb{P}^N for suitable N . Let $C(X) \subset \mathbb{P}^{N+1}$ be the cone over X .

Fix $r \geq 1$ and let D_r be the sum of r distinct divisors of the form $\{\text{point}\} \times Z \subset X$. Let $H \subset X$ be a general hyperplane section. Then $H \sim_{\mathbb{Q}} -(K_X + \frac{1}{r}D_r)$, that is, $(X, \frac{1}{r}D_r)$ is (numerically) anticanonically embedded. Thus, by (2.35), $(C(H), \frac{1}{r}C(H \cap D_r))$ is lc and there is a locally stable family with general fiber $(X, \frac{1}{r}D_r)$ and special fiber $(C(H), \frac{1}{r}C(H \cap D_r))$.

However, $C(H \cap D_r)$ is not a flat deformation of D_r . Indeed, if $D_{r_i} (\cong Z)$ is any irreducible component of D_r , then $C(H \cap D_{r_i})$ is a flat deformation of D_{r_i} . Thus $\Pi_i C(H \cap D_{r_i})$ is a flat deformation of $D_r = \Pi_i D_{r_i}$. Note further that $\Pi_i C(H \cap D_{r_i})$ is the normalization of $C(H \cap D_r)$, and the normalization map is $r : 1$ over the vertex of the cone. Thus

$$\begin{aligned} \chi(D_r, \mathcal{O}_{D_r}) &= \sum_i \chi(D_{r_i}, \mathcal{O}_{D_{r_i}}) = \\ &= \sum_i \chi(C(H \cap D_{r_i}), \mathcal{O}_{C(H \cap D_{r_i})}) \\ &\geq \chi(C(H \cap D_r), \mathcal{O}_{C(H \cap D_r)}) + (r - 1). \end{aligned}$$

Therefore $C(H \cap D_r)$ can not be a flat deformation of D_r for $r > 1$. We pick up at least $r - 1$ embedded points.

EXAMPLE 2.42 (Counterexample to (2.34.3)). Set $X := C_a(\mathbb{P}^1 \times \mathbb{P}^n, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(1, a))$ for some $0 < a < n + 1$. Let $D \subset X$ be the cone over a smooth divisor in $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(1, n + 1 - a)|$. Then (X, D) is canonical and $K_X + D$ is Cartier.

Let $\pi : (X, D) \rightarrow \mathbb{A}^1$ be a general projection. Then π is locally stable and its central fiber is the cone $X_0 = C_a(H, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(1, a)|_H)$ where $H \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(1, a)|$ is a smooth divisor.

We claim that if $2a > n + 1$ then $r_m : \omega_{X/\mathbb{A}^1}^{[m]}|_{X_0} \rightarrow \omega_{X_0}^{[m]}$ is not surjective for $m \gg 1$.

Indeed, we can write this map as

$$\sum_{r \geq 0} H^0(\mathbb{P}^1 \times \mathbb{P}^n, \mathcal{O}(r - 2m, ra - (n + 1)m)) \rightarrow \sum_{r \geq 0} H^0(H, \mathcal{O}(r - 2m, ra - (n + 1)m)|_H)$$

and r_m is surjective iff

$$H^1(\mathbb{P}^1 \times \mathbb{P}^n, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(r - 2m, ra - (n + 1)m)) = 0$$

for every $r \geq -1$. Choose $r = 2m - 2$. Then, by the Künneth formula, this group is

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2a(m - 1) - m(n - 1))).$$

This is nonzero iff $2a \geq \frac{m}{m-1}(n - 1)$.

The following example, related to [Pat13], shows that the relative dualizing sheaf does not commute with base change in general.

EXAMPLE 2.43. We give an example of a flat family of normal varieties $Y \rightarrow U$ such that ω_{Y_0} is locally free for some $0 \in U$ yet $\omega_{Y/U}$ is not locally free along Y_0 .

We start with a smooth, projective variety X such that $H^1(X, \mathcal{O}_X) \neq 0$ but $H^0(X, \omega_X) = H^1(X, \omega_X) = 0$. For example, we can take $X = C \times \mathbb{P}^n$ where C is a smooth curve of genus > 0 and $n \geq 2$.

Let L_0 be a very ample line bundle such that $L_0 \otimes \omega_X$ is ample. All line bundles algebraically equivalent to L_0 are parametrized by $\text{Pic}^L(X)$.

Pick a smooth divisor $D \subset X$ linearly equivalent to L_0 . Our example will be the family of cones

$$Y_L := \text{Spec}_k \sum_m H^0(D, (L \otimes \omega_X)^m|_D),$$

parametrized by a suitable open set $[L_0] \in U \subset \text{Pic}^L(X)$.

The Y_L form a flat family iff the $h^0(D, (L \otimes \omega_X)^m|_D)$ are all constant on U . To compute these, consider the exact sequence

$$0 \rightarrow (L \otimes \omega_X)^m(-D) \rightarrow (L \otimes \omega_X)^m \rightarrow (L \otimes \omega_X)^m|_D \rightarrow 0.$$

Since $(L \otimes \omega_X)^m(-D)$ is numerically equivalent to $\omega_X \otimes (L \otimes \omega_X)^{m-1}$, its higher cohomologies vanish. Thus $h^0(D, (L \otimes \omega_X)^m|_D)$ is independent of L for $m \geq 2$. If $m = 1$ then $(L_0 \otimes \omega_X)(-D) \cong \omega_X$ and we assumed that $H^0(X, \omega_X) = H^1(X, \omega_X) = 0$. Thus $H^0(D, (L \otimes \omega_X)|_D) = H^0(X, L \otimes \omega_X) = 0$ holds for all L in a neighborhood of $[L_0]$; this condition defines our U .

The cones Y_L form the fibers of a flat morphism $Y \rightarrow U$. By (2.35) or [Kol13b, 3.14.4], ω_{Y_L} is locally free iff $L = L_0$. Thus $\omega_{Y/U}$ is not locally free along Y_{L_0} yet $\omega_{Y_{L_0}}$ is locally free.

2.4. Stable families

Next we define the notion of stable families over a regular 1-dimensional base scheme and establish, in characteristic 0, the valuative criteria of separatedness and properness.

DEFINITION 2.44. Let $f: (X, \Delta) \rightarrow C$ be a family of pairs (2.2) over a regular 1-dimensional scheme C . We say that $f: (X, \Delta) \rightarrow C$ is *stable* if

(2.44.1) f is locally stable (2.3),

(2.44.2) f is proper and

(2.44.3) $K_{X/C} + \Delta$ is f -ample.

Note that if f is locally stable then $K_X + \Delta$ is \mathbb{R} -Cartier, so f -ampleness makes sense. As we remarked in (2.3), if C is over a field of characteristic zero, then being stable is preserved by base change $C' \rightarrow C$. This is expected to hold in general, but it is not known. See (2.56) for an important special case.

More generally, whenever the notion of local stability is defined later over a scheme S , then $f: (X, \Delta) \rightarrow S$ is called *stable* if the above 3 conditions are satisfied. (Thus we have to make sure that local stability implies that $K_{X/S} + \Delta$ makes sense and is \mathbb{R} -Cartier.)

The relationship between locally stable morphisms and stable morphisms parallels the connection between smooth varieties and their canonical models.

PROPOSITION 2.45. *Let $f: (Y, \Delta_Y) \rightarrow B$ be a locally stable proper morphism over a 1-dimensional regular scheme B . Assume that the generic fibers are normal,*

of general type and f has a canonical model $g: (X, \Delta_X) \rightarrow B$. Then $f: (X, \Delta_X) \rightarrow B$ is stable.

Furthermore, if B is over a field of characteristic zero, then taking the canonical model commutes with flat base changes $\pi: B' \rightarrow B$.

Proof. First, $K_X + \Delta_X$ is g -ample by definition (1.38) and (X, Δ_X) is lc.

Let $b \in B$ be any closed point and Y_b (resp. X_b) the fibers over b . Since f is locally stable, $(Y, Y_b + \Delta_Y)$ is lc. Since any fiber is f -linearly trivial, we conclude using [Kol13b, 1.28] that $(X, X_b + \Delta_X)$ is also lc. Thus g is locally stable, hence stable.

In characteristic zero, being locally stable commutes with base change (2.3), thus the last assertion follows from (2.47). \square

As discussed in (1.20), the separatedness and properness criteria of the (not yet defined) moduli functor/stack of stable morphisms involve the extensions of a stable family defined over an open subset $C^\circ \subset C$ to a stable family defined over C .

2.46 (Separatedness and Properness). Let C be a regular 1-dimensional scheme, $C^\circ \subset C$ an open and dense subscheme and $f^\circ: (X^\circ, \Delta^\circ) \rightarrow C^\circ$ a stable morphism. We aim to prove the following two properties.

Separatedness: $f^\circ: (X^\circ, \Delta^\circ) \rightarrow C^\circ$ has at most one extension to a stable morphism $f: (X, \Delta) \rightarrow C$.

Properness: There is a finite surjection $\pi: B \rightarrow C$ such that the pull back

$$\pi^* f^\circ: (X^\circ \times_C B, \Delta^\circ \times_C B) \rightarrow \pi^{-1}(C^\circ)$$

extends to a stable morphism $f_B: (X_B, \Delta_B) \rightarrow B$.

Next we show that separatedness holds over regular 1-dimensional schemes and properness holds for stable morphisms in characteristic 0. In both cases the proof relies on theorems which, for later applications, we state in rather general forms.

Proof of separatedness.

We start with a variant of (1.27) which holds over arbitrary base schemes and then conclude that separatedness holds for stable morphisms.

THEOREM 2.47. *Let $f_i: (X^i, \Delta^i) \rightarrow B$ be two proper morphisms from slc pairs to an irreducible, Noetherian scheme B . Assume that*

(2.47.1) *every irreducible component of X^i dominates B ,*

(2.47.2) *every divisor E^i over X^i satisfying $a(E^i, X^i, \Delta^i) < 0$ dominates B and*

(2.47.3) *$K_{X^i} + \Delta^i$ is f_i -ample.*

Then every isomorphism of the generic fibers

$$\phi: (X_{k(B)}^1, \Delta_{k(B)}^1) \cong (X_{k(B)}^2, \Delta_{k(B)}^2)$$

extends to an isomorphism

$$\Phi: (X^1, \Delta^1) \cong (X^2, \Delta^2).$$

Proof. Let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of ϕ and $\Gamma' \subset \Gamma$ the union of those irreducible components that dominate B . Let $Y \rightarrow \Gamma'$ be the partial normalization (10.73) that is an isomorphism over the generic point of B and such that every irreducible component of the non-normal locus of Y dominates B . Let

$p_i: Y \rightarrow X^i$ and $f: Y \rightarrow B$ be the projections. The p_i are isomorphisms over the generic point of B by construction. We aim to prove that the p_i are isomorphisms. Since the X^i are demi-normal, it is enough to prove the latter after normalization. Thus we may assume that the X^i are normal.

As in (1.27), we use the log canonical class to compare the X^i . If F^i is an irreducible component of Δ^i then $a(F^i, X^i, \Delta^i) = -\text{coeff}_{F^i} \Delta^i < 0$, thus every irreducible component of Δ^i dominates B by assumption (2). In particular, $(p_1)_*^{-1} \Delta^1 = (p_2)_*^{-1} \Delta^2$; let us denote this divisor by Δ_Y . Write

$$K_Y + \Delta_Y \sim p_i^*(K_{X^i} + \Delta^i) + E_i, \quad (2.47.4)$$

where E_i is p_i -exceptional and does not dominate B . Hence E_i is effective by assumption (2).

Subtracting the $i = 1, 2$ cases of (2.47.4) from each other we get that

$$E_1 - E_2 \sim p_2^*(K_{X^2} + \Delta^2) - p_1^*(K_{X^1} + \Delta^1), \quad (2.47.5)$$

Thus $E_1 - E_2$ is p_1 -nef and $-(p_1)_*(E_1 - E_2) = (p_1)_*(E_2)$ is effective. Thus $E_2 - E_1$ is effective by (11.50). Using π_2 shows that $E_1 - E_2$ is effective, hence $E_1 = E_2$. Thus

$$p_1^*(K_{X^1} + \Delta^1) \sim p_2^*(K_{X^2} + \Delta^2). \quad (2.47.6)$$

If a proper, irreducible curve $C \subset Y$ is contracted by p_1 , then it is not contracted by p_2 . Thus we get a contradiction

$$0 = p_1^*(K_{X^1} + \Delta^1) \cdot C = p_2^*(K_{X^2} + \Delta^2) \cdot C > 0. \quad (2.47.7)$$

Therefore there are no such curves C , hence the p_i are isomorphisms. \square

COROLLARY 2.48 (Separatedness for stable maps). *Let $f_i: (X^i, \Delta^i) \rightarrow B$ be two stable morphisms over a 1-dimensional, regular scheme B . Let*

$$\phi: (X_{k(B)}^1, \Delta_{k(B)}^1) \cong (X_{k(B)}^2, \Delta_{k(B)}^2)$$

be an isomorphism of the generic fibers. Then ϕ extends to an isomorphism

$$\Phi: (X^1, \Delta^1) \cong (X^2, \Delta^2).$$

Proof. Observe that (2.47.1) holds since the f_i are flat, (2.47.2) was proved in (2.14), and (2.47.3) holds by definition. Thus (2.47) implies (2.48). \square

As a consequence of (2.48) we obtain that $\text{Aut}(X, \Delta)$ is finite for a stable pair (X, Δ) in arbitrary characteristic, using (2.15.2). (See [Uen75, Sec.14] for other approaches over \mathbb{C} .) We prove a more general form of it in (8.47).

Proof of properness.

The following result verifies the valuative criterion of properness for slc morphisms.

THEOREM 2.49 (Valuative-properness for stable maps). *Let C be a smooth curve over a field of characteristic 0 and $C^\circ \subset C$ an open and dense subset. Let $f^\circ: (X^\circ, \Delta^\circ) \rightarrow C^\circ$ be a stable morphism.*

Then there is a finite surjection $\pi: B \rightarrow C$ such that the pull back

$$f_B^\circ := \pi^* f^\circ: (X^\circ \times_C B, \Delta^\circ \times_C B) \rightarrow \pi^{-1}(C^\circ)$$

extends to a stable morphism $f_B: (X_B, \Delta_B) \rightarrow B$.

Proof. We begin with the case when X° is normal. Start with $f^\circ: (X^\circ, \Delta^\circ) \rightarrow C^\circ$ and extend it to a proper flat morphism $f_1: (X_1, \Delta_1) \rightarrow C$ where X_1 is normal. In general (X_1, Δ_1) is no longer lc.

By [Kol13b, 10.46], there is a log resolution $g_1: Y_1 \rightarrow X_1$ such that $(g_1^{-1})_* \Delta_1 + \text{Ex}(g_1) + Y_{1c}$ is an snc divisor for every $c \in C$. In general, the fibers of $f_1 \circ g_1: Y_1 \rightarrow C$ are not reduced, hence $g_1: (Y_1, (g_1^{-1})_* \Delta_1 + \text{Ex}(g_1)) \rightarrow C$ is not locally stable.

Let B be a smooth curve and $\pi: B \rightarrow C$ a finite surjection. Let $X_2 \rightarrow X_1 \times_C B$ and $Y_2 \rightarrow Y_1 \times_C B$ denote the normalizations and $g_2: Y_2 \rightarrow X_2$ the induced morphism. Let Δ_2 be the pull back of $\Delta_1 \times_C B$ to X_2 .

Note that

$$f_2 \circ g_2: (Y_2, (g_2^{-1})_* \Delta_2 + \text{Ex}(g_2)) \rightarrow B$$

is a log resolution over the points where π is étale, but Y_2 need not be smooth everywhere. However, by (2.51), $(Y_2, (g_2^{-1})_* \Delta_2 + \text{Ex}(g_2) + \text{red } Y_{2b})$ is lc for every $b \in B$.

By (2.52), one can choose $\pi: B \rightarrow C$ such that every fiber of $f_2 \circ g_2$ is reduced. With such a choice, $f_2 \circ g_2$ is locally stable.

If the generic fiber $(X_g^\circ, \Delta_g^\circ)$ is klt, then, using (2.14) and after shrinking C° , we may assume that (X°, Δ°) is klt. Pick $0 < \epsilon \ll 1$. Then $(Y_2, \Delta_2 + (1-\epsilon)\text{Ex}(g_2))$ is also klt and so by (11.28.1) it has a canonical model $f_B: (X_B, \Delta_B) \rightarrow B$ which is stable by (2.45).

We are almost done, except that, by construction, $f_B: (X_B, \Delta_B) \rightarrow B$ is isomorphic to the pull-back of $f^\circ: (X^\circ, \Delta^\circ) \rightarrow C^\circ$ only over a possibly smaller dense open subset. However, by (2.48), this implies that this isomorphism holds over the entire C° .

The argument is the same if (X°, Δ°) is lc, but we need to take the canonical model of $(Y_2, \Delta_2 + \text{Ex}(g_2))$. The latter is dlt but not klt. Here we rely on (11.28.2).

Next we show how the semi-log-canonical case can be reduced to the log canonical case.

Let $\bar{X}^\circ \rightarrow X^\circ$ be the normalization with conductor $\bar{D}^\circ \subset \bar{X}^\circ$. As we noted in (2.3), we get a stable morphism

$$\bar{f}^\circ: (\bar{X}^\circ, \bar{\Delta}^\circ + \bar{D}^\circ) \rightarrow C^\circ. \quad (2.49.4)$$

By the already completed normal case, we get $B \rightarrow C$ such that the pull-back of (2.49.4) extends to a stable morphism

$$\bar{f}_B: (\bar{X}_B, \bar{\Delta}_B + \bar{D}_B) \rightarrow B. \quad (2.49.5)$$

Finally, (2.55) shows that (2.49.5) is the normalization of a stable morphism $f_B: (X_B, \Delta_B) \rightarrow B$, which is the required extension of the pull-back of $f^\circ: (X^\circ, \Delta^\circ) \rightarrow C^\circ$. \square

We have used the following 3 lemmas during the proof. The first one will be strengthened in (2.80).

LEMMA 2.50. *Let B be a smooth curve over a field of characteristic 0 and $f: (X, D + \Delta) \rightarrow B$ a locally stable (resp. stable) morphism where D is a \mathbb{Z} -divisor. Let $n: D^n \rightarrow D$ be the normalization. Then $f \circ n: (D^n, \text{Diff}_{D^n} \Delta) \rightarrow B$ is also locally stable (resp. stable).*

Proof. For any $b \in B$, the fiber X_b is a Cartier divisor, thus

$$\text{Diff}_{D^n}(\Delta + X_b) = (\text{Diff}_{D^n} \Delta) + X_b|_{D^n} = (\text{Diff}_{D^n} \Delta) + D_b^n.$$

Together with adjunction (11.20), this shows that $f_D: (D^n, \text{Diff}_{D^n} \Delta) \rightarrow B$ is locally stable. Since $D^n \rightarrow D$ is finite and

$$K_{D^n} + \text{Diff}_{D^n} \Delta \sim_{\mathbb{Q}} n^*(K_X + D + \Delta),$$

we see that if $K_X + D + \Delta$ is f -ample then $K_{D^n} + \text{Diff}_{D^n}$ is $f \circ n$ -ample. Hence if f is stable then so is $f \circ n: (D^n, \text{Diff}_{D^n} \Delta) \rightarrow B$. \square

LEMMA 2.51. *Let C be a smooth curve over a field of characteristic 0, $f: X \rightarrow C$ a flat morphism and Δ a \mathbb{R} -divisor on X . Assume that $(X, \text{red } X_c + \Delta)$ is lc for every $c \in C$. Let B be a smooth curve, $g: B \rightarrow C$ a quasi-finite morphism, $g_Y: Y \rightarrow X \times_C B$ the normalization and $\Delta_Y := g_Y^* \Delta$.*

Then $(Y, \text{red } X_b + \Delta_Y)$ is lc for every $b \in B$.

Proof. Pick $c \in C$ and let $b_i \in B$ be its preimages. By the Hurwitz formula

$$K_Y + \Delta_Y + \sum_i \text{red } Y_{b_i} = g_X^*(K_X + \Delta + \text{red } X_c).$$

By assumption, $(X, \Delta + \text{red } X_c)$ is lc for every $c \in C$. Hence, by (11.13.3), $(Y, \Delta_Y + \sum_i \text{red } Y_{b_i})$ is also lc. \square

LEMMA 2.52. *Let $f: X \rightarrow T$ be a flat morphism from a normal scheme to a 1-dimensional regular scheme T . Let S be another 1-dimensional regular scheme and $\pi: S \rightarrow T$ a quasi-finite morphism. Let $Y \rightarrow X \times_T S$ be the normalization and $f_Y: Y \rightarrow S$ the projection. Assume that*

- (2.52.1) *for every $s \in S$, the multiplicity of every irreducible component of $X_{\pi(s)}$ divides the ramification index of π at s and*
 (2.52.2) *π is tamely ramified everywhere.*

Then every fiber of $f_Y: Y \rightarrow S$ is reduced.

Proof. The claim is local, so pick points $0_S \in S$ and $0_T := \pi(0_S) \in T$.

We want to study how the multiplicities of the irreducible components of the fiber over 0_T change under base extension. We can focus on one such irreducible component and pass to any open subset of X that is not disjoint from the chosen component. We can thus think of X as a hypersurface $X \subset \mathbb{A}_T^n$ defined by an equation $f \in \mathcal{O}_T[x_1, \dots, x_n]$. The central fiber X_0 is defined by $\bar{f} = 0$ where \bar{f} is the mod t reduction of f . By focusing at a generic point of X_0 , after an étale coordinate change we may assume that $\bar{f} = x_1^m$ where m is the multiplicity of X_0 . We can thus write $f = x_1^m - t \cdot u(\mathbf{x}, t)$. Since X is normal (hence regular) at the generic point of X_0 , we see that u is not identically zero along X_0 .

Let s be a local coordinate at 0_S . We can write $\pi^* t = s^e v(s)$ where e is the ramification index of π at 0_S and v is a unit at 0_S . Consider now the fiber product $X_S := X \times_T S \rightarrow S$. It is defined by the equation

$$x_1^m = s^e \cdot u(\mathbf{x}, s^e v(s)) \cdot v(s).$$

Note that X_S is not normal along $(s = x_1 = 0)$ if $m, e > 1$.

We construct its normalization by repeatedly blowing up. This is especially simple if e is a multiple of m . Write $e = md$ and set $x'_1 := xs^{-d}$. Then we get $Y \subset \mathbb{A}_S^n$ (with coordinates x'_1, x_2, \dots, x_n) defined by

$$x'_1{}^m = u(x'_1 s^d, x_2, \dots, x_n, s^e v(s)) \cdot v(s),$$

and the central fiber Y_0 is defined by the equation

$$x'_1{}^m = u(0, x_2, \dots, x_n, 0) \cdot v(0),$$

where the right hand side is not identically zero.

If the characteristic of $k(0_S)$ does not divide m , then the projection $Y_0 \rightarrow \mathbb{A}_{x_2, \dots, x_n}^{n-1}$ is generically étale and Y_0 is smooth at its generic points. In this case, Y is the normalization of X_S (at least generically along Y_0) and the central fiber of $Y \rightarrow S$ has multiplicity 1. \square

Note that the proof of (2.52) does not work if the characteristic of $k(0_S)$ divides m . Then $Y_0 \rightarrow \mathbb{A}_{x_2, \dots, x_n}^{n-1}$ is inseparable. If $u(0, x_2, \dots, x_n, 0)$ is not a p th power over the algebraic closure of $k(0_S)$, then Y_0 is geometrically integral, hence generically nonsingular. In this case, Y is the normalization of X_S and the central fiber of $Y \rightarrow S$ has multiplicity 1.

However, if $u(0, x_2, \dots, x_n, 0)$ is a p th power, then Y_0 is not generically reduced. In this case Y need not be normal, and further blow-ups may be needed to reach the normalization. In any case, usually one does not get a reduced fiber. The situation seems rather complicated, even for families of curves [AW71]. A weaker result is in (2.61).

Gluing of slc pairs.

At the end of the proof of (2.49) we needed to reconstruct an slc pair from its normalizations. The technical background for this is discussed in [Kol13b, Chaps. 5 and 9]. In the current setting we aim to compactify an slc pair (X°, Δ°) by first normalizing it, then obtaining an lc compactification of the normalization and finally descending the lc compactification to a compactification of (X°, Δ°) . A precise version of this process is the following.

2.53 (Compactification problem for slc pairs). Consider a diagram

$$\begin{array}{ccc} (\bar{X}^\circ, \bar{\Delta}^\circ + \bar{D}^\circ) & \xhookrightarrow{\bar{\iota}} & (\bar{X}, \bar{\Delta} + \bar{D}) \\ \pi^\circ \downarrow & & \\ (X^\circ, \Delta^\circ) & & \end{array} \quad (2.53.1)$$

where (X°, Δ°) is an slc pair, π° its normalization and $\bar{\iota}$ an open embedding with dense image. We say that (2.53.1) defines a partial compactification of (X°, Δ°) if (2.53.1) can be extended to a diagram

$$\begin{array}{ccc} (\bar{X}^\circ, \bar{\Delta}^\circ + \bar{D}^\circ) & \xhookrightarrow{\bar{\iota}} & (\bar{X}, \bar{\Delta} + \bar{D}) \\ \pi^\circ \downarrow & & \downarrow \pi \\ (X^\circ, \Delta^\circ) & \xhookrightarrow{\iota} & (X, \Delta) \end{array} \quad (2.53.2)$$

where (X, Δ) is demi-normal, π is its normalization and ι an open embedding.

Note that (X, Δ) is unique. [Kol13b, Sec.9.4] contains a series of examples where (X, Δ) does not exist.

THEOREM 2.54. *Let (X°, Δ°) be an slc pair over a field of characteristic 0 and consider a diagram (2.53.1). Let $n: \bar{D}^n \rightarrow \bar{D}$ denote the normalization and assume that the involution τ° on $(\bar{D}^\circ)^n$ extends to an involution τ on \bar{D}^n .*

(2.54.1) *If none of the lc centers of $(\bar{X}, \bar{\Delta} + \bar{D})$ is disjoint from \bar{X}° then (2.53.1) has an extension to a diagram (2.53.2).*

(2.54.2) *If none of the log centers of $(\bar{X}, \bar{\Delta} + \bar{D})$ is disjoint from \bar{X}° then (X, Δ) is slc.*

Proof. Our aim is to construct (X, Δ) as the geometric quotient [Kol13b, 9.4] by the gluing relation generated by the relation $(n, n \circ \tau): \bar{D}^n \rightrightarrows \bar{X}$ as in [Kol13b, 5.31].

We assume that $(\bar{X}, \bar{\Delta} + \bar{D})$ is lc. By assumption none of the lc centers of $(\bar{X}, \bar{\Delta} + \bar{D})$ is contained in $\bar{X} \setminus \bar{X}^\circ$ and, over \bar{X}° , we have a finite equivalence relation whose quotient is X° . Thus [Kol13b, 9.55] implies that $(n, n \circ \tau)$ generates a finite equivalence relation on $(\bar{X}, \bar{\Delta} + \bar{D})$. Therefore, by [Kol13b, 5.33], there is a demi-normal scheme (X, Δ) that contains (X°, Δ°) as an open subscheme, proving (1).

By inversion of adjunction (11.20), every irreducible component of $\text{Diff}_{\bar{D}^n}$ lies over a log center of $(\bar{X}, \bar{\Delta} + \bar{D})$. Thus if none of the log centers of $(\bar{X}, \bar{\Delta} + \bar{D})$ is disjoint from \bar{X}° then none of the irreducible components of $\text{Diff}_{\bar{D}^n}$ is disjoint from \bar{X}° . Thus $\text{Diff}_{\bar{D}^n}$ is $\bar{\tau}$ -invariant and therefore [Kol13b, 5.38] shows that (X, Δ) is slc. \square

COROLLARY 2.55. *Let B be a smooth curve over a field of characteristic 0 and $B^\circ \subset B$ a dense open subset. Let $f^\circ: (X^\circ, \Delta^\circ) \rightarrow B^\circ$ be a stable morphism. Let $\bar{X}^\circ \rightarrow X^\circ$ be the normalization with conductor $\bar{D}^\circ \subset \bar{X}^\circ$.*

Assume that $\bar{f}^\circ: (\bar{X}^\circ, \bar{\Delta}^\circ + \bar{D}^\circ) \rightarrow B^\circ$ extends to a stable morphism $\bar{f}: (\bar{X}, \bar{\Delta} + \bar{D}) \rightarrow B$.

Then $f^\circ: (X^\circ, \Delta^\circ) \rightarrow B^\circ$ also extends to a stable morphism $f: (X, \Delta) \rightarrow B$.

Proof. Let $n: \bar{D}^n \rightarrow \bar{D}$ denote the normalization. By (2.50),

$$\bar{f} \circ n: (\bar{D}^n, \text{Diff}_{\bar{D}^n} \bar{\Delta}) \rightarrow B$$

is also stable. In particular, by (2.48) the involution τ° of $(\bar{D}^\circ)^n$ extends to an involution τ on \bar{D}^n .

By (2.14), none of the log centers of $(\bar{X}, \bar{\Delta} + \bar{D})$ is contained in $\bar{X} \setminus \bar{X}^\circ$. Thus the assumption of (2.54.2) hold, hence we get (X, Δ) . Finally f° extends to a proper morphism $f: (X, \Delta) \rightarrow B$ by the universal property of geometric quotients. \square

Base change in positive characteristic.

As we noted in (2.15), it is not known whether being locally stable commutes with base change in positive characteristic. However, the next result shows that this holds for all families obtained as in (2.49).

THEOREM 2.56. *Let $h: C' \rightarrow C$ be a quasi-finite morphisms of regular schemes of dimension 1 and $f: X \rightarrow C$ a proper morphism from a regular scheme X to C whose fibers are simple normal crossing divisors. Then $X' := X \times_C C'$ has canonical singularities and*

$$\sum_{m \geq 0} f'_* \omega_{X'/C'}^{\otimes m} \cong h^* \sum_{m \geq 0} f_* \omega_{X/C}^{\otimes m}. \quad (2.56.1)$$

Proof. Note that (2.56.1) is just the claim that push-forward commutes with flat base change $h: C' \rightarrow C$. The substantial part is the assertion that X' has canonical singularities, hence the proj of $\sum_{m \geq 0} f'_* \omega_{X'/C'}^{\otimes m}$ is also the relative canonical model of any resolution of X' .

Pick a point $x \in X$ and set $c = f(x)$. We may assume that C and C' are the spectra of a DVRs with local parameters t and s . Thus the Henselisation of (x, X) can be given as a hypersurface

$$(x_1 \cdots x_m = t) \subset (\mathbf{A}_C^n, 0), \quad (2.56.2)$$

where \mathbf{A}_C^n denotes the Henselisation of \mathbb{A}_k^n at $(0, 0)$.

If $h^*t = \phi(s)$ then (x', X') can be given as a hypersurface

$$(x_1 \cdots x_m = \phi(s)) \subset (\mathbf{A}_{C'}^n, 0). \quad (2.56.3)$$

Thus the main claim is that the singularity defined by (2.56.3) is canonical.

If we are over a field then (2.56.3) defines a toric singularity and we are done, essentially as in (4.70.3). We check below that although there is no torus action on the base C , we can compute the simplest blow-ups suggested by toric geometry and everything works out as expected.

(Note, however, that although the pair $(\mathbb{A}_k^n, (x_1 \cdots x_n = 0))$ is lc, this is not a completely toric question. We need to understand all exceptional divisors over \mathbb{A}_k^n , not just the toric ones; see [Kol13b, 2.11].) \square

LEMMA 2.57. *Let T be a DVR with local parameter t and residue field k and \mathbf{A}_T^n the Henselisation of \mathbb{A}_T^n at $(0, 0)$. Let $m \leq n$ and e be natural numbers and ϕ a regular function on \mathbf{A}_T^n . Set*

$$X := X(m, n, e, \phi) = (x_1 \cdots x_m = t^e + t^{e+1}\phi(x_1, \dots, x_n)) \subset (\mathbf{A}_T^n, 0) \quad (2.57.1)$$

and let D be the divisor $(t = 0) \subset X$. Then the pair (X, D) is log canonical and X is canonical, near the origin.

Proof. If $\text{char } k = 0$, this immediately follows from (2.10), so the main point is that it also holds for any DVR.

If $m = 0$ or $e = 0$ then X is empty and we are done. Otherwise we can set $x'_m := x_m(1 + t\phi)^{-1}$ to get the simpler equation $x_1 \cdots x_m = t^e$. For inductive purposes we introduce a new variable s and work with the more general systems

$$\begin{aligned} X &:= (x_1 \cdots x_m - s^e = x_{m+1} \cdots x_{m+r}s - t = 0) \subset (\mathbf{A}_T^{n+1}, 0) \\ D &:= (t = 0), \quad \text{where } 0 \leq r \leq n - m. \end{aligned} \quad (2.57.2)$$

The case $r = 0$ corresponds to (2.57.1). We use induction on m and e .

Let E be an exceptional divisor over X and v the corresponding valuation. Assume first that $v(x_1) \geq v(s)$. We blow up $(x_1 = s = 0)$. In the affine chart where $x'_1 := x_1/s$ we get the new equations

$$x'_1 x_2 \cdots x_m - s^{e-1} = x_{m+1} \cdots x_{m+r}s - t = 0 \quad (2.57.2)$$

defining (X', D') . A local generator of $\omega_{X'/T}(D')$ is

$$\frac{1}{t} \cdot \frac{dx_2 \wedge \cdots \wedge dx_n}{x_2 \cdots x_{m+r}}, \quad (2.57.3)$$

which is unchanged by pull-back.

Such operations reduce e , until we reach a situation where $v(x_i) < v(s)$ for every i . If $v(x_i) = 0$ for some i and $i \neq m$ then x_i is nonzero at the generic point of center X E . Thus we can set $x'_m := x_i x_m$ and reduce the value of m . Thus we may assume that $v(x_i) > 0$ for $i = 1, \dots, m$. Since $\sum v(x_i) = e \cdot v(s)$, we conclude that $e < m$. If $e \geq 2$ then we may assume that $v(x_e)$ is the smallest. Set $x'_i = x_i/x_e$ for $i = 1, \dots, e-1$ and $s' := s/x_e$. We get new equations

$$x'_1 \cdots x'_{e-1} x_{e+1} \cdots x_m - (s')^e = x_e x_{m+1} \cdots x_{m+r} s' - t = 0 \quad (2.57.4)$$

defining (X', D') and the value of m dropped. The pull-back of the form (2.57.3) is

$$\begin{aligned} & \frac{1}{t} \cdot \frac{d(x_e x'_2) \wedge \cdots \wedge d(x_e x'_{e-1}) \wedge dx_e \wedge \cdots \wedge dx_n}{(x_e x'_2) \cdots (x_e x'_{e-1}) x_e \cdots x_{m+r}} \\ &= \frac{1}{t} \cdot \frac{dx'_2 \wedge \cdots \wedge dx'_{e-1} \wedge dx_e \wedge \cdots \wedge dx_n}{x'_2 \cdots x'_{e-1} x_e \cdots x_{m+r}}, \end{aligned} \quad (2.57.5)$$

which is again a local generator of $\omega_{X'/T}(D')$.

Eventually we reach the situation where $e = 1$. We can now eliminate s and, after setting $r + m \mapsto m$, rewrite the system as

$$\begin{aligned} X &:= (x_1 \cdots x_m = t) \subset (\mathbf{A}_T^n, 0) \\ D &:= (t = 0). \end{aligned} \quad (2.57.6)$$

Now X is regular, this case was treated in [Kol13b, 2.11]. \square

Other extension theorems.

We discuss a collection of other results about extending 1-parameter families of varieties or pairs. These can be useful in many situations.

2.58 (Extending a stable family without base change).

Let C be a smooth curve over a field of characteristic 0, $C^\circ \subset C$ an open and dense subscheme and $f^\circ: (X^\circ, \Delta^\circ) \rightarrow C^\circ$ a stable morphism. Here we consider the question of how to extend f° to a proper morphism $f: X \rightarrow C$ in a ‘nice’ way without a base change. For simplicity assume that X° is normal.

We can take any extension of f° to a proper morphism $f_1: X_1 \rightarrow C$, then take a log resolution of $(X_2, \Delta_2) \rightarrow (X_1, \Delta_1)$ and finally the canonical model of (X_2, Δ_2) using [Kol13b, 1.30.7] We have proved:

Claim 2.58.1. There is a unique extension $f: (X, \Delta) \rightarrow C$ such that (X, Δ) is lc and $K_X + \Delta$ is f -ample. \square

This model has the problem that its fibers over the points $C \setminus C^\circ =: \{c_1, \dots, c_r\}$ can be pretty complicated. A slight twist improves the fibers considerably. Instead of starting with the above (X_1, Δ_1) , we take a log resolution $(X_2, \Delta_2 + \sum \text{red } X_{2,c_i})$ of $(X_1, \Delta_1 + \sum \text{red } X_{1,c_i})$ and its canonical model over C . We need to apply [Kol13b, 1.30.7] to $(X_2, \Delta_2 + \sum \text{red } X_{2,c_i} - \epsilon \sum X_{2,c_i})$ and use [Kol13b, 1.28] to obtain the following.

Claim 2.58.2. There is a unique extension $f: (X, \Delta) \rightarrow C$ such that $(X, \Delta + \sum \text{red } X_{c_i})$ is lc and $K_X + \Delta + \sum \text{red } X_{c_i}$ is f -ample. By adjunction, in this case $(\text{red } X_{c_i}, \text{Diff } \Delta)$ is slc. \square

A variant of this starts with any extension (X_1, Δ_1) and then takes a dlt modification of $(X_1, \Delta_1 + \sum \text{red } X_{1,c_i})$ as in [Kol13b, 1.36].

Claim 2.58.3. There is a dlt modification $(Y^\circ, \Delta_Y^\circ) \rightarrow (X^\circ, \Delta^\circ)$ and an extension of it to $g: (Y, \Delta_Y) \rightarrow C$ such that $(Y, \Delta + \sum \text{red } Y_{c_i})$ is dlt. \square

Taking a minimal model of the above $g: (Y, \Delta_Y) \rightarrow C$ yields another useful version.

Claim 2.58.4. There is a dlt modification $(Y^\circ, \Delta_Y^\circ) \rightarrow (X^\circ, \Delta^\circ)$ and an extension of it to $g: (Y, \Delta_Y) \rightarrow C$ such that $(Y, \Delta + \sum \text{red } Y_{c_i})$ is dlt and $K_X + \Delta + \sum \text{red } X_{c_i}$ is f -nef. \square

Finally, if we are willing to change X° drastically, [Kol13b, 10.46] gives the following.

Claim 2.58.5. There is a log resolution $(Y^\circ, \Delta_Y^\circ) \rightarrow (X^\circ, \Delta^\circ)$ and an extension of it to $g: (Y, \Delta_Y) \rightarrow C$ such that $(Y, \Delta_Y + \text{red } Y_c)$ is snc for every $c \in C$. \square

Let us also mention the following very strong variant of (2.58.5), traditionally called the ‘semi-stable reduction theorem.’ We do not use it, and one of the points of our proof of (2.49) was to show that the much easier (2.51) and (2.52) are enough for our purposes.

THEOREM 2.59. [KKMSD73] *Let C be a smooth curve over a field of characteristic 0, $f: X \rightarrow C$ a flat morphism of finite type and D a divisor on X . Then there is a smooth curve B , a finite surjection $\pi: B \rightarrow C$ and a log resolution $g: Y \rightarrow X \times_C B$ such that for every $b \in B$,*

$$(2.59.1) \quad g_*^{-1}(D \times_C B) + \text{Ex}(g) + Y_b \text{ is an snc divisor and}$$

$$(2.59.2) \quad Y_b \text{ is reduced.} \quad \square$$

The positive or mixed characteristic analogs of (2.59) are not known, but the following result on ‘semi-stable alterations’ holds in general.

THEOREM 2.60. [dJ96, Sec.6] *Let T be a 1-dimensional regular scheme, $f: X \rightarrow T$ a flat morphism of finite type whose generic fiber is geometrically reduced. Then there is a 1-dimensional regular scheme S , a finite surjection $\pi: S \rightarrow T$ and a generically finite, separable, proper morphism $g: Y \rightarrow X \times_T S$ such that for every $s \in S$, Y_s is a reduced snc divisor.* \square

The following variant of (2.52) is an easy consequence of (2.60). (See [BLR90, Chap.IV] for similar results.)

COROLLARY 2.61. *Let $f: X \rightarrow T$ be a flat morphism of finite type from a pure dimensional scheme to a 1-dimensional regular scheme T . Then there is a 1-dimensional regular scheme S and a finite morphism $\pi: S \rightarrow T$ such that every fiber of the projection of the normalization $\overline{X} \times_T \overline{S} \rightarrow S$ is generically reduced.* \square

2.5. Cohomology of the structure sheaf

In studying moduli questions, it is very useful to know that certain numerical invariants are locally constant. In this section we study the deformation invariance of (the dimension of) certain cohomology groups. The key to this is the Du Bois property of slc pairs. The definition of Du Bois singularities is rather complicated, but fortunately for our applications we need to know only the following two facts.

2.62 (Properties of Du Bois singularities). Let M be a complex analytic variety. Since constant functions are analytic, there is an injection of sheaves $\mathbb{C}_M \hookrightarrow \mathcal{O}_M^{\text{an}}$. Taking cohomologies we get

$$H^i(M, \mathbb{C}) \rightarrow H^i(M, \mathcal{O}_M^{\text{an}}).$$

If X is projective over \mathbb{C} and X^{an} the corresponding analytic variety, then, by the GAGA theorems (cf. [Ser56] or [Har77, App.B]), $H^i(X^{\text{an}}, \mathcal{O}_X^{\text{an}}) \cong H^i(X, \mathcal{O}_X)$.

If X is also smooth, Hodge theory tells us that

$$H^i(X^{\text{an}}, \mathbb{C}) \rightarrow H^{0,i}(X^{\text{an}}, \mathbb{C}) \cong H^i(X^{\text{an}}, \mathcal{O}_X^{\text{an}}) \cong H^i(X, \mathcal{O}_X)$$

is surjective. Du Bois singularities were essentially defined to preserve this surjectivity [DB81, Ste83]. (There does not seem to be a good definition of Du Bois singularities in positive characteristic; see however [KK20].) Thus we have the following.

Property 2.62.1. Let X be a proper variety over \mathbb{C} with Du Bois singularities. Then the natural maps

$$H^i(X^{\text{an}}, \mathbb{C}) \rightarrow H^i(X^{\text{an}}, \mathcal{O}_X^{\text{an}}) \cong H^i(X, \mathcal{O}_X).$$

are surjective.

Next we need to know which singularities are Du Bois. Over a field of characteristic 0, rational singularities are Du Bois; see [Kol95b, 12.9] and [Kov99] but for our applications the key result is the following. The normal, projective case is proved in [KK10] and extended to the non-normal case in [Kol13b, 6.32]. The quasi-projective version is in [KK20].

Property 2.62.2. Let (X, Δ) be an slc pair over \mathbb{C} . Then X has Du Bois singularities.

These are the only facts we need to know about Du Bois singularities.

The main use of (2.62.1) is through the following base-change theorem, due to [DJ74, DB81].

THEOREM 2.63. *Let S be a Noetherian scheme over a field of characteristic 0 and $f: X \rightarrow S$ a flat, proper morphism. Assume that the fiber X_s is Du Bois for some $s \in S$. Then there is an open neighborhood $s \in S^\circ \subset S$ such that, for all i ,*

(2.63.1) *$R^i f_* \mathcal{O}_X$ is locally free and compatible with base change over S° and*

(2.63.2) *$s \mapsto h^i(X_s, \mathcal{O}_{X_s})$ is a locally constant function on S° .*

Proof. By Cohomology and Base Change [Har77, III.12.11], the theorem is equivalent to proving that the restriction maps

$$\phi_s^i: R^i f_* \mathcal{O}_X \rightarrow H^i(X_s, \mathcal{O}_{X_s}) \quad (2.63.3)$$

are surjective for every i . By the Theorem on Formal Functions [Har77, III.11.1], it is enough to prove this when S is replaced by any 0-dimensional scheme S_n whose closed point is s .

Thus assume from now on that we have a flat, proper morphism $f_n: X_n \rightarrow S_n$, $s \in S_n$ is the only closed point and X_s is Du Bois. Then $H^0(S_n, R^i f_* \mathcal{O}_X) = H^i(X_n, \mathcal{O}_{X_n})$, hence we can identify the ϕ_s^i with the maps

$$\psi^i: H^i(X_n, \mathcal{O}_{X_n}) \rightarrow H^i(X_s, \mathcal{O}_{X_s}). \quad (2.63.4)$$

By the Lefschetz principle we may assume that $k(s) \cong \mathbb{C}$ and then both sides of (2.63.4) are unchanged if we replace X_n by the corresponding analytic space X_n^{an} . Let \mathbb{C}_{X_n} (resp. \mathbb{C}_{X_s}) denote the sheaf of locally constant functions on X_n (resp. X_s) and $j_n: \mathbb{C}_{X_n} \rightarrow \mathcal{O}_{X_n}$ (resp. $j_s: \mathbb{C}_{X_s} \rightarrow \mathcal{O}_{X_s}$) the natural inclusions. We have a commutative diagram

$$\begin{array}{ccc} H^i(X_n, \mathbb{C}_{X_n}) & \xrightarrow{\alpha^i} & H^i(X_s, \mathbb{C}_{X_s}) \\ j_n^i \downarrow & & \downarrow j_s^i \\ H^i(X_n, \mathcal{O}_{X_n}) & \xrightarrow{\psi^i} & H^i(X_s, \mathcal{O}_{X_s}). \end{array}$$

Note that α^i is an isomorphism since the inclusion $X_s \hookrightarrow X_n$ is a homeomorphism and j_s^i is surjective since X_s is Du Bois. Thus ψ^i is also surjective. \square

DEFINITION 2.64. A scheme Y is said to be *potentially slc* or *slc-type* if, for every point $y \in Y$, there is an effective \mathbb{R} -divisor Δ_y on Y such that (Y, Δ_y) is slc at y .

Let $f: X \rightarrow S$ be a flat morphism. We say that f has *potentially slc fibers over closed points* if the fiber X_s is potentially slc for every closed point $s \in S$.

One can similarly define the notion *potentially klt*, and so on.

In our final applications, the Δ_s usually come as the restriction of a global divisor Δ to X_s , but here we do not assume this.

If (X_s, Δ_s) is semi-log-canonical then X_s is Du Bois by (2.62.2), hence (2.63) implies the following.

COROLLARY 2.65. *Let S be a Noetherian scheme over a field of characteristic 0 and $f: X \rightarrow S$ a proper and flat morphism with potentially slc fibers over closed points. Then, for all i ,*

(2.65.1) *$R^i f_* \mathcal{O}_X$ is locally free and compatible with base change and*

(2.65.2) *if S is connected, then $h^i(X_s, \mathcal{O}_{X_s})$ is independent of $s \in S$.* \square

We can derive from (2.65) similar results for other line bundles. A line bundle L on X is called *f -semi-ample* if there is an $m > 0$ such that L^m is f -generated by global sections. That is, the natural map $f^*(f_*(L^m)) \rightarrow L^m$ is surjective. Equivalently, L^m is the pull-back of a relatively ample line bundle by a suitable morphism $X \rightarrow Y$.

COROLLARY 2.66. *Let S be a Noetherian, connected scheme over a field of characteristic 0 and $f: X \rightarrow S$ a proper and flat morphism with potentially slc fibers over closed points. Let L be an f -semi-ample line bundle on X . Then, for all i ,*

(2.66.1) *$R^i f_*(L^{-1})$ is locally free and compatible with base change and*

(2.66.2) *$h^i(X_s, L_{X_s}^{-1})$ is independent of $s \in S$.*

Proof. The question is local on S , thus we may assume that S is local with closed point s . Chose $m > 0$ such that L^m is f -generated by global sections. Since S is affine, L^m is generated by global sections. By (2.12), there is a finite morphism $\pi: Y \rightarrow X$ such that $\pi_* \mathcal{O}_Y = \sum_{r=0}^{m-1} L^{-r}$ and $f \circ \pi: (Y, \pi^{-1} \Delta) \rightarrow S$ also has potentially slc fiber over s . Thus, by (2.65),

$$R^i(f \circ \pi)_* \mathcal{O}_Y = \sum_{r=0}^{m-1} R^i f_*(L^{-r})$$

is locally free and compatible with arbitrary base change. Thus the same holds for every summand. \square

COROLLARY 2.67. [KK10, KK20] *Let S be a Noetherian, connected scheme over a field of characteristic 0 and $f: X \rightarrow S$ a proper, flat morphism of finite type. Assume that all fibers are potentially slc and X_s is CM for some $s \in S$. Then all fibers of f are CM.*

For arbitrary flat morphisms $\pi: X \rightarrow S$, the set of points $x \in X$ such that the fiber $X_{\pi(x)}$ is CM at x is open (10.2), but usually not closed. (Many such examples can be constructed using [Kol13b, 3.9–11].) If π is proper, then the set

$\{s \in S: X_s \text{ is CM}\}$ is open in S (10.3). Thus the key point of (2.67) is to show that, in our case, this set is also closed.

More generally, under the assumptions of (2.67), if one fiber of f is S_k for some k then all fibers of f are S_k , see [KK10, 1.3] and [KK20].

Proof. We prove the projective case; see [KK20] for the proper one.

Let L be an f -ample line bundle on X . If X_s is CM for some $s \in S$, then, by [KM98, 5.72] $H^i(X_s, L_{X_s}^{-r}) = 0$ for $r \gg 1$ and $i < \dim X_s$. Thus by (2.66), the same vanishing holds for every $s \in S$. Hence, using [KM98, 5.72] in the other direction, we conclude that X_s is CM for every $s \in S$. \square

The next theorem implies that $\omega_{X/S}$ exists and commutes with base change for locally stable morphisms. For projective morphisms it was proved in [KK10], the general case is settled in [KK20].

THEOREM 2.68. *Let S be a Noetherian scheme over a field of characteristic 0 and $f: X \rightarrow S$ a flat morphism of finite type with potentially slc fibers over closed points. Then $\omega_{X/S}$ exists and is compatible with base change. That is, for any $g: T \rightarrow S$ the natural map*

$$g_X^* \omega_{X/S} \rightarrow \omega_{X_T/T} \quad \text{is an isomorphism,} \quad (2.68.1)$$

where $g_X: X_T := X \times_S T \rightarrow X$ is the first projection.

We give a detailed proof of the projective case below; this is sufficient for almost all applications in this book. For the general case we refer to [KK20].

The existence of $\omega_{X/S}$ is easy and, as we see in (2.69.1–3), it holds under rather weak restrictions. Compatibility with base change is not automatic; see [Pat13] and (2.43) for some examples.

As we explain in (2.69.4–5), once the definition of $\omega_{X/S}$ is set up right, (2.68) becomes an easy consequence of (2.66).

2.69 (The relative dualizing sheaf II). The best way to define the relative dualizing sheaf is via general duality theory as in [Har66, AK70, Con00]. It is, however, worthwhile to observe that a slight modification of the treatment in [Har77] gives the relative dualizing sheaf in the following cases.

Assumptions. S is an arbitrary Noetherian scheme and $f: X \rightarrow S$ a projective morphism of pure relative dimension n (2.72). (We do not assume flatness.)

Weak duality for \mathbb{P}_S^n 2.69.1. Let $P = \mathbb{P}_S^n$ with projection $g: P \rightarrow S$ and set $\omega_{P/S} := \wedge^n \Omega_{P/S}$.

The proof of [Har77, III.7.1] shows that there is a natural isomorphism, called the *trace map*, $t: R^n g_* \omega_{P/S} \cong \mathcal{O}_S$ and, for any coherent sheaf F on X , there is a natural isomorphism

$$g_* \mathcal{H}om_P(F, \omega_{P/S}) \cong \mathcal{H}om_S(R^n g_* F, \mathcal{O}_S).$$

Note that if S is a point then $g_* \mathcal{H}om_P = \mathcal{H}om_P$, thus we recover the usual formulation of [Har77, III.7.1].

Construction of $\omega_{X/S}$ 2.69.2. Let $f: X \rightarrow S$ be a projective morphism of pure relative dimension n . We construct $\omega_{X/S}$ first locally over S . Once we establish weak duality, the proof of [Har77, III.7.2] shows that a relative dualizing sheaf is unique up to unique isomorphism, hence the local pieces glue together to produce

$\omega_{X/S}$. Working locally over S , using Noether normalization we can assume that there is a finite morphism $\pi: X \rightarrow P = \mathbb{P}_S^n$. Set

$$\omega_{X/S} := \mathcal{H}om_P(\pi_* \mathcal{O}_X, \omega_{P/S}). \quad (2.69.2.a)$$

If f is flat with CM fibers over S then $\pi_* \mathcal{O}_X$ is locally free and so is $\pi_* \omega_{X/S}$. Thus $\omega_{X/S}$ is also flat over S with CM fibers and it commutes with base change. We discuss a local version of this in (2.69.7).

Weak duality for X/S 2.69.3. Let $f: X \rightarrow S$ be a projective morphism of pure relative dimension n (2.72). Use [Har77, Exrc.III.6.10] to show that there is a trace map

$$t: R^n f_* \omega_{X/S} \rightarrow \mathcal{O}_S$$

and for any coherent sheaf F on X there is a natural isomorphism

$$f_* \mathcal{H}om_X(F, \omega_{X/S}) \cong \mathcal{H}om_S(R^n f_* F, \mathcal{O}_S).$$

If F is locally free, this is equivalent to the isomorphism

$$f_*(\omega_{X/S} \otimes F^{-1}) \cong \mathcal{H}om_S(R^n f_* F, \mathcal{O}_S).$$

(Note that $M \mapsto \mathcal{H}om_S(M, \mathcal{O}_S)$ is a duality for locally free coherent \mathcal{O}_S -sheaves but not for all coherent sheaves. In particular, the torsion in $R^n f_* F$ is invisible on the left hand side $f_*(\omega_{X/S} \otimes F^{-1})$. Duality over a base scheme is less symmetric than over a field.)

Flatness of $\omega_{X/S}$ 2.69.4. Let L be relatively ample on X/S . By the proof of [Har77, III.9.9] $\omega_{X/S}$ is flat over S iff $f_*(\omega_{X/S} \otimes L^m)$ is locally free for $m \gg 1$; see also (3.20). If this holds then $\omega_{X/S}$ is the coherent \mathcal{O}_X -sheaf associated to

$$\sum_{m \geq m_0} f_*(\omega_{X/S} \otimes L^m)$$

as a module over the \mathcal{O}_S -algebra $\sum_{m \geq 0} f_*(L^m)$.

Applying weak duality with $F = L^{-m}$ we see that these hold if $R^n f_*(L^{-m})$ is locally free for $m \gg 1$. The latter is satisfied in two important cases.

- (a) $f: X \rightarrow S$ is flat with CM fibers. Then $R^i f_*(L^{-m}) = 0$ for $i < n$ and $m \gg 1$, hence $R^n f_*(L^{-m})$ is locally free of rank $(-1)^n \chi(X_s, L_s^{-m})$ for $m \gg 1$.
- (b) $f: X \rightarrow S$ is flat with potentially slc fibers. Then $R^n f_*(L^{-m})$ is locally free for $m \geq 0$ by (2.66).

Base change properties of $\omega_{X/S}$ 2.69.5. Let $f: X \rightarrow S$ be a projective morphism of pure relative dimension n and L relatively ample. We claim that the following are equivalent.

- (a) $\omega_{X/S}$ commutes with base change as in (2.68.1).
- (b) $R^n f_*(L^{-m})$ is locally free for $m \gg 0$.

To see this first note that (2.69.3–4) show that $\omega_{X/S}$ commutes with base change iff $\mathcal{H}om_S(R^n f_*(L^{-m}), \mathcal{O}_S)$ is locally free and commutes with base change for $m \gg 0$. Finally show that a coherent sheaf M is locally free iff $\mathcal{H}om_S(M, \mathcal{O}_S)$ is locally free and commutes with base change.

Warning on general duality 2.69.6. If F is locally free, then we get a natural pairing

$$R^i f_*(\omega_{X/S} \otimes F^{-1}) \times R^{n-i} f_*(F) \rightarrow R^n f_* \omega_{X/S} \rightarrow \mathcal{O}_S,$$

but this is not a perfect pairing, not even if $f: X \rightarrow S$ is smooth.

If f is CM, one should not expect this pairing to be perfect unless both sheaves on the left are locally free and commute with base change.

More on the CM case 2.69.7. Let $f: X \rightarrow S$ be a projective morphism of pure relative dimension n . We already noted in (2.69.2) that if f is flat with CM fibers over S then the same holds for $\omega_{X/S}$. We consider what happens if f is not everywhere CM. By (10.2) there is a largest open subset $X^{\text{cm}} \subset X$ such that $f|_{X^{\text{cm}}}$ is flat with CM fibers. Assume for simplicity that $X_s \cap X^{\text{cm}}$ is dense in X_s and $s \in S$ is local. Then, for every $x \in X_s \cap X^{\text{cm}}$ one can choose a finite morphism $\pi: X \rightarrow P = \mathbb{P}_S^n$ such that $\pi^{-1}(\pi(x)) \subset X^{\text{cm}}$. Thus $\pi_*\mathcal{O}_X$ is locally free at $\pi(x)$ and so is $\pi_*\omega_{X/S}$. Thus we have proved that the restriction of $\omega_{X/S}$ to X^{cm} is

- (a) flat over S with CM fibers and
- (b) commutes with base change.

This is actually true for all finite type morphisms, one just needs to find a local analog of the projection π (see Section 10.7) and show that (2.69.2.a) holds if π is finite; see [Con00] for details.

COROLLARY 2.70. *Let S be a Noetherian scheme over a field of characteristic 0 and $f: X \rightarrow S$ a proper and flat morphism with potentially slc fibers over closed points. Let L be an f -semi-ample line bundle on X . Then, for all i ,*

(2.70.1) $R^i f_*(\omega_{X/S} \otimes L)$ is locally free and compatible with base change and

(2.70.2) $h^i(X_s, \omega_{X_s} \otimes L_s)$ is independent of $s \in S$.

In particular, for $L = \mathcal{O}_X$ we get that

(2.70.3) $R^i f_*\omega_{X/S}$ is locally free and compatible with base change and

(2.70.4) $h^i(X_s, \omega_{X_s})$ is independent of $s \in S$.

If the fibers X_s are CM, then $H^i(X_s, \omega_{X_s} \otimes L_s)$ is dual to $H^{n-i}(X_s, L_s^{-1})$ and (2.70) follows from (2.66). If the fibers X_s are not CM, the relationship between (2.70) and (2.66) is not so clear.

Proof. Let us start with the case $i = 0$. By weak duality (2.69.3),

$$f_*(\omega_{X/S} \otimes L) \cong \text{Hom}_S(R^n f_*(L^{-1}), \mathcal{O}_S),$$

where $n = \dim(X/S)$. By (2.66), $R^n f_*(L^{-1})$ is locally free and compatible with base change, hence so is $f_*(\omega_{X/S} \otimes L)$. Thus (2.70.1) holds for $i = 0$. Next we use this and induction on n to get the $i > 0$ cases.

Choose M very ample on X such that $R^i f_*(\omega_{X/S} \otimes L \otimes M) = 0$ for $i > 0$, and this also holds after any base change. Working locally on S , as in the proof of (2.66), let $H \subset X$ be a general member of $|M|$ such that $H \rightarrow S$ is also flat with potentially slc fibers (2.12). The push-forward of the sequence

$$0 \rightarrow \omega_{X/S} \otimes L \rightarrow \omega_{X/S} \otimes L \otimes M \rightarrow \omega_{H/S} \otimes L \rightarrow 0$$

gives isomorphisms

$$R^i f_*(\omega_{X/S} \otimes L) \cong R^{i-1} f_*(\omega_{H/S} \otimes L) \quad \text{for } i \geq 2.$$

Using induction, these imply that (2.70.1) holds for $i \geq 2$.

The beginning of the push-forward is an exact sequence

$$0 \rightarrow f_*(\omega_{X/S} \otimes L) \rightarrow f_*(\omega_{X/S} \otimes L \otimes M) \rightarrow f_*(\omega_{H/S} \otimes L) \rightarrow R^1 f_*(\omega_{X/S} \otimes L) \rightarrow 0.$$

We already proved that the first 3 terms are locally free. In general, this does not imply that the last term is locally free, but this implication holds if S is the spectrum of an Artin ring (2.71).

In general, pick any point $s \in S$ with maximal ideal sheaf m_s . Set $A_n := \mathcal{O}_{s,S}/m_s^n$ and $X_n := \text{Spec}(\mathcal{O}_X/f^*m_s^n)$. By the above considerations,

$$H^1(X_n, (\omega_{X/S} \otimes L)|_{X_n})$$

is a free A_n -module and the restriction maps

$$H^1(X_n, (\omega_{X/S} \otimes L)|_{X_n}) \otimes_{A_n} k(s) \rightarrow H^1(X_s, \omega_{X_s} \otimes L_s)$$

are isomorphisms. By the Theorem on Formal Functions [Har77, III.11.1], this implies that $R^1f_*(\omega_{X/S} \otimes L)$ is locally free and commutes with base change. \square

2.71. Let (A, m) be a local Artin ring. Let F be a free A -module and $j: A \hookrightarrow F$ an injection. We claim that $j(A)$ is a direct summand of F . Indeed, let $r \geq 1$ be the smallest natural number such that $m^r A = 0$. Note that $m^{r-1}m = 0$. If $j(A) \subset mF$ then $m^{r-1}A = 0$, a contradiction. Thus $j(A)$ is a direct summand of F . By induction this shows that any injection between free A -modules is split. This also implies that if

$$0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n \rightarrow 0$$

is an exact sequence of A -modules and all but one of them are free then they are all free.

2.72 (Pure dimensional morphisms). A finite type morphism $f: X \rightarrow S$ is said to have *pure relative dimension* n if for every integral scheme T and every $h: T \rightarrow S$, every irreducible component of $X \times_S T$ has dimension $\dim T + n$. We also say that f is *pure dimensional* if it is pure of relative dimension n for some n . It is enough to check this property for all cases when T is the spectrum of a DVR.

Applying the definition when T is a point shows that if f has pure relative dimension n then every fiber of f has pure dimension n , but the converse does not always hold. For instance, let C be a curve and $\pi: \bar{C} \rightarrow C$ the normalization. If C is nodal then π does not have pure relative dimension 0 since $\bar{C} \times_C \bar{C}$ contains 2 isolated points; if C is cuspidal then it does. However, the converse does hold in several important cases.

Claim 2.72.1. Let $f: X \rightarrow S$ be a finite type morphism whose fibers have pure dimension n . Then f has pure relative dimension n iff it is universally open. Thus both properties hold if f is flat.

Proof. Both properties can be checked after base change to spectra of DVRs. In the latter case the equivalence is clear and flatness implies both. \square

The following is called *Chevalley's criterion*, see [Gro60, IV.14.4.1].

Claim 2.72.2. Let $f: X \rightarrow S$ be a finite type morphism whose fibers have pure dimension n . Assume that S is normal (or geometrically unibranch) and X is irreducible. Then f is universally open.

Proof. By an easy limit argument, it is enough to check openness after base change for finite type, affine morphisms $S' \rightarrow S$; see [Gro60, IV.8.10.1]. We may thus assume that $S' \subset \mathbb{A}_S^n$ for some n . The restriction of an open morphism to the preimage of a closed subset is also open, thus it is enough to show that the natural

morphism $f^{(n)} : \mathbb{A}_Y^n \rightarrow \mathbb{A}_S^n$ is open for every n . If S is normal then so is \mathbb{A}_S^n , thus it is enough to show that, under the assumptions of (2.72.2), the map f is open.

To see openness, let $U \subset X$ be an open set and $x \in U$ a closed point. We need to show that $f(U)$ contains an open neighborhood of $s := f(x)$. Let $x \in W \subset X$ be an irreducible component of a complete intersection of n Cartier divisors such that x is an isolated point of $W \cap X_s$. It is enough to prove that $f(U \cap W)$ contains an open neighborhood of s . After extending $W \rightarrow S$ to a proper morphism and Stein factorization, we are reduced to showing that (2.72.2) holds for finite morphisms.

Since $f(U)$ is constructible, it is open iff it is closed under generalization. The latter holds by the going-down theorem; see for instance [AM69, 5.16]. \square

Remark 2.72.3. The dimension can have unexpected behavior for morphisms that are not of finite type. For example, $\text{Spec}_k k(x)$ has dimension 0, but, after base change to $\text{Spec}_k k(y)$, we get the 1-dimensional $k(y)$ -scheme $\text{Spec}_{k(y)} k(x) \otimes_k k(y)$.

2.6. Families of divisors I

Assumptions. In this Section we work with arbitrary schemes.

We saw in (2.68) that for locally stable morphisms $g: (X, \Delta) \rightarrow C$ the relative dualizing sheaf $\omega_{X/C}$ commutes with base change. We also saw in (2.42) that its powers $\omega_{X/C}^{[m]}$ usually do not commute with base change. Here we consider this question for a general divisor D : What does it mean to restrict a divisor D on X to a fiber X_c and how are the two sheaves $\mathcal{O}_X(D)|_{X_c}$ and $\mathcal{O}_{X_c}(D|_{X_c})$ related?

2.73 (One-parameter families of divisors). Let T be a regular 1-dimensional scheme and $f: X \rightarrow T$ a flat, proper morphism. For simplicity assume for now that X is normal. Let D be an effective Weil divisor on X . Under what conditions can we view D as giving a ‘reasonable’ family of Weil divisors on the fibers of f ?

We can view D as a subscheme of X and, if $\text{Supp } D$ does not contain any irreducible component of any fiber X_t , then $f|_D: D \rightarrow T$ is flat, hence the fibers D_t form a flat family of subschemes of the fibers X_t . The D_t may have embedded points, ignoring them gives a well-defined effective Weil divisor on the fiber X_t . Let us denote it temporarily by $[D_t]$. Understanding the difference between the subscheme D_t and the divisor $[D_t]$ is the key to dealing with many issues. As a rule of thumb, D defines a ‘nice’ family of divisors iff $D_t = [D_t]$ for every $t \in T$.

It can happen that D_t is contained in $\text{Sing } X_t$ for some t . These are the cases when the correspondence between Weil divisors and rank 1 reflexive sheaves breaks down. Fortunately, this does not happen for locally stable families. That is, we can restrict to the cases when the D_t are Mumford divisors, that is, when X_t is smooth at all generic points of D_t (4.20.4).

It is now time to drop the normality assumption, and work with divisors in one of the following general setting. (Further generalizations will be considered in Sections 5.8 and 9.4.) We start with the absolute version.

- (1.a) X is a pure dimensional, reduced scheme and $H \subset X$ a Cartier divisor such that X is S_3 along H .
- (1.b) D is a Mumford divisor along H (4.78), that is, a Weil divisor on X such that $\text{Supp } D$ contains neither an irreducible component of H nor a codimension 1 irreducible component of $\text{Sing } H$.

(1.c) There is a closed subscheme $Z \subset X$ such that $D|_{X \setminus Z}$ is a Cartier divisor and $\text{codim}_H(H \cap Z) \geq 2$. (If X is excellent, this is implied by the previous 2.)

In the relative version, we assume the following.

(2.a) T is a regular, 1-dimensional, irreducible scheme and $f: X \rightarrow T$ is a flat, pure dimensional morphism whose fibers are reduced and S_2 .

(2.b) D is a relative Mumford divisor (4.78).

(2.c) There is a closed subscheme $Z \subset X$ such that $D|_{X \setminus Z}$ is a Cartier divisor and $\text{codim}_{X_t}(X_t \cap Z) \geq 2$ for every $t \in T$. (If X is excellent, this is implied by the previous 2.)

Under these conditions, $[D_t]$ (resp. $[D_H]$) is defined as the unique Weil divisor on X_t (resp. H) that agrees with the restriction of the Cartier divisor $D|_{X \setminus Z}$ to $X_t \setminus Z$ (resp. $H \setminus Z$).

If D is effective, it can be identified with a subscheme of pure codimension 1 of X and then D_t denotes the fiber of this subscheme over $t \in T$. Similarly, $D_H := H \cap D$. As we noted before, D_t and $[D_t]$ differ only in the former possibly having some embedded points.

PROPOSITION 2.74 (Absolute version). *Notation and assumptions as in (2.73.1.a–c) with D effective. The following conditions are equivalent.*

(2.74.1) \mathcal{O}_D has depth ≥ 2 at every point of $H \cap Z$.

(2.74.2) D_H has no embedded points.

(2.74.3) $D_H = [D_H]$.

(2.74.4) $\mathcal{O}_X(-D)$ has depth ≥ 3 at every point of $H \cap Z$.

(2.74.5) $\mathcal{O}_X(-D)|_H$ is S_2 .

(2.74.6) The restriction map $r_0: \mathcal{O}_X(-D)|_H \rightarrow \mathcal{O}_H(-[D_H])$ is an isomorphism.

(2.74.7) The following sequence is exact

$$0 \rightarrow \mathcal{O}_X(D - H) \rightarrow \mathcal{O}_X(H) \rightarrow \mathcal{O}_H(-[D_H]) \rightarrow 0.$$

Proof. Let s be a local equation of H . Then s is not a zero divisor on \mathcal{O}_D and $\mathcal{O}_{D_0} = \mathcal{O}_D/(s)$. Thus (1) \Leftrightarrow (2) and (3) is just a reformulation of (2). A similar argument gives that (4) \Leftrightarrow (5). Since $\mathcal{O}_X(-D)$ is S_2 , r_0 is an injection and an isomorphism outside Z . Since $\mathcal{O}_H(-[D_0])$ is S_2 by definition, it is the S_2 -hull of $\mathcal{O}_X(-D)|_H$; see (9.13.4). Thus r_0 is surjective $\Leftrightarrow r_0$ is an isomorphism $\Leftrightarrow \mathcal{O}_X(-D)|_H$ is S_2 . This proves (5) \Leftrightarrow (6).

Since \mathcal{O}_X has depth ≥ 3 at every codimension ≥ 2 point of H , the exact sequence

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

and an easy lemma (10.27) shows that (1) \Leftrightarrow (4). \square

PROPOSITION 2.75 (Relative version). *Notation and assumptions as in (2.73.2.a–c) with D effective. Let $0 \in T$ be a closed point and $g \in T$ the generic point. The following conditions are equivalent.*

(2.75.1) \mathcal{O}_D has depth ≥ 2 at every point of $X_0 \cap Z$.

(2.75.2) D_0 has no embedded points.

(2.75.3) $D_0 = [D_0]$.

(2.75.4) $\mathcal{O}_X(-D)$ has depth ≥ 3 at every point of $X_0 \cap Z$.

(2.75.5) $\mathcal{O}_X(-D)|_{X_0}$ is S_2 .

(2.75.6) *The restriction map $r_0: \mathcal{O}_X(-D)|_{X_0} \rightarrow \mathcal{O}_{X_0}(-[D_0])$ is an isomorphism.*

If f is projective and $\mathcal{O}_X(1)$ is f -ample then these are also equivalent to:

(2.75.7) $\chi(X_0, \mathcal{O}_{X_0}(-[D_0])(m)) = \chi(X_g, \mathcal{O}_{X_g}(-D_g)(m))$ for all $m \in \mathbb{Z}$.

If $\dim(X_0 \cap Z) = 0$ then these are further equivalent to:

(2.75.8) $\chi(X_0, \mathcal{O}_{X_0}(-[D_0])) = \chi(X_g, \mathcal{O}_{X_g}(-D_g))$.

Proof. The equivalences of (1–6) follow from (2.74). If f is projective then, since $\mathcal{O}_X(-D)$ is flat over T ,

$$\begin{aligned} \chi(X_g, \mathcal{O}_{X_g}(-D_g)(m)) &= \chi(X_g, \mathcal{O}_X(m) \otimes \mathcal{O}_X(-D)|_{X_g}) \\ &= \chi(X_0, \mathcal{O}_X(m) \otimes \mathcal{O}_X(-D)|_{X_0}). \end{aligned}$$

Therefore the difference of the two sides in (7) is $\chi(X_0, \mathcal{O}_{X_0}(m) \otimes Q)$ where $Q := \text{coker } r_0$. Thus $Q = 0$ iff equality holds in (7), hence (6) \Leftrightarrow (7).

If $\dim(X_0 \cap Z) = 0$ then Q has 0-dimensional support, thus

$$\chi(X_0, \mathcal{O}_{X_0}(m) \otimes Q) = \chi(X_0, Q) = H^0(X_0, Q),$$

so, in this case, (7) is equivalent to (8). \square

Note that (2.75) shows that one can go rather freely between effective divisors and their ideal sheaves when studying restrictions. Much of the above results on ideal sheaves generalize to arbitrary sheaves; these are worked out in Sections 5.8 and 9.4.

As shown by (2.75.4), the conditions (2.75) are all preserved by linear equivalence. However, they are not preserved by sums of divisors.

EXAMPLE 2.76. Consider a family of smooth quadrics $Q \subset \mathbb{P}^3 \times \mathbb{A}^1$ degenerating to the quadric cone Q_0 . Take four families of lines L^i, M^i such that $L_0^1, L_0^2, M_0^1, M_0^2$ are 4 distinct lines in Q_0 , $L_c^1 \neq L_c^2$ are in one family of lines on Q_c and $M_c^1 \neq M_c^2$ are in the other family for $c \neq 0$. Note that

$$(Q, \frac{1}{2}(L^1 + L^2 + M^1 + M^2)) \rightarrow \mathbb{A}^1$$

is a locally stable family.

Each of the 4 families of lines L^i, M^i is a flat family of Weil divisors.

For pairs of lines, flatness is more complicated. $L^1 + L^2$ is *not* a flat family (the flat limit has an embedded point at the vertex) but $L^i + M^j$ is a flat family for every i, j . The union of any 3 of them, for instance $L^1 + L^2 + M^1$ is again a flat family and so is $L^1 + L^2 + M^1 + M^2$.

Next we give examples of divisors and divisorial sheaves that satisfy the equivalent conditions of (2.75). We state them using the equivalent form (2.75.6).

PROPOSITION 2.77. *Let $f: (X, \Delta) \rightarrow C$ be a locally stable morphism to a smooth curve defined over a field of characteristic 0 and $c \in C$ a closed point.*

(2.77.1) *If B is a \mathbb{Q} -Cartier \mathbb{Z} -divisor then $\mathcal{O}_X(-B)|_{X_c} \cong \mathcal{O}_{X_c}(-B|_{X_c})$.*

(2.77.2) *If $\Delta = 0$ then $\omega_{X/C}^{[m]}|_{X_c} \cong \omega_{X_c}^{[m]}$ for every $m \in \mathbb{Z}$.*

(2.77.3) *If $m\Delta$ is a \mathbb{Z} -divisor then*

$$(\omega_{X/C}^{[m]}(m\Delta))|_{X_c} \cong \omega_{X_c}^{[m]}(m\Delta|_{X_c}).$$

(2.77.4) *If $m\Delta$ is a \mathbb{Z} -divisor then*

$$(\omega_{X/C}^{[m+1]}(m\Delta))|_{X_c} \cong \omega_{X_c}^{[m+1]}(m\Delta|_{X_c}).$$

(2.77.5) Assume that $\Delta = \sum(1 - \frac{1}{r_i})D_i$ for some $r_i \in \mathbb{N}$. Then, for every $m \in \mathbb{Z}$,

$$(\omega_{X/C}^{[m]}(\lfloor m\Delta \rfloor))|_{X_c} \cong \omega_{X_c}^{[m]}(\lfloor m\Delta \rfloor|_{X_c}).$$

(2.77.6) Assume that $\Delta = \sum c_i D_i$ and $1 - \frac{1}{m} \leq c_i \leq 1$ for every i . Then

$$(\omega_{X/C}^{[m]}(\lfloor m\Delta \rfloor))|_{X_c} \cong \omega_{X_c}^{[m]}(\lfloor m\Delta \rfloor|_{X_c}). \quad (2.77.6.m)$$

(2.77.7) The set $\{m \in \mathbb{N} : (2.77.6.m) \text{ holds}\}$ has positive density.

(2.77.8) All the above claims also hold for $\omega_{X/C}^{[m]}(\lfloor m\Delta \rfloor + B)$, where B is any \mathbb{Q} -Cartier \mathbb{Z} -divisor.

Proof. Let D be a Weil divisor on X as in (2.73.2–4). Assume that there is an effective \mathbb{R} -divisor $\Delta' \leq \Delta$ and an \mathbb{R} -Cartier \mathbb{R} -divisor L such that $D \sim_{\mathbb{R}} \Delta' + L$. Then $\mathcal{O}_X(-D)$ satisfies the equivalent conditions of (2.75) by (11.7).

In cases (1–3) we can take $\Delta' = 0$ and $L := -m(K_{X/C} + \Delta)$ and in case (4) we use $\Delta' = \Delta$ and $L := -(m+1)(K_{X/C} + \Delta)$.

For (5–6) we employ $\Delta' = m\Delta - \lfloor m\Delta \rfloor$ and $L := -m(K_{X/C} + \Delta)$. The assumptions on the coefficients of Δ ensure that $\Delta' \leq \Delta$. (Note that if $m\Delta - \lfloor m\Delta \rfloor \leq \Delta$ for every m then in fact every coefficient of Δ is of the form $1 - \frac{1}{r}$ for some $r \in \mathbb{N}$.) Claim (7) follows from (11.41.2).

Finally note that if $D \sim_{\mathbb{R}} \Delta' + L$ and B is \mathbb{Q} -Cartier then $D + B \sim_{\mathbb{R}} \Delta' + L$, so the proofs are unchanged when we add B . \square

These results are close to being optimal. For instance, under the assumptions of (2.77.3), if n is different from m and $m+1$ then the two sheaves

$$(\omega_{X/C}^{[n]}(mD))|_{X_c} \quad \text{and} \quad \omega_{X_c}^{[n]}(mD|_{X_c})$$

are frequently different, see (2.40.3). In general, as shown by (2.42), even the two sheaves $(\omega_{X/C}^{[m]})|_{X_c}$ and $\omega_{X_c}^{[m]}$ can be different if $\Delta \neq 0$. However, a considerable generalization of the cases (2.77.5–6) is proved in Section 2.7.

2.7. Boundary with coefficients $> \frac{1}{2}$

Assumptions. In this Section we work with varieties over a field of characteristic 0.

2.78 (Boundaries and embedded points). Consider a locally stable morphism $f: (X, \Delta = \sum a_i D^i) \rightarrow C$ to a smooth curve C . It is very tempting to think of each fiber (X_c, Δ_c) as a compound object $(X_c, D_c^i: i \in I, a_i: i \in I)$ consisting of the scheme X_c , the divisors $D_c^i := D^i|_{X_c}$ and their coefficients a_i . Two problems make this simple picture questionable.

(2.78.1) Different D_c^i may have an irreducible component E_c in common. Our definition of the fiber says that we should treat E_c as a divisor with coefficient $\sum_{i \in I} \text{coeff}_E D_c^i$. The individual D_c^i do not seem to be part of the data any more.

(2.78.2) The D_c^i may have embedded points. Do we ignore them or do we take them into consideration?

One could hope that the first problem (2.78.1) is just a matter of bookkeeping, but this does not seem to be the case, as shown by the examples (2.76). Similar examples were given in (2.40). In both cases the coefficients in Δ were $\leq \frac{1}{2}$.

The aim of this section is to show that these examples were optimal; the problems (2.78.1–2) do not occur if the coefficients in Δ are all $> \frac{1}{2}$. We start with the case when the coefficients are 1.

Given a locally stable map $f: (X, \Delta) \rightarrow C$ it is not true that the lc centers of the fibers (X_c, Δ_c) form a flat family. Indeed, there are many cases when the generic fiber is smooth but a special fiber is not klt. However, as we show next, the specialization of an lc center on the generic fiber becomes a union of lc centers on a special fiber. Set theoretically this follows from adjunction (11.20) and (11.25.4), but now we prove this even scheme theoretically.

THEOREM 2.79. *Let C be a smooth curve over a field of characteristic 0, $f: (X, \Delta) \rightarrow C$ a locally stable morphism and $Z \subset X$ any union of lc centers of (X, Δ) . Then $f|_Z: Z \rightarrow C$ is flat with reduced fibers and, for every $c \in C$, the fiber Z_c is a union of lc centers of (X_c, Δ_c) (scheme theoretically).*

Proof. Z is reduced, and by (2.14), every irreducible component of Z dominates C . Thus $f|_Z: Z \rightarrow C$ is flat. We can write its fibers as $Z_c = X_c \cap Z$. Since $X_c + Z$ is a union of lc centers of $(X, X_c + \Delta)$, it is seminormal (11.25.2) and $X_c \cap Z$ is reduced by (11.25.3). The last claim follows from (11.23). \square

In the divisorial case we can say more.

COROLLARY 2.80. *Let C be a smooth curve over a field of characteristic 0 and $f: (X, \Delta) \rightarrow C$ a locally stable (resp. stable) morphism. Let $\{D_i: i \in I\}$ be irreducible components of $\lfloor \Delta \rfloor$ and set $D := \cup_{i \in I} D_i$. Then*

$$f|_D: (D, \text{Diff}_D(\Delta - D)) \rightarrow C$$

is locally stable (resp. stable).

Proof. We have already proved in (2.50) that we have a locally stable (resp. stable) morphism on the normalization of D . Using (2.11) it remains to prove that D is demi-normal. The fibers of $f|_D: D \rightarrow C$ are reduced, hence S_1 , so D is S_2 . In codimension 1 D has only nodes by (11.23.5). hence it is demi-normal. \square

When the coefficients are in $(\frac{1}{2}, 1]$, we start with a simple result.

2.81 (Restriction and rounding down). Let $f: (X, \Delta = \sum_{i \in I} a_i D^i) \rightarrow C$ be a locally stable family over a 1-dimensional, regular scheme.

By (2.4), (X_c, Δ_c) is slc, hence every component of Δ_c appears with coefficient ≤ 1 . For a divisor $A \subset X_c$,

$$1 \geq \text{coeff}_A \Delta_c = \sum_{i \in I} a_i \cdot \text{coeff}_A D_c^i.$$

Since the $\text{coeff}_A D_c^i$ are natural numbers, we get the following properties.

- (2.81.1) If $a_i > \frac{1}{2}$ then every irreducible component of D_c^i has multiplicity 1.
- (2.81.2) If $a_i + a_j > 1$ and $i \neq j$ then the divisors D_c^i and D_c^j have no irreducible components in common.

Next let $\Theta = \sum_j b_j B^j$ be an \mathbb{R} -divisor on X . If every irreducible component of B_c^j has multiplicity 1 and the different restrictions have no irreducible components

in common, then $\Theta_c = \sum_j b_j B_c^j$ is the irreducible decomposition of Θ_c . Combining with (1–2) we get:

Claim 2.81.3. Assume that $\text{Supp } \Theta \subset \text{Supp}(\Delta^{>1/2})$. Then

(a) $\text{coeff}(\Theta|_H) \subset \text{coeff } \Theta$ and

(b) $[\Theta|_H] = [\Theta]|_H$. □

Applying this to $\Theta = m\Delta$ gives the following.

Corollary 2.81.4. If $\text{coeff } \Delta \subset (\frac{1}{2}, 1]$ then $[m\Delta_c] = [m\Delta]_c$ for every m . □

The next result of [Kol14] solves the embedded point problem (2.78.2) when all the occurring coefficients are $> \frac{1}{2}$.

THEOREM 2.82. *Let $f: (X, \Delta = \sum_{i \in I} a_i D_i) \rightarrow C$ be a locally stable morphism to a smooth curve over a field of characteristic 0. Let $J \subset I$ be any subset such that $a_j > \frac{1}{2}$ for every $j \in J$ and set $D_J := \cup_{j \in J} D_j$. Then*

(2.82.1) $f|_{D_J}: D_J \rightarrow C$ is flat with reduced fibers,

(2.82.2) D is S_2 and

(2.82.3) $\mathcal{O}_X(-D)$ is S_3 along every closed fiber.

Proof. Note that each D_i is a log center of (X, Δ) (11.24) and $\text{mld}(D_i, X, \Delta) = 1 - a_i$ by (11.22.2). Thus $\text{mld}(D_J, X, \Delta) < \frac{1}{2}$.

Let X_c be any fiber of f . Then $(X, X_c + \Delta)$ is slc and

$$\text{mld}(D_i, X, X_c + \Delta) = \text{mld}(D_i, X, \Delta) < \frac{1}{2},$$

since none of the D_i is contained in X_c . Each irreducible component of X_c is a log canonical center of $(X, X_c + \Delta)$ (11.23), thus $\text{mld}(X_c, X, X_c + \Delta) = 0$. Therefore, $\text{mld}(D_J, X, X_c + \Delta) + \text{mld}(X_c, X, X_c + \Delta) < \frac{1}{2}$.

We can apply (11.25.3) to $(X, X_c + \Delta)$ with $W = D_J$ and $Z = X_c$ to conclude that $X_c \cap D_J$ is reduced. This proves (1) which in turn implies (2–3) by (2.75). □

For the plurigenera, we have the following generalization of (2.77.5–6).

THEOREM 2.83. [Kol18a] *Let C be a smooth curve over a field of characteristic 0 and $f: (X, \Delta) \rightarrow C$ a locally stable morphism with normal generic fiber. Assume that $\text{coeff } \Delta \subset (\frac{1}{2}, 1]$. Then, for every $c \in C$ and $m \in \mathbb{Z}$,*

$$\omega_{X/C}^{[m]}([m\Delta])|_{X_c} \cong \omega_{X_c}^{[m]}([m\Delta_c]).$$

Method of proof. If $mK_X + [m\Delta]$ is \mathbb{Q} -Cartier, then this follows from (11.7); see also [Ale08, Kol11a] and [Kol13b, 7.20]. Thus we aim to construct a birational modification $X' \rightarrow X$ such that $mK_{X'} + [m\Delta']$ is \mathbb{Q} -Cartier, and then descend from X' to X .

More generally, let $g: Y \rightarrow X$ be a proper morphism of normal varieties, F a coherent sheaf on X , $H \subset X$ a Cartier divisor and $H_Y := g^*H$. Assuming that F is S_m along H_Y , we would like to understand when g_*F is S_m along H . If (the local equation of) H_Y is not a zero divisor on F then the sequence

$$0 \rightarrow F(-H_Y) \rightarrow F \rightarrow F|_{H_Y} \rightarrow 0 \tag{2.83.1}$$

is exact. By push-forward we get the exact sequence

$$0 \rightarrow g_*F(-H) \rightarrow g_*F \rightarrow g_*(F|_{H_Y}) \rightarrow R^1 g_*F(-H_Y) \cong \mathcal{O}_X(-H) \otimes R^1 g_*F \tag{2.83.2}$$

Thus, by (10.27), g_*F is S_m along H if

- (3.a) $R^1g_*F = 0$ and
- (3.b) $g_*(F|_{H_Y})$ is S_{m-1} along H .

In many cases, for instance if g is an isomorphism outside H_Y , these conditions are also necessary.

In our case we choose $F = \mathcal{O}_{X'}(mK_{X'} + \lfloor m\Delta' \rfloor)$ and then we need that

- (4.a) $R^1g_*\mathcal{O}_{X'}(mK_{X'} + \lfloor m\Delta' \rfloor) = 0$,
- (4.b) $g_*(\mathcal{O}_{X'}(mK_{X'} + \lfloor m\Delta' \rfloor)|_{H_Y})$ is S_2 along H and
- (4.c) $g_*\mathcal{O}_{X'}(mK_{X'} + \lfloor m\Delta' \rfloor) \cong \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)$.

For us (4.c) will be easy to satisfy. Using a Kodaira-type vanishing theorem, (4.a) needs some semipositivity condition on $(m-1)K_{X'} + \lfloor m\Delta' \rfloor$. By contrast, (11.51) suggests that (4.b) needs some negativity condition on $mK_{X'} + \lfloor m\Delta' \rfloor$.

The next result grew out of trying to satisfy the assumptions of both the relative Kodaira-type vanishing theorem and (11.51). Once one writes the assumptions down, there seems to be a unique way of satisfying them. The proof of (2.83) is then given in (2.86).

PROPOSITION 2.84. *Let (X, Δ) be an lc pair and Δ_1, Δ_2 effective divisors such that $\Delta_1 + \Delta_2 \leq \Delta$. Let B be a Weil \mathbb{Z} -divisor such that $B \sim_{\mathbb{R}} K_X + L + \Delta_1 - c\Delta_2$ where L is \mathbb{R} -Cartier and $c \geq 0$. Then there is a small, lc modification $\pi : (X', \Delta') \rightarrow (X, \Delta)$ such that*

- (2.84.1) B' is \mathbb{Q} -Cartier,
- (2.84.2) $K_{X'} + \Delta'_1$ is \mathbb{R} -Cartier,
- (2.84.3) $\text{Ex}(\pi) \subset \text{Supp}(\Delta' - \Delta'_1)$,
- (2.84.4) none of the lc centers of (X', Δ'_1) are contained in $\text{Ex}(\pi)$,
- (2.84.5) $-\Delta'_2$ is \mathbb{R} -Cartier and π -nef,
- (2.84.6) $\pi_*\mathcal{O}_{X'}(B') = \mathcal{O}_X(B)$,
- (2.84.7) $R^i\pi_*\mathcal{O}_{X'}(B') = 0$ for $i > 0$ and
- (2.84.8) $H^i(X, \mathcal{O}_X(B)) = H^i(X, \mathcal{O}_{X'}(B'))$.

Proof. We construct $\pi : (X', \Delta') \rightarrow (X, \Delta)$ in 2 steps. First we apply (11.31) to $(X, \Delta - \Delta_2)$. We get $\tau_1 : (X^*, \Delta^*) \rightarrow (X, \Delta)$ such that

$$-\Delta_2^* \sim_{\mathbb{R}} K_{X^*} + \Delta^* - \Delta_2^* - \tau_1^*(K_X + \Delta)$$

is τ_1 -ample and its support contains the exceptional set of τ_1 . Since $\Delta_1^* \leq \Delta^* - \Delta_2^*$, we can next apply (11.31) to (X^*, Δ_1^*) to get $\tau_2 : (X', \Delta') \rightarrow (X^*, \Delta^*)$. Set $\pi := \tau_1 \circ \tau_2$. By construction $\Delta'_2 = \tau_2^*\Delta_2^*$ and $K_{X'} + \Delta'_1$ are \mathbb{R} -Cartier, and so is $B' \sim_{\mathbb{R}} (K_{X'} + \Delta'_1) + \pi^*L - c\Delta'_2$.

Using (11.51) with $N = -B'$ and $B = 0$ gives (6). Furthermore, $\text{Ex}(\pi)$ is contained in $\text{Supp}(\Delta'_2) \cup \text{Supp}(\Delta' - \Delta'_1)$. Since $\Delta'_2 \leq \Delta' - \Delta'_1$, this implies that $\text{Ex}(\pi) \subset \text{Supp}(\Delta' - \Delta'_1)$. Thus none of the lc centers of (X', Δ'_1) are contained in $\text{Ex}(\pi)$. Since $-\Delta'_2 = -\tau_2^*\Delta_2^*$ is π -nef, so is $\pi^*L - \Delta'_2$ and these in turn imply that $R^i\pi_*\mathcal{O}_{X'}(B') = 0$ for $i > 0$ by (11.33). Finally the Leray spectral sequence shows (8). \square

Now we come to the core of the proof.

PROPOSITION 2.85. *Let $(X, S + \Delta)$ be an lc pair where S is \mathbb{Q} -Cartier. Let B be a Weil \mathbb{Z} -divisor that is Mumford along S (4.78) and Δ_3 an effective \mathbb{R} -divisor such that*

$$(2.85.1) \quad B \sim_{\mathbb{R}} -\Delta_3,$$

$$(2.85.2) \quad \Delta_3 \leq \lceil \Delta^{(>1/2)} \rceil \text{ and}$$

$$(2.85.3) \quad \lfloor \Delta_3 \rfloor = 0.$$

Then $\mathcal{O}_X(B)$ is S_3 along S .

Proof. Assume first that B is \mathbb{Q} -Cartier. Then (X, Δ) is also an slc pair and none of its lc centers are contained in S by (11.23.8). Hence $\mathcal{O}_X(B)$ is S_3 along S by (11.7).

Next assume that B is arbitrary but S is Cartier with equation $s = 0$. The question is local on X , so we may assume that $K_X + \Delta \sim_{\mathbb{R}} 0$. By assumption (1)

$$B \sim_{\mathbb{R}} -\Delta_3 \sim_{\mathbb{R}} K_X + \Delta - \Delta_3. \quad (2.85.4)$$

Set $\Delta_1 := (\Delta - \Delta_3)^{\geq 0}$ and $\Delta_2 := \epsilon(\Delta - \Delta_3)^{< 0}$ for some $0 < \epsilon \ll 1$. It is clear that $\text{Supp } \Delta_1, \text{Supp } \Delta_2$ have no common irreducible components and $\Delta_1 + \Delta_2 \leq \Delta$ for $0 < \epsilon \ll 1$. Furthermore, $B \sim_{\mathbb{R}} K_X + \Delta_1 - c\Delta_2$ with $c := \epsilon^{-1}$. Using these Δ_1, Δ_2 in (2.84) we obtain $\pi : (X', \Delta') \rightarrow (X, \Delta)$.

Note that B' is \mathbb{Q} -Cartier by (2.84.1), $(X', S' + \Delta')$ is lc and $\text{Ex}(\pi) \subset \text{Supp}(\Delta' - \Delta'_1)$ by (2.84.3). Thus none of the lc centers of $(X', S' + \Delta'_1)$ are contained in $\text{Ex}(\pi)$, in particular, S' is smooth at the generic points of all exceptional divisors of $\pi_S := \pi|_{S'} : S' \rightarrow S$. Thus B' is also Mumford along S' , hence, as we proved at the beginning, $\mathcal{O}_{X'}(B')$ is S_3 along S' . Thus the sequence

$$0 \rightarrow \mathcal{O}_{X'}(B' - S') \rightarrow \mathcal{O}_{X'}(B') \rightarrow \mathcal{O}_{S'}(B'|_{S'}) \rightarrow 0 \quad (2.85.5)$$

is exact by (2.74). Since $R^1\pi_*\mathcal{O}_{X'}(B') = 0$ by (2.84.7), pushing (2.85.5) forward and using (2.84.6) gives an exact sequence

$$0 \rightarrow \mathcal{O}_X(B - S) \rightarrow \mathcal{O}_X(B) \rightarrow (\pi_S)_*\mathcal{O}_{S'}(B'|_{S'}) \rightarrow 0. \quad (2.85.6)$$

Again by (2.74), $\mathcal{O}_X(B)$ is S_3 along S iff $(\pi_S)_*\mathcal{O}_{S'}(B'|_{S'})$ is S_2 . The latter is equivalent to

$$(\pi_S)_*\mathcal{O}_{S'}(B'|_{S'}) \stackrel{?}{=} \mathcal{O}_S(B|_S). \quad (2.85.7)$$

By assumption $B + \Delta_3 \sim_{\mathbb{R}} 0$, hence $B' + \Delta'_3 \sim_{\mathbb{R}} 0$. Thus $(B' + \Delta'_3)|_{S'} \sim_{\mathbb{R}} 0$ and using (11.51) with $N = -(B' + \Delta'_3)|_{S'}$ and $B = 0$ gives that

$$(\pi_S)_*\mathcal{O}_{S'}(\lfloor B'|_{S'} + \Delta'_3|_{S'} \rfloor) = \mathcal{O}_S(\lfloor B|_S + \Delta_3|_S \rfloor). \quad (2.85.8)$$

Since S' is Cartier, $B'|_{S'}$ is a \mathbb{Z} -divisor, so

$$\lfloor B'|_{S'} + \Delta'_3|_{S'} \rfloor = B'|_{S'} + \lfloor \Delta'_3|_{S'} \rfloor \quad \text{and} \quad \lfloor B|_S + \Delta_3|_S \rfloor = B|_S + \lfloor \Delta_3|_S \rfloor.$$

Next (2.81.3) implies that $\lfloor \Delta_3|_S \rfloor = \lfloor \Delta_3 \rfloor|_S = 0$ and $\lfloor \Delta'_3|_{S'} \rfloor = \lfloor \Delta'_3 \rfloor|_{S'} = 0$. Thus

$$(\pi_S)_*\mathcal{O}_{S'}(B'|_{S'}) = \mathcal{O}_S(B|_S), \quad (2.85.9)$$

and we are done with this case.

Finally, if S is only \mathbb{Q} -Cartier, a suitable cyclic cover reduces everything to the Cartier case, as in (11.15). \square

2.86 (Proof of (2.83)). We may assume that X is affine and $K_X + \Delta \sim_{\mathbb{R}} 0$. Pick a fiber X_c and let $x \in X_c$ be a point of codimension 1. Then either X_c and X are both smooth at x or X_c has a node and $x \notin \text{Supp } \Delta$. Thus $mK_X + \lfloor m\Delta \rfloor$ is Cartier at x , hence a general divisor $B \sim mK_X + \lfloor m\Delta \rfloor$ is Mumford along X_c .

We apply (2.85) to B with $\Delta_3 := m\Delta - \lfloor m\Delta \rfloor$. Thus

$$B \sim mK_X + \lfloor m\Delta \rfloor = m(K_X + \Delta) - \Delta_3 \sim_{\mathbb{R}} -\Delta_3.$$

It is clear that $\Delta_3 \leq \lceil \Delta^{(>1/2)} \rceil = \text{Supp } \Delta$. So the assumptions of (2.85) are satisfied and $\mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor) \cong \mathcal{O}_X(B)$ is S_3 along X_c . By (2.75) this implies (2.83). \square

2.8. Grothendieck–Lefschetz-type theorems

The following theorem was conjectured in [Kol13a] and proved there in the lc case. For normal schemes the proof is given in [BdJ14], aside from possible p -torsion in characteristic $p > 0$. The general case is established in [Kol16a] and [dJ15].

THEOREM 2.87. *Let $(x \in X)$ be an excellent, local scheme of pure dimension ≥ 4 over a field such that $\text{depth}_x \mathcal{O}_X \geq 3$. Let $x \in D \subset X$ be a Cartier divisor. Then the restriction map*

$$r_D^X: \text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(x, D) \quad \text{is an injection.}$$

Recall that the *local Picard group* $\text{Pic}^{\text{loc}}(x, X)$ is defined as $\text{Pic}(X \setminus \{x\})$.

We only discuss the situation when X is normal and essentially of finite type over a field k ; this is the only case that we use in this book. The non-normal case is reduced to the normal one in [Kol16a]. The general setting can be reduced to the finite type cases by a short but subtle approximation argument; see [BdJ14, Sec.1.4] for details.

The argument in this section proves a weaker version: the kernel of $r_D^X: \text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(x, D)$ is torsion. That is, if L is a line bundle on $U := X \setminus \{x\}$ such that $L_D := L|_{U \cap D} \cong \mathcal{O}_{U \cap D}$ then $L^m \cong \mathcal{O}_U$ for some $m > 0$. Then we show in the next section that in fact L is trivial.

The proof is somewhat roundabout. The main step is to prove a variant of (2.87) in characteristic p ; see (2.91). Then in (2.92) we reduce everything to positive characteristic and lift back to characteristic 0 using (2.88).

During the proof we need several general results on cohomology groups of sheaves over quasi-affine schemes, these are recalled in (10.28).

Our discussions present these steps in the reverse order. The reason is that the proof of (2.88) is the simplest, showing the key ideas. The proof of (2.91) follows the same path but with several technical detours.

THEOREM 2.88. *Let $(x \in X)$ be a local scheme such that $\text{depth}_x \mathcal{O}_X \geq 4$. Let $x \in D \subset X$ be a Cartier divisor. Set $U := X \setminus \{x\}$ and $U_D := D \setminus \{x\}$. Let L be a coherent, rank 1, S_2 sheaf on U such that $L_D := L|_{U_D} \cong \mathcal{O}_{U_D}$. Then $L \cong \mathcal{O}_U$.*

Proof. Let t be a defining equation of D and consider the exact sequence

$$0 \rightarrow L \xrightarrow{t} L \xrightarrow{r} L_D \cong \mathcal{O}_{U_D} \rightarrow 0.$$

Take cohomologies to get

$$\begin{array}{ccccccc} H^0(U, L) & \xrightarrow{t} & H^0(U, L) & \xrightarrow{r} & H^0(U_D, L_D \cong \mathcal{O}_{U_D}) & \rightarrow & \\ H^1(U, L) & \xrightarrow{t} & H^1(U, L) & \rightarrow & H^1(U_D, L_D \cong \mathcal{O}_{U_D}). & & \end{array} \quad (2.88.1)$$

In order to prove that t is injective we start on the right hand side. The cohomology sequence of

$$0 \rightarrow \mathcal{O}_U \xrightarrow{t} \mathcal{O}_U \rightarrow \mathcal{O}_{U_D} \rightarrow 0$$

contains the piece

$$H^1(U, \mathcal{O}_U) \rightarrow H^1(U_D, \mathcal{O}_{U_D}) \rightarrow H^2(U, \mathcal{O}_U). \quad (2.88.2)$$

Since $\text{depth}_x \mathcal{O}_X \geq 4$, (10.28.2–3) imply that $H^i(U, \mathcal{O}_U) = 0$ for $1 \leq i \leq 2$, hence the two sides of (2.88.2) are 0. Thus $H^1(U_D, \mathcal{O}_{U_D}) = 0$ and so the second line of (2.88.1) shows that $t: H^1(U, L) \rightarrow H^1(U, L)$ is surjective. By (10.28.7), $H^1(U, L)$ has finite length since $\dim U \geq 3$, so

$$H^1(U, L) \xrightarrow{t} H^1(U, L) \quad \text{is an isomorphism.}$$

(Note that by (10.28.5), multiplication by t is nilpotent on $H^i(U, \mathcal{O}_U)$ for $i > 0$, thus in fact $H^1(U, L) = 0$.) Thus (2.88.1) shows that the restriction map

$$r: H^0(U, L) \rightarrow H^0(U_D, L_D) \cong H^0(U_D, \mathcal{O}_{U_D}) \quad \text{is surjective.}$$

In particular, the constant 1 section of $H^0(U_D, \mathcal{O}_{U_D})$ lifts to a section $s \in H^0(U, L)$. Thus $L \cong \mathcal{O}_U$ by (2.89). \square

LEMMA 2.89. *Let X be a pure dimensional, S_2 scheme, $D \subset X$ a Cartier divisor and $W \subset D$ a subscheme such that $\text{codim}_D W \geq 2$. Let L be a rank 1, torsion free sheaf on X that is locally free along $D \setminus W$ and s a section of L such that $s|_{D \setminus W}$ is nowhere zero. Then L is trivial and s is nowhere zero in a neighborhood of D .*

Proof. The section s gives an exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{s} L \rightarrow Q \rightarrow 0.$$

By (9.8) every associated prime of Q has codimension 1 in X . Thus $D \cap \text{Supp } Q$ has codimension 1 in D . Therefore D is disjoint from $\text{Supp } Q$ and L is trivial on $X \setminus \text{Supp } Q$. \square

The next example, following [BdJ14] and [Kol13a, 12], shows that (2.88) fails if $\text{depth}_x X = 3$; see also (2.43). As we see afterwards, one can say more if L is locally free on U .

EXAMPLE 2.90. Let (A, Θ) be a principally polarized Abelian variety over a field k . Let $C_a(A, \Theta)$ be the affine cone over A with vertex v . It is easy to compute that $\text{depth}_v C_a(A, \Theta) = 2$, see [Kol13b, 3.12]. Set $X := C_a(A, \Theta) \times \text{Pic}^\circ(A)$ with $f: X \rightarrow \text{Pic}^\circ(A)$ the second projection. Since $L(\Theta)$ has a unique section for every $L \in \text{Pic}^\circ(A)$, there is a unique divisor D_A on $A \times \text{Pic}^\circ(A)$ whose restriction to $A \times \{[L]\}$ is the above divisor. By taking the cone we get a divisor D_X on X .

For $L \in \text{Pic}^\circ(A)$, let $D_{[L]}$ denote the restriction of D_X to the fiber $C_a(A, \Theta) \times \{[L]\}$ of f . We see that

$$(2.90.1) \quad D_{[L]} \text{ is Cartier iff } L \cong \mathcal{O}_A.$$

$$(2.90.2) \quad mD_{[L]} \text{ is Cartier iff } L^m \cong \mathcal{O}_A.$$

$$(2.90.3) \quad D_{[L]} \text{ is not } \mathbb{Q}\text{-Cartier for very general } L \in \text{Pic}^\circ(A).$$

The next result proves that, at least in characteristic p , the kernel of the restriction map between the local Picard groups is torsion.

THEOREM 2.91. [\[BdJ14\]](#) *Let $(x \in X)$ be a normal, excellent, local scheme of characteristic $p > 0$ and dimension ≥ 4 . Let $x \in D \subset X$ be a Cartier divisor. Set $U := X \setminus \{x\}$ and $U_D := D \setminus \{x\}$.*

Let L be a line bundle on U such that $L_D := L|_{U_D} \cong \mathcal{O}_{U_D}$. Then $L^m \cong \mathcal{O}_U$ for some $m > 0$.

Proof. In order to emphasize the similarities, we follow the proof of (2.88) as closely as possible, even though this is somewhat repetitive. In a few places, we need to add technical details to establish results that were obvious under the assumptions of (2.88).

Our main effort goes to proving that there is a normal scheme V and a finite, surjective morphism $\pi: V \rightarrow U$ such that $\pi^*L \cong \mathcal{O}_V$.

We do not know a priori which finite surjective morphism to take, so we work with their direct limit. That is, let \mathcal{O}_X^+ denote the normalization of \mathcal{O}_X in an algebraic closure of the function field of X . We view \mathcal{O}_X^+ as a quasi-coherent sheaf of X . Note that \mathcal{O}_X^+ is the direct limit of the structure sheaves of the normalizations of X in finite degree algebraic extensions of its function field. Set $\mathcal{O}_U^+ := \mathcal{O}_X^+|_U$.

The key result we use is that \mathcal{O}_X^+ is CM, which is a hard theorem due to [\[HH92\]](#). This is the only point where we use that the characteristic is positive. (In characteristic 0 the sheaves \mathcal{O}_X^+ are never CM if $\dim X \geq 3$.)

We use Grothendieck's characterization of CM sheaves by local cohomology groups (see [\[Gro67, Sec.3\]](#) or [\[BH93, 3.5.7\]](#)):

$$H_x^j(X, \mathcal{O}_X^+) = 0 \quad \text{for } j < \dim X. \quad (2.91.1)$$

As in (10.28.2–3) this implies that

$$H^i(U, \mathcal{O}_U^+) = 0 \quad \text{for } 1 \leq i \leq \dim X - 2. \quad (2.91.2)$$

This is the only consequence of the CM property we use.

Let $(t = 0)$ be an equation of D and consider the exact sequence

$$0 \rightarrow L \xrightarrow{t} L \xrightarrow{r} L_D \cong \mathcal{O}_{U_D} \rightarrow 0. \quad (2.91.3)$$

If the constant 1 section of $L_D \cong \mathcal{O}_{U_D}$ can be lifted to a section of L , then $L \cong \mathcal{O}_U$ and we are done. This holds if $H^1(U, L) = 0$, but the latter usually fails. However, as we noted before, we need this only after some finite base change. The groups $H^1(V, \pi^*L)$ usually do not vanish for any finite cover $\pi: V \rightarrow U$, but, rather surprisingly, vanishing holds for their direct limit.

Thus we tensor (2.91.3) with \mathcal{O}_U^+ to get

$$0 \rightarrow L \otimes \mathcal{O}_U^+ \xrightarrow{t} L \otimes \mathcal{O}_U^+ \xrightarrow{r} \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+ \rightarrow 0. \quad (2.91.4)$$

While \mathcal{O}_U^+ is not flat, it is torsion free, so (2.91.4) is still left exact. Next we take cohomologies to get

$$\begin{array}{ccccccc} H^0(U, L \otimes \mathcal{O}_U^+) & \xrightarrow{t} & H^0(U, L \otimes \mathcal{O}_U^+) & \xrightarrow{r} & H^0(U_D, \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+) & \rightarrow & \\ H^1(U, L \otimes \mathcal{O}_U^+) & \xrightarrow{t} & H^1(U, L \otimes \mathcal{O}_U^+) & \rightarrow & H^1(U_D, \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+) & & \end{array} \quad (2.91.5)$$

Note that all of these cohomology groups are naturally $H^0(X, \mathcal{O}_X)$ -modules. (They are even $H^0(X, \mathcal{O}_X^+)$ -modules, but we will not use this richer structure.)

Our next aim is to prove that the second row of (2.91.3) is identically zero. Again we start on the right hand side. The cohomology sequence of

$$0 \rightarrow \mathcal{O}_U^+ \xrightarrow{t} \mathcal{O}_U^+ \rightarrow \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+ \rightarrow 0$$

contains the piece

$$H^1(U, \mathcal{O}_U^+) \rightarrow H^1(U_D, \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+) \rightarrow H^2(U, \mathcal{O}_U^+).$$

The two sides are 0 by (2.91.2) since $\dim X \geq 4$. Thus $H^1(U_D, \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+) = 0$ and so

$$H^1(U, L \otimes \mathcal{O}_U^+) \xrightarrow{t} H^1(U, L \otimes \mathcal{O}_U^+) \quad \text{is surjective.}$$

Equivalently, $H^1(U, L \otimes \mathcal{O}_U^+)$ is t -divisible. (This does not yet imply vanishing since $H^1(U, L \otimes \mathcal{O}_U^+)$ is not a coherent \mathcal{O}_X -module.)

Next we establish that $H^1(U, L \otimes \mathcal{O}_U^+)$ is killed by t^r for $r \gg 1$. Together with t -divisibility, this proves that $H^1(U, L \otimes \mathcal{O}_U^+) = 0$.

Here we use that L is locally free. Since U is quasi-affine, for every point $x' \in U$ there is a global section g of L not vanishing at x' . This gives an exact sequence

$$0 \rightarrow \mathcal{O}_U \xrightarrow{g} L \rightarrow Q_g \rightarrow 0$$

where multiplication by g kills Q_g . Tensoring with \mathcal{O}_X^+ and taking cohomologies we get

$$H^1(U, \mathcal{O}_U^+) \xrightarrow{g} H^1(U, L \otimes \mathcal{O}_U^+) \rightarrow H^1(U, Q_g \otimes \mathcal{O}_U^+).$$

The group on the left is 0 by (2.91.2) and the one on the right is killed by g . Thus multiplication by g kills $H^1(U, L \otimes \mathcal{O}_U^+)$. Using this for every $x' \in U$, we get that the annihilator of $H^1(U, L \otimes \mathcal{O}_U^+)$ is an $m_{x, X}$ -primary ideal. In particular, t^r is in the annihilator of $H^1(U, L \otimes \mathcal{O}_U^+)$ for $r \gg 1$.

These observations together imply that $H^1(U, L \otimes \mathcal{O}_U^+) = 0$. Thus (2.91.3) shows that the restriction map

$$r: H^0(U, L \otimes \mathcal{O}_U^+) \rightarrow H^0(U_D, \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+) \quad \text{is surjective.}$$

In particular, the constant 1 section of $H^0(U_D, \mathcal{O}_{U_D} \otimes \mathcal{O}_U^+)$ lifts to a section $s \in H^0(U, L \otimes \mathcal{O}_U^+)$.

Since \mathcal{O}_U^+ is the direct limit of the structure sheaves of the normalizations of $U \subset X$ in finite degree algebraic extensions, we conclude that there is a normal scheme V and a finite, surjective morphism $\pi: V \rightarrow U$ such that the constant 1 section of

$$H^0(V_D, \pi^* L_D \cong \pi^* \mathcal{O}_{U_D} = \mathcal{O}_{V_D})$$

lifts to a section

$$s_V \in H^0(V, \pi^* L).$$

By (2.89) this implies that $\pi^* L$ is a trivial line bundle on V .

Taking the norm (cf. [Kol13b, 2.40]) then gives that

$$\mathcal{O}_U \cong \text{norm}_{V/U} \mathcal{O}_V \cong \text{norm}_{V/U} \pi^* L \cong L^{\deg V/U}. \quad \square$$

Next we prove a weaker version of (2.87), which, as we noted in the discussion after the statement, can be used to settle the general case as well.

PROPOSITION 2.92. *Let $(x \in X)$ be a normal, local scheme of finite type over a field k of characteristic 0. Let $x \in D \subset X$ be a Cartier divisor. Set $U := X \setminus \{x\}$ and $U_D := D \setminus \{x\}$. Let L be a line bundle on U such that $L_D := L|_{U_D} \cong \mathcal{O}_{U_D}$. Assume that $\dim X \geq 4$ and $\text{depth}_x \mathcal{O}_X \geq 3$.*

Then $L^m \cong \mathcal{O}_U$ for some $m > 0$.

Proof. We use reduction to positive characteristic; see for instance [KM98, p.14] for a more detailed exposition.

There is a finitely generated \mathbb{Z} -algebra $R \subset k$, an R -scheme of finite type X^R , a section $x^R \subset X^R$, a Cartier divisor $x^R \subset D^R \subset X^R$, a line bundle L^R on $U^R := X^R \setminus x^R$ such that $L_D^R := L^R|_{U^D} \cong \mathcal{O}_{U^D}$ where $U_D^R := U^R \cap X^D$. Furthermore, after base change to k and localizing at x we recover the original X , $x \in D \subset X$ and L .

The assumptions of (2.92) are open in families, hence, after inverting finitely many elements of R , we may assume that the following holds.

Let $P \subset R$ be any prime ideal and X^P , $x^P \in D^P \subset X^P$ and L^P the fiber over $\text{Spec } R/P$ of X^R , $x^R \in D^R \subset X^R$ and L^R . Then, after localizing at a generic point of x^P , the assumptions of (2.92) are satisfied, except that $k(P)$ need not have characteristic 0. (The only non-obvious assertion is that normality and the depth are preserved. For these see [Gro60, IV.12.1.6] or (10.2).)

Choose P to be a minimal prime ideal sitting over a prime $p \in \mathbb{Z}$ and localize R at P and X^R at a generic point of x^P . Set $T := \text{Spec } R_P$. Denote the closed point of T by p . We thus have

- (2.92.1) a scheme X^T that is flat over T with normal fibers,
- (2.92.2) a section $x: T \rightarrow X^T$ such that $\text{depth}_{x_p} \mathcal{O}_{X_p^T} = \text{depth}_x \mathcal{O}_X$,
- (2.92.3) a relative Cartier divisor $x(T) \subset D^T \subset X^T$,
- (2.92.4) a line bundle L^T on $U^T := X^T \setminus x(T)$ such that the restriction of L^T to $U_p^T := U^T \cap X_p$ is trivial and

over the generic point of T we recover the original $x \in D \subset X$ and L .

Since the closed fiber has positive characteristic, we can apply (2.91) to show that the restriction of $(L^T)^m$ to U_p^T is trivial for some $m > 0$. Then we use (2.88) to obtain that $(L^T)^m$ is trivial on U^T , hence also on the generic fiber. Thus L^m is trivial for some $m > 0$; proving the assertion in the first case. \square

2.9. Torsion in Grothendieck–Lefschetz-type theorems

Here we complete most of the proof of (2.87) that the kernel is trivial. We proved in (2.91) that the kernel of the restriction map $\text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(x, D)$ is torsion. Now we aim to prove that it is also torsion free. For prime to $\text{char } k(x)$ torsion this is proved in [Gro68, XIII]. In the global setting a short proof is given in [Koll16a], we recall it in (5.53). The general case is due to [dJ15]; see (5.54–5.59).

THEOREM 2.93. *Let $(x \in X)$ be a Noetherian, local scheme and $x \in D \subset X$ a Cartier divisor. Assume that $\text{depth}_x \mathcal{O}_X \geq 3$. Then*

$$\ker[r_D^X: \text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(x, D)] \text{ is torsion free.}$$

Note that, unlike the previous results, this works already when the dimension is ≥ 3 .

Here we discuss two proofs of the weaker claim that the kernel does not contain m -torsion if $\text{char } k(x) \nmid m$, which is enough for all applications in this book. The general case is postponed to Section 5.8.

PROPOSITION 2.94. *Let $(x \in X)$ be a Noetherian, local scheme and $x \in D \subset X$ the support of a Cartier divisor. Assume that $X \setminus Z$ is connected for every closed*

subset Z of dimension ≤ 1 . Then

$$\ker[r_D^X : \text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(x, D)]$$

is torsion free if $\text{char } k(s) = 0$ and is p^∞ -torsion if $\text{char } k(s) = p > 0$.

Proof. Set $U := X \setminus \{x\}$ and $U_D := U \cap D$. Let L be a line bundle on U . Assume that $L|_{U_D} \cong \mathcal{O}_{U_D}$ and pick the smallest $m > 0$ such that $L^m \cong \mathcal{O}_U$. By assumption m is not divisible by $\text{char } k(x)$. We prove that $L \cong \mathcal{O}_U$.

This isomorphism gives a cyclic cover $\pi: \tilde{X} \rightarrow X$ that is étale over U (11.14) and has the same connectedness property. If $L_D \cong \mathcal{O}_{U_D}$ then $\pi^{-1}(D)$ is geometrically reducible and its irreducible components meet only at $\pi^{-1}(x)$. This is impossible by (2.95). \square

We have used the following.

LEMMA 2.95. [Gro68, XIII.2.1] *Let $(x \in X)$ be a Noetherian, local scheme and $x \in D \subset X$ the support of a Cartier divisor. Assume that $X \setminus Z$ is connected for every closed subset Z of dimension $\leq i + 1$. Then $D \setminus Z$ is connected for every closed subset Z of dimension $\leq i$.* \square

Next we discuss a relative variant of (2.93) with an infinitesimal proof, which is basically just an adaptation of the method of [Gro68, XIII.2.1].

THEOREM 2.96. *Let (s, S) be a local scheme, $f: X \rightarrow S$ a flat morphism and $x \in X_s$ a point such that $\text{depth}_x X_s \geq 2$. Set $U := X \setminus \{x\}$ and let L be a line bundle on U . Assume that $L|_{U_s}$ is trivial and L^m is trivial for some m not divisible by $\text{char } k(s)$. Then L is trivial.*

Proof. Set $S_n := \text{Spec}_S \mathcal{O}_S/m_s^n$. First we use (2.97) to show that the restriction of L to $X_n := X \times_S S_n$ is trivial for every n . Then (2.98.1) implies that L itself is trivial. \square

LEMMA 2.97. *Let (A, m) be a local Artin k -algebra and $J \subset m$ an ideal such that $mJ = 0$. Let $f: X \rightarrow \text{Spec } A$ be a flat morphism and $x \in X$ a point such that $\text{depth}_x \mathcal{O}_{X_0} \geq 2$. Set $X_J := X \times_{\text{Spec } A} \text{Spec}(A/J)$. Then the kernel of the restriction map $\text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(x, X_J)$ is a k -vector space (possibly infinite dimensional).*

In particular, if L is an element of the kernel and L^m is trivial for some m not divisible by $\text{char } k$ then L is trivial.

Proof. We have an exact sequence

$$0 \rightarrow J \otimes_k \mathcal{O}_{U_0} \xrightarrow{\tau} \mathcal{O}_U^* \rightarrow \mathcal{O}_{U_J}^* \rightarrow 1$$

where $\tau(g) = 1 + g$ for a local section g of $J\mathcal{O}_U \cong J \otimes_k \mathcal{O}_{U_0}$.

A global section of $\mathcal{O}_{U_J}^*$ extends to a global section of \mathcal{O}_{X_J} since $\text{depth}_x \mathcal{O}_{X_J} = \text{depth}_x \mathcal{O}_{X_0} \geq 2$ and then it lifts to a section of \mathcal{O}_X , which is necessarily nowhere zero by (2.89). Thus the cohomology sequence gives

$$0 \rightarrow J \otimes_k H^1(U, \mathcal{O}_{U_0}) \rightarrow \text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(x, X_J). \quad \square$$

PROPOSITION 2.98. *Let (s, S) be a local scheme with maximal ideal m . Let $f: X \rightarrow S$ be a flat morphism with S_2 -fibers, $X_n := \text{Spec}_X \mathcal{O}_X/m^{n+1}\mathcal{O}_X$ the n th infinitesimal neighborhood of $X_0 := X_s$ and $Z \subset X$ a subscheme that is finite over S with natural injections $j: X \setminus Z \hookrightarrow X$ and $j_n: X_n \setminus Z_n \hookrightarrow X_n$. Let L be an*

invertible sheaf on $X \setminus Z$ and $L_n := L|_{X_n \setminus Z_n}$. Assume that one of the following holds.

(2.98.1) $(j_n)_*(L_n)$ is locally free for every $n \geq 0$.

(2.98.2) $(j_0)_*(L_0)$ is locally free and $R^1(j_0)_*(L_0) = 0$.

Then j_*L is invertible in a neighborhood of Z_0 .

Proof. We may assume that \mathcal{O}_S is m -adically complete and, possibly after passing to a smaller neighborhood of Z_0 , we may assume that f is affine and $(j_0)_*(L_0) \cong \mathcal{O}_{X_0}$. For every n we have an exact sequence

$$0 \rightarrow (m_0^n/m_0^{n+1}) \otimes L_0 \rightarrow L_n \rightarrow L_{n-1} \rightarrow 0.$$

Pushing it forward we get an exact sequence

$$\begin{aligned} 0 \rightarrow (m_0^n/m_0^{n+1}) \otimes (j_0)_*(L_0) &\rightarrow (j_n)_*(L_n) \xrightarrow{r_n} (j_{n-1})_*(L_{n-1}) \rightarrow \\ &\rightarrow (m_0^n/m_0^{n+1}) \otimes R^1(j_0)_*(L_0). \end{aligned}$$

If $(j_n)_*(L_n)$ is locally free then so is its restriction to X_{n-1} and r_n gives a map of locally free sheaves

$$\bar{r}_n: (j_n)_*(L_n)|_{X_{n-1}} \rightarrow (j_{n-1})_*(L_{n-1})$$

that is an isomorphism on $X_{n-1} \setminus Z_{n-1}$. Since $\text{depth}_{Z_{n-1}} X_{n-1} \geq 2$, this implies that \bar{r}_n is an isomorphism and so r_n is surjective. The vanishing of $R^1(j_0)_*(L_0)$ also implies that r_n is surjective. Thus each $(j_n)_*(L_n)$ is locally free along X_n and the constant 1 section of $(j_0)_*(L_0) \cong \mathcal{O}_{X_0}$ lifts back to a nowhere zero global section of $\varprojlim (j_n)_*(L_n)$. Hence $\varprojlim (j_n)_*(L_n) \cong \mathcal{O}_X$ by (2.89).

Furthermore, we have a natural map $j_*L \rightarrow \varprojlim (j_n)_*(L_n) \cong \mathcal{O}_X$ that is an isomorphism on $X \setminus Z$. Since $\text{depth}_Z j_*L \geq 2$, this implies that $j_*L \cong \mathcal{O}_X$. \square

The next example shows that going from formal triviality to triviality is not automatic.

EXAMPLE 2.99. Let $(e, E) \cong (e, E')$ be an elliptic curve. Set $X := (E \setminus \{e\}) \times E'$ and $p: X \rightarrow E'$ the second projection. Let $\Delta \subset X$ be the diagonal and $L = \mathcal{O}_X(\Delta)$.

For $p \in E' \setminus \{e\}$ the line bundle $L|_{X_p}$ is a nontrivial element of

$$\text{Pic}(X_p \setminus \{e\}) \cong \text{Pic}(E \setminus \{e\}) \cong \text{Pic}^\circ(E).$$

but $L|_{X_e}$ is trivial.

Let $X_m \subset X$ denote the m th infinitesimal neighborhood of the fiber $X_0 := X_e$. We have exact sequences

$$H^1(X_0, \mathcal{O}_{X_0}) \rightarrow H^1(X_{m+1}, \mathcal{O}_{X_{m+1}}^*) \rightarrow H^1(X_m, \mathcal{O}_{X_m}^*) \rightarrow H^2(X_0, \mathcal{O}_{X_0}).$$

Since $X_0 \cong E \setminus \{e\}$ is affine, this shows that

$$\text{Pic}(X_m \setminus \{e\}) \cong \text{Pic}(E \setminus \{e\}) \cong \text{Pic}^\circ(E).$$

Thus $L|_{X_m}$ is trivial for every m .

Families of stable varieties

We have defined stable and locally stable families over 1-dimensional regular schemes in Sections 2.1 and 2.4. The first task in this Chapter is to define these notions for families over more general base schemes. It turns out that this is much easier if there is no boundary divisor Δ . Since this case is of considerable interest, we treat it here before delving into the general setting in the next Chapter. While restricting to the special case saves quite a lot of foundational work, the key parts of the proofs of the main theorems stay the same. To avoid repetition, we outline the proofs here but leave the detailed discussions to Chapter 4.

In Section 3.1 we review the theory of Chow varieties and Hilbert schemes. In general these suggest different answers to what a ‘family of varieties’ or a ‘family of divisors’ should be. The main conclusions, (3.11) and (3.13), can be summarized in the following principles.

- A *family of S_2 varieties* should be a **flat** morphism whose geometric fibers are reduced, connected and satisfy Serre’s condition S_2 .
- Flatness is **not** the right condition for divisors on the fibers.

Note that both stability and local stability should be preserved by pull-back. Together with the earlier definitions for 1-parameter families given in (2.3) and (2.44), we get necessary conditions for a family to be stable or locally stable. The next definition declares these conditions to be also sufficient.

DEFINITION 3.1 (Local stability and stability I). Let S be reduced scheme and $f : X \rightarrow S$ a flat morphism whose geometric fibers are reduced and S_2 .

Then $f : X \rightarrow S$ is called *stable* (resp. *locally stable*) iff the family obtained by base change $f_T : X_T \rightarrow T$ is stable as in (2.44) (resp. locally stable as in (2.3)) whenever T is the spectrum of a DVR and $T \rightarrow S$ a morphism.

As we mentioned above, these conditions are clearly necessary, but it seems quite surprising that this definition works. We see in Section 4.1 that, for locally stable families of *pairs*, one needs to make further assumptions about the boundary divisor. We establish the equivalence of (3.1) with the more traditional definitions in (3.37). The definition over arbitrary schemes is given in (3.40).

Let now $f : X \rightarrow S$ be a flat, projective family of S_2 varieties. It turns out that, starting in relative dimension 3, the set of points

$$\{s \in S : X_s \text{ is semi-log-canonical}\}$$

is not even locally closed; see (3.42) for an example. In order to describe the situation, in Section 3.2 we study functors that are representable by a locally closed decomposition (10.84).

We start the study of families of non-Cartier divisors in Section 3.3. As we noted above, this is one of the key new technical issues of the theory.

In Section 3.4 we use a representability theorem (3.36) to clarify the definition of stable and locally stable families, the main result (3.37) gives 5 equivalent definitions of local stability, leading to the definition of stability and local stability over arbitrary bases (3.40). In Section 3.5 we bring these results together in (3.43) to prove the first main theorem of the chapter.

THEOREM 3.2 (Local stability is representable). *Let S be a scheme over a field of characteristic 0 and $f : X \rightarrow S$ a projective morphism. Then there is a locally closed partial decomposition (10.84) $j : S^{\text{ls}} \rightarrow S$ such that the following holds.*

Let W be a scheme and $q : W \rightarrow S$ a morphism. Then the family obtained by base change $f_W : X_W \rightarrow W$ is locally stable iff q factors as $q : W \rightarrow S^{\text{ls}} \rightarrow S$.

Since ampleness is an open condition for a \mathbb{Q} -Cartier divisor, (3.2) implies the following.

COROLLARY 3.3 (Stability is representable). *Let S be a scheme over a field of characteristic 0 and $f : X \rightarrow S$ a projective morphism. Then there is a locally closed partial decomposition $j : S^{\text{stab}} \rightarrow S$ such that the following holds.*

Let W be a reduced scheme and $q : W \rightarrow S$ a morphism. Then the family obtained by base change $f_W : X_W \rightarrow W$ is stable iff q factors as $q : W \rightarrow S^{\text{stab}} \rightarrow S$. \square

One can now jump ahead to Section 6.1 and see that we have the main ingredients in place to construct the coarse moduli space of stable varieties.

To formulate it, let \mathcal{SV} (for stable varieties) denote the functor that associates to a scheme S the set of all stable families $f : X \rightarrow S$, up-to isomorphism.

In order to get a moduli space of finite type, we fix the relative dimension n and the volume $v = \text{vol}(K_{X_s}) := (K_{X_s}^n)$ of the fibers. This gives the subfunctor $\mathcal{SV}(n, v)$. The proof of the following will be given in (6.17).

THEOREM 3.4 (Moduli space of stable varieties). *Let S be a base scheme of characteristic 0 and fix n, v . Then the functor $\mathcal{SV}(n, v)$ has a coarse moduli space $\text{SV}(n, v) \rightarrow S$, which is projective over S .*

Assumptions. In Sections 3.1–3.3 we work with arbitrary schemes but, starting with Section 3.4, the main results are fully proved only over a field of characteristic 0.

However, most of the methods that use the 1-parameter theorems of Chapter 2 and extend them to higher dimensional families, work over arbitrary bases.

3.1. Chow varieties and Hilbert schemes

What is a good family of algebraic varieties? Historically 2 answers emerged to this question. The first one originates with Cayley [Cay60], with a detailed presentation given in [HP47, Chap.X]. The corresponding moduli space is usually called the Chow variety. The second one is due to Grothendieck [Gro62]; it is the theory of Hilbert schemes. For both of them see [Kol96, Chap.I], [Ser06] or the original sources for details.

For the purposes of the following general discussion, a variety is a proper, geometrically reduced and pure dimensional k -scheme.

The theory of Chow varieties suggests the following.

DEFINITION 3.5 (Cayley-Chow variant). A *Cayley-Chow family* of varieties over a reduced base scheme S is a proper, pure dimensional (2.72) morphism $f : X \rightarrow S$, whose fibers X_s are generically geometrically reduced for every $s \in S$.

This is called an *algebraic family of varieties* in [Har77, p.263]. More general Cayley-Chow families are defined in [Kol96, Sec.I.3].

It seems hard to make a precise statement but one can think of Cayley-Chow families as being ‘topologically flat.’ That is, any topological consequence of flatness also holds for Cayley-Chow families. This holds for the Zariski topology but also for the Euclidean topology if we are over \mathbb{C} .

There are 2 disadvantages of Cayley-Chow families. First, basic numerical invariants, for example the arithmetic genus of curves can jump in a Cayley-Chow family. Second, the topological nature of the definition implies that we completely ignore the nilpotent structure of S . In fact, it really does not seem possible to define what a Cayley-Chow family should be over an Artinian base scheme S .

The theory of Hilbert schemes was introduced to solve these problems. It suggests the following definition.

DEFINITION 3.6 (Hilbert-Grothendieck variant). A *Hilbert-Grothendieck family* of varieties is a proper, flat morphism $f : X \rightarrow S$ whose fibers X_s are geometrically reduced and pure dimensional.

Here S is allowed to be non-reduced.

Note that every Hilbert-Grothendieck family is also a Cayley-Chow family and technically it is much better to have a Hilbert-Grothendieck family than a Cayley-Chow family. However, there are many Cayley-Chow families that are not flat.

3.7 (Universal families). Both Cayley-Chow and Hilbert-Grothendieck families are preserved by pull-backs, thus they form a functor. In both cases this functor has a fine moduli space if we work with families that are subvarieties of a given scheme Y/S .

We use $\text{Univ}(\ast)$ to denote a universal family, the target specifies which functor we work with.

Let us thus fix a scheme Y that is projective over a base scheme S . For general existence questions the key case is $Y = \mathbb{P}_S^N$. For any closed subscheme $Y \subset \mathbb{P}_S^N$, the Chow variety (resp. the Hilbert scheme) of Y is naturally a subvariety (resp. subscheme) of the Chow variety (resp. the Hilbert scheme) of \mathbb{P}_S^N and the corresponding universal family is obtained by restriction. (See (3.15) or [Kol96, Secs.I.5] for some cases when Y/S is not projective.)

Chow variety 3.7.1. (See Section 4.7 or [Kol96, Sec.I.3] for details and (3.14) for comments on seminormality.) There is a seminormal S -scheme $\text{Chow}^\circ(Y/S)$ and a universal family

$$\text{Univ}^\circ(Y/S) \rightarrow \text{Chow}^\circ(Y/S) \quad (3.7.1.1)$$

that represents the functor $\text{Chow}^\circ(Y/S)$ of Cayley-Chow subfamilies of Y over seminormal S -schemes. That is, given a seminormal S -scheme $q : T \rightarrow S$,

$$\text{Chow}^\circ(Y/S)(T) := \left\{ \begin{array}{l} \text{closed subsets } X \subset Y \times_S T \text{ such that} \\ X \rightarrow T \text{ is a Cayley-Chow family of varieties} \end{array} \right\}. \quad (3.7.1.2)$$

($\text{Chow}^\circ(Y/S)$ is the ‘open’ part of the full $\text{Chow}(Y/S)$, as defined in [Kol96, Sec.I.3].) If we also fix a relatively very ample line bundle $\mathcal{O}_Y(1)$ then we can

write

$$\text{Chow}^\circ(Y/S) = \coprod_n \text{Chow}_n^\circ(Y/S) = \coprod_{n,d} \text{Chow}_{n,d}^\circ(Y/S), \quad (3.7.1.3)$$

where Chow_n° parametrizes varieties of dimension n and $\text{Chow}_{n,d}^\circ$ parametrizes varieties of dimension n and of degree d . Each $\text{Chow}_{n,d}^\circ(Y/S)$ is of finite type but usually still reducible.

Hilbert scheme 3.7.2. (See [Kol96, Sec.I.1] or [Ser06] for details.) There is an S -scheme $\text{Hilb}^\circ(Y/S)$ and a universal family

$$\text{Univ}^\circ(Y/S) \rightarrow \text{Hilb}^\circ(Y/S) \quad (3.7.2.1)$$

that represents the functor of Hilbert-Grothendieck families

$$\text{Hilb}^\circ(Y/S)(T) := \left\{ \begin{array}{l} \text{closed subschemes } X \subset Y \times_S T \text{ such that} \\ X \rightarrow T \text{ is a flat family of varieties} \end{array} \right\}. \quad (3.7.2.2)$$

More generally, there is an S -scheme $\text{Hilb}(Y/S)$ and a universal family

$$\text{Univ}(Y/S) \rightarrow \text{Hilb}(Y/S) \quad (3.7.2.3)$$

that represents the functor

$$\text{Hilb}(Y/S)(T) := \left\{ \begin{array}{l} \text{closed subschemes } X \subset Y \times_S T \\ \text{such that } X \rightarrow T \text{ is flat} \end{array} \right\}. \quad (3.7.2.4)$$

We can write

$$\text{Hilb}(Y/S) = \coprod_n \text{Hilb}_n(Y/S) = \coprod_H \text{Hilb}_H(Y/S), \quad (3.7.2.5)$$

where Hilb_n parametrizes subschemes of (not necessarily pure) dimension n and Hilb_H parametrizes subschemes with Hilbert polynomial $H(t)$. Each $\text{Hilb}_H(Y/S)$ is projective but usually still reducible.

3.8 (Comparing Chow and Hilb). Given a subscheme $X \subset Y$ of dimension $\leq n$, we get an n dimensional cycle $[X] = \sum_i m_i [X_i]$ where X_i are the n -dimensional irreducible components and m_i is the length of \mathcal{O}_X at the generic point of X_i . (Thus we completely ignore the lower dimensional irreducible components.)

If $m_i = 1$ for every i then $[X] = \sum_i [X_i]$ can be identified with a point in $\text{Chow}^\circ(Y/S)$. In order to make this map everywhere defined, we need to extend the notion of Cayley-Chow families to allow fibers that are formal linear combinations of varieties; see [Kol96, Sec.I.3] for details. The end result is an everywhere defined map $\text{Hilb}_n(Y/S) \dashrightarrow \text{Chow}_n(Y/S)$. Since $\text{Hilb}_n(Y/S)$ is a scheme but $\text{Chow}_n(Y/S)$ is a seminormal variety, it is better to think of it as a morphism defined on the seminormalization

$$\mathfrak{R}_C^H : \text{Hilb}_n(Y/S)^{\text{sn}} \rightarrow \text{Chow}_n(Y/S). \quad (3.8.1)$$

This is a very complicated morphism. As written, its fibers have infinitely many irreducible components for $n \geq 1$ since we can just add disjoint 0-dimensional subschemes to any variety $X \subset Y$ to get new subschemes with the same underlying variety. Even if we restrict to pure dimensional subschemes we get fibers with infinitely many irreducible components. This happens for instance for the fiber over $m[L] \in \text{Chow}_{1,m}(\mathbb{P}^3)$ where $L \subset \mathbb{P}^3$ is a line and $m \geq 2$.

It is much more interesting to understand what happens on

$$\overline{\text{Hilb}}_n^\circ(Y/S) := \text{closure of } \text{Hilb}_n^\circ(Y/S) \text{ in } \text{Hilb}_n(Y/S). \quad (3.8.2)$$

That is, $\overline{\text{Hilb}}_n^\circ(Y/S)$ parametrizes n -dimensional subschemes that occur as limits of varieties. It turns out that the restriction of the Hilbert-to-Chow map

$$\mathfrak{R}_C^H : \overline{\text{Hilb}}_n^\circ(Y/S)^{\text{sn}} \rightarrow \text{Chow}_n(Y/S) \quad (3.8.3)$$

is a local isomorphism at many points. For smooth varieties this is quite clear from the definition of Chow-forms. Classical writers seem to have been fully aware of various equivalent versions, but I did not find an explicit formulation. The normal case, due to [Hir58], is more subtle and in fact quite surprising; see [Har77, III.9.11] for its usual form and (10.63) for a stronger version. These imply the following comparison of Hilbert schemes and Chow varieties.

THEOREM 3.9. *Using the notation of (3.8) let $s \in S$ be a point and $X_s \subset Y_s$ a normal, projective subvariety of dimension n . Then the Hilbert-to-Chow morphism*

$$\mathfrak{R}_C^H : \overline{\text{Hilb}}_n^\circ(Y/S)^{\text{sn}} \rightarrow \text{Chow}_n(Y/S)$$

is a local isomorphism over $[X_s] \in \text{Chow}_n(Y/S)$. \square

Informally speaking, for normal varieties the Cayley-Chow theory is equivalent to the Hilbert-Grothendieck theory, at least over seminormal base schemes.

By contrast, $\text{Hilb}(Y/S)$ and $\text{Chow}(Y/S)$ are different near the class of a singular curve. For example, let $C \subset \mathbb{P}^3$ be a planar, nodal cubic. Then $[C] \in \text{Chow}_1(\mathbb{P}^3)$ is contained in 1 irreducible component of $\text{Hilb}_1(\mathbb{P}^3)$ but in 2 different irreducible components of $\text{Chow}_1(\mathbb{P}^3)$. A general member of one component is a planar, smooth cubic. This component parametrizes flat deformations. A general member of the other component is a smooth, rational, non-planar cubic. The arithmetic genus jumps, so these deformations are not flat. Thus we see that \mathfrak{R}_C^H is not a local isomorphism over $[C] \in \text{Chow}_1(\mathbb{P}^3)$, but this is explained by the change of the genus. It turns out that once we correct for the genus change, (3.9) becomes stronger.

DEFINITION 3.10. Let $X \subset \mathbb{P}^N$ be a closed subscheme of pure dimension n . Let $X \cap L$ denote the intersection of X with $n - 1$ general hyperplanes. Then

$$1 - \chi(X \cap L, \mathcal{O}_{X \cap L})$$

is independent of L . It is called the *sectional genus* of X . (The sectional genus is a linear combination of the 2 highest coefficients of the Hilbert polynomial of X . Knowing the degree of X and its sectional genus is equivalent to knowing the 2 highest coefficients of its Hilbert polynomial.)

It is easy to see that the sectional genus is a constructible and upper semicontinuous function on $\text{Chow}_n^\circ(Y/S)$; see (5.29). Thus there are locally closed subschemes $\text{Chow}_{n,*,g}^\circ(Y/S) \subset \text{Chow}_n^\circ(Y/S)$ that parametrize geometrically reduced cycles with sectional genus g ; see (10.84). (The $*$ stands for the degree which we ignore in these formulas. Also, one can not define the sectional genus for cycles with multiplicities (4.71) though this can easily be corrected.) We can now define the Chow variety parametrizing families with locally constant sectional genus as

$$\text{Chow}_n^{\text{sg}}(Y/S) := \coprod_{n,g} \text{Chow}_{n,*,g}^\circ(Y/S)^{\text{sn}},$$

the disjoint union of the seminormalizations of the $\text{Chow}_{n,*,g}^\circ(Y/S)$.

The sectional genus is constant in a flat family, and we get the following strengthening of (3.9); see (5.29) and (10.63).

THEOREM 3.11. *Using the notation of (3.8) let $s \in S$ be a point and $X_s \subset Y_s$ a geometrically reduced, pure dimensional, projective, S_2 subvariety of dimension n . Then the Hilbert-to-Chow map*

$$\mathfrak{R}_C^H : \overline{\text{Hilb}}_n^\circ(Y/S)^{\text{sn}} \rightarrow \text{Chow}_n^{\text{sg}}(Y/S)$$

is a local isomorphism over $[X_s] \in \text{Chow}_n^{\text{sg}}(Y/S)$.

We can informally summarize these considerations as follows.

PRINCIPLE 3.12. *For geometrically reduced, pure dimensional, projective, S_2 varieties, the Cayley-Chow theory is equivalent to the Hilbert-Grothendieck theory over seminormal base schemes, once we correct for the sectional genus.*

We are studying not just varieties but semi-log-canonical pairs (X, Δ) . The underlying variety is demi-normal, hence geometrically reduced and S_2 . Thus (3.12) says that even if we start with the more general Cayley-Chow families, we end up with flat morphisms $f : X \rightarrow S$ with S_2 fibers. The latter is a class that is well behaved over arbitrary base schemes.

However, the divisorial part is harder to understand. Although we have seen only a few examples supporting it, the following counterpart of (3.12) turns out to give the right picture.

PRINCIPLE 3.13. *For stable families of semi-log-canonical pairs (X, Δ) the Hilbert-Grothendieck theory is optimal for the underlying variety X but the Cayley-Chow theory is the ‘right’ one for the divisorial part Δ .*

3.14 (Comment on seminormality). Hilbert schemes work well over any base scheme, but in [Kol96] the theory of Cayley–Chow families is developed only over seminormal bases. Following the methods of Section 4.7, it is possible to work out the Cayley–Chow theory of geometrically reduced cycles over reduced bases. In characteristic 0 it might be possible to do this for all cycles [Bar75, BM20], but examples of Nagata [Nag55] suggest that in positive characteristic the restriction to seminormal bases may be necessary.

3.15 (Non-projective cases). Let Y be an arbitrary scheme over S . We define $\mathcal{Hilb}(Y/S)(T)$ as the set of all subschemes $X \subset Y \times_S T$ that are proper and flat over T . [Art69] proves that if $Y \rightarrow S$ is locally of finite presentation then the Hilbert functor is represented by a morphism $\text{Hilb}(Y/S) \rightarrow S$ that is also locally of finite presentation. However, in general $\text{Hilb}(Y/S)$ is not a scheme but an algebraic space over S . More generally, $Y \rightarrow S$ is allowed to be an algebraic space.

Most likely similar results hold for $\text{Chow}(Y/S)$ but I am not aware of complete references. See [Kol96, Sec.I.5] for further discussions. The complex analytic generalization is worked out in [BM20].

3.2. Representable properties

Let \mathcal{P} be a property of schemes that is *invariant under base field extensions*. (That is, if K/k is a field extension and X_k is a k -scheme then X_k satisfies \mathcal{P} iff X_K satisfies \mathcal{P} .) For a morphism $f : X \rightarrow S$ one can then consider the set

$$S(\mathcal{P}) := \{s \in S : X_s \text{ satisfies } \mathcal{P}\}.$$

Note that $S(\mathcal{P})$ depends on $f : X \rightarrow S$, so we use the notation $S(\mathcal{P}, X/S)$ if the choice of $f : X \rightarrow S$ is not clear.

In nice situations, $S(\mathcal{P})$ is an open or closed or at least locally closed subset of S . For example satisfying Serre's condition S_m is an open condition for proper, flat morphisms by (10.3) and being singular is a closed condition.

Similarly, if $f : X \rightarrow S$ is a proper morphism of relative dimension 1 then

$$S(\text{stable curve}) := \{s \in S : X_s \text{ is a stable curve}\}$$

is an open subset of S . However, we see in (3.42) that if $f : X \rightarrow S$ is a proper, flat morphism of relative dimension ≥ 3 then

$$S(\text{stable variety}) := \{s \in S : X_s \text{ is a stable variety}\}$$

is not even a locally closed subset of S in general.

We already noted in Section 1.4 that flat morphisms with stable fibers do not give the right moduli problem in higher dimensions and one should look at stable families instead. Thus our main interest is not in the set $S(\text{stable variety})$ but in the class of morphisms $q : T \rightarrow S$ for which the pulled-back family $f_T : X_T \rightarrow T$ is stable. We then hope to prove that this happens in a predictable way. The following definition formalizes this.

DEFINITION 3.16. Let \mathcal{P} be a property of morphisms that is preserved by pull-back. That is, if $X \rightarrow S$ satisfies \mathcal{P} and $q : T \rightarrow S$ is a morphism then $f_T : X_T \rightarrow T$ also satisfies \mathcal{P} . Depending on the situation, pull-back can mean the usual fiber product $X_T := X \times_S T$ or a modified version of it, like the S_2 pull-back to be defined in (3.26), the various versions of divisorial pull-back to be defined in Chapter 4 or the Cayley-Chow pull-back of cycles [Kol96, I.3.18].

We can associate to \mathcal{P} the functor of \mathcal{P} -pull-backs defined for morphisms $W \rightarrow S$ by setting

$$\text{Property}(\mathcal{P})(W) := \begin{cases} 1 & \text{if } X_W \rightarrow W \text{ satisfies } \mathcal{P}, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases} \quad (3.16.1)$$

Thus a morphism $i_{\mathcal{P}} : S^{\mathcal{P}} \rightarrow S$ represents \mathcal{P} -pull-backs iff the following hold.

$$(3.16.2) \quad f^{\mathcal{P}} : X^{\mathcal{P}} := X_{S^{\mathcal{P}}} \rightarrow S^{\mathcal{P}} \text{ satisfies } \mathcal{P} \text{ and}$$

$$(3.16.3) \quad \text{if } f_W : X_W \rightarrow W \text{ satisfies } \mathcal{P} \text{ then } q \text{ factors as } q : T \rightarrow S^{\mathcal{P}} \rightarrow S, \text{ and the factorization is unique.}$$

It is also of interest to understand what happens if we focus on special classes of bases. Let \mathcal{R} be a property of schemes. We say that $i_{\mathcal{P}} : S^{\mathcal{P}} \rightarrow S$ represents \mathcal{P} -pull-backs for \mathcal{R} -schemes if $S^{\mathcal{P}}$ satisfies \mathcal{R} and (3) holds whenever W satisfies \mathcal{R} . In this section we are mostly interested in the properties $\mathcal{R} = (\text{reduced})$, $\mathcal{R} = (\text{seminormal})$ and $\mathcal{R} = (\text{normal})$.

If (3) holds for all $T = (\text{spectrum of a field})$ then $i_{\mathcal{P}} : S^{\mathcal{P}} \rightarrow S$ is geometrically injective (10.83). If (3) holds for all schemes then $i_{\mathcal{P}}$ is a monomorphism (10.83).

In many cases of interest \mathcal{P} is invariant under base field extensions and then $i_{\mathcal{P}} : S^{\mathcal{P}} \rightarrow S$ is also residue field preserving (10.83).

If $X \rightarrow S$ is projective then we are frequently able to prove that $i_{\mathcal{P}} : S^{\mathcal{P}} \rightarrow S$ is a locally closed partial decomposition (10.84).

If $i_{\mathcal{P}} : S^{\mathcal{P}} \rightarrow S$ represents \mathcal{P} -pull-backs and $i_{\mathcal{P}}$ is of finite type (this will always be the case for us) then

$$S(\mathcal{P}) = \{s : X_s \text{ satisfies } \mathcal{P}\} = i_{\mathcal{P}}(S^{\mathcal{P}})$$

is a constructible subset of S . Constructibility is much weaker than representability but we will frequently need constructibility in our proofs of representability.

Let us give some basic examples of representable properties.

EXAMPLE 3.17. Let $f : X \rightarrow S$ be a proper morphism and $m \in \mathbb{N}$. We claim that both of the following functors are representable by a locally closed decomposition for reduced schemes but not representable for all schemes.

- (3.17.1) The functor of pull-backs whose fibers have pure dimension m .
 (3.17.2) The functor of pull-backs of pure relative dimension m (2.72).

EXAMPLE 3.18 (Simultaneous normalization). Sometimes it is best to focus not on a property of a morphism but on a property of its “improvement.” We say that $f : X \rightarrow S$ has *simultaneous normalization* if there is a finite morphism $\pi : \bar{X} \rightarrow X$ such that $\pi_s : \bar{X}_s \rightarrow X_s$ is the normalization for every $s \in S$ and $f \circ \pi : \bar{X} \rightarrow S$ is flat. For example, consider the family of quadrics

$$X := (x_0^2 - x_1^2 + u_2x_2^2 + u_3x_3^2 = 0) \subset \mathbb{P}_{\mathbf{x}}^3 \times \mathbb{A}_{\mathbf{u}}^2.$$

Then the functor of simultaneous normalizations is represented by

$$\{(0, 0)\} \amalg (\mathbb{A}_{\mathbf{u}}^2 \setminus \{(0, 0)\}) \rightarrow \mathbb{A}_{\mathbf{u}}^2.$$

In general, we have the following result, due to [CHL06, Kol11b].

Theorem 3.18.1. Let $f : X \rightarrow S$ be a proper morphism whose fibers X_s are generically geometrically reduced. Then there is a morphism $\pi : S^n \rightarrow S$ such that for any $g : T \rightarrow S$, the fiber product $X \times_S T \rightarrow T$ has a simultaneous normalization iff g factors through $\pi : S^n \rightarrow S$. \square

Flatness is representable.

Let $f : X \rightarrow S$ be a morphism and F a coherent sheaf on X . Given any $q : W \rightarrow S$, we get

$$\begin{array}{ccccc} X \times_S W & =: & X_W & \xrightarrow{q_X} & X \\ & & f_W \downarrow & & \downarrow f \\ & & W & \xrightarrow{q} & S. \end{array}$$

The *functor of flat pull-backs* of F is defined as

$$\text{Flat}(F)(W) := \begin{cases} 1 & \text{if } q_X^* F \text{ is flat over } W, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

One of the most useful representation theorems is the following. The projective case is proved in [Mum66, Lect.8], the proper case is in [Art69].

THEOREM 3.19 (Flattening decomposition theorem). *Let $f : X \rightarrow S$ be a proper morphism and F a coherent sheaf on X . Then the functor of flat pull-backs $\text{Flat}(F)(*)$ is represented by a finite type monomorphism $i^{\text{flat}} : S^{\text{flat}} \rightarrow S$.*

If f is projective then i^{flat} is a locally closed decomposition. \square

One can frequently check flatness using the following numerical criterion which is proved, but not fully stated, in [Har77, III.9.9]. (See also (9.57) for a more precise variant of the last part.)

THEOREM 3.20. *Let $f : X \rightarrow S$ be a projective morphism with relatively ample $\mathcal{O}_X(1)$ and F a coherent sheaf on X . The following are equivalent.*

(3.20.1) F is flat over S .

(3.20.2) $f_*(F(m))$ is locally free for $m \gg 1$.

If S is reduced then these are also equivalent to the following.

(3.20.3) $s \mapsto \chi(X_s, F_s(m))$ is a locally constant function on S . \square

COROLLARY 3.21. *Using the notation of (3.20) assume that S is reduced. Then F is flat over S iff q_X^*F is flat over T whenever T is the spectrum of a DVR and $q : T \rightarrow S$ a morphism. \square*

The local version of (3.21) is also true, but its proof is harder, see [Gro60, IV.11.6, IV.11.8].

THEOREM 3.22. *Let S be a reduced scheme, $f : X \rightarrow S$ a morphism of finite type and F a coherent sheaf on X . Let $x \in X$ be a point and $s = f(x)$. Then F is flat over S at x iff q_X^*F is flat over T along $q_X^{-1}(x)$ for every local morphism $q : (0, T) \rightarrow (s, S)$ from the spectrum of a DVR to S .*

Moreover, it is enough to check this for finitely many local morphisms $q_i : (0, T_i) \rightarrow (s, S)$ whose images together dominate S . \square

REMARK 3.23. Being pure dimensional is an open property for flat, proper morphisms. Thus, using (3.19) we obtain that for any projective morphism $f : X \rightarrow S$ we have a locally closed partial decomposition

$$f^{\text{fp}} : S^{\text{fp}} \rightarrow S$$

that represents flat and pure dimensional pull-backs of f . Next let \mathcal{P} be a property that implies flat and pure dimensional. Assume that $q : T \rightarrow S$ is a morphism such that $f_T : X_T \rightarrow T$ satisfies \mathcal{P} . Then $f_T : X_T \rightarrow T$ is also flat and pure dimensional, hence $q : T \rightarrow S$ factors through f^{fp} . This shows that

$$S^{\mathcal{P}} = (S^{\text{fp}})^{\mathcal{P}}.$$

In particular, if we want to prove that $S^{\mathcal{P}} \rightarrow S$ exists for all projective morphisms, then it is enough to show that it exists for all flat, pure dimensional and projective morphisms. More generally, if $\mathcal{P}_1 \Rightarrow \mathcal{P}_2$ and $S^{\mathcal{P}_2}$ exists then

$$S^{\mathcal{P}_1} = (S^{\mathcal{P}_2})^{\mathcal{P}_1}. \quad (3.23.1)$$

3.3. Divisorial sheaves

We frequently have to deal with divisors $D \subset X$ that are not Cartier, hence the corresponding sheaves $\mathcal{O}_X(D)$ are not locally free. Understanding families of such sheaves is a key aspect of the moduli problem. Many of the results proved here are developed for arbitrary coherent sheaves in Chapter 9.

DEFINITION 3.24 (Divisorial sheaves). A coherent sheaf L on a scheme X is called a *divisorial sheaf* if L is S_2 and there is a closed subset $Z \subset X$ of codimension ≥ 2 such that $L|_{X \setminus Z}$ is locally free of rank 1.

Set $U := X \setminus Z$ and let $j : U \hookrightarrow X$ denote the natural injection. Then $L = j_*(L|_U)$ by (9.8), thus L is uniquely determined by $L|_U$.

Divisorial sheaves for a group, with

$$L_1 \hat{\otimes} L_2 := j_*(L_1|_U \otimes L_2|_U) = (L_1 \otimes L_2)^{[**]}.$$

For powers we use the notation $L^{[m]}$.

The prime examples we have in mind are the following.

Let X be a normal scheme and D a Weil divisor on X . Then $\mathcal{O}_X(D)$ is a divisorial sheaf and we can take $Z = \text{Sing } X$.

Let X be a demi-normal scheme. Then ω_X is a divisorial sheaf and we can take Z to be the non-nodal locus of X .

If $\dim X = 1$ then $Z = \emptyset$ and a divisorial sheaf is the same as an invertible sheaf.

We are mostly interested in the cases when X itself is demi-normal, but the definition makes sense in general, although with unexpected properties. For example, \mathcal{O}_X is a divisorial sheaf iff X is S_2 .

DEFINITION 3.25 (Mostly flat families of divisorial sheaves). Let $f : X \rightarrow S$ be a pure dimensional morphism. A coherent sheaf F is called a *mostly flat family of divisorial sheaves* if there is a closed subset $Z \subset X$ with complement $U := X \setminus Z$ such that

- (3.25.1) $Z \cap X_s$ has codimension ≥ 2 in X_s for every $s \in S$,
- (3.25.2) $f|_U : U \rightarrow S$ is flat over S with pure, S_2 fibers,
- (3.25.3) $F|_U$ is locally free of rank 1, and
- (3.25.4) $\text{depth}_Z F \geq 2$.

The last assumption and (9.8) imply that $F = j_*(F|_U)$. Furthermore, if G is a coherent sheaf that satisfies (1–3) then $j_*(G|_U)$ satisfies (1–4). (This needs a mild technical condition which holds if X is excellent, see (10.25).) We call $j_*(G|_U)$ the *relative hull* of G and denote it by G^H . (Hulls of more general sheaves will be defined and studied in Chapter 9.) The natural map

$$r^G : G \rightarrow j_*(G|_U) = G^H \quad (3.25.5)$$

is an isomorphism iff $\text{depth}_Z G \geq 2$.

If $\dim X/S = 1$ then $Z = \emptyset$ and a mostly flat family of divisorial sheaves is the same as a flat family of invertible sheaves.

DEFINITION 3.26 (S_2 pull-back). Let $f : X \rightarrow S$ be a morphism and F a mostly flat family of divisorial sheaves on X . If $q : W \rightarrow S$ is any morphism then we get

$$\begin{array}{ccccc} X \times_S W & =: & X_W & \xrightarrow{q_X} & X \\ & & f_W \downarrow & & \downarrow f \\ & & W & \xrightarrow{q} & S. \end{array} \quad (3.26.1)$$

Thus we also have $U_W := q_X^{-1}(U)$ with injection $j_W : U_W \hookrightarrow X_W$, $Z_W := q_X^{-1}(Z)$ and $F_W := q_X^*F$. Note that F_W satisfies the conditions (3.25.1–3), so its hull $F_W^H := (F_W)^H$ is a mostly flat family of divisorial sheaves. We call F_W^H the *S_2 pull-back* of F . (If confusion is likely, we use $(F_W)^H$ to denote the hull of the pull-back and $(F^H)_W$ to denote pull-back of the hull F^H .) As in (3.25.5), we are especially interested in the map

$$r_W^F : F_W = q_X^*F \rightarrow (j_W)_*(F_W|_{U_W}) = F_W^H. \quad (3.26.2)$$

We have already encountered these maps in (2.75.6) when $W = \{s\}$ is a point

$$r_s^F : F_s \rightarrow (j_s)_*(F|_{U_s}) = F_s^H. \quad (3.26.3)$$

A mostly flat family of divisorial sheaves F is called a *flat family of divisorial sheaves* if it satisfies the following equivalent conditions.

(3.26.4) F is flat over S and the maps r_W^F defined in (3.26.2) are isomorphisms for every $q : W \rightarrow S$.

(3.26.5) The maps r_s^F in (3.26.3) are surjective for every closed point $s \in S$.

It is clear that (4) \Rightarrow (5) and the converse is proved in (10.62).

The following two observations are useful.

(3.26.6) If $g \in S$ is a generic point then F_g is S_2 , hence r_g^F is an isomorphism by (9.8). Thus F is a flat family of divisorial sheaves over some dense, open subset $S^\circ \subset S$ by (10.2).

(3.26.7) If F is a flat family of divisorial sheaves then every pull-back of it is also a flat family of divisorial sheaves, and there is no need to take the hull of the pull-back.

For applications the key point is to understand when a mostly flat family of divisorial sheaves is a flat family of divisorial sheaves. The main result is the following special case of (9.64).

THEOREM 3.27. *Let $f : X \rightarrow S$ be a projective morphism with relatively ample $\mathcal{O}_X(1)$ and L a mostly flat family of divisorial sheaves on X . Then there is a locally closed decomposition $j : S^{\text{H-flat}} \rightarrow S$ such that for every morphism $q : W \rightarrow S$ the following are equivalent.*

(3.27.1) L_W^H is a flat family of divisorial sheaves on X_W ,

(3.27.2) q factors as $q : W \rightarrow S^{\text{H-flat}} \rightarrow S$.

The following analog of (3.20) is quite important.

THEOREM 3.28. *Let S be a reduced scheme, $f : X \rightarrow S$ a projective morphism with relatively ample $\mathcal{O}_X(1)$ and L a mostly flat family of divisorial sheaves on X . Then L is a flat family of divisorial sheaves iff $s \mapsto \chi(X_s, L_s^H(*))$ is locally constant on S . \square*

Remark 3.28.1. Recall that by (3.20) a coherent sheaf G is flat over S iff $s \mapsto \chi(X_s, G_s(*))$ is locally constant on S . However, the assumptions of (3.28) are quite different since L_s^H is not assumed to be the fiber of L over s . In fact, usually there is no coherent sheaf on X whose fiber over s is isomorphic to L_s^H for every $s \in S$.

Nonetheless, at the end the key point is to compare the Euler characteristics of the sheaves

$$r_s^L : L_s \rightarrow L_s^H$$

occurring in (3.26.3); see also (2.75). The map r_s^L is an isomorphism over U_s , but both its kernel and the cokernel can be nontrivial and they have opposite contributions to the Euler characteristic.

Another consequence of (3.27) is also important.

COROLLARY 3.29. *Let $f : X \rightarrow S$ be a flat, projective morphism with S_2 fibers and relatively ample $\mathcal{O}_X(1)$. Let L be a mostly flat family of divisorial sheaves on X . Then there is a locally closed partial decomposition $j : S^{\text{inv}} \rightarrow S$ with the following property.*

Let $q : W \rightarrow S$ be a morphism. Then L_W^H is a flat family of invertible sheaves on X_W iff q factors as $q : W \rightarrow S^{\text{inv}} \rightarrow S$.

Proof. For flat morphisms with S_2 fibers, a flat family of invertible sheaves is also a flat family of divisorial sheaves. Thus if L_W^H is a flat family of invertible sheaves then q factors through $S^{\text{H-flat}} \rightarrow S$. So, by (3.23.1),

$$S^{\text{inv}} = (S^{\text{H-flat}})^{\text{inv}}.$$

For a flat family of sheaves being invertible is an open condition, thus S^{inv} is an open subscheme of $S^{\text{H-flat}}$. \square

The following semicontinuity result follows from (9.58), but it is easy to establish it without the general machinery.

LEMMA 3.30. *Let $f : X \rightarrow S$ be a proper morphism and L a mostly flat family of divisorial sheaves. Then*

(3.30.1) $s \mapsto h^0(X_s, L_s^H)$ is a constructible and upper semicontinuous function.

Furthermore, if $\mathcal{O}_X(1)$ is relatively ample then

(3.30.2) $s \mapsto \chi(X_s, L_s^H(*))$

is also constructible and upper semicontinuous, where for polynomials we use the ordering $f(*) \preceq g(*) \Leftrightarrow f(t) \leq g(t) \forall t \gg 1$.

Remark 3.30.3. If a coherent sheaf F is flat over S then $s \mapsto h^0(X_s, F_s)$ is constructible and upper semicontinuous on S . However, as in (3.28.3), our assumptions are different since L_s^H is not the fiber of L over s .

Proof. In order to prove constructibility, we may replace S by a locally closed decomposition of it. Such a decomposition is provided by the following result, which is a weak version of (3.27).

Claim 3.30.4. There is a locally closed decomposition $i : S' \rightarrow S$ such that $(i_X^* L)^H$ is a flat family of divisorial sheaves.

To prove this, note that the generic fibers L_g are S_2 , hence, by (10.3), there is a dense open subset $S^\circ \subset S$ such that every fiber over S° is S_2 . Thus L is a flat family of divisorial sheaves over S° . We can now replace S by $S^\circ \amalg (S \setminus S^\circ)$ and finish by Noetherian induction. \square

After replacing S by S' , we may assume that L is flat over S . Then (3.30.1–2) become the usual upper semicontinuity claims for coherent sheaves that are flat over S .

A constructible function is upper semicontinuous iff it is upper semicontinuous after base change to any DVR. Thus we may assume from now on that $S = T$ is the spectrum of a DVR with closed point 0 and generic point g .

In this case L is flat over T and S_2 . Thus its central fiber L_0 is S_1 . In particular, the restriction map (3.26.3) $r_0^F : L_0 \rightarrow L_0^H$ is an injection and we get that

$$h^0(X_g, L_g) \leq h^0(X_0, L_0) \leq h^0(X_0, (L_0)^H). \quad (3.30.5)$$

This proves (1) and applying it to twists of L gives (2). \square

COROLLARY 3.31. *Let S be a reduced scheme and $f : X \rightarrow S$ a flat, proper morphism with S_2 fibers. Let L be a mostly flat family of divisorial sheaves on X such that $L_s^H \cong \mathcal{O}_{X_s}$ for every $s \in S$. Then L is a line bundle on X , $f_* L$ is a line bundle on S and $L \cong f^*(f_* L)$.*

Proof. Note that $\chi(X_s, L_s^H(*)) = \chi(X_s, \mathcal{O}_{X_s}(*))$ is locally constant since f is flat. Thus L is a flat family of divisorial sheaves by (3.28) so L is a line bundle on X . Hence, by Grauert's theorem [Har77, III.12.9], f_*L is locally free of rank 1 and $L \cong f^*(f_*L)$. \square

In many cases we need to know when a mostly flat family of divisorial sheaves L is locally free. A necessary condition is that each L_s^H be locally free, but, as we saw in (1.42), this is not sufficient. Thus, as an intermediate step, we will be interested in the set

$$\{s \in S : L_s^{[m_s]} \text{ is locally free for some } m_s > 0\}.$$

We see in (4.19) that this set is not constructible in general. The following lemma is sometimes useful.

LEMMA 3.32. *Let $f : X \rightarrow S$ be a flat, finite type morphism with S_2 fibers and L a mostly flat family of divisorial sheaves on X . Assume that $L_s^{[m_s]}$ is locally free for some $m_s > 0$ for every $s \in S$. Then there is a common $m > 0$ such that $L_s^{[m]}$ is locally free for every $s \in S$.*

Note that we do not claim that $L^{[m]}$ itself is locally free.

Proof. Let $g \in S$ be a generic point. Then $L_g^{[m_g]}$ is locally free for some $m_g \in \mathbb{N}$, thus the same holds in an open neighborhood of $g \in S$. We finish by Noetherian induction. \square

3.33 (Hilbert function of divisorial sheaves). Let X be a proper scheme of dimension n and L, M line bundles on X . The Hirzebruch-Riemann-Roch theorem computes $\chi(X, L \otimes M^r)$ as a polynomial of r . Its leading terms are

$$\chi(X, L \otimes M^r) = \frac{r^n}{n!}(M^n) + \frac{r^{n-1}}{2(n-1)!}((\tau_1(X) + 2L) \cdot M^{n-1}) + \dots \quad (3.33.1)$$

Assume next that L is a torsion free sheaf that is locally free outside a subset $Z \subset X$ of codimension ≥ 2 . By blowing up L we get a proper birational morphism $\pi : X' \rightarrow X$ and a line bundle L' such that $\pi_*L' = L$. Thus we can compute $\chi(X, L \otimes M^r)$ as $\chi(X', L' \otimes \pi^*M^r)$, modulo an error term which involves the sheaves $R^i\pi_*L'$. These may be hard to control, but they are supported on Z , hence the $\chi(X, R^i\pi_*L' \otimes M^r)$ all have degree $\leq n-2$. Thus we again obtain the HRR formula (3.34.1). If X is demi-normal then $\tau_1(X) = -K_X$, hence we get the usual form

$$\chi(X, L \otimes M^r) = \frac{r^n}{n!}(M^n) - \frac{r^{n-1}}{2(n-1)!}((K_X - 2L) \cdot M^{n-1}) + \dots \quad (3.33.2)$$

If, in addition, $L^{[m]}$ is locally free for some $m > 0$, then applying (3.34.2) to $L \mapsto L^{[a]}$ for all $0 \leq a < m$ and $M = L^{[m]}$ we end up with the expected formula

$$\chi(X, L^{[r]}) = \frac{r^n}{n!}(L^n) - \frac{r^{n-1}}{2(n-1)!}(K_X \cdot L^{n-1}) + (\text{lower order terms}). \quad (3.33.3)$$

Note further that (3.33.2) shows that $\chi(X, L^{[r]})$ is a polynomial on any translate of $m\mathbb{Z}$. We can thus write

$$\chi(X, L^{[r]}) = \frac{r^n}{n!}(L^n) - \frac{r^{n-1}}{2(n-1)!}(K_X \cdot L^{n-1}) + \sum_{i=0}^{n-2} a_i(r)r^i, \quad (3.33.4)$$

where the $a_i(r)$ are periodic functions that depend on X and L .

3.34 (Hilbert function of slc varieties). Let X be a proper, slc variety of dimension n . We are especially interested in

$$r \mapsto \chi(X, \omega_X^{[r]}), \quad (3.34.1)$$

which we call the *Hilbert function* of ω_X .

Comment on the terminology. It might be more natural to call $r \mapsto h^0(X, \omega_X^{[r]})$ the Hilbert function and (3.34.1) the Hilbert polynomial. However, (3.34.1) is not a polynomial in general. For stable varieties the two variants differ only for $r = 1$, see (3.34.3).)

By (3.33.4) we can write the Hilbert function as

$$\chi(X, \omega_X^{[r]}) = \frac{r^n}{n!} (K_X^n) - \frac{r^{n-1}}{2(n-1)!} (K_X^n) + \sum_{i=0}^{n-2} a_i(r) r^i, \quad (3.34.2)$$

where the $a_i(r)$ are periodic functions with period = $\text{index}(X)$, that depend on X .

If ω_X is ample and the characteristic is 0, then (11.33) implies that, for $r \geq 2$,

$$\begin{aligned} h^i(X, \omega_X^{[r]}) &= 0, \quad \text{hence} \\ h^0(X, \omega_X^{[r]}) &= \chi(X, \omega_X^{[r]}). \end{aligned} \quad (3.34.3)$$

3.4. Local stability I

DEFINITION 3.35 (Relative canonical class). Let $f : X \rightarrow S$ be a flat, projective family of demi-normal varieties. The relative dualizing sheaf $\omega_{X/S}$ was constructed in (2.69).

Let $Z \subset X$ be the subset where the fibers are neither smooth nor nodal and set $j : U := X \setminus Z \hookrightarrow X$. Then $f|_U$ is flat with CM, even Gorenstein fibers. Thus, by (2.69.7), $\omega_{U/S}$ is locally free, commutes with base change and $X_s \cap Z$ has codimension ≥ 2 for every fiber X_s . Thus $\omega_{X/S} = j_* \omega_{U/S}$, hence $\omega_{X/S}$ is a mostly flat family of divisorial sheaves. The corresponding divisor class is denoted by $K_{X/S}$.

We define the reflexive powers of $\omega_{X/S}$ by the formula

$$\omega_{X/S}^{[m]} := j_*(\omega_{U/S}^m). \quad (3.35.1)$$

Thus $\omega_{X/S}^{[m]} \cong \mathcal{O}_X(mK_{X/S})$. In particular, $mK_{X/S}$ is Cartier iff $\omega_{X/S}^{[m]}$ is locally free.

Note that if $\omega_{X/S}^{[m]}$ is locally free then we have natural isomorphisms

$$\omega_{X/S}^{[m+n]} \cong \omega_{X/S}^{[m]} \otimes \omega_{X/S}^{[n]} \quad (3.35.2)$$

for every $n \in \mathbb{Z}$.

All these also hold for flat, finite type morphisms (that are not necessarily projective) by (2.69.7).

If the fibers of $f : X \rightarrow S$ are slc then $\omega_{X/S}$ is a flat family of divisorial sheaves by (2.68). However, its reflexive powers are usually only mostly flat over S . Applying (3.29) to $\omega_{X/S}^{[m]}$ gives the following, which turns out to be the key to our treatment of local stability over reduced schemes.

COROLLARY 3.36. *Let $f : X \rightarrow S$ be a flat, projective family of demi-normal varieties and fix $m \in \mathbb{Z}$. Then there is a locally closed decomposition $j : S^{[m]} \rightarrow S$ such that the following holds.*

Let $q : W \rightarrow S$ be a morphism. Then $\omega_{X_W/W}^{[m]}$ is a flat family of divisorial sheaves iff q factors as $q : W \rightarrow S^{[m]} \rightarrow S$. \square

In applications of (3.36) a frequent problem is that $S^{[m]}$ depends on m , even if we choose m to be large and divisible; see (3.41) for such an example.

We are now ready to prove that the preliminary definition (3.1) of local stability over reduced schemes is equivalent to the more conceptual one, which is (3.37.1).

The proof is complete over a field of characteristic 0. However, if the 3 versions of local stability (2.4.1–3) are equivalent over arbitrary DVRs, then the theorem also holds over arbitrary noetherian base schemes.

THEOREM 3.37 (Local stability over reduced schemes). *Let S be a reduced scheme over a field of characteristic 0 and $f : X \rightarrow S$ a flat family of demi-normal varieties. Then the following are equivalent.*

- (3.37.1) $\omega_{X/S}^{[m]}$ is a flat family of divisorial sheaves for every $m \in \mathbb{Z}$ and the fibers X_s are slc for all points $s \in S$.
- (3.37.2) $\omega_{X/S}^{[m]}$ is a flat family of invertible sheaves for some $m > 0$ and the fibers X_s are slc for all points $s \in S$.
- (3.37.3) $K_{X/S}$ is \mathbb{Q} -Cartier and the fibers X_s are slc for all points $s \in S$.
- (3.37.4) $K_{X/S}$ is \mathbb{Q} -Cartier and X_s is slc for all closed points $s \in S$.
- (3.37.5) $f_T : X_T \rightarrow T$ is locally stable (2.3) whenever T is the spectrum of a DVR and $q : T \rightarrow S$ is a morphism.

Proof. Assume (1) and pick $s \in S$. Since X_s is slc, $\omega_{X_s}^{[m_s]}$ is locally free for some $m_s > 0$. In a flat family of sheaves being invertible is an open condition, thus $\omega_{X/S}^{[m_s]}$ is a flat family of invertible sheaves in an open neighborhood $X_s \subset U_s \subset X$. Finitely many of these U_{s_i} cover X , and then $m = \text{lcm}\{m_{s_i}\}$ works for (2). Assertions (2) and (3) say the same using different terminology and (3) \Rightarrow (4) is clear.

Next assume (4) and let $s_g \in S$ be a non-closed point with a closed specialization $s \in S$. Choose a spectrum of a DVR T and a morphism $q : T \rightarrow S$ that maps the generic point to s_g and the closed point of T to s . We get $f_T : X_T \rightarrow T$ such that $K_{X_T/T}$ is \mathbb{Q} -Cartier and the special fiber is slc. Thus the generic fiber is also slc by (2.4), hence X_s is slc. This shows that (4) \Rightarrow (3). If $K_{X/S}$ is \mathbb{Q} -Cartier then so is any pull-back $K_{X_T/T}$. Thus (4) \Rightarrow (5) also follows from (2.4).

It remains to show that (5) \Rightarrow (1). If (5) holds then all fibers are slc and we need to prove that $\omega_{X/S}^{[m]}$ is a flat family of divisorial sheaves. This is a local question on S , hence we may assume that $(0 \in S)$ is local.

Let us discuss first the case when f is projective. By (3.36) the property

$$\mathcal{P}^{[m]}(W) := (\omega_{X_W/W}^{[m]} \text{ is a flat family of divisorial sheaves})$$

is representable by a locally closed decomposition $i_m : S^{[m]} \rightarrow S$. We aim to prove that i_m is an isomorphism.

For each generic point $g_i \in S$ choose a local morphism $q_i : (0_i \in T_i) \rightarrow (0 \in S)$ that maps the generic point $t_i \in T_i$ to g_i . By assumption $X_{T_i} \rightarrow T_i$ is locally stable, hence $\omega_{X_{T_i}/T_i}^{[m]}$ is a flat family of divisorial sheaves by (2.77.2). Thus g_i factors through $i_m : S^{[m]} \rightarrow S$. Therefore $i_m : S^{[m]} \rightarrow S$ is an isomorphism by (10.84.2), completing the proof for projective morphisms.

The above argument also works in the non-projective case, provided $i_m : S^{[m]} \rightarrow S$ exists. As we discuss in Section 9.9, the latter is unlikely. However, if S is local, complete, and we aim to represent flat divisorial pull-backs for local morphisms, then $i_m : S^{[m]} \rightarrow S$ exists, see (9.72) for details. The rest of the argument now works as before.

We discuss a more general case in (4.48), but the key arguments are in (4.37). \square

COROLLARY 3.38. *Let $f : X \rightarrow S$ be a flat morphism of finite type with demi-normal fibers such that $K_{X/S}$ is \mathbb{Q} -Cartier. Then*

$$S^{\text{slc}} := \{s : X_s \text{ is slc}\} \subset S \text{ is open.} \quad (3.38.1)$$

Proof. By (10.5), a set $U \subset S$ is open iff it is closed under generalization and U contains a dense open subset of \bar{s} for every $s \in U$.

For S^{slc} , the first of these follows from (2.4). In order to see the second, assume first that X_s is lc. Then mK_{X_s} is Cartier for some $m > 0$ hence $mK_{X/S}$ is Cartier over an open neighborhood of $s \in U_s \subset \bar{s}$. Next consider a log resolution $p_s : Y_s \rightarrow X_s$. It extends to a simultaneous log resolution $p^\circ : Y^\circ \rightarrow X^\circ$ over a suitable $U_s^\circ \subset \bar{s}$. Thus, if $E^\circ \subset Y^\circ$ is any exceptional divisor, then $a(E_t, X_t) = a(E^\circ, X^\circ) = a(E_s, X_s)$ for every $t \in U_s^\circ$. This shows that all fibers over U_s° are lc.

If X_s is not normal, one can use either a simultaneous semi-log-resolution [Kol13b, Sec.10.4] or normalize first, apply the above argument and descend to X , essentially by definition (11.11). \square

The following is a direct consequence of (2.93).

COROLLARY 3.39. *Let S be a reduced scheme and $f : X \rightarrow S$ a locally stable morphism. Then $\omega_{X/S}^{[m]}$ is locally free at a point $x \in X_s$ iff $\omega_{X_s}^{[m]}$ is locally free at x . \square*

Focusing on the property (3.37.1), we get the Kollár–Shepherd-Barron definition of local stability [KSB88].

DEFINITION 3.40 (Local stability and stability II). Let S be a scheme over a field of characteristic 0 and $f : X \rightarrow S$ a flat morphism with demi-normal fibers. Then f is *locally stable* iff

- (3.40.1) the fibers X_s are slc for every $s \in S$, and
- (3.40.2) $\omega_{X/S}^{[m]}$ is a flat family of divisorial sheaves for every $m \in \mathbb{Z}$.

Furthermore, f is *stable* iff, in addition

- (3.40.3) f is proper, and
- (3.40.4) $\omega_{X/S}$ is f -ample.

EXAMPLE 3.41. Following [Pat13], we give an example of a flat family of normal varieties $Y \rightarrow U$ such that ω_{Y_0} is locally free for some $0 \in U$ yet $\omega_{Y/U}$ is not locally free along Y_0 . Furthermore, $\{u : K_{Y_u} \text{ is } \mathbb{Q}\text{-Cartier}\}$ is a countable dense subset of U .

We start with a smooth, projective variety X such that $H^1(X, \mathcal{O}_X) \neq 0$ but $H^0(X, \omega_X) = H^1(X, \omega_X) = 0$. For example, we can take $X = C \times \mathbb{P}^n$ where C is a smooth curve of genus > 0 and $n \geq 2$.

Let L_0 be a very ample line bundle such that $L_0 \otimes \omega_X$ is ample. All line bundles algebraically equivalent to L_0 are parametrized by $\mathbf{Pic}^L(X)$, a connected component of $\mathbf{Pic}(X)$.

Choose a smooth divisor $D \subset X$ linearly equivalent to L_0 . Our example will be the family of cones

$$Y_L := \text{Spec}_k \sum_m H^0(D, (L \otimes \omega_X)^m|_D),$$

parametrized by a suitable open set $[L_0] \in U \subset \mathbf{Pic}^L(X)$.

The Y_L form a flat family iff the $h^0(D, (L \otimes \omega_X)^m|_D)$ are all constant on U . To compute these, consider the exact sequence

$$0 \rightarrow (L \otimes \omega_X)^m(-D) \rightarrow (L \otimes \omega_X)^m \rightarrow (L \otimes \omega_X)^m|_D \rightarrow 0.$$

By Kodaira vanishing, the higher cohomologies of the first 2 sheaves vanish, except for $L \otimes \omega_X(-D)$. We assumed that $H^0(X, \omega_X) = H^1(X, \omega_X) = 0$. Thus, by semicontinuity,

$$H^0(X, L \otimes \omega_X(-D)) = H^1(X, L \otimes \omega_X(-D)) = 0$$

for all L in a neighborhood of $[L_0]$; this condition defines our U . (If $X = C \times \mathbb{P}^n$ then actually $U = \mathbf{Pic}^L(X)$.) Hence $h^0(D, L \otimes \omega_X|_D)$ is independent of L for $[L] \in U$. The cones Y_L are the fibers of a flat morphism $Y \rightarrow U$.

By [Kol13b, 3.14.4], ω_{Y_L} is locally free iff ω_D is a power of $L \otimes \omega_X|_D$ and the latter is isomorphic to $\omega_D \otimes (L \otimes L_0^{-1})$. Thus ω_{Y_L} is locally free iff $L|_D \cong L_0|_D$. By the Lefschetz theorem this holds iff $L \cong L_0$. Thus $\omega_{Y/U}$ is not locally free along Y_{L_0} yet $\omega_{Y_{L_0}}$ is locally free.

Similarly we get that $\omega_{Y_L}^{[m]}$ is locally free iff $L^m \cong L_0^m$, so $\{L : K_{Y_L} \text{ is } \mathbb{Q}\text{-Cartier}\}$ is a countable dense subset of U .

3.5. Stability is representable I

We start with an example showing that being locally stable is not an open condition, not even a locally closed one.

EXAMPLE 3.42. In $\mathbb{P}_x^5 \times \mathbb{A}_{st}^2$ consider the family of varieties given by the equations

$$X := \left(\text{rank} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \leq 1 \right).$$

We claim that the fibers X_{st} are normal, projective with rational singularities and for every s, t the following equivalences hold:

(3.42.1) X_{st} is lc $\Leftrightarrow X_{st}$ is klt $\Leftrightarrow K_{X_{st}}$ is \mathbb{Q} -Cartier $\Leftrightarrow 3K_{X_{st}}$ is Cartier \Leftrightarrow either $(s, t) = (0, 0)$ or $st \neq 0$.

All these become clear once we show that there are 3 types of fibers.

(3.42.2) If $st \neq 0$ then, after a linear coordinate change, we get that

$$X_{st} \cong X_{11} \cong \left(\text{rank} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_4 & x_5 & x_3 \end{pmatrix} \leq 1 \right).$$

This is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$, hence smooth. The self-intersection of its canonical class is -54 .

(3.42.3) If $s = t = 0$ then we get the fiber

$$X_{00} := \left(\text{rank} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \leq 1 \right).$$

This is the cone (with \mathbb{P}^1 as vertex-line) over the rational normal curve $C_3 \subset \mathbb{P}^3$. The singularity along the vertex-line is isomorphic to $\mathbb{A}^2/\frac{1}{3}(1,1) \times \mathbb{A}^1$, hence log terminal. The canonical class of X_{00} is $-\frac{8}{3}H$, where H is the hyperplane class and its self-intersection is $-512/9 < -54$.

(3.42.4) Otherwise either s or t (but not both) are zero. After possibly permuting s, t and a linear coordinate change we get the fiber

$$X_{0t} \cong X_{01} \cong \left(\text{rank} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_4 & x_3 \end{pmatrix} \leq 1 \right).$$

This is the cone over the degree 3 surface $S_3 \cong \mathbb{F}_1 \hookrightarrow \mathbb{P}^4$. Its canonical class is not \mathbb{Q} -Cartier at the vertex, so this is not lc.

Thus the best one can hope for is that local stability is representable. From now on the base scheme is assumed to be over a field of characteristic 0. (See (4.61) for a list of problems in positive characteristic.)

3.43 (Proof of (3.2). Being flat is representable by (3.19), thus, using (3.23.1), we may assume from now on that $f : X \rightarrow S$ is flat. By (2.72) we may also assume that it has pure relative dimension n . For flat morphisms being demi-normal is an open condition by (10.41), hence, again using (3.23.1), we may assume that $f : X \rightarrow S$ is flat and its fibers are demi-normal of pure relative dimension n . This part of the argument works over any base scheme S .

Now we come to a surprisingly subtle part of the argument. By definition, if X_s is slc then K_{X_s} is \mathbb{Q} -Cartier, thus the next natural step would be to consider the following.

Question 3.43.1. Is $\{s \in S : K_{X_s} \text{ is } \mathbb{Q}\text{-Cartier}\}$ a constructible subset of S ?

We saw in (3.41) that this is not the case, not even for families of normal varieties. We thus need to jump ahead to deal with the slc property.

Claim 3.43.2. $\{s \in S : X_s \text{ is slc}\}$ is a constructible subset of S .

Actually, it is better to establish a different version that controls the denominators of the canonical divisors. Recall that the *index* of an slc variety Y , denoted by $\text{index}(Y)$, is the smallest positive integer m such that mK_Y is Cartier.

The key property turns out to be the following, which is an immediate consequence of (4.51). This allows us to avoid the dependence on the extra constant m when applying (3.36). (This result can also be thought of as a local variant of [HMX18].)

Lemma 3.43.3. Let $f : X \rightarrow S$ be a flat, proper family of demi-normal varieties. Then

$$\{\text{index}(X_s) : X_s \text{ is slc}\} \text{ is a finite set.} \quad \square$$

Let now M be an upper bound of the indices of slc fibers. For every $1 \leq m \leq M$ we apply (3.36) to get a locally closed partial decomposition $j_m : S^{[m]} \rightarrow S$ and

$$X^{[m]} := X \times_S S^{[m]} \rightarrow S^{[m]}$$

such that $\omega_{X^{[m]}/S^{[m]}}^{[m]}$ is a flat family of divisorial sheaves.

Let $q : W \rightarrow S$ be a morphism such that $q_W : X_W \rightarrow W$ is locally stable. Then $\omega_{X_W/W}^{[m]}$ is a flat family of divisorial sheaves, hence q factors as

$$q : W \xrightarrow{q_m} S^{[m]} \rightarrow S.$$

Conversely, if such a factorization exists then $\omega_{X_W/W}^{[m]}$ is a flat family of divisorial sheaves. Taking the product for $1 \leq m \leq M$ we get the following.

Claim 3.43.4. $q : W \rightarrow S$ factors as

$$q : W \xrightarrow{q^M} S^{[1]} \times_S \cdots \times_S S^{[M]} \rightarrow S$$

iff $\omega_{X_W/W}^{[m]}$ is a flat family of divisorial sheaves for every $1 \leq m \leq M$. \square

The following now finishes the proof of the theorem.

Claim 3.43.5. $S^{\text{ls}} = S^{[1]} \times_S \cdots \times_S S^{[M]}$.

Proof. By definition $\omega_{X^{\text{ls}}/S^{\text{ls}}}^{[m]}$ is a flat family of divisorial sheaves for every m , so, by (3.43.4), there is a natural map

$$S^{\text{ls}} \rightarrow S^{[1]} \times_S \cdots \times_S S^{[M]}.$$

Conversely, set $S^* := S^{[1]} \times_S \cdots \times_S S^{[M]}$. We know that $\omega_{X^*/S^*}^{[m]}$ is a flat family of divisorial sheaves for every $1 \leq m \leq M$. We claim that in fact this holds for every $m \in \mathbb{Z}$.

This is a local question, so pick $s \in S$. By (3.43.3) there is a $1 \leq m_s \leq M$ such that $\omega_{X_s}^{[m_s]}$ is locally free. Thus $\omega_{X^*/S^*}^{[m_s]}$ is locally free along X_s by (3.39). In particular,

$$\omega_{X^*/S^*}^{[m+rm_s]} \cong \omega_{X^*/S^*}^{[m]} \otimes (\omega_{X^*/S^*}^{[m_s]})^{\otimes r}$$

for every $r \in \mathbb{Z}$ by (3.35.2). Since every $m' \in \mathbb{Z}$ can be written as $m + rm_s$ for some $1 \leq m \leq M$, we see that $\omega_{X^*/S^*}^{[m']}$ is a flat family of divisorial sheaves for every $m' \in \mathbb{Z}$. This gives the inverse map

$$S^{[1]} \times_S \cdots \times_S S^{[M]} = S^* \rightarrow S^{\text{ls}}. \quad \square$$

Stable pairs over reduced base schemes

So far we have identified stable pairs (X, Δ) as the basic objects of our moduli problem, defined stable and locally stable families of pairs over 1-dimensional regular schemes in Chapter 2 and in Chapter 3 we treated families of varieties over reduced base schemes. Here we unite the two by discussing stable and locally stable families over reduced base schemes. Some of the final results apply only over seminormal base schemes.

After stating the main results in Section 4.1 we give a series of examples in Section 4.2. The technical core of the chapter is the treatment of various notions of families of divisors given in Section 4.3. The behavior of generically \mathbb{R} -Cartier divisors is studied in Section 4.4.

In Section 4.5 we finally define stable and locally stable families over reduced base schemes and prove that local stability is a representable property. Families over a smooth base scheme are especially well behaved; their properties are discussed in the short Section 4.6.

The universal family of Mumford divisors is constructed in Section 4.7

Assumptions. In the foundational Sections 4.1–4.4 we work with arbitrary schemes, but, for the applications to stable morphisms presented in Sections 4.5–4.6, we usually need to assume that the base scheme is over a field of characteristic 0.

4.1. Statement of the main results

In the study of locally stable families of pairs over reduced base schemes the key step is to give the ‘correct’ definition for the divisor component for families of pairs.

TEMPORARY DEFINITION 4.1. A *family of pairs* (with \mathbb{Z} -coefficients) of dimension n over a reduced scheme is an object

$$f : (X, D) \rightarrow S \tag{4.1.1}$$

consisting of a morphism of schemes $f : X \rightarrow S$ and an effective ‘divisor’ D satisfying the following properties.

4.1.2 (Flatness for X). *The morphism $f : X \rightarrow S$ is flat, of finite type, of pure relative dimension n and with geometrically reduced fibers.* This is the expected condition from the point of view of moduli theory, following the Principles (3.12) and (3.13). (Note, however, that (X, Δ) slc does not imply that X is slc, so maybe we are just lucky that this is the right condition. Later we will consider some cases where f is assumed to be flat only outside a codimension ≥ 2 set on each fiber.)

4.1.3 (Equidimensionality for $\text{Supp } D$). *The nonempty fibers of $\text{Supp } D \rightarrow S$ have pure dimension $n - 1$.* This implies that every irreducible component of $\text{Supp } D$ dominates an irreducible component of S and $\text{Supp } D$ does not contain any

irreducible component of any fiber of f . If S is normal then this condition holds iff $\text{Supp } D \rightarrow S$ has pure relative dimension $n - 1$ by (2.72.2), but in general our assumption is weaker. We noted in (2.40) that $D \rightarrow S$ need not be flat for locally stable families. So we start with the above weak assumption and strengthen it later as needed.

So far we have not said what a ‘divisor’ is. Working on a normal variety X , by an effective ‘divisor’ D we usually mean either a Weil divisor or a divisorial subscheme, that is, a pure, codimension 1 subscheme. The two versions are equivalent since X is regular at the generic points of D ; see (4.20) for details. If (X, Δ) is an slc pair, then X is smooth at all generic points of $\text{Supp } \Delta$. So if D is an effective divisor supported on $\text{Supp } \Delta$, the 2 viewpoints are again interchangeable.

This condition was first codified in Mumford’s observation that, in order to get a good moduli theory of pointed curves (C, P) , the curve should be nodal and the marked points $P = \{p_1, \dots, p_n\}$ should be smooth points of C . It turns out that such generic smoothness is a crucial condition technically and it is very hard to do anything without it. So we make it part of the definition for families of pairs.

4.1.4 (Mumford condition—generic smoothness along D). *The morphism f is smooth at generic points of $X_s \cap \text{Supp } D$ for every $s \in S$.* Equivalently, for each $s \in S$, none of the irreducible components of $X_s \cap \text{Supp } D$ is contained in $\text{Sing}(X_s)$.

This means that from now on we can identify effective Weil divisors with divisorial subschemes. The usual notation uses Weil divisors.

After these preliminary, mostly obvious assumptions, now we come to the heart of the matter.

We would like the notion of families of pairs to give a functor, so for any morphism $g : W \rightarrow S$ we need to define the pulled-back family. We have a fiber product diagram

$$\begin{array}{ccc} X \times_S W & \xrightarrow{g_X} & X \\ f_W \downarrow & & \downarrow f \\ W & \xrightarrow{g} & S. \end{array} \quad (4.1.5)$$

It is clear that we should take $X_W := X \times_S W$ with morphism $f_W : X_W \rightarrow W$. The definition of D_W is less clear since pull-backs of Cartier and of Weil divisors are not compatible in general; see (4.12). Both of these suggest a definition of D_W .

4.1.6 (Generically Cartier pull-back) Assume that D is a relative *generically Cartier* divisor. That is, there is an open subset $U \subset X$ such that $D|_U$ is Cartier and U contains the generic points of $X_s \cap \text{Supp } D$ for every s . We can then define the *generically Cartier pull-back* of D as the closure of $D|_U \times_S W \subset U \times_S W \subset X \times_S W$. This is clearly functorial.

4.1.7 (Divisorial pull-back) For any subscheme $Z \subset X$ and morphism $h : Y \rightarrow X$, define the *divisorial pull-back* as either the divisorial subscheme $\text{Div}(h^{-1}(Z))$ or the Weil divisor $\text{Weil}(h^{-1}(Z))$ associated to it, see (4.20.6) for formal definitions. (The 2 versions are equivalent if Y is regular at all codimension 1 generic points of $h^{-1}(Z)$.) We can thus start with D , view it as a divisorial subscheme $D \subset X$ and then set

$$D_W := g^{[*]}(D) := \text{Div}(g_X^{-1}(D)) \quad \text{or} \quad \text{Weil}(g_X^{-1}(D)). \quad (4.1.7.a)$$

Note that condition (4.1.4) is crucial here in identifying the two versions with each other.

Warning. Note that, in general, $\mathcal{O}_{D_W} \neq g_X^* \mathcal{O}_D$ and $\mathcal{O}_{X_W}(-D_W) \neq g_X^* \mathcal{O}_X(-D)$, so when the scheme structure is crucial, we (aim to) carefully distinguish these objects. Divisorial pull-back does not preserve linear equivalence, it is not even additive; see (4.12).

4.1.8 (Well-defined families of pairs I). *We say that $f : (X, D) \rightarrow S$ is a well-defined family if it satisfies the assumptions (4.1.2–4) and the divisorial pull-back defined in (4.1.7) is a functor for reduced schemes. That is*

$$h^{[*]}(g^{[*]}(D)) = (g \circ h)^{[*]}(D) \quad (4.1.8.a)$$

for all morphisms of reduced schemes $h : V \rightarrow W$ and $g : W \rightarrow S$.

In any concrete situation the conditions (4.1.2–4) should be easy to check, but (4.1.8) requires computing $g^{[*]}(D)$ for all morphisms $W \rightarrow S$. The following variant is much easier to verify.

4.1.9 (Well-defined specializations). *We say that $f : (X, D) \rightarrow S$ as in (4.1.8) has well-defined specializations if the following special case of (4.1.8) holds.*

Let T be the spectrum of a DVR, with closed point $\iota : t_0 \hookrightarrow T$ and $g : T \rightarrow S$ a morphism that maps the generic point of T to a generic point of S . Then $\iota^{[]}(g^{[*]}(D))$ depends only on $\{t_0\} \rightarrow S$.*

It turns out that (4.1.8) is automatic if S is normal and (4.1.9) \Rightarrow (4.1.8) if S is weakly normal. The next theorem is an immediate consequence of (4.25).

THEOREM 4.2. *Let $f : (X, D) \rightarrow S$ be a family of pairs satisfying the conditions (4.1.2–4). If S is normal then D is generically Cartier (4.1.6) and gives a well-defined family (4.1.8).*

Over non-normal base schemes it is usually quite easy to check well-definedness using the normalization. The weakly normal case is especially simple; see (4.30.5) for proof.

THEOREM 4.3. *Let S be a reduced scheme with normalization $\bar{S} \rightarrow S$. Let $f : (X, D) \rightarrow S$ be a projective family of pairs satisfying the assumptions (4.1.2–4). and*

$$\bar{f} : (\bar{X}, \bar{D}) := (X, D) \times_S \bar{S} \rightarrow \bar{S}$$

the corresponding family over \bar{S} . (The latter is well-defined by (4.2).) Then $f : (X, D) \rightarrow S$ is well-defined in either of the following cases.

(4.3.1) *There are natural isomorphisms $\tau^{[*]}(X, D) \cong \bar{\tau}^{[*]}(\bar{X}, \bar{D})$ for every geometric point $\tau : s \rightarrow S$ and for every lifting $\bar{\tau} : s \rightarrow \bar{S}$.*

(4.3.2) *S is weakly normal and the fiber $\bar{\tau}^{[*]}(\bar{X}, \bar{D})$ is independent of the lifting $\bar{\tau} : s \rightarrow \bar{S}$ for every geometric point $\tau : s \rightarrow S$.*

A further good news is that, over reduced schemes, the two versions (4.1.6) and (4.1.8) are equivalent to each other and also to several other natural conditions. The common theme is that we need to understand only the codimension 1 behavior of $f : (X, D) \rightarrow S$. The following theorem is proved in (4.30).

THEOREM 4.4. *Let $f : (X, D) \rightarrow S$ be a family of pairs satisfying the conditions (4.1.2–4) over a reduced scheme S . Viewing D as a divisorial subscheme (4.20), the following are equivalent.*

(4.4.1) *The family is well-defined as in (4.1.8).*

(4.4.2) D is a relative, generically Cartier divisor (4.1.6).

(4.4.3) $D \rightarrow S$ is flat at the generic points of $X_s \cap \text{Supp } D$ for every $s \in S$.

If f is projective then these are also equivalent to

(4.4.4) $s \mapsto \deg D_s$ is a locally constant function on S .

If S is also weakly normal, then these are further equivalent to

(4.4.5) D has well-defined specializations (4.1.9).

Next we turn to the case that we are really interested in, when the boundary Δ is a \mathbb{Q} or \mathbb{R} -divisor. We can apply (4.1.8) in 3 basic ways. First we consider 2 versions that are natural, work for normal base schemes, but not in general.

TEMPORARY DEFINITION 4.5. Let $f : (X, \Delta) \rightarrow S$ be a family of pairs over a reduced scheme S satisfying the conditions (4.1.2–4), where $\Delta = \sum a_i D^i$ is an effective Weil \mathbb{R} -divisor and the D^i are irreducible Weil divisors.

4.5.1 (Component-wise version). We require that each $(X, D^i) \rightarrow S$ be a well-defined family as in (4.1.8). For any $g : W \rightarrow S$ the pulled-back family can then be defined as

$$g_{cw}^{[*]}(X, \Delta) := (X_W, \Delta_W := \sum a_i g^{[*]}(D^i)).$$

Over a normal base this definition works well, essentially by (4.2). However, otherwise it is too restrictive and, in most cases, the other variants work better.

4.5.2 (Common denominator version). Assume that Δ is a \mathbb{Q} -divisor and choose a common denominator N for the numbers a_i . Then $N\Delta$ is an effective \mathbb{Z} -divisor.

We require that $f : (X, N\Delta) \rightarrow S$ be a well-defined family as in (4.1.8). For any $g : W \rightarrow S$ the pulled-back family can then be defined as

$$g_N^{[*]}(X, \Delta) := (X_W, \Delta_W := \frac{1}{N} g^{[*]}(N\Delta)).$$

By (4.4) the requirement (4.1.8) is satisfied iff $N\Delta$ is a relative, generically Cartier divisor. We prove in (4.39) that the resulting notion is independent of the choice of N in characteristic 0, but not in characteristic $p > 0$. Thus, in characteristic 0, we get a well-defined family iff Δ is a relative, generically \mathbb{Q} -Cartier divisor.

There are important examples of moduli problems where we have a common denominator in mind, thus the dependence on N is not a problem; see Section 8.1 for details. However, in other cases a common denominator seems an artificial choice; see (4.13).

The following is the version we adopt in this book; we discuss it in more detail in (4.29–4.30). The price we pay is that it works only in characteristic 0.

DEFINITION 4.6 (Well-defined families of pairs II). Let Δ be an effective, relative generically \mathbb{R} -Cartier divisor. By definition, (11.34.3) Δ can be written as $\Delta = \sum_j d_j \Theta_j$ where $d_j > 0$ and each Θ_j is an effective, relative, generically Cartier divisor. So, for any $g : W \rightarrow S$, the pulled-back family can be defined as

$$g_C^{[*]}(X, \Delta) := (X_W, \Delta_W := \sum_j d_j g^{[*]} \Theta_j).$$

This is independent of the way of writing Δ as $\sum_j d_j \Theta_j$.

Comment. Working with reduced schemes in characteristic 0, this is the optimal definition.

However, in characteristic $p > 0$ this definition leads to nonsensical moduli functors, already for 4 points on \mathbb{P}^1 ; as we discussed in (1.82–1.83).

It is not clear what the optimal choice is in characteristic $p > 0$, so for now we try to keep our options open and study both (4.5.2) and (4.6).

If g is flat on a dense open subset of W then (4.5.1–2) and (4.6) give the same pull-back, but otherwise, already when the base scheme S is a reduced curve, everything can go wrong with these definitions. That is, they differ from each other, (4.5.2) does depend on the choice of N and neither one is functorial in general; see Section 4.2 for such examples.

The next result gives necessary and sufficient criteria by comparing the fibers over geometric points $\tau : s \rightarrow S$ with the fibers over geometric points of the normalization $\bar{S} \rightarrow S$ for all possible liftings $\bar{\tau} : s \rightarrow \bar{S}$ of τ .

THEOREM 4.7. *Let S be a reduced, excellent scheme and $f : (X, D) \rightarrow S$ a projective family of pairs satisfying the assumptions (4.1.2–4). Assume that*

(4.7.1) *either we are over a field of characteristic 0,*

(4.7.2) *or S is weakly normal.*

Then the 3 versions (4.5.1), (4.5.2) and (4.6) are equivalent to each other.

Part (1) is proved in (4.39) and (2) in (4.30). The examples in Section 4.2 show that in characteristic p we have only the implications

$$(4.5.1) \Rightarrow (4.5.2) \Rightarrow (4.6).$$

Moreover, the condition (4.5.2) does depend on the choice of common denominator N , but only on the largest power of p that divides N by (4.39).

We have defined stable and locally stable families over a DVR in (2.3), and being locally stable should be preserved by pull-back. We can thus define these notions in general by imposing the following valuative criterion; see (4.48) for an equivalent but conceptually more complete version.

TEMPORARY DEFINITION 4.8. Let S be a reduced scheme over a field of characteristic 0 and $f : (X, \Delta) \rightarrow S$ a well-defined family of pairs as in (4.6). Assume that Δ is an effective, relative, generically \mathbb{R} -Cartier divisor.

Then $f : (X, \Delta) \rightarrow S$ is called *stable* (resp. *locally stable*) iff the family obtained by base change $f_T : (X_T, \Delta_T) \rightarrow T$ is stable as in (2.44) (resp. locally stable as in (2.3)) whenever T is the spectrum of a DVR and $T \rightarrow S$ a morphism.

Let now $f : (X, \Delta) \rightarrow S$ be a family of pairs. It turns out that, starting in relative dimension 3, the set of points

$$\{s \in S : (X_s, \Delta_s) \text{ is semi-log-canonical}\}$$

is neither open nor closed; see (3.42) for an example. Thus the strongest result one can hope for is the following.

THEOREM 4.9 (Local stability is representable). *Let S be a reduced, excellent scheme over a field of characteristic 0 and $f : (X, \Delta) \rightarrow S$ a well-defined, projective family of pairs. Assume that Δ is an effective, relative, generically \mathbb{R} -Cartier divisor. Then there is a locally closed partial decomposition $j : S^{\text{ls}} \rightarrow S$ such that the following holds.*

Let W be any reduced scheme and $q : W \rightarrow S$ a morphism. Then the family obtained by base change $f_W : (X_W, \Delta_W) \rightarrow W$ is locally stable iff q factors as $q : W \rightarrow S^{\text{ls}} \rightarrow S$.

A stable morphism is locally stable and stability is an open condition for a locally stable morphism. Thus (4.9) implies the following.

COROLLARY 4.10 (Stability is representable). *Using the notation and assumptions as in (4.9), there is a locally closed partial decomposition $j : S^{\text{stab}} \rightarrow S$ such that the following holds.*

Let W be any reduced scheme and $q : W \rightarrow S$ a morphism. Then the family obtained by base change $f_W : (X_W, \Delta_W) \rightarrow W$ is stable iff q factors as $q : W \rightarrow S^{\text{stab}} \rightarrow S$. \square

Next we turn to the moduli functor $\mathcal{SP}^{\text{red}}$ that associates to a reduced scheme S the set of all stable families $f : (X, \Delta) \rightarrow S$, up-to isomorphism. (Here \mathcal{SP} stands for stable pairs and the superscript $^{\text{red}}$ indicates that we work with reduced schemes.)

To be precise, we fix the dimension n of the fibers, a common denominator m for all coefficients occurring in Δ and the volume v . Our objects are stable pairs $(X, \Delta = \frac{1}{m}D)$ where D is a \mathbb{Z} -divisor and $v = \text{vol}(K_{X_s} + \Delta_s) := ((K_{X_s} + \Delta_s)^n)$. This gives the functor

$$\mathcal{SP}^{\text{red}}(n, m, v) : \{\text{reduced } S\text{-schemes}\} \rightarrow \{\text{sets}\}.$$

By [HMX18], there is an $M = M(n, m, v)$ such that $M(K_{X_s} + \Delta_s)$ is Cartier for every $(X_s, \Delta_s) \in \mathcal{SP}^{\text{red}}(n, m, v)(\text{point})$. However, this is not needed for the purposes of the theorem; one could just add M as a new variable and suppress it in the notation. We can now state the second main theorem of this Chapter.

THEOREM 4.11 (Existence of reduced moduli spaces). *Let S be an excellent base scheme of characteristic 0 and fix n, m, v . Then the functor $\mathcal{SP}^{\text{red}}(n, m, v)$ has a coarse moduli space*

$$\text{SP}^{\text{red}}(n, m, v) \rightarrow S,$$

which is a reduced scheme whose irreducible components are proper over S .

Moreover—though this can be made precise only later—the space $\text{SP}^{\text{red}}(n, m, v)$ is the reduced subscheme of the ‘true’ moduli space $\text{SP}(n, m, v)$ of stable pairs.

4.2. Examples

We start with a series of examples related to (4.7).

EXAMPLE 4.12. Consider the cone $Q := (xy - z^2) \subset \mathbb{A}^3$ and one chart of its resolution $\pi : \mathbb{A}_{uv}^2 \rightarrow Q$ given by $(u, v) \mapsto (uv^2, u, uv)$. $D := (x = z = 0)$ is a Weil divisor and its divisorial pull-back is $\pi^{[*]}(D) = \text{Weil}(uv^2, uv) = (uv = 0)$. Note that $2D$ is also a Cartier divisor with equation $(x = 0)$. Then $\pi^*(2D) = (uv^2 = 0) = (u = 0) + 2(v = 0)$. Thus

$$2\pi^{[*]}(D) \neq \pi^*(2D).$$

EXAMPLE 4.13. Consider the A_4 singularity $S := (xy - z^5 = 0) \subset \mathbb{A}^3$. Its class group is $\mathbb{Z}/5$ generated by $D_1 := (x = z = 0)$. Set $D_2 := (x - z^2 = y - z^3 = 0)$. We check that

$$3D_1 + D_2 = (x - z^3 = 0) \quad \text{and} \quad D_1 + 2D_2 = (x - 2z^3 + yz = 0)$$

are both Cartier. We can represent $D_1 + D_2$ as a linear combination of Cartier divisors in several ways as

$$D_1 + D_2 = \frac{1}{5}(5D_1) + \frac{1}{5}(5D_2) = \frac{1}{3}(3D_1 + D_2) + \frac{2}{15}(5D_2) = \frac{1}{2}(D_1 + 2D_2) + \frac{1}{10}(5D_1),$$

involving different denominators.

EXAMPLE 4.14. Let $S = (xy = 0) \subset \mathbb{A}^2$ and $X = (xy = 0) \subset \mathbb{A}^3$. Consider the divisors $D_x := (y = z - 1 = 0)$ and $D_y := (x = z + 1 = 0)$. We get a family

$$f : (X, D_x + D_y) \rightarrow S \quad (4.14.1)$$

that satisfies the assumptions (4.1.2–4).

We compute the ‘fiber’ of the above family over the origin in 3 different ways and get 3 different results.

First restrict the family to the x -axis. The pull back of X becomes the plane \mathbb{A}_{xz}^2 . The divisor D_x pulls back to $(z - 1 = 0)$ but the pull back of the ideal sheaf of D_y is the maximal ideal $(x, z + 1)$. It has no divisorial part, so restriction to the x -axis gives the pair

$$(\mathbb{A}_{xz}^2, (z - 1 = 0)) \rightarrow \mathbb{A}_x^1. \quad (4.14.2)$$

Similarly, restriction to the y -axis gives the pair

$$(\mathbb{A}_{yz}^2, (z + 1 = 0)) \rightarrow \mathbb{A}_y^1. \quad (4.14.3)$$

If we restrict these to the origin, we get

$$(\mathbb{A}_z^1, (z - 1 = 0)) \quad \text{and} \quad (\mathbb{A}_z^1, (z + 1 = 0)). \quad (4.14.4)$$

Finally, if we restrict to the origin of S in one step then we get the pair

$$(\mathbb{A}_z^1, (z - 1 = 0) + (z + 1 = 0)). \quad (4.14.5)$$

Thus we have 3 different pairs in (4.14.4–5) that can claim to be the fiber of (4.14.1) over the origin.

In the above example the problem is visibly set-theoretic, but there can be problems even when the set theory works out. For example, with X, S as above, consider the family

$$f : (X, \frac{1}{2}D'_x + \frac{1}{2}D'_y) \rightarrow S, \quad (4.14.6)$$

where $D'_x := (y = (z - 1)(z^2 + 2z + 1 - x) = 0)$ and $D'_y := (x = (z + 1)(z^2 - 2z + 1 - y) = 0)$. Computing as above we again get 3 different pairs as in (4.14.4–5) that can claim to be the fiber of (4.14.6) over the origin:

$$(\mathbb{A}_z^1, \frac{1}{2}P + Q), (\mathbb{A}_z^1, P + \frac{1}{2}Q) \quad \text{and} \quad (\mathbb{A}_z^1, P + Q), \quad (4.14.7)$$

where $P := (z - 1 = 0)$ and $Q := (z + 1 = 0)$.

In the above example the problem is that the restrictions of D'_x and D'_y to the z -axis have different multiplicities. The next example shows that even when the multiplicities are the same, there can be scheme-theoretic problems.

EXAMPLE 4.15. Set $X = (x^2 - y^2 = u^2 - v^2) \subset \mathbb{A}^4$, $D = (x - u = y - v = 0) \cup (x + u = y + v = 0)$ and $f : (X, D) \rightarrow \mathbb{A}_{uv}^2$ the coordinate projection. The irreducible components of D intersect only at the origin and D is not Cartier there.

Let L_c be the line $(v = cu)$ for some c . Restricting the family to L_c we get $X_c = (x^2 - y^2 = (1 - c^2)u^2) \subset \mathbb{A}^3$ and the divisor becomes $D_c = (x - u = y - cu = 0) \cup (x + u = y + cu = 0)$. Observe that D_c is a Cartier divisor with defining equation

$cx = y$. (Note that base change does not commute with union, so $D \times_{\mathbb{A}^2} L_c$ has an embedded point at the origin.)

Thus although D is not Cartier at the origin, after base change to a general line we get a Cartier divisor. For all of these base changes, D_c has multiplicity 2 at the origin. (These claims actually hold after base change to any smooth curve.)

However, the origin is a singular point of the fiber, and if we restrict D_c to the fiber over the origin, the resulting scheme structure varies with c .

This would be a very difficult problem to deal with, but for a stable pair (X, Δ) we are in a better situation since the irreducible components of Δ are not contained in $\text{Sing } X$.

EXAMPLE 4.16. Let B be a smooth projective curve of genus ≥ 1 with an involution σ and $b_1, b_2 \in B$ a pair of points interchanged by σ . Let C' be another smooth curve with two points $c'_1, c'_2 \in C'$. Start with the trivial family $(B \times C', \{b_1\} \times C' + \{b_1\} \times C') \rightarrow C'$ and then identify $c'_1 \sim c'_2$ and $(b, c'_1) \sim (\sigma(b), c'_2)$ for every $b \in B$. We get an étale locally trivial stable morphism $(S, D_1 + D_2) \rightarrow C$. Here C is a nodal curve with node $\tau : \{c\} \rightarrow C$. The fiber over the node is $(B, [b_1] + [b_2])$.

However, the fiber of each D_i over c is $[b_1] + [b_2]$, hence the component-wise pull-back (4.5.1) is $\tau_{cw}^{[*]} = (B, 2[b_1] + 2[b_2])$.

EXAMPLE 4.17. Set $C := (xy(x - y) = 0) \subset \mathbb{A}_{xy}^2$ and $X := (xy(x - y) = 0) \subset \mathbb{A}_{xyz}^3$. For any $c \in k$ consider the divisor

$$D_c := (x = z = 0) + (y = z = 0) + (x - y = z - cx = 0).$$

The pull-back of D_c to any of the irreducible components of X is Cartier, it intersects the central fiber at the origin of the z -axis and with multiplicity 1. Nonetheless, we claim that D_c is Cartier only for $c = 0$.

Indeed, assume that $h(x, y, z) = 0$ is a local equation of D_c . Then $h(x, 0, z) = 0$ is a local equation of the x -axis and $h(0, y, z) = 0$ is a local equation of the y -axis. Thus $h = az + (\text{higher terms})$. Restricting to the $(x - y = 0)$ plane we get that $c = 0$.

Note also that if $\text{char } k = 0$ and $c \neq 0$ then no multiple of D_c is a Cartier divisor. To see this note that if $f(x, y, z) = 0$ is a local defining equation of mD_c on X then $\partial^{m-1} f / \partial z^{m-1}$ vanishes on D_c . Its restriction to the z -axis vanishes at the origin with multiplicity 1. We proved above that this is not possible.

The situation is different if $\text{char } k > 0$, see (4.46).

EXAMPLE 4.18. Consider the cusp $C := (x^2 = y^3) \subset \mathbb{A}_{xy}^2$ and the trivial curve family $Y := C \times \mathbb{A}_z^1 \rightarrow C$. Let $D \subset Y$ be the Cartier divisor given by the equation $y = z^2$. Then $D \rightarrow C$ is flat of degree 2. Furthermore, D is reducible with irreducible components $D^\pm := \text{image of } t \mapsto (t^3, t^2, \pm t)$.

Note that $D^\pm \cong \mathbb{A}_t^1$ and the projections $D^\pm \rightarrow C$ corresponds to the ring extension $k[t^3, t^2] \hookrightarrow k[t]$. Thus the projections $D^\pm \rightarrow C$ are not flat and the fiber of $D^\pm \rightarrow C$ over the origin has length 2 for both D^+ and D^- .

Thus if we compute the fiber of $D = D^+ \cup D^- \rightarrow C$ over the origin $(0, 0) \in C$ using the common denominator $N = 1$ as in (4.5.2), then we get the point $(0, 0, 0)$ with multiplicity 2. However, if we compute the fiber component-wise (4.5.1) then we get the point $(0, 0, 0)$ with multiplicity 4.

Thus $(Y, D^+) \rightarrow C$ is not stable but it becomes stable after pull-back to C^n .

Arguing as in (4.17) shows that the D^\pm are not \mathbb{Q} -Cartier in characteristic 0. The situation is again more complicated if $\text{char } k > 0$, see (4.47).

The next examples discuss the variation of the \mathbb{Q} -Cartier property in families of divisors. Related positive results are in Section 4.5.

EXAMPLE 4.19. Let $C \subset \mathbb{P}^2$ be a smooth cubic curve and $S_C \subset \mathbb{P}^3$ the cone over it. For $p \in C$ let $L_p \subset S_C$ denote the ruling over p . Note that L_p is \mathbb{Q} -Cartier iff p is a torsion point, that is, $3m[p] \sim \mathcal{O}_C(m)$ for some $m > 0$. The latter is a countable dense subset of the moduli space of the lines $\text{Chow}_{1,1}(S_C) \cong C$.

In the above example the surface is not \mathbb{Q} -factorial and the curve L_p is sometimes \mathbb{Q} -Cartier, sometimes not. Next we give a similar example of a flat family of lc surfaces $S \rightarrow B$ such that $\{b : S_b \text{ is } \mathbb{Q}\text{-factorial}\} \subset B$ is a countable set of points. Thus being \mathbb{Q} -factorial is not a constructible condition.

Let $C \subset \mathbb{P}^2$ be a smooth cubic curve. Pick 11 points $P_1, \dots, P_{11} \in C$ and set $P_{12} = -(P_1 + \dots + P_{11})$. Then there is a quartic curve D such that $C \cap D = P_1 + \dots + P_{12}$. Thus the linear system $|\mathcal{O}_{\mathbb{P}^2}(4)(-P_1 - \dots - P_{12})|$ blows up the points P_i and contracts C . Its image is a degree 4 surface $S = S(P_1, \dots, P_{11})$ in \mathbb{P}^3 with a single simple elliptic singularity. If $C = (f_3(x, y, z) = 0)$ and $D = (f_4(x, y, z) = 0)$ then

$$S \cong (f_3(x, y, z)w + f_4(x, y, z) = 0) \subset \mathbb{P}^3.$$

At the point $(x = y = z = 0)$ the singularity of S is analytically isomorphic to the cone S_C and S is smooth elsewhere iff the points P_1, \dots, P_{12} are distinct. If this holds then the class group of S is generated by the image L of a line in \mathbb{P}^2 and the images E_1, \dots, E_{12} of the 12 exceptional curves. They satisfy a single relation $3L = E_1 + \dots + E_{12}$. Note that E_i is \mathbb{Q} -Cartier iff P_i is a torsion point.

If we vary $P_1, \dots, P_{11} \in C$ we get a flat family of lc surfaces parametrized by

$$\pi : \mathbf{S} \rightarrow C^{11} \setminus (\text{diagonals}),$$

with universal divisors $\mathbf{E}_i \subset \mathbf{S}$. We see that

(4.19.1) $E_i(P_1, \dots, P_{11})$ is \mathbb{Q} -Cartier iff P_i is a torsion point and

(4.19.2) $S(P_1, \dots, P_{11})$ is \mathbb{Q} -factorial iff P_i is a torsion point for every i .

4.3. Families of divisors II

At least 3 different notions of effective divisors are commonly used in algebraic geometry and our discussions in Section 4.1 show that other variants are also necessary.

4.20 (Five notions of divisors). Let X be an arbitrary scheme.

(4.20.1) An effective *Cartier divisor* is a subscheme $D \subset X$ such that, for every $x \in D$, the ideal sheaf of $\mathcal{O}_X(-D)$ is locally generated by a non-zerodivisor $s_x \in \mathcal{O}_{x,X}$, called a *local equation* of D .

(4.20.2) A *divisorial subscheme* $D \subset X$ such that \mathcal{O}_D has no embedded points and $\text{Supp } D$ has pure codimension 1 in X .

(4.20.3) A divisorial subscheme D is called an effective *generically Cartier divisor* if it is Cartier at its generic points.

(4.20.4) A divisorial subscheme D is called an effective *Mumford divisor* if X is regular at generic points of D . Thus a Mumford divisor is also a generically Cartier divisor.

(4.20.5) A *Weil divisor* (in traditional terminology) is a formal, finite linear combination $D = \sum_i m_i D_i$ where $m_i \in \mathbb{Z}$ and the D_i are integral subschemes of codimension 1 in X . We say that D is effective if $m_i \geq 0$ for every i .

If A is an abelian group then an *Weil A -divisor* is a formal, finite linear combination $D = \sum_i a_i D_i$ where $a_i \in A$. We will only use the cases $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Thus Weil \mathbb{Z} -divisor = traditional Weil divisor; we use the terminology ‘Weil \mathbb{Z} -divisor’ if the coefficient group is not clear. (A Weil \mathbb{Z} -divisor is sometimes called an integral Weil divisor, but the latter could also mean the Weil divisor corresponding to an integral subscheme of codimension 1.)

Note that usually divisorial subschemes and Weil divisors are used only when X is irreducible or at least pure dimensional, but the definition makes sense in general.

If X is smooth then the 5 variants are equivalent to each other, but in general they are different.

Usually we think of Cartier divisor as the most restrictive notion. If X is S_2 then every effective Cartier divisor is a divisorial subscheme, but this does not hold if X is not S_2 ; see (4.20.9). This is good to keep in mind but it will not be a problem for us.

Let $W \subset X$ be a closed subscheme. We can associate to it both a divisorial subscheme and a Weil divisor by the rules

$$\begin{aligned} \text{Div}(W) &:= \mathcal{O}_W / (\text{torsion in codimension} \geq 2) \quad \text{and} \\ \text{Weil}(W) &:= \sum_i \text{length}_{g_i}(\mathcal{O}_{g_i, W}) \cdot [D_i], \end{aligned} \tag{4.20.6}$$

where in the first case we take the quotient by the subsheaf of those sections whose support has codimension ≥ 2 in X , and in the second case $D_i \subset \text{Supp } W$ are the irreducible components of codimension 1 in X with generic points $g_i \in D_i$. In particular, this associates an effective Weil divisor to any effective Cartier divisor or divisorial subscheme.

Thus, if X is S_2 then we have the basic relations among effective divisors

$$\left(\begin{array}{c} \text{Cartier} \\ \text{divisors} \end{array} \right) \subset \left(\begin{array}{c} \text{Mumford} \\ \text{divisors} \end{array} \right) \subset \left(\begin{array}{c} \text{generically} \\ \text{Cartier divisors} \end{array} \right) \subset \left(\begin{array}{c} \text{divisorial} \\ \text{subschemes} \end{array} \right).$$

Assume next that X is regular at a codimension 1 point $g \in X$. Then $\mathcal{O}_{g, X}$ is a DVR, hence an ideal in it is uniquely determined by its colength. Thus we have the following.

Claim 4.20.7. If X is a normal scheme then 4 of the notions agree for effective divisors

$$\left(\begin{array}{c} \text{Mumford} \\ \text{divisors} \end{array} \right) = \left(\begin{array}{c} \text{generically} \\ \text{Cartier divisors} \end{array} \right) = \left(\begin{array}{c} \text{divisorial} \\ \text{subschemes} \end{array} \right) = \left(\begin{array}{c} \text{Weil} \\ \text{divisors} \end{array} \right).$$

We are mainly interested in slc pairs (X, Δ) , thus the underlying schemes X are deminormal but not normal. Fortunately, X is smooth at the generic points of Δ . Thus for our purposes we can always imagine that the identifications (4.20.7) hold.

Convention 4.20.8. Let X be a scheme and $W \subset X$ a subscheme. Assume that X is regular at all 1-dimensional generic points of W . Then we will frequently identify $\text{Div}(W)$, the divisorial subscheme associated to W and $\text{Weil}(W)$, the Weil divisor associated to W and denote this common object by $[W]$.

We can thus usually harmlessly identify divisorial subschemes and Weil divisors. However—and this is one of the basic difficulties of the theory—it is quite

problematic to keep the identification between *families* of divisorial subschemes and *families* of Weil divisors.

Example 4.20.9. Let $S \subset \mathbb{A}^4$ be the union of the planes $(x_1 = x_2 = 0)$ and $(x_3 = x_4 = 0)$. For $c \neq 0$ consider the Cartier divisors $D_c := (x_1 + cx_3 = 0)$. For any c , the corresponding divisorial subscheme is the union of 2 lines $(x_1 = x_2 = x_3 = 0) \cup (x_1 = x_3 = x_4 = 0)$, hence independent of c . However the D_c are different Cartier divisors for different $c \in k$. Indeed, $(x_1 + c'x_3)/(x_1 + cx_3)$ is a non-regular rational function that is constant c'/c the first plane and 1 on the second.

Note that S is seminormal and the D_c are Mumford.

Corresponding to the 5 notions of divisors, there are 5 main notions of families of divisors.

Relative Weil divisors.

DEFINITION 4.21. Let $f : X \rightarrow S$ a morphism whose fibers have pure dimension n . A Weil divisor $W = \sum m_i W_i$ is called a *relative Weil divisor* if the fibers of $f|_{W_i} : W_i \rightarrow f(W_i)$ have pure dimension $n - 1$ for every i .

We are interested in defining the divisorial fibers of $W \rightarrow S$. A typical example is (4.15), where the multiplicity of the scheme-theoretic fiber jumps over the origin. It is, however, quite natural to say that the ‘correct’ fiber is the origin with multiplicity 2, the only problem we have is that scheme theory miscounts the multiplicity. The following theorem, proved in [Kol96, 3.17], says that this is indeed frequently the case. As with many results about Chow varieties, all the essential ideas are in [HP47, Chap.X].

THEOREM 4.22. Let S be a normal scheme, $f : X \rightarrow S$ a projective morphism and $Z \subset X$ a closed subscheme such that $f|_Z : Z \rightarrow S$ has pure relative dimension m . Then there is a section $\sigma_Z : S \rightarrow \text{Chow}_m(X/S)$ with the following properties.

- (4.22.1) Let $g \in S$ be the generic point. Then $\sigma_Z(g) = [Z_g]$, the cycle associated to the generic fiber of $f|_Z : Z \rightarrow S$ as in (3.8).
- (4.22.2) $\text{Supp}(\sigma_Z(s)) = \text{Supp}(Z_s)$ for every $s \in S$.
- (4.22.3) $\sigma_Z(s) = [Z_s]$ if $f|_Z$ is flat at all generic points of Z_s .
- (4.22.4) $s \mapsto (\sigma_Z(s) \cdot L^m)$ is a locally constant function of $s \in S$, for any line bundle L on X . □

Example (4.14) shows that (4.22) does not hold if S is only seminormal. The notion of well-defined families of algebraic cycles is designed to avoid similar problems, leading to the definition of the Cayley-Chow functor; see [Kol96, Sec.I.3–4] for details.

Flat families of divisorial subschemes.

Let $X \rightarrow S$ be a morphism and $D \subset X$ a subscheme. If $\text{Supp } D$ does not contain any irreducible component of a fiber X_s , then $\mathcal{O}_{D \cap X_s}/(\text{torsion in codimension } \geq 2)$ is (the structure sheaf of) a divisorial subscheme of X_s . This notion, however, frequently does not have good continuity properties, as illustrated by (4.15).

We would like to have a notion of flat families of divisorial subschemes where both the structure sheaf \mathcal{O}_D and the ideal sheaf $\mathcal{O}_X(-D)$ are well behaved. This seems possible only if $X \rightarrow S$ is well behaved, but then the two aspects turn out to be equivalent.

DEFINITION–LEMMA 4.23. Let $f : X \rightarrow S$ be a flat morphism of pure relative dimension n with S_2 -fibers and $D \subset X$ a closed subscheme of relative dimension $n - 1$ over S . We say that $f|_D : D \rightarrow S$ is a *flat family of divisorial subschemes* if the following equivalent conditions hold.

- (4.23.1) $f|_D : D \rightarrow S$ is flat with pure (9.2) fibers of dimension $n - 1$.
 (4.23.2) $\mathcal{O}_X(-D)$ is flat over S with S_2 fibers.

Proof. We have a surjection $\mathcal{O}_X \rightarrow \mathcal{O}_D$ and if both of these sheaves are flat then so is the kernel $\mathcal{O}_X(-D)$. If the kernel is flat then $\mathcal{O}_{X_s}(-D_s) \cong \mathcal{O}_X(-D)|_{X_s}$ is also the kernel of $\mathcal{O}_{X_s} \rightarrow \mathcal{O}_{D_s}$. Since \mathcal{O}_{X_s} is S_2 , we see that $\mathcal{O}_{X_s}(-D_s)$ is S_2 iff \mathcal{O}_{D_s} is pure of dimension $n - 1$.

Conversely, assume (2). For any $T \rightarrow S$ the pull-back map $q_T^* \mathcal{O}_X(-D) \rightarrow q_T^* \mathcal{O}_X$ is an isomorphism over $X_T \setminus D_T$. Since $\mathcal{O}_X(-D)$ is flat with S_2 fibers, $q_T^* \mathcal{O}_X(-D)$ does not have any sections supported on D_T . Thus the pulled-back sequence

$$0 \rightarrow q_T^* \mathcal{O}_X(-D) \rightarrow q_T^* \mathcal{O}_X \rightarrow q_T^* \mathcal{O}_D \rightarrow 0$$

is exact. Therefore $\mathrm{Tor}_1^S(\mathcal{O}_T, \mathcal{O}_D) = 0$ hence \mathcal{O}_D is flat over S and we already noted that then it has pure fibers of dimension $n - 1$. \square

Relative Cartier divisors.

DEFINITION–LEMMA 4.24. Let $f : X \rightarrow S$ be a morphism, $x \in X$ a point such that f is flat at x and set $s := f(x)$. A subscheme $D \subset X$ is a *relative Cartier divisor* at $x \in X$ if the following equivalent conditions hold.

- (4.24.1) D is flat over S at x and $D_s := D|_{X_s}$ is a Cartier divisor on X_s at x .
 (4.24.2) D is a Cartier divisor on X at x and a local equation $g_x \in \mathcal{O}_{x,X}$ of D restricts to a non-zerodivisor on the fiber X_s .
 (4.24.3) D is a Cartier divisor on X at x and it does not contain any irreducible component of X_s that passes through x .

If these hold for all $x \in D$ then D is a *relative Cartier divisor*. If $f : X \rightarrow S$ is also proper then the functor of relative Cartier divisors is represented by an open subscheme of the Hilbert scheme $\mathrm{CDiv}(X/S) \subset \mathrm{Hilb}(X/S)$; see [Kol96, I.1.13] for the easy details.

If (2) holds then D is flat by (4.23). The other nontrivial claim is that (1) implies that D is a Cartier divisor on X at x . We may assume that $(x \in X)$ is local. A defining equation g_s of D_s lifts to an equation g of D . We have the exact sequence

$$0 \rightarrow I_D/(g) \rightarrow \mathcal{O}_X/(g) \rightarrow \mathcal{O}_D \rightarrow 0.$$

Here $\mathcal{O}_X/(g)$ and \mathcal{O}_D are both flat, hence so is $I_D/(g)$. Restricting to X_s we get

$$0 \rightarrow (I_D/(g))_s \rightarrow \mathcal{O}_{X_s}/(g_s) \xrightarrow{\cong} \mathcal{O}_{D_s} \rightarrow 0.$$

Thus $I_D/(g) = 0$ by the Nakayama lemma and so g is a defining equation of D . \square

Relative Cartier divisors form a very well behaved class, but in applications we frequently have to handle 2 problems. It is not always easy to see which divisors are Cartier and we also need to deal with divisors that are not Cartier.

On a smooth variety every divisor is Cartier, thus if X itself is smooth then a divisor D is relatively Cartier iff its support does not contain any of the fibers. In the relative setting, we usually focus on properties of the morphism f . Thus

we would like to have similar results for smooth morphisms. The next result of [Ram63, Sam62] shows that this is indeed true if S is normal; see also [Gro60, IV.21.14.1]. Note that by (4.28) some kind of normality assumption is necessary.

THEOREM 4.25. *Let S be a normal scheme, $f : X \rightarrow S$ a smooth morphism and D a Weil divisor on X . Assume that D does not contain any irreducible component of a fiber. Then D is a Cartier divisor, hence a relative Cartier divisor.*

Proof. Assume to the contrary that the non-Cartier locus of D is nonempty. Let $x \in X$ be a generic point of it and set $s = f(x)$. If $s \in S$ has codimension 1 then S is regular at s , hence X is regular at x and so D is Cartier at x , a contradiction.

Thus $s \in S$ has codimension ≥ 2 , hence $\text{depth}_s S \geq 2$. This case now follows from (4.26). \square

THEOREM 4.26. *Let $f : (x, X) \rightarrow (s, S)$ be a smooth morphism of local schemes such that $\text{depth}_s S \geq 2$. Let D be a Weil divisor on X . Assume that D is Cartier on $X \setminus \{x\}$ and its support does not contain any irreducible component of X_s . Then D is Cartier.*

Proof. We start with the special case when $k(x) = k(s)$ and f has relative dimension 1. Then $f|_D : D \rightarrow S$ is quasi-finite at x , so $f|_D$ is flat at x by (10.53). Thus D is a relative Cartier divisor at x by (4.24.1).

Next assume that $k(x) = k(s)$ still holds but f has relative dimension $n > 1$. Since f is smooth, over a neighborhood of x it can be written as a composite

$$f : (x, X) \xrightarrow{\tau} ((0, s), \mathbb{A}_S^n) \xrightarrow{\pi} S,$$

where τ is étale and π is the structure projection. Composing with any of the coordinate projections we factor f as

$$f : (x, X) \xrightarrow{g} ((0, s), \mathbb{A}_S^{n-1}) \rightarrow S,$$

where g is smooth of relative dimension 1. If D does not contain the fiber of g passing through x then D is a Cartier divisor by the already discussed 1-dimensional case.

To find such a g , assume first that $k(s)$ is infinite. Let $L \subset \mathbb{A}_S^n$ be a general line through the origin. Then $\pi_s^{-1}(L) \not\subset D_s$. Thus if we choose the projection $\mathbb{A}_S^n \rightarrow \mathbb{A}_S^{n-1}$ to have kernel L over s then the argument proves that D is a Cartier divisor at x .

If $k(s)$ is finite then consider the trivial lifting $f^{(1)} : X \times \mathbb{A}^1 \rightarrow S \times \mathbb{A}^1$. By the previous argument $D \times \mathbb{A}^1$ is a Cartier divisor at the generic point of $\{x\} \times \mathbb{A}^1$, hence D is a Cartier divisor at x by (4.27).

Finally (10.47) shows how to reduce the general case when $k(x) \neq k(s)$ to the special case where $k(x) = k(s)$ by a simple base change. \square

LEMMA 4.27. *Let $(R, m_R) \rightarrow (S, m_S)$ be a flat extension of local rings and $I_R \subset R$ an ideal. Then I_R is principal iff $I_R S$ is principal.*

Proof. One direction is clear. Conversely, assume that $I_R S$ is principal, thus $I_R S / m_S I_R S \cong S / m_S$. Let r_1, \dots, r_n be generators of I_R . They also generate $I_R S$ hence at least one of them, say r_1 , is not contained in $m_S I_R S$. Thus $(r_1) \subset I_R$ is a sub-ideal such that $r_1 S = I_R S$. Since $(R, m_R) \rightarrow (S, m_S)$ is faithfully flat, this implies that $(r_1) = I_R$. \square

EXAMPLE 4.28. We give 2 examples showing that in (4.25) we do need normality of S .

Set $S_n := \text{Spec } k[x, y]/(xy)$ and $X_n = \text{Spec } k[x, y, z]/(xy)$. Then (x, z) defines a Weil divisor which is not Cartier.

Set $S_c := \text{Spec } k[x^2, x^3]$ and $X_c = \text{Spec } k[x^2, x^3, y]$. Then $(y^2 - x^2, y^3 - x^3)$ defines a Weil divisor which is not Cartier.

Relative generically Cartier divisors.

DEFINITION 4.29. Let $f : X \rightarrow S$ be a morphism. A subscheme $D \subset X$ is a *relative, generically Cartier, effective divisor* or a *family of generically Cartier, effective divisors* over S if there is an open subset $U \subset X$ such that

- (4.29.1) f is flat over U with S_2 fibers,
- (4.29.2) $\text{codim}_{X_s}(X_s \setminus U) \geq 2$ for every $s \in S$,
- (4.29.3) $D|_U$ is a relative Cartier divisor (4.24) and
- (4.29.4) D is the closure of $D|_U$.

If $U \subset X$ denotes the largest open set with these properties then $Z := X \setminus U$ is the *non-Cartier locus* of D .

Thus $\mathcal{O}_X(mD)$ is a mostly flat family of divisorial sheaves on X (3.25) for any $m \in \mathbb{Z}$. Conversely, if L is a mostly flat family of divisorial sheaves on X and h a global section of it that does not vanish on any irreducible component of any fiber, then $(h = 0)$ is a family of generically Cartier, effective divisors over S .

Let $q : W \rightarrow S$ be any morphism. We have a fiber product diagram

$$\begin{array}{ccc} X_W & \xrightarrow{q_X} & X \\ f_W \downarrow & & \downarrow f \\ W & \xrightarrow{q} & S. \end{array} \quad (4.29.5)$$

Then $q_X^*(D|_U)$ is a well-defined relative Cartier divisor on $U_W := q_X^{-1}(U)$; let $D_W \subset X_W$ denote its closure. It agrees with the divisorial pull-back of D defined in (4.1.7.a). Since the pull-back of Cartier divisors is functorial, this shows that a family of relative, generically Cartier divisors is a well-defined family of divisors.

The next result shows that the converse is also true.

THEOREM 4.30. *Let S be a reduced scheme and $f : (X, D) \rightarrow S$ a projective family of pairs satisfying the assumptions (4.1.2–4). Then the following are equivalent.*

- (4.30.1) *The family is well-defined using the divisorial pull-back $g^{[*]}$ (4.1.7.a).*
- (4.30.2) *D is a relative, generically Cartier divisor on X .*
- (4.30.3) *$f|_D : D \rightarrow S$ is flat at generic points of D_s for every $s \in S$.*
- (4.30.4) *$s \mapsto \deg D_s$ is locally constant on S .*

If S is weakly normal, then these are also equivalent to

- (4.30.5) *D has well-defined specializations (4.1.9).*

Proof. All 5 conditions are local on S and can be checked on a general relative hyperplane section of X ; see (4.32), (4.31) and (10.46).

Thus we may assume that $X \rightarrow S$ has relative dimension 1, hence f is smooth along $\text{Supp } D$. We view D as a divisorial subscheme of X .

Applying (3.20) to $F := f_*\mathcal{O}_D$ (with $X = S$) we see that (4) holds iff \mathcal{O}_D is flat over S . By (4.24) the latter holds iff D a relative Cartier divisor. Thus (2) \Leftrightarrow (3) \Leftrightarrow (4).

Let $\tau_s : s \rightarrow S$ be a geometric point. By construction, $\deg \tau_s^{[*]}D = \dim_{k(s)} \mathcal{O}_{D_s}$, thus $s \mapsto \deg \tau_s^{[*]}D$ is locally constant iff (4) holds.

Finally, let T be the spectrum of a DVR and $h : T \rightarrow S$ a morphism that maps the closed point to $s \in S$ and the generic point to a generalization g of s . Then $h^{[*]}D$ is flat over T of degree $\deg_{k(g)} \mathcal{O}_{D_g}$. Thus if $\bar{\tau}_s : s \rightarrow T$ is a lifting of τ then

$$\deg \bar{\tau}_s^{[*]}h^{[*]}D = \deg_{k(g)} \mathcal{O}_{D_g}.$$

This shows that $(X, D) \rightarrow S$ is a well-defined family of 0-cycles iff $s \mapsto \deg \tau_s^{[*]}D$ is locally constant, proving that (1) \Leftrightarrow (4). Also, (1) \Rightarrow (5) holds for any S .

Finally we prove that (5) \Rightarrow (2). If S is normal then (4.25) shows that D is relative Cartier divisor and we are done. In the weakly normal case essentially the same proof as in (4.25) works, but we need to use different references. First (10.54) shows that D is flat over S and then D is relatively Cartier by (4.24.1). \square

The following two Bertini-type results are frequently useful. The first is an immediate consequence of (10.46) and the second follows from (10.13.1).

PROPOSITION 4.31. *Let $(0 \in S)$ be a local scheme, $X \subset \mathbb{P}_S^N$ a quasi-projective S -scheme with fibers of pure dimension ≥ 2 and $D \subset X$ a relative divisorial subscheme. Then D is a generically Cartier family of divisors on X iff $D|_H$ is a generically Cartier family of divisors on $X \cap H$ for general $H \in |\mathcal{O}_{\mathbb{P}_S^N}(1)|$. \square*

LEMMA 4.32. *Let $f : (X, D) \rightarrow S$ be a family of pairs that satisfies the assumptions (4.1.2–4) and S excellent. For a morphism $g : T \rightarrow S$ let $g^{[*]}$ denote one of the pull-back constructions defined in (4.5.1–2) or (4.6). Then there are finitely many points $\{x_i : i \in I\}$ of X (depending on X, D and g) such that if $H \subset X$ is a relative Cartier divisor that does not contain any of the points x_i then*

$$(g^{[*]}(X, D))|_{H_T} \cong g^{[*]}(H, D|_H)$$

where H_T denotes the preimage of H in $X \times_S T$. \square

Representability for divisorial pull-backs.

Let $f : (X, D) \rightarrow S$ be a family of generically Cartier divisors. We study those morphisms $q : W \rightarrow S$ for which the divisorial pull-back D_W is flat or relatively Cartier.

DEFINITION 4.33. Let S be a scheme, $f : X \rightarrow S$ a flat, projective morphism with S_2 fibers and $D \subset X$ a family of generically Cartier divisors. Let $f : X \rightarrow S$ be a flat morphism and D a relative Mumford divisor on X . The functor of *Cartier divisorial pull-backs* is

$$\text{Car}_D(q : T \rightarrow S) = \begin{cases} 1 & \text{if } q^{[*]}D \rightarrow T \text{ is Cartier, and} \\ 0 & \text{otherwise.} \end{cases}$$

One defines analogously the functor of \mathbb{Q} - or \mathbb{R} -Cartier pull-backs and the functor of *flat, divisorial pull-backs*, denoted by $DFlat_D$.

We prove that the Cartier and the flat pull-backs are represented by a locally closed partial decomposition $S' \rightarrow S$. This, however, does not hold for \mathbb{Q} -Cartier

divisorial pull-backs (4.19). To remedy this we introduce the notion of numerically \mathbb{R} -Cartier divisors later in (4.55).

Since the distinction is important, in the remainder of this subsection we use D_s to denote the fiber of D over $s \in S$ and $[D_s] = \text{pure}(D_s)$ to denote the pure fiber, as in (4.20.8). Thus $[D_s] = \text{Div}(D_s)$ in the notation of (4.20.6).

The first result is another version of (3.28).

THEOREM 4.34. *Let S be a scheme, $f : X \rightarrow S$ a flat, projective morphism with S_2 fibers and $D \subset X$ a family of generically Cartier divisors. Then the functor of flat, divisorial pull-backs is represented by a locally closed decomposition $j^{\text{dflat}} : S^{\text{dflat}} \rightarrow S$.*

Equivalently, for every S -scheme $q : W \rightarrow S$, the divisorial pull-back $f_W : (X_W, D_W) \rightarrow W$ is a flat family of divisorial subschemes (4.23) iff q factors as $q : W \rightarrow S^{\text{dflat}} \rightarrow S$.

Note also that, as in (3.28.1), the subset of S

$$\{s : D \text{ is flat along } \text{Supp } D_s\}$$

is open, but we want the ‘corrected’ restrictions $[D_s]$ (4.20.6) to form a flat family.

Proof. This is a special case of (9.64), but let us outline here a shorter proof that works when S is reduced.

First we claim that $s \mapsto \chi(X_s, \mathcal{O}_{[D_s]}(*))$ is constructible and upper semicontinuous on S . For this we may replace S with its seminormalization, hence the claim follows from (3.30). Thus we get a locally closed decomposition $j : S' \rightarrow S$ such that $s \mapsto \chi(X_s, \mathcal{O}_{[D_s]}(*))$ is locally constant on S' .

If $f_W : (X_W, D_W) \rightarrow W$ is a flat family of divisorial subschemes then $W \rightarrow S$ factors through $j : S' \rightarrow S$. Thus it remains to prove that $D' := j_X^{[*]} D \subset X' := X \times_S S'$ is a flat family of divisorial subschemes. The latter follows from (4.35). \square

PROPOSITION 4.35. *Let $f : X \rightarrow S$ be a flat, projective morphism with S_2 fibers and $D \subset X$ a family of generically Cartier divisors. Assume in addition that S is reduced and $\mathcal{O}_X(1)$ is relatively ample. The following are equivalent.*

(4.35.1) $f : (X, D) \rightarrow S$ is a flat family of divisorial subschemes,

(4.35.2) $s \mapsto \chi(X_s, \mathcal{O}_{X_s}(-[D_s])(*))$ is locally constant on S and

(4.35.3) $s \mapsto \chi(X_s, \mathcal{O}_{[D_s]}(*))$ is locally constant on S .

Proof. The last two assertions are equivalent since

$$\chi(X_s, \mathcal{O}_{X_s}(-[D_s])(*)) = \chi(X_s, \mathcal{O}_{X_s}(*)) - \chi(X_s, \mathcal{O}_{[D_s]}(*)).$$

If (1) holds then the $\mathcal{O}_{[D_s]}$ are fibers of the flat sheaf \mathcal{O}_D , hence (1) \Rightarrow (3). The converse follows from (9.71). \square

The representability of Cartier pull-backs now follows easily. Example (4.15) shows that (4.34) and (4.36) both can fail if D is not a generically Cartier family.

COROLLARY 4.36. *Let $f : X \rightarrow S$ be a flat, projective morphism with S_2 fibers and D a family of generically Cartier, not necessarily effective divisors on X .*

(4.36.1) *The functor of Cartier divisorial pull-backs (4.33) is represented by a locally closed partial decomposition $j^{\text{car}} : S^{\text{car}} \rightarrow S$.*

(4.36.2) *If f is smooth then $S^{\text{car}} \hookrightarrow S$ and $\text{red } S^{\text{car}} = \text{red } S$.*

In general $i(S^{\text{car}}) \subset S$ is neither open nor closed; see (3.42).

Proof. Let H be an ample Cartier divisor on X . Then $D + mH$ is linearly equivalent to an effective family of generically Cartier divisors for $m \gg 1$. Thus we may as well assume that D is effective.

An effective, relatively Cartier family is also a flat family of divisors, and we proved in (4.34) that flat divisorial pull-backs are represented by a locally closed decomposition $S^{\text{dflat}} \rightarrow S$. As we noted in (3.23.1), we can replace S by S^{dflat} and henceforth consider only the special case when D is flat over S .

For a flat family of divisorial subschemes being Cartier is an open condition, thus S^{car} is an open subset of S^{dflat} , proving (1).

Assume next that f is flat and let $q : T \rightarrow S$ be a DVR. Then X_T is regular, hence D_T is Cartier. Thus $S^{\text{car}} \rightarrow S$ is a proper monomorphism, hence a closed embedding by (10.83.1). It is also surjective, proving (2). \square

As shown by (4.19), the analogous result does not hold for \mathbb{Q} - or \mathbb{R} -Cartier divisors; see, however, (4.38).

Next we give a valuative criterion for Cartier divisors. The generically Cartier assumption is necessary, as shown by (4.15).

THEOREM 4.37 (Valuative criterion for Cartier divisors). *Let S be a reduced scheme, $f : X \rightarrow S$ a flat morphism of finite type with S_2 fibers and D a family of generically Cartier divisors on X . Assume that either f is projective or S is excellent. Then following are equivalent.*

- (4.37.1) D is a Cartier (resp. \mathbb{Q} - or \mathbb{R} -Cartier) divisor.
 (4.37.2) For every morphism $q : T \rightarrow S$ from the spectrum of a DVR to S , the divisorial pull-back $D_T \subset X_T$ is a Cartier (resp. \mathbb{Q} - or \mathbb{R} -Cartier) divisor.

Proof. It is clear that (1) implies (2). Thus assume that (2) holds.

Assume first that f is projective and start with the Cartier case. Consider the locally closed embedding $j^{\text{car}} : S^{\text{car}} \rightarrow S$ given by (4.36). Since every $q : T \rightarrow S$ factors through $j^{\text{car}} : S^{\text{car}} \rightarrow S$, we see that i is proper and surjective, hence an isomorphism (10.84.2).

Assume next that D is \mathbb{Q} -Cartier and (2) holds. By assumption $m_s D_s$ is Cartier for some $m_s > 0$ for every $s \in S$. By (3.32) there is a common m such that mD_s is Cartier for every $s \in S$.

Let T be the spectrum of a DVR mapping to S . Then D_T is \mathbb{Q} -Cartier by assumption, thus mD_T is Cartier by (2.93). We are thus reduced to the already settled Cartier case.

In the \mathbb{R} -Cartier case write $D = \sum d_i D^i$ where the D^i are \mathbb{Q} -divisors and the $d_i \in \mathbb{R}$ are linearly independent over \mathbb{Q} . Note that this decomposition of D is preserved by any base change. By (11.34.2) the D_T^i are \mathbb{Q} -Cartier for every $q : T \rightarrow S$, hence the D^i are \mathbb{Q} -Cartier by the first part. Thus D is \mathbb{R} -Cartier.

Consider next the case when f is non-projective. Pick any point $x \in X$ and its image $s := f(x)$. Let \hat{S} denote the completion of S at s ; it is reduced since S is excellent. Then D is Cartier (resp. \mathbb{Q} - or \mathbb{R} -Cartier) at x iff this holds after base change to \hat{S} . Thus it is enough to show that (2) \Rightarrow (1) whenever S is complete.

In the Cartier case we use (9.72) to get $i : S^u \hookrightarrow S$. Let $(0, T)$ be the spectrum of a DVR and $q : (0, T) \rightarrow (0, S)$ a local morphism. By assumption (2) $\mathcal{O}_{X_T}(D_T)$ is locally free, hence it is flat with S_2 fibers over T . Thus $\mathcal{O}_{X_T}(D_T)$ is a universal hull

by (9.27). Therefore q factors through S^u . Since this holds for every $q : T \rightarrow S$, we conclude that $S^u = S$.

The \mathbb{Q} - and \mathbb{R} -Cartier cases are handled as before. \square

COROLLARY 4.38. *Let S be a reduced scheme, $f : X \rightarrow S$ a flat, projective morphism with S_2 fibers and D a family of generically \mathbb{Q} -Cartier (resp. \mathbb{R} -Cartier) divisors on X . Let $S^* \subset S$ be a constructible subset and assume that D_s is \mathbb{Q} -Cartier (resp. \mathbb{R} -Cartier) for every $s \in S^*$.*

Then there is a locally closed partial decomposition $j^{\text{qcar}} : S^{\text{qcar}} \rightarrow S$ (resp. $j^{\text{rcar}} : S^{\text{rcar}} \rightarrow S$) such that the following holds.

(4.38.1) *Let $q : W \rightarrow S$ be reduced S -scheme such that $q(W) \subset S^*$. Then the divisorial pull-back $D_W \subset X_W$ is \mathbb{Q} -Cartier (resp. \mathbb{R} -Cartier) iff q factors as $q : W \rightarrow S^{\text{qcar}} \rightarrow S$ (resp. $q : W \rightarrow S^{\text{rcar}} \rightarrow S$).*

Proof. We may assume that S^* is dense in S . Start with the \mathbb{Q} -Cartier case. Using noetherian induction we see that there is an $m > 0$ such that mD_s is Cartier for every $s \in S^*$. As in the proof of (4.37) there are largest, dense, open subsets $S_1^{\text{car}} \subset S_2^{\text{car}} \subset \dots$ such that $r! \cdot mD$ is Cartier over S_2^{car} . The union of all of them is the open stratum of $S^{\text{qcar}} \rightarrow S$. Induction then gives the other strata.

In the \mathbb{R} -Cartier case we again write $D = \sum d_i D^i$ where the D^i are \mathbb{Q} -divisors and the $d_i \in \mathbb{R}$ are linearly independent over \mathbb{Q} . We already have locally closed partial decompositions $j_i^{\text{qcar}} : S_i^{\text{qcar}} \rightarrow S$ using D^i , and $j^{\text{rcar}} : S^{\text{rcar}} \rightarrow S$ is their fiber product over S . \square

4.4. Generically \mathbb{Q} -Cartier divisors

In the study of lc and slc pairs, \mathbb{Q} -Cartier divisors are more important than Cartier divisors. Unfortunately, as (3.41) and (4.19) show, having \mathbb{Q} -Cartier divisorial pull-backs is not a representable condition in general. See, however, (4.54) for a positive result.

We have also seen many examples of Weil \mathbb{Z} -divisors that are \mathbb{Q} -Cartier but not Cartier. By contrast, we show that if a relative Weil \mathbb{Z} -divisor is generically \mathbb{Q} -Cartier then it is generically Cartier in characteristic 0.

One of the consequences of this is that the common denominator pull-back $g_N^{[*]}(X, \Delta)$ defined in (4.5.2) is independent of the denominator N in characteristic 0.

Let $f : (X, D) \rightarrow S$ be a family of pairs and D a relative Weil \mathbb{Z} -divisor on X .

Since we are interested in generic properties, we can focus on a generic point x of $D \cap X_s$. If the assumption (4.1.4) holds then f is smooth at x . Thus we may as well assume that f is smooth (but not proper).

If S is normal then D is a Cartier divisor by (4.25), thus here our main interest is in those cases where S is reduced but not normal. Examples (4.14) and (4.16) show that then D need not be Cartier in general. However, the next result shows that if some multiple of D is Cartier, then so is D , at least in characteristic 0. This also completes the proof of (4.7.3).

Positive characteristic counter examples are given in (4.18) and (4.46).

PROPOSITION 4.39. *Let S be a reduced scheme, $f : X \rightarrow S$ a smooth morphism and D a relative Weil \mathbb{Z} -divisor on X . Assume that mD is Cartier at a point $x \in X$ and $\text{char } k(x) \nmid m$. Then D is Cartier at x .*

Proof. By noetherian induction we may assume that D is Cartier on $X \setminus \{x\}$. Using (10.47) and (4.27), it is enough to prove this when $k(x) = k(f(x))$. We may also assume that $f : (x, X) \rightarrow (s, S)$ is a local morphism of local, henselian schemes and $k(x) = k(s)$ is separably closed. By (11.14) $mD \sim 0$ determines a cyclic cover $\tilde{X} \rightarrow X$ that is étale over $X \setminus \{x\}$ whenever $\text{char } k(x) \nmid m$. In our case $\tilde{X} \rightarrow X$ is trivial by (4.40) hence D is Cartier at x . \square

LEMMA 4.40. *Let $f : (x, X) \rightarrow (s, S)$ be a smooth, local morphism of henselian, local schemes. Assume that $k(x) = k(s)$ is separably closed, f has relative dimension ≥ 1 and S has dimension ≥ 1 . Then $\hat{\pi}_1(X \setminus \{x\}) = 1$.*

Proof. Over \mathbb{C} , a topological proof is given in (4.41). A similar algebraic argument is the following.

Set $X^\circ := X \setminus \{x\}$ and let $\tilde{X}^\circ \rightarrow X^\circ$ be a finite étale cover. Let (t, T) be the spectrum of a DVR and $(t, T) \rightarrow (s, S)$ a local morphism that maps the generic point of T isomorphically to a generic point of S . By pull-back we get a finite étale cover $\tilde{X}_T^\circ \rightarrow X_T^\circ$. Since X_T is regular and of dimension ≥ 2 , the purity of branch loci implies that $\tilde{X}_T^\circ \rightarrow X_T^\circ$ is trivial. In particular, $\tilde{X}^\circ \rightarrow X^\circ$ is trivial on every irreducible component $X_i^\circ \subset X^\circ$.

Thus $\tilde{X}^\circ \rightarrow X^\circ$ is also trivial on X_s° . If $\tilde{X}_s^\circ = \cup_j Z_j^\circ$ then, for every i, j there is a unique irreducible component $\tilde{X}_{ij}^\circ \subset \tilde{X}^\circ$ that dominates X_i° and contains Z_j° . Furthermore $\tilde{X}_j^\circ := \cup_i \tilde{X}_{ij}^\circ$ is a connected component of \tilde{X}° that maps isomorphically onto X° . Thus $\tilde{X}^\circ \rightarrow X^\circ$ is trivial. \square

4.41 (Links and smooth morphisms). Let $f : X \rightarrow S$ be a smooth morphism of complex spaces of relative dimension $n \geq 1$. We describe the topology of the link of a point $x \in X$ in terms of the topology of the link of $s := f(x) \in S$.

We can write $S \subset \mathbb{C}_{\mathbf{z}}^N$ such that s is the origin and $X \subset S \times \mathbb{C}_{\mathbf{t}}^n$ where x is the origin. Intersecting S with a sphere of radius ϵ centered at s we get L_S , the link of $s \in S$. The intersection of S with the corresponding ball of radius ϵ is homeomorphic to the cone C_S over L_S .

The link L_X of $x \in X$ can be obtained as the intersection of X with the level set $\max\{\sum |z_i|^2, \sum |t_j|^2\} = \epsilon^2$. Thus L_X is homeomorphic to the amalgamation of

$$\begin{aligned} L_S \times \mathbb{D}^{2n} &= \{(\mathbf{z}, \mathbf{t}) : \sum |z_i|^2 = \epsilon^2, \sum |t_j|^2 \leq \epsilon^2\} \quad \text{and of} \\ C_S \times \mathbb{S}^{2n-1} &= \{(\mathbf{z}, \mathbf{t}) : \sum |z_i|^2 \leq \epsilon^2, \sum |t_j|^2 = \epsilon^2\}, \quad \text{glued along} \\ L_S \times \mathbb{S}^{2n-1} &= \{(\mathbf{z}, \mathbf{t}) : \sum |z_i|^2 = \epsilon^2, \sum |t_j|^2 = \epsilon^2\}. \end{aligned}$$

Let L_S^i be the connected components of L_S . Note that $\pi_1(L_S^i \times \mathbb{S}^{2n-1}) \cong \pi_1(L_S^i) \times \pi_1(\mathbb{S}^{2n-1})$. The first factor gets killed in $\pi_1(C_S \times \mathbb{S}^{2n-1})$, the second is trivial if $n \geq 2$ and gets killed in $\pi_1(L_S^i \times \mathbb{D}^{2n})$ if $n = 1$. Thus L_X is simply connected for $n \geq 1$.

The cohomology of L_X can be computed from the Mayer-Vietoris sequence. Using that $H^i(L_S \times \mathbb{D}^{2n}, \mathbb{Z}) = H^i(L_S, \mathbb{Z})$ and $H^i(C_S \times \mathbb{S}^{2n-1}, \mathbb{Z}) = H^i(\mathbb{S}^{2n-1}, \mathbb{Z})$, for H^2 the key pieces are

$$\begin{aligned} &\rightarrow H^1(L_S, \mathbb{Z}) + H^1(\mathbb{S}^{2n-1}, \mathbb{Z}) \xrightarrow{\sigma^1} H^1(L_S \times \mathbb{S}^{2n-1}, \mathbb{Z}) \\ \rightarrow H^2(L_X, \mathbb{Z}) &\rightarrow H^2(L_S, \mathbb{Z}) + H^2(\mathbb{S}^{2n-1}, \mathbb{Z}) \xrightarrow{\sigma^2} H^2(L_S \times \mathbb{S}^{2n-1}, \mathbb{Z}). \end{aligned}$$

The Künneth formula gives that the σ^i are injections and σ^1 is an isomorphism if $n \geq 2$. In this case $H^2(L_X, \mathbb{Z}) = 0$. If $n = 1$ then

$$\begin{aligned} H^2(L_X, \mathbb{Z}) &\cong \operatorname{coker}[H^1(\mathbb{S}^1, \mathbb{Z}) \rightarrow H^0(L_X, \mathbb{Z}) \times H^1(\mathbb{S}^1, \mathbb{Z})] \\ &\cong H^0(L_S, \mathbb{Z})/\mathbb{Z}. \end{aligned} \quad (4.41.1)$$

We have thus proved the following.

Claim 4.41.2. $f : X \rightarrow S$ be a smooth morphism of complex spaces, L_X the link of a point $x \in X$ and $s := f(x)$. Assume that $\dim_x X > \dim_s S \geq 1$.

Then L_X is simply connected. Furthermore, $H^2(L_X, \mathbb{Z}) = 0$ iff either $n \geq 2$ or the link of $s \in S$ is connected. \square

Next we compute the local Picard groups in more detail, starting with the weakly normal case, which can be viewed as a further generalization of (4.25) and (4.26). See [Sta15, Tag 0BSK] for the *strict Henselization* S^{sh} of a local scheme ($s \in S$).

THEOREM 4.42. *Let $(s \in S)$ be a local, weakly normal pair (10.76) and $f : X \rightarrow S$ a smooth morphism. Let $x \in X_s$ be a point. Then,*

(4.42.1) *if $\operatorname{codim}(x \in X_s) \geq 2$ then $\operatorname{Pic}^{\text{loc}}(x, X) = 0$, and*

(4.42.2) *if $\operatorname{codim}(x \in X_s) = 1$ then $\operatorname{Pic}^{\text{loc}}(x, X)$ is free of $\leq r - 1$ generators, where r is the number of connected components of $S^{\text{sh}} \setminus \{s\}$.*

Proof. By (10.47.6) it is enough to consider the case when $k(x)/k(s)$ is finite and purely inseparable. We may factor f as

$$f : (x, X) \xrightarrow{g} ((s, 0), S \times \mathbb{A}^{d-1}) \xrightarrow{\pi} (s, S)$$

If $d - 1 \geq 1$ then $(S \times \mathbb{A}^{d-1}) \setminus \{(s, 0)\}$ is connected. Thus establishing (2) for $g : (x, X) \rightarrow ((s, 0), S \times \mathbb{A}^{d-1})$ yields (1).

So assume from now on that $d = 1$. Every class in $\operatorname{Pic}^{\text{loc}}(x, X)$ can be represented by an effective divisor D that does not contain X_s . Then $f|_D : D \rightarrow S$ is finite and flat over $S \setminus \{s\}$. We claim that the isomorphism (4.42.1) is given by $[D] \mapsto (\operatorname{rank}_{T_i} f_* \mathcal{O}_D : i \in I)$. The key point is to show that if $f_* \mathcal{O}_D$ has constant rank d on $S \setminus \{s\}$ then D is Cartier.

By assumption $f|_D : D \rightarrow S$ is flat over $S \setminus \{s\}$ and has constant degree. Thus $f|_D$ is flat by (10.54) hence D is Cartier by (4.24). \square

Next we compute how the local Picard group changes as we pass from a pair ($s \in S$) to its weak normalization.

THEOREM 4.43. *Let $\pi : (s' \in S') \rightarrow (s \in S)$ be a finite, local morphism of reduced, local schemes that is an isomorphism outside the closed points. Assume that s, s' are not associated points and $k(s')/k(s)$ is purely inseparable. Let $X \rightarrow S$ be a flat morphism, $x \in X_s$ a non-generic point and $x' \in X' := X \times_S S'$ its preimage. Then the pull-back map $\pi^* : \operatorname{Pic}^{\text{loc}}(x, X) \rightarrow \operatorname{Pic}^{\text{loc}}(x', X')$ is*

(4.43.1) *an injection if $\operatorname{depth}_x X_s \geq 2$, and*

(4.43.2) *a finite length extension of $k(s)$ -vector spaces if X_s is smooth.*

Complement 4.43.3. The $k(s)$ -vector spaces are infinite dimensional if $\dim X_s = 1$ and an extension of $k(s)$ -vector spaces is a $k(s)$ -vector space if $\operatorname{char} k(s) = 0$.

Proof. We use (10.75) to write the weak normalization morphism as a composite of simpler partial weak normalizations. We deal with partial seminormalizations in (4.44) and then with residue field extensions in (4.45). \square

LEMMA 4.44. *Using the notation of (4.43), assume in addition that $k(s') = k(s)$. Then π^* is*

(4.44.1) *an injection if $\text{depth}_x X_s \geq 2$, and*

(4.44.2) *a finite length extension of $k(s)$ -vector spaces in general.*

Proof. Using (10.75) it is enough to prove the lemma in the special case when $m \cdot \mathcal{O}_{S'} \subset \mathcal{O}_S$.

Set $U := X \setminus \{x\}$ and $U' := X' \setminus \{x'\}$. We claim that there is an exact sequence

$$1 \rightarrow \mathcal{O}_U^\times \rightarrow \mathcal{O}_{U'}^\times \xrightarrow{\tau} (m'/m) \otimes \mathcal{O}_{U_s} \rightarrow 0, \quad (4.44.3)$$

defined as follows.

Since $\text{depth}_x X' \geq 2$, any local section h' of $\mathcal{O}_{U'}^\times$ extends to a local section \bar{h}' of $\mathcal{O}_{X'}$. Then $X_s \cong X'_{s'}$ shows that there is a local section h of \mathcal{O}_U such that $\bar{h}'|_{X'_{s'}} = h|_{X_s}$. Now we set

$$\tau : h' \mapsto 1 - h/h' \in m' \mathcal{O}_{U'} / m \mathcal{O}_U \cong (m'/m) \otimes \mathcal{O}_{U_s}.$$

To see that it is well-defined, note that any other choice of h is of the form $h(1+g)$ where $g \in m \mathcal{O}_U$. Then

$$(1 - h/h') - (1 - h(1+g)/h') = g/h' \in m \mathcal{O}_{U'} = m \mathcal{O}_U.$$

From (4.44.1) we get the exact sequence

$$H^0(U', \mathcal{O}_{U'}^\times) \xrightarrow{\tau} H^0(U_s, (m'/m) \otimes \mathcal{O}_{U_s}) \rightarrow \text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(x', X'). \quad (4.44.4)$$

We claim that

$$\text{coker}(\tau) = (m'/m) \otimes H^0(U_s, \mathcal{O}_{U_s}) / H^0(X_s, \mathcal{O}_{X_s}).$$

Indeed, let ϕ_0^* be any global section of $(m'/m) \otimes \mathcal{O}_{U_s}$ that extends to a global section ϕ_0 of $(m'/m) \otimes \mathcal{O}_{X_s}$. Then it lifts to a section ϕ of $m' \mathcal{O}_{X'}$. Now $h' := 1 - \phi$ and $h := 1$ show that $\tau(h') = \phi_0^*$.

Finally note that $\text{depth}_x X_s \geq 2$ iff $H^0(U_s, \mathcal{O}_{U_s}) = H^0(X_s, \mathcal{O}_{X_s})$. Otherwise their quotient is a $k(s)$ -vector space that is finite dimensional if $\dim X_s = 2$ but infinite dimensional if $\dim X_s = 1$. An extension of $k(s)$ -vector spaces is a $k(s)$ -vector space if $\text{char } k(s) = 0$ and a unipotent group in general; cf. [Bor91, §10]. \square

Next we deal with residue field extensions.

LEMMA 4.45. *Using the notation of (4.43), assume in addition that $m_{s',s'} = m_{s,s}$. Then $\pi^* : \text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(x', X')$ is an injection if $\text{depth}_x X_s \geq 2$ or if X_s is smooth.*

Proof. The exact sequence (4.44.4) becomes

$$1 \rightarrow \mathcal{O}_U^\times \rightarrow \mathcal{O}_{U'}^\times \xrightarrow{\tau} \mathcal{O}_{U_{s'}}^\times / \mathcal{O}_{U_s}^\times \rightarrow 0, \quad (4.45.1)$$

If $\text{depth}_x X_s \geq 2$ then $H^0(U_{s'}, \mathcal{O}_{U_{s'}}^\times) = H^0(X_{s'}, \mathcal{O}_{X_{s'}}^\times)$ and all these sections lift to $H^0(X, \mathcal{O}_X^\times)$.

If X_s is smooth and of dimension 1 then $H^0(U_{s'}, \mathcal{O}_{U_{s'}}^\times) \cong H^0(X_{s'}, \mathcal{O}_{X_{s'}}^\times) \times \mathbb{Z}$. and $H^0(U_s, \mathcal{O}_{U_s}^\times) \cong H^0(X_s, \mathcal{O}_{X_s}^\times) \times \mathbb{Z}$. The \mathbb{Z} -factors cancel out in the quotient $\mathcal{O}_{U_{s'}}^\times / \mathcal{O}_{U_s}^\times$ and the sections in $H^0(X_{s'}, \mathcal{O}_{X_{s'}}^\times)$ lift to $H^0(X, \mathcal{O}_X^\times)$ as before. \square

The next examples show that p -torsion does occur in (4.43) whenever the residue field has positive characteristic.

EXAMPLE 4.46. In (4.17) we studied the trivial family $X := (xy(x-y) = 0) \subset \mathbb{A}_{xyz}^3$ over the curve $C := (xy(x-y) = 0) \subset \mathbb{A}_{xy}^2$. We proved that for $c \neq 0$ the divisor $D_c := (x = z = 0) + (y = z = 0) + (x - y = z - cx = 0)$ is not Cartier, yet its pull-back to the normalization is Cartier.

Here we note that if $\text{char } k = p > 0$ then $z^p - c^p xy^{p-1} = 0$ shows that pD_c is a Cartier divisor.

EXAMPLE 4.47. In (4.18) we considered the cusp $C := (x^2 = y^3) \subset \mathbb{A}_{xy}^2$, the trivial curve family $Y := C \times \mathbb{A}_z^1 \rightarrow C$ and the Weil divisor $D^+ := \text{image of } t \mapsto (t^3, t^2, t)$.

We proved that D^+ is not \mathbb{Q} -Cartier in characteristic 0 but the equation

$$pD^+ = (xy^{(p-3)/2} = z^p)$$

shows that it is \mathbb{Q} -Cartier in characteristic $p > 0$.

4.5. Stability is representable II

Assumption. In this Section we work over a field of characteristic 0.

Let $f : (X, \Delta) \rightarrow S$ be a well-defined family of pairs as in (4.6). In (3.37) we gave 5 equivalent definitions locally stable families of varieties. Now we extend these to families of pairs. The main difference is that the natural analog of (3.37.1) is no longer equivalent to the others; see Section 2.6 for some cases when it is.

DEFINITION–THEOREM 4.48. Let S be a reduced scheme and $f : (X, \Delta) \rightarrow S$ a projective, well-defined family of pairs. Then $f : (X, \Delta) \rightarrow S$ is *locally stable* or *slc* if the following equivalent conditions hold.

- (4.48.1) $K_{X/S} + \Delta$ is \mathbb{R} -Cartier and (X_s, Δ_s) is slc for all points $s \in S$.
- (4.48.2) $K_{X/S} + \Delta$ is \mathbb{R} -Cartier and (X_s, Δ_s) is slc for all closed points $s \in S$.
- (4.48.3) $f_T : (X_T, \Delta_T) \rightarrow T$ is locally stable whenever T is the spectrum of a DVR and $q : T \rightarrow S$ is a morphism.

Such a family is called *stable* if, in addition, $K_{X/S} + \Delta$ is f -ample.

Proof. The arguments are essentially the same as in (3.37). It is clear that (1) \Rightarrow (2). The converse and (2) \Rightarrow (3) both follow from (2.4).

If (3) holds then all fibers are slc and $K_{X_T} + \Delta_T$ is \mathbb{R} -Cartier for every $q : T \rightarrow S$. Thus $K_{X/S} + \Delta$ is \mathbb{R} -Cartier by (4.37). \square

We can now state the main result of this section which can be thought of as a local variant of [HMX18]. Eventually we remove the reduced assumption in Chapter 8.

THEOREM 4.49. *Let $f : (X, \Delta) \rightarrow S$ be a projective, well-defined family of pairs. Then the functor of locally stable divisorial pull-backs is represented by a locally closed partial decomposition $i^{\text{lst}} : S^{\text{lst}} \rightarrow S$ for reduced schemes.*

As in (3.1), a proper morphism $f : (X, \Delta) \rightarrow S$ is called stable iff it is locally stable and the divisor $K_{X/S} + \Delta$ is \mathbb{R} -Cartier and f -ample. Since ampleness is an open condition for an \mathbb{R} -Cartier divisor (11.44.3), (4.49) implies the analogous result for stable morphisms.

COROLLARY 4.50. *Let $f : (X, \Delta) \rightarrow S$ be a projective, well-defined family of pairs. Then the functor of stable divisorial pull-backs is represented by a locally closed partial decomposition $i^{\text{stab}} : S^{\text{stab}} \rightarrow S$ for reduced schemes. \square*

We start the proof of (4.49), which will be completed in (4.53), with a weaker version.

LEMMA 4.51. *Let $f : (X, \Delta) \rightarrow S$ be a proper, well-defined family of pairs. Then there is a finite collection of locally closed subschemes $S_i \subset S$ such that*

- (4.51.1) $f_i : X_i := X \times_S S_i \rightarrow S_i$ is locally stable for every i ,
- (4.51.2) $K_{X_i/S_i} + \Delta_{S_i}$ is \mathbb{R} -Cartier and
- (4.51.3) a fiber (X_s, Δ_s) is slc iff $s \in \cup_i S_i$.

In particular, $\{s : (X_s, \Delta_s) \text{ is slc}\} \subset S$ is constructible.

Proof. Being demi-normal is an open condition by (10.41) and slc implies demi-normal by definition. Thus we may assume that all fibers are demi-normal and S is irreducible with generic point g . Throughout the proof we use $S^\circ \subset S$ to denote a dense open subset which we shrink whenever necessary.

First we treat morphisms whose generic fiber X_g is normal.

Case 1: (X_g, Δ_g) is lc. Then $K_{X_g} + \Delta_g$ is \mathbb{R} -Cartier, hence $K_{X/S} + \Delta$ is \mathbb{R} -Cartier over an open neighborhood of g . Next consider a log resolution $p_g : Y_g \rightarrow X_g$. It extends to a simultaneous log resolution $p^\circ : Y^\circ \rightarrow X^\circ$ over a suitable $S^\circ \subset S$. Thus, if $E^\circ \subset Y^\circ$ is any exceptional divisor, then $a(E_s, X_s, \Delta_s) = a(E^\circ, X^\circ, \Delta^\circ) = a(E_g, X_g, \Delta_g)$. This shows that all fibers over S° are lc.

Case 2: (X_g, Δ_g) is not lc. Note that the previous argument works if $K_{X_g} + \Delta_g$ is \mathbb{R} -Cartier. Indeed, then there is a divisor E with $a(E_g, X_g, \Delta_g) < -1$ and this shows that $a(E_s, X_s, \Delta_s) < -1$ for $s \in S^\circ$. However when $K_{X_g} + \Delta_g$ is not \mathbb{R} -Cartier then the discrepancy $a(E_g, X_g, \Delta_g)$ is not defined. We could try to prove that $K_{X_s} + \Delta_s$ is not \mathbb{R} -Cartier for $s \in S^\circ$ but this is not true in general; see (4.19).

Thus we use the notion of numerically Cartier divisors (4.55) instead. If $K_{X_g} + \Delta_g$ is not numerically Cartier then, by (4.58), $K_{X_s} + \Delta_s$ is also not numerically Cartier over an open subset $S^\circ \ni g$. Thus (X_s, Δ_s) is not lc for $s \in S^\circ$.

If $K_{X_g} + \Delta_g$ is numerically Cartier then the notion of discrepancy makes sense (4.55) and, again using (4.58), the above arguments show that if (X_g, Δ_g) is numerically lc (resp. not numerically lc) then the same holds for (X_s, Δ_s) for s in a suitable open subset $S^\circ \ni g$. We complete Case 2 by noting that being numerically lc is equivalent to being lc by (4.56).

An alternate approach to the previous case is the following. By (4.57) the log canonical modification (5.16) $\pi_g : (Y_g, \Theta_g) \rightarrow (X_g, \Delta_g)$ exists and it extends to a simultaneous log canonical modification $\pi : (Y, \Theta) \rightarrow (X, \Delta)$ over an open subset $S^\circ \subset S$. By the arguments of Case 1, (Y_s, Θ_s) is lc for $s \in S^\circ$ and the relative ampleness of the log canonical class is also an open condition. Thus $\pi_s : (Y_s, \Theta_s) \rightarrow (X_s, \Delta_s)$ is the log canonical modification for $s \in S^\circ$. By assumption π_g is not an isomorphism, so none of the π_s are isomorphisms. Therefore none of the fibers over S° are lc.

If X_g is not normal, the proofs mostly work the same using a simultaneous semi-log resolution [Kol13b, Sec.10.4]. However, for Case 2 it is more convenient to use the following argument.

Let $\pi_g : \bar{X}_g \rightarrow X_g$ denote the normalization. Over an open subset $S^\circ \ni g$ it extends to a simultaneous normalization $(\bar{X}, \bar{D} + \bar{\Delta}) \rightarrow S$. If $(\bar{X}_g, \bar{D}_g + \bar{\Delta}_g)$ is not lc then $(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s)$ is not lc for $s \in S^\circ$, hence (X_s, Δ_s) is not slc, essentially by definition; see [Kol13b, 5.10].

Using the already settled normal case, it remains to deal with the situation when $(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s)$ is lc for every $s \in S^\circ$. By [Kol13b, 5.38], (X_s, Δ_s) is slc iff $\text{Diff}_{\bar{D}_s} \bar{\Delta}_s$ is τ_s -invariant. The different can be computed on any log resolution as the intersection of the birational transform of \bar{D}_s with the discrepancy divisor. Thus $\text{Diff}_{\bar{D}_s} \bar{\Delta}_s$ is also locally constant over an open set S° . Therefore, if $\text{Diff}_{\bar{D}_g} \bar{\Delta}_g$ is not τ_g -invariant then $\text{Diff}_{\bar{D}_s} \bar{\Delta}_s$ is also not τ_s -invariant for $s \in S^\circ$. Hence (X_s, Δ_s) is not slc for every $s \in S^\circ$.

In both cases we complete the proof by Noetherian induction. \square

The following consequence of (4.51) is quite useful, though it could have been proved before it as in (3.38).

COROLLARY 4.52. *Let $f : (X, \Delta) \rightarrow S$ be a proper, well-defined family of pairs. Assume in addition that $K_{X/S} + \Delta$ is \mathbb{R} -Cartier. Then $\{s : (X_s, \Delta_s) \text{ is slc}\} \subset S$ is open.*

Proof. By (4.51) this set is constructible. A constructible set $U \subset S$ is open iff it is closed under generalization, that is, $x \in U$ and $x \in \bar{y}$ implies that $y \in U$. This follows from (2.4). \square

4.53 (Proof of (4.49)). Let $S_i \subset S$ be as in (4.51). By restriction we get slc families $f_i : (X_i, \Delta_i) \rightarrow S_i$. We apply (4.38) to the family $f : (X, K_{X/S} + \Delta) \rightarrow S$ to obtain $S^{\text{rcar}} \rightarrow S$ such that, for every reduced S -scheme $q : T \rightarrow S$, the pulled-back divisor $K_{X_T/T} + \Delta_T$ is \mathbb{R} -Cartier iff q factors as $q : T \rightarrow S^{\text{rcar}} \rightarrow S$.

Assume now that $f_T : (X_T, \Delta_T) \rightarrow T$ is slc. Then $K_{X_T/T} + \Delta_T$ is \mathbb{R} -Cartier, hence q factors through $S^{\text{rcar}} \rightarrow S$. As we observed in (3.23), this implies that $S^{\text{slc}} = (S^{\text{rcar}})^{\text{slc}}$. By definition $K_{X^{\text{rcar}}/S^{\text{rcar}}} + \Delta$ is \mathbb{R} -Cartier, thus (4.52) implies that $S^{\text{slc}} = (S^{\text{rcar}})^{\text{slc}}$ is an open subscheme of S^{rcar} . \square

We showed in (4.19) that being \mathbb{Q} -Cartier or \mathbb{R} -Cartier is not a constructible condition. The next result shows that the situation is better for boundary divisors of lc pairs.

COROLLARY 4.54. *Let $f : (X, \Delta) \rightarrow S$ be a proper, flat family of pairs with slc fibers. Let D be an effective divisor on X . Assume that*

(4.54.1) *either $\text{Supp } D \subset \text{Supp } \Delta$,*

(4.54.2) *or $\text{Supp } D$ does not contain any of the log canonical centers of any of the fibers (X_s, Δ_s) .*

Then $\{s : D_s \text{ is } \mathbb{R}\text{-Cartier}\} \subset S$ is constructible.

Proof. Choose $0 < \epsilon \ll 1$. In the first case $(X_s, \Delta_s - \epsilon D_s)$ is slc iff D_s is \mathbb{R} -Cartier. In the second case $(X_s, \Delta_s + \epsilon D_s)$ is slc iff D_s is \mathbb{R} -Cartier. Thus, in both cases, (4.51) implies our claim. \square

Numerically Cartier divisors.

DEFINITION 4.55. Let $g : Y \rightarrow S$ be a proper morphism. An \mathbb{R} -Cartier divisor D is called *numerically g -trivial* if $(C \cdot D) = 0$ for every curve $C \subset Y$ that is contracted by g .

Let X be a normal scheme and $p : Y \rightarrow X$ a resolution. An \mathbb{R} -divisor D on X is called *numerically \mathbb{R} -Cartier* if there is a p -exceptional \mathbb{R} -divisor E_D such that $E_D + p_*^{-1}D$ is numerically p -trivial. It follows from (11.50) that such an E_D is unique. (See (4.60) for a different variant of this definition.) If D is a \mathbb{Q} -divisor then E_D is also a \mathbb{Q} -divisor since its coefficients are solutions of a linear system of equations. Such a D is called *numerically \mathbb{Q} -Cartier*.

If $g : X \rightarrow S$ is proper then a numerically \mathbb{R} -Cartier divisor D is called *numerically g -trivial* if $E_D + p_*^{-1}D$ is numerically $g \circ p$ -trivial on Y .

Being numerically \mathbb{R} -Cartier is preserved by \mathbb{R} -linear equivalence. We can thus define when a linear equivalence class $|D|$ is numerically \mathbb{R} -Cartier, though the divisors E_D depend on $D \in |D|$. It is easy to see that these notions are independent of the resolution.

For $K_X + \Delta$ we can make a canonical choice. Thus we see that $K_X + \Delta$ is numerically \mathbb{R} -Cartier iff there is a p -exceptional \mathbb{R} -divisor $E_{K+\Delta}$ such that $E_{K+\Delta} + K_Y + p_*^{-1}\Delta$ is numerically p -trivial.

If $K_X + \Delta$ is numerically \mathbb{R} -Cartier then one can define the *discrepancy* of any divisor E over X by

$$a(E, X, \Delta) := a(E, Y, E_{K+\Delta} + p_*^{-1}\Delta).$$

We can thus define when a pair (X, Δ) is *numerically lc*. This concept was useful in the proof of (4.51). There are many divisors that are numerically \mathbb{R} -Cartier but not \mathbb{R} -Cartier, however, the next result says that the notion of numerically lc pairs does not give anything new.

THEOREM 4.56. [HX16, 1.6] *A numerically lc pair is lc.*

Outline of proof. This is surprisingly complicated, using many different ingredients. We start with the numerically \mathbb{Q} -Cartier case.

For clarity, let us concentrate on the very special case when (X, Δ) is dlt, except at a single point $x \in X$. All the key ideas appear in this case but we avoid a more technical inductive argument.

Let $f : (Y, E + \Delta_Y) \rightarrow (X, \Delta)$ be a \mathbb{Q} -factorial, dlt modification (as in [Kol13b, 1.34]) where E is the exceptional divisor dominating x and Δ_Y is the birational transform of Δ . Let $\Delta_E := \text{Diff}_E \Delta_Y$. Then (E, Δ_E) is a semi-dlt pair such that $K_E + \Delta_E$ is numerically trivial. Next we need a global version of the theorem.

Claim 4.56.1. Let (E, Δ_E) be semi-slc pair such that $K_E + \Delta_E$ is \mathbb{Q} -Cartier and numerically trivial. Then $K_E + \Delta_E \sim_{\mathbb{Q}} 0$.

The first general proof is in [Gon13], but special cases go back to [Kaw85, Fuj00]. We discuss a very special case: E is smooth and $\Delta = 0$. The following argument is from [CKP12, Kaw13].

We assume that $\mathcal{O}_E(K_E) \in \text{Pic}^\tau(E)$ but after passing to an étale cover of E we have that $\mathcal{O}_E(K_E) \in \text{Pic}^\circ(E)$. Note that $H^n(E, \mathcal{O}_E(K_E)) = 1$ where $n = \dim E$. Next we use a theorem of [Sim93] which says that the cohomology groups of line bundles in Pic° jump along torsion translates of subtori. However

$$H^n(E, L) = 1 \Leftrightarrow H^0(E, L^{-1}(K_E)) = 1 \Leftrightarrow L \cong \mathcal{O}_E(K_E).$$

Thus $\mathcal{O}_E(K_E)$ is a torsion element of $\text{Pic}^\circ(E)$. \square

It remains to lift information from the exceptional divisor E to the dlt model Y . To this end consider the exact sequence

$$0 \rightarrow \mathcal{O}_Y(m(K_Y + E + \Delta_Y) - E) \rightarrow \mathcal{O}_Y(m(K_Y + E + \Delta_Y)) \rightarrow \mathcal{O}_E(m(K_E + \Delta_E)) \rightarrow 0.$$

Note that $m(K_Y + E + \Delta_Y) - E - (K_Y + \Delta_Y) \sim_{\mathbb{Q},f} 0$, thus

$$R^1 f_* (\mathcal{O}_Y(m(K_Y + E + \Delta_Y) - E)) = 0$$

by [Kol13b, 10.38.1] (or the even stronger [Fuj14, 1.10]). Hence a nowhere zero global section of $\mathcal{O}_E(m(K_E + \Delta_E))$ lifts back to a global section of $\mathcal{O}_Y(m(K_Y + E + \Delta_Y))$ that is nowhere zero near E . Thus $\mathcal{O}_X(m(K_X + \Delta)) \cong f_* \mathcal{O}_Y(m(K_Y + E + \Delta_Y))$ is free in a neighborhood of x . Thus completes the numerically \mathbb{Q} -Cartier case.

The \mathbb{R} -Cartier case is reduced to the numerically \mathbb{Q} -Cartier setting using (11.38) as follows.

Let $f : (Y, \Delta_Y) \rightarrow (X, \Delta)$ be a log resolution. Pick curves C_m that span $N_1(Y/X)$ and apply (11.38) to (Y, Δ_Y) . Thus for $n \gg 1$ we get $K_Y + \Delta_Y = \sum_j \lambda_j (K_Y + \Delta_Y^j)$ where the Δ_Y^j are \mathbb{Q} -divisors and (Y, Δ_Y^j) is lc. Also, since $(C_m \cdot (K_Y + \Delta_Y)) = 0$, (11.38.6.a) shows that $(C_m \cdot (K_Y + \Delta_Y^j)) = 0$. Thus each $(X, f(\Delta_Y^j))$ is a numerically \mathbb{Q} -Cartier lc pair. They are thus lc, and so is (X, Δ) by (11.3.4). \square

The following was used to give an alternate proof of one of the steps in (4.51). C. Xu pointed out that it can be proved using the arguments of [OX12].

THEOREM 4.57. *Let X be a normal variety and Δ a boundary such that $K_X + \Delta$ is numerically \mathbb{R} -Cartier. Then (X, Δ) has a log canonical modification (5.15). \square*

The advantage of the concept of numerically \mathbb{R} -Cartier divisors is that we have better behavior in families.

PROPOSITION 4.58. *Let $f : X \rightarrow S$ be a proper morphism with normal fibers over a field of characteristic 0 and D a generically Cartier family of divisors on X . Then*

$$\{s \in S : D_s \text{ is numerically } \mathbb{R}\text{-Cartier}\}$$

is a constructible subset of S .

Proof. Let $g \in S$ be a generic point. We show that if D_g is numerically \mathbb{R} -Cartier (resp. not numerically \mathbb{R} -Cartier) then the same holds for all D_s in an open neighborhood $s \in S^\circ \subset S$. Then we finish by Noetherian induction.

To see our claim, consider a log resolution $p_g : Y_g \rightarrow X_g$. It extends to a simultaneous log resolution $p^\circ : Y^\circ \rightarrow X^\circ$ over a suitable open neighborhood $g \in S^\circ \subset S$.

If D_g is numerically \mathbb{R} -Cartier then there is a p_g -exceptional \mathbb{R} -divisor E_g such that $E_g + (p_g)_*^{-1} D_g$ is numerically p_g -trivial. This E_g extends to a p -exceptional \mathbb{R} -divisor E and $E + p_*^{-1} D$ is numerically p -trivial over an open neighborhood $g \in S^\circ \subset S$ by (4.59). Thus D_s is numerically \mathbb{R} -Cartier for $s \in S^\circ$.

Assume next that D_g is not numerically \mathbb{R} -Cartier. Let E_g^i be the p -exceptional divisors. Then there are proper curves $C_g^j \subset Y_g$ that are contracted by p_g and such that $(p_g)_*^{-1} D_g$, viewed as a linear function on $\oplus_j \mathbb{R}[C_g^j]$, is linearly independent of

the E_g^i . Both the divisors E_g^i and the curves C_g^j extend to give divisors E_s^i and the curves C_s^j over an open neighborhood $g \in S^\circ \subset S$. Thus $(p_s)_*^{-1}D_s$, viewed as a linear function on $\oplus_j \mathbb{R}[C_s^j]$, is linearly independent of the E_s^i , hence D_s is not numerically \mathbb{R} -Cartier for $s \in S^\circ$. \square

LEMMA 4.59. *Let $p : Y \rightarrow X$ be a morphism of proper S -schemes and D an \mathbb{R} -Cartier divisor on Y . Then*

$$S^{\text{nt}} := \{s \in S : D_s \text{ is numerically } p_s\text{-trivial}\}$$

is an open subset of S .

Proof. We check Nagata's openness criterion (10.5)

Let us start with the special case when $X = S$. Pick points $s_1 \in \overline{s_2} \subset S$. A curve $C_2 \subset Y_{s_2}$ specializes to $C_1 \subset Y_{s_1}$ and if $(D_{s_1} \cdot C_1) = 0$ then $(D_{s_2} \cdot C_2) = 0$.

Next assume that D_{s_2} is numerically p_{s_2} -trivial. By (11.34.2), $D_{s_2} = \sum a_i A_{s_2}^i$ where the $A_{s_2}^i$ are numerically p_{s_2} -trivial \mathbb{Q} -divisors. Thus each $m A_{s_2}^i$ is algebraically equivalent to 0 for some $m > 0$; see [Laz04, I.4.38]. We can spread out this algebraic equivalence to obtain that there is an open subset $U \subset \overline{s_2}$ such that $m D_s$ is algebraically (and hence numerically) equivalent to 0 on all fibers $s \in U$.

Applying this to $Y \rightarrow X$ shows that

$$X^{\text{nt}} := \{x \in X : D_x \text{ is numerically trivial on } Y_x\}$$

is an open subset of X . Thus $S^{\text{nt}} = S \setminus \pi_X(X \setminus X^{\text{nt}})$ is an open subset of S , where $\pi_X : X \rightarrow S$ is the structure map. \square

REMARK 4.60. On a normal surface every \mathbb{R} -divisor is numerically \mathbb{R} -Cartier. This observation is used in (11.47) to define intersection numbers of divisors on normal surfaces, but in higher dimensions one needs a different version of numerically \mathbb{R} -Cartier in order to define intersection numbers with curves.

Let X be a proper and normal variety. Let us say that a divisor D on X is *strongly numerically \mathbb{R} -Cartier* if there is a p -exceptional \mathbb{R} -divisor E_D such that $E_D + p_*^{-1}D$ is *strongly numerically p -trivial*, that is, $(Z \cdot (E_D + p_*^{-1}D)) = 0$ for every (not necessarily effective) 1-cycle Z on Y such that $p_*[Z] = 0$.

For example, let $E \subset \mathbb{P}^2$ be a smooth cubic and $S \subset \mathbb{P}^3$ the cone over it. For $p \in E$ let $L_p \subset S$ denote the line over p . Set $X := S \times E$ and consider the divisors D_1 , swept out by the lines $L_{p_0} \times \{p\}$ for some fixed $p_0 \in E$, and D_2 , swept out by the lines $L_p \times \{p\}$ for $p \in E$. Then $D_1 - D_2$ is numerically Cartier but not strongly numerically Cartier. To see the latter, compute that if F is the exceptional divisor obtained by blowing up the singular locus then $F \cdot (D_1' - D_2')^2 = -2$ where D_i' denotes the birational transform of D_i .

One can define the intersection of a 1-cycle $W \subset X$ with a strongly numerically \mathbb{R} -Cartier divisor D by the formula

$$(D \cdot W) = p_*(W_Y \cdot (E_D + p_*^{-1}D)),$$

where $W_Y \subset Y$ is any 1-cycle such that $p_*(W_Y) = W$. (We can set $W_Y = p_*^{-1}W$ if the latter is defined but there is always a 1-cycle W_Y such that $p_*(W_Y) = dW$ for some $d > 0$.)

If D is \mathbb{R} -Cartier outside a finite set of points then D is strongly numerically \mathbb{R} -Cartier iff it is numerically \mathbb{R} -Cartier, and this case can be understood in terms of the local Picard groups of X as follows.

Assume that $\dim X \geq 3$ and D is Cartier except at a point $x \in X$. There is a local Picard scheme $\mathbf{Pic}^{\text{loc}}(x, X)$ which is an extension of a finitely generated local Néron-Severi group with a connected algebraic group $\mathbf{Pic}^{\text{loc-}o}(x, X)$; see [Bou78] or [Kol16a] for details. Then D is numerically \mathbb{R} -Cartier iff $[D] \in \mathbf{Pic}^{\text{loc-}\tau}(x, X)$ where $\mathbf{Pic}^{\text{loc-}\tau}(x, X)/\mathbf{Pic}^{\text{loc-}o}(x, X)$ is the torsion subgroup of the local Néron-Severi group.

The local Picard scheme also exists in positive characteristic, thus one can turn the above equivalence into a definition of numerically \mathbb{R} -Cartier divisors in positive characteristic. However, it is not clear how to prove various theorems, including (4.58), using this definition.

Over \mathbb{C} one can also consider those line bundles on the smooth locus of X that extend to a topological line bundle on X . It is not clear how this notion compares with the above algebraic ones.

4.61 (Comments on positive characteristic). There are numerous problems with the arguments of this section in positive characteristic.

To start with, the 3 versions of the basic definition (4.48) are not known to be equivalent. Clearly (4.48.1) implies the other two; we can adopt it as the definition in general.

The discussion in (10.40) has gaps in characteristic 2, but these can be fixed.

Case 1 of the proof of (4.51) uses generic smoothness. For surfaces the structure of slc singularities in any characteristic is worked out in [Kol13b, Sec.3.3], and this can be used instead of generic smoothness. Probably one can do something similar in higher dimensions as well. We also use many properties of the different that are not known in positive characteristic.

The discussion on numerically \mathbb{R} -Cartier divisors uses resolution of singularities. A treatment without resolution would be desirable.

Finally (4.56) is also not known in positive characteristic; see however [CT20].

4.6. Stable families over smooth base schemes

All the results of the previous sections apply to families $p : (X, \Delta) \rightarrow S$ over a smooth base scheme, but the smooth case has other interesting features as well. One can then obtain results about families over other base schemes by working over a resolution of singularities of the base. The following can be viewed as a direct generalization of (2.4).

THEOREM 4.62. *Let $(0 \in S)$ be a smooth, local scheme and $D_1 + \cdots + D_r \subset S$ an snc divisor such that $D_1 \cap \cdots \cap D_r = \{0\}$. Let $p : (X, \Delta) \rightarrow (0 \in S)$ be a pure dimensional morphism and Δ a \mathbb{R} -divisor on X such that $\text{Supp } \Delta \cap \text{Sing } X_0$ has codimension ≥ 2 in X_0 . The following are equivalent.*

- (4.62.1) $p : (X, \Delta) \rightarrow S$ is slc.
- (4.62.2) $K_{X/S} + \Delta$ is \mathbb{R} -Cartier, p is flat and (X_0, Δ_0) is slc.
- (4.62.3) $K_{X/S} + \Delta$ is \mathbb{R} -Cartier, X is S_2 and $(\text{pure}(X_0), \Delta_0)$ is slc.
- (4.62.4) $(X, \Delta + p^*D_1 + \cdots + p^*D_r)$ is slc.

Proof. (1) \Rightarrow (2) holds by definition and (2) \Rightarrow (3) since both S and X_0 are S_2 (9.6). If (3) holds then (10.63) shows that p is flat and X_0 is pure, hence (3) \Rightarrow (2). Next we show that (2) \Leftrightarrow (4) using induction on r . Both implications are trivial if $r = 0$.

Assume (4). Then $K_X + \Delta + p^*D_1 + \cdots + p^*D_r$ is \mathbb{R} -Cartier at x hence so is $K_X + \Delta$. Set $D_Y := p^*D_r$. Adjunction (11.20) shows that

$$(D_Y, \Delta|_{D_Y} + p^*D_1|_{D_Y} + \cdots + p^*D_{r-1}|_{D_Y})$$

is slc at x , hence (X_0, Δ_0) is slc at x by induction. The local equations of the p^*D_i form a regular sequence at x by (4.67), hence p is flat at x .

Conversely, assume that (2) holds. By induction

$$(D_Y, \Delta|_{D_Y} + p^*D_1|_{D_Y} + \cdots + p^*D_{r-1}|_{D_Y})$$

is slc at x hence inversion of adjunction (11.20) shows that $(X, \Delta + p^*D_1 + \cdots + p^*D_r)$ is slc at x . \square

COROLLARY 4.63. *Let S be a smooth scheme and $p : (X, \Delta) \rightarrow S$ a morphism. Then $p : (X, \Delta) \rightarrow S$ is locally stable iff the pair $(X, \Delta + p^*D)$ is slc for every snc divisor $D \subset S$. \square*

COROLLARY 4.64. *Let S be a smooth, irreducible scheme and $p : (X, \Delta) \rightarrow S$ a locally stable morphism. Then every log center of (X, Δ) dominates S .*

Proof. Let E be a divisor over X such that $a(E, X, \Delta) < 0$ and let $Z \subset S$ denote the image of E in S . If $Z \neq S$ then, possibly after replacing S by an open subset, we may assume that Z is contained in a smooth divisor $D \subset S$. Thus $(X, \Delta + p^*D)$ is slc by (4.63). However, $a(E, X, \Delta + p^*D) \leq a(E, X, \Delta) - 1 < -1$, a contradiction. \square

COROLLARY 4.65. *Let S be a smooth scheme and $p : (X, \Delta) \rightarrow S$ a projective, locally stable morphism. Let $p^w : (X^w, \Delta^w) \rightarrow S$ denote a weak canonical model and $p^c : (X^c, \Delta^c) \rightarrow S$ the canonical model of $p : (X, \Delta) \rightarrow S$ (cf. [KM98, 3.50] or [Kol13b, 1.19]). Then*

(4.65.1) $p^w : (X^w, \Delta^w) \rightarrow S$ is locally stable and

(4.65.2) $p^c : (X^c, \Delta^c) \rightarrow S$ is stable.

Proof. Let $D \subset S$ be an snc divisor. By (4.63) $(X, \Delta + p^*D)$ is lc and $p^w : (X^w, \Delta^w + (p^*D)^w) \rightarrow S$ is also a weak canonical model over S by [Kol13b, 1.28]. Thus $(X^w, \Delta^w + (p^*D)^w)$ is also slc. Next we claim that $(p^*D)^w = (p^w)^*D$. This is clear away from the exceptional set of ϕ^{-1} which has codimension ≥ 2 in X^w . Thus $(p^*D)^w$ and $(p^w)^*D$ are 2 divisors that agree outside a codimension ≥ 2 subset, hence they agree. Now we can use (4.63) again to conclude that $p^w : (X^w, \Delta^w) \rightarrow S$ is locally stable.

A weak canonical model is a canonical model iff $K_{X^w/S} + \Delta^w$ is p^w -ample and the latter is also what makes a locally stable morphism stable. \square

4.66. The above proof of (4.65.2) is short but it does not illuminate the structure of $p^c : (X^c, \Delta^c) \rightarrow S$. In particular, it does not answer either of the following.

Question 4.66.1. Using the notation of (4.65), assume that $p : (X, \Delta) \rightarrow S$ is trivial over a closed subset $Z \subset S$. Is $p^c : (X^c, \Delta^c) \rightarrow S$ also trivial over Z ?

Question 4.66.2. Let S be a normal scheme and $p : (X, \Delta) \rightarrow S$ a projective, locally stable morphism. Is there a stable morphism $p^* : (X^*, \Delta^*) \rightarrow S$ that agrees with the canonical model constructed in (4.65) over the smooth locus $S^{\text{sm}} \subset S$?

LEMMA 4.67. *Let $(y \in Y, \Delta + D_1 + \cdots + D_r)$ be slc. Assume that the D_i are Cartier divisors with local equations $(s_i = 0)$. Then the s_i form a regular sequence.*

Proof. We use induction on r . Since Y is S_2 , s_r is a non-zerodivisor at y . By adjunction $(y \in D_r, \Delta|_{D_r} + D_1|_{D_r} + \cdots + D_{r-1}|_{D_r})$ is slc, hence the restrictions $s_1|_{D_r}, \dots, s_{r-1}|_{D_r}$ form a regular sequence at x . Thus s_1, \dots, s_r is a regular sequence at y . \square

The following result of [Kar00] is a generalization of (2.49) from 1-dimensional to higher dimensional bases. As we see in (4.69), it implies that every irreducible component of the moduli space of stable pairs is proper, at least in characteristic 0.

THEOREM 4.68. *Let U be a k -variety and $f_U : (X_U, \Delta_U) \rightarrow U$ a stable morphism. Then there is projective, generically finite, dominant morphism $\pi : V \rightarrow U$ and a compactification $V \hookrightarrow \bar{V}$ such that the pull-back $(X_U, \Delta_U) \times_U V$ extends to a stable morphism $f_{\bar{V}} : (X_{\bar{V}}, \Delta_{\bar{V}}) \rightarrow \bar{V}$.*

Proof. We may assume that U is irreducible with generic point g .

Assume first that the generic fiber of f_U is normal and geometrically irreducible. Let $(Y_g, \Delta_g^Y) \rightarrow (X_g, \Delta_g)$ be a log resolution. It extends to a simultaneous log resolution $(Y_{U_0}, \Delta_{U_0}^Y) \rightarrow (X_{U_0}, \Delta_{U_0})$ over an open subset $U_0 \subset U$. By (4.70.2) there is a projective, generically finite, dominant morphism $\pi : V_0 \rightarrow U_0$ and a compactification $V_0 \hookrightarrow \bar{V}$ such that the pull-back $(Y_{U_0}, \Delta_{U_0}^Y) \times_{U_0} V_0$ extends to a locally stable morphism $g_{\bar{V}} : (Y_{\bar{V}}, \Delta_{\bar{V}}^Y) \rightarrow \bar{V}$.

We can harmlessly replace \bar{V} by a resolution of it. Thus we may assume that \bar{V} is smooth and there is an open subset $V \subset \bar{V}$ such that the rational map $\bar{\pi}|_V : V \dashrightarrow U$ is a proper morphism.

Since $g_{\bar{V}}$ is a projective, locally stable morphism, the relative canonical model $f_{\bar{V}} : (X_{\bar{V}}, \Delta_{\bar{V}}) \rightarrow \bar{V}$ of $g_{\bar{V}} : (Y_{\bar{V}}, \Delta_{\bar{V}}^Y) \rightarrow \bar{V}$ exists by [HX13] and it is stable by (4.65.2).

By construction $(X_{\bar{V}}, \Delta_{\bar{V}})$ and $(X_U, \Delta_U) \times_U V$ are isomorphic over $V_0 \subset V$, but (2.47) implies that in fact they are isomorphic over V . This completes the case when the generic fiber of f_U is normal.

In general, we can first pull back everything to the Stein factorization of $X^n \rightarrow U$ where X^n is the normalization of X . The previous step now gives $f_{\bar{V}} : (X_{\bar{V}}^n, \Delta_{\bar{V}}^n) \rightarrow \bar{V}$. Finally (4.64) shows that (2.54) applies and we get $f_{\bar{V}} : (X_{\bar{V}}, \Delta_{\bar{V}}) \rightarrow \bar{V}$. \square

COROLLARY 4.69. *Let k be a field of characteristic 0 and assume that the coarse moduli space of stable pairs SP exists, is separated and locally of finite type.*

Then every irreducible component of SP is proper over k .

Proof. Let M be an irreducible component of SP with generic point g_M . By assumption there is a field extension $K \supset k(g_M)$ and a stable K -variety (X_K, Δ_K) corresponding to g_M .

Since it takes only finitely many equations to define a stable pair, we may also assume that $K/k(g_M)$ is finitely generated, hence K/k is also finitely generated.

By (4.68) there is a smooth, projective k -variety \bar{Y} and a stable family $\bar{f} : (\bar{Y}, \bar{\Delta}_{\bar{Y}}) \rightarrow \bar{V}$ such that $k(\bar{Y})$ is a finite field extension of K and the generic fiber of \bar{f} is isomorphic to $(X_K, \Delta_K)_{k(\bar{Y})}$.

The image of the corresponding moduli morphism $\phi : \bar{Y} \rightarrow \text{SP}$ contains g_M and it is proper. It is thus the closure of g_M , which is M . So M is proper. \square

4.70 (Weakly semistable reduction). A higher dimensional generalization of the Semistable reduction theorem of [KKMSD73] (see also (2.59)) is proved in [AK00]. The general question is the following.

Problem 4.70.1. Let $f : X \rightarrow S$ be a morphism. Find a proper, dominant, generically finite morphism $S' \rightarrow S$ and a proper morphism $f' : X' \rightarrow S'$ such that X' is birational to the main component of $X \times_S S'$, and the local structure of f' is as ‘nice’ as possible.

If $\dim S = 1$ then, as shown by (2.59), we can achieve that f' is flat and its fibers are reduced snc divisors, but a similar expectation would be overly optimistic if $\dim S > 1$ [Kar00].

The methods of [KKMSD73] are toric, and this suggests to look for a generalization where f' is toric. However, if S is not unirational then one certainly can not take S' to be toric, so the best one can hope in this direction is that f' is toroidal, where a map $p : U \rightarrow V$ is *toroidal* if for each point $u \in U$ with image $v := p(u)$, the map of formal completions $\hat{p} : \hat{U}_u \rightarrow \hat{V}_v$ is isomorphic to the formal completion of a toric morphism. The following is proved in [AK00, Thm.0.3 and Lem.6.1]; the last claim follows from (4.70.3).

Theorem 4.70.2. Let S be a scheme of finite type over a field of characteristic 0 and $f : X \rightarrow S$ a proper morphism. Then there is a proper, dominant, generically finite morphism $S' \rightarrow S$ and a proper morphism $f' : X' \rightarrow S'$ such that

- (a) X' is birational to the main components of $X \times_S S'$,
- (b) f' is flat with reduced fibers,
- (c) f' is toroidal,
- (d) S' is smooth, X' is canonical, $K_{X'}$ is Cartier and
- (e) f' is locally stable. □

Note on terminology. Such morphisms are called weakly semistable in [AK00]; this is a much stronger condition than being locally stable. The terminology of [AK00] does not match ours.

Example 4.70.3.(Log centers on toric varieties) (See [Ful93] or [Oda88] for introductions to toric varieties.)

Let X be a normal, toric variety and D the sum of the torus-invariant Weil divisors. Then $K_X + D \sim 0$ and (X, D) is lc. Furthermore the log centers are exactly the torus-invariant irreducible subvarieties.

To prove these, we may assume that the base field is algebraically closed. Thus $X \setminus D \cong \mathbb{G}_m^n$ and

$$\sigma_X := \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$$

is a torus invariant n -form with simple poles along D . This shows that $K_X + D \sim 0$.

Next let $p : (Y, D_Y) \rightarrow (X, D)$ be a toric resolution of (X, D) . Since $K_Y + D_Y \sim 0 \sim p^*(K_X + D)$, we conclude that all exceptional divisors have discrepancy -1 . Thus (X, D) is lc and the log centers are exactly the images of the exceptional divisors. The latter are the torus-invariant irreducible subvarieties.

4.7. Mumford divisors

On a normal variety, our basic objects are Weil divisors. On a non-normal variety, we work with Weil divisors that are not contained in the singular locus.

It has been long understood that these give the correct theory of generalized Jacobians [Ser59]. Their first appearance in the moduli theory of curves may be Mumford's definition of pointed stable curves given in [Knu83, Def.1.1]. The definition of Mumford divisors—given in (4.1.4) and (4.20)—codifies long understood basic properties of the divisorial part of families of stable pairs, though with a new name introduced in [Kol18c].

Here we consider the a relative version that is compatible with Cayley-Chow forms in a very strong way (4.79). This enables us to construct a universal family of Mumford divisors (4.86), which is a key step in the construction of the moduli space of stable pairs.

We start by recalling the foundational properties of Chow varieties, as treated in [Kol96, Secs.I.3–4], and then discuss the ideal of Chow equations. Many of the basic ideas are taken from [HP47, Chap.X], which is the standard classical treatment.

We focus on the classical theory, which is over fields, but this approach works out for seminormal base schemes in characteristic 0; see [Kol96, Secs.I.3–4]. A closer inspection reveals that the theory works well over arbitrary bases for geometrically reduced cycles, but it is not clear how to work with non-reduced cycles in general, but the methods work very well for Mumford divisors over arbitrary bases.

The end result is that the functor of Mumford divisors is representable over reduced bases (4.79). We prove a temporary version in the scheme theoretic setting (4.86); we will give its final form in (7.3). These are key ingredients of the main theorems of Chapter 8.

DEFINITION 4.71. A d -cycle on a scheme X is a finite linear combination $Z := \sum_i m_i [V_i]$, where $m_i \in \mathbb{Z}$ and the V_i are d -dimensional irreducible, reduced subschemes. We usually tacitly assume that the V_i are distinct and $m_i \neq 0$. Then the V_i are called the *irreducible components* of Z and the m_i the *multiplicities*. A d -cycle is called *effective* if $m_i \geq 0$ for every i and *reduced* if all its multiplicities equal 1.

If $W \subset X$ is a subscheme of dimension $\leq d$ then we can associate to it a d -cycle

$$[W] := \sum_i (\text{length}_{w_i} \mathcal{O}_W) \cdot [W_i], \quad (4.71.1)$$

where $W_i \subset W$ are the d -dimensional irreducible components with generic points $w_i \in W_i$. If W is reduced and pure dimensional then $[W]$ determines W ; we will not always distinguish them clearly. However, if W is nonreduced, then it carries much more information than $[W]$. The only exception is when W is a Mumford divisor.

If X is projective and L is an ample line bundle on X , then the *degree* of a d -cycle $Z = \sum_i m_i [V_i]$ is defined as $\deg_L Z := \sum_i m_i \deg_L V_i = \sum_i m_i (L^d \cdot V_i)$.

Assume that X is a scheme over a field k and K/k a field extension. If $V \subset X$ is a d -dimensional irreducible, reduced subvariety then $V_K \subset X_K$ is a d -dimensional subscheme which may be reducible and, if $\text{char } k > 0$, may be non-reduced. If $Z = \sum m_i V_i$ is a d -cycle, we set

$$Z_K := \sum m_i [(V_i)_K]. \quad (4.71.2)$$

Z is called *geometrically reduced* if $Z_{\bar{k}}$ is reduced. If $\text{char } k = 0$ then reduced is the same as geometrically reduced.

Assume that X is projective and choose an embedding $X \hookrightarrow \mathbb{P}^n$. Then every d -cycle on X is also a d -cycle on \mathbb{P}^n . Cayley-Chow theory focuses primarily on cycles in \mathbb{P}^n

4.72 (Cayley-Chow correspondence over fields I). Fix a projective space \mathbb{P}^n over a field k with dual projective space $\check{\mathbb{P}}^n$. Points in $\check{\mathbb{P}}^n$ are hyperplanes in \mathbb{P}^n .

For $d \leq n - 1$ we have an incidence correspondence

$$\mathbf{I}^{(n,d)} := \{(p, H_0, \dots, H_d) : p \in H_0 \cap \dots \cap H_d\} \subset \mathbb{P}^n \times (\check{\mathbb{P}}^n)^{d+1}, \quad (4.72.1)$$

which comes with the coordinate projections

$$\mathbb{P}^n \xleftarrow{\pi_1} \mathbf{I}^{(n,d)} \xrightarrow{\pi_2} (\check{\mathbb{P}}^n)^{d+1} \xrightarrow{\sigma_i} (\check{\mathbb{P}}^n)^d, \quad (4.72.2)$$

where π_1 is a $(\check{\mathbb{P}}^{n-1})^{d+1}$ -bundle and σ_i deletes the i th component. The projection π_2 is a \mathbb{P}^{n-d-1} -bundle over a dense open subset. For a closed subscheme $Y \subset \mathbb{P}^n$ set $\mathbf{I}_Y^{(n,d)} := \pi_1^{-1}(Y)$.

Let $Z \subset \mathbb{P}^n$ be an irreducible, geometrically reduced, closed subvariety of dimension d . Its *Cayley-Chow hypersurface* is defined as

$$\begin{aligned} \text{Ch}(Z) &:= \pi_2(\mathbf{I}_Z^{(n,d)}) \\ &= \{(H_0, \dots, H_d) \in (\check{\mathbb{P}}^n)^{d+1} : Z \cap H_0 \cap \dots \cap H_d \neq \emptyset\}. \end{aligned} \quad (4.72.3)$$

An equation of $\text{Ch}(Z)$ is called a *Cayley-Chow form*. Next note that

$$\mathbf{I}_Z^{(n,d)} \cap \pi_2^{-1}(H_0, \dots, H_d) = Z \cap H_0 \cap \dots \cap H_d. \quad (4.72.4)$$

In particular, a general $H_0 \cap \dots \cap H_d$ is disjoint from Z and a general $H_0 \cap \dots \cap H_d$ containing a smooth point $p \in Z$ meets Z only at p (scheme theoretically). Thus we see the following.

Claim 4.72.5. Let Z be a geometrically reduced d -cycle. Then $\pi_2 : \mathbf{I}_Z^{(n,d)} \rightarrow \text{Ch}(Z)$ is birational and $\text{Ch}(Z)$ is a hypersurface in $(\check{\mathbb{P}}^n)^{d+1}$. \square

For any H_0, \dots, H_{d-1} the fiber of the coordinate projection $\sigma_d : \text{Ch}(Z) \rightarrow (\check{\mathbb{P}}^n)^d$ is $\check{\mathbb{P}}^n$ if $\dim(Z \cap H_0 \cap \dots \cap H_{d-1}) \geq 1$; otherwise it is the set of hyperplanes that contain one of the points of $Z \cap H_0 \cap \dots \cap H_{d-1}$. Similarly for all the other σ_i . Thus we proved the following.

Claim 4.72.6. Let Z be a geometrically reduced d -cycle of degree r . Then a general geometric fiber of any of the projections $\sigma_i : \text{Ch}(Z) \rightarrow (\check{\mathbb{P}}^n)^d$ is the union of r distinct hyperplanes in $\check{\mathbb{P}}^n$. In particular, the projections are generically smooth and $\text{Ch}(Z)$ has multidegree (r, \dots, r) . \square

For $p \in \mathbb{P}^n$ let \check{p} denote the set of hyperplanes passing through p . Then $p \in Z$ iff $\check{p} \times \dots \times \check{p} \subset \text{Ch}(Z)$. This leads us to the definition of the inverse of the map $Z \mapsto \text{Ch}(Z)$. Let $D \subset (\check{\mathbb{P}}^n)^{d+1}$ be a geometrically reduced subscheme. (In practice, D will always be a hypersurface.) Define

$$\text{Ch}_{\text{set}}^{-1}(D) := \{p : \check{p} \times \dots \times \check{p} \subset D\} \subset \mathbb{P}^n. \quad (4.72.7)$$

For now we view $\text{Ch}_{\text{set}}^{-1}(D)$ as a reduced subscheme; scheme theoretic versions will be discussed in (4.81).

It is easy to see that $\dim \text{Ch}_{\text{set}}^{-1}(D) \leq d$, and an irreducible hypersurface D is of *Cayley-Chow type* of Z if $\dim \text{Ch}_{\text{set}}^{-1}(D) = d$. An arbitrary hypersurface D is of *Cayley-Chow type* if all of its irreducible components are. The basic correspondence of Cayley-Chow theory is the following, see [Kol96, I.3.24.5].

Claim 4.72.8. Fix n, d, r and a base field k . Then the maps Ch and $\text{Ch}_{\text{set}}^{-1}$ provide a one-to-one correspondence between

$$\left\{ \begin{array}{l} \text{geometrically reduced} \\ d\text{-cycles of degree } r \text{ in } \mathbb{P}^n \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \text{geometrically irreducible} \\ \text{Cayley-Chow type hypersurfaces of} \\ \text{degree } (r, \dots, r) \text{ in } (\check{\mathbb{P}}^n)^{d+1} \end{array} \right\}.$$

Proof. We already saw the \Rightarrow part. To see the converse, observe the inclusion $\text{Ch}(\text{Ch}_{\text{set}}^{-1}(D)) \subset D$. Thus if $Z \subset \text{Ch}_{\text{set}}^{-1}(D)$ is any subvariety of dimension d then $\text{Ch}(Z) \subset D$, hence $\text{Ch}(Z)$ is an irreducible component of D . Thus $D = \text{Ch}(\text{Ch}_{\text{set}}^{-1}(D))$. \square

Ideal of Chow equations.

Let $Z \subset \mathbb{P}^n$ be a pure dimensional subscheme or a cycle. The Chow equations are the ‘most obvious’ equations of Z . They generate a homogeneous ideal (or an idea sheaf) which was studied in various forms in [Cat92, DS95, Kol99].

Its relationship with the scheme-theoretic $\text{Ch}_{\text{sch}}^{-1}$ will be given in (4.83).

4.73 (Element-wise powers of ideals). Let R be a ring, $I \subset R$ an ideal and $m \in \mathbb{N}$. Set

$$I^{[m]} := (r^m : r \in I).$$

These ideals have been studied mostly when $\text{char } k = p > 0$ and q is a power of p ; one of the early occurrences is in [Kun76]. In these cases $I^{[q]}$ is called a *Frobenius power* of I . Other values of the exponent are also interesting, the following properties follow from (4.73.4). We assume for simplicity that R is a k -algebra.

(4.73.1) If I is principal then $I^{[m]} = I^m$.

(4.73.2) If $\text{char } k = 0$ then $I^{[m]} = I^m$.

(4.73.3) If $m < \text{char } k$ then $I^{[m]} = I^m$.

(4.73.4) If k is infinite then $(r_1, \dots, r_n)^{[m]} = ((\sum c_i r_i)^m : c_i \in k)$.

Note that (2) is close to being optimal. For example, if $I = (x, y) \subset k[x, y]$ and $\text{char } k = p \geq 3$ then

$$(x, y)^{[p+1]} = (x^{p+1}, x^p y, x y^p, y^{p+1}) \subsetneq (x, y)^{p+1}.$$

Claim 4.73.5. Let k be an infinite field. Then

$$\langle (c_1 x_1 + \dots + c_n x_n)^m : c_i \in k \rangle = \langle x_1^{i_1} \dots x_n^{i_n} : \binom{m}{i_1 \dots i_n} \neq 0 \rangle.$$

Here $\binom{m}{i_1 \dots i_n}$ denotes the multinomial coefficient, that is, the coefficient of $x_1^{i_1} \dots x_n^{i_n}$ in $(x_1 + \dots + x_n)^m$.

Proof. The containment \subset is clear. If the 2 sides are not equal then the left hand side is contained in some hyperplane of the form $\sum \lambda_I x^I = 0$, but this would give a nontrivial polynomial identity $\sum \binom{m}{i_1 \dots i_n} \lambda_I c^I = 0$ for the c_i . \square

4.74 (Ideal of Chow equations). Let Z be a d -cycle of degree r in \mathbb{P}^n . Let $\rho : \mathbb{P}^n \dashrightarrow \mathbb{P}^{d+1}$ be a projection that is defined along Z . Then $\rho_*(Z)$ is a d -cycle in \mathbb{P}^{d+1} , thus it can be identified with a hypersurface; hence with a homogeneous polynomial $\phi(Z, \rho)$ of degree r . Its pull-back to \mathbb{P}^n is a homogeneous polynomial $\Phi(Z, \rho)$ of degree r . Together they generate the *ideal sheaf of Chow equations* $I^{\text{ch}}(Z) \subset \mathcal{O}_{\mathbb{P}^n}$.

Over a finite field k there may not be any projections defined along Z . The above definition gives $I^{\text{ch}}(Z)$ over \bar{k} , and it is clearly defined over k .

The embedded primes of $I^{\text{ch}}(Z)$ are quite hard to understand, so frequently we focus on the *Chow hull* of the cycle Z :

$$\text{CHull}(Z) := \text{pure}(\text{Spec } \mathcal{O}_{\mathbb{P}^n} / I^{\text{ch}}(Z)).$$

It is sometimes convenient to know that any Zariski dense set of projections generate $I^{\text{ch}}(Z)$. That is, if $P \subset \text{Gr}(n - d, n + 1)$ is Zariski dense then

$$I^{\text{ch}}(Z) = (\Phi(Z, \rho) : \rho \in P).$$

It is enough to show that this holds in every Artin quotient $\sigma : \mathcal{O}_X \twoheadrightarrow A$. Let $B \subset A$ be the ideal generated by $\sigma(\Phi(Z, \rho) : \rho \in P)$. All the $\sigma(\Phi(Z, \rho))$ are points of an irreducible subvariety $G \subset A$ obtained as an image of $\text{Gr}(n - d, n + 1)$. By assumption $G \cap B$ contains the points $\sigma(\Phi(Z, \rho) : \rho \in P)$, hence it is dense in G . So $G \subset B$, since B is Zariski closed, if we think of A as a k -vectorspace.

Claim 4.74.1. Let Z be a geometrically reduced cycle. Then $I^{\text{ch}}(Z) \subset I_Z$ and the 2 agree along the smooth locus of Z .

Proof. Let $p \in Z$ be a smooth point and $v \in T_p \mathbb{P}^n \setminus T_p Z$. A general projection $\rho : \mathbb{P}^n \dashrightarrow \mathbb{P}^{d+1}$ maps $\langle T_p Z, v \rangle$ isomorphically onto $T_{\rho(p)} \mathbb{P}^{d+1}$. Then $d\Phi(Z, \rho)$ is nonzero on v . Thus the $\Phi(Z, \rho)$ generate I_Z in a neighbourhood of p . \square

For the nonreduced case, we need a definition.

DEFINITION–LEMMA 4.75. Let $Z \subset \mathbb{P}^n$ be an irreducible, d -dimensional subscheme such that $\text{red } Z$ is geometrically reduced. Its *width* is defined in the following equivalent ways.

- (4.75.1) The *projection width* of Z is the generic multiplicity of $\pi(Z)$ for a general projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{d+1}$.
- (4.75.2) The *power width* of Z is the smallest m such that $I_{\text{red } Z}^{[m]} \cdot \mathcal{O}_Z$ is generically 0 along Z .

In general, we first take a purely inseparable field extension K/k such that $\text{red}(Z_K)$ is geometrically reduced and define the width of Z as the width of Z_K .

For example, it is easy to see that the width of $\text{Spec } k[x, y]/(x, y)^m$ is m and the width of $\text{Spec } k[x, y]/(x^m, y^m)$ is $2m - 1$.

Proof. For a general projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{d+1}$ let ϕ_π be an equation of $\pi(\text{red } Z)$ and Φ_π its pull-back to \mathbb{P}^n . Then Z has projection width m iff $\Phi_\pi^m \cdot \mathcal{O}_Z$ is generically 0 for every π and m is the smallest such. Since the Φ_π generically generate $I_{\text{red } Z}$, this holds iff $I_{\text{red } Z}^{[m]} \cdot \mathcal{O}_Z$ is generically 0 and m is the smallest. Thus the projection width equals the power width. \square

PROPOSITION 4.76. *Let $Z_i \subset \mathbb{P}^n$ be distinct, geometrically irreducible cycles of the same dimension. Then*

$$\text{CHull}(\sum m_i Z_i) = \text{pure}(\text{Spec } \mathcal{O}_{\mathbb{P}^n} / \cap_i I(Z_i)^{[m_i]})$$

Proof. The equations of the projections $\phi(\sum Z_i, \rho)$ (as in (4.74)) generate $I_{\sum Z}$ at its smooth points. So if $p \in Z_i$ is a smooth point of $\sum Z$, then $I(Z_i)^{[m_i]}$ agrees with $I^{\text{ch}}(\sum m_i Z_i)$ at p by (4.73.4). \square

The following consequence of (4.76) is key to our study of Mumford divisors.

COROLLARY 4.77. *Let k be an infinite field, $X \subset \mathbb{P}_k^n$ a reduced subscheme of pure dimension $d + 1$ and $D \subset X$ a Mumford divisor. Then*

$$\text{pure}(X \cap \text{CHull}(D)) = D. \quad (4.77.1)$$

Proof. The containment \supset is clear, hence equality can be checked after a field extension. Then we can write $D = \sum m_i D_i$ where the D_i are geometrically irreducible and reduced. Then (4.76) says that

$$\text{CHull}(D) = \text{pure}(\text{Spec } \mathcal{O}_{\mathbb{P}_k^n} / \cap_i I(D_i)^{[m_i]}).$$

Let $g_i \in D_i$ be the generic point and R_i its local ring in \mathbb{P}_k^n . Let $J_i \subset R_i$ be the ideal defining X and (J_i, h_i) the ideal defining D_i . The ideal defining the left hand side of (4.77.1) is then $(J_i + (J_i, h_i)^{[m_i]})/J_i$. This is the same as $(h_i)^{[m_i]}$, as an ideal in R_i/J_i , which equals $(h_i^{m_i})$ by (4.73.1). \square

Relative Mumford divisors.

DEFINITION 4.78. Let S be a scheme and $f : X \rightarrow S$ a morphism of pure relative dimension n that is flat in codimension 1 on every fiber. A *relative, effective Mumford divisor* on X is a relative, generically Cartier, effective divisor D (4.29) such that, for every $s \in S$, the fiber X_s is smooth at all generic points of D_s .

Let S' be another reduced scheme and $h : S' \rightarrow S$ a morphism. Then the divisorial pull-back $h^{[*]}D$ is again a relative Mumford divisor on $X \times_S S' \rightarrow S'$. So we get the *functor of Mumford divisors*.

In some cases we use an absolute version of it. Let X be a scheme and $Z \subset X$ a subscheme. A divisor D is *Mumford along Z* if X and Z are both regular at every generic point of $Z \cap \text{Supp } D$.

The following result—whose proof will be given after (4.86.5)—turns a relative, effective Mumford divisor into a flat family of Cartier divisors on another morphism, but only over reduced base schemes.

THEOREM 4.79. *Let S be a reduced scheme, $f : X \rightarrow S$ a projective morphism that is flat in codimension 1 on every fiber and $j : X \hookrightarrow \mathbf{P}_S$ an embedding into a \mathbb{P}^N -bundle. Then the maps Ch and Ch_X^{-1} —to be defined in (4.80) and (4.85.3)—provide a one-to-one correspondence*

$$\left\{ \begin{array}{l} \text{relative Mumford} \\ \text{divisors on } X \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \text{flat families of Cayley-Chow forms} \\ \text{of Mumford divisors on } X \end{array} \right\} \quad (4.79.1)$$

Comments 4.79.2. There are 2 remarkable aspects of this equivalence. First, the left hand side depends only on $X \rightarrow S$, while the right hand side is defined in terms of an embedding $j : X \hookrightarrow \mathbf{P}_S$.

Second, on the left we have families that are usually not flat, on the right families of hypersurfaces in a product of projective spaces that (locally on S) can be given by a single multihomogeneous equation. These are the simplest possible non-smooth flat families.

The correspondence (4.79.1) fails very badly over non-reduced bases. We see in (7.15) that, if $S = \text{Spec } \mathbb{C}[\epsilon]$, then the left hand side is infinite dimensional, but the right hand side is finite dimensional. Nonetheless, we will be guided by (4.79.1) in general. The rough plan is that we declare the right hand side to give the correct answer and then work backwards to see what additional conditions we

need to impose on relative Mumford divisors in order to get an equivalence. This leads us to the notion of C-flatness (7.40).

Independence of the choice of an embedding $j : X \hookrightarrow \mathbf{P}_S$ then becomes a major issue, which will be solved only in Chapter 7.

4.80 (Definition of Ch). In order to construct $\text{Chow}_{d,r}(\mathbb{P}_S^n)$, the Chow variety of degree r cycles of dimension d in \mathbb{P}_S^n , we start with the incidence correspondence as in (4.72)

$$\begin{array}{ccc} & \text{Inc}_S(\text{point}, (\check{\mathbb{P}}^n)^{d+1}) & \\ \sigma \swarrow & & \searrow \tau \\ \mathbb{P}_S^n & & (\check{\mathbb{P}}^n)_S^{d+1}. \end{array} \quad (4.80.1)$$

Note that here σ is a $(\check{\mathbb{P}}^{n-1})^{d+1}$ -bundle. The fibers of τ are linear spaces of dimension $\geq n - d - 1$ and τ is a \mathbb{P}^{n-d-1} -bundle over a dense open subset.

Let now $D \subset \mathbb{P}_S^n$ be a generically flat family d -dimensional subschemes. Assume also that the generic embedding dimension of D_s is $\leq n + 1$ for every $s \in S$. (This is satisfied iff each D_s is a Mumford divisor on some $X \subset \mathbb{P}_s^n$; a more general definition of $\text{Ch}(D)$ will be given in (7.49).) Then we set

$$\text{Ch}(D) := \tau_*(\sigma^{-1}(D)). \quad (4.80.2)$$

Claim 4.80.3. With the above assumptions, $\tau : \sigma^{-1}(D) \rightarrow \text{Ch}(D)$ is a local isomorphism on a dense open subset U such that $U \cap D_s$ is dense in D_s for every $s \in S$.

Proof. Pick $p \in D_s$ such that T_{D_s} has dimension $d+1$ at p . If $L_s \supset p$ is a general linear subspace of dimension $n - d - 1$, then $L_s \cap D_s = \{p\}$, scheme theoretically. This is exactly the fiber of $\tau : \sigma^{-1}(D) \rightarrow \text{Ch}(D)$ over any (H_0, \dots, H_d) for which $L_s = H_0 \cap \dots \cap H_n$. \square

Corollary 4.80.4. With the above assumptions, $\text{Ch}(D)$ is a generically flat family of Cartier divisors. If S is reduced, then $\text{Ch}(D)$ is a flat family of Cartier divisors.

Proof. By assumption, D is a generically flat family, hence so is $\sigma^{-1}(D)$ since σ is smooth. The first part is now immediate from (4.80.3). The second claim then follows from (4.36.2). \square

4.81 (Definition of $\text{Ch}_{\text{sch}}^{-1}$). Although $\text{Ch}(D)$ is not a flat family of Cartier divisor in general, we decide that from now on we are only interested in the cases when it is flat. Thus let

$$H^{\text{cc}} \subset (\check{\mathbb{P}}_S^n)^{d+1} \quad (4.81.1)$$

be a relative hypersurface of multidegree (r, \dots, r) . We first define its *scheme-theoretic Cayley-Chow inverse*, denoted by $\text{Ch}_{\text{sch}}^{-1}(H^{\text{cc}})$. We should think of this as a very rough first approximation of the ‘correct’ Cayley-Chow inverse.

Working with the diagram (4.80.1) consider the restriction of the left hand projection

$$\bar{\sigma}_{n,d,r} : (\text{Inc}_S \cap \tau_{n,d,r}^{-1}(H^{\text{cc}})) \rightarrow \mathbb{P}_S^n. \quad (4.81.2)$$

Fix now $s \in S$, a point $p_s \in \mathbb{P}_s^n$. Note that the preimage of p_s consists of all $(d+1)$ -tuples (H_0, \dots, H_d) such that $p \in H_i$ for every i and $(H_0, \dots, H_d) \in H_s^{\text{cc}}$. In particular, if Z is a d -cycle of degree r on \mathbb{P}_S^n and $H^{\text{cc}} = \text{Ch}(Z)$ is its Cayley-Chow hypersurface, then $\bar{\sigma}_{n,d,r}$ is a $(\check{\mathbb{P}}_S^{n-1})^{d+1}$ -bundle over $\text{Supp } Z$.

The key observation is that this property alone is enough to define $\text{Ch}_{\text{sch}}^{-1}$ and to construct the Chow variety. So we define

$$\text{Ch}_{\text{sch}}^{-1}(H^{\text{cc}}) \subset \mathbb{P}_S^n \quad (4.81.3)$$

as the unique, largest, closed subscheme over which $\bar{\sigma}_{n,d,r}$ is a $(\check{\mathbb{P}}^{n-1})^{d+1}$ -bundle. (Its existence is a special case of (3.19), but we derive its equations in (4.82.3).)

The set-theoretic behavior of the projection

$$\rho : \text{Ch}_{\text{sch}}^{-1}(H^{\text{cc}}) \rightarrow S \quad (4.81.4)$$

is described in (4.72). The fibers have dimension $\leq d$, and $Z_s \subset \mathbb{P}_s^n$ is a d -dimensional irreducible component iff $\text{Ch}(Z_s)$ is an irreducible component of H_s^{cc} . It is now not hard to see that there is a maximal closed subset $S^{\text{cc}} \subset S$ over which H^{cc} is the Cayley-Chow hypersurface of a family of d -cycles; see [Kol96, ??].

However, we can not get the correct scheme structure on S^{cc} since the scheme structure of the fibers of $\rho : \text{Ch}_{\text{sch}}^{-1}(H^{\text{cc}}) \rightarrow S$ is not ‘correct.’ Before we move ahead, we need to understand this scheme structure.

4.82 (Scheme structure of $\text{Ch}_{\text{sch}}^{-1}(H^{\text{cc}})$). Let S be a scheme and $H^{\text{cc}} := (F = 0) \subset (\check{\mathbb{P}}^n)_S^{d+1}$ a hypersurface of multidegree (r, \dots, r) . We aim to write down equations for $\text{Ch}_{\text{sch}}^{-1}(F = 0)$.

Choose coordinates $(x_0 : \dots : x_n)$ on \mathbb{P}_S^n and dual coordinates $(\check{x}_{0j} : \dots : \check{x}_{nj})$ on the j th copy of $\check{\mathbb{P}}_S^n$ for $j = 0, \dots, d$. So $F = F(\check{x}_{ij})$ is a homogeneous polynomial of multidegree (r, \dots, r) . For notational simplicity we compute in the affine chart $\mathbb{A}_S^n = \mathbb{P}_S^n \setminus (x_0 = 0)$.

For $(x_1, \dots, x_n) \in \mathbb{A}_S^n$, the hyperplanes H in the j th copy of $\check{\mathbb{P}}_S^n$ that pass through (x_1, \dots, x_n) are all written in the form

$$\left(-\sum_{i=1}^n x_i \check{x}_{ij} : \check{x}_{1j} : \dots : \check{x}_{nj}\right). \quad (4.82.1)$$

Let $M(\check{x}_{ij})$ be all the monomials in the \check{x}_{ij} for $1 \leq i \leq n, 0 \leq j \leq d$. Then we can write

$$\begin{aligned} & F\left(-\sum_{i=1}^n x_i \check{x}_{i0} : \check{x}_{10} : \dots : \check{x}_{n0}; \dots; -\sum_{i=1}^n x_i \check{x}_{id} : \check{x}_{1d} : \dots : \check{x}_{nd}\right) \\ & =: \sum_M F_M(x_1, \dots, x_n) M(\check{x}_{ij}). \end{aligned} \quad (4.82.2)$$

Since the monomials $M(\check{x}_{ij})$ are linearly independent, this vanishes for all \check{x}_{ij} iff $F_M = 0$ for every M . Equivalently:

Claim 4.82.3. The subscheme $\text{Ch}_{\text{sch}}^{-1}(F = 0) \cap \mathbb{A}_S^n$ is given by the equations

$$F_M(x_1, \dots, x_n) = 0 \quad \text{for all monomials } M. \quad \square$$

Assume now that $(F = 0) = \text{Ch}(Y)$. If we fix $\check{x}_{ij} = c_{ij}$, then these give the matrix of a linear projection $\pi_c : \mathbb{A}_S^n \rightarrow \mathbb{A}_S^{d+1}$, and the corresponding Chow equation of Y is

$$\sum_M F_M(x_1, \dots, x_n) M(c_{ij}) = 0. \quad (4.82.4)$$

Thus we proved the following.

THEOREM 4.83. *Let $Z \subset \mathbb{P}_k^n$ be a d -cycle of degree r . Then $\text{Ch}_{\text{sch}}^{-1}(\text{Ch}(Z)) \subset \mathbb{P}_k^n$ is the subscheme defined by the ideal of Chow equations $I^{\text{ch}}(Z)$. \square*

Note that we proved a little more, that if the residue field of S is infinite then $I^{\text{ch}}(Y)|_{\mathbb{A}_S^n}$ is generated by the Chow equations of the linear projections $\pi_{\mathbf{e}} : \mathbb{A}_S^n \rightarrow \mathbb{A}_S^{n+1}$. A priori we would need to use the more general projections (7.37.4), but this is just a matter of choosing the hyperplane at infinity.

Combining (4.83) and (4.77) gives the following.

COROLLARY 4.84. *Let k be a field, $X \subset \mathbb{P}_k^n$ a subscheme of pure dimension $d+1$ and $D \subset X$ a Mumford divisor of degree r . Then*

$$\text{pure}(X \cap \text{Ch}_{\text{sch}}^{-1}(\text{Ch}(D))) = D. \quad \square$$

4.85 (Construction of $\text{MDiv}(X)$). As we noted in (4.79.2), we construct $\text{MDiv}(X)$ by starting on the right hand side of (4.79.1)

Let S be ascheme, $f : X \rightarrow S$ a projective morphism of pure dimension d that is flat in codimension 1 on every fiber, and $j : X \hookrightarrow \mathbf{P}_S$ an embedding into a \mathbb{P}^n -bundle.

We fix the intended degree to be r and let $\mathbf{P}_{n,d,r} = |\mathcal{O}_{(\check{\mathbb{P}}^n)^{d+1}}(r, \dots, r)|$ be the linear system of hypersurfaces of multidegree (r, \dots, r) in $(\check{\mathbb{P}}^n)^{d+1}$ with universal hypersurface

$$\mathbf{H}_{n,d,r}^{\text{cc}} \subset (\check{\mathbb{P}}^n)^{d+1} \times \mathbf{P}_{n,d,r}. \quad (4.85.1)$$

Thus (4.80.1) extends to

$$\begin{array}{ccc} \text{Inc}_S(\text{point}, (\check{\mathbb{P}}^n)^{d+1}) \times_S \mathbf{P}_{n,d,r} & & \\ \sigma_{n,d,r} \swarrow & & \searrow \tau_{n,d,r} \\ \mathbb{P}_S^n \times_S \mathbf{P}_{n,d,r} & & (\check{\mathbb{P}}^n)^{d+1} \times_S \mathbf{P}_{n,d,r} \end{array} \quad (4.85.2)$$

As in (4.81) we get

$$\text{Ch}_{\text{sch}}^{-1}(\mathbf{H}_{n,d,r}^{\text{cc}}) \subset \mathbb{P}_S^n \times_S \mathbf{P}_{n,d,r}. \quad (4.85.3)$$

We are interested in d -cycles that lie on X , so we should take

$$\text{Ch}_X^{-1}(\mathbf{H}_{n,d,r}^{\text{cc}}) := \text{Ch}_{\text{sch}}^{-1}(\mathbf{H}_{n,d,r}^{\text{cc}}) \cap (X \times_S \mathbf{P}_{n,d,r}) \subset \mathbb{P}_S^n \times_S \mathbf{P}_{n,d,r}. \quad (4.85.3)$$

By (4.84), if $D_s \subset X_s$ is a Mumford divisor of degree r then the fiber of the coordinate projection

$$\rho_{n,d,r} : \text{Ch}_X^{-1}(\mathbf{H}_{n,d,r}^{\text{cc}}) \rightarrow \mathbf{P}_{n,d,r} \quad (4.85.4)$$

over $[\text{Ch}(D_s)]$ is D_s (aside from possible embedded points).

This leads us to the following.

DEFINITION–THEOREM 4.86. Let S be a scheme, $f : X \rightarrow S$ a projective morphism of pure dimension d that is flat in codimension 1 on every fiber, and $j : X \hookrightarrow \mathbf{P}_S$ an embedding into a \mathbb{P}^n -bundle. Let $\text{MDiv}_r(X)$ be the unique, largest, locally closed subscheme $\text{MDiv}_r(X) \subset \mathbf{P}_{n,d,r}$ such that $\rho_{n,d,r}$ (as in (4.85.4)) is a generically flat family of degree r Mumford divisors.

Thus, over $\text{MDiv}_r(X)$ we have

(4.86.1) $\text{Univ}_r^{\text{mum}}(X) \rightarrow \text{MDiv}_r(X)$, a universal, generically flat family of degree r Mumford divisors, and

(4.86.2) $\mathbf{H}_r^{\text{cc}} \subset (\check{\mathbb{P}}^n)_S^{d+1}$, a flat family of multidegree (r, \dots, r) hypersurfaces, such that

(4.86.3) $\text{Ch}(\text{Univ}_r^{\text{mum}}(X)) = \mathbf{H}_r^{\text{cc}}$, and

(4.86.4) $v\text{-pure}(\text{Ch}_X^{-1}(\mathbf{H}_r^{\text{cc}})) = \text{Univ}_r^{\text{mum}}(X)$.

Proof. As we noted in (4.72), every fiber of $\rho_{n,d,r}$ has dimension $\leq d$. The condition

$$\{H_s^{\text{cc}} : \dim(\text{Sing } X_s \cap \text{Supp } \text{Ch}_{\text{sch}}^{-1}(H_s^{\text{cc}})) \geq d\}$$

defines a closed subset of $\mathbf{P}_{n,d,r}$, let $\mathbf{P}_{n,d,r}^\circ$ denote its complement.

If $H_s^{\text{cc}} \in \mathbf{P}_{n,d,r}^\circ$ then (the pure part of) $\rho_{n,d,r}^{-1}(H_s^{\text{cc}})$ is a Mumford divisor of degree $\leq r$. Furthermore, equality holds only when $H_s^{\text{cc}} = \text{Ch}(D_s)$ for some Mumford divisor D_s and

$$\text{pure}(\rho_{n,d,r}^{-1}[D_s]) = D_s.$$

The upper semicontinuity of the dimension and the degree imply that there is a maximal closed subset

$$\text{MDiv}_r^{\text{red}}(X) \subset \mathbf{P}_{n,d,r}^\circ \quad (4.86.5)$$

over which $\deg \rho_{n,d,r}^{-1}(H_s^{\text{cc}}) = r$.

Note that this completes the proof of (4.79). Indeed, first, over a reduced base, every generically flat family of Mumford divisors D gives a flat family of Cayley-Chow hypersurfaces by (4.80.4), so $\text{Ch}(D)$ gives a family of hypersurfaces in $\text{MDiv}_r^{\text{red}}(X)$. Second, $\text{MDiv}_r^{\text{red}}(X)$ was chosen so that Ch_X^{-1} is a generically flat family of Mumford divisors over it.

The scheme structure is defined using (4.87). \square

Warning 4.86.6. In the non-reduced case, we do not have an independent definition of the left hand side of the correspondence in (4.79). In particular, the resulting $\text{MDiv}(X)$ a priori depends on the projective embedding $j : X \hookrightarrow \mathbf{P}_S$. We write $\text{MDiv}(X \subset \mathbf{P}_S)$ if we want to emphasize this.

In (7.3) we construct a variant, denoted by $\text{KDiv}(X)$, that does not depend on the embedding. A positive answer to Question 7.45 would imply that $\text{MDiv}(X \subset \mathbf{P}_S) = \text{KDiv}(X)$.

We have used the following variant of the Flattening Decomposition Theorem of [Mum66, Lec.8].

PROPOSITION 4.87. *Let $f : X \rightarrow S$ be a projective morphism and F a coherent sheaf on X . Let n be the maximal fiber dimension of $\text{Supp } F \rightarrow S$. There is a locally closed decomposition $j_F^{\text{g-flat}} : S_F^{\text{g-flat}} \rightarrow S$ such that F_W is generically flat over W in dimension n iff $W \rightarrow S$ factors through $j_F^{\text{g-flat}}$.*

Proof. Let $S^\circ \subset S$ be the largest open subscheme over which $\text{Supp } F \rightarrow S$ has fiber dimension $< n$. Depending on the precise definition, S° is the trivial component of $S_F^{\text{g-flat}}$.

The main interest is in the components where the fiber dimension is n . We may thus replace X by $\text{Supp } F$. The question is local on S . By Noether normalization we may assume that there is a finite morphism $\pi : X \rightarrow \mathbb{P}_S^n$. Note that F_W is generically flat over W in dimension n iff the same holds for $\pi_* F_W$. We may thus assume that $X = \mathbb{P}_S^n$.

Applying [Mum66, Lec.8] to the identity $X \rightarrow X$ and F , we get a decomposition $\Pi_i X_i \rightarrow X = \mathbb{P}_S^n$ where every $F|_{X_i}$ is flat, hence locally free of rank i .

Let $Z \subset \mathbb{P}_S^n$ be a closed subscheme. Applying [Mum66, Lec.8] to the projection $\mathbb{P}_S^n \rightarrow S$ and \mathcal{O}_Z , we see that there is a unique largest subscheme $S(Z) \subset S$ such that $S(Z) \times_S \mathbb{P}_S^n \subset Z$. For a locally closed subscheme $Z \subset \mathbb{P}_S^n$ set $S(Z) = S(\bar{Z}) \setminus S(\bar{Z} \setminus Z)$, where \bar{Z} denotes the scheme-theoretic closure of $Z \subset \mathbb{P}_S^n$.

Note that $S(Z)$ is the largest subscheme $T \subset S$ with the following property:

(4.87.1) There is an open subscheme $\mathbf{P}_T^\circ \subset \mathbb{P}_T^n$ that contains the generic point of \mathbb{P}_t^n for every $t \in T$ and such that $\mathbf{P}_T^\circ \subset Z$.

We claim that $\coprod_{i>0} S(X_i)$ is the union of those components of $S_F^{\text{g-flat}}$ for which the fiber dimension of $\text{Supp } F \rightarrow S$ is n .

First, $F|_{X_i}$ is locally free of rank i , so the restriction of F to $S(X_i) \times_S \mathbb{P}_S^n$ is locally free, hence flat, at every generic point of every fiber.

Conversely, let W be a connected scheme and $q : W \rightarrow S$ a morphism such that F_W is generically flat over W in dimension n . Since F_w is generically free for every $w \in W$, this implies that F_W is locally free at the generic point of every fiber. Let $\mathbf{P}_W^\circ \subset \mathbb{P}_W^n$ be the open set where F_W is locally free. By assumption the closure of \mathbf{P}_W° equals \mathbb{P}_W^n .

Since \mathbf{P}_W° contains the generic point of every fiber \mathbb{P}_w^n , it is connected. Thus F has constant rank, say i , on \mathbf{P}_W° . Therefore, the restriction of q to \mathbf{P}_W° lifts to $\tilde{q} : \mathbf{P}_W^\circ \rightarrow X_i$, which in turn extends to the closures $\bar{q} : \mathbb{P}_W^n \rightarrow \bar{X}_i$. Thus \bar{q} gives $q_W : W \rightarrow S(X_i)$ in view of (4.87.1). \square

Numerical flatness and stability criteria

The aim of this chapter is to prove several characterizations of stable and locally stable families $f : (X, \Delta) \rightarrow S$. An earlier result, established in (3.37), has two assumptions:

- every fiber (X_s, Δ_s) is semi-log-canonical and
- $K_{X/S} + \Delta$ is \mathbb{Q} -Cartier.

In many applications the first of these is given, but the second one can be quite subtle.

Note that such difficulties arise already for surfaces, even if $\Delta = 0$. Indeed, we saw in Section 1.4 that there are flat, projective families $g : X \rightarrow C$ of surfaces with quotient singularities that are not locally stable. In these cases every fiber is log terminal, but $K_{X/C}$ is not \mathbb{Q} -Cartier, although its restriction to every fiber $K_{X/C}|_{X_c} = K_{X_c}$ is \mathbb{Q} -Cartier.

In all the examples in Section 1.4, this unexpected behavior coincides with a jump in the self-intersection number of the canonical class of the fiber. Our aim is to prove that this is always the case, as shown by the following simplified version of the main theorem.

THEOREM 5.1 (Numerical criterion of stability, weak form). *Let S be a connected, reduced scheme over a field of characteristic 0 and $f : X \rightarrow S$ a proper morphism of pure relative dimension n . Assume that all fibers are semi-log-canonical with ample canonical class K_{X_c} . Then*

(5.1.1) $s \mapsto (K_{X_s}^n)$ is an upper semicontinuous function on S and

(5.1.2) $f : X \rightarrow S$ is stable iff the above function is constant.

If $f : X \rightarrow S$ is stable then $K_{X/S}$ is \mathbb{Q} -Cartier, hence $(K_{X_s}^n)$ is clearly independent of $s \in S$, but the converse is surprising. General theory says that stability holds iff the Hilbert function $\chi(X_s, \mathcal{O}_{X_s}(mK_{X_s}))$ is independent of $s \in S$. Thus (5.1.2) asserts that if the leading coefficient of the Hilbert function is independent of s , then the same holds for the whole Hilbert function. We collect many similar results in this chapter; see [Kol15] for other such examples.

The main theorems are stated in Section 5.1. Related results on simultaneous canonical models and modifications are discussed in Section 5.2. The key claim is that, for families of slc pairs, local stability can fail only in relative codimension two and it can be characterized by the constancy of just 1 intersection number. A similar numerical condition characterizes Cartier divisors on flat families.

A series of examples in Section 5.3 shows that the assumptions of the theorems are likely to be optimal in characteristic 0. All the results are expected to hold in positive and mixed characteristic as well, but very few of the proofs apply to these cases. Numerical criteria for stability in codimension ≤ 1 are discussed in Section 5.4.

For all of the main theorems the key step is to establish them for families over smooth curves. This is done in Section 5.5. The numerical criterion of global stability and a weaker version of local stability are derived in Section 5.5. The existence of simultaneous canonical models is studied in Section 5.6 and we treat simultaneous canonical modifications in Section 5.7.

Going from families over smooth curves to families over higher dimensional singular bases turns out to be quite quick, but several of the arguments, presented in Section 5.9, rely heavily on the techniques and results of Chapter 9.

Assumptions. For all the main theorems of this Chapter we work with varieties over a field of characteristic 0, but the background results worked out in Section 5.8 are established for excellent schemes.

5.1. Statements of the main theorems

We develop a series of criteria to characterize locally stable (4.48) or stable (2.44) morphisms using a few, simple, numerical invariants of the fibers.

We follow the general set-up of (5.1) but we strengthen it in 3 ways:

- We add a boundary divisor Δ .
- We assume only that f is flat in codimension 1 on each fiber. The reason for this is that many natural constructions (for instance flips, taking cones or ramified covers) do not preserve flatness. Thus we frequently end up with morphisms that are not known to be flat everywhere. This is rarely a problem when the base space is a smooth curve, but it becomes a serious issue over higher dimensional singular bases.
- We deal with local stability as well. A weak variant, involving several intersection numbers, is quite similar to the global case, but the sharper form requires different considerations.

For the main results of this Chapter we work with the following set-up, which is a slight generalization of (3.25) and (4.1).

ASSUMPTION 5.2. Let $f : X \rightarrow S$ be a proper morphism of pure relative dimension n (2.72) and $Z \subset X$ a closed subset with complement $U := X \setminus Z$ such that the following hold.

$$(5.2.1) \quad \text{codim}_{X_s}(Z \cap X_s) \geq 2 \text{ for every } s \in S,$$

$$(5.2.2) \quad f|_U : U \rightarrow S \text{ is flat and}$$

$$(5.2.3) \quad \text{depth}_Z X \geq 2.$$

Sheaf versions of these assumptions are studied in Section 5.8.

Given $f : X \rightarrow S$ and $U = X \setminus Z$ as above, we also consider effective \mathbb{R} -divisors $\Delta = \sum b_i B_i$ on X where the B are generically Cartier divisors (4.29). In applications to moduli problems we usually know that

$$(5.2.4) \quad f|_U : (U, \Delta|_U) \rightarrow S \text{ is locally stable.}$$

In this case let $\pi_s : (\bar{X}_s, \bar{D}_s + \bar{\Delta}_s) \rightarrow (X_s, \Delta_s)$ denote the normalization of a fiber where $\bar{D}_s \subset \bar{X}_s$ is the conductor (11.10). Thus

$$K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s \sim_{\mathbb{R}} \pi_1^*(K_{X_s} + D_s + \Delta_s) \quad (5.2.5)$$

and it makes sense to ask whether $(\bar{X}_s, \bar{D}_s + \bar{\Delta}_s)$ is lc or not.

We aim to give numerical criteria applicable to any morphism satisfying (5.2.1–4). The first such result generalizes (5.1) to pairs.

THEOREM 5.3 (Numerical criterion of stability). *We use the notation of (5.2). In addition to (5.2.1–3) assume that S is a reduced scheme over a field of char 0. Assume further that*

(5.3.1) $f|_U : (U, \Delta|_U) \rightarrow S$ is locally stable,

(5.3.2) (X_g, Δ_g) is slc for all generic points $g \in S$,

(5.3.3) every fiber has lc normalization $\pi_s : (\bar{X}_s, \bar{D}_s + \bar{\Delta}_s) \rightarrow (X_s, \Delta_s)$ and

(5.3.4) $K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s$ is ample for every $s \in S$.

Then

(5.3.5) $s \mapsto (K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s)^n$ is an upper semicontinuous function on S and

(5.3.6) $f : (X, \Delta) \rightarrow S$ is stable iff the above function is locally constant.

The local stability version of (5.3) is the following.

THEOREM 5.4 (Numerical criterion of local stability). *We use the notation of (5.2). In addition to (5.2.1–3) assume that S is a reduced scheme over a field of char 0 and H is a relatively ample Cartier divisor class on X . Assume further that*

(5.4.1) $f|_U : (U, \Delta|_U) \rightarrow S$ is locally stable,

(5.4.2) (X_g, Δ_g) is slc for all generic points $g \in S$ and

(5.4.3) every fiber has lc normalization $\pi_s : (\bar{X}_s, \bar{D}_s + \bar{\Delta}_s) \rightarrow (X_s, \Delta_s)$.

Then

(5.4.4) $s \mapsto (\pi_s^* H^{n-2} \cdot (K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s)^2)$ is upper semicontinuous and

(5.4.5) $f : (X, \Delta) \rightarrow S$ is locally stable iff the above function is locally constant.

Under the assumptions of (5.4) the functions $(\pi_s^* H^n)$ and $(\pi_s^* H^{n-1} \cdot (K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s))$ are always locally constant but the functions $(\pi_s^* H^{n-i} \cdot (K_{\bar{X}_s} + \bar{D}_s + \bar{\Delta}_s)^i)$ are neither upper nor lower semicontinuous for $i \geq 3$.

A key part of the proof of (5.4) is to show that local stability is essentially a 2-dimensional question. The following is a strong form of this claim.

THEOREM 5.5 (Local stability is automatic in codimension ≥ 3). [**Kol13a**] *Using the notation and assumptions of (5.2.1–3) let S be a reduced scheme of char 0. Assume that*

(5.5.1) $\text{codim}_{X_s}(Z \cap X_s) \geq 3$ for every $s \in S$,

(5.5.2) $f|_U : (U, \Delta|_U) \rightarrow S$ is locally stable,

(5.5.3) (X_g, Δ_g) is slc for all generic points $g \in S$ and

(5.5.4) every fiber has lc normalization $\pi_s : (\bar{X}_s, \bar{D}_s + \bar{\Delta}_s) \rightarrow (X_s, \Delta_s)$.

Then $f : (X, \Delta) \rightarrow S$ is locally stable.

One can also restate this as a converse of the Bertini-type result (2.12).

COROLLARY 5.6. *Notation and assumptions as in (5.4). Assume in addition that the relative dimension is $n \geq 3$ and $f|_H : (H, \Delta|_H) \rightarrow S$ is locally stable, where $H \subset X$ is a relatively ample Cartier divisor. Then $f : (X, \Delta) \rightarrow S$ is also locally stable. \square*

Comment. As we noted in (2.13), (11.20) implies that $f : (X, H + \Delta) \rightarrow S$, and hence also $f : (X, \Delta) \rightarrow S$, are locally stable in a neighborhood of H . The unexpected new claim is that local stability holds everywhere.

A variant of (5.3) holds for arbitrary divisors and for non-slc fibers but we have to assume that f is flat with S_2 fibers. On the other hand, this holds over any field. We state the general form treated in [Kol16a] but in this book we prove only the special case when $f : X \rightarrow S$ has normal fibers.

THEOREM 5.7 (Numerical criterion for relative line bundles). [Kol16a] *Let S be a reduced scheme over a field, $f : X \rightarrow S$ a flat, proper morphism of pure relative dimension n with S_2 fibers, and $Z \subset X$ a closed subset such that $\text{codim}_{X_s}(Z \cap X_s) \geq 2$ for every $s \in S$. Let H be an f -ample line bundle on X .*

Let L_U be an invertible sheaf on $U := X \setminus Z$ and assume that, for every $s \in S$, the restriction $L_U|_{U_s}$ extends to an invertible sheaf L_s on X_s . Then

(5.7.1) $s \mapsto (H_s^{n-2} \cdot L_s^2)$ is an upper semicontinuous function on S , and

(5.7.2) L_U extends to an invertible sheaf L on X iff the above function is locally constant.

Furthermore, if L_s is ample for every s , then

(5.7.3) $s \mapsto (L_s^n)$ is an upper semicontinuous function on S , and

(5.7.4) L_U extends to an f -ample invertible sheaf L on X iff the above function is locally constant.

Most likely (5.7) holds over any reduced scheme S , but a key step (2.87) is known only over fields.

5.2. Simultaneous canonical models and modifications

We also aim to get numerical criteria for the existence of simultaneous canonical models and canonical modifications. That is, given a morphism $f : X \rightarrow S$, we would like to know when the canonical models (or the canonical modifications) of the fibers form a flat family; see (5.9) and (5.16) for the precise definitions.

There are two distinct definitions of canonical models.

DEFINITION 5.8 (Canonical models). Let (X, Δ) be a proper lc pair such that $K_X + \Delta$ is big. As usual (see [KM98, 3.50], [Kol13b, 1.19] or (11.26)) its *canonical model* is given by a birational contraction

$$\phi : (X, \Delta) \dashrightarrow (X^c, \Delta^c := \phi_* \Delta), \quad (5.8.1)$$

where (X^c, Δ^c) is lc, $K_{X^c} + \Delta^c$ is ample and

$$\sum_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)) \cong \sum_{m \geq 0} H^0(X^c, \mathcal{O}_{X^c}(mK_{X^c} + \lfloor m\Delta^c \rfloor)).$$

On the other hand, if X is a proper variety with arbitrary singularities, then one frequently defines the *canonical model* of X as the canonical model of a resolution of X . We denote the latter variant by X^{cr} .

More generally, let X be a proper, pure dimensional scheme over a field. The *canonical model of resolutions* of X , denoted by X^{cr} , is obtained as follows. We start with a resolution $X^r \rightarrow \text{red } X$ and then take the disjoint union of the canonical models of those irreducible components that are of general type. With a slight abuse of terminology, there is a natural map

$$\phi : X \dashrightarrow X^{\text{cr}} \quad (5.8.2)$$

which is birational on the general type components and not defined on the others.

If X has log canonical singularities then both variants are defined but note that $X^c \cong X^{\text{cr}}$ iff X has only canonical singularities.

Aside. One can also define the canonical model of resolutions of a pair (X, Δ) as long as none of the irreducible components of the boundary Δ is contained in $\text{Sing } X$. We start with a resolution $p : X' \rightarrow X$ such that $p_*^{-1}\Delta$ has smooth support and then take the canonical model of $(X', p_*^{-1}\Delta)$ to get $(X^{\text{cr}}, \Delta^{\text{cr}})$.

We see in (5.21) that this notion does not seem to behave well in families.

DEFINITION 5.9 (Simultaneous canonical model). Let $f : (X, \Delta) \rightarrow S$ be a morphism as in (5.2) such that every fiber has log canonical normalization $\pi_s : (\bar{X}_s, \bar{\Delta}_s) \rightarrow (X_s, \Delta_s)$. Its *simultaneous canonical model* is a diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^{\text{sc}} \\ f \searrow & & \swarrow f^{\text{sc}} \\ & S & \end{array} \tag{5.9.1}$$

where $f^{\text{sc}} : (X^{\text{sc}}, \Delta^{\text{sc}}) \rightarrow S$ is stable and

$$\phi_s \circ \pi_s : (\bar{X}_s, \bar{\Delta}_s) \dashrightarrow (X_s^{\text{sc}}, \Delta_s^{\text{sc}})$$

is the canonical model, as in (5.8.1), for every $s \in S$.

For a pure dimensional proper morphism $f : X \rightarrow S$ the *simultaneous canonical model of resolutions* $f^{\text{sc}} : X^{\text{sc}} \rightarrow S$ is defined analogously. Here we require that $\phi_s : X_s \dashrightarrow X_s^{\text{sc}}$ be the canonical model of resolutions (5.8.2) for every $s \in S$.

Some known cases 5.9.2. Assume that $f : X \rightarrow S$ is flat, projective and X_s has canonical singularities and a canonical model for some $s \in S$. Then a simultaneous canonical model exists over an open neighborhood $s \in S^\circ \subset S$. In increasing generality this was proved in [KM92, 12.5.1], [Siu98, Kaw99] and [Nak04, Chap.VI]; see also [Kol21] for the complex analytic cases. We discuss a version of this in (5.41). The $\Delta \neq 0$ case is more subtle, as shown by the Examples 5.21–5.23.

Comment on the conductor 5.9.3. Note that we do not add the conductor of π_s to $\bar{\Delta}_s$. If the fibers are normal in codimension 1 then the divisorial part of the conductor is 0, hence the above notion is the only sensible one. In general, however, one has a choice and the simultaneous slc model, to be defined in (5.44), may be a better concept.

We give criteria for the existence of simultaneous canonical models in terms of the volume (10.30) of the canonical class of the fibers. Note that if Y is a proper scheme of dimension n then $\text{vol}(K_{Y^r})$ is independent of the choice of the resolution $Y^r \rightarrow Y$ and it equals the self-intersection number $((K_{Y^{\text{cr}}})^n)$. Similarly, if (Y, Δ) is log canonical then $\text{vol}(K_Y + \Delta) = ((K_{Y^c} + \Delta^c)^n)$.

THEOREM 5.10 (Numerical criterion for simultaneous canonical models I). *Let S be a seminormal scheme of char 0 and $f : X \rightarrow S$ a proper morphism of pure relative dimension n . Then*

- (5.10.1) $s \mapsto \text{vol}(K_{X_s^r})$ is a lower semicontinuous function on S and
- (5.10.2) $f : X \rightarrow S$ has a simultaneous canonical model of resolutions iff this function is locally constant (and positive).

The key case, when S is a smooth curve, is settled in (5.37), the general case is in (5.62). This is a surprising result on two accounts. First, cohomology groups almost always vary *upper* semicontinuously; the *lower* semicontinuity in this setting

was first observed and proved in [Nak86, Nak87]. Second, usually it is easy to generalize similar proofs from smooth varieties to klt or lc pairs, but here adding any boundary can ruin the argument and the conclusion, as shown by the Examples 5.21–5.23. Example 5.20 shows that S needs to be seminormal.

The following is a similar result for normal lc pairs, but the lower semicontinuity of (5.10) changes to upper semicontinuity.

THEOREM 5.11 (Numerical criterion for simultaneous canonical models II). *We use the notation of (5.2). In addition to (5.2.1–3) assume that S is a seminormal scheme of char 0. Assume furthermore that*

(5.11.1) $f|_U : U \rightarrow S$ is smooth (hence every fiber is irreducible),

(5.11.2) every fiber has lc normalization $\pi_s : (\bar{X}_s, \bar{\Delta}_s) \rightarrow (X_s, \Delta_s)$ and

(5.11.3) the canonical models $\phi_s : (\bar{X}_s, \bar{\Delta}_s) \dashrightarrow (X_s^c, \Delta_s^c)$ exist.

Then

(5.11.4) $s \mapsto \text{vol}(K_{\bar{X}_s} + \bar{\Delta}_s)$ is an upper semicontinuous function on S and

(5.11.5) $f : (X, \Delta) \rightarrow S$ has a simultaneous canonical model iff this function is locally constant.

The proof is given in (5.39), and (5.62).

One should think of (5.11) as a generalization of (5.3) but there are differences. In (5.11) we allow only fibers that are smooth in codimension 1 and S is assumed seminormal, not just reduced. (The extra assumption (3) is expected to hold always.) However, the key difference is in the proofs given in Section 5.9. While the proof of (5.3) uses only the basic theory of hulls and husks, we rely on the existence of moduli spaces of pairs in order to establish (5.11).

Both (5.10) and (5.11) apply to $f : X \rightarrow S$ iff the normalizations of the fibers have canonical singularities. In this case f is locally stable (2.8) and the plurigenera—and hence the volume—are locally constant [Siu98, Kaw99].

A key ingredient of the proof of (5.10–5.11) is the following characterization of canonical models. We prove a more general version of it in (10.35).

PROPOSITION 5.12. *Let X be a smooth proper variety of dimension n . Let Y be a normal, proper variety birational to X and D an effective \mathbb{R} -divisor on Y such that $K_Y + D$ is \mathbb{R} -Cartier, nef and big. Then*

(5.12.1) $\text{vol}(K_X) \leq \text{vol}(K_Y + D) = (K_Y + D)^n$ and

(5.12.2) equality holds iff $D = 0$ and Y has canonical singularities (thus Y is a weak canonical model (cf. [Kol13b, 1.19]) of X).

For surfaces, the existence criterion of simultaneous canonical modifications is proved in [KSB88, Sec.2]. In higher dimensions we need to work with a sequence of intersection numbers and with their lexicographic ordering.

DEFINITION 5.13. Let X be a proper scheme of dimension n and A, B \mathbb{R} -Cartier divisors on X . Their sequence of intersection numbers is

$$I(A, B) := ((A^n), \dots, (A^{n-i} \cdot B^i), \dots, (B^n)) \in \mathbb{R}^{n+1}.$$

DEFINITION 5.14. The lexicographic ordering of length $n + 1$ real sequences is denoted by

$$(a_0, \dots, a_n) \preceq (b_0, \dots, b_n).$$

This holds if either $a_i = b_i$ for every i or there is an $r \leq n$ such that $a_i = b_i$ for $i < r$ but $a_r < b_r$. For polynomials we define an ordering

$$f(t) \preceq g(t) \Leftrightarrow f(t) \leq g(t) \forall t \gg 0.$$

Note that

$$\sum_i a_i t^{n-i} \preceq \sum_i b_i t^{n-i} \Leftrightarrow (a_0, \dots, a_n) \preceq (b_0, \dots, b_n).$$

If we have proper schemes X, X' of dimension n and \mathbb{R} -Cartier divisors A, B on X and A', B' on X' then

$$I(A, B) \preceq I(A', B') \Leftrightarrow (mA + B)^n \leq (mA' + B')^n \forall m \gg 0.$$

We will consider functions that associate a sequence or a polynomial to all points of a scheme X . Using the above definitions it makes sense to ask if such a function is *lexicographically upper/lower semicontinuous* or not.

5.15 (Canonical and log canonical modification). Let Y be a scheme over a field k . (We allow Y to be reducible and non-reduced but in applications usually pure dimensional.) Its *canonical modification* $p : Y^{\text{can}} \rightarrow Y$ is the canonical modification of its normalization $\pi : \bar{Y} \rightarrow Y$; that is, Y^{can} has canonical singularities, $Y^{\text{can}} \rightarrow \bar{Y}$ is proper, birational and $K_{Y^{\text{can}}}$ is ample over \bar{Y} .

Let Δ be an effective divisor on Y . We define the *canonical modification* $p : (Y^{\text{can}}, \Delta^{\text{can}}) \rightarrow (Y, \Delta)$ as the canonical modification of the normalization $(\bar{Y}, \bar{\Delta} := \pi^* \Delta)$, provided that it makes sense. That is, the pull-back $\pi^* \Delta$ should be defined as in (5.2) and its irreducible components should have coefficient ≤ 1 . If these hold then $\bar{p} : Y^{\text{can}} \rightarrow \bar{Y}$ is the unique proper, birational morphism such that $(Y^{\text{can}}, \Delta^{\text{can}} := (\bar{p})_*^{-1} \bar{\Delta})$ is canonical and $K_{Y^{\text{can}}} + \Delta^{\text{can}}$ is ample over \bar{Y} ; see [Kol13b, 1.31].

The *log canonical modification* $p : (Y^{\text{lc}}, \Delta^{\text{lc}}) \rightarrow (Y, \Delta)$ is defined similarly. The change is that $(Y^{\text{lc}}, \Delta^{\text{lc}} + E^{\text{lc}})$ is log canonical and $K_{Y^{\text{can}}} + \Delta^{\text{can}} + E^{\text{lc}}$ is ample over \bar{Y} , where E^{lc} denotes the reduced exceptional divisor of \bar{p} .

Log canonical modifications are conjectured to exist. Currently this is known in the following cases.

(5.15.1) $K_Y + \Delta$ is \mathbb{R} -Cartier.

(5.15.2) There is a Cartier divisor D such that $\text{Supp } D \subset \text{Supp } \Delta$ and the log canonical modification exists over $Y \setminus \text{Supp } D$.

Hint of proof. For the first claim, see [OX12] or (4.57). For the second, we take a log resolution $(Y', E + \Delta') \rightarrow (Y, \Delta)$ and apply [HX13, HX16] (see also (11.28.2)) to $(Y', E + \Delta' - \epsilon p^* D) \rightarrow Y$. \square

DEFINITION 5.16 (Simultaneous canonical modification). Let $f : X \rightarrow S$ be a morphism of pure relative dimension n and $\Delta = \sum a_i D_i$ a generically \mathbb{Q} -Cartier effective divisor on Y . A *simultaneous canonical modification* is a proper morphism $p : (Y, \Delta^Y) \rightarrow (X, \Delta)$ such that $f \circ p : (Y, \Delta^Y) \rightarrow S$ is locally stable and

$$p_s : (Y_s, (\Delta^Y)_s) \rightarrow (X_s, \Delta_s)$$

is the canonical modification for every $s \in S$.

A *simultaneous log canonical modification* is defined analogously.

In the following result we definitely need to assume that the base scheme is seminormal; see (5.24) for some examples.

THEOREM 5.17 (Numerical criterion for simultaneous canonical modification).

We use the notation of (5.2). In addition to (5.2.1–4) assume that S is a seminormal scheme of char 0 and H is a relatively ample Cartier divisor class on X . For $s \in S$ let $\pi_s : (X_s^{\text{can}}, \Delta_s^{\text{can}}) \rightarrow (X_s, \Delta_s)$ denote the canonical modification of the fiber (X_s, Δ_s) . Then

(5.17.1) $s \mapsto I(\pi_s^* H_s, K_{X_s^{\text{can}}} + \Delta_s^{\text{can}})$ is a lexicographically lower semicontinuous function on S and

(5.17.2) $f : (X, \Delta) \rightarrow S$ has a simultaneous canonical modification iff this function is locally constant.

There is also a similar condition for simultaneous log canonical and semi-log-canonical modifications (5.45) but these only apply when $K_{X/S} + \Delta$ is \mathbb{Q} -Cartier.

5.3. Examples

Here we present a series of examples that show that the assumptions of the Theorems in Sections 5.1–5.2 are close to being optimal, except that the characteristic 0 assumption is probably superfluous.

Examples related to Theorems 5.3, 5.4 and 5.7.

The following is the simplest example illustrating the difference between being Cartier and fiber-wise Cartier.

EXAMPLE 5.18. Consider the family of quadrics

$$X = (x^2 - y^2 + z^2 - t^2 w^2 = 0) \subset \mathbb{P}_{xyzw}^3 \times \mathbb{A}_t \quad \text{and} \quad D = (x - y = z - tw = 0).$$

Here X_0 is a quadric cone and X_t is a smooth quadric for $t \neq 0$. The divisor D is Cartier, except at the origin, where it is not even \mathbb{Q} -Cartier. However D_0 is a line on a quadric cone, hence $2D_0 = (x - y = 0)$ is Cartier. It is easy to compute that

$$L = \mathcal{O}_X(-2D) = (x - y, z - tw)^2 \cdot \mathcal{O}_X$$

is locally free outside the origin, not locally free at the origin, but the hull of its restriction

$$L_0^H := \mathcal{O}_{X_0}(-2D_0) = (x - y) \cdot \mathcal{O}_{X_0}$$

is locally free. The natural restriction map gives an identification

$$\mathcal{O}_X(-2D)|_{X_0} = (x, y, z) \cdot \mathcal{O}_{X_0}(-2D_0) \subset \mathcal{O}_{X_0}(-2D_0).$$

Note that the self-intersection number of the fibers of D also jumps. For $t \neq 0$ we have $(D_t^2) = 0$, but $(D_0^2) = 1/2$.

It is harder to get examples where the self-intersections in (5.7) are locally constant, yet the divisor is not Cartier, but, as we see next, this can happen even for the canonical class. Thus in (5.7) one needs to assume that the fibers of f are S_2 and in (5.3) that the fibers are slc.

EXAMPLE 5.19. (See (2.35) or [Kol13b, 3.8–14] for the notation and basic results on cones.) Let $X \subset \mathbb{P}^N$ be a smooth, projective variety of dimension n and $L_X = \mathcal{O}_X(1)$. Let $C(X) := C_p(X, L_X)$ denote the projective cone over X with vertex v and natural ample line bundle $L_{C(X)}$. Let $H \subset X$ be a smooth hyperplane section and $C(H) := C_p(H, L_H)$ the projective cone over H . Note that

$$(L_X^n) = (L_{C(X)}^{n+1}) = (L_H^{n-1}) = (L_{C(H)}^n).$$

The canonical class of $C(X)$ is Cartier iff $K_X \sim mc_1(L_X)$ for some $m \in \mathbb{Z}$. In this case $K_{C(X)} \sim (m-1)c_1(L_{C(X)})$.

We can think of H as sitting in $X \subset C(X)$. The pencil of hyperplanes containing $H \subset C(X)$ gives a morphism of the blow-up $p : Y := B_H C(X) \rightarrow \mathbb{P}^1$ such that $Y_t \cong X$ for $t \neq 0$ and the normalization \bar{Y}_0 of Y_0 is isomorphic to $C(H)$. However, if $H^1(X, \mathcal{O}_X) \neq 0$ then Y_0 is not normal. For instance, this happens if X is the product of non-hyperelliptic curves of genus ≥ 2 with its canonical embedding. Thus, if these hold, then

- (5.19.1) Y_t is smooth and K_{Y_t} is ample for $t \neq 0$,
- (5.19.2) $K_{\bar{Y}_0}$ is locally free and ample,
- (5.19.3) the normalization $\bar{Y}_0 \rightarrow Y_0$ is an isomorphism except at v ,
- (5.19.4) $(K_{Y_t}^n) = (K_{\bar{Y}_0}^n)$ (where $n = \dim X$), yet
- (5.19.5) Y_0 is not normal.

This shows that (5.3) needs some assumptions about the singularities of the normalizations of the fibers. However, in this example K_Y is Cartier.

One can get another example where the canonical class of the total space is not Cartier as follows.

Note first that we can also obtain the family $Y \rightarrow \mathbb{P}^1$ by starting with $X \times \mathbb{P}^1$, blowing up $H \times \{0\}$ and contracting the birational transform of $X \times \{0\}$.

Assume next that $\text{Pic}(X)$ is positive dimensional. After a suitable base change ($c \in C$) \rightarrow ($0 \in \mathbb{P}^1$), there is a line bundle M_C on $X \times C$ that is trivial on $X \times \{c\}$ but only numerically trivial on $X \times \{c'\}$ for general $c' \in C$. After blowing up $H \times \{c\}$ and contracting the birational transform of $X \times \{c\}$ we get $(v \in Y) \rightarrow (c \in C)$ and a line bundle M on $Y \setminus \{v\}$ such that M is trivial on $Y_c \setminus \{v\}$ but only numerically trivial on $Y_{c'}$ for general $c' \in C$.

Let $Z \rightarrow Y$ be a double cover ramified along a general section of $M^2 \otimes p_1^* L_X^{2m}$ for $m \gg 1$. Then we get a morphism of a normal variety Z to a smooth curve $p_Z : Z \rightarrow C$ such that

- (5.19.6) Z_t is smooth and K_{Z_t} is ample for $t \neq 0$,
- (5.19.7) $K_{\bar{Z}_0}$ is locally free and ample,
- (5.19.8) the normalization $\bar{Z}_0 \rightarrow Z_0$ is an isomorphism except at a point v ,
- (5.19.9) $(K_{Z_t}^n) = (K_{\bar{Z}_0}^n)$, yet
- (5.19.10) K_Z is not Cartier at v .

Examples related to Theorems 5.10 and 5.11.

The next example shows that (5.10) fails if S is not seminormal.

EXAMPLE 5.20. Let S be a local, reduced but not seminormal scheme with seminormalization $S' \rightarrow S$. Choose an embedding of S' into the moduli space of automorphism-free curves of genus g for some g . Let $p' : X' \rightarrow S'$ be the resulting smooth family. This induces a family $p : X' \rightarrow S' \rightarrow S$ that satisfies the assumptions of (5.10). However, there is no simultaneous canonical model since $p' : X' \rightarrow S'$ does not descend to $p : X \rightarrow S$.

The next examples show that there does not seem to be a log version of (5.10) for families with reducible fibers, not even for families of curves.

EXAMPLE 5.21. Let $g : S \rightarrow C$ be a smooth family of curves and $D_i \subset S$ a set of n disjoint sections. Set $\Delta := \sum d_i D_i$. Pick a point $0 \in C$, the fiber over it is $(S_0, \sum d_i [p_i])$ where $p_i = S_0 \cap D_i$. The ‘log volume’ is $2g(S_0) - 2 + \sum d_i$.

Let $\pi : S^1 \rightarrow S$ be the blow up of all the points p_i with exceptional curves E_i and set $\Delta^1 := \pi_*^{-1} \Delta$. The central fiber of $g^1 : (S^1, \Delta^1) \rightarrow C$ is $(S_0^1, 0) + \sum_i (E_i, d_i [p_i])$. Its normalization consists of S_0 (with no boundary points) and $E_i \cong \mathbb{P}^1$, each with one marked point of multiplicity d_i . Thus the ‘log volume’ of the central fiber is now $2g(S_0) - 2$; the effect of the boundary vanished.

One can try to compensate for this, as in (5.31), by adding the double point divisor \bar{D}_0 . This variant of the ‘log volume’ is now $2g(S_0) - 2 + n$. This formula remembers only the number of the sections, not their coefficients. Even worse, we can blow up m other points on S_0 , then the ‘log volume’ formula gives $2g(S_0) - 2 + n + m$.

In general, there does not seem to be a sensible and birationally invariant way to define the ‘log volume’ of degenerations. For families of curves one can use the degree of the log canonical class; this gives negative contribution for some of the components. I do not know whether something similar can be done in higher dimension or not.

The next series of examples shows that, even for locally stable morphisms, the canonical models of the fibers need not form a flat family.

EXAMPLE 5.22. Let $f : X \rightarrow B$ be a locally stable family of surfaces. Assume for simplicity that the fibers have only quotient singularities.

Let $g : X \rightarrow Z$ be a flipping contraction. (For concrete examples, see [KM98, 2.7] or the exhaustive list in [KM92].) Thus there is a closed point $0 \in B$ such that g is an isomorphism over $B \setminus \{0\}$. Over the special point we have a birational contraction $g_0 : X_0 \rightarrow Z_0$ that contracts an irreducible curve $C \subset X_0$ to a point. Moreover $(C \cdot K_{X_0}) = (C \cdot K_X) < 0$, thus Z_0 is again log terminal and the contraction $g_0 : X_0 \rightarrow Z_0$ is a step in the MMP for X_0 .

However, since $g : X \rightarrow Z$ a flipping contraction, the special fiber of the flip $g^+ : X^+ \rightarrow Z$ is another surface $X_0^+ \rightarrow Z_0$ with a new exceptional curve $C^+ \subset X_0^+$ such that $(C^+ \cdot K_{X_0^+}) = (C^+ \cdot K_{X^+}) > 0$. Thus X_0^+ is not the canonical model of X_0 and $X_0 \dashrightarrow X_0^+$ is not even a correct step of the minimal model program.

It is easy to write down examples when $g^+ : X^+ \rightarrow Z$ is the canonical model of $g : X \rightarrow Z$; thus we get many examples without simultaneous canonical models.

In the above examples X_0 is log terminal but never canonical. There are further counter examples when (X, Δ) is canonical but $\Delta \neq 0$.

EXAMPLE 5.23. Set $Y := (xy + z^2 - s^2) \subset \mathbb{A}^4$ and $X := B_{(x,z-s)} Y$ with 4th projection $\pi : X \rightarrow \mathbb{A}_s^1$. The central fiber X_0 is the minimal resolution of the quadric cone $Y_0 := (xy + z^2) \subset \mathbb{A}^3$ with exceptional curve $E_0 \subset X_0$. Let D_1 be the birational transform of $(y = z + s = 0) \subset Y$. Note that D_1 is smooth but $D_1|_{X_0} = E_0 + L_0$ where L_0 denotes the birational transform of the line $L_Y := (y = z + s = 0)$. Thus $(X_0, \epsilon D_1|_{X_0})$ is canonical if $\epsilon \leq \frac{1}{2}$ and terminal if $\epsilon < \frac{1}{2}$. Furthermore,

$$(E_0 \cdot D_1)_X = (E_0 \cdot E_0)_{X_0} + (E_0 \cdot L_0)_{X_0} = -2 + 1 = -1.$$

For any $0 < \epsilon \leq 1$ the canonical model of $(X, \epsilon D_1) \rightarrow Y$ is given by the flop $X^+ := B_{(x,z+s)} Y$. Note that $X_0^+ \cong X_0$ and under this isomorphism $D_1^+|_{X_0^+} = L_0$. Thus E_0 is not contained in $\text{Supp } D_1^+$.

Therefore $(X_0^+, \epsilon D_1^+|_{X_0^+})$ is its own canonical model, but the canonical model of $(X_0, \epsilon D_1|_{X_0})$ is $(Y_0, \epsilon L_Y)$.

We see that the map $X_0 \dashrightarrow X_0^+$ is the identity, the problem is the unexpected change in the boundary divisor D_1 .

One can obtain from the above local example a global one as follows. Compactify Y as

$$\bar{Y} := (xy + z^2 - t^2s^2) \subset \mathbb{P}_{xyzt}^3 \times \mathbb{A}_s^1$$

with $\bar{\pi} : \bar{Y} \rightarrow \mathbb{A}_s^1$ the projection. Set $\bar{X} := B_{(x,z-ts)}\bar{Y}$ and let \bar{D}_1 be the birational transform of $(y = z + ts = 0) \subset \bar{Y}$. Let $\bar{D}_2, \dots, \bar{D}_5$ denote the pull-back of 4 general hyperplanes in \mathbb{P}^3 . Fix $0 < \epsilon < \frac{1}{8}$ and consider

$$(\bar{X}, \bar{\Delta} := 4\epsilon D_1 + (\frac{1}{2} - \frac{\epsilon}{4})(\bar{D}_2 + \dots + \bar{D}_5)).$$

Every fiber of $\bar{X} \rightarrow \mathbb{A}_s^1$ is terminal.

The central fiber is the minimal resolution of the quadric cone. Since the pull-back of the hyperplane class is $E_0 + 2L_0$, the boundary divisor $\bar{\Delta}_0$ is linearly equivalent to

$$4\epsilon(E_0 + L_0) + (2 - \epsilon)(E_0 + 2L_0) = (2E_0 + 4L_0) + 2\epsilon E_0 + \epsilon(E_0 + 2L_0).$$

The canonical class of the quadric is $-2(\text{hyperplane class})$, thus we get that

$$K_{\bar{X}_0} + \bar{\Delta}_0 \sim 2\epsilon E_0 + \epsilon(E_0 + 2L_0),$$

thus $(\bar{X}_0, \bar{\Delta}_0)$ is of general type.

The general fiber is a smooth quadric; choose the two families of lines A, B such that D_1 restricts to A . Then the boundary divisor $\bar{\Delta}_g$ is linearly equivalent to

$$4\epsilon A + (2 - \epsilon)(A + B) = 2(A + B) + 2\epsilon A - 2\epsilon B.$$

Therefore

$$K_{\bar{X}_g} + \bar{\Delta}_g \sim 2\epsilon A - 2\epsilon B,$$

hence its Kodaira dimension is $-\infty$.

Examples related to Theorem 5.17.

In (5.17) the base scheme is assumed to be seminormal. The reason for this is that canonical modifications do have unexpected infinitesimal deformations.

EXAMPLE 5.24 (Deformation of canonical modifications). We give an example of a normal, projective variety with isolated singularities and canonical modification $X^c \rightarrow X$ such that the trivial deformation of X can be lifted to a nontrivial deformation of X^c .

Consider the isolated hypersurface singularity

$$X := X_{n,r} := (x_1^r + \dots + x_n^r + x_{n+1}^{r+1} = 0) \subset \mathbb{A}_k^{n+1}.$$

Let $p : Y := B_0X \rightarrow X$ denote the blow-up of the origin. Then Y is smooth, the exceptional divisor is the cone $E \cong (x_1^r + \dots + x_n^r = 0) \subset \mathbb{P}^n$ and $N_{E|Y} \cong \mathcal{O}_E(-1)$. We compute that $a(E, X_{n,r}, 0) = n - r$. Thus $X_{n,r}$ is canonical iff $r \leq n$ and Y is the canonical modification for $r > n$.

We claim that $p : Y \rightarrow X$ has a nontrivial deformation over $X \times_k \text{Spec } k[\epsilon]$. The trivial deformation is obtained by blowing up

$$(x_1 = \dots = x_{n+1} = 0) \subset X \times_k \text{Spec } k[\epsilon].$$

The nontrivial deformation is obtained by blowing up

$$Z := (x_1 = \cdots = x_n = x_{n+1} - \epsilon = 0) \subset X \times_k \text{Spec } k[\epsilon].$$

We need to check that X is equimultiple along the blow-up center. This is more transparent if we introduce a new coordinate $y := x_{n+1} - \epsilon$. Then the equations become

$$Z := (x_1 = \cdots = x_n = y = 0) \subset (x_1^r + \cdots + x_n^r + y^{r+1} + (r+1)\epsilon y^r = 0),$$

thus $X \times_k \text{Spec } k[\epsilon]$ is clearly equimultiple along Z .

Note that $E \subset Y$ has a unique extension E_ϵ to a deformation Y_ϵ of Y since $H^1(E, N_{E|Y}) = 0$. The blow-up ideal is then the push-forward of the ideal sheaf of E_ϵ . Thus different blow-up ideals give different deformations of Y .

The following examples show that the existence of simultaneous canonical modifications is more complicated for pairs.

EXAMPLE 5.25. In \mathbb{P}^2 consider a line $L \subset \mathbb{P}^2$ and a family of degree 8 curves C_t such that C_0 has 4 nodes on L plus an ordinary 6-fold point outside L and C_t is smooth and tangent to L at 4 points for $t \neq 0$.

Let $\pi_t : S_t \rightarrow \mathbb{P}^2$ denote the double cover of \mathbb{P}^2 ramified along C_t . Note that $K_{S_t} = \pi_t^* \mathcal{O}(1)$, thus $(K_{S_t}^2) = 2$. For each t , the preimage $\pi_t^{-1}(L)$ is a union of 2 curves $D_t + D'_t$. Our example is the family of pairs (S_t, D_t) . We claim that

(5.25.1) there is a log canonical modification $(S_t^{\text{lc}}, D_t^{\text{lc}}) \rightarrow (S_t, D_t)$ for every t and

(5.25.2) $(K_{S_t^{\text{lc}}} + D_t^{\text{lc}})^2 = 1$ for every t yet

(5.25.3) there is no simultaneous log canonical modification.

If $t \neq 0$ then S_t is smooth and D_t is smooth. Furthermore D_t, D'_t meet transversally at 4 points, thus $(D_t \cdot D'_t) = 4$. Using $((D_t + D'_t)^2) = 2$, we obtain that $(D_t^2) = -3$. Thus $(K_{S_t} + D_t)^2 = 1$.

If $t = 0$ then S_0 is singular at 5 points. D_0, D'_0 meet transversally at 4 singular points of type A_1 , thus $(D_0 \cdot D'_0) = 2$. This gives that $(D_0^2) = -1$. Thus $(K_{S_0} + D_0)^2 = 3$. The pair (S_0, D_0) is lc away from the preimage of the 6-fold point. Let $q : T_0 \rightarrow S_0$ denote the minimal resolution of this point. The exceptional curve E is smooth, has genus 2 and $(E^2) = -2$. Thus $K_{T_0} = q^* K_{S_0} - 2E$ hence $(T_0, E + D_0)$ is the log canonical modification of (S_0, D_0) and

$$(K_{T_0} + E + D_0)^2 = (q^* K_{S_0} - E + D_0)^2 = (K_{S_0} + D_0)^2 + (E^2) = 1.$$

Thus $(K_{S_t^{\text{lc}}} + D_t^{\text{lc}})^2 = 1$ for every t .

Nonetheless, the log canonical modifications do not form a flat family. Indeed, such a family would be a family of surfaces with ordinary nodes, so the relative canonical class would be a Cartier divisor. However, $(K_{S_t}^2) = 2$ for $t \neq 0$ but $(K_{T_0}^2) = (q^* K_{S_0} - 2E)^2 = -6$.

EXAMPLE 5.26. We start with a family of quadric surfaces $Q_t \subset \mathbb{P}^3$ where Q_0 is a cone and Q_t is smooth for $t \neq 0$. We take 6 families of lines L_t^i such that for $t = 0$ we have 6 distinct lines and for $t \neq 0$ two of them L_t^1, L_t^2 are from one ruling of the quadric, the other 4 from the other ruling.

Finally S_t denotes the double cover of Q_t ramified along the 6 lines $L_t^1 + \cdots + L_t^6$. For $t \neq 0$ the surface S_t has ordinary nodes and $(K_{S_t}^2) = 0$.

For $t = 0$ the surface S_0 has a unique singular point. Its minimal resolution $q : T_0 \rightarrow S_0$ is a double cover of \mathbb{F}_2 ramified along 6 fibers. Thus $(K_{T_0}^2) = -4$. Thus the canonical modifications do not form a flat family. The log canonical modification of S_0 is (T_0, E_0) where E_0 is the q -exceptional curve. Thus $(K_{T_0} + E_0)^2 = 0$.

The numerical condition is satisfied but the log canonical modifications do not form a flat family since $T_0 = S_0^{\text{lc}}$ is smooth but $S_t^{\text{lc}} = S_t$ is singular for $t \neq 0$.

However, there is a flat family that is a weaker variant of a simultaneous log canonical modification.

This is obtained by replacing the singular quadric Q_0 with its resolution $Q'_0 \cong \mathbb{F}_2$. Let $E \subset \mathbb{F}_2$ denote the -2 -section and $|F|$ the ruling. One can arrange that L_t^1, L_t^2 degenerate to $F^i + E$ for $F^i \in |F|$ and the others degenerate to fibers F^j . This way a flat limit of the double covers S_t is obtained as the double cover of \mathbb{F}_2 ramified along $F^1 + \cdots + F^6 + 2E$. This is a semi-log-canonical surface whose normalization is the log canonical modification of S_0 .

5.4. Flatness and stability criteria in codimension 1

As a preliminary step we characterize those morphisms that are locally stable in codimension ≤ 1 on each fiber. In applications this is rarely an issue, but it is instructive to see which arguments work or fail.

EXAMPLE 5.27. Let $f : X \rightarrow C$ be a projective morphism from a normal variety of dimension $n + 1$ to a smooth curve and H a relatively ample divisor class on X . We would like to understand when f is stable in terms of numerical invariants of the normalizations of the fibers $\pi_c : \tilde{X}_c \rightarrow X_c$. The simplest invariant is the self-intersection number $(\pi_c^* H)^n$ which describes the codimension 0 behavior of f . It is clear that $c \mapsto (\pi_c^* H)^n$ is a

(5.27.1) lower semicontinuous function on C and

(5.27.2) it is locally constant iff the fibers are generically reduced.

The following result is a reformulation of [Kol96, I.6.5].

THEOREM 5.28 (Smoothness criterion in codimension 0). *Let S be a weakly normal scheme, $f : X \rightarrow S$ a projective morphism of pure relative dimension n (2.72) and H an f -ample divisor class. Assume that X is reduced and for $s \in S$ let $\pi_s : \tilde{X}_s \rightarrow X_s$ denote the normalization of the fiber. Then $s \mapsto (\pi_s^* H)^n$ is a lower semicontinuous function on S , and it is locally constant iff there is a closed subset $Z_1 \subset X$ such that*

(5.28.1) $\dim_s(Z_1 \cap X_s) \leq n - 1$ for every $s \in S$ and

(5.28.2) $f : (X \setminus Z_1) \rightarrow S$ is smooth. □

Similarly, the next results says that the codimension 1 behavior of a morphism is described by the degree and the sectional genus (3.10) of its fibers.

PROPOSITION 5.29. *Let X, S be reduced schemes and $f : X \rightarrow S$ a projective morphism of pure relative dimension 1 with generically geometrically reduced fibers. Then*

(5.29.1) *the function $s \mapsto \chi(\text{red } X_s)$ is constructible, lower semicontinuous, and*

(5.29.2) *f is flat with reduced fibers iff this function locally constant on S .*

Proof. Assume that f is flat. Then $s \mapsto \chi(X_s)$ is locally constant. Since X is reduced, so are the generic fibers. Thus f has reduced fibers iff $s \mapsto \chi(\text{red } X_s)$ is

locally constant. These also show both claims when S is regular and of dimension 1. By generic flatness and Noetherian induction we also see that $s \mapsto \chi(\text{red } X_s)$ is constructible. Then lower semicontinuity can be checked over DVR's, and this was already done.

Assume next that $s \mapsto \chi(\text{red } X_s)$ is locally constant. Note that f is flat over S at every generic point of every fiber X_s by (5.28) or by (10.48). The rest is a special case of (9.71). \square

The following example illustrates the necessity of the assumptions in (5.28) and (5.29).

EXAMPLE 5.30. Set $S := (uv = 0) \subset \mathbb{A}^2$ and let X be the union of

$$X_u := (xy + uz^2 = 0) \subset \mathbb{P}_{xyz}^2 \times \mathbb{A}_u^1 \quad \text{and} \quad X_v := (x^2 + vyz = 0) \subset \mathbb{P}_{xyz}^2 \times \mathbb{A}_v^1.$$

Let $\pi : X \rightarrow S$ be the coordinate projection. Then X, S are weakly normal and S_2 , π has relative dimension 1 and $(\deg(\text{red } X_s), \chi(\text{red } X_s)) = (2, 1)$ for every $s \in S$, yet π is not flat.

The fiber over $(0, 0)$ is not generically reduced and π does not have pure relative dimension 1, so (5.28) and (5.29) are not contradicted.

In order to characterize stability, consider a normal surface S and a flat, proper family of curves $f : S \rightarrow T$ where T is the spectrum of a DVR with closed point 0 and generic point t . In order to describe stability using the normalization of the fibers, we need to take the singularities of the central fiber into account. In the stable case the correct formula adds 1 for each point on \bar{S}_0 whose image is a singular point of the fiber S_0 . So let us define the divisor of singularities $\bar{D}_0 \subset \bar{S}_0$ as the sum of all the points $p \in \bar{S}_0$ such that $\text{red } S_0$ is singular at $\pi_0(p)$. Let us say that the fiber S_0 is *pre-stable* if $K_{\bar{S}_0} + \bar{D}_0$ is ample. We have the following stability criterion.

LEMMA 5.31. *Let S be a normal surface, $(0, T)$ the spectrum of a DVR and $f : S \rightarrow T$ a proper morphism with pre-stable central fiber. Then*

$$(5.31.1) \quad \deg(K_{\bar{S}_0} + \bar{D}_0) \leq \deg(K_{S_t}) \quad \text{and}$$

$$(5.31.2) \quad f : S \rightarrow T \text{ is stable iff equality holds.}$$

Proof. In order to prove these write $S_0 = \sum e_i E_i$ with normalizations $\pi_i : F_i \rightarrow E_i$. The key is to understand that we have to work with the divisor $K_S + \sum E_i$. We use intersection numbers as in (11.47).

Since $\sum E_i$ is disjoint from the generic fiber,

$$\begin{aligned} \deg(K_{S_t}) &= (K_S \cdot S_t) = ((K_S + \sum E_i) \cdot S_t) \\ &= \sum_j e_j ((K_S + \sum E_i) \cdot E_j) \\ &= \sum_j e_j \deg \pi_j^*(K_S + \sum E_i). \end{aligned} \tag{5.31.3}$$

As in (11.17) we can write

$$\pi_j^*(K_S + \sum E_i) = K_{F_j} + \text{Diff}_{F_j}(\sum_{i \neq j} E_i).$$

Using (11.19) we obtain that

$$\deg \pi_j^*(K_S + \sum E_i) \geq \deg(K_{F_j} + \bar{D}_j) \tag{5.31.4}$$

and equality holds iff all singularities of $\text{red } S_0$ are ordinary nodes. We can thus continue the inequalities (5.31.3) to get that

$$\begin{aligned} \deg(K_{S_t}) &= \sum_j e_j \deg \pi_j^*(K_S + \sum E_i) \\ &\geq \sum_j e_j \deg(K_{F_j} + \bar{D}_j) \\ &\geq \sum_j \deg(K_{F_j} + \bar{D}_j) \\ &= \deg(K_{\bar{S}_0} + \bar{D}_0), \end{aligned} \tag{5.31.5}$$

and the first inequality is an equality iff all the singularities of $\text{red } S_0$ are ordinary nodes and the second inequality is an equality iff S_0 is reduced. \square

Although (5.31) is promising, the normality assumption on S makes dimension induction difficult and the following example shows that its natural analog fails if T is replaced by a nodal curve.

EXAMPLE 5.32. Let $p(x)$ be a polynomial of degree $2d$ without multiple roots and pick $a_1 \neq a_2$ that are not roots of p . Let C_i denote the compactification (smooth at infinity) of the singular hyperelliptic curve ($y^2 = (x - a_i)^4 p(x)$) and C_0 the compactification of ($y^2 = (x - a_1)^4 (x - a_2)^4 p(x)$). (Thus C_i has a tacnode at $x = a_i$ for $i = 1, 2$, C_0 has 2 tacnodes and they are smooth elsewhere.) For $i = 1, 2$ let $\pi'_i : S'_i \rightarrow \mathbb{A}_{t_i}^1$ be a general smoothing of C_i . The generic fiber has genus $d+1$. There are natural finite maps $C_i \rightarrow C_0$ that pinch the points $(a_{3-i}, \pm(a_i - a_{3-i})^2 \sqrt{p(a_i)})$ together. Doing the same pinching on S'_i we get surfaces $\pi_i : S_i \rightarrow \mathbb{A}_{t_i}^1$ where the central fiber is C_0 (plus some embedded points). We can identify the reduced central fibers to get a reducible surface $S = S_1 \amalg_{C_0} S_2$ and a morphism $\pi : S \rightarrow (t_1 t_2 = 0) \subset \mathbb{A}_{t_1 t_2}^2$. Note that although the S_i are not seminormal, S itself is seminormal and satisfies Serre's condition S_2 .

The normalization \bar{C}_0 of C_0 has genus $d-1$. The divisor of singularities consists of the 4 preimages of the points $(x = a_i)$. Thus for $t \neq (0, 0)$ we have

$$\deg(K_{S_t}) = 2d \quad \text{and} \quad \deg(K_{\bar{S}_{(0,0)}} + \bar{D}_{(0,0)}) = 2d - 4 + 4 = 2d.$$

Hence the numerical stability criterion of (5.31) does not extend to seminormal, S_2 surfaces over nodal curves.

A satisfactory analog of (5.31) over higher dimensional normal bases is proved in [Koll1b, 14–15].

We can thus expect that, for families that are locally stable in codimension 1, there are results connecting the intersection numbers $((\pi_0^* H)^{n-i} \cdot (K_{\bar{X}_0} + \bar{D}_0)^i)$ with the higher codimension behavior of f . There are two surprising twists.

- The *lower* semicontinuity in (5.28) and (5.31) switches to *upper* semicontinuity.
- In most cases we need only *one* more intersection number to take care of all codimensions.

5.5. Deformations of slc pairs

So far we have focused on locally stable deformations of slc pairs. The next result, due to [KSB88], connects arbitrary flat deformations (X_t, Δ_t) of an slc pair (X_0, Δ_0) to locally stable deformations of a suitable birational model $f_0 : (Y_0, \Delta_0^Y) \rightarrow (X_0, \Delta_0)$. We then compare various numerical invariants of (X_0, Δ_0) and of (X_t, Δ_t) by going through (Y_0, Δ_0^Y) . This implies a weaker version of (5.4).

THEOREM 5.33. [KSB88] *Let (X, Δ) be a normal pair and $g : X \rightarrow C$ a flat morphism of pure relative dimension n to a smooth pointed curve ($0 \in C$) such that X_0 is nodal in codimension 1 and its normalization $(\bar{X}_0, \text{Diff}_{\bar{X}_0} \Delta)$ is lc.*

Assume that the log canonical modification $f : (Y, \Delta^Y + Y_0 + E) \rightarrow (X, X_0 + \Delta)$ exists¹, where $\Delta^Y + Y_0$ is the birational transform of $\Delta + X_0$ and E is f -exceptional.

Then, possibly after shrinking ($0 \in C$), the following hold.

(5.33.1) *f is small (that is, $E = 0$) and $f(\text{Ex}(f))$ is precisely the locus where g is not locally stable.*

(5.33.2) *$g \circ f : (Y, \Delta^Y) \rightarrow C$ is locally stable.*

(5.33.3) *For every f_0 -exceptional divisor $F \subset Y_0$, the divisor Y_0 is normal at the generic point of F and $a(F, \bar{X}_0, \text{Diff}_{\bar{X}_0} \Delta) < 0$.*

Furthermore, if $K_X + \Delta$ is \mathbb{R} -Cartier on the generic fiber, then

(5.33.4) *the log canonical modification f is known to exist,*

(5.33.5) *f is an isomorphism over $C \setminus \{0\}$, and*

(5.33.6) *g is locally stable over $C \setminus \{0\}$.*

Proof. Let $\pi_X : \bar{X}_0 \rightarrow X_0$ and $\pi_Y : \bar{Y}_0 \rightarrow Y_0$ be the normalizations. Then f_0 lifts to $\bar{f}_0 : \bar{Y}_0 \rightarrow \bar{X}_0$. Write $K_{\bar{Y}_0} + \Delta_{\bar{Y}_0} \sim_{\mathbb{Q}} \bar{f}_0^*(K_{\bar{X}_0} + \text{Diff}_{\bar{X}_0} \Delta)$. By adjunction,

$$\begin{aligned} \pi_Y^*(K_Y + \Delta^Y + Y_0 + E) &\sim_{\mathbb{Q}} K_{\bar{Y}_0} + \text{Diff}_{\bar{Y}_0}(\Delta^Y + E) \\ &\sim_{\mathbb{Q}} \bar{f}_0^*(K_{\bar{X}_0} + \text{Diff}_{\bar{X}_0} \Delta) + (\text{Diff}_{\bar{Y}_0}(\Delta^Y + E) - \Delta_{\bar{Y}_0}). \end{aligned}$$

Since X_0 has only nodes at codimension 1 points, X is canonical at codimension 1 points of X_0 (11.9) and f is an isomorphism near these points. Thus $\text{Diff}_{\bar{Y}_0}(\Delta^Y + E) - \Delta_{\bar{Y}_0}$ is \bar{f}_0 -exceptional and \bar{f}_0 -ample. By (11.50) this implies that every \bar{f}_0 -exceptional divisor appears in $\text{Diff}_{\bar{Y}_0}(\Delta^Y + E) - \Delta_{\bar{Y}_0}$ with strictly negative coefficient.

Every divisor in $Y_0 \cap E$ appears in $\text{Diff}_{\bar{Y}_0}(\Delta^Y + E)$ with coefficient ≥ 1 by (11.19). On the other hand, since $(\bar{X}_0, \text{Diff}_{\bar{X}_0} \Delta)$ is lc by assumption, every exceptional divisor appears in $\Delta_{\bar{Y}_0}$ with coefficient ≤ 1 . Thus $Y_0 \cap E = \emptyset$ and, possibly after shrinking C , there are no exceptional divisors in $f : Y \rightarrow X$, hence f is small.

Thus Y_0 is a complete fiber of $g \circ f : Y \rightarrow C$. Since $(Y, \Delta^Y + Y_0)$ is lc, this implies that $g \circ f : (Y, \Delta^Y) \rightarrow C$ is locally stable in a neighborhood of Y_0 by (2.4).

Let $\bar{F} \subset \bar{Y}_0$ be any \bar{f}_0 -exceptional divisor. Since it appears in $\text{Diff}_{\bar{Y}_0}(\Delta^Y) - \Delta_{\bar{Y}_0}$ with negative coefficient, it must appear in $\Delta_{\bar{Y}_0}$ with positive coefficient and in $\text{Diff}_{\bar{Y}_0}(\Delta^Y)$ with coefficient < 1 . By (11.19) the latter implies that Y_0 is smooth at the generic point of $\pi_Y(\bar{F})$, proving (3).

If $K_X + \Delta$ is \mathbb{R} -Cartier then we apply (5.15.2) with $D = X_0$ to get (4).

Finally let $x \in X \setminus X_0$ be a point where $K_X + \Delta$ is \mathbb{Q} -Cartier. Since f is small, $K_Y + \Delta^Y \sim_{\mathbb{Q}} f^*(K_X + \Delta)$ over a neighborhood of x . Since $K_Y + \Delta^Y$ is f -ample, f is an isomorphism over a neighborhood of x . This proves the second assertion in (1) and also completes (5-6). \square

5.34 (Proof of (5.3)). We prove (5.3) when the base S is the spectrum of a DVR. By (4.48), this implies the case when S is higher dimensional, provided f is assumed to be flat with S_2 fibers.

¹This is conjecturally automatic; see (5.15).

As a preliminary step, we replace (X, Δ) by its normalization. This leaves the assumptions and the numerical conclusion unchanged. By (2.55), a demi-normal pair $(X, \Delta) \rightarrow C$ with slc generic fibers is slc iff its normalization is lc. Thus the conclusion is also unchanged.

Thus assume that X is normal. The conclusions are local on C , so pick a point $0 \in C$ and let $f : (Y, \Delta^Y + Y_0) \rightarrow (X, X_0 + \Delta)$ be the log canonical modification as in (5.33). Let $\pi_Y : \bar{Y}_0 \rightarrow Y_0$ be the normalization and $\bar{f}_0 : \bar{Y}_0 \rightarrow \bar{X}_0$ the induced birational morphism. We apply (10.31.3–4) to

$$D_Y := K_{\bar{Y}_0} + \text{Diff}_{\bar{Y}_0} \Delta^Y \quad \text{and} \quad D_X := K_{\bar{X}_0} + \text{Diff}_{\bar{X}_0} \Delta = K_{\bar{X}_0} + \bar{D}_0 + \bar{\Delta}_0.$$

The assumptions are satisfied since

$$(\bar{f}_0)_*(K_{\bar{Y}_0} + \text{Diff}_{\bar{Y}_0} \Delta^Y) = K_{\bar{X}_0} + \text{Diff}_{\bar{X}_0} \Delta$$

and $K_{\bar{Y}_0} + \text{Diff}_{\bar{Y}_0} \Delta^Y$ is \bar{f}_0 -ample. Using the volume of divisors (10.30), this implies that

$$(K_{\bar{X}_0} + \text{Diff}_{\bar{X}_0} \Delta)^n = \text{vol}(K_{\bar{X}_0} + \text{Diff}_{\bar{X}_0} \Delta) \geq \text{vol}(K_{\bar{Y}_0} + \text{Diff}_{\bar{Y}_0} \Delta^Y),$$

and equality holds iff \bar{f}_0 is an isomorphism. Furthermore, since $K_Y + \Delta^Y$ is \mathbb{Q} -Cartier,

$$\text{vol}(K_{\bar{Y}_0} + \text{Diff}_{\bar{Y}_0} \Delta^Y) \geq \text{vol}(K_{\bar{Y}_c} + \Delta^Y|_{\bar{Y}_c}) = (K_{\bar{Y}_c} + \bar{\Delta}_c)^n$$

for general $c \neq 0$ and $(\bar{Y}_c, \bar{\Delta}_c) = (X_c, \Delta_c)$ by (5.33.4). Combining the inequalities shows that

$$(K_{\bar{X}_0} + \bar{D}_0 + \bar{\Delta}_0)^n \geq (K_{\bar{X}_c} + \Delta_c)^n \quad \text{for general } c \neq 0$$

and equality holds iff \bar{f}_0 , and hence f , are isomorphisms over $0 \in C$. \square

The same method can be used to prove a weaker version of the numerical criterion of local stability over smooth curves. This establishes (5.4) for families of surfaces over a smooth curve. I do not know how to use these methods to complete the proof of (5.4) for higher dimensional families. We will derive (5.4) from (5.7) instead; see (5.51) for the key step.

PROPOSITION 5.35 (Weak numerical criterion of local stability). *Let C be a smooth curve of char 0 and $f : (X, \Delta) \rightarrow C$ a morphism satisfying the assumptions (5.4.1–3). Then*

- (5.35.1) $c \mapsto I(\pi_c^* H, K_{\bar{X}_c} + \bar{D}_c + \bar{\Delta}_c)$ is lexicographically upper semicontinuous and
(5.35.2) $f : (X, \Delta) \rightarrow C$ is locally stable iff the above function is locally constant.

Note that the first two numbers in the sequence $I(\pi_c^* H, K_{\bar{X}_c} + \bar{D}_c + \bar{\Delta}_c)$ equal $(H^n \cdot X_c)$ and $(H^{n-1} \cdot (K_X + \Delta) \cdot X_c)$, hence they are always locally constant. The first interesting number is $(\pi_c^* H^{n-2} \cdot (K_{\bar{X}_c} + \bar{D}_c + \bar{\Delta}_c)^2)$ which is thus an upper semicontinuous function on C by (1).

Proof. As in (5.34) we may assume that X is normal. Let $f : (Y, \Delta^Y + Y_0) \rightarrow (X, X_0 + \Delta)$ be the log canonical modification and $\bar{f}_0 : \bar{X}_0 \rightarrow \bar{Y}_0$ the induced birational morphism between the normalizations. Here we apply (10.31.1–2) to $K_{\bar{Y}_0} + \text{Diff}_{\bar{Y}_0} \Delta^Y$ and $K_{\bar{X}_0} + \text{Diff}_{\bar{X}_0} \Delta$ to obtain that

$$I(\pi_0^* H, K_{\bar{X}_0} + \text{Diff}_{\bar{X}_0} \Delta) \succeq I(\bar{f}_0^* \pi_0^* H, K_{\bar{Y}_0} + \text{Diff}_{\bar{Y}_0} \Delta^Y),$$

and equality holds iff \bar{f}_0 is an isomorphism. Since $K_Y + \Delta^Y$ is a \mathbb{Q} -Cartier divisor,

$$I(\bar{f}_0^* \pi_0^* H, K_{\bar{Y}_0} + \text{Diff}_{\bar{Y}_0} \Delta^Y) = I(\pi_c^* H, K_{\bar{Y}_c} + \Delta^Y|_{\bar{Y}_c}) = I(\pi_c^* H, K_{\bar{X}_c} + \bar{\Delta}_c)$$

for general $c \neq 0$. Thus

$$I(\pi_0^* H, K_{\bar{X}_0} + \bar{D}_0 + \bar{\Delta}_0) \succeq I(\pi_c^* H, K_{\bar{X}_c} + \bar{\Delta}_c) \quad \text{for general } c \neq 0$$

and equality holds iff \bar{f}_0 , and hence f , are isomorphisms. \square

5.36 (Start of the proof of (5.5)). We prove (5.5) when the base S is the spectrum of a DVR. By (4.48), this implies the case when S is higher dimensional, provided f is assumed to be flat with S_2 fibers.

As in (5.34) we may assume that X is normal. Thus, for suitable $m > 0$ we have $U := X \setminus \{x\}$ and a line bundle $L := \mathcal{O}_U(mK_U + m\Delta)$ whose restriction to $U_D := U \cap X_0 = X_0 \setminus \{x\}$ is trivial. If f is flat with S_2 fibers then $\text{depth}_x \mathcal{O}_X \geq 3$ and we can apply (2.92) to conclude.

Even if X_0 is not S_2 , we are in the situation studied in (5.33), hence there is a proper, birational, small morphism $f : Y \rightarrow X$ such that $K_Y + \Delta^Y$ is \mathbb{Q} -Cartier and f -ample.

For the rest of the argument it is not important that we are dealing with $K_X + \Delta$. Thus, for suitable $m > 0$ we have an f -ample line bundle $M := \mathcal{O}_X(mK_Y + m\Delta^Y)$ on Y such that $M|_{f^{-1}(U)} \cong L$. The key new additional information is that $\dim f^{-1}(x) \leq \dim X - 2$.

Next we use reduction modulo p as in (2.92) but we have to keep track of $f : Y \rightarrow X$ as well. In our case, in addition to (2.92.1–4) we also have a proper, birational morphism $f^T : Y^T \rightarrow X^T$ that is an isomorphism over U^T and an f^T -ample line bundle M^T such that $M^T|_{U^T} \cong L^T$ and $\dim(f_p^T)^{-1}(x_p) = \dim f^{-1}(x)$.

As before, (2.91) shows that $(L_p^T)^m \cong \mathcal{O}_{U_p^T}$ for some $m > 0$. Thus $(M_p^T)^m$ and $\mathcal{O}_{Y_p^T} \cong ((f_p^T)^* \mathcal{O}_{X_p^T})^m$ are 2 invertible sheaves on Y_p^T that are isomorphic over the open subset $Y_p^T \setminus (f_p^T)^{-1}(x_p)$. If $(f_p^T)^{-1}(x_p)$ has codimension ≥ 2 then $(M_p^T)^m \cong \mathcal{O}_{Y_p^T}$. Since M_p^T is f_p^T -ample, this is only possible if f_p^T is an isomorphism. Then f^T and hence f are also isomorphisms and so $M^m \cong \mathcal{O}_Y$ shows that $L^m \cong \mathcal{O}_U$. \square

5.6. Simultaneous canonical models

In this section we consider the existence of simultaneous canonical models.

5.37 (Proof of (5.10) over curves). Let B be a smooth curve of char 0 and $f : X \rightarrow B$ a morphism of pure relative dimension n .

First we prove that $b \mapsto \text{vol}(K_{X_b^r})$ is a lower semicontinuous function on B .

If we replace X by a resolution $X^r \rightarrow X$ then $\text{vol}(K_{X_b^r})$ is unchanged for general fibers and it can only increase for special fibers. There are two possible sources for an increase. First, the resolution may introduce new divisors of general type. Second, if X is not normal, an irreducible component of a fiber may be replaced by a finite cover of it. The latter increases the volume by (10.37).

Thus it is enough to check lower semicontinuity when X is smooth and all fibers are snc. If the volume of the general fiber is 0 then the volume of every fiber is 0 by (5.38), hence we may assume that general fibers are of general type.

Let F be the union of all singular fibers and $f^c : X^c \rightarrow B$ the relative canonical model of $(X, \text{red } F) \rightarrow B$ as in (2.58.2). An irreducible component $E \subset F$ may get contracted. However, when this happens, then $K_E + (\text{red } F - E)|_E = (K_X +$

$\text{red } F)|_E$, and hence also K_E , are ≤ 0 on the fibers of the contraction. Such divisors contribute 0 to the volume. Thus we can check lower semicontinuity on $f^c : X^c \rightarrow B$.

Pick $b \in B$, let $\sum e_i E_i := X_b^c$ denote the fiber over b and $\pi_i : \bar{E}_i \rightarrow E_i$ the normalizations. As in (11.17) write $\pi_i^*(K_{X^c} + \text{red } F^c) = K_{\bar{E}_i} + \bar{D}_i$ where $\bar{D}_i = \text{Diff}_{\bar{E}_i}(\sum_{j \neq i} E_j)$. Let $g \in B$ be a point not contained in $f^c(F)$. Then F^c is disjoint from X_g^c and we have

$$\begin{aligned} (K_{X_g^c})^n &= ((K_{X^c} + \text{red } F^c)^n \cdot X_g^c) \\ &= ((K_{X^c} + \text{red } F^c)^n \cdot X_b^c) \\ &= \sum_i e_i (K_{\bar{E}_i} + \bar{D}_i)^n \geq \sum_i (K_{\bar{E}_i} + \bar{D}_i)^n. \end{aligned} \quad (5.37.1)$$

Next we use (5.12) to obtain that $(K_{\bar{E}_i} + \bar{D}_i)^n \geq (K_{E_i^{\text{cr}}})^n$. Putting these together we see that

$$\text{vol}(K_{X_r}) = (K_{X_g^c})^n \geq \sum_i (K_{E_i^{\text{cr}}})^n = \text{vol}(K_{X_b^r}),$$

proving the lower semicontinuity assertion. Furthermore, by (5.12), equality holds iff $D_i = 0$, the E_i have canonical singularities and $e_i = 1$ for every i . If $D_i = 0$ then E_i is the only irreducible component of its fiber by (11.19). Thus X_b^c is reduced, irreducible and has canonical singularities. So $f^c : X^c \rightarrow B$ is the simultaneous canonical model of $f : X \rightarrow B$. \square

LEMMA 5.38. *Let $f : X \rightarrow B$ be a projective morphism to a smooth curve B such that $\text{vol}(K_{X_r})$ is zero for the generic fiber X_g . Then $\text{vol}(K_{X_b^r})$ is zero for every $b \in B$.*

Proof. The proof in (5.37) gives this if a resolution of X has a minimal model over B . This is not fully known, so we have to find a way to go around it.

As in (5.37), we can reduce to the case when X is smooth and $\text{red } F_b$ is an snc divisor for every b . Let now H be a general, smooth relatively ample divisor. Fix some $\epsilon > 0$ and run MMP for $(X, \text{red } F + \epsilon H) \rightarrow B$. The boundary is now big, so [BCHM10] applies. If we end with a Fano contraction then every irreducible component of every fiber is uniruled, hence of volume 0. Otherwise we get a relative minimal model $(X^m, \text{red } F^m + \epsilon H^m) \rightarrow B$. Arguing as in (5.37.1) we get that

$$(K_{X_g^m} + \epsilon H_g^m) \geq \sum_i (K_{\bar{E}_i} + \bar{D}_i)^n \quad \text{where} \quad \bar{D}_i = \text{Diff}_{\bar{E}_i}(\epsilon H^m + \sum_{j \neq i} E_j).$$

Next (5.12) shows that $(K_{\bar{E}_i} + \bar{D}_i)^n \geq (K_{E_i^{\text{cr}}})^n$, so

$$\text{vol}(K_{X_g^m} + \epsilon H_g^m) \geq \text{vol}(K_{X_b^r})$$

for every $b \in B$ and $\epsilon > 0$. Finally, letting $\epsilon \rightarrow 0$, the continuity of the volume [Laz04, 2.2.44] gives that $\text{vol}(K_{X_g^m}) = \text{vol}(K_{X_g^r}) \geq \text{vol}(K_{X_b^r})$, as required. \square

Next we prove (5.11) when the base is a smooth curve.

5.39 (Proof of (5.11) over curves). Let B be a smooth curve over a field of char 0 and $f : (X, \Delta) \rightarrow B$ a flat morphism whose fibers are irreducible and smooth outside a codimension ≥ 2 subset. We may replace X by its normalization. Thus we may assume to start with that X is normal and then the generic fiber is lc.

Assume first that f is locally stable. We prove that $b \mapsto \text{vol}(K_{X_b} + \Delta_b)$ is an upper semicontinuous function on S and $f : (X, \Delta) \rightarrow B$ has a simultaneous canonical model iff this function is locally constant.

To see these let $f^c : (X^c, \Delta^c) \rightarrow B$ denote the canonical model of $f : (X, \Delta) \rightarrow B$, it exists by [Kol13b, 1.30.7]. For every $b \in B$ we need to understand the difference between

- $((X^c)_b, (\Delta^c)_b)$, the fiber of f^c over b and
- $((X_b)^c, (\Delta_b)^c)$, the canonical model of the fiber (X_b, Δ_b) of f over b .

These two are the same for general $g \in B$ but they can be different for some special points in B .

Let $\phi : X \dashrightarrow X^c$ denote the natural birational map. Since the fibers of f are irreducible, they can not be contracted, thus ϕ induces birational maps $\phi_b : X_b \dashrightarrow (X^c)_b$. Let Z_b denote the normalization of the closure of the graph of ϕ_b with projections $X_b \xleftarrow{g} Z_b \xrightarrow{h} (X^c)_b$. The key computation, done in (5.40), shows that

$$g^*(K_{X_b} + \Delta_b) \sim_{\mathbb{Q}} h^*(K_{(X^c)_b} + (\Delta^c)_b) + F_b \quad (5.39.1)$$

where F_b is effective. This implies that

$$\text{vol}(K_{X_b} + \Delta_b) = \text{vol}(g^*(K_{X_b} + \Delta_b)) \geq \text{vol}(h^*(K_{(X^c)_b} + (\Delta^c)_b)) = \text{vol}(K_{(X^c)_b} + (\Delta^c)_b).$$

Note further that since $f^c : (X^c, \Delta^c) \rightarrow B$ is flat and $K_{X^c} + \Delta^c$ is f^c -ample, its restriction to different fibers have the same volume. Thus

$$\text{vol}(K_{(X^c)_b} + (\Delta^c)_b) = \text{vol}(K_{(X^c)_g} + (\Delta^c)_g) = \text{vol}(K_{X_g} + \Delta_g)$$

for generic $g \in B$. Putting the two together shows that

$$\text{vol}(K_{X_b} + \Delta_b) \geq \text{vol}(K_{X_g} + \Delta_g) \quad (5.39.2)$$

and, by (10.38), equality holds iff F_b is h -exceptional, in which case $((X^c)_b, (\Delta^c)_b)$ is the canonical model of (X_b, Δ_b) . This proves both claims.

In the general case, when $f : (X, \Delta) \rightarrow B$ is not locally stable, we first use (5.33) to construct $h : (\bar{X}, \bar{\Delta}) \rightarrow (X, \Delta)$ such that the composite $f \circ h : (\bar{X}, \bar{\Delta}) \rightarrow B$ is locally stable. Thus (5.39.2) applies and we get that

$$\text{vol}(K_{\bar{X}_b} + \bar{\Delta}_b) \geq \text{vol}(K_{X_g} + \Delta_g). \quad (5.39.3)$$

Note that $h_b : (\bar{X}_b, \bar{\Delta}_b) \rightarrow (X_b, \Delta_b)$ is birational by (5.33) and $K_{\bar{X}_b} + \bar{\Delta}_b$ is h_b -ample. Thus (10.31.1) implies that

$$\text{vol}(X_b, \Delta_b) \geq \text{vol}(\bar{X}_b, \bar{\Delta}_b). \quad (5.39.4)$$

Putting (5.39.3) and (5.39.4) together shows the upper semicontinuity of the volume.

It remains to show that if equality holds in (5.39.3) and (5.39.4) then there is a simultaneous canonical model. We already proved that if equality holds in (5.39.3) then $f \circ h : (\bar{X}, \bar{\Delta}) \rightarrow B$ has a simultaneous canonical model $(\bar{X}_b^c, \bar{\Delta}_b^c)$. Next we show that if equality holds in (5.39.4) then $(\bar{X}_b^c, \bar{\Delta}_b^c)$ is also the simultaneous canonical model of $f : (X, \Delta) \rightarrow B$. Equivalently, that $(\bar{X}_b, \bar{\Delta}_b)$ and (X_b, Δ_b) have isomorphic canonical models. The latter follows from (10.38) but it can also be obtained by applying the simpler (10.31) to the (normalization of the closure of the) graph of $(\bar{X}_b, \bar{\Delta}_b) \dashrightarrow (X_b^c, \Delta_b^c)$. \square

LEMMA 5.40. *Let $(X, D + \Delta)$ be lc where D is a reduced Weil divisor and $\Delta = \sum a_i D_i$ is an \mathbb{R} -divisor. Let $f : X \rightarrow S$ be a proper morphism and $\phi :$*

$(X, D + \Delta) \dashrightarrow (X^c, D^c + \Delta^c)$ the relative canonical model. If none of the irreducible components of D are contracted by ϕ , we get a birational map of the normalizations

$$\phi_{\bar{D}} : (\bar{D}, \text{Diff}_{\bar{D}} \Delta) \dashrightarrow (\bar{D}^c, \text{Diff}_{\bar{D}^c} \Delta^c), \quad \text{and}$$

$$(5.40.1) \quad a(E, \bar{D}, \text{Diff}_{\bar{D}} \Delta) \leq a(E, \bar{D}^c, \text{Diff}_{\bar{D}^c} \Delta^c) \text{ for every divisor } E \text{ over } \bar{D} \text{ and}$$

$$(5.40.2) \quad (\phi_{\bar{D}})_* \text{Diff}_{\bar{D}} \Delta \geq \text{Diff}_{\bar{D}^c} \Delta^c.$$

Proof. Let Y be the normalization of the main component of the fiber product $X \times_S X^c$ with projections $X \xleftarrow{g} Y \xrightarrow{h} X^c$. By definition,

$$g^*(K_X + D + \Delta) \sim_{\mathbb{Q}} h^*(K_{X^c} + D^c + \Delta^c) + F \quad (5.40.3)$$

where F is effective. Let D_Y denote the birational transform of D on Y . Restricting to D_Y we get

$$(g|_{D_Y})^*(K_D + \text{Diff}_D \Delta) \sim_{\mathbb{Q}} (h|_{D_Y})^*(K_{D^c} + \text{Diff}_{D^c} \Delta^c) + F|_{D_Y}$$

and $F|_{D_Y}$ is also effective. This proves (1) and (2) is a special case. \square

Looking at the above proof shows that the existence of simultaneous canonical models is part of the following more general problem.

QUESTION 5.41. Let $(X, D + \Delta)$ be an lc pair and $(X^c, D^c + \Delta^c)$ its canonical model. What is the relationship between

- the canonical model of $(D, \text{Diff}_D \Delta)$ and
- $(D^c, \text{Diff}_{D^c} \Delta^c)$?

Examples 5.21–5.23 show that these two are usually different. An even simpler smooth example is the following.

Start with a smooth variety X' , a smooth divisor $D' \subset X'$ and another smooth divisor $C' \subset D'$. Assume that $K_{X'} + D'$ is ample. Set $X := B_{C'} X'$ with exceptional divisor E and let $D \subset X$ denote the birational transform of D' . Then $(X, D + E)$ is an lc pair whose canonical model is (X', D') and $(D', 0)$ is its own canonical model. However, $(D, \text{Diff}_D E) \cong (D', C')$ is different from $(D', 0)$.

Note that for suitable choices we can arrange that $K_{D'} + C'$ is ample (in which case (D', C') is its own canonical model) or that $K_{D'} + C'$ is negative on C' (in which case the canonical model of (D', C') is obtained by contracting C' to a point).

The following is proved in [AK19b, Thm.7].

THEOREM 5.42. Let $(X, D + \Delta)$ be an lc pair that is projective over a base scheme S with relatively ample divisor H , where all divisors in D appear with coefficient 1. Set $(X^0, D^0 + \Delta^0) := (X, D + \Delta)$ and for $i = 1, \dots, m$ let

$$\phi^i : (X^{i-1}, D^{i-1} + \Delta^{i-1}) \dashrightarrow (X^i, D^i + \Delta^i)$$

be the steps of the $(X, D + \Delta)$ -MMP with scaling of H . Assume that the intersection of D with the exceptional locus of $\phi^m \circ \dots \circ \phi^1 : X \dashrightarrow X^m$ does not contain any log center of $(X, D + \Delta)$. Let $\rho : \bar{D} \rightarrow D$ be the normalization.

Then the induced maps

$$\phi_{\bar{D}}^i : (\bar{D}^{i-1}, \text{Diff}_{\bar{D}} \Delta^{i-1}) \dashrightarrow (\bar{D}^i, \text{Diff}_{\bar{D}} \Delta^i)$$

form the steps of the MMP starting with $(\bar{D}^0, \text{Diff}_{\bar{D}} \Delta^0) := (\bar{D}, \text{Diff}_{\bar{D}} \Delta)$ and with scaling of $\rho^* H$. \square

5.7. Simultaneous canonical modifications

If S is smooth then the simultaneous canonical modification of $f : (X, \Delta) \rightarrow S$ is also the canonical modification of (X, Δ) . This suggests that one should consider the canonical modification of (X, Δ) and try to prove that it is a simultaneous canonical modification.

5.43 (Proof of (5.17) over curves). Let C be a smooth curve and $f : (X, \Delta) \rightarrow C$ a flat, projective morphism of pure relative dimension n that satisfies the assumptions of (5.17).

Each $c \mapsto (\pi_c^* H_c^{n-i} \cdot (K_{X_c^{\text{can}}} + \Delta_0^{\text{can}})^i)$ is a constructible function on C . Thus, in order to prove (5.17.1) we may assume that C is the spectrum of a DVR with closed point $0 \in C$ and generic point $g \in C$. We may also assume that X is reduced, thus f is flat.

By (5.28), $(\pi_0^* H_0^n) \leq (\pi_g^* H_g^n)$ and equality holds iff X_0 is generically reduced. It is thus enough to deal with the latter case. Then X is generically normal along X_0 and we can replace X by its normalization without changing any of the assumptions or conclusions. We may now also assume that X is irreducible.

Let $\pi : (Y, \Delta^Y = \pi_*^{-1} \Delta) \rightarrow (X, \Delta)$ denote the canonical modification.

Write $Y_0 = \sum_i e_i E_i$ where $e_0 = 1$ and E_0 is the birational transform of X_0 . (For now E_0 is allowed to be reducible.) Set $E := \text{red } Y_0 = \sum E_i$. Let $\tau : \bar{E}_0 \rightarrow E_0$ denote the normalization and write $\tau^*(K_Y + E + \Delta^Y) = K_{\bar{E}_0} + D_0$ where $D_0 = \text{Diff}_{\bar{E}_0}(E - E_0 + \Delta^Y)$ as in (11.17). Choose $m \geq 0$ such that $K_Y + E + \Delta^Y + m\pi^* H$ is ample over C . We claim the following sequence of (in)equalities.

$$\begin{aligned}
& (K_{X_g^{\text{can}}} + \Delta_g^{\text{can}} + m\pi_g^* H)^n \\
&= (K_{Y_g} + \Delta_g^Y + m\pi_g^* H)^n \\
&= (K_Y + \Delta^Y + m\pi^* H)^n \cdot [Y_g] \\
&= (K_Y + E + \Delta^Y + m\pi^* H)^n \cdot [Y_g] \\
&= (K_Y + E + \Delta^Y + m\pi^* H)^n \cdot [Y_0] \\
&= \sum_i e_i ((K_Y + E + \Delta^Y + m\pi^* H)|_{E_i})^n \\
&\geq (K_{\bar{E}_0} + D_0 + m\pi_0^* H)^n \\
&\geq \text{vol}(K_{X_0^{\text{can}}} + \Delta_0^{\text{can}} + m\pi_0^* H) \\
&= (K_{X_0^{\text{can}}} + \Delta_0^{\text{can}} + m\pi_0^* H)^n.
\end{aligned} \tag{5.43.1}$$

The first equality holds since (Y_g, Δ_g^Y) is the canonical model of (X_g, Δ_g) , hence $\Delta_g^{\text{can}} = \Delta_g^Y$. The second equality is clear. We are allowed to add E in the fourth row since it is disjoint from Y_g . We can then replace Y_g by Y_0 since they are algebraically equivalent and compute the latter one component at a time. $K_Y + E + \Delta^Y + m\pi^* H$ is ample, thus if we keep only the summands corresponding to E_0 , we get the first inequality, which is an equality iff $Y_0 = E_0$.

The second inequality follows from (10.35), once we check that $\sigma_*^{-1} \Delta_0 \leq D_0$ where $\sigma := \pi_0 \circ \tau : \bar{E}_0 \rightarrow \bar{X}_0$ is the natural map. Since D_0 is effective, this is clear for σ -exceptional divisors. Otherwise, either π is an isomorphism over the generic point of a divisor D_0^i (hence D_0^i has the same coefficients in $\sigma_*^{-1} \Delta_0$ and D_0) or $\sigma_*^{-1} D_0^i$ is contained in another irreducible component of $\text{red } Y_0$. In this case $\sigma_*^{-1} D_0^i$ appears in D_0 with coefficient 1 and in $\sigma_*^{-1} \Delta_0$ with coefficient ≤ 1 by assumption. This proves the second inequality and, by (10.35), if equality holds then $D_0 = \sigma_*^{-1} \Delta_0$. The last equality is a general property of ample divisors.

As we noted in (5.14), the inequality proved in (5.43.1) is equivalent to

$$I(\pi_g^* H_g, K_{X_g^{\text{can}}} + \Delta_g^{\text{can}}) \succeq I(\pi_0^* H_0, K_{X_0^{\text{can}}} + \Delta_0^{\text{can}})$$

which proves (5.17.1).

If equality holds everywhere in (5.43.1) then $Y_0 = E_0$, $D_0 = \sigma_*^{-1} \Delta_0$ and (\bar{E}_0, D_0) is canonical. On the other hand, D_0 is the sum of $\sigma_*^{-1} \Delta_0$ and of the conductor of $\bar{E}_0 \rightarrow E_0 = Y_0$. Thus the conductor is 0, hence Y_0 is normal in codimension 1 (hence irreducible), $D_0 = (\pi_0)_*^{-1} \Delta_0$ and $(Y_0, (\pi_0)_*^{-1} \Delta_0)$ is canonical in codimension 1. Thus Y_0 is normal and $(Y_0, (\pi_0)_*^{-1} \Delta_0)$ is canonical by (2.4). Since $K_{Y_0} + D_0$ is ample over X_0 , these show that $(Y_0, (\pi_0)_*^{-1} \Delta_0)$ is the canonical modification of (X_0, Δ_0) . Thus the canonical modification of (X, Δ) is also the simultaneous canonical modification, proving (5.17.2) over curves. \square

In analogy with (5.16), we can define simultaneous slc modifications.

DEFINITION 5.44. Let (X, Δ) be a pair over a field k that is slc in codimension 1. Its *semi-log-canonical modification* is a proper, birational morphism $\pi : (X^{\text{slc}}, \Delta^{\text{slc}}) \rightarrow (X, \Delta)$ such that π is an isomorphism over codimension 1 points of X , $\Delta^{\text{slc}} = \pi_*^{-1} \Delta + E$ where E contains every π -exceptional divisor with coefficient 1, $K_{X^{\text{slc}}} + \Delta^{\text{slc}}$ is π -ample and $(X^{\text{slc}}, \Delta^{\text{slc}})$ is slc.

If X is normal, then the semi-log-canonical modification is automatically normal and it agrees with the log-canonical modification.

In general lc modifications are conjectured to exist but there are slc pairs without slc modification, see [Kol13b, 1.40]. In both cases existence is known when $K_X + \Delta$ is \mathbb{R} -Cartier, see [OX12].

Let $f : (X, \Delta) \rightarrow S$ be a morphism that satisfies the conditions (5.2.1–4). A *simultaneous slc modification* is a proper morphism $\pi : (Y, \Delta^Y) \rightarrow (X, \Delta)$ such that $f \circ \pi : (Y, \Delta^Y) \rightarrow S$ is locally stable and $\pi_s : (Y_s, \Delta_s^Y) \rightarrow (X_s, \Delta_s)$ is the slc modification for every $s \in S$.

We get the following variant of (5.17).

THEOREM 5.45. *Let C be a smooth curve, $f : (X, \Delta) \rightarrow C$ a projective morphism satisfying (5.2.1–4). Assume that $K_X + \Delta$ is \mathbb{R} -Cartier and, for $c \in C$ let $\pi_c : (X_c^{\text{slc}}, \Delta_c^{\text{slc}}) \rightarrow (X_c, \Delta_c)$ be the slc modification. Then*

(5.45.1) $c \mapsto I(\pi_c^* H_c^{n-2}, K_{X_c^{\text{slc}}} + \Delta_c^{\text{slc}})$ is a lexicographically lower semicontinuous function on C and

(5.45.2) $f : (X, \Delta) \rightarrow C$ has a simultaneous slc modification iff this function is locally constant.

Proof. Using (2.55) we may assume that X is normal. Next we closely follow the proof of (5.43).

Let $\pi : (Y, \Delta^Y) \rightarrow (X, \Delta)$ denote the log-canonical modification; this exists by (5.15). Note that here $\Delta^Y = \pi_*^{-1} \Delta + F$ where F is the sum of all π -exceptional divisors that dominate C .

Write $Y_0 = \sum_i e_i E_i$ where $e_0 = 1$ and E_0 is the birational transform of X_0 . Let $\tau : \bar{E}_0 \rightarrow E_0$ denote the normalization and write $\tau^*(K_Y + Y_0 + \Delta^Y) = K_{\bar{E}_0} + D_0$. Choose $m \geq 0$ such that $K_Y + Y_0 + \Delta^Y + m\pi^* H$ is ample over C . As in the proof

of (5.43) we get that

$$\begin{aligned} (K_{X_g^{\text{lc}}} + \Delta_g^{\text{lc}} + m\pi_g^*H)^n &\geq (K_{\bar{E}_0} + D_0 + m\pi_0^*H)^n \quad \text{and} \\ \text{vol}(K_{X_0^{\text{lc}}} + \Delta_0^{\text{lc}} + m\pi_0^*H) &= (K_{X_0^{\text{lc}}} + \Delta_0^{\text{lc}} + m\pi_0^*H)^n. \end{aligned}$$

It remains to prove that $(K_{\bar{E}_0} + D_0 + m\pi_0^*H)^n \geq \text{vol}(K_{X_0^{\text{lc}}} + \Delta_0^{\text{lc}} + m\pi_0^*H)$.

We have $\sigma : \bar{E}_0 \rightarrow X_0$ and we can apply (10.36) provided every σ -exceptional divisor $\bar{F}_0 \subset \bar{E}_0$ appears in D_0 with coefficient 1.

By the definition of lc modifications, every divisor F_i that is exceptional for $Y \rightarrow X$ appears in Δ^Y with coefficient 1. If $K_X + \Delta$ is \mathbb{Q} -Cartier then the exceptional set of $Y \rightarrow X$ has pure codimension 1. In this case $\tau(\bar{F}_0)$ is contained in a divisor that is exceptional for $Y \rightarrow X$. Thus, by adjunction, \bar{F}_0 appears in D_0 with coefficient 1.

If (X_0, Δ_0) is slc at a point x_0 then (X, Δ) is also slc at x_0 by inversion of adjunction (11.20) hence π is a local isomorphism over x_0 . Thus $\pi_0 : (Y_0, \Delta_0^Y) \rightarrow (X_0, \Delta_0)$ is an isomorphism over codimension 1 points of X_0 .

The rest of the proof works as before. □

If $K_X + \Delta$ is not \mathbb{Q} -Cartier then it can happen that an exceptional divisor $\bar{F}_0 \subset \bar{E}_0$ is not contained in any exceptional divisor of $X^{\text{lc}} \rightarrow X$. In such cases we lose control of the coefficient of \bar{F} in D_0 . This occurs in (5.25) over the 4 singular points that lie on D_0 .

5.8. Mostly flat families of line bundles

So far we have studied morphisms that were known to be stable in codimension 1. Next we turn to investigating sheaves that are known to be invertible in codimension 1; a topic we already encountered in Section 2.6. This leads to the proofs of (5.7) and (5.4). Many of the results proved here are developed for arbitrary coherent sheaves in Chapter 9.

DEFINITION 5.46 (Mostly flat families of line bundles). Let $f : X \rightarrow S$ be a morphism and L a mostly flat family of divisorial sheaves (3.25). Thus there is a closed subscheme $Z \subset X$ with complement $j : U := X \setminus Z \hookrightarrow X$ such that $Z \cap X_s$ has codimension ≥ 2 in X_s for every $s \in S$, $f|_U : U \rightarrow S$ is flat over S with pure, S_2 fibers and $L|_U$ is a line bundle.

We say that L is a *mostly flat families of line bundles* if the S_2 -hulls

$$L_s^H = (j_s)_*(L|_{U_s}) \tag{5.46.1}$$

are locally free over the S_2 -hull of the fibers of f

$$X_s^H = \text{Spec}_{X_s}(j_s)_*(\mathcal{O}_{U_s}). \tag{5.46.2}$$

(If U_s is normal, which is the main case, then X_s^H is the normalization of X_s .)

We may as well assume that $\mathcal{O}_X = j_*\mathcal{O}_U$ (equivalently, that $\text{depth}_Z \mathcal{O}_X \geq 2$) and then (10.4) implies that there is a dense open subset $S^\circ \subset S$ such that L is a line bundle on $X^\circ := f^{-1}(S^\circ)$.

A mostly flat family of line bundles L on X is called *fiber-wise ample* if L_s^H is ample for every $s \in S$.

EXAMPLE 5.47. Let $f' : X' \rightarrow \mathbb{A}^1$ be a family of degree 4 surfaces in \mathbb{P}^3 such that X'_0 contains a line ℓ but the Picard number of X'_t is 1 for some $t \neq 0$. Then $\ell \subset X'$ can be contracted and we get $\pi : X' \rightarrow X$ and $f : X \rightarrow \mathbb{A}^1$. (Usually X

is only an analytic or algebraic space.) Here X_0 is a K3 surface with a node. Set $L := \pi_* \mathcal{O}_X(2)$.

Then L is a mostly flat family of fiber-wise ample line bundles yet f itself is not projective.

Our aim is to find conditions to ensure that a mostly flat family of line bundles is a flat family of line bundles. We start with 1-parameter families.

5.48 (Euler characteristic and specialization). Let $(0, T)$ be the spectrum of a DVR, $f : X \rightarrow T$ a proper morphism of pure relative dimension n . We can usually harmlessly assume that X is S_2 . Thus the generic fiber X_g is S_2 and the special fiber X_0 is S_1 . Let L be a mostly flat family of line bundles on X .

By the assumptions (3.25) L is S_2 and there is a subset $Z_0 \subset X_0$ of codimension ≥ 2 , called the *degeneracy set* of L , such that L is locally free on $X \setminus Z_0$ and $X_0 \setminus Z_0$ is S_2 .

L_0 is also S_1 , hence $L_0 \rightarrow L_0^H$ is an injection. By semicontinuity we have $h^0(X_0, L_0^H) \geq h^0(X_0, L_0) \geq h^0(X_g, L_g)$. Applying this inequality to powers of L we obtain that

$$\text{vol}(L_0^H) = \lim \frac{h^0(X_0, (L_0^H)^{\otimes m})}{m^n/n!} \geq \lim \frac{h^0(X_g, L_g^{\otimes m})}{m^n/n!} = \text{vol}(L_g). \quad (5.48.1)$$

If L is fiber-wise ample then the volume equals the self-intersection number, thus

$$((L_0^H)^n) \geq ((L_g^H)^n). \quad (5.48.2)$$

In order to get more precise information, note that we have an exact sequence

$$0 \rightarrow L_0 \xrightarrow{r_0^L} L_0^H \rightarrow Q \rightarrow 0 \quad (5.48.3)$$

which defines the sheaf Q whose support is contained in the degeneracy set Z_0 . Thus

$$\begin{aligned} \chi(X_0, L_0^H) &= \chi(X_0, L_0) + \chi(X_0, Q) \\ &= \chi(X_g, L_g) + \chi(X_0, Q). \end{aligned} \quad (5.48.4)$$

Let $\mathcal{O}_X(1)$ be an ample line bundle. Twisting (5.48.3) by $\mathcal{O}_X(m)$ and taking Euler characteristic we proved the following

Claim 5.48.5. $\chi(X_0, L_0^H(m)) \geq \chi(X_g, L_g(m))$ and equality holds iff $Q = 0$. Equivalently, iff $r_0^L : L_0 \rightarrow L_0^H$ is an isomorphism. If X_0 is S_2 then this is further equivalent to L being locally free. \square

If the degeneracy set Z_0 is finite then $Q = 0$ iff $\chi(X_0, Q) = 0$, hence

$$L_0 = L_0^H \iff \chi(X_0, L_0^H) = \chi(X_g, L_g). \quad (5.48.6)$$

As before, if X_0 is S_2 then this is further equivalent to L being locally free.

LEMMA 5.49. *Let $f : X \rightarrow S$ be a proper morphism of pure relative dimension n , A a relatively ample line bundle on X and L a mostly flat family of fiber-wise ample line bundles. Then*

(5.49.1) $s \mapsto (A_s^i \cdot (L_s^H)^{n-i})$ is constructible and upper semicontinuous for every i , and

(5.49.2) if $((L_s^H)^n)$ is constant (as a function of s) then so is every $(A_s^i \cdot (L_s^H)^{n-i})$.

Proof. As we noted in (5.46), there is a dense open subset $S^\circ \subset \text{red } S$ such that $L|_{S^\circ}$ is a line bundle. Thus the functions $s \mapsto (A_s^i \cdot (L_s^H)^{n-i})$ are locally constant on S° and so constructible on S by Noetherian induction. Together with (5.48.2) this implies semicontinuity for $i = 0$.

For $i > 0$ we prove (1) by induction on n . We may assume that S is local and A is relatively very ample. Let $Y \subset X$ be a hypersurface cut out by a general section of A . By a Bertini-type theorem (10.11) the restriction $L|_Y$ is a mostly flat family of fiber-wise ample line bundles on $Y \rightarrow S$. Furthermore

$$(A_s^i \cdot (L_s^H)^{n-i}) = (Y_s \cdot A_s^{i-1} \cdot (L_s^H)^{n-i}) = ((A|_Y)_s^{i-1} \cdot ((L|_Y)_s^H)^{n-i}), \quad (5.49.3)$$

and the latter is constructible and upper semicontinuous by induction.

In order to see (2) note that $L^m \otimes A^{-1}$ is also a mostly flat family of fiber-wise ample line bundles for $m \gg 1$, and

$$m^n \cdot ((L_s^H)^n) = \sum_i \binom{n}{i} (A_s^i \cdot ((L^m \otimes A^{-1})_s^H)^{n-i}). \quad (5.49.4)$$

By (1) all summands on the right are constructible and upper semicontinuous. Therefore, if the sum is constant as a function of s , then so is every summand. Finally note that

$$(((L^m \otimes A^{-1})_s^H)^n) = \sum_i (-1)^i m^{n-i} \binom{n}{i} (A_s^i \cdot (L_s^H)^{n-i}). \quad (5.49.5)$$

If the left side is constant for $m \gg 1$, as a function of s , then every summand on the right is constant. \square

Remark 5.49.6. Let $f : X \rightarrow S$ be a proper morphism of pure relative dimension n and L a line bundle on X . It is not well understood under what conditions is the function $s \mapsto \text{vol}(L_s)$ constructible; see [Les14, PS13].

5.50 (Proof of (5.7)). The assertions (5.7.1) and (5.7.3) are proved in (5.49.1). Furthermore, (5.49.2) shows that (5.7.2) implies (5.7.4).

Thus it remains to prove (5.7.2). We start with the case when S is the spectrum of a DVR; this implies the general case by (4.37).

Our argument has 3 parts. The first step, when the relative dimension is 2, is done in (5.52).

The next step is induction on the dimension. We may assume that S is local and A is relatively very ample. Let $Y \subset X$ be a general hypersurface cut out by a general section of A . Then (10.11) ensures that $L^H|_Y = (L|_Y)^H$. The restriction $L|_Y$ is a mostly flat family of fiber-wise ample line bundles on $Y \rightarrow S$ and, as we noted in (5.49.3),

$$(A_s^{n-2} \cdot (L_s^H)^2) = ((A|_Y)_s^{n-3} \cdot ((L|_Y)_s^H)^2).$$

Thus, by induction, $L^H|_Y$ is a line bundle. This implies that L^H is a line bundle along Y . Therefore L^H is a line bundle, except possibly at finitely many points $Z \subset X$.

Finally we need to exclude this finite set Z when the fiber dimension is at least 3. This follows from (2.93), which we have not proved yet.

Alternatively, we can use (2.92) and conclude that $L^{[m]}$ is a line bundle for some $m > 0$. Then a short global argument given in (5.53) shows that L itself is locally free. \square

5.51 (Start of the proof of (5.4)). Note that (5.4.4) follows from (5.49.1).

Next we consider (5.4.5) when S is the spectrum of a DVR; the general setting is postponed to (5.61).

Thus assume that we have a smooth curve C and $f : (X, \Delta) \rightarrow C$ satisfying the assumptions (5.4.1–3) and such that

$$c \mapsto (\pi_c^* H^{n-2} \cdot (K_{\bar{X}_c} + \bar{D}_c + \bar{\Delta}_c)^2)$$

is a constant function on C . We aim to prove that $K_X + \Delta$ is \mathbb{Q} -Cartier.

As a first step, we replace (X, Δ) by its normalization. This leaves the assumptions and the numerical conclusion unchanged. By (2.55), a demi-normal pair $(X, \Delta) \rightarrow C$ with slc generic fibers is slc iff its normalization is lc. Thus the conclusion is also unchanged.

It would seem that we should use (5.7). However, a key assumption of (5.7) is that every fiber is S_2 ; this is true but not obvious in our case. Thus we consider two separate cases.

If $n = 2$ then the weak numerical criterion (5.35) implies (5.4). However, for $n \geq 3$ the weak numerical criterion also involves the terms $(\pi_c^* H^{n-i} \cdot (K_{\bar{X}_c} + \bar{D}_c + \bar{\Delta}_c)^i)$ for $i \geq 3$; these are unknown to us.

Instead, using the already established $n = 2$ case and the Bertini-type result (10.11) as in (5.50), we may assume that $f : (X, \Delta) \rightarrow C$ is locally stable outside a subset of codimension ≥ 3 . We can now apply (5.5), or rather, the special case proved in (5.36), to complete the argument. \square

PROPOSITION 5.52. *Let T be an irreducible, regular, 1-dimensional scheme and $f : X \rightarrow T$ a flat, proper morphism of relative dimension 2 with S_2 fibers. Let L be a mostly flat family of line bundles on X . Then*

(5.52.1) $t \mapsto (L_t^H \cdot L_t^H)$ is upper semicontinuous and

(5.52.2) L is locally free on X iff the above function is constant.

Proof. If L is locally free then $(L_t^H \cdot L_t^H) = (L \cdot L \cdot [X_t])$ is independent of $t \in T$. To see the converse we may assume that T is local with closed point $0 \in T$ and generic point $g \in T$. Note that L is locally free, except possibly at a finite set $Z_0 \subset X_0$, and $L_g^H \cong L_g$.

For each $t \in T$, the Euler characteristic is a quadratic polynomial

$$\chi(X_t, (L_t^H)^{\otimes m}) = a_t m^2 + b_t m + c_t,$$

and we know from Riemann–Roch that $a_t = \frac{1}{2}(L_t^H \cdot L_t^H)$ and $c_t = \chi(X_t, \mathcal{O}_{X_t})$. Furthermore, (5.48.4) implies that

$$a_0 m^2 + b_0 m + c_0 \geq a_g m^2 + b_g m + c_g \quad \text{for every } m \in \mathbb{Z}. \quad (5.52.3)$$

For $m \gg 1$ the quadratic terms dominate, which gives that

$$(L_0^H \cdot L_0^H) = 2a_0 \geq 2a_g = (L_g \cdot L_g). \quad (5.52.4)$$

Assume now that $(L_0^H \cdot L_0^H) = (L_g \cdot L_g)$. Then $a_0 = a_g$ thus (5.52.3) implies that

$$b_0 m + c_0 \geq b_g m + c_g \quad \text{for every } m \in \mathbb{Z}. \quad (5.52.5)$$

For $m \gg 1$ this implies that $b_0 \geq b_g$ and for $m \ll -1$ that $-b_0 \geq -b_g$. Thus $b_0 = b_g$ and $c_0 = c_g$ also holds since f is flat. Therefore we have equality in (5.52.3).

Thus L is a flat family of locally free sheaves by (5.48.6). \square

Local extension problems.

We are now ready to complete the proofs of (2.93) and (5.7).

In both cases, the only remaining case is when some reflexive tensor power $L^{[m]}$ is locally free on X and in Section 2.9 we even settled the case when $\text{char } k(x) \nmid m$.

Below we give first the global argument of [Kol16a] and then discuss how it was localized in [dJ15].

PROPOSITION 5.53. *Let T be the spectrum of a DVR with closed point $0 \in T$ and generic point $g \in T$. Let $f : X \rightarrow T$ be a projective morphism with S_2 fibers and L a mostly flat family of line bundles such that $L^{[m]}$ is locally free for some $m > 0$. Then L is locally free.*

Proof. We claim an equality of the Hilbert polynomials

$$\chi(X_0, (L_0^H)^{\otimes r}) = \chi(X_g, L_g^{\otimes r}). \quad (5.53.1)$$

Since both sides are polynomials in r , it is sufficient to prove that they are equal for all multiples of m .

Note that $L^{[m]}|_{X_0}$ and $(L_0^H)^{\otimes m}$ are both locally free sheaves that agree outside a codimension 2 subset, hence they are isomorphic. Thus

$$\begin{aligned} \chi(X_0, (L_0^H)^{\otimes rm}) &= \chi(X_0, (L^{[m]}|_{X_0})^{\otimes r}) = \\ &= \chi(X_g, (L^{[m]}|_{X_g})^{\otimes r}) = \chi(X_g, L_g^{\otimes rm}), \end{aligned} \quad (5.53.2)$$

where the last equality holds since L_g is a line bundle (5.46). In particular we conclude that

$$\chi(X_0, L_0^H) = \chi(X_g, L_g). \quad (5.53.3)$$

Let now $\mathcal{O}_{X/S}(1)$ be an f -ample invertible sheaf. We can apply the same argument to any $L(t)$ to obtain that

$$\chi(X_0, L_0^H(t)) = \chi(X_g, L_g(t)) \quad \forall t \in \mathbb{Z}. \quad (5.53.4)$$

By (5.48.5) this implies that L is locally free. \square

5.54 (Extension problems). Let (x, X) be a local Noetherian scheme, $s \in m_x$ a non-zero-divisor and $D := (s = 0)$. Set $U := X \setminus \{x\}$ and $U_D := U \cap D$. Let F be a coherent sheaf on U such that s is a non-zero-divisor on F . There are 2 natural quotient maps associated to this set-up. First, $r_U : F \rightarrow F/sF$ gives $h_U : H^0(U, F) \rightarrow H^0(U, F/sF)$. Second, let $j : U \hookrightarrow X$ denote the injection, then r_U extends to a map $r_X := j_* r_U : j_* F \rightarrow j_*(F/sF)$. It is clear that h_U is surjective iff r_X is, thus we can formulate our question in 2 equivalent forms.

Local extension problem 5.54.1. When are the above maps

$$h_U : H^0(U, F) \rightarrow H^0(U, F/sF) \quad \text{and} \quad r_X : j_* F \rightarrow j_*(F/sF) \quad \text{surjective?}$$

A key observation of [dJ15] is that while $j_* F \rightarrow j_*(F/sF)$ is the case we need, one can run induction on the dimension if $j_* F$ and $j_*(F/sF)$ are replaced by suitable subsheaves $F_X \subset j_* F$ and $F_D \subset j_*(F/sF)$. This leads to the following definition.

A *extension problem* associated to $r_U : F \rightarrow F/sF$ consists of coherent sheaves F_X on X and F_D on D plus a map $r_X : F_X \rightarrow F_D$ such that

$$(r_X : F_X \rightarrow F_D)|_U = (r_U : F \rightarrow F/sF). \quad (5.54.2)$$

Observe that r_X induces a morphism

$$r_D : F_X|_D \rightarrow F_D \quad (5.54.3)$$

that is an isomorphism over U_D . Thus both its kernel and cokernel have finite length. Set

$$\delta(F_X, F_D) := \text{length}(\text{coker } r_D) - \text{length}(\text{ker } r_D). \quad (5.54.4)$$

Claim 5.54.5. Using the above notation, assume that $x \notin \text{Ass}(F_X)$. Then $\delta(F_X, F_D)$ depends only on F_D and F , not on F_X .

This suggests that the role of F_X is not very important, hence from now on we will think of an extension problem as a pair (F, F_D) and we choose F_X later in a convenient way. Thus from now on we write

$$\delta(F, F_D) := \delta(F_X, F_D) \quad (5.54.6)$$

for any suitable choice of F_X .

Proof. The assumption $x \notin \text{Ass}(F_X)$ implies that F_X can be identified with a subsheaf of j_*F and if F'_X, F''_X give such extensions problems then so does their intersection. Thus it is enough to check that $\delta(F'_X, F_D) = \delta(F''_X, F_D)$ where $m_x F'_X \subset F''_X \subset F'_X$. Then, as we go from $F'_X \rightarrow F_D$ to $F''_X \rightarrow F_D$, the length of the kernel and of the cokernel both increase by $\text{length}(F'_X/F''_X)$. \square

Next we show that the original question of surjectivity of the restriction map $H^0(U, F) \rightarrow H^0(U_D, F/sF)$ between infinite dimensional vector spaces is equivalent to the vanishing of δ .

Claim 5.54.7. Assume that j_*F and $j_*(F/sF)$ are both coherent. Then $H^0(U, F) \rightarrow H^0(U_D, F/sF)$ is surjective iff $\delta(F, j_*(F/sF)) = 0$.

Proof. We can choose $F_X := j_*F$. Then $\text{depth}_x(F_X/sF_X) \geq 1$, hence $r_X : F_X/sF_X \rightarrow j_*(F/sF)$ is injective. Thus $\delta(F, j_*(F/sF)) = 0$ iff r_X is an isomorphism. Every global section of F/sF extends to a global section of $j_*(F/sF)$ and then lifts to a global section of j_*F since X is affine. \square

5.55 (Maps of extension problems). A *map* of extension problems

$$\alpha : (F, F_D) \rightarrow (G, G_D) \quad (5.55.1)$$

is a pair of maps sitting in a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\alpha_U} & G \\ r^F \downarrow & & \downarrow r^G \\ F_D & \xrightarrow{\alpha_D} & G_D. \end{array} \quad (5.55.2)$$

We do *not* assume that α_U extends to a map between F_X and G_X . However, we can always choose F_X and G_X so that such an extension exists. Indeed, α_U gives a map $\bar{\alpha}_U : G_X \rightarrow j_*F$. We can thus replace G_X by $\bar{\alpha}_U^{-1}(F_X)$ to get $\alpha_X : G_X \rightarrow F_X$.

Correspondingly, an *exact sequence* of extension problems

$$0 \rightarrow (F, F_D) \rightarrow (G, G_D) \rightarrow (H, H_D) \rightarrow 0 \quad (5.55.3)$$

is a commutative diagram of 2 exact sequences

$$\begin{array}{ccccccccc} 0 & \rightarrow & F & \rightarrow & G & \rightarrow & H & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & F_D & \rightarrow & G_D & \rightarrow & H_D & \rightarrow & 0. \end{array} \quad (5.55.4)$$

As before, we do not assume exactness for F_X, G_X, H_X . However, we claim that one can always choose F_X, G_X, H_X such that the sequences

$$\begin{array}{ccccccccc} 0 & \rightarrow & F_X & \rightarrow & G_X & \rightarrow & H_X & \rightarrow & 0 & \text{and} \\ & & \downarrow & & \downarrow & & \downarrow & & & \\ 0 & \rightarrow & F_X/sF_X & \rightarrow & G_X/sG_X & \rightarrow & H_X/sH_X & \rightarrow & 0 \end{array} \quad (5.55.5)$$

are also exact. Indeed, first we choose G_X , then set $F_X := \ker[G_X \rightarrow j_*H]$. A problem is that there may not be a map $F_X \rightarrow F_D$, but such a map exists if we first replace G_X by $m_x^r G_X$ for some $r \gg 1$. Finally set $H_X := G_X/F_X$.

LEMMA 5.56. *Consider an exact sequence of extension problems as in (5.55.3). Then*

$$\delta(G, G_D) = \delta(F, F_D) + \delta(H, H_D).$$

Proof. Choose F_X, G_X, H_X such that the sequences in (5.55.5) are also exact. Then the claim follows from (5.54.5) and the snake lemma applied to

$$\begin{array}{ccccccccc} 0 & \rightarrow & F_X/sF_X & \rightarrow & G_X/sG_X & \rightarrow & H_X/sH_X & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \square \\ 0 & \rightarrow & F_D & \rightarrow & G_D & \rightarrow & H_D & \rightarrow & 0. \end{array}$$

DEFINITION 5.57. We say that (L, L_D) is a *line bundle extension problem* if L is a line bundle on U and L_D is a line bundle on D . Since (x, D) is local, in fact $L_D \cong \mathcal{O}_D$.

If $\text{depth}_x D \geq 2$ then L_D is uniquely determined by $L|_{U_D}$, thus, in this case, line bundle extension problems are in one-to-one correspondence with the kernel of $\text{Pic}^{\text{loc}}(x, X) \rightarrow \text{Pic}^{\text{loc}}(x, D)$.

If (F, F_D) is an extension problem then $(F \otimes L, F_D \otimes L_D)$ is also an extension problem. Note that $F_D \cong F_D \otimes L_D$, thus we are changing the sheaf on U but keeping the sheaf on D fixed.

Now we come to the key point: δ behaves like a Hilbert polynomial.

PROPOSITION 5.58. *Let (F, F_D) be an extension problem and (L, L_D) a line bundle extension problem. Then*

$$n \mapsto \delta(F \otimes L^n, F_D \otimes L_D^n)$$

is a polynomial of degree $\leq \dim(\text{Supp } F_X) - 1$.

Proof. We use induction on $\dim(\text{Supp } F_X)$. If $\dim(\text{Supp } F_X) \leq 1$ then $U \cap \text{Supp } F_X$ is affine and L is trivial on $\text{Spec}_U(\mathcal{O}_U/\text{Ann } F)$. Thus $F \otimes L^n \cong F$ and $F_D \otimes L_D^n \cong F_D$ shows that $\delta(F \otimes L^n, F_D \otimes L_D^n)$ is constant.

Next we claim that for every (F, F_D) there is an exact sequence of extension problems

$$0 \rightarrow (F, F_D) \rightarrow (F \otimes L, F_D \otimes L_D) \rightarrow (Q, Q_D) \rightarrow 0.$$

To see this note that U is quasi-affine, so we can choose a global section g of L that does not vanish at any of the associated points of F or of F/sF . Thus we get injections

$$1_U \otimes g : F \rightarrow F \otimes L \quad \text{and} \quad 1_D \otimes g : (F/sF) \rightarrow (F/sF) \otimes L.$$

Thus $1_D \otimes g$ gives a map $F_D \rightarrow j_*(F/sF) \otimes L_D$ whose image need not be contained in $F_D \otimes L_D$. However, this can be rectified if we replace g by hg for suitable $h \in m_x$.

Thus we get an injection $(F, F_D) \rightarrow (F \otimes L, F_D \otimes L_D)$ and (Q, Q_D) is defined as its cokernel.

Tensoring with a line bundle extension problem is clearly exact, thus we also have

$$0 \rightarrow (F \otimes L^{n-1}, F_D \otimes L_D^{n-1}) \rightarrow (F \otimes L^n, F_D \otimes L_D^n) \rightarrow (Q \otimes L^{n-1}, Q_D \otimes L_D^{n-1}) \rightarrow 0.$$

By (5.54.9) and induction this shows that

$$\delta(F \otimes L^n, F_D \otimes L_D^n) - \delta(F \otimes L^{n-1}, F_D \otimes L_D^{n-1})$$

is a polynomial of degree $\leq \dim(\text{Supp } F_X) - 2$. Thus $\delta(F \otimes L^n, F_D \otimes L_D^n)$ is a polynomial of degree $\leq \dim(\text{Supp } F_X) - 1$. \square

5.59 (Proof of (2.93)). Let L be a line bundle on U such that $L|_{U_D} \cong \mathcal{O}_{U_D}$ and $L^m \cong \mathcal{O}_U$ for some $m > 0$.

We apply (5.58) to the trivial extension problem $(\mathcal{O}_U, \mathcal{O}_D)$. Since $\text{depth}_x D \geq 2$, every isomorphism $\mathcal{O}_{U_D} \cong \mathcal{O}_{U_D}$ is multiplication by a unit, so the actual choice of the isomorphism $\mathcal{O}_U|_{U_D} \cong \mathcal{O}_{U_D}$ does not matter.

Thus we obtain that $n \mapsto \delta(L^n, L_D^n)$ is a polynomial function of n .

If $L^n \cong \mathcal{O}_U$ then $(L^n, L_D^n) \cong (\mathcal{O}_U, \mathcal{O}_D)$, thus $\delta(L^n, L_D^n) = 0$ whenever m divides n . A polynomial with infinitely many roots is identically zero, thus $\delta(L^n, L_D^n) = 0$ for every n . In particular $\delta(L, L_D) = 0$. Thus the constant 1 section of $L_D \cong \mathcal{O}_D$ lifts to a global section of L by (5.54.5) and so $L \cong \mathcal{O}_U$ by (2.89). \square

5.9. Families over higher dimensional bases

Here we complete the proofs of Theorems 5.3–5.17. In all cases the first part asserts that a certain constructible function on the base scheme S is upper or lower semicontinuous. For constructible functions semicontinuity can be checked along spectra of DVR's and this was already done in all cases.

The remaining part is to show that if our functions are locally constant on S , then certain constructions produce a flat family of varieties or sheaves. Again, in all cases we have already checked that this holds when the base is a smooth curve.

Going from curves to arbitrary reduced base schemes is quickest in the following example.

5.60 (Proof of Theorem 5.1). We already proved the case when S is the spectrum of a DVR in (5.34). As we noted above, this implies (5.1.1) in general. Thus it remains to prove that if $s \mapsto (K_{X_s}^n)$ is constant then $f : X \rightarrow S$ is stable.

In view of (5.34) we know that $f_T : X_T \rightarrow T$ is stable for every $T \rightarrow S$ where T is the spectrum of a DVR. Thus $f : X \rightarrow S$ is stable by (4.48). \square

We aim to argue similarly for Theorems 5.3, 5.4 and 5.5. Note that in these cases we can not apply (5.7) since f is not assumed to be flat, and its fibers are not assumed to be S_2 . We need the theory of hulls and husks—to be explained in Chapter 9—which was developed with exactly such situations in mind.

5.61 (Proof of Theorems 5.3–5.5). Let $\pi : \text{Hull}(\mathcal{O}_X) \rightarrow S$ denote the hull (9.63) of \mathcal{O}_X . We aim to show that π is an isomorphism.

By (9.64), π is a locally closed decomposition.

Let T be the spectrum of a DVR and $g : T \rightarrow S$ a morphism that maps the generic point of T to a generic point of S . We apply (5.34), (5.51) or (5.36) to the

divisorial pull-back $f_T : (X_T, \Delta_T) \rightarrow T$. We conclude that $f_T : (X_T, \Delta_T) \rightarrow T$ is stable (resp. locally stable).

Thus $g : T \rightarrow S$ factors uniquely through $\pi : \text{Hull}(\mathcal{O}_X) \rightarrow S$, hence π is proper. $\pi : H \rightarrow S$ is an isomorphism by (10.85). In particular, $f : X \rightarrow S$ is flat with S_2 fibers. Thus the fibers are slc by assumption and (11.11.2.a).

Now we can apply (4.37) to conclude that $K_{X/S} + \Delta$ is \mathbb{R} -Cartier, hence $f : (X, \Delta) \rightarrow S$ is stable (resp. locally stable). \square

We are trying to use similar arguments for the other theorems, but there is no scheme to which the previous methods could be applied to. We go around this problem by using the moduli space of pairs and pull-back the universal family. This needs an artificial way of rigidifying the situation which is, however, quite efficient.

5.62 (Proof of Theorems 5.10–5.11). Both claims were already established over the spectrum of a DVR, see (5.37) and (5.39). This implies the semicontinuity assertions in both cases.

It remains to show that if the volume is constant then $f : X \rightarrow S$ (resp. $f : (X, \Delta) \rightarrow S$) has a simultaneous canonical model.

Consider the moduli space of marked stable pairs $\pi : \text{MSP} \rightarrow S$ and set

$$W := \{(X_s^c, \Delta_s^c) : s \in S\} \subset \text{MSP}.$$

In order to prove that W is a closed subset, first we claim that it is constructible. This is clear since the canonical model over a generic point of S extends to a canonical model over an open subset of S and we can finish by Noetherian induction. Thus closedness needs to be checked over spectra of DVR's, and the latter follows from (5.37) and (5.39).

Thus W is a scheme and the projection π induces a geometric bijection $W \rightarrow S$ which is finite by (5.37) and (5.39). Thus $W \rightarrow S$ is an isomorphism since we assumed that S is seminormal.

If each (X_s^c, Δ_s^c) is rigid, then $W \subset \text{MSP}^{\text{rigid}}$ and there is a universal family $\text{Univ}^{\text{rigid}} \rightarrow \text{MSP}^{\text{rigid}}$ by (8.53). Therefore the pull-back of the universal family $\text{Univ}^{\text{rigid}}$ to W gives the simultaneous canonical model over $S \cong W$.

We have no reason to assume that the (X_s^c, Δ_s^c) are rigid, but we can make the proof work by rigidifying $f : (X, \Delta) \rightarrow S$. To make this simpler, note that a simultaneous canonical model is unique, hence it is enough to construct it étale locally.

After replacing S by an étale neighborhood $(s', S') \rightarrow (s, S)$, we may assume that there are r sections $\sigma_i : S' \rightarrow X'$ such that $(X'_{s'}, \Delta'_{s'}, \sigma_1(s'), \dots, \sigma_r(s'))$ is rigid, and the $\sigma_i(s')$ are smooth points of $X_{s'} \setminus \text{Supp } \Delta_{s'}$ such that $(X'_{s'}, \Delta'_{s'}) \dashrightarrow (X_s^c, \Delta_s^c)$ is a local isomorphism at these points.

By (8.48), after further shrinking S' we may assume that the same holds at every $t \in S'$. Using the moduli of pointed stable pairs MpSP (8.24) and (8.53.1), we can now run the previous argument over S' for

$$W' := \{(X'_t, \Delta'_t, \sigma_1(t), \dots, \sigma_r(t)) : t \in S'\} \subset \text{MpSP}^{\text{rigid}},$$

to prove that the simultaneous canonical model exists over S' . \square

5.63 (Proof of Theorem 5.17). The proof follows very closely the arguments in (5.62). Both claims were already established over the spectrum of a DVR, see (5.43). This implies the semicontinuity assertion in general.

In order to complete the proof of (5.17) it remains to show that if $s \mapsto I(\pi_s^* H_s, K_{X_s^{\text{can}}})$ is constant then $f : (X, \Delta) \rightarrow S$ has a simultaneous canonical modification. Since the simultaneous canonical modification is unique, it is sufficient to construct it étale locally over S . So pick a point $s_0 \in S$, in the sequel we are free to replace S by smaller neighborhoods of s_0 .

Choose $m > 0$ such that $K_{X_s^{\text{can}}} + m\pi_s^* H_s$ is ample for every $s \in S$. Next choose a general $D \in |mH|$ such that $(X_{s_0}^{\text{can}}, \Delta_{s_0}^{\text{can}} + \pi_{s_0}^* D_{s_0})$ is log canonical. We claim that, possibly after shrinking S , $(X_s^{\text{can}}, \Delta_s^{\text{can}} + \pi_s^* D_s)$ is log canonical for every $s \in S$. By (4.51) this condition defines a constructible subset of S and, by (5.43), it contains every generalization of s_0 . Thus it contains an open neighborhood of s_0 . Thus $(X_s^{\text{can}}, \Delta_s^{\text{can}} + \pi_s^* D_s)$ is a stable pair for every $s \in S$.

Consider the moduli space of marked stable pairs $\pi : \text{MSP} \rightarrow S$ and set

$$W := \{(X_s^{\text{can}}, \Delta_s^{\text{can}} + \pi_s^* D_s) : s \in S\} \subset \text{MSP}.$$

In order to prove that W is a closed subset, first we claim that it is constructible. This is clear since the canonical modification over a generic point of S extends to a canonical modification over an open subset of S and we can finish by Noetherian induction. Thus closedness needs to be checked over spectra of DVR's, and the latter follows from (5.43).

Thus W is a scheme and the projection π induces a geometric bijection $W \rightarrow S$ which is finite by (5.43). Thus $W \rightarrow S$ is an isomorphism since S is assumed seminormal.

For general D the pairs $(X_s^{\text{can}}, \Delta_s^{\text{can}} + \pi_s^* D_s)$ should be rigid, and then the pull-back of the universal family to W gives the simultaneous canonical modification over $S \cong W$. Technically it may be easier to rigidify using étale-local sections as in (5.62). \square

Moduli problems with flat divisorial part

So far we have identified stable pairs (X, Δ) as the basic objects of our moduli problem, the 1-parameter families that we want to allow and worked out the reduced part of the moduli spaces. Now we come to the next step of identifying the stable families over an arbitrary base scheme.

In this Chapter we consider several special cases whose solution is easier since we are able to treat the underlying variety X and the boundary divisor Δ as separate objects that are both flat over the base. This is achieved by imposing one of 4 different types of restrictions on the coefficients occurring in Δ .

- (No boundary) Stable varieties X with $\Delta = 0$.
- (Standard boundary) The coefficients in Δ are in the ‘diminished standard coefficient’ set $\{1 - \frac{1}{3}, 1 - \frac{1}{4}, 1 - \frac{1}{5}, \dots, 1\}$.
- (Major boundary) The coefficients in Δ are all $> \frac{1}{2}$.
- (Generic boundary) The coefficients in Δ are \mathbb{Q} -linearly independent.

These examples cover many cases; the most jarring omission is that none of these allow $\frac{1}{2}$ as a coefficient.

After a general discussion of moduli problems in Section 6.1, we treat 2 notions of stability for stable varieties in Sections 6.2–6.3. The first of these—introduced in [KSB88]—starts with the proposal that all plurigenera should be deformation invariant. The second—introduced in [Vie95]—posits that all sufficiently divisible plurigenera should be deformation invariant. The 2 versions agree over reduced base schemes.

Both of these versions can be extended to pairs (X, Δ) as long as Δ is a standard or major boundary as above.

In Section 6.4 we discuss another variant—due to [Ale15]—that works if the coefficients in Δ are sufficiently general. This is especially natural when the boundary arises as a small perturbation of a basic situation.

The infinitesimal deformation theory of stable varieties is not yet well understood, but a large part of the first order theory for surfaces is treated in [AK19a]. After a general discussion of first order deformations of singular varieties in Section 6.5, we work out in detail the theory for cyclic quotient surface singularities in Section 6.6. These are the simplest non-canonical singularities, and they show that the 2 versions outlined in Sections 6.2–6.3 differ from each other already over $\text{Spec } k[\epsilon]$.

6.1. Introduction to moduli of stable pairs

Based on the outline in Section 1.2, we discuss the plan that we use to treat many moduli problems in algebraic geometry. The following version is designed to

work best for the moduli of stable pairs (X, Δ) , but we also mention some of the similarities to related moduli problems.

The method first deals with stable pairs with an embedding into a fixed projective space, and then removes the effect of the embedding.

Defining the moduli problem.

STEP 6.1 (Objects of the moduli problem). At the beginning we have to decide which objects and families our moduli problem should cover. This is usually done in 3 stages.

Interior objects over algebraically closed fields 6.1.1. As the very first step we have to decide what kind of objects we want to parametrize. Probably the first non-linear moduli problem considered was elliptic curves, followed by smooth projective curves of higher genus and their close relatives, Abelian varieties. The study of the moduli of higher dimensional smooth projective varieties was systematically undertaken first by Matsusaka. His approach focuses on polarized pairs (X, L) , where X is a variety and L an ample divisor or divisor class. Here our main aim is to study canonical models of varieties and pairs of general type; equivalently, normal stable pairs (X, Δ) .

It is expected that once we understand the moduli of varieties, it should be relatively easy to work out the moduli theory of related compound objects. For example one could also consider varieties with a group action $G \times X \rightarrow X$, pointed varieties (X, p_1, \dots, p_r) , maps between varieties ($g : X \rightarrow Y$), or various combinations of these.

Boundary objects over algebraically closed fields 6.1.2. By now the answers are mostly well-established, but historically this was a difficult and very non-trivial step for several of the moduli problems. The compactification of the moduli of smooth curves by stable curves was discovered by [DM69].

For surfaces the need to work with canonical models (instead of minimal models) seems to have become clear rather early, but the choice of stable surfaces for boundary points was proposed only in [KSB88].

It should be noted that the distinction between interior and boundary points is not always clear cut. While everyone agrees that smooth curves give the interior points and nodal curves the boundary points of \bar{M}_g , for surfaces one may view either canonical models or only smooth canonical models as interior points. Also this usage of ‘boundary’ does not correspond to any meaning of the word in topology.

Although historically the development went in the other direction, for a logical treatment of a moduli problem it is better to settle on the right class of interior and boundary objects at the beginning, and then gradually prove that they have the required properties.

Objects over arbitrary fields 6.1.3. For stable pairs, the definitions of (6.1.1–2) carry over to arbitrary fields, but in a few examples new questions emerge.

For pointed schemes (X, p_1, \dots, p_r) it may be better to replace the set of closed points $\{p_1, \dots, p_r\}$ by a zero dimensional subscheme $Z \subset X$ of length r . A more subtle problem appears for polarizations, due to the difference between $\text{Pic}(X_k)$ and $\mathbf{Pic}(X_k)(k)$, where $\mathbf{Pic}(X_k)(k)$ is the set of k -points of the Picard scheme of X_k ; see [BLR90, Sec.8.1] for a discussion. This will not be a major issue for us.

Finally there can be other problems caused by inseparable extensions in positive characteristic.

Conclusion 6.1.4. We are working with stable varieties (1.41) and, more generally, with stable pairs (X, Δ) as defined in (2.1). There seems to be full agreement about these being the right objects.

STEP 6.2 (Families of the moduli problem). In many moduli problems, it is considered obvious that the families are determined by the objects: one should work with flat families whose fibers are among our objects. Then the traditional approach is to determine families over $\text{Spec } k[\epsilon]$, and, more generally, over Artin base schemes. This is usually called *obstruction theory*, see [Art76, Ser06, Har10] for introductions to various cases.

However, for stable varieties and pairs, flat families with stable fibers do *not* give a sensible moduli theory, and we need to proceed differently.

Families over DVRs 6.2.1. In Chapter 2 we defined and described stable families over smooth curves and 1-dimensional regular schemes. The advantage of this setting is that the total space of a family is also a locally stable pair, so minimal model theory can be applied both to the fibers and to the total space.

Families over reduced bases 6.2.2. For stable varieties, we proved in (3.37) that stable families over DVRs determine stable families over reduced base schemes. We needed to work quite a bit harder to extend the theory to stable families of pairs over reduced base schemes in Chapter 4, but the end result is the same, at least in characteristic 0: the families over DVRs determine the families over reduced base schemes. (See Section 1.9 for a discussion of the special problems in positive characteristic.)

Families over arbitrary bases 6.2.3. This is where the picture becomes rather complicated. For stable varieties there have been different proposals for about 30 years, we discuss these in Sections 6.2–6.3. These were proved to be non-equivalent in [AK19a], see Section 6.6.

We believe that the notion of KSB stability—treated in Section 6.2—gives the optimal answer for stable varieties.

For pairs the situation is quite complicated. The problem is that while KSB stability has an obvious generalization to pairs, it does not always hold for families over smooth curves; see (2.40) for the first such examples. Thus insisting on it frequently gives non-proper moduli spaces. Still, KSB stability is expected to work well for pairs (X, Δ) if all the coefficients in Δ are $> \frac{1}{2}$; we discuss these in (6.23) and (6.27).

Another approach, by Alexeev, gives a good theory if the coefficients in Δ are sufficiently general real numbers, see (6.40).

However, there was not even a plausible proposal for the general theory before [Koll19]. We work out the details of it in Chapter 7.

Conclusion 6.2.4. We are not aware of any other proposed definition that might work in general, but it is too soon to tell whether the theory of Chapter 7 is the final word on the subject. We comment on some of the issues in the next Step.

Once we have settled on the right objects and families, we need to start working on producing all families and constructing the moduli spaces.

Universal families.

We would like to have a ‘sensible’ way to obtain all stable varieties and pairs and their stable families. It is not a priori clear what this means.

For example, Noether normalizations says that every variety of dimension n is obtained as the normalization of a hypersurface in \mathbb{P}^{n+1} . We can thus start working through all hypersurfaces and describe their normalizations.

For curves this is not a bad approach. Classical authors developed much of the theory by thinking of smooth curves as normalizations of plane curves with nodes. However, the approach becomes harder and harder as the genus increases. The problem is that even if a curve is general, the nodal set of its plane representatives are always in very special position. There are some cases of surfaces where such a description is very useful, for example, Enriques obtained his namesake surfaces in 1896 as sextics in \mathbb{P}^3 that are double along the edges of a tetrahedron. However, for most surfaces Noether normalization introduces very complicated singular sets that hide the geometry of the surface. Also, it is quite hard to decide when the normalizations of 2 hypersurfaces are isomorphic to each other and there is no ‘optimal’ representation. This approach does not seem very helpful in general; see, however, the proof of Noether’s formula in [GH78, Sec.4.6].

Thus we aim to find projective embeddings of varieties that do not depend on too many auxiliary choices.

STEP 6.3 (Rigidification by embedding). A global coordinate system on a space V is a way of associating a string of numbers (called coordinates) to any point of V . Equivalently, a choice of a map $V \rightarrow \mathbb{R}^n$. We prefer to work with projective objects, so for us the natural choice is to use homogeneous coordinates. Equivalently, we fix an algebraic morphism $X \rightarrow \mathbb{P}^N$. (There is a slight notational issue here. Although we almost always construct \mathbb{P}^n as $\text{Proj } k[x_0, \dots, x_N]$, we usually emphasize that there are no natural coordinates on it. By contrast, with rigidification we do think of the target \mathbb{P}^N as having either fixed coordinates or at least no automorphisms.)

For varieties the most frequently used approach is to use an embedding ($X \hookrightarrow \mathbb{P}^N$), though in special cases finite maps $X \rightarrow \mathbb{P}^N$ or maps to other targets—weighted projective spaces or \mathbb{P}^N -bundles over curves—can give better insight.

Thus we choose a very ample line bundle L on X , a subspace $V^{N+1} \subset H^0(X, L)$ and a basis of V^{N+1} (up to a multiplicative constant). In practice it is much better to eliminate the second of these choices by taking $V = H^0(X, L)$. That is, we work with embeddings ($X \hookrightarrow \mathbb{P}^N$) whose image is *linearly normal*. The rigidification involves 2 types of choices.

Discrete choice 6.3.1. A very ample line bundle L . (We use this terminology although $\mathbf{Pic}(X)$ is not always discrete).

If C is a stable curve then ω_C^r is very ample for $r \geq 3$. If S is a canonical model of a surface of general type, then ω_S is an ample line bundle and ω_S^r is very ample for $r \geq 5$ by [Bom73, Eke88]. Thus again we get an embedding of S into a projective space whose dimension depends only on the coefficients of the Hilbert polynomial $\chi(\omega_S^r)$, namely (K_S^2) and $\chi(\mathcal{O}_S)$.

The situation is more complicated for stable surfaces. These can have singularities where ω_S is not locally free. Even worse, for any $m \in \mathbb{N}$ there are stable surfaces S_m and canonical 3-folds X_m such that $\omega_{S_m}^{[m]}$ (resp. $\omega_{X_m}^{[m]}$) is not locally free

at some point $x_m \in S_m$. Thus every section of $\omega_{S_m}^{[m]}$ vanishes at x_m and $H^0(X, \omega_{S_m}^{[m]})$ does not define an embedding of S_m .

We skirt this problem by fixing $m > 0$ and aiming to construct a moduli space for those stable varieties for which $\omega_S^{[m]}$ is locally free and very ample. Similarly, if (X, Δ) is a stable pair and Δ is a \mathbb{Q} -divisor, we can take $L = \omega_X^{[m]}(m\Delta)$ for some $m > 0$. Thus L is indeed a discrete choice for us.

Then we need to show in Step 6.8 that, if m is sufficiently divisible (depending on other numerical invariants), then the moduli theory we get is independent of m .

There does not seem to be a similarly natural choice of L if Δ is an \mathbb{R} -divisor. We have to work around this in Section 8.2.

Continuous choice 6.3.2. A basis in $H^0(X, L)$. The different continuous choices are equivalent to each other under the natural group action by $\mathrm{PGL}(H^0(X, L))$. We eliminate the effect of this choice in Step 6.5.

Aside. For smooth varieties over \mathbb{C} , the use of topological rigidifiers can be very powerful; leading to the Teichmüller space for curves and to Griffiths's theory of period domains [Gri03]. These work well for smooth varieties, but have many open problems for their degenerations. For flat families of stable varieties $f : X \rightarrow S$, the topological type, or even dimension of $H^*(X_s(\mathbb{C}), \mathbb{C})$ need not be a locally constant function on S . It does not seem to be possible to make sense of a topological rigidification in general.

Moduli of embedded varieties 6.3.3. Once we have a rigidification, we construct moduli spaces of more general embedded objects. Instead of embedded stable varieties ($X \hookrightarrow \mathbb{P}^N$) of dimension n one can work either with n -cycles (Cayley-Chow approach) or, which works better for us, with all subschemes ($X \subset \mathbb{P}^N$) (Hilbert-Grothendieck approach). Thus we start with the universal family over the Hilbert scheme on n -dimensional subschemes

$$\pi : \mathrm{Univ}_n(\mathbb{P}^N) \rightarrow \mathrm{Hilb}_n(\mathbb{P}^N).$$

We encounter a severe problem when we try to extend this method to pairs (X, Δ) .

Moduli of embedded pairs 6.3.4. We need to construct the universal family of relative Mumford divisors

$$\mathrm{MDiv}(\mathrm{Univ}_n(\mathbb{P}^N)/\mathrm{Hilb}_n(\mathbb{P}^N)).$$

The traditional approaches try to obtain this as a subscheme of either

$$(6.3.4.a) \quad \mathrm{Chow}_{n-1}(\mathrm{Univ}_n(\mathbb{P}^N)/\mathrm{Hilb}_n(\mathbb{P}^N)), \text{ or of}$$

$$(6.3.4.b) \quad \mathrm{Hilb}_{n-1}(\mathrm{Univ}_n(\mathbb{P}^N)/\mathrm{Hilb}_n(\mathbb{P}^N)).$$

As we noted in (6.2), the Chow version works over reduced schemes, but neither works in general. This was a long-standing conundrum in the theory.

Conclusion 6.3.5. We have the universal family of embedded varieties, but we hit a problem with pairs. The notion of K-flatness—to be worked out in Chapter 7—was introduced to solve this.

Here we will take an easier path, and in Sections 6.2–6.4 we consider several cases when (3.b) applies, see (6.13) for details.

Construction of the moduli space.

Assume now that the above steps have been completed. Then, instead of our original moduli problem, we have solved a related one that also includes a rigidification and has many more objects. In order to get back to our original problem, we need to remove the non-stable objects and then see how to undo the effects of rigidification.

STEP 6.4 (Representability). The previous steps gave us the universal family of relative Mumford divisors

$$\text{MDiv}(\text{Univ}_n(\mathbb{P}^N)/\text{Hilb}_n(\mathbb{P}^N)),$$

but this is way too big. The Hilbert scheme contains objects that are very singular, possibly even nonreduced and disconnected. We are interested only in stable families.

To understand this more generally, let $X \rightarrow S$ be a proper, flat morphism. The set of fibers that are geometrically reduced (or geometrically normal, or geometrically demi-normal or smooth) form an open subset $S^\circ \subset S$.

However, as we saw in Section 3.5, the set of stable fibers $S^{\text{stable}} \subset S$ is not open, not even locally closed. Nonetheless, as we proved in Section 3.5 for stable varieties and in Section 4.5 for stable pairs, stable families are parametrized by a locally closed partial subdivision $S^{\text{stab}} \rightarrow S$. Thus, although the set of all stable subvarieties of \mathbb{P}^N do not form a locally closed subvariety of $\text{Chow}(\mathbb{P}^N)$ or of $\text{Hilb}(\mathbb{P}^N)$, the functor of all stable families of subvarieties of \mathbb{P}^N is represented by a locally closed partial subdivision $\text{Chow}^{\text{stab}}(\mathbb{P}^N) \rightarrow \text{Chow}(\mathbb{P}^N)$ or $\text{Hilb}^{\text{stab}}(\mathbb{P}^N) \rightarrow \text{Hilb}(\mathbb{P}^N)$. Thus we have $\text{Hilb}_n^{\text{stab}}(\mathbb{P}^N)$ with induced universal family

$$\pi^{\text{stab}} : \text{Univ}_n^{\text{stab}}(\mathbb{P}^N) \rightarrow \text{Hilb}_n^{\text{stab}}(\mathbb{P}^N). \quad (6.4.1)$$

Let us now remember the choice we made in Step 6.3.2. We have 2 line bundles on $\text{Univ}_n^{\text{stab}}(\mathbb{P}^N)$:

- M , the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$, and
- our choice L , which is the m th reflexive power of the relative dualizing sheaf.

We are interested in those families $S \rightarrow \text{Hilb}_n^{\text{stab}}(\mathbb{P}^N)$ where the pullbacks of M and L are the ‘same.’ This is again a representable condition by (9.68).

Thus we get the universal family of m -canonically embedded stable varieties in \mathbb{P}^N . Let us denote it by

$$\pi_m^{\text{stab}} : \text{Univ}_{n,m}^{\text{stab}}(\mathbb{P}^N) \rightarrow \text{Hilb}_{n,m}^{\text{stab}}(\mathbb{P}^N). \quad (6.4.2)$$

The arguments for pairs are very similar.

Conclusion 6.4.3. For each $m > 0$ we obtain universal families of m -canonically embedded stable varieties and pairs. However, m and the embedding are artificial choices, we still need to undo their effect.

(In practice we need to be more precise here and control various properties of the embedding (like linear normality, vanishing of certain cohomology groups) but these turn out to be technical issues, see Section 8.3.)

STEP 6.5 (Quotients by group actions). Let us deal next with the continuous choice in the rigidification, which is a basis in $H^0(X, L)$ for varieties and a basis in $H^0(X, F \otimes L)$ for sheaves. As we noted in (6.3.2), the different continuous choices are equivalent to each other under a PGL-action.

This gives a group action on the moduli of rigidified objects, and the moduli space of the non-rigidified objects is the space of orbits of this action. This leads us to the question whether the orbit space of a group action $m : G \times Y \rightarrow Y$ on a variety or scheme Y is itself a variety or scheme. In general the answer is no but a series of works starting with Matsusaka, Mumford and Seshadri in the 60's and culminating with [Kol97, KM97] says that if $m : G \times Y \rightarrow Y \times Y$ is proper then the orbit space Y/G is naturally an algebraic space; see [Ols16] for a general treatment. This is slightly weaker than being a scheme but it will not cause any problems for us.

So, aside from the slight difference between schemes and algebraic spaces, we consider the quotient problem solved. See Section 8.5 for more details.

6.6 (Conclusion of Steps 6.1–6.5). Let us start with the case when Δ is a \mathbb{Q} -divisor. Fix the dimension n , the volume v and a finite set $\mathbf{a} = \{a_i \in [0, 1] \cap \mathbb{Q} : i \in I\}$; the latter will be called a *marking* in Section 8.1.

Let $\mathcal{SP}(\mathbf{a}, n, v)$ denote the functor of stable pairs $(X, \Delta = \sum a_i D_i)$ (where the D_i are effective \mathbb{Z} -divisors) of dimension n and volume v .

For $m \in \mathbb{N}$ let $\mathcal{SP}(\mathbf{a}, n, v, m) \subset \mathcal{SP}(\mathbf{a}, n, v)$ be those pairs for which $m(K_X + \Delta)$ is very ample.

So far we have constructed the coarse moduli spaces $\mathrm{SP}(\mathbf{a}, n, v, m)$; they are of finite type. (Note that $h^0(X, \mathcal{O}_X(mK_X + m\Delta))$ can be bounded using the volume by Matsusaka's inequality (11.42.3).)

Here n, v are locally constant in stable families, so fixing them is no restriction. However, m is usually not deformation invariant. If $m_1 | m_2$ then $\mathcal{SP}(\mathbf{a}, n, v, m_1) \subset \mathcal{SP}(\mathbf{a}, n, v, m_2)$, thus we obtain $\mathcal{SP}(\mathbf{a}, n, v)$ as the direct limit of the $\mathcal{SP}(\mathbf{a}, n, v, m)$. It is more convenient to consider the increasing subsequence

$$\cdots \subset \mathcal{SP}(\mathbf{a}, n, v, m!) \subset \mathcal{SP}(\mathbf{a}, n, v, (m+1)!) \subset \cdots$$

whose union is $\mathcal{SP}(\mathbf{a}, n, v)$.

It is not hard to see that this union is locally finite, thus we have obtained $\mathcal{SP}(\mathbf{a}, n, v)$ as an algebraic space that is locally of finite type.

Now that we have constructed our moduli spaces $\mathcal{SP}(\mathbf{a}, n, v)$, we should study their properties.

STEP 6.7 (Separatedness and valuative-properness). Since these notions depend only on families over DVR's, these will always hold for us. The discussion in (1.20) needs no amplification.

The next 2 topics merit a treatment of their own; here we give only a few comments and the main references to the literature.

STEP 6.8 (Boundedness). We aim to prove that $\mathrm{SP}(\mathbf{a}, n, v)$ is actually of finite type. Equivalently, that $\mathrm{SP}(\mathbf{a}, n, v) = \mathrm{SP}(\mathbf{a}, n, v, m)$ for some m (depending on \mathbf{a}, n, v).

We discussed the case of stable varieties in (1.21), but there are some changes for pairs. The main one is that the Hilbert function $\chi(X, \omega_X^{[r]}([r\Delta]))$ is no longer deformation invariant, but its (rescaled) leading coefficient

$$\mathrm{vol}(X, \Delta) = (K_X + \Delta)^{\dim X}$$

and the constant coefficient $\chi(X, \mathcal{O}_X)$ are. This is why we use only the volume in the definition of $\mathrm{SP}(\mathbf{a}, n, v)$ (6.5.1).

An infinite union is of finite type only if it eventually stabilizes, so one can formulate our question independent of moduli theory as follows. It was proved by [Ale93] for surfaces and by [HMX18] in general.

Strong boundedness theorem 6.8.1. Assume that the a_i are rational. Then there is an $m = m(\mathbf{a}, n, v)$ such that $mK_X + m\Delta$ is a very ample Cartier divisor for every $(X, \Delta) \in \mathcal{SP}(\mathbf{a}, n, v)$. Therefore, the moduli space $\mathcal{SP}(\mathbf{a}, n, v)$ is of finite type. \square

If some of the a_i are irrational, then usually $mK_X + m\Delta$ is never a \mathbb{Z} -divisor. The natural correction would be to use $mK_X + \lfloor m\Delta \rfloor$, but there are examples when it is never Cartier (11.41.3). Thus the best one can hope for is the following.

Strong boundedness conjecture 6.8.2. There are $m = m(\mathbf{a}, n, v)$ and $a_{im} = a_{im}(\mathbf{a}, n, v)$ such that $mK_X + \sum a_{im}D_i$ is a very ample Cartier divisor for every $(X, \Delta) \in \mathcal{SP}(\mathbf{a}, n, v)$.

This is currently not known, but most likely the methods of [Bir21] will lead to a proof.

The following variant is much easier to prove (4.69) and is sufficient for most applications.

Weak boundedness theorem 6.8.3. Every irreducible component of $\mathcal{SP}(\mathbf{a}, n, v)$ is of finite type. \square

STEP 6.9 (Projectivity). Once we know that the connected (or irreducible) components are proper, we would like to show that they are projective. In cases when GIT works, it gives (quasi)projectivity right away, but the general quotient theorems of [Kol97, KM97] do not give projectivity, in fact there are many quotients that are not quasi-projective [Kol06].

So we need to find some ample line bundles on our moduli spaces. Let $f : X \rightarrow S$ be a stable morphism. The only divisorial sheaves that we can always write down on X are $\omega_{X/S}^{[m]}$, and these give the sheaves $\det f_*\omega_{X/S}^{[m]}$ on S . It is not hard to work out that these are actually line bundles, so let us hope that some of these are ample.

It was Iitaka who realized that the sheaves $f_*\omega_{X/S}^{[m]}$ should always have semi-positivity properties, at least in characteristic 0, [Iit72]. These properties were established and applied to prove many of Iitaka's conjectures by many authors; see [Mor87] for a survey. These methods were used to prove projectivity statements for the moduli of stable surfaces in [Kol90]. Extending it to the higher dimensional cases turned out to be quite difficult. It was done by [Fuj18] for stable varieties and by [KP17] for stable pairs.

The situation is more complicated in positive characteristic, but the surface case was settled by [Pat14, Pat17].

Conclusion 6.9.1. In all cases the outcome is that every proper subset of the moduli space is projective. Thus we consider the projectivity question solved.

Let us now summarize in a checklist the properties that we would like to see.

6.10 (Good moduli theories). A moduli theory \mathbf{S} is given by specifying the objects over fields and the families. We are mainly studying those cases whose objects are various subsets of all stable pairs.

For example, the most classical example is *Curves*, whose objects are stable curves and whose families are all flat, proper morphisms with stable curves as fibers.

In Chapter 4 we established that optimal definitions over reduced base spaces and proved its properties. However, there seem to be several possible theories over non-reduced base schemes, so let us list the properties that we aim for.

We say that \mathbf{S} is a *good moduli theory* if the following hold.

- (6.10.1) \mathbf{S} is separated (1.20). Since this depend only on families over DVR's, this always holds for us by (2.48).
- (6.10.2) \mathbf{S} is valuative-proper (1.20). The positive answer is given by (2.49), but we need to check that the central fiber also satisfies the additional assumptions that we have in \mathbf{S} .
- (6.10.3) Embedded moduli spaces exist (6.3.3–4). This is the main point where having a flat divisorial part makes the theory much simpler; see (6.12–6.13) for details.
- (6.10.4) Representability as in (6.4).
- (6.10.5) Boundedness in the weaker form (6.8.3). Together with valuative-properness, this means that the irreducible components of the corresponding moduli spaces are proper.

For the main results of this Chapter we work with the following set-up, which is a slight generalization of (3.25) and (4.1).

6.11 (Basic set-up). We consider flat families of demi-normal schemes with flat families of Mumford divisors. That is, our objects are proper morphisms $f : X \rightarrow S$ of pure relative dimension n , and subschemes $\{D_i \subset X : i \in I\}$ satisfying the following conditions.

- (6.11.1) f is flat with demi-normal fibers,
- (6.11.2) the D_i are relative Mumford divisors, and
- (6.11.3) the $D_i \rightarrow S$ are flat with divisorial subschemes as fibers.

Note that (3) means that the D_i satisfy the equivalent conditions of (2.75).

Furthermore, if D is a relative Mumford divisor, then $\mathcal{O}_X(D)$ is a mostly flat family of divisorial sheaves (3.25). We will repeatedly apply (3.27) such sheaves.

Next, fix distinct, positive real numbers $\{d_i : i \in I\}$. Then $f : (X, \sum d_i D_i) \rightarrow S$ is family of pairs as in (5.2).

We say that this $f : (X, \sum d_i D_i) \rightarrow S$ is *reduced-stable* (resp. *locally reduced-stable*) if its restriction to $\text{red } S$ is stable (resp. locally stable). It is clear that stable implies reduced-stable, thus the main question we aim to address is the following.

Question 6.11.4. What additional restrictions should be imposed in order to get a stable (resp. locally stable) family over a non-reduced scheme S ?

Comments 6.11.5. As I understand it, there may be several different good answers to this question. These in turn give different moduli spaces, though all of them have the same underlying reduced subspace.

Also, as we noted in (2.40–2.42), requiring the D_i to be flat over S means that we do not even get all stable families over smooth curves when $a_i < \frac{1}{2}$. So, while our answers cover many important special cases, substantially new ideas will be needed to get the full theory.

6.12 (Advantages of flat divisorial parts). The cases considered in this chapter have 4 major advantages. The first 3 come from using option (6.3.4.b), that is, flatness, for the divisorial part.

- (6.12.1) One can define the families using only flatness; thus we avoid the notion of K -flatness, which is defined and studied in Chapter 7.
- (6.12.2) Hilbert schemes give a quick way to write down the universal family of Mumford divisors.
- (6.12.3) The pluricanonical sheaves all commute with base change, as in (2.77). This is not crucial, but it helps us avoid some artificial choices.

The last one may be an accidental consequence of our choices.

- (6.12.4) There is a natural way of writing the boundary as a linear combination of \mathbb{Z} -divisors, thus we avoid the notion of marking, to be introduced in Section 8.4.

The key advantage turns out to be (6.12.2), which takes care of Step 6.3.4. So let us discuss it in detail.

6.13 (Universal family of flat Mumford divisors). Let $g : X \rightarrow S$ be a flat, projective morphism. Consider the relative Hilbert scheme of $\text{Hilb}(X/S)$. It parametrizes flat families of closed subschemes of $X \rightarrow S$. Thus it has a largest open subscheme that parametrizes subschemes $B_s \subset X_s$ of pure codimension 1, without embedded points, such that X_s is regular at the generic points of B_s . This is the universal family of flat, Mumford divisors on X/S . Let us denote it by

$$\text{MDiv}(X/S) \rightarrow S,$$

When we wish to parametrize r such divisors, the universal family is given by the r -fold fiber product

$$\text{MDiv}(X/S) \times_S \cdots \times_S \text{MDiv}(X/S),$$

which we abbreviate as $\times_S^r \text{MDiv}(X/S)$.

We want to apply this to the Hilbert scheme with its universal family. Although not strictly necessary, it is convenient to pass to the a largest open subscheme $\text{Hilb}^\circ(\mathbb{P}_S^N) \subset \text{Hilb}(\mathbb{P}_S^N)$ over which the fibers of u are demi-normal and of pure dimension n . Thus we have

$$u : \text{Univ}_n^\circ(\mathbb{P}_S^N) \rightarrow \text{Hilb}_n^\circ(\mathbb{P}_S^N). \quad (6.13.1)$$

The universal family of flat, Mumford divisors is

$$\text{MDiv}(\text{Univ}_n^\circ(\mathbb{P}_S^N) / \text{Hilb}_n^\circ(\mathbb{P}_S^N)) \rightarrow \text{Hilb}_n^\circ(\mathbb{P}_S^N). \quad (6.13.2)$$

If we need r such divisors, the universal family we want is given by the r -fold fiber product

$$\times_{\text{Hilb}_n^\circ(\mathbb{P}_S^N)}^r \text{MDiv}(\text{Univ}_n^\circ(\mathbb{P}_S^N) / \text{Hilb}_n^\circ(\mathbb{P}_S^N)). \quad (6.13.3)$$

As in (6.3.3), we can now use (4.50) to show that the functor of stable pairs is representable by a monomorphism, and (9.68) takes care of the condition of being embedded by a given multiple of $K_X + \Delta$.

In the next 3 sections we give various stability notions, and then check that they all give a good moduli theory as in (6.10).

6.2. Kollár–Shepherd-Barron stability

This notion of stability is obtained by imposing the strongest possible properties that are satisfied by 1-parameter stable families. For surfaces, this was accomplished in [KSB88]. There were 2 reasons why the original paper dealt only with surfaces. First, the existence of stable limits relies on the minimal model program, which

was only available for families of surfaces at that time. It was, however, clear that this part should work in all dimensions. Second, the proof of the representability (6.17) relied on detailed properties of deformations of lc singularities of surfaces. The theory of hulls and husks, to be discussed in Chapter 9, was then developed mainly to prove representability.

We discuss two versions. First the classical setting of stable varieties without boundary divisors, and then a generalization where we allow boundary divisors with coefficients $1 - \frac{1}{n}$.

Kollár–Shepherd-Barron stability without boundary.

6.14 (Stable objects). The stable objects are geometrically reduced, proper k -schemes X with slc singularities such that K_X is ample.

6.15 (Stable families). A family $f : X \rightarrow S$ is *KSB-stable* if the following hold.

(6.15.1) f is flat with slc fibers.

(6.15.2) $\omega_{X/S}^{[m]}$ commutes with base change for every $m \geq 0$.

(6.15.3) f is proper and $\omega_{X/S}^{[M]}$ is an f -ample line bundle for some $M > 0$.

The first 2 of these conditions define *locally KSB-stable* families.

6.16 (Explanation). As far as I know, this definition imposes the strongest restrictions on stable families, thus it gives the smallest scheme structure on the moduli space of stable varieties.

We see in Section 6.3 that assumption (6.15.2) can be weakened, leading to a moduli space with the same underlying reduced space but with a possibly larger nilpotent structure. The difference between the 2 versions is explored in Section 6.6.

THEOREM 6.17. *KSB-stability, as defined in (6.14–6.15) is a good moduli theory (6.10).*

Proof. As we already noted, only conditions (6.10.2–4) need checking. Valuative-properness is (2.49), the existence of embedded moduli spaces is a trivial special case of (6.13), and representability is a restatement of (3.3). \square

Let us also note another good property of this case.

PROPOSITION 6.18. *For KSB-stable families as in (6.14–6.15), the Hilbert function $\chi(X, \omega_X^{[m]})$ and the plurigenera $h^0(X, \omega_X^{[m]})$ are deformation invariant.*

Proof. For the Hilbert function, this follows from the assumption (6.15.2).

If $m \geq 2$ then the higher cohomologies of $\omega_X^{[m]}$ vanish by (11.33). For $m = 1$ we use (2.70). \square

Kollár–Shepherd-Barron stability with standard coefficients.

DEFINITION 6.19. Let Δ be an effective \mathbb{R} -divisor such that $1 \geq \text{coeff}_D \Delta > \frac{1}{2}$ for every irreducible divisor $D \subset \text{Supp } \Delta$. Then there is a unique way of writing $\Delta = \sum_i a_i D_i$ where the D_i are reduced divisors, $a_i > \frac{1}{2}$ for every i and $a_i \neq a_j$ for $i \neq j$. We call this the *reduced normal form* of Δ .

6.20 (Stable objects). We parametrize pairs $(X, \Delta = \sum_i a_i D_i)$ in reduced normal form such that

(6.20.1) (X, Δ) is slc,

$$(6.20.2) \quad a_i \in \{1 - \frac{1}{3}, 1 - \frac{1}{4}, \dots, 1\},$$

(6.20.3) X is projective and $K_X + \Delta$ is ample.

6.21 (Stable families). A family $f : (X, \Delta = \sum_i a_i D_i) \rightarrow S$ is *KSB-stable* if the following hold.

(6.21.1) $f : (X, \Delta) \rightarrow S$ is a flat family of pairs as in (6.11).

(6.21.2) The fibers (X_s, Δ_s) satisfy (6.20.1–2).

(6.21.3) $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$ commutes with base change for every $m \geq 0$.

(6.21.4) f is proper and $\omega_{X/S}^{[M]}(M\Delta)$ is an f -ample line bundle for some $M > 0$.

The first 3 of these conditions define *locally KSB-stable* families.

6.22 (Explanation). These conditions are rather straightforward generalizations of (6.15.1–3). The main question is: why the restriction on the coefficients?

It follows from (2.77.4) that condition (6.21.3) is satisfied if the coefficients of Δ are standard. We proved in (2.82) that if the coefficients are $> \frac{1}{2}$ then the scheme-theoretic specialization of the boundary divisors are reduced, and the different $(D_i)_s$ have no common irreducible components. In particular, $\lfloor m\Delta \rfloor_s = \lfloor m\Delta_s \rfloor$ for every $s \in S$. That is, valuative-properness holds. We are thus assuming that both of these conditions are satisfied.

The case of all pairs $(X, \Delta = \sum_i a_i D_i)$ where $\frac{1}{2} < a_i \leq 1$ for every i is studied in (6.25–6.26).

THEOREM 6.23. *KSB-stability with standard coefficients, as defined in (6.20–6.21) is a good moduli theory (6.10).*

Proof. As before only (6.10.2–4) need checking. We noted above that valuative-properness holds, and the existence of embedded moduli spaces follows from (6.13).

For representability, the proof of (3.3)—given in (3.43)—carries over with minor changes. This is, however, an important technical point, so let us go through the details.

By (4.51) the set of slc fibers $S^{\text{slc}} := \{s \in S : (X_s, \Delta_s) \text{ is slc}\}$ is constructible, hence $\{\text{index}(K_{X_s} + \Delta_s) : s \in S^{\text{slc}}\}$ is bounded. Let M be a common multiple of these values.

We apply (3.27) to each $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$. We get monomorphisms $j_m : S^m \rightarrow S$ with the following property:

(6.23.1.m) Let $q : T \rightarrow S$ be a morphism. Then $\omega_{X_T/T}^{[m]}(\lfloor m\Delta \rfloor)$ commutes with further base changes iff q factors as $q : T \rightarrow S^m \rightarrow S$.

If $X_T \rightarrow T$ is KSB-stable then $\omega_{X_T/T}^{[M]}(\lfloor M\Delta \rfloor) = \omega_{X_T/T}^{[M]}(M\Delta)$ is locally free, hence it commutes with further base changes. Thus q factors through S^M .

Next (4.52) shows that the stable fibers are parametrized by an open subscheme of $\text{red } S^M$; let $S^\circ \subset S^M$ be the corresponding open subscheme of S^M . Thus $X \times_S S^\circ \rightarrow S^\circ$ satisfies (6.21.1–2) and (6.21.4).

We still need to deal with the infinitely many conditions (6.21.3.m). However, once $\omega_{X_T/T}^{[M]}(\lfloor M\Delta \rfloor)$ is a line bundle, then

$$\omega_{X_T/T}^{[m+rM]}(\lfloor (m+rM)\Delta \rfloor) \cong \omega_{X_T/T}^{[m]}(\lfloor m\Delta \rfloor) \otimes (\omega_{X_T/T}^{[M]}(\lfloor M\Delta \rfloor))^{\otimes r}. \quad (6.23.1)$$

Thus (6.21.3) holds for every m since it holds for all $1 \leq m \leq M$. Therefore

$$S^{\text{KSB}} = S^\circ \times_S S^1 \times_S \cdots \times_S S^{M-1}. \quad \square$$

PROPOSITION 6.24. [Kol18a, Cor.3] *For KSB-stable families with standard coefficients as in (6.20–6.21), the Hilbert function $\chi(X, \omega_X^{[m]}(\lfloor m\Delta \rfloor))$ and the plurigenera $h^0(X, \omega_X^{[m]}(\lfloor m\Delta \rfloor))$ are deformation invariant.*

Proof. For the Hilbert function, this follows from the assumption (6.21.3). For the plurigenera, write

$$mK_X + \lfloor m\Delta \rfloor = K_X + (\lfloor m\Delta \rfloor - (m-1)\Delta) + (m-1)(K_X + \Delta).$$

Since the coefficients are standard, $0 \leq \lfloor m\Delta \rfloor - (m-1)\Delta \leq \Delta$, hence (11.33) applies and the higher cohomologies vanish for $m \geq 2$. For $m = 1$ we use (2.70). \square

Kollár–Shepherd-Barron stability with major coefficients.

6.25 (Stable objects). We parametrize pairs $(X, \Delta = \sum_i a_i D_i)$ in reduced normal form (6.19) such that

$$(6.25.1) \quad (X, \Delta) \text{ is slc,}$$

$$(6.25.2) \quad a_i \in (\frac{1}{2}, 1] \cap \mathbb{Q},$$

$$(6.25.3) \quad X \text{ is projective and } K_X + \Delta \text{ is ample.}$$

6.26 (Stable families). A family $f : (X, \Delta = \sum_i a_i D_i) \rightarrow S$ is *KSB-stable* if the following hold.

$$(6.26.1) \quad f : (X, \Delta) \rightarrow S \text{ is a flat family of pairs as in (6.11).}$$

$$(6.26.2) \quad \text{The fibers } (X_s, \Delta_s) \text{ satisfy (6.25.1–2).}$$

$$(6.26.3) \quad \omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor) \text{ commutes with base change for every } m \geq 0.$$

$$(6.26.4) \quad f \text{ is proper and } K_{X/S} + \Delta \text{ is an } f\text{-ample } \mathbb{R}\text{-divisor.}$$

The first 3 of these conditions define *locally KSB-stable* families.

CONJECTURE 6.27. *KSB-stability with major coefficients, as defined in (6.25–6.26) is a good moduli theory (6.10).*

6.28 (Comments). The proof should closely follow (6.23), but there are some unresolved issues here.

We proved in (2.82) that, if the coefficients are $> \frac{1}{2}$, then the scheme-theoretic specialization of the boundary divisors are reduced, so assuming that the D_i are flat divisorial sheaves is correct. Thus valuative-properness holds.

However, we can not guarantee the assumption (6.26.3) over reduced bases. Following [Kol18a], we outlined a proof in (2.83) when the general fibers are normal, and [Kol18b] treats all families of surfaces.

For representability we again follow the proofs in (3.43) and (6.23). Note that these start with a monomorphism that makes

$$\omega_{X/S}^{[M]}(\lfloor M\Delta \rfloor) = \omega_{X/S}^{[M]}(M\Delta)$$

locally free. Such an $M > 0$ exists if the a_i are rational, but not if some of them are irrational. This is the main reason why the \mathbb{Q} -divisor cases is different from the \mathbb{R} -divisor ones.

Actually, it turns out that when Δ is not a \mathbb{Q} -divisor, we can extract several independent line bundles from it, so in some sense this case is simpler than for

\mathbb{Q} -divisors. The purest form of this is discussed in Section 6.4; the general setting is postponed to Chapter 8.

The following is a direct consequence of (6.26.3).

PROPOSITION 6.29. *For KSB-stable families with major coefficients as in (6.25–6.26), the Hilbert function $\chi(X, \omega_X^{[m]}(\lfloor m\Delta \rfloor))$ is deformation invariant.* \square

Unlike in the earlier cases, the plurigenera $h^0(X, \omega_X^{[m]}(\lfloor m\Delta \rfloor))$ need not be deformation invariant, see [Kol18a, 40–43].

6.3. Strict Viehweg stability

6.30 (Stable objects). The same as in (6.14): reduced, proper k -schemes X with slc singularities such that K_X is ample.

6.31 (Stable families). A family $f : X \rightarrow S$ is *SV-stable* if the following hold.

(6.31.1) f is flat with slc fibers.

(6.31.2) For every m , $\omega_{X/S}^{[m]}$ is locally free at $x \in X_s$ iff $\omega_{X_s}^{[m]}$ is locally free at $x \in X_s$.

(6.31.3) f is proper and $\omega_{X/S}^{[M]}$ is an f -ample line bundle for some $M > 0$.

The first 2 of these conditions define *locally SV-stable* families.

6.32 (Explanation). The original version, adopted in [Vie95] and frequently called *V-stability*, assumes (6.31.2) only for some $m > 0$. By (2.96) the latter is equivalent to SV-stability in characteristic 0, but has some rather unexpected properties in positive characteristic, see Section 1.9.

It is actually not obvious that 1-parameter families do satisfy (6.31.2) in positive characteristic; a key result in this direction is (5.53).

Already for families of surfaces with quotient singularities this definition gives a large nilpotent structure on the moduli space of stable varieties, even when KSB-stability gives a smooth moduli space, see Section 6.6.

SV-stability may be a more natural notion for pairs.

Strict Viehweg stability with major coefficients.

6.33 (Stable objects). We parametrize pairs $(X, \Delta = \sum_i a_i D_i)$ in reduced normal form such that

(6.33.1) (X, Δ) is slc,

(6.33.2) $a_i \in (\frac{1}{2}, 1] \cap \mathbb{Q}$ for every i ,

(6.33.3) X is projective and $K_X + \Delta$ is ample.

The first 2 of these conditions define *locally stable* pairs.

6.34 (Stable families). A family $f : (X, \Delta = \sum_i a_i D_i) \rightarrow S$ is *SV-stable* if the following hold.

(6.34.1) $f : X \rightarrow S$ is flat and the fibers of $f|_{D_i} : D_i \rightarrow S$ are reduced subschemes of pure codimension 1 for every i .

(6.34.2) The fibers $(X_s, \sum_i a_i (D_i)_s)$ are stable as in (6.33).

(6.34.3) $\omega_{X/S}^{[m]}(m\Delta)$ is locally free along X_s iff $\omega_{X_s}^{[m]}(m\Delta_s)$ is locally free.

(6.34.4) f is proper and $\omega_{X/S}^{[M]}(M\Delta)$ is an f -ample line bundle for some $M > 0$.

The first 3 of these conditions define *locally stable* families.

6.35 (Explanation). These conditions are rather straightforward generalizations of (6.31) and (6.26).

THEOREM 6.36. *SV-stability with major coefficients, as defined in (6.33–6.34) is a good moduli theory (6.10).*

Proof. The arguments given in (6.28) work since we no longer require the condition (6.26.3) that gave us trouble there.

Representability is actually simpler than in the proof of (6.23). We work only with the locally free $\omega_{X/S}^{[M]}(M\Delta)$, and ignore the other $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$. \square

6.4. Alexeev stability

6.37 (Stable objects). We parametrize pairs $(X, \Delta = \sum_i a_i D_i)$ in reduced normal form such that

(6.37.1) (X, Δ) is slc,

(6.37.2) $1, a_1, \dots, a_r$ are \mathbb{Q} -linearly independent,

(6.37.3) X is projective and $K_X + \Delta$ is ample.

6.38 (Stable families). A family $f : (X, \Delta = \sum_i a_i D_i) \rightarrow S$ is *A-stable* if the following hold.

(6.38.1) $f : (X, \Delta) \rightarrow S$ is a flat family of pairs as in (6.11).

(6.38.2) the fibers (X_s, Δ_s) are stable as in (6.37).

(6.38.3) $\omega_{X/S}^{[m_0]}(\sum m_i D_i)$ commute with base change for every m_i .

(6.38.4) f is proper and $K_{X/S} + \Delta$ is an f -ample \mathbb{R} -divisor.

The first 3 of these conditions define *locally A-stable* families.

6.39 (Explanation). The 2 new features are the \mathbb{Q} -linear independence in (6.37) and (6.38.3).

Let us start with \mathbb{Q} -linear independence. As a simple example, let X be a smooth, projective variety and $\sum D_i$ an snc divisor with index set $\{i \in I\}$. Then $(X, \sum a_i D_i)$ is an lc pair for every $a_i \in [0, 1]$. So we can ask how the answers to various questions—for example the ampleness of $K_X + \sum a_i D_i$, or the steps of the MMP—depend on the a_i .

In many cases the answer is that $[0, 1]^I$ admits a rational chamber decomposition, such that the answers depend only on the chamber we are in, not the particular choice of the $\{a_i : i \in I\}$ inside the chamber. There is reason to expect that if a point $\{a'_i : i \in I\}$ lies in an open chamber, then $K_X + \sum a'_i D_i$ exhibits generic—hence simplest—behavior.

Since the chambers are polyhedra with rational vertices, a point $\{a'_i : i \in I\}$ whose coordinates are \mathbb{Q} -linearly independent, must lie in an open chamber. Thus condition (6.37) is a convenient way to guarantee that we encounter the generic behavior.

A hint about (6.38.3) was already given in the proof of (6.27). By (11.34), the \mathbb{Q} -linear independence assumption implies that $K_{X/S} + \sum_i a_i D_i$ is \mathbb{R} -Cartier iff $K_{X/S}$ and the D_i are \mathbb{Q} -Cartier. Thus all the $m_0 K_{X/S} + \sum m_i D_i$ are \mathbb{Q} -Cartier \mathbb{Z} -divisors.

Assume for a moment that S is reduced. If $f : (X, \Delta = \sum_i a_i D_i) \rightarrow S$ is locally A-stable and $K_{X/S}$ is \mathbb{Q} -Cartier, then $f : X \rightarrow S$ is locally stable. Thus,

once the D_i are \mathbb{Q} -Cartier, all the sheaves in (6.38.3) commute with base change (apply (11.7) with $\Delta' = 0$).

In the spirit of KSB-stability, it is thus best to require (6.38.3) over any base scheme. This gives a moduli space with many flat universal sheaves, and, as we see next, it also helps with the proof of existence.

Note also that we do not assume anything about the sheaves $\omega_{X/S}^{[m]}([m\Delta])$. If the $a_i < \frac{1}{2}$ then these frequently do not commute with base change (2.40–2.42). Although (11.41) shows that infinitely many of them do commute, I do not know how to predict which ones.

THEOREM 6.40. *A-stability, as defined in (6.37–6.38) is a good moduli theory (6.10).*

Proof. As before, separatedness and valuative-properness holds. The idea of the proof of the existence of embedded moduli spaces is the following. The chamber structure mentioned in (6.39) suggests that, if we pick a rational point (a'_1, \dots, a'_r) in the interior of the chamber, then the pairs $(X, \sum a_i D_i)$ and $(X, \sum a'_i D_i)$ have the same moduli theory. We can thus work with the rational-coefficient pairs $(X, \sum a'_i D_i)$ as in (6.13). This is basically what we do, but the details are more complicated. See (8.16) for a full treatment.

Representability needs a somewhat different proof. The set of slc fibers is constructible by (4.51), hence there are $M_i > 0$ such that $M_0 K_{X_s}$ and the $M_i D_i|_{X_s}$ are Cartier whenever (X_s, Δ_s) is slc. To simplify notation, set $D_0 := K_{X/S}$.

Applying (3.27) to the $M_i D_i$, and taking fiber product, we get a monomorphism $S^1 \rightarrow S$ such that the sheaves $\mathcal{O}_{X^1}(M_i D_i^1)$ commute with further base changes $q: T \rightarrow S^1$. We can now pass to an open subset $S^2 \subset S^1$ such that the $\mathcal{O}_{X^2}(M_i D_i^2)$ are locally free.

We next perform the same steps for each $\mathcal{O}_{X^2}(\sum m_i D_i^2)$ for every $0 \leq m_i \leq M_i$. We get a monomorphism $S^3 \rightarrow S^2$ such that these sheaves commute with further base changes $q: T \rightarrow S^3$. As in (6.17.1), then the $\mathcal{O}_{X^3}(\sum m_i D_i^3)$ commute with further base changes for every $m_i \in \mathbb{Z}$.

Next (3.38) shows that the stable fibers are parametrized by an open subscheme $S^4 \subset \text{red } S^3$. Thus $S^A \subset S^3$ is the open subscheme of S^3 for which $\text{red } S^A = S^4$. \square

6.5. First order deformations

In this section we study first order infinitesimal deformations of normal varieties. We describe the deformations of the smooth locus and then try to understand when a deformation of the smooth locus extends to a deformation of the whole variety. The final aim is to get an explicit obstruction theory for lifting sections of powers of the dualizing sheaf. This turns out to be given by the classical notion of divergence.

6.41 (First order thickening). Let k be a field and R a k -algebra. Consider the algebra $R[\epsilon]$ where ϵ is a new variable satisfying $\epsilon^2 = 0$. It is flat over $k[\epsilon]$ and $R[\epsilon] \otimes_{k[\epsilon]} k \cong R$. Thus we can think of $R[\epsilon]$ as the trivial first order deformation of R .

Let $v: R \rightarrow R$ be a k -linear derivation. Then

$$\alpha_v: r_1 + \epsilon r_2 \mapsto r_1 + \epsilon(v(r_1) + r_2) \quad (6.41.1)$$

defines an automorphism of $R[\epsilon]$ that is trivial modulo (ϵ) . Conversely, every automorphism of $R[\epsilon]$ that is trivial modulo (ϵ) arises this way. (The product (or Leibnitz) rule for v is equivalent to the multiplicativity of α_v .)

Let X be a k -scheme. The trivial first order deformation of X is

$$X[\epsilon] := X \times_k \text{Spec}_k k[\epsilon]. \quad (6.41.2)$$

As in (6.41.1), every derivation $v : \mathcal{O}_X \rightarrow \mathcal{O}_X$ defines an automorphism α_v of $X[\epsilon]$ that is trivial modulo (ϵ) . This gives an exact sequence

$$0 \rightarrow \text{Hom}(\Omega_X^1, \mathcal{O}_X) \rightarrow \text{Aut}(X[\epsilon]) \rightarrow \text{Aut}(X) \rightarrow 1. \quad (6.41.3)$$

If X is smooth, or at least normal, then $\mathcal{H}om(\Omega_X^1, \mathcal{O}_X)$ is the tangent sheaf T_X of X , hence we can rewrite the sequence as

$$0 \rightarrow H^0(X, T_X) \xrightarrow{\alpha} \text{Aut}(X[\epsilon]) \rightarrow \text{Aut}(X) \rightarrow 1. \quad (6.41.4)$$

Aside. On a differentiable manifold M one can identify the Lie algebra of all vector fields with the Lie algebra of the automorphism group. If X is a smooth variety, then this identification works if X is proper but not otherwise. For instance, an affine curve C of genus ≥ 1 has only finitely many automorphisms but $H^0(C, T_C)$ is infinite dimensional. Infinitesimal thickenings restore the connection between vector fields and automorphisms.

6.42 (Locally trivial first order deformations). Let k be a field and X a k -scheme. A *deformation* of X over $A := \text{Spec}_k k[\epsilon]$ is a flat A -scheme X' together with an isomorphism $X' \times_A \text{Spec}_k k \cong X$. The set of isomorphism classes of first order deformations is denoted by $T^1(X)$. It is easy to see that $T^1(X)$ is naturally a k -vector space whose zero is the trivial deformation $X[\epsilon]$, but this is not very important for us now. See [Art76] or [Har10] for detailed discussions.

We say that X' is *locally trivial* if there is an affine cover $X = \cup_i X_i$ such that each X'_i is a trivial deformation of X_i .

We aim to classify all locally trivial first order deformations of arbitrary k -schemes X , but our main interest is in cases when X is smooth and quasi-projective.

Let $X = \cup_i X_i$ be an affine cover. This gives an affine cover $X' = \cup_i X'_i$ and we assume that each X'_i is a trivial deformation of X_i . Fix trivializations $\phi_i : X'_i \cong X_i[\epsilon]$. Over $X'_{ij} := X'_i \cap X'_j$ we have 2 trivializations, these differ by an automorphism

$$\alpha_{ij} := \phi_j^{-1} \circ \phi_i : X'_{ij} \rightarrow X'_{ij}, \quad (6.42.1)$$

which is the identity on X_{ij} . By (6.41.1) the automorphisms α_{ij} correspond to $v_{ij} \in \text{Hom}(\Omega_{X_{ij}}^1, \mathcal{O}_{X_{ij}})$ and these form a 1-cocycle $D := \{v_{ij}\}$. Changing the trivializations changes the cocycle by a coboundary. Thus we get a well defined element

$$D = D(X') \in H^1(X, \mathcal{H}om(\Omega_X^1, \mathcal{O}_X)). \quad (6.42.2)$$

The construction can be reversed. It is left to the reader to check that $D(X')$ is independent of the choices we made. The final outcome is the following.

Claim 6.42.3. Let X be a k -scheme. There is a one-to-one correspondence, denoted by $D \mapsto X_D$, between

- (a) elements of $H^1(X, \mathcal{H}om(\Omega_X^1, \mathcal{O}_X))$ and
- (b) locally trivial deformations of X over $\text{Spec}_k k[\epsilon]$, up-to isomorphism.

Furthermore, if X is normal then $H^1(X, \mathcal{H}om(\Omega_X^1, \mathcal{O}_X)) = H^1(X, T_X)$. \square

Next we check that every first order deformation of a smooth variety Y is locally trivial. To see this we may assume that Y is affine. Then Y' is also affine and we can fix a vector space isomorphism $k[Y'] \cong k[Y] \otimes k[\epsilon]$. Pick a point $p \in Y$, local coordinates y_1, \dots, y_n and their trivial lifts $y'_1, \dots, y'_n \in k[Y']$. Any other $z \in k[Y]$ satisfies a monic, separable equation $F(z, \mathbf{y}) = 0$. We claim that z has a unique lift $z' \in k[Y']$ such that $F(z', \mathbf{y}') = 0$. To see this pick any lift z^* . Then $F(z^*, \mathbf{y}') = \epsilon G(z)$ for some $G(z) \in k[Y]$. We are looking for z' in the form $z' = z^* + \epsilon g$ where $g \in k[Y]$. Since $F(z^* + \epsilon g, \mathbf{y}') = \epsilon G(z) + \epsilon g \cdot \partial F(z, \mathbf{y})/\partial z$, we see that $g = -G(z)(\partial F(z, \mathbf{y})/\partial z)^{-1}$ is the unique solution. We do this for a finite set of generators $\{z_i\}$ of $k[Y]$ to get a trivialization in a neighborhood where all the $\partial F_i(z, \mathbf{y})/\partial z$ are invertible.

Combining with (6.42.3), this proves the following. (See [Har77, Exrc.II.8.6] for a slightly different proof.)

Claim 6.42.4. Every deformation of a smooth, affine variety over $k[\epsilon]$ is trivial. \square

6.43 (Arbitrary first order deformations). Let k be a field and X a normal k -variety. Let $U \subset X$ be the smooth locus, $Z \subset X$ the singular locus and $j : U \hookrightarrow X$ the natural injection.

Let $X' \rightarrow \text{Spec}_k k[\epsilon]$ be a flat deformation of X . By restriction it induces a flat deformation U' of U . Note that U' uniquely determines X' . Indeed, $\text{depth}_Z \mathcal{O}_X \geq 2$ since X is normal, hence $\text{depth}_Z \mathcal{O}_{X'} \geq 2$ since $\mathcal{O}_{X'}$ is an extension of 2 copies of \mathcal{O}_X . Therefore $\mathcal{O}_{X'} = j_* \mathcal{O}_{U'}$ by (9.8). Thus we have an injection

$$T^1(X) \hookrightarrow T^1(U) = H^1(U, T_U).$$

Following [Sch71], our plan is to study $T^1(X)$ by first describing $T^1(U)$ and then understanding which $D \in H^1(U, T_U)$ correspond to a deformation of X ; see also [vE90]. The second step is accomplished in (6.46).

DEFINITION 6.44. Let X be a k -scheme. Given $v \in \text{Hom}(\Omega_X^1, \mathcal{O}_X)$, *differentiation* by v is defined as the composite

$$v(\cdot) : \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{v} \mathcal{O}_X. \tag{6.44.1}$$

Let x_1, \dots, x_n be (analytic or étale) local coordinates at a smooth point of X and write $v = \sum_i v_i \frac{\partial}{\partial x_i}$. Then the above maps are

$$v : f \mapsto \sum_i \frac{\partial f}{\partial x_i} dx_i \mapsto \sum_i v_i \frac{\partial f}{\partial x_i}.$$

Thus if X is smooth and v is identified with a section of T_X , then (6.44.1) agrees with the usual definition.

Next let $D \in H^1(X, \mathcal{H}om(\Omega_X^1, \mathcal{O}_X))$ and choose a representative 1-cocycle $D = \{v_{ij}\}$ using an affine cover $X = \cup X_i$. For any $s \in H^0(X, \mathcal{O}_X)$ the derivatives $\{v_{ij}(s|_{X_{ij}})\}$ form a 1-cocycle with values in \mathcal{O}_X . This defines $D(s) \in H^1(X, \mathcal{O}_X)$. We think of it either as a *cohomological differentiation* map

$$D : H^0(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X) \tag{6.44.2}$$

or as a k -bilinear map

$$H^1(X, \mathcal{H}om(\Omega_X^1, \mathcal{O}_X)) \times H^0(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X). \tag{6.44.3}$$

If X is normal then we can rewrite this as

$$H^1(X, T_X) \times H^0(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X). \quad (6.44.4)$$

Let X_D be the deformation of X corresponding to D . Its structure sheaf sits in an exact sequence

$$0 \rightarrow \epsilon \mathcal{O}_X \rightarrow \mathcal{O}_{X_D} \rightarrow \mathcal{O}_X \rightarrow 0. \quad (6.44.5)$$

Taking cohomology we see that D in (6.44.2) is the connecting map

$$H^0(X_D, \mathcal{O}_{X_D}) \rightarrow H^0(X, \mathcal{O}_X) \xrightarrow{D} H^1(X, \mathcal{O}_X). \quad (6.44.6)$$

Warning 6.44.7. Note that although $H^0(X, \mathcal{O}_X)$ and $H^1(X, \mathcal{O}_X)$ are both $H^0(X, \mathcal{O}_X)$ -modules, the map D is usually not an $H^0(X, \mathcal{O}_X)$ -module homomorphism. Indeed, the constant section $1_X \in H^0(X, \mathcal{O}_X)$ always lifts, hence $D(1_X) = 0$. Thus D is an $H^0(X, \mathcal{O}_X)$ -module homomorphism iff it is identically 0.

We can summarize the above considerations as follows.

LEMMA 6.45. *Let X be a k -scheme, $D \in H^1(X, \mathcal{H}om(\Omega_X^1, \mathcal{O}_X))$ and X_D the corresponding deformation of X . Then a global section $s \in H^0(X, \mathcal{O}_X)$ lifts to $s_D \in H^0(X_D, \mathcal{O}_{X_D})$ iff $D(s) \in H^1(X, \mathcal{O}_X)$ is zero. \square*

COROLLARY 6.46. *Let X be a normal, affine variety and $U \subset X$ its smooth locus. Let U_D be the deformation of U corresponding to $D \in H^1(U, T_U)$. Then*

(6.46.1) *U_D extends to a flat deformation X_D of X iff $D : H^0(U, \mathcal{O}_U) \rightarrow H^1(U, \mathcal{O}_U)$ (as in (6.44.2)) is identically 0.*

(6.46.2) *$T^1(X)$ is the left kernel of $H^1(U, T_U) \times H^0(U, \mathcal{O}_U) \rightarrow H^1(U, \mathcal{O}_U)$.*

Proof. Assume that U_D extends to a flat deformation X_D of X . Since X is affine, so is X_D and so $H^0(X_D, \mathcal{O}_{X_D}) \rightarrow H^0(X, \mathcal{O}_X)$ is surjective. Thus $D : H^0(U, \mathcal{O}_U) \rightarrow H^1(U, \mathcal{O}_U)$ is identically 0 by (6.45).

Conversely, if $D : H^0(U, \mathcal{O}_U) \rightarrow H^1(U, \mathcal{O}_U)$ is identically 0 then $H^0(U_D, \mathcal{O}_{U_D}) \rightarrow H^0(U, \mathcal{O}_U)$ is surjective and $H^0(U, \mathcal{O}_U) = H^0(X, \mathcal{O}_X)$ since X is normal. We can then take $X_D := \text{Spec}_k H^0(U_D, \mathcal{O}_{U_D})$. This proves the first claim and the second is a reformulation of it. \square

REMARK 6.47. If X is not affine, one can restate (6.46) as follows. $D \in H^1(U, T_U)$ gives a k -linear map $D : \mathcal{O}_X = j_* \mathcal{O}_U \rightarrow R^1 j_* \mathcal{O}_U = \mathcal{H}_Z^2(\mathcal{O}_X)$ where $Z := X \setminus U$ is the singular locus. Then U_D extends to a flat deformation X_D of X iff $D : \mathcal{O}_X \rightarrow \mathcal{H}_Z^2(\mathcal{O}_X)$ is identically 0.

6.48 (Lie derivative). Let M be a smooth, real manifold and v a vector field on M . By integrating v we get a 1-parameter family of diffeomorphisms ϕ_t of M . The *Lie derivative* of a covariant tensor field S is defined as

$$L_v S := \frac{d}{dt} (\phi_t^* S)_{t=0}. \quad (6.48.1)$$

In local coordinates $\{y_i\}$ write $v = \sum_i v_i \frac{\partial}{\partial y_i}$. The Lie derivatives of a function s and of a 1-form dy_j are given by the formulas

$$L_v s = v(s) = \sum_i v_i \frac{\partial s}{\partial y_i} \quad \text{and} \quad L_v(dy_j) = dv_j. \quad (6.48.2)$$

Since functions and 1-forms generate the algebra of covariant tensors, the Lie derivative is uniquely determined by the formulas (6.48.2). One can extend the definition to all tensors by duality.

We can transplant this definition to algebraic geometry as follows.

Let Y be a smooth variety over a field k and $v \in H^0(Y, T_Y)$ a vector field. By (6.41.4) v can be identified with an automorphism α_v of $Y[\epsilon]$. We write Ω_Y for the module of derivations (frequently denoted by Ω_Y^1). The covariant tensors are sections of the algebra $\sum_{m \geq 0} \Omega_Y^{\otimes m}$.

Let $S \in H^0(Y, \sum_{m \geq 0} \Omega_Y^{\otimes m})$ be a covariant tensor on Y . It has a trivial extension to $Y[\epsilon]$; denote it by $S[\epsilon]$. Thus $\alpha_v^*(S[\epsilon])$ is a global section of $\sum_{m \geq 0} \Omega_{Y[\epsilon]}^{\otimes m}$. Since α_v is the identity on X , $\alpha_v^*(S[\epsilon]) - S[\epsilon]$ is divisible by ϵ and we can define the *Lie derivative* of S by the formula

$$\alpha_v^*(S[\epsilon]) = S[\epsilon] + \epsilon L_v S. \tag{6.48.3}$$

Expanding the identity $\alpha_v^*(S_1[\epsilon] \otimes S_2[\epsilon]) = \alpha_v^*(S_1[\epsilon]) \otimes \alpha_v^*(S_2[\epsilon])$ shows that the Lie derivative is a k -linear derivation of the tensor algebra

$$L_v : \sum_{m \geq 0} \Omega_Y^{\otimes m} \rightarrow \sum_{m \geq 0} \Omega_Y^{\otimes m}. \tag{6.48.4}$$

The Lie derivative preserves natural quotient bundles of $\Omega_X^{\otimes m}$. Thus we get similar maps L_v for symmetric and skew-symmetric tensors. Our main interest is in powers of ω_X . The corresponding map

$$L_v : \omega_Y^m \rightarrow \omega_Y^m \tag{6.48.5}$$

is obtained using the identification $\Omega_Y^{\otimes n} \rightarrow \Omega_Y^n = \omega_Y$ where $n = \dim Y$.

From (6.41.1) we see that

$$\alpha_v^*(s[\epsilon]) = s[\epsilon] + \epsilon v(s) \quad \text{and} \quad \alpha_v^*(dy_j) = d(\alpha_v^*(y_j)) = dy_j + \epsilon dv_j. \tag{6.48.6}$$

Comparing with (6.48.2) we see that the algebraic definition coincides with the differential geometry definition.

6.49 (Cartan formula). This is an identity which holds for exterior forms S

$$L_v(S) = d(v \lrcorner S) + v \lrcorner dS, \tag{6.49.1}$$

where \lrcorner denotes *contraction* or *inner product* by a vector field $v \in H^0(Y, T_Y)$ obtained as follows. We have the contraction map $T_Y \otimes \Omega_Y^m \rightarrow \Omega_Y^{m-1}$, thus every $v \in H^0(Y, T_Y)$ gives the \mathcal{O}_Y -linear map

$$v \lrcorner : \Omega_Y^m \rightarrow \Omega_Y^{m-1}. \tag{6.49.2}$$

In (analytic or étale) local coordinates y_1, \dots, y_n write $v = \sum_i v_i \frac{\partial}{\partial y_i}$. Then

$$v \lrcorner (dy_1 \wedge \dots \wedge dy_m) = \sum_r (-1)^{r-1} v_r \cdot dy_1 \wedge \dots \wedge \widehat{dy_r} \wedge \dots \wedge dy_m, \tag{6.49.3}$$

where the hat indicates that we omit that term.

To prove (6.49.1), one first checks that $S \mapsto d(v \lrcorner S) + v \lrcorner dS$ is also a derivation. Thus it is sufficient to verify (6.49.1) for a generating set of exterior forms. For functions and for dy_j we recover the identities (6.48.2).

6.50. As in (6.44), let Y be a smooth k -variety. Pick $D \in H^1(Y, T_Y)$ and choose a representative 1-cocycle $D = \{v_{ij}\}$ using an affine cover $Y = \cup Y_i$. For any $S \in H^0(Y, \Omega_Y^{\otimes m})$ the Lie derivatives $\{L_{v_{ij}}(S|_{Y_{ij}})\}$ form a 1-cocycle with values in $\Omega_Y^{\otimes m}$. This defines

$$L_D(S) \in H^1(Y, \Omega_Y^{\otimes m}), \tag{6.50.1}$$

which we view as a *cohomological differentiation* map

$$L_D : \sum H^0(Y, \Omega_Y^{\otimes m}) \rightarrow \sum H^1(Y, \Omega_Y^{\otimes m}). \quad (6.50.2)$$

As we noted in (6.48), the map L_D respects natural quotient bundles of $\Omega_Y^{\otimes m}$. Thus we get similar maps for symmetric and skew-symmetric tensors and for powers of ω_Y

$$L_D : \sum H^0(Y, \omega_Y^m) \rightarrow \sum H^1(Y, \omega_Y^m). \quad (6.50.3)$$

For $m = 0$ the map $L_D : H^0(Y, \omega_Y^0) \rightarrow H^1(Y, \omega_Y^0)$ agrees with the map $D : H^0(Y, \mathcal{O}_Y) \rightarrow H^1(Y, \mathcal{O}_Y)$ defined in (6.44.2).

As in (6.44.7), L_D is a k -linear differentiation which is usually not $H^0(Y, \mathcal{O}_Y)$ -linear. However, if the map $D : H^0(Y, \mathcal{O}_Y) \rightarrow H^1(Y, \mathcal{O}_Y)$ is zero then L_D is $H^0(Y, \mathcal{O}_Y)$ -linear; this holds both for the general case (6.50.2) and the special one (6.50.3).

Arguing as in (6.45) we obtain the following lifting criterion.

LEMMA 6.51. *Let Y be a smooth k -variety and Y_D a first order deformation of Y . Then $S \in H^0(Y, \Omega_Y^{\otimes m})$ lifts to $S_D \in H^0(Y_D, \Omega_{Y_D}^{\otimes m})$ iff $L_D(S) \in H^1(Y, \Omega_Y^{\otimes m})$ is zero. \square*

Divergence.

Next we consider what the previous method gives for ω_Y and its powers using (6.50.3).

6.52 (Divergence). Let Y be a smooth k -variety, $\sigma \in H^0(Y, \omega_Y^m)$ and $v \in H^0(Y, T_Y)$. Then σ and $L_v \sigma$ are both sections of the line bundle ω_Y^m , hence their quotient is a rational function, called the *divergence* of v with respect to σ ,

$$\nabla_\sigma v := \frac{L_v \sigma}{\sigma}. \quad (6.52.1)$$

(Most books seem to use this terminology only when σ is a nowhere 0 section of ω_Y and σ is frequently suppressed in the notation.)

In order to compute this, start with a section σ of ω_Y . Since $d\sigma = 0$, Cartan's formula (6.49) shows that $L_v : \omega_Y \rightarrow \omega_Y$ is the composite map

$$L_v : \omega_Y = \Omega_Y^n \xrightarrow{v \lrcorner} \Omega_Y^{n-1} \xrightarrow{d} \Omega_Y^n = \omega_Y. \quad (6.52.2)$$

In local coordinates y_1, \dots, y_n assume that $\sigma = dy_1 \wedge \dots \wedge dy_n$ and $v = \sum_i v_i \frac{\partial}{\partial y_i}$. Contraction by v sends σ to

$$\sum_i (-1)^{i-1} v_i dy_1 \wedge \dots \wedge \widehat{dy_i} \wedge \dots \wedge dy_n. \quad (6.52.3)$$

Exterior differentiation now gives that

$$L_v \sigma = d(v \lrcorner \sigma) = \sum_i \frac{\partial v_i}{\partial y_i} \cdot \sigma. \quad (6.52.4)$$

That is, the usual formula holds for the divergence:

$$\nabla_\sigma v = \nabla_{\mathbf{y}} v := \sum_i \frac{\partial v_i}{\partial y_i}. \quad (6.52.5)$$

For powers of ω_Y this gives the next formula.

LEMMA 6.53. *Let Y be a smooth k -variety of dimension n . Let $v \in H^0(Y, T_Y)$ be a vector field, $s \in H^0(Y, \mathcal{O}_Y)$ a function and $\sigma \in H^0(Y, \omega_Y)$ an n -form. Then*

$$\nabla_{(s\sigma^m)} v = \frac{v(s)}{s} + m \nabla_\sigma v. \quad (6.53.1)$$

Proof. This is really just the assertion that the Lie derivative is a derivation, but it is instructive to do the local computations.

The claimed identities are local, so we may work with local coordinates y_1, \dots, y_n and assume that $\sigma = dy_1 \wedge \dots \wedge dy_n$. Write $v = \sum_i v_i \frac{\partial}{\partial y_i}$. We need to compute how the isomorphism α_v acts on $s\sigma^m$. It sends y_i to $y_i + \epsilon v(y_i) = y_i + \epsilon v_i$, thus

$$\alpha_v^*(dy_i) = (1 + \epsilon \frac{\partial v_i}{\partial y_i}) dy_i + \epsilon (\sum_{j \neq i} \frac{\partial v_i}{\partial y_j} dy_j). \quad (6.53.2)$$

Next we wedge these together. Any two epsilon terms wedge to 0 since $\epsilon^2 = 0$. Thus $\epsilon (\sum_{j \neq i} \frac{\partial v_i}{\partial y_j} dy_j)$ gets killed unless it is wedged with all the other dy_j , but the result is then zero in the exterior algebra. Hence the only term that survives is

$$\begin{aligned} \prod_i (1 + \epsilon \frac{\partial v_i}{\partial y_i}) \cdot dy_1 \wedge \dots \wedge dy_n &= (1 + \epsilon \sum_i \frac{\partial v_i}{\partial y_i}) \cdot dy_1 \wedge \dots \wedge dy_n \\ &= (1 + \epsilon \nabla_{\mathbf{y}} v) \cdot dy_1 \wedge \dots \wedge dy_n. \end{aligned} \quad (6.53.3)$$

Thus we get that $s\sigma^m$ is mapped to

$$\begin{aligned} (s + \epsilon v(s)) (1 + m\epsilon \nabla_{\mathbf{y}} v) \cdot \sigma^m \\ = (s + \epsilon v(s) + m\epsilon s \nabla_{\mathbf{y}} v) \cdot \sigma^m \\ = s\sigma^m + \epsilon \cdot (\frac{v(s)}{s} + m \nabla_{\mathbf{y}} v) \cdot s\sigma^m. \quad \square \end{aligned} \quad (6.53.4)$$

NOTATION 6.54. Let X be a normal, affine k -variety and X_D a flat deformation of X over $k[\epsilon]$ corresponding to $D \in T^1(X)$. Let $U \subset X$ be the smooth locus. By (6.43) we can think of D as a cohomology class $D \in H^1(U, T_U)$. By (6.44.2) D induces a map

$$D : H^0(U, \mathcal{O}_U) \rightarrow H^1(U, \mathcal{O}_U) \quad (6.54.1)$$

which is identically zero by (6.46.2). There is a natural exact sequence

$$0 \rightarrow \omega_U^m \rightarrow \omega_{U_D}^m \rightarrow \omega_U^m \rightarrow 0. \quad (6.54.2)$$

Taking cohomologies gives an exact sequence

$$H^0(U_D, \omega_{U_D}^m) \rightarrow H^0(U, \omega_U^m) \xrightarrow{\delta_m} H^1(U, \omega_U^m). \quad (6.54.3)$$

As we noted at the end of (6.50), δ_m is $H^0(U, \mathcal{O}_U)$ -linear since D in (6.54.1) is 0.

It was observed in [Ste88] that, for cyclic quotients, the deformation obstruction computed in [EV85] equals the divergence. The next result shows that this is a general phenomenon.

THEOREM 6.55. *Let $X, U \subset X, D = \{v_{ij}\} \in H^1(U, T_U)$ and X_D be as above (6.54). Assume that $\omega_U^{[m]}$ has a nowhere 0 section σ_m for some $m > 0$ such that $\text{char } k \nmid m$. Set $\nabla_{\sigma_m} D := \{\nabla_{\sigma_m}(v_{ij})\} \in H^1(U, \mathcal{O}_U)$. Then*

$$(6.55.1) \quad \nabla D := \frac{1}{m} \nabla_{\sigma_m} D \in H^1(U, \mathcal{O}_U) \text{ is independent of the choice of } m \text{ and } \sigma_m.$$

$$(6.55.2) \quad \text{The boundary map } \delta_m : H^0(U, \omega_U^m) \rightarrow H^1(U, \omega_U^m) \text{ defined in (6.54.3) is multiplication by } m \nabla D.$$

$$(6.55.3) \quad \omega_U^{[m]} \text{ is free} \Leftrightarrow \text{it is locally free} \Leftrightarrow \nabla D = 0 \text{ in } H^1(U, \mathcal{O}_U).$$

Proof. Choose affine charts $\{U_i\}$ on U such that $D = \{v_{ij}\}$ and $\sigma_m|_{U_i} = s_{ij} \sigma_{ij}^m$ for some $\sigma_{ij} \in H^0(U_{ij}, \omega_{U_{ij}})$. Any other section of ω_U^m can be written as $g\sigma_m$ where $g \in H^0(U, \mathcal{O}_U)$. Using (6.53) we obtain that

$$\nabla_{\sigma_m} D = \{\nabla_{\sigma_m}(v_{ij})\} = \left\{ \frac{v_{ij}(s_{ij})}{s_{ij}} + m \nabla_{\sigma_{ij}}(v_{ij}) \right\}. \quad (6.55.4)$$

Similarly, we get that

$$\nabla_{g\sigma_m} D = \left\{ \frac{v_{ij}(gs_{ij})}{gs_{ij}} + m\nabla_{\sigma_{ij}}(v_{ij}) \right\}. \quad (6.55.5)$$

Since

$$\frac{v_{ij}(gs_{ij})}{gs_{ij}} = \frac{v_{ij}(g)}{g} + \frac{v_{ij}(s_{ij})}{s_{ij}}, \quad (6.55.6)$$

subtracting (6.55.4) from (6.55.5) yields

$$\nabla_{g\sigma_m} D - \nabla_{\sigma_m} D = \frac{1}{g} D(g) \in H^1(U, \mathcal{O}_U). \quad (6.55.7)$$

As we noted in (6.54), $D(g) = 0$ in $H^1(U, \mathcal{O}_U)$. Thus $\nabla_{g\sigma_m} D = \nabla_{\sigma_m} D$ (as classes in $H^1(U, \mathcal{O}_U)$). Independence of the choice of m is shown by the formula

$$\nabla_{(\sigma_m^r)} D = \left\{ \frac{v_{ij}(s_{ij}^r)}{s_{ij}^r} + rm\nabla_{\sigma_{ij}}(v_{ij}) \right\} = r \cdot \left\{ \frac{v_{ij}(s_{ij})}{s_{ij}} + m\nabla_{\sigma_{ij}}(v_{ij}) \right\}. \quad (6.55.8)$$

Thus ∇D is well defined and this proves (1–2).

Finally, $\omega_{U_D}^{[m]}$ is free iff σ_m lifts to a section of $\omega_{X_D}^{[m]}$ and $\nabla D \cdot \sigma_m$ is the lifting obstruction. This implies (3). \square

REMARK 6.56. Let $x \in X$ be an isolated normal singularity and $U := X \setminus \{x\}$. Then $H^1(U, \mathcal{O}_U) = H_x^2(X, \mathcal{O}_X)$ and $H^1(U, T_U) = H_x^2(X, T_X)$. Thus if $\omega_U^m \cong \mathcal{O}_U$ for some $m > 0$ then the divergence can be thought of as a map

$$\nabla : T^1(X) \rightarrow H_x^2(X, \mathcal{O}_X).$$

If $\text{depth}_x \mathcal{O}_X \geq 3$ then $H_x^2(X, \mathcal{O}_X) = 0$ by Grothendieck's vanishing theorem (10.28.5), thus in this case the divergence vanishes and sections of ω_U^m lift to all first order deformations. This, however, already follows from (6.54.3) since $H^1(U, \omega_U^m) = H^1(U, \mathcal{O}_U) = H_x^2(X, \mathcal{O}_X) = 0$.

If X is lc and ω_X is locally free, then sections of ω_X lift to any deformation by [KK20], see also (2.68). By (6.55) this implies that $\nabla : T^1(X) \rightarrow H^1(U, \mathcal{O}_U)$ is the zero map.

This should either have a direct proof or some interesting consequences.

Next we give explicit forms of the maps in the general theory for $X := \mathbb{A}^2$ and $U := \mathbb{A}^2 \setminus \{(0, 0)\}$. At first this seems quite foolish to do since we already know that a smooth affine variety has only trivial infinitesimal deformations. However, we will be able to use these computations to get very detailed information about deformations of 2-dimensional cyclic quotient singularities; a very interesting subject.

NOTATION 6.57. Let k be a field, $X = \mathbb{A}_{xy}^2$ and $U := X \setminus \{(0, 0)\}$. Using the affine charts $U_0 := U \setminus (x = 0)$, $U_1 := U \setminus (y = 0)$ and $U_{01} := U \setminus (xy = 0)$ we compute that

$$H^1(U, \mathcal{O}_U) = \left\langle \frac{1}{x^i y^j} : i, j \geq 1 \right\rangle \quad (6.57.1)$$

and also that

$$H^1(U, T_U) = \left\langle \frac{1}{x^i y^j} \cdot \frac{\partial}{\partial x}, \frac{1}{x^i y^j} \cdot \frac{\partial}{\partial y} : i, j \geq 1 \right\rangle.$$

Note that $H^1(U, \mathcal{O}_U)$ is naturally a quotient of

$$H^0(U_{01}, \mathcal{O}_{U_{01}}) = k[x^i y^j : i, j \in \mathbb{Z}];$$

the basis in (6.57.1) depends on the choice of coordinates x, y . Similarly, $H^1(U, T_U)$ is naturally a quotient of $H^0(U_{01}, T_{U_{01}})$.

It is very convenient computationally that the diagonal subgroup $\mathbb{G}_m^2 \subset \mathrm{GL}_2$ acts on these cohomology groups and subsequent constructions are \mathbb{G}_m^2 -equivariant. In order to keep track of this action it is better to use the \mathbb{G}_m^2 -invariant differential operators

$$\partial_x := x \frac{\partial}{\partial x} \quad \text{and} \quad \partial_y := y \frac{\partial}{\partial y}. \quad (6.57.2)$$

Thus $\partial_x(x^r y^s) = r x^{r-1} y^s$, $\partial_y(x^r y^s) = s x^r y^{s-1}$ and

$$H^1(U, T_U) = \left\langle \frac{\partial_x}{x^i y^j} : i \geq 2, j \geq 1 \right\rangle \oplus \left\langle \frac{\partial_y}{x^i y^j} : i \geq 1, j \geq 2 \right\rangle. \quad (6.57.3)$$

The \mathbb{G}_m^2 -eigenspaces in $H^1(U, T_U)$ are usually 2-dimensional

$$\left\langle \frac{\partial_x}{x^i y^j}, \frac{\partial_y}{x^i y^j} \right\rangle \quad \text{for} \quad i, j \geq 2. \quad (6.57.4.a)$$

The 1-dimensional eigenspaces are

$$\left\langle \frac{\partial_x}{x^i y} \right\rangle \quad \text{and} \quad \left\langle \frac{\partial_y}{x y^j} \right\rangle \quad \text{for} \quad i, j \geq 2. \quad (6.57.4.b)$$

The pairing $H^1(U, T_U) \times H^0(U, \mathcal{O}_U) \rightarrow H^1(U, \mathcal{O}_U)$ defined in (6.44.3) is especially transparent using the bases (6.57.1–4) since

$$\frac{a\partial_x - b\partial_y}{x^i y^j}(x^r y^s) = (ar - bs) \cdot x^{r-i} y^{s-j}. \quad (6.57.5)$$

This is identically 0 as an element of $H^0(U_{01}, \mathcal{O}_{U_{01}})$ iff $ar - bs = 0$. It is more important to know when this is 0 as an element of $H^1(U, \mathcal{O}_U)$. The latter holds iff

- (6.a) either $ar - bs = 0$ or
- (6.b) $r \geq i$ or $s \geq j$.

This easily implies that the left kernel of $H^1(U, T_U) \times H^0(U, \mathcal{O}_U) \rightarrow H^1(U, \mathcal{O}_U)$ is trivial, hence $T^1(\mathbb{A}^2) = 0$ by (6.46.2); but this we already knew.

Combining (6.51) and (6.53) gives the following.

LEMMA 6.58. *Using the above notation, let $D \in H^1(U, T_U)$ and U_D the corresponding deformation. Then $f(dx \wedge dy)^m$ lifts to a section of $\omega_{U_D}^m$ iff*

$$D(f) + mf\nabla D \in H^1(U, \mathcal{O}_U) \quad \text{vanishes.} \quad \square \quad (6.58.1)$$

We are thus interested in computing the kernels of the operators

$$(D, f) \mapsto D(f) + mf\nabla D.$$

We start by describing the kernel of ∇ .

6.59 (Computing the divergence). Set $D := (a\partial_x - b\partial_y)x^{-i}y^{-j}$. By explicit computation,

$$\nabla \left(\frac{a\partial_x - b\partial_y}{x^i y^j} \right) = -\frac{a(i-1) - b(j-1)}{x^i y^j}. \quad (6.59.1)$$

Thus ∇D is identically zero iff $a(i-1) - b(j-1) = 0$. If D is a nonzero element of $H^1(U, T_U)$ then $i, j > 0$ and then ∇D is 0 as an element of $H^1(U, \mathcal{O}_U)$ iff it is identically zero.

If $(i, j) = (1, 1)$ then $\nabla D = 0$ but then D vanishes in $H^1(U, T_U)$. If $\nabla D = 0$ and $i = 1, j > 1$ then $b = 0$ and again D vanishes in $H^1(U, T_U)$. Thus we conclude that

$$\ker[H^1(U, T_U) \xrightarrow{\nabla} H^1(U, \mathcal{O}_U)] = \left\langle \frac{(j-1)\partial_x - (i-1)\partial_y}{x^i y^j} : i, j \geq 2 \right\rangle. \quad (6.59.2)$$

COROLLARY 6.60. *Let $D \in H^1(U, T_U)$. Then $D(xy), \nabla D \in H^1(U, \mathcal{O}_U)$ are both 0 iff D is contained in the subspace*

$$K_{VW} := \left\langle \frac{\partial_x - \partial_y}{(xy)^i} : i \geq 2 \right\rangle \subset H^1(U, T_U).$$

Proof. Corresponding to the 2 cases in (6.57.6.a–b), the kernel of the map $D \mapsto D(xy) \in H^1(U, \mathcal{O}_U)$ is a direct sum of 2 subspaces

$$K_1 := \left\langle \frac{\partial_x - \partial_y}{x^i y^j} : i, j \geq 2 \right\rangle \quad \text{and} \quad K_2 := \left\langle \frac{\partial_y}{x y^j}, \frac{\partial_x}{x^i y} : i, j \geq 2 \right\rangle. \quad (6.60.1)$$

Combining this with (6.59.2) gives the claim. □

6.6. Deformations of cyclic quotient singularities

The results in this section are about cyclic quotient singularities of surfaces, but for the basic definitions we need to assume only the following.

NOTATION 6.61. X is a pure dimensional, S_2 scheme over a field k such that ω_X is locally free outside a closed subset $Z \subset X$ of codimension ≥ 2 and $\omega_X^{[m]}$ is locally free for some $m > 0$. The smallest such $m > 0$ is called the *index* of ω_X . Both of these conditions are satisfied by schemes with slc singularities.

Let $(0, T)$ be a local scheme such that $k(0) \cong k$ and $p : X_T \rightarrow T$ a flat deformation of $X \cong X_0$. As in (2.6), for every $r \in \mathbb{Z}$ we have natural restriction maps

$$\mathcal{R}^{[r]} : \omega_{X_T/T}^{[r]}|_{X_0} \rightarrow \omega_{X_0}^{[r]}. \quad (6.61.1)$$

These maps are isomorphisms over $X \setminus Z$ and we are interested in understanding those cases when they are isomorphisms over X . By (9.27), if T is Artinian, then

$$\mathcal{R}^{[r]} \text{ is an isomorphism} \Leftrightarrow \mathcal{R}^{[r]} \text{ is surjective} \Leftrightarrow \omega_{X_T/T}^{[r]} \text{ is flat over } T. \quad (6.61.2)$$

DEFINITION 6.62. Let $p : X_T \rightarrow T$ be a flat deformation as in (6.61).

(6.62.1) We call $p : X_T \rightarrow T$ a *qG-deformation* if the conditions (6.61.2) hold for every r . It is enough to check these for $r = 1, \dots, \text{index}(\omega_X)$. (qG is short for ‘Quotient of Gorenstein,’ but this is misleading if $\dim X \geq 3$.)

These deformations were introduced and studied in [KSB88] as the class most suitable for compactifying the moduli of varieties of general type. A list of lc surface singularities with qG-smoothings is given in [KSB88]. In the key case of cyclic quotient singularities the list was earlier established in [Wah80, 2.7], though there they are viewed as examples of W-deformations (see below).

(6.62.2) We call $p : X_T \rightarrow T$ a *Viehweg-type deformation* (or V-deformation) if the conditions (6.61.2) hold for every r divisible by $\text{index}(\omega_X)$. It is enough to check this for $r = \text{index}(\omega_X)$. These deformations are used in the monograph [Vie95]. Actually, [Vie95] considers the—a priori weaker—condition: $\mathcal{R}^{[r]}$ is an isomorphism for some $r > 0$ divisible by $\text{index}(\omega_X)$. One can see that in this case

(6.61.2) holds for every r divisible by $\text{index}(\omega_X)$, at least in characteristic 0; see (2.96). The two notions are different in positive characteristic by (4.44).

(6.62.3) We call $p : X_T \rightarrow T$ a *Wahl-type deformation* (or W-deformation) if the conditions (6.61.2) hold for $r = -1$. These deformations were considered in [Wah80, Wah81] and called ω^* -constant deformations there.

(6.62.4) We call $p : X_T \rightarrow T$ a VW-deformation if it is both a V-deformation and a W-deformation.

It is clear that every qG-deformation is also a VW-deformation. Understanding the precise relationship between these 4 classes has been a long standing open problem, especially for quotient singularities of surfaces. For reduced base spaces we have the following, which is a combination of (2.77) and (3.37).

THEOREM 6.63. *A flat deformation of an slc variety over a reduced, local scheme of characteristic 0 is a V-deformation iff it is a qG-deformation.*

This raised the possibility that every V-deformation of an slc singularity is also a qG-deformation over arbitrary base schemes. It would be enough to check this for Artinian bases. Here we focus on first order deformations and prove that these 2 classes are quite different from each other.

DEFINITION 6.64. Let X be a scheme satisfying the conditions of (6.61). Let $T^1(X)$ denote the set of isomorphism classes of deformations of X over $\text{Spec}_k k[\epsilon]$. This is a (possibly infinite dimensional) k -vector space. Let $T^1_{\text{qG}}(X) \subset T^1(X)$ denote the space of first order qG-deformations, $T^1_V(X)$ the space of first order V-deformations, $T^1_W(X)$ the space of first order W-deformations and $T^1_{\text{VW}}(X)$ the space of first order VW-deformations. We have obvious inclusions

$$T^1_{\text{qG}}(X) \subset T^1_{\text{VW}}(X) \subset T^1_V(X), T^1_W(X) \subset T^1(X),$$

but the relationship between $T^1_V(X)$ and $T^1_W(X)$ is not clear.

These $T^1_*(X)$ are the tangent spaces to the corresponding miniversal deformation spaces; we denote these by $\text{Def}_{\text{qG}}(X), \text{Def}_V(X)$ and so on. See [Art76] or [Loo84] for precise definitions and introductions or (2.25–2.29) for details on surface quotient singularities.

6.65 (Cyclic quotient singularities). Let $\frac{1}{n}(1, q)$ denote the cyclic group action

$$g : (x, y) \mapsto (\eta x, \eta^q y),$$

where η is a primitive n th root of unity. We always assume that $\text{char } k \nmid n$ and $(n, q) = 1$; then the action is free outside the origin on $\mathbb{A}^2 = \text{Spec } k[x, y]$. The ring of invariants is

$$R_{nq} := k[x, y]^G = k[x^i y^j : i, j \geq 0, i + qj \equiv 0 \pmod n], \tag{6.65.1}$$

and the corresponding quotient singularity is

$$S_{n,q} := \mathbb{A}^2 / \frac{1}{n}(1, q) = \text{Spec}_k R_{nq}. \tag{6.65.2}$$

While we work with this affine model, all the results apply to its localization, Henselisation or completion at the origin.

We can also choose $\eta' = \eta^q$ as our primitive n th root of unity. This shows the isomorphism

$$S_{n,q} \cong S_{n,q'} \quad \text{where } qq' \equiv 1 \pmod n. \tag{6.65.3}$$

Note that $q \equiv q' \pmod n$ iff $n \mid q^2 - 1$. These are the simplest log terminal surface singularities.

Various ways of studying such singularities go back a long time. The first relevant work might be [Jun08], followed by [Hir53] and [Bri68a].

In (6.71) we give an algorithm that yields an explicit, minimal generating set of R_{nq} . The number of generators is the embedding dimension.

For us the embedding dimension is the most natural invariant, but traditionally the multiplicity is considered the basic one. For cyclic quotients, more generally, for rational surface singularities, these are related by the formula

$$\text{embdim}(S_{n,q}) = \text{mult}(S_{n,q}) + 1, \quad (6.65.4)$$

see [Art66, Bri68a].

We completely describe first order qG-, V- and W-deformations of cyclic quotient singularities. The main conclusion is that qG-deformations and V-deformations are quite different over Artinian bases; the proof is given in (6.83).

THEOREM 6.66. *Let $S_{n,q} := \mathbb{A}^2/\frac{1}{n}(1, q)$ be as in (6.65). Then*

$$\dim T_V^1(S_{n,q}) - \dim T_{VW}^1(S_{n,q}) = \text{embdim}(S_{n,q}) - 4 \quad \text{or} \quad \text{embdim}(S_{n,q}) - 5.$$

In particular, if $\text{embdim}(S_{n,q}) \geq 6$ then $S_{n,q}$ has V-deformations that are not VW-deformations, hence also not qG-deformations.

By contrast, qG-deformations and VW-deformations are quite close to each other, as shown by the next result, proved in (6.85).

THEOREM 6.67. *Let $S_{n,q} := \mathbb{A}^2/\frac{1}{n}(1, q)$ be as in (6.65).*

$$(6.67.1) \quad \text{If } (n, q+1) = 1, \text{ then } \text{Def}_{\text{qG}}(S_{n,q}) = \text{Def}_{VW}(S_{n,q}) = \{0\}.$$

$$(6.67.2) \quad \text{If } S_{n,q} \text{ admits a qG-smoothing, then } \text{Def}_{\text{qG}}(S_{n,q}) = \text{Def}_{VW}(S_{n,q}).$$

$$(6.67.3) \quad \text{In general, } \dim T_{\text{qG}}^1(S_{n,q}) \leq \dim T_{VW}^1(S_{n,q}) \leq \dim T_{\text{qG}}^1(S_{n,q}) + 1.$$

COROLLARY 6.68. *The cyclic quotient singularities for which every V-deformation is a qG-deformation are the following.*

$$(6.68.1) \quad \text{Double points: } \mathbb{A}^2/\frac{1}{n}(1, n-1) \text{ for } n \geq 1.$$

$$(6.68.2) \quad \text{Triple points: } \mathbb{A}^2/\frac{1}{ab-1}(1, ab-b-1) \text{ for } a, b \geq 2.$$

$$(6.68.3) \quad \text{Quadruple points: } \mathbb{A}^2/\frac{1}{a(ab-2)}(1, (ab-2)(a-1)-1) \text{ for } a, b \geq 2.$$

(The list includes all triple points but only some of the quadruple points.)

Next we discuss what the general theory of the previous section says about deformations of 2-dimensional quotient singularities. The results are very explicit for cyclic quotient singularities.

6.69 (Deformation of quotients). Let k be a field, X an affine k -scheme that is S_2 , $x \in X$ a closed point and $U := X \setminus \{x\}$. Let G be a finite group acting on X such that x is a G -fixed point and the action is free on U . The quotient map $\pi_U : U \rightarrow U/G$ is finite and étale. This extends to a finite map $\pi_X : X \rightarrow X/G$ which is ramified at x .

$\mathcal{O}_{U/G}$ is identified with the G -invariant subsheaf $(\pi_*\mathcal{O}_U)^G$ and similarly $\omega_{U/G}$ is identified with $(\pi_*\omega_U)^G$. (For the latter we need that the action is free). Thus

we get that

$$\begin{aligned} H^0(U/G, \mathcal{O}_{U/G}) &= H^0(U, \mathcal{O}_U)^G = H^0(X, \mathcal{O}_X)^G \quad \text{and} \\ H^0(U/G, \omega_{U/G}^{[m]}) &= H^0(U, \omega_U^{[m]})^G = H^0(X, \omega_X^{[m]})^G. \end{aligned} \tag{6.69.1}$$

If $\text{char } k \nmid |G|$ then the G -invariant subsheaf is a direct summand, hence by taking cohomologies we similarly see that

$$H^1(U/G, \mathcal{O}_{U/G}) = H^1(U, \mathcal{O}_U)^G \quad \text{and} \quad H^1(U/G, T_{U/G}) = H^1(U, T_U)^G. \tag{6.69.2}$$

If $D \in H^1(U, T_U)$ is G -invariant then the deformation U_D descends to a deformation $(U/G)_D$ of U/G and these give all first order deformations of U/G . If $H^0(U/G, \mathcal{O}_{U/G})$ is flat over $k[\epsilon]$ then its spectrum gives a flat deformation of X/G and every flat deformation of X/G that is locally trivial on U/G arises this way.

Thus, using (6.46) we get the following fundamental observation.

THEOREM 6.70. [Sch71] *Let k be a field, X a smooth, affine k -variety, $x \in X$ a closed point and $U := X \setminus \{x\}$. Let G be a finite group acting on X such that x is a G -fixed point, the action is free on U and $\text{char } k \nmid |G|$. Then $T^1(X/G)$ is the left kernel of the pairing*

$$H^1(U, T_U)^G \times H^0(U, \mathcal{O}_U)^G \rightarrow H^1(U, \mathcal{O}_U)^G \tag{6.70.1}$$

defined in (6.44). More generally, if X is normal, the left kernel corresponds to those flat deformations of X/G that are locally trivial on U/G . \square

Next we compute the terms in (6.70.1) for cyclic quotient singularities.

NOTATION 6.71. As in (6.65), let $R_{nq} \subset k[x, y]$ denote the ring of invariants of the cyclic group action

$$g : (x, y) \mapsto (\eta x, \eta^q y).$$

We assume that $\text{char } k \nmid n$ and $(n, q) = 1$.

Our aim is to describe the generators of R_{nq} . Most of the following formulas can be found in [Rie74]; see [Ste13] for an introduction and many examples.

The group action preserves the monomials, hence R_{nq} has a generating set consisting of monomials. A non-minimal generating set can be constructed as follows. For any $0 < j < n$ let $0 < \gamma_j < n$ be the unique integer such that $\gamma_j + qj \equiv 0 \pmod n$. Then

$$x^n, x^{\gamma_1}y, x^{\gamma_2}y^2, \dots, x^{\gamma_{n-1}}y^{n-1}, y^n$$

is a generating set of R_{nq} . We know that $\gamma_1 = n - q$ and $\gamma_{n-1} = q$. This is a minimal generating set of R_{nq} as a $k[x^n, y^n]$ -module, but usually not as a k -algebra. Indeed, $x^{\gamma_i}y^i$ divides $x^{\gamma_j}y^j$ if $\gamma_i < \gamma_j$ and $i < j$. In any concrete case one can use this observation to get a minimal set of algebra generators.

We label the monomials of the minimal generating set as $M_i = x^{a_i}y^{b_i}$, ordered by increasing y -powers

$$M_0 = x^n, M_1 = x^{n-q}y = x^{a_1}y^{b_1}, M_2 = x^{a_2}y^{b_2}, \dots, M_r = y^n. \tag{6.71.1}$$

At the same time the a_i form a decreasing sequence. Indeed, if $b_i < b_j$ and $a_i \leq a_j$ then M_i divides M_j so the sequence would not be minimal.

From (6.72.2) we obtain that there are relations of the form

$$M_i^{c_i} = M_{i-1}M_{i+1} \quad \text{for } i = 1, \dots, r - 1. \tag{6.71.2}$$

This tells us that the a_i and the c_i are recursively defined by

$$a_0 = n, a_1 = n - q, c_i = \lceil a_{i-1}/a_i \rceil, a_{i+1} = c_i a_i - a_{i-1}. \quad (6.71.3)$$

Similarly, $b_0 = 0, b_1 = 1$ and $b_{i+1} = c_i b_i - b_{i-1}$. These imply that $(a_i, a_{i+1}) = (b_i, b_{i+1}) = 1$ for every i and that the c_i are computed by the modified continued fraction expansion

$$\frac{n}{n-q} = c_1 - \frac{1}{c_2 - \frac{1}{c_3 - \frac{1}{c_4 - \frac{1}{\dots}}}} \quad (6.71.4)$$

The following observations about the a_i, b_i, c_i are quite useful. The first 2 follow from the original construction of the M_i , the 3rd from (6.71.5) and the last one is equivalent to (6.72.3).

- (5.a) $a_{i-1} = \min\{\alpha > 0 : \exists x^\alpha y^\beta \in R_{nq} \text{ such that } \beta < b_i\}$ for $i > 0$.
- (5.b) $b_{i+1} = \min\{\beta > 0 : \exists x^\alpha y^\beta \in R_{nq} \text{ such that } \alpha < a_i\}$ for $i < r$.
- (5.c) $c_i - 1 = \lfloor \frac{a_{i-1}}{a_i} \rfloor = \lfloor \frac{b_{i+1}}{b_i} \rfloor$ for $0 < i < r$.
- (4.d) $a_i b_{i+1} - a_{i+1} b_i = n$ for $0 \leq i < r$.

Note that $r + 1$ is the embedding dimension of S_{nq} and r is its multiplicity. Thus $r = 2$ iff $M_1 = M_{r-1} = xy$ and hence we have the A_{n-1} -singularity $\mathbb{A}^2/\frac{1}{n}(1, -1)$. These are exceptional for many of the subsequent formulas, so we assume from now on that $r \geq 3$.

6.72 (Cones and semigroups). Let $v_0, v_1 \in \mathbb{Z}^2$ be primitive vectors and $C := \mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_1 \subset \mathbb{R}^2$ the closed cone spanned by them. Let $\bar{C}(\mathbb{Z})$ be the closed, convex hull of $(\mathbb{Z}^2 \cap C) \setminus \{(0, 0)\}$ and $N(C)$ the part of the boundary of $\bar{C}(\mathbb{Z})$ that connects v_0 and v_1 . Let $m_0 = v_0, m_1, \dots, m_{r-1}, m_r = v_1$ be the integral points in $N(C)$ as we move from v_0 to v_1 . We leave it to the reader to prove that

- (6.72.1) the m_i generate the semigroup $\mathbb{Z}^2 \cap C$,
- (6.72.2) there are natural numbers $c_1, \dots, c_{r-1} \geq 2$ such that $c_i m_i = m_{i-1} + m_{i+1}$ holds for every i and
- (6.72.3) the triangles with vertices $\{(0, 0), m_i, m_{i+1}\}$ all have the same area.

Thus $R(C)$, the semigroup algebra of $\mathbb{Z}^2 \cap C$, is generated by m_0, \dots, m_s .

For $1 \leq q < n$ and $(n, q) = 1$ consider the cone C_{nq} spanned by $v_0 = (1, 0)$ and $v_1 = (q, n)$. Then

$$\mathbb{Z}^2 \cap C_{nq} = \langle \left(\frac{i}{n}, \frac{j}{n}\right) : i, j \geq 0, i + qj \equiv 0 \pmod{n} \rangle.$$

Thus we see that the semigroup algebra $R(C_{nq})$ is isomorphic to the algebra of invariants R_{nq} defined in (6.65). (It is not hard to see that, up-to the action of $\text{SL}(2, \mathbb{Z})$, every rational cone in \mathbb{R}^2 is of the form C_{nq} .)

6.73 (Computing $T^1(S_{nq})$). Continuing with the notation of (6.69–6.71) we see that $D \in H^1(U, T_U)^G$ is in $T^1(S_{nq})$ iff $D(M_i) = 0 \in H^1(U, \mathcal{O}_U)$ for every i .

Since the pairing (6.70.1) is \mathbb{G}_m^2 -equivariant, it is sufficient to consider one eigenspace at a time. As in (6.57.4.a–b), the eigenspaces in $H^1(U, T_U)^G$ are usually 2-dimensional and of the form

$$\left\langle \frac{\partial_x}{M}, \frac{\partial_y}{M} \right\rangle \quad (6.73.1)$$

where M is a monomial in the M_i -s involving both x, y . The exceptions are 1-dimensional subspaces. For every $s \geq 0$ we have two of them

$$\left\langle \frac{\partial_x}{M_0^s M_1} \right\rangle \quad \text{and} \quad \left\langle \frac{\partial_y}{M_{r-1} M_r^s} \right\rangle. \quad (6.73.2)$$

Thus we can write $D = (\alpha \partial_x - \beta \partial_y)/M$. Note that

$$D(x^a y^b) = (\alpha a - \beta b) \frac{x^a y^b}{M}, \quad (6.73.3)$$

thus if $a < \text{ord}_x M$ and $b < \text{ord}_y M$ then this is zero in $H^1(U, \mathcal{O}_U)$ iff $\beta/\alpha = a/b$. Thus if M is divisible by at least 2 different monomials M_i, M_j for $0 < i, j < r$ then $D(M_i) = 0$ and $D(M_j) = 0$ imply that we need to satisfy both of the equations $\beta/\alpha = a_i/b_i$ and $\beta/\alpha = a_j/b_j$, a contradiction. We get a similar contradiction for the eigenspaces (6.73.2) if $s > 0$. We are left with the cases when $M = M_i^s$ for some $0 < i < r$. If $s \geq 2$ then $D(M_i) = 0$ implies that $D = (b_i \partial_x - a_i \partial_y)/M_i^s$. Then $b_i a_j - a_i b_j \neq 0$ for $j \neq i$ hence $D(M_j) = (b_i a_j - a_i b_j)(M_j/M_i^s)$ vanishes in $H^1(U, \mathcal{O}_U)$ iff $s a_i \leq a_j$ or $s b_i \leq b_j$. If $j < i$ then $b_j < b_i$, hence $s a_i \leq a_j$ must hold. Since the a_j form a decreasing sequence, we need $s a_i \leq a_{i-1}$. Similarly, $s b_j \leq b_{j+1}$. By (6.71.5.c) these are equivalent to $s \leq c_i - 1$.

We have thus proved the following.

PROPOSITION 6.74. [Rie74, Pin77] *Let $M_i = x^{a_i} y^{b_i}$ for $i = 0, \dots, r$ be the generators of R_{nq} as in (6.71.1). Then $T^1(S_{nq}) \subset H^1(U, T_U)$ has a basis consisting of*

$$\left\{ \frac{\partial_x}{M_1}, \frac{\partial_y}{M_{r-1}} \right\} \quad \text{and} \quad \left\{ \frac{\partial_x}{M_i}, \frac{\partial_y}{M_i} : 2 \leq i \leq r-2 \right\}, \quad (6.74.1)$$

plus the possibly empty set

$$\left\{ \frac{b_i \partial_x - a_i \partial_y}{M_i^s} : 1 \leq i \leq r-1, 2 \leq s \leq c_i - 1 \right\} \quad (6.74.2)$$

where $c_i = \lceil \frac{a_i-1}{a_i} \rceil = \lceil \frac{b_i+1}{b_i} \rceil$ is defined in (6.71.2).

6.75 (Powers of ω). Fix any $m \in \mathbb{Z}$. Then $H^0(U, \omega_U^m)$ has a basis consisting of $M(dx \wedge dy)^m$ where M is any monomial. Thus $H^0(S_{nq}, \omega_{S_{nq}}^{[m]}) = H^0(U/G, \omega_{U/G}^m)$ has a basis consisting of

$$\{x^a y^b (dx \wedge dy)^m : a + qb \equiv -m(1+q) \pmod{n}\}. \quad (6.75.1)$$

For $D \in T^1(S_{nq})$ let S_D denote the corresponding deformation. By (6.58) $x^a y^b (dx \wedge dy)^m f$ lifts to a section of $\omega_{S_D}^{[m]}$ iff

$$D(x^a y^b) + m x^a y^b \nabla D = 0 \in H^1(U, \mathcal{O}_U). \quad (6.75.2)$$

It is enough to check (6.75.2) for a minimal generating set of $H^0(S_{nq}, \omega_{S_{nq}}^{[m]})$ as an R_{nq} -module. In any given case this can be worked out by hand, but there are 2 instances where the answer is simple.

(6.75.3) If $n \mid (q+1)m$ then $H^0(S_{nq}, \omega_{S_{nq}}^m)$ is cyclic with generator $1 \cdot (dx \wedge dy)^m$.

(6.75.4) If $m = -1$ then $xy(dx \wedge dy)^{-1}$ is G -invariant. Thus every other $x^a y^b (dx \wedge dy)^{-1}$ is a multiple of it, save for powers of x or y . Thus $\omega_{S_{nq}}^{-1}$ has 3 generating sections:

$$\frac{xy}{dx \wedge dy}, \quad \frac{x^{q+1}}{dx \wedge dy}, \quad \frac{y^{q'+1}}{dx \wedge dy}.$$

6.76 (V-deformations). If $n \mid (q+1)m$ then $1 \cdot (dx \wedge dy)^m$ is a generator by (6.75.3) thus the condition (6.75.2) is equivalent to $\nabla D = 0$.

Therefore $T_V^1(S_{nq})$ equals the intersection of $T^1(S_{nq})$ with the kernel of ∇ . The former was computed in (6.74) the latter in (6.59.2). Thus we see that a basis of $T_V^1(S_{nq})$ is

$$\left\{ \frac{(b_i - 1)\partial_x - (a_i - 1)\partial_y}{M_i} : 2 \leq i \leq r - 2 \right\} \quad (6.76.1.a)$$

and, if M_i is a power of xy for some i , then we have to add

$$\left\{ \frac{\partial_x - \partial_y}{M_i^s} : 2 \leq s \leq c_i - 1 \right\}. \quad (6.76.1.b)$$

6.77 (W-deformations). By (6.75.4), $\omega_{X/G}^{-1}$ has 3 generating sections. Thus, by (6.75.2), D corresponds to a W-deformation iff

$$(1.a) \quad D(xy) - xy\nabla D = 0,$$

$$(1.b) \quad D(x^{q+1}) - x^{q+1}\nabla D = 0 \text{ and } D(y^{q'+1}) - y^{q'+1}\nabla D = 0.$$

The first of these conditions is especially strong. We do not compute it here, rather go directly to the next case where the answer is simpler.

6.78 (VW-deformations). Combining (6.76) and (6.77) we get the description of VW-deformations. These satisfy the conditions

$$(1.a) \quad \nabla D = 0,$$

$$(1.b) \quad D(xy) = 0,$$

$$(1.c) \quad D(x^{q+1}) = 0 \text{ and } D(y^{q'+1}) = 0.$$

We computed the subspace K_{VW} where (1.a) and (1.b) both hold in (6.60). It is spanned by the derivations $(\partial_x - \partial_y)(xy)^{-i}$ for $i \geq 2$. Comparing this with (6.74) we get the following.

Claim 6.78.2. If $T_{VW}^1(S_{nq}) \neq 0$ then R_{nq} has a minimal generator of the form $M_i = (xy)^a$. \square

In order to put this into a cleaner form, assume that $(xy)^s$ is the smallest G -invariant power of xy . Note that $(xy)^n = M_0 M_r$ is G -invariant but it is not one of the M_i . We have $s(q+1) \equiv 0 \pmod n$, thus if $s < n$ then $b := (n, q+1) > 1$. We have thus shown the following.

Claim 6.78.3. If $(n, q+1) = 1$ then $T_{qG}^1(S_{nq}) = T_{VW}^1(S_{nq}) = 0$ and $\dim T_V^1(S_{nq}) = r - 3$. \square

Claim 6.78.4. Assume that $M_i = (xy)^a$ for some i (so $a_i = b_i = a$). Then the space of VW-deformations is spanned by

$$\left\{ \frac{\partial_x - \partial_y}{M_i^s} : 1 \leq s \leq \min\left\{c_i - 1, \frac{q+1}{a}, \frac{q'+1}{a}\right\} \right\}.$$

Proof. The first restriction on s we get from (6.74.2). The condition $D(x^{q+1}) = 0$ is equivalent to $sa \leq q+1$ and $D(y^{q'+1}) = 0$ is equivalent to $sa \leq q'+1$. These give the last 2 restrictions. \square

We thus need to compare the 2 upper bounds occurring in (6.76.1.b) and (6.78.4). The key is the following general estimate.

LEMMA 6.79. *Using the notation of (6.71) we have*

$$\frac{n}{a_i b_i} \leq \frac{a_{i-1}}{a_i}, \frac{b_{i+1}}{b_i} < \frac{n}{a_i b_i} + 1.$$

Proof. Note that $n = a_i b_{i+1} - a_{i+1} b_i$ by (6.71.5.d). Dividing by $a_i b_i$ we get that

$$\frac{n}{a_i b_i} = \frac{b_{i+1}}{b_i} - \frac{a_{i+1}}{a_i}.$$

Since the a_i form a decreasing sequence, $\frac{a_{i+1}}{a_i} < 1$. \square

The final estimate connecting (6.76.1.b) and (6.78.4) is easier to state using a different system of indexing the singularities.

NOTATION 6.80. Set $b = (n, q+1)$ and write $n = ab$, $q+1 = bc$ where $(a, c) = 1$. The inverse (modulo ab) of $bc - 1$ is written as $bc' - 1$. We thus have the singularity

$$S_{abc} := S_{nq} = \mathbb{A}^2 / \frac{1}{ab}(1, bc - 1) \cong \mathbb{A}^2 / \frac{1}{ab}(1, bc' - 1) \quad (6.80.1)$$

Note that $(xy)^a$ is the smallest G -invariant power of xy but it need not be among the generators M_i ; see (6.82).

COROLLARY 6.81. *Assume in addition that $M_i = (xy)^a$ for some i . Then*

$$\lfloor \frac{b}{a} \rfloor \leq \min\{c_i - 1, \frac{q+1}{a}, \frac{q'+1}{a}\} \leq c_i - 1 \leq \lfloor \frac{b}{a} \rfloor + 1. \quad (6.81.1)$$

Proof. First we claim that

$$\frac{b}{a} \leq \min\{\frac{a_{i-1}}{a_i}, \frac{b_{i+1}}{b_i}, \frac{q+1}{a}, \frac{q'+1}{a}\} \leq \min\{\frac{a_{i-1}}{a_i}, \frac{b_{i+1}}{b_i}\} < \frac{b}{a} + 1. \quad (6.81.2)$$

To see this note that $q = bc - 1$, $q' = bc' - 1$. Thus $b \leq q + 1$, $q' + 1$, so it is enough to show that

$$\frac{b}{a} \leq \min\{\frac{a_{i-1}}{a_i}, \frac{b_{i+1}}{b_i}\} < \frac{b}{a} + 1.$$

Since $n = ab$ and $a = a_i = b_i$, the latter is equivalent to (6.79). Taking the round-down gives (1) using (6.71.5.c). \square

EXAMPLE 6.82. Assume that $x^\alpha y^\beta$ is G -invariant. From $\alpha + \beta(bc - 1) \equiv 0 \pmod{ab}$ we see that $\alpha \equiv \beta \pmod{b}$. Thus if $0 < \alpha, \beta \leq 2b$ then either $\alpha = \beta$ or $\alpha = \beta \pm b$.

It turns out that if $a \leq b$ then we can write down these invariants explicitly. Corresponding to the first case we have $(xy)^a$ (and its square). In order to get the other cases, let $0 < e < a$ (resp. $0 < e' < a$) be the unique solution of $ec \equiv -1 \pmod{a}$ (resp. $e'c' \equiv -1 \pmod{a}$). Then $(b + e) + e(bc - 1) = b(ec + 1) \equiv 0 \pmod{ab}$ and $e'(bc' - 1) + (b + e') = b(e'c' + 1) \equiv 0 \pmod{ab}$. Thus we get the minimal generators

$$M_{i-1} = x^{b+e} y^e, \quad M_i = x^a y^a, \quad M_{i+1} = x^{e'} y^{b+e'}.$$

This gives that

$$c_i - 1 = \lfloor \frac{b+e}{a} \rfloor = \lfloor \frac{b+e'}{a} \rfloor.$$

Fixing a, b we can choose any $0 < e < a$ such that $(a, e) = 1$ and then solve for c . Thus we see that if $b \equiv 0 \pmod{a}$ then $\lfloor \frac{b}{a} \rfloor = c_i - 1$ for every e and if $b \equiv -1 \pmod{a}$ then $\lfloor \frac{b}{a} \rfloor = c_i - 2$ for every e but otherwise both are possible for suitable choice of e .

We see in (6.84) that the condition $a \leq b$ holds iff S_{abc} has a nontrivial qG -deformation, so this is a natural class to consider.

6.83 (Proof of (6.66)). Comparing (6.76) and (6.78) we see that the derivations listed in (6.76.1) give V-deformations but not W-deformations. The only possible exception occurs if $M_i = (xy)^a$ for some i . Thus we have 2 cases.

If $M_i = (xy)^a$ does not occur then $\dim T_V^1(S_{nq}) = \dim T_{VW}^1(S_{nq}) + r - 3$.

If $M_i = (xy)^a$ for some i then (6.76.1) gives $r - 4$ basis vectors that give V-deformations but not W-deformations. By (6.81), there is at most 1 derivation as in (6.76.2) that gives a V-deformation that is not a W-deformation. \square

6.84 (qG-deformations). From (6.58) and (6.71.2)) we see that D corresponds to a qG-deformation iff $D(x^i y^j) + m x^i y^j \nabla D = 0$ whenever $i + j(bc - 1) \equiv -mbc \pmod{ab}$.

First we use this for $1 \cdot (dx \wedge dy)^{ab}$ to conclude that $\nabla D = 0$. Second, we note that since $(a, c) = 1$, the congruence $i + j(bc - 1) \equiv -mbc \pmod{ab}$ holds for some m iff $i \equiv j \pmod{b}$. The ring of such monomials is generated by x^b, xy, y^b . Thus D gives a first order qG-deformation iff

- (1.a) $\nabla D = 0$,
- (1.b) $D(xy) = 0$,
- (1.c) $D(x^b) = 0$ and $D(y^b) = 0$.

We thus get that $T_{qG}^1(S_{abc})$ is spanned by the derivations

$$\left\{ \frac{\partial_x - \partial_y}{(xy)^{as}} : 1 \leq s \leq \lfloor b/a \rfloor \right\}. \quad (6.84.2)$$

The corresponding deformations were written down in [Wah80, 2.7]. The canonical cover of S_{abc} is

$$(uv - w^b = 0) \cong \mathbb{A}^2 / \frac{1}{b}(1, bc - 1) = \mathbb{A}^2 / \frac{1}{b}(1, -1), \quad (6.84.3)$$

hence $u = x^b, v = y^b, w = xy$ and

$$S_{abc} \cong (uv - w^b = 0) / \frac{1}{a}(1, bc - 1, c). \quad (6.84.4)$$

Thus we get explicit qG-deformations of S_{abc} :

$$(uv - w^b - t_1 w^{b-a} - \dots - t_r w^{b-ra} = 0) / \frac{1}{a}(1, bc - 1, c). \quad (6.84.5)$$

To make this \mathbb{G}_m^2 -equivariant, the \mathbb{G}_m^2 -action on t_i should be the same as on $(xy)^{ai}$. Thus (6.84.5) describes a smooth subscheme T of $\text{Def}_{qG}(S_{abc}) \subset \text{Def}(S_{abc})$ and $\dim T = \lfloor b/a \rfloor$. By (6.84.2), the tangent space of $\text{Def}_{qG}(S_{abc})$ has dimension $\lfloor b/a \rfloor$, so $T = \text{Def}_{qG}(S_{abc})$ and $\text{Def}_{qG}(S_{abc})$ is smooth.

In particular, there is a nontrivial 1-parameter qG-deformation iff $a \leq b$ and there is a qG-smoothing iff $a|b$. Note that $a \leq b$ is equivalent to $ab \leq b^2$ and we have proved the following.

Claim 6.84.6. The singularity S_{nq} has

- (a) a qG-smoothing iff $n|(q+1)^2$ and
- (b) a nontrivial qG-deformation iff $n \leq (n, q+1)^2$. Furthermore,
- (c) $\dim T_{qG}^1(S_{nq}) = \lfloor b/a \rfloor = \lfloor (n, q+1)^2/n \rfloor$. \square

If $a|b$ then write $b = ad$. We get the singularities

$$W_{adc} := \frac{1}{a^2 d}(1, adc - 1) \cong (uv - w^{ad} = 0) / \frac{1}{a}(1, -1, c). \quad (6.84.7)$$

In this case $b/a = c_i - 1$ hence the above arguments give the following.

Claim 6.84.8. For the singularities $W_{adc} = \mathbb{A}^2/\frac{1}{a^2d}(1, adc - 1)$ every VW-deformation is a qG-deformation. \square

6.85 (Proof of (6.67)). Note that (6.67.1) follows from (6.78.3) and (6.67.2) from (6.84.8) for first order deformations. Since $\text{Def}_{\text{qG}}(S_{n,q})$ is smooth by (2.29) or by the explicit description (6.84.5), equality of the tangent spaces $T_{\text{qG}}^1(S_{n,q}) = T_{\text{VW}}^1(S_{n,q})$ implies that $\text{Def}_{\text{qG}}(S_{n,q}) = \text{Def}_{\text{VW}}(S_{n,q})$.

In order to prove (6.67.3) we consider 2 cases. If R_{nq} does not have a minimal generator of the form $M_i = (xy)^a$ then $T_{\text{VW}}^1(S_{nq}) = T_{\text{qG}}^1(S_{nq}) = \{0\}$ by (6.78.4).

Otherwise, we have proved in (6.84) that

$$\dim T_{\text{qG}}^1(\mathbb{A}^2/\frac{1}{ab}(1, bc - 1)) = \lfloor \frac{b}{a} \rfloor$$

and (6.81) shows that

$$\dim T_{\text{VW}}^1(\mathbb{A}^2/\frac{1}{ab}(1, bc - 1)) = \min\{c_i - 1, \frac{q+1}{a}, \frac{q'+1}{a}\} \leq \lfloor \frac{b}{a} \rfloor + 1. \quad \square$$

EXAMPLE 6.86. We list those cyclic quotient singularities for which every V-deformation is a qG-deformation. We go by multiplicity, noting that $\text{embdim } S_{nq} = \text{mult } S_{nq} + 1$ by (6.65.4).

Double points 6.86.1. These are the A_n singularities; every deformation is a qG-deformation.

Triple points 6.86.2. For cyclic quotient triple points the minimal generators of its coordinate ring are $x^n, x^{n-q}y, xy^{n-q'}, y^n$. Thus $\frac{n}{n-q}$ has a 2-step continued fraction expansion involving c_1, c_2 . Setting $c_1 = e, c_2 = d$ we have the singularities

$$\mathbb{A}^2/\frac{1}{ed-1}(1, ed - d - 1).$$

with invariants

$$x^{ed-1}, x^d y, xy^e, y^{ed-1}.$$

By (6.76) we have $T_{\text{V}}^1 = T_{\text{qG}}^1 = 0$.

Quadruple points 6.86.3. By (6.66) and (6.83), every cyclic quotient singularity of multiplicity 4 has a V-deformation that is not a qG-deformation, unless M_2 (6.71.1) is a power of xy . Thus in this case the minimal generators of its coordinate ring are

$$x^n, x^{n-q}y, x^a y^a, xy^{n-q'}, y^n.$$

The equation $M_2^{c_2} = M_1 M_3$ now implies that $q = q'$. Thus $\frac{n}{n-q}$ has a 3-step continued fraction expansion involving $c_1, c_2, c_3 = c_1$. By expanding it we see that $c_1 = a$. Setting $c_2 = d$ the singularity is

$$\mathbb{A}^2/\frac{1}{a(ad-2)}(1, (ad-2)(a-1) - 1)$$

and the minimal generators of the ring of invariants are

$$x^{a(ad-2)}, x^{ad-1}y, x^a y^a, xy^{ad-1}, y^{a(ad-2)}.$$

Thus $\lfloor (ad-2)/a \rfloor = d-1 = c_2 - 1$ and hence, by (6.76) and (6.84), $T_{\text{V}}^1 = T_{\text{qG}}^1$ is spanned by

$$\left\{ \frac{\partial_x - \partial_y}{(xy)^{as}} : 1 \leq s \leq d-1 \right\}.$$

These singularities admit a qG-smoothing iff $a = 2$. Then, after replacing $d - 1$ by d , the normal form becomes

$$\mathbb{A}^2 / \frac{1}{4d}(1, 2d - 1).$$

Together with the A_n -series, these are the only cyclic quotient singularities with a qG-smoothing for which every V-deformation is a qG-deformation.

Higher multiplicity points 6.86.4. By (6.66), every cyclic quotient singularity of multiplicity ≥ 5 has V-deformations that are not qG-deformations.

Cayley flatness

There are 2 traditional notions of what a ‘family of varieties’ is: the older Cayley-Chow variant (3.5) and the currently ubiquitous Hilbert-Grothendieck variant (3.6), which puts flatness at the center.

For stable varieties, the Hilbert-Grothendieck approach gives the correct moduli theory. That is, a stable morphism $X \rightarrow S$ is a flat morphism with additional properties, as in Section 6.2.

A major problem in the moduli theory of stable pairs is that, while the underlying varieties X form flat families, the divisorial parts Δ do not. Neither of the two main traditional methods of parametrizing varieties or schemes gives the right answer for the divisorial part.

- Cayley-Chow theory works only over reduced base schemes.
- Hilbert-Grothendieck theory works only when the coefficients of Δ satisfy various restrictions, as in Sections 6.2 and 6.4.

In this chapter we develop a theory—called K-flatness—that interpolates between these two, managing to keep from both of them the properties that we need. The objects that we parametrize are divisors—so the strong geometric flavor of Cayley-Chow theory is preserved—but one can work over Artinian base schemes. The latter is one of the key advantages of the theory of Hilbert schemes. Quite unexpectedly, the new theory behaves better than either of the classical approaches in several aspects; see especially (7.4–7.5).

One might say that the main new result is Definition 7.1; we discuss its origin and relationship to the classical theory of Chow varieties in (7.2). The rest of this chapter is then devoted to proving that it has all the hoped-for properties.

K-flatness turns out to have many very good properties; see (7.3–7.5) for a detailed discussion. (Actually, we end up with several variants, but we conjecture them to be equivalent; see Section 7.4.)

The definition of K-flatness and its main properties are discussed in Section 7.1, while Section 7.2 reviews divisor theory over Artin schemes. The key notion of divisorial support is introduced and studied in Section 7.3.

Several versions of K-flatness are investigated in Section 7.4. For our treatment, technically the most important is C-flatness, which is treated in detail in Section 7.5. The main results are proved in Section 7.6.

Sections 7.7–7.9 are devoted to examples. First we show that a K-flat deformation of a normal variety is flat. Then we describe first order K-flat deformations of plane curves in Section 7.8 and of seminormal curves in Section 7.9. While the computations are somewhat lengthy, the answers are quite nice in both cases.

Assumptions. In this Chapter we work with \mathbb{Q} -schemes. Almost all the proofs work for arbitrary schemes, the main problem is with (7.4.6). This is very similar

to the difficulties encountered in Section 1.9, so it is more significant than seems at first sight.

7.1. K-flatness

We eventually introduce several closely related (possibly equivalent) notions in (7.40). The most natural one is C-flatness, which is closest to the ideas of Cayley. Aiming to create a notion that is independent of projective embeddings led to K-flatness. Conveniently, K is also the first syllable of Cayley.

DEFINITION 7.1 (K-flatness). Let $f : X \rightarrow S$ be a projective morphism of pure relative dimension n . A relative Mumford divisor $D \subset X$ is *K-flat* over S iff one of the following—increasingly more general—conditions hold.

(7.1.1) (S local with infinite residue field) For every finite morphism $\pi : X \rightarrow \mathbb{P}_S^n$, $\pi_*D \subset \mathbb{P}_S^n$ is a relative Cartier divisor.

(7.1.2) (S local) q^*D is K-flat over S' for some (equivalently every) flat, local morphism $q : S' \rightarrow S$, where S' has infinite residue field.

(7.1.3) (S arbitrary) D is K-flat over every localization of S .

Let us start with some comments on the definition.

(7.1.4) The definition of π_*D is not always obvious; in essence Section 7.3 is mainly devoted to establishing it. However, π_*D equals the scheme-theoretic image of D if $\text{red } D \rightarrow \text{red}(\pi(D))$ is birational and π is étale at every generic point of every fiber D_s (7.30.2). It is sufficient to check condition (7.1.1) for such morphisms $\pi : X \rightarrow \mathbb{P}_S^n$.

(7.1.5) If S is not local, then there may not be any finite morphisms $\pi : X \rightarrow \mathbb{P}_S^n$; see (7.7) for an example. This is one reason for the 3 step definition.

(7.1.6) The infinite residue field extensions in (7.1.2) are necessary in some cases; see for example (7.84.9).

(7.1.7) The definition of K-flatness is global in nature, but we show that it is in fact local on X (7.55).

(7.1.8) We eventually define K-flatness also for families of coherent sheaves in (7.40). This turns out to be quite convenient technically. However, while the images π_*D carry a lot of information about a Mumford divisor D , much of the sheaf information is lost. Thus it is unlikely that K-flatness can be useful for studying the moduli of sheaves.

As we already noted, we do not claim that this is the only possible definition, so it may be helpful to outline what led us to it.

7.2 (Why this definition?). Cayley's idea—developed in the papers [Cay60, Cay62]—is to associate to a subvariety $Y^{n-1} \subset \mathbb{P}_k^N$ a hypersurface

$$\text{Ch}(Y) := \{L \in \text{Gr}(N-n, \mathbb{P}_k^N) : Y \cap L \neq \emptyset\} \subset \text{Gr}(N-n, \mathbb{P}_k^N),$$

we call it the Cayley-Chow hypersurface. He clearly had in mind to apply this also to families of subvarieties. In modern terminology, the end result is that, at least over seminormal bases in characteristic 0, there is a one-to-one correspondence

$$\left\{ \begin{array}{l} \text{well defined families} \\ \text{of subvarieties} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{flat families of} \\ \text{Cayley-Chow hypersurfaces} \end{array} \right\}; \quad (7.2.1)$$

see Section 4.7 or [Kol96, Sec.I.3] for details.

The correspondence (7.2.1) works well for reduced, pure dimensional subschemes, but for an arbitrary subscheme $Z \subset \mathbb{P}^N$, its Cayley-Chow hypersurface $\text{Ch}(Z)$ detects only $\text{red } Z$ and the multiplicities of Z at the maximal dimensional generic points. This is where the role of X and the Mumford condition become crucial: a Mumford divisor $D \subset X$ is uniquely determined by $\text{red } D$ and the multiplicities.

We know how to define flatness in general, so we try to make the above equivalence into a definition over an arbitrary base scheme. So let $f : X \rightarrow S$ be a flat, projective morphism, say with reduced fibers of pure dimension n . We fix an embedding $X \hookrightarrow \mathbb{P}_S^N$ and let $D \subset X$ be a Mumford divisor. We say that D is C-flat over S iff $\text{Ch}(D/S)$ is flat over S . (This needs a suitable extension of the definition of $\text{Ch}(D/S)$ to allow for multiple fibers; see (7.40) for details.)

There are 2 immediate disadvantages of C-flatness. Cayley-Chow hypersurfaces are unwieldy objects, and the resulting notion is very much tied to the choice of an embedding $X^n \hookrightarrow \mathbb{P}_k^N$.

One can think of a Cayley-Chow hypersurface $\text{Ch}(D/S)$ as encoding the images $\pi(D)$ for all linear projections $\pi : \mathbb{P}_S^N \dashrightarrow \mathbb{P}_S^n$. (This also goes back to Cayley, and is worked out in [Cat92, DS95, Kol99].) One can show that the Cayley-Chow hypersurface $\text{Ch}(D/S)$ is flat over S iff $\pi(D) \subset \mathbb{P}_S^n$ is flat over S , for all *linear* projections $\pi : \mathbb{P}_S^N \dashrightarrow \mathbb{P}_S^n$ that are finite on $\text{Supp } D$; see (7.50). (In fact, by (7.50), it is enough to check this for a dense set of projections. We need S to be local with infinite residue field to ensure that there are enough projections.)

This suggests 3 different generalizations of C-flatness. We can work with

- projective morphisms $f : X \rightarrow S$, and all finite $\pi : X \rightarrow \mathbb{P}_S^n$,
- affine morphisms $f : U \rightarrow S$, and all finite $\pi : U \rightarrow \mathbb{A}_S^n$, or
- morphisms of complete, local schemes $f : \widehat{X} \rightarrow \widehat{S}$, and all finite $\pi : \widehat{X} \rightarrow \widehat{\mathbb{A}}_S^n$.

The affine version has the problem that, even if S is local, there might not be any finite morphisms $\pi : U \rightarrow \mathbb{A}_S^n$; see (10.67.7) for such examples. Working with complete, local schemes would be the best theoretically, but several of the technical problems remain unresolved. This leaves us with projective morphisms, which is our definition of K-flatness.

We conjecture that C-flatness, K-flatness and formal K-flatness are equivalent, giving a very robust concept.

The key technical result (7.43) shows that K-flatness is equivalent to C-flatness for every Veronese embedding

$$X \hookrightarrow \mathbb{P}_S^N \xrightarrow{v_m} \mathbb{P}_S^M, \quad \text{where } M = \binom{N+m}{m} - 1;$$

we call the resulting notion stable C-flatness.

If, as conjectured, K-flatness and formal K-flatness are equivalent, this would show that our notion is truly about the singularities in families of divisors. The equivalence of C-flatness and K-flatness would be very helpful computationally, but does not seem to be theoretically significant.

Good properties of K-flatness.

K-flat families have several good properties. Some of them are needed for the moduli theory of stable pairs, but others, for example (7.5), come as bonus.

The functoriality of K-flatness is not obvious. Indeed, let $T \subset S$ be a closed subscheme. Then a finite morphism $\pi_T : X_T \rightarrow \mathbb{P}_T^n$ need not extend to a finite

morphism $\pi_S : X_S \rightarrow \mathbb{P}_S^n$. Thus flatness of all $\pi_S(X_S)$ does not directly imply that $\pi_T(X_T)$ is also flat.

Nonetheless, we prove in (7.43) and (7.53) that being K-flat is preserved by arbitrary base changes and it descends from faithfully flat base changes. Thus we get the functor $KDiv(X/S)$ of K-flat, relative Mumford divisors on X/S . If we have a fixed relatively ample divisor H on X , then $KDiv_d(X/S)$ denotes the functor of K-flat, relative Mumford divisors of degree d .

We have a disjoint union decomposition $KDiv(X/S) = \cup_d KDiv_d(X/S)$. The main result is the following, to be proved in (7.68).

THEOREM 7.3. *Let $f : X \rightarrow S$ be a projective morphism of pure relative dimension n . Then the functor $KDiv_d(X/S)$ of K-flat, relative Mumford divisors of degree d is representable by a separated S -scheme of finite type $KDiv_d(X/S)$.*

Complement 7.3.1. If f is flat with normal fibers then $KDiv_d(X/S)$ is proper over S , but otherwise usually $KDiv_d(X/S)$ is not proper. This is, however, not a problem for the moduli of stable pairs.

7.4 (Properties of K-flatness). We list a series of good properties of K-flatness.

Comparison with flatness 7.4.1. K-flatness is a generalization of flatness and it is equivalent to it for smooth morphisms and for normal divisors.

- If $f|_D : D \rightarrow S$ is flat then D is K-flat; see (7.57).
- If $f : X \rightarrow S$ is smooth, then D is K-flat $\Leftrightarrow D$ is flat over $S \Leftrightarrow D$ is a relative Cartier divisor; see (7.56).
- Assume that D is K-flat, $D_s \subset X_s$ has multiplicity 1 and $\text{red}(D_s)$ is normal. Then $f|_D : D \rightarrow S$ is flat along D_s by (7.70). When S is a DVR, this is essentially Hironaka's flatness theorem, as in [Har77, III.9.11]. K-flatness makes it work over an arbitrary base.

All of these properties also hold locally on X . Hence, the notion of K-flatness gives something new only at the points where f is not smooth and $f|_D$ is not flat.

Reduced base schemes 7.4.2. If S is reduced then every relative Mumford divisor is K-flat; see (7.31). In retrospect, this is the reason why the moduli theory of pairs could be developed over reduced base schemes without the notion of K-flatness in Chapter 4.

Artinian base schemes 7.4.3. A divisor $D \subset X$ is K-flat over S iff $D_A \subset X_A$ is K-flat over A for every Artin subscheme $A \subset S$; see (7.47).

Thus one can fully understand K-flatness by studying it over reduced bases (as in Chapter 4) and over Artinian base schemes.

Push-forward 7.4.4. Let $f : X \rightarrow S$ and $g : Y \rightarrow S$ be projective morphisms of pure relative dimension n and $\tau : X \rightarrow Y$ a finite morphism. Let $D \subset X$ be a K-flat relative Mumford divisor such that τ_*D is also a relative Mumford divisor. (That is, g is smooth at generic points of $\tau(D_s)$ for every s .) Then τ_*D is also K-flat, see (7.48). (See (7.1.7) and Section 7.3 for the correct definition of τ_*D .)

A similar property fails for flatness; combine (7.7.3) and the arguments in (7.48).

Additivity 7.4.5. Let $f : X \rightarrow S$ be a projective morphism of pure relative dimension n and $D_1, D_2 \subset X$ relative Mumford divisors. If the D_i are K-flat then so is $D_1 + D_2$, see (7.48). This again fails for flatness; see (7.7.3).

Multiplicativity 7.4.6. Let $f : X \rightarrow S$ be a projective morphism of pure relative dimension n and $D \subset X$ a relative Mumford divisor. Let $m > 0$ be relatively prime to the residue characteristics. Then D is K-flat iff mD is K-flat, see (7.48).

By contrast, if A is Artinian, nonreduced, with residue field k of characteristic $p > 0$, then the divisors D on \mathbb{P}_A^2 such that pD is K-flat (= relative Cartier) but D is not K-flat, span an infinite dimensional k -vector space; see (7.9.6–7). This is a major conceptual difficulty in positive characteristic.

Linear equivalence 7.4.7. K-flatness is preserved by linear equivalence, see (7.36). (Note that flatness is not preserved by linear equivalence (7.7.4).)

K-flatness does not depend on X 7.4.8. It is well understood that in the theory of pairs (X, Δ) one can not separate the underlying variety X from the divisorial part Δ . For example, if X is a surface with quotient singularities only, and $D \subset X$ is a smooth curve, then the pair (X, D) is plt if $D \subset \text{Sing } X = \emptyset$, but not even lc in some other cases. It really matters how exactly D sits inside X .

Thus it is unexpected that K-flatness depends only on the divisor D , not on the ambient variety X , though maybe this is less surprising if one thinks of K-flatness as a variant of flatness.

On the other hand, not all K-flat deformations (7.69) of D are realized on deformations of a given X . For example, for deformations of the pair $(\mathbb{A}^2, D_1 := (xy = 0))$, K-flatness is equivalent to flatness by (7.4.2).

On the other hand, there are deformations of the pair $((xy = z^2), (z = 0))$ that induce a K-flat but non-flat deformation of $D_2 := (xy - z^2 = z = 0) \cong D_1$. A typical example is

$$((xy = z^2 - t^2), (x = z + t = 0) \cup (y = z - t) = 0) \subset \mathbb{A}_{xyz}^3 \times \mathbb{A}_t^1.$$

Now we come to a property that is quite unexpected, but makes the whole theory much easier to use: K-flatness is essentially a property of surface pairs (S, D) . Thus K-flatness is mostly about families of singular curves.

THEOREM 7.5 (Bertini theorems, up and down). *Let $f : X \rightarrow S$ be a projective morphism of pure relative dimension n and D a Mumford divisor on X . Assume that $n \geq 3$ and let $|H|$ be a basepoint-free linear system on X . Then D is K-flat iff $D|_H$ is K-flat for general $H \in |H|$.*

This is established by combining (7.60–7.62) with (7.43). As a consequence, K-flatness is really a question about families of surfaces and curves on them. There are similar theorems for families of stable pairs, see (5.6) or [Kol13a, BdJ14, Kol16a].

This reduction to surfaces is very helpful conceptually, but also computationally since we have rather complete lists of singularities of log canonical surface pairs (X, Δ) , at least when the coefficients of Δ are not too small.

Another variant of the phenomenon, that higher codimension points sometimes do not matter much, is the Hironaka-type flatness theorem (10.63).

7.6 (Problems and questions about K-flatness). There are also some difficulties with K-flatness. I believe that they do not effect the general moduli theory of stable pairs, but they make some of the proofs convoluted and explicit computations lengthy.

The definition is not formal-local 7.6.1. One expects K-flatness to be a formal-local property on X , but there are some (hopefully only technical) problems with

this. See (7.44) and (7.63) for partial results. This is probably the main open foundational question.

Hard to compute 7.6.2. The definition of K-flatness is quite hard to check, since for $X \subset \mathbb{P}^N$ we need to check not just linear projections $\mathbb{P}_S^N \dashrightarrow \mathbb{P}_S^n$ (7.39) but all morphisms $X \rightarrow \mathbb{P}_S^n$ involving all linear systems on X .

It is, however, possible that checking general linear projections is in fact sufficient; see (7.50) and (7.45) for a precise formulation.

In the examples in Sections 7.8–7.9, the computation of the restrictions imposed by general linear projections is the hard part. From the resulting answers it is then easy read off what happens for all morphisms $X \rightarrow \mathbb{P}_S^n$. It would be good to work out more space curves $C \subset \mathbb{A}^3$.

Tangent space and obstruction theory 7.6.3. I do not know how to write down the tangent space of $\mathrm{KDiv}(X/S)$. A handful of examples are computed in Sections 7.8–7.9, but they do not seem to suggest any general pattern. The obstruction theory of K-flatness is completely open.

Bounding the torsion 7.6.4. Over a DVR, every K-flat deformation (7.69) of a variety X is a flat deformation of some scheme X' such that $\mathrm{red} X' = X$. By (7.4.1), the torsion subsheaf $\mathrm{tors} \mathcal{O}_{X'} \subset \mathcal{O}_{X'}$ is supported on $\mathrm{Sing} X$. It would be good to get an a priori bound on the size of $\mathrm{tors} \mathcal{O}_{X'}$.

To be precise, let (A, m, k) be an Artin scheme and $C_A \rightarrow \mathrm{Spec} A$ a K-flat deformation of a reduced, pointed curve (c, C) that is flat on $C \setminus \{c\}$. Let C_k be the central fiber (thus $C = \mathrm{red} C_k$) and $I = \ker[\mathcal{O}_{C_A} \rightarrow \mathcal{O}_C]$. Thus $\mathrm{tors} C_k = I/m\mathcal{O}_{C_A}$ (and $C_A \rightarrow \mathrm{Spec} A$ is flat iff $\mathrm{tors} C_k = 0$ by (7.11)).

Question 7.6.5. What is the best bound for $\mathrm{tors} C_k$, depending only on C ?

Universal deformation spaces 7.6.6. While we are mainly interested in divisors that lie on a particular family of varieties $X \rightarrow S$, the following seems also natural, in view of (7.4.8). (See (7.75.7) for a nonreduced example.)

Question 7.6.7. Let D be a reduced, projective scheme over a field k . Is there a universal deformation space for its K-flat deformations?

EXAMPLES 7.7. The first example shows that the space of first order deformations of the smooth divisor $(x = 0) \subset \mathbb{A}^2$, that are Cartier away from the origin, is infinite dimensional. Thus working with generically flat divisors does not give a sensible moduli space.

(7.7.1) Let $g(y^{-1}) \in y^{-1}k[y^{-1}]$ be a polynomial of degree n . Then

$$x + g(y^{-1})\epsilon \in k[x, y, y^{-1}, \epsilon]_{(x,y)}$$

defines a relative Cartier divisor D_g° whose restriction to the closed fiber is $(x = 0)$. One can check (7.15) that, if $g_1 \neq g_2$, then $D_{g_1}^\circ$ and $D_{g_2}^\circ$ give different elements of the Picard group. Consider the ideal

$$I_g = (x^2, xy^n + y^n g(y^{-1})\epsilon, \epsilon x) \subset k[x, y, \epsilon]_{(x,y)} \quad \text{and set} \quad D_g = \mathrm{Spec} k[x, y, \epsilon]/I_g.$$

Note that $y^n g(y^{-1})$ is invertible in $k[x, y, \epsilon]_{(x,y)}$, hence

$$k[x, y, \epsilon]_{(x,y)}/(x^2, xy^n + y^n g(y^{-1})\epsilon, \epsilon x) \cong k[x, y, \epsilon]_{(x,y)}/(x^2).$$

Thus D_g is the scheme-theoretic closure of D_g° and we see that D_g is Cartier away from the origin, $(I_g, \epsilon)/(\epsilon) = (x^2, xy^n)$, D_g has no embedded points, and $D_{g_1} \sim D_{g_2}$ iff $g_1 = g_2$.

More general computations are done in (7.21).

(7.7.2) Let C be a smooth projective curve and E a vector bundle over C of rank $n + 1 \geq 2$ and of degree 0. We claim that usually there is no finite morphism $\pi : \mathbb{P}_C(E) \rightarrow \mathbb{P}^n \times C$.

Indeed, let $p_0, \dots, p_{n+1} \in \mathbb{P}^n$ be the coordinate vertices plus $(1: \dots : 1)$. Then $C_i := \pi^{-1}(\{p_i\} \times C)$ are $n+2$ disjoint multi-sections of $\mathbb{P}_C(E) \rightarrow C$. Pick $p : D \rightarrow C$ that factors through all of the $C_i \rightarrow C$. Then $\mathbb{P}_D(p^*E)$ has $n+2$ disjoint sections in linearly general position, hence $\mathbb{P}_D(p^*E) \cong \mathbb{P}^n \times D$. Equivalently, $p^*D \cong L \otimes \mathcal{O}_D^{n+1}$ for some line bundle L of degree 0.

This can not happen for most line bundles. The simplest example is $E = \mathcal{O}_{\mathbb{P}^1}(1) + \mathcal{O}_{\mathbb{P}^1}(-1)$. More generally, such a line bundle has to be semi-stable. If E is stable, hence comes from a representation $\pi_1(C) \rightarrow \mathrm{U}(n+1)$, then its image in $\mathrm{PU}(n+1)$ must be finite.

(7.7.3) Let X be the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. Let $D_1, D_2 \subset X$ be cones over $\{p_i\} \times \mathbb{P}^2$ and $\pi : X \dashrightarrow \mathbb{P}^2$ be a general projection.

Note that the restrictions $D_i \dashrightarrow \mathbb{P}^2$ are flat, but $D_1 \cup D_2 \dashrightarrow \mathbb{P}^2$ and $2D_1 \dashrightarrow \mathbb{P}^2$ are not flat at the vertex of the cone.

This example even occurs on a log canonical 4-fold. If D_3 is the cone over $\mathbb{P}^1 \times C_6$, where $C_6 \subset \mathbb{P}^2$ is a smooth sextic, then $(X, D_1 + D_2 + \frac{1}{2}D_3)$ is log canonical

(7.7.4) Let $A \subset \mathbb{P}^n$ be a projectively normal Abelian variety of dimension ≥ 2 and $C_A \subset \mathbb{P}^{n+1}$ the cone over it. Let $\pi : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^2$ be a general projection. Let $H \subset C_A$ be a hyperplane section. If H does not pass through the vertex then $H \cong A$ is smooth and $\pi|_H : H \dashrightarrow \mathbb{P}^2$ is flat.

If H does pass through the vertex v , then $\mathrm{depth}_v H = 1$ (cf. [Kol13b, 3.10]), hence $\pi|_H : H \dashrightarrow \mathbb{P}^2$ is not flat at v .

7.2. Infinitesimal study of Mumford divisors

In this section we review the divisor theory of nonreduced schemes. The standard reference books treat Cartier divisors in detail, but for us the interesting cases are precisely when the divisors fail to be Cartier. We start with the general theory and at the end give explicit formulas for some cases that we need later.

7.8. The infinitesimal method to study families of objects in algebraic geometry posits that we should proceed in 3 broad steps.

- Study families over Artin schemes.
- Inverse limits then give families over complete local schemes.
- For arbitrary local schemes, descend properties from the completion.

This approach has been very successful for proper varieties and coherent sheaves on them. One of the problems we have with general (possibly non-flat) families of divisors is that the global and the infinitesimal computations do not match up; in fact they say the opposite in some cases. We discuss 2 instances of this:

- Relative Cartier divisors on non-proper varieties.
- Generically flat families of divisors on surfaces.

The surprising feature is that the two behave quite differently. We state 2 special cases of the results where the contrasts between Artin and DVR bases are especially striking; see (7.9) for details.

Claim 7.8.1. Let $\pi : X \rightarrow (s, S)$ be a smooth, affine morphism to a local scheme.

- (a) If S is Artin then the restriction map $\text{Pic}(X) \rightarrow \text{Pic}(X_s)$ is an isomorphism.
- (b) If $S = \text{Spec } k[[t]]$ then $\text{Pic}(X)$ is frequently infinite dimensional.

Thus there can be many nontrivial line bundles on X over $\text{Spec } k[[t]]$, but we do not see them when working over $\text{Spec } k[[t]]/(t^m)$; see (7.9.3) and (7.13) for details.

Claim 7.8.2. Let $\pi : X \rightarrow (s, S)$ be a smooth morphism of relative dimension 2 to a local scheme S .

- (a) If S is Artin and non-reduced, then the relative class group $\text{Cl}(X/S)$ (7.14) is infinite dimensional.
- (b) If $S = \text{Spec } k[[t]]$ then every divisor $D \subset X$ is Cartier.

As an example, one easily computes that

$$\text{Cl}(\mathbb{P}_{k[[t]]}^2) \cong \mathbb{Z} \quad \text{but} \quad \text{Cl}(\mathbb{P}_{k[[t]]/(t^m)}^2) \cong \mathbb{Z} + k^\infty \quad \text{for } m \geq 2.$$

Note that if $D \subset X$ is a relative Mumford divisor over $S = \text{Spec } k[[t]]/(t^m)$ then it is Cartier on an open set $X^\circ \subset X$ whose complement has codimension ≥ 2 . Thus the study of $\text{Cl}(X/S)$ is pretty much equivalent to the study of $\text{CDiv}(X^\circ/S)$ for every such X° .

Relative Picard group, examples.

7.9 (Picard group over Artin schemes). Let (A, m, k) be a local Artin ring and $X_A \rightarrow \text{Spec } A$ a flat morphism. Let $(\epsilon) \subset A$ be an ideal such that $I \cong k$ and set $B = A/(\epsilon)$. We have an exact sequence

$$0 \rightarrow \mathcal{O}_{X_k} \xrightarrow{e} \mathcal{O}_{X_A}^* \rightarrow \mathcal{O}_{X_B}^* \rightarrow 1, \quad (7.9.1)$$

where $e(h) = 1 + h\epsilon$ is the exponential map. We use its long exact cohomology sequence and induction on length A to compute $\text{Pic}(X_A)$. There are 3 cases that are especially interesting for us.

Claim 7.9.2. Let $X_A \rightarrow \text{Spec } A$ be a flat, affine morphism. Then the restriction map $\text{Pic}(X_A) \rightarrow \text{Pic}(X_k)$ is an isomorphism.

Proof. We use the exact sequence

$$H^1(X_k, \mathcal{O}_{X_k}) \rightarrow \text{Pic}(X_A) \rightarrow \text{Pic}(X_B) \rightarrow H^2(X_k, \mathcal{O}_{X_k}). \quad (7.9.3)$$

Since X is affine, the two groups at the ends vanish, hence we get an isomorphism in the middle. Induction completes the proof. \square

Claim 7.9.4. Let $X_A \rightarrow \text{Spec } A$ be a flat, proper morphism. Assume that $H^0(X_k, \mathcal{O}_{X_k}) = k$. Then the kernel of the restriction map $\text{Pic}(X_A) \rightarrow \text{Pic}(X_k)$ is a unipotent group scheme of dimension $\leq h^1(X_k, \mathcal{O}_{X_k}) \cdot (\text{length } A - 1)$, and equality holds if $H^2(X_k, \mathcal{O}_{X_k}) = 0$. (In fact, if $\text{char } k = 0$, then the kernel is a k -vector space and equality holds even if $H^2(X_k, \mathcal{O}_{X_k}) \neq 0$; see [BLR90, Chap.8].)

Proof. By [Har77, III.12.11], $H^0(X_A, \mathcal{O}_{X_A}) \rightarrow H^0(X_B, \mathcal{O}_{X_B})$ is surjective, and so is $H^0(X_A, \mathcal{O}_{X_A}^*) \rightarrow H^0(X_B, \mathcal{O}_{X_B}^*)$. Thus we get the exactness of

$$0 \rightarrow H^1(X_k, \mathcal{O}_{X_k}) \rightarrow \text{Pic}(X_A) \rightarrow \text{Pic}(X_B) \rightarrow H^2(X_k, \mathcal{O}_{X_k}). \quad \square \quad (7.9.5)$$

Claim 7.9.6. Let $X_A \rightarrow \text{Spec } A$ be a flat, affine morphism and $Z \subset X_A$ a closed subset of codimension ≥ 2 . Set $X_A^\circ := X_A \setminus Z$. Assume that X_k is S_2 . Then the kernel of the restriction map $\text{Pic}(X_A^\circ) \rightarrow \text{Pic}(X_k^\circ)$ is a unipotent group scheme of dimension $\leq h^1(X_k^\circ, \mathcal{O}_{X_k^\circ}) \cdot (\text{length } A - 1)$.

Proof. Since X_k is S_2 , $H^0(X_k^\circ, \mathcal{O}_{X_k^\circ}) \cong H^0(X_k, \mathcal{O}_{X_k})$ and similarly for X_A . Thus $H^0(X_A^\circ, \mathcal{O}_{X_A^\circ}^*) \rightarrow H^0(X_B^\circ, \mathcal{O}_{X_B^\circ}^*)$ is surjective, and the rest of the argument works as in (7.9.5). □

Remark 7.9.7. Although (7.9.6) is very similar to (7.9.4), a key difference is that in (7.9.6) the group $H^1(X_k^\circ, \mathcal{O}_{X_k^\circ})$ can be infinite dimensional. Indeed, $H^1(X_k^\circ, \mathcal{O}_{X_k^\circ}) \cong H_Z^2(X_k, \mathcal{O}_{X_k})$ and it is

- (a) infinite dimensional if $\dim X_k = 2$,
- (b) finite dimensional if X_k is S_2 and $\text{codim}_{X_k} Z \geq 3$, and
- (c) 0 if X_k is S_3 and $\text{codim}_{X_k} Z \geq 3$.

See, for example, Section 10.3 for these claims.

The following immediate consequence of (7.9.7.c) is especially useful for us; see also (2.98).

Corollary 7.9.8. Let $X \rightarrow S$ be a smooth morphism, $D \subset X$ a closed subscheme and $Z \subset X$ a closed subset. Assume that

- (a) D is a relative Cartier divisor on $X \setminus Z$,
- (b) D has no embedded points in Z and
- (c) $\text{codim}_{X_s} Z_s \geq 3$ for every $s \in S$.

Then D is a relative Cartier divisor. □

The following is a reformulation of (2.98.1) and a special case of (4.34).

LEMMA 7.10. *Let $X \rightarrow S$ be a flat morphism with S_2 fibers and D a divisorial subscheme. Let $U \subset X$ be an open subscheme such that $D|_U$ is relatively Cartier and $\text{codim}_{X_s}(X_s \setminus U_s) \geq 2$ for every $s \in S$.*

Then D is relatively Cartier iff the divisorial pull-back $\tau^{[]}D$ (4.1.7) is relatively Cartier for every Artin subscheme $\tau : A \hookrightarrow S$.* □

Over Artin rings, we have the following flatness criterion. For a coherent sheaf F let $\text{emb}(F)$ denote the largest subsheaf whose support is the union of the (closures of the) embedded points of F .

LEMMA 7.11. *Let (A, m, k) be a local Artin ring, $g : X \rightarrow \text{Spec } A$ a morphism and F a coherent sheaf on X . Assume that F is generically flat over A and $\text{emb } F = 0$. Then F is flat over A iff $\text{emb}(F_k) = 0$.*

Proof. Choose $0 \neq \epsilon \in m$ such that $m\epsilon = 0$. If F is flat over A then $\epsilon F \cong F_k$, thus we get an injection $\epsilon : \text{emb}(F_k) \hookrightarrow \text{emb}(F)$. Thus if $\text{emb}(F) = 0$ then so is $\text{emb}(F_k)$.

Conversely, assume that $\text{emb}(F_k) = 0$. We may assume that X is affine. By induction on length A we may assume that $(F/\epsilon F)/\text{emb}(F/\epsilon F)$ is flat over $A/(\epsilon)$. We claim that $\text{emb}(F/\epsilon F) \subset \text{emb}(F)$.

By assumption $(F/\epsilon F)/\text{emb}(F/\epsilon F)$ is a free $A/(\epsilon)$ module. Choose basis elements f_λ and lift them back to $\tilde{f}_\lambda \in H^0(X, F)$.

Let $(\epsilon F)^{(1)} \subset F$ be the preimage of $\text{emb}(F/\epsilon F)$. Pick now $h \in (\epsilon F)^{(1)}$. The image of h in (F/mF) is 0, so $h = \sum a_i g_i$ for some $a_i \in m, g_i \in F$. Write each g_i in the f_λ basis modulo ϵ . Thus we have

$$g_i \equiv \sum_\lambda c_{i\lambda} \tilde{f}_\lambda \pmod{(\epsilon F)^{(1)}}.$$

Since $m(\epsilon F)^{(1)} = 0$, we get that

$$h = \sum_\lambda (\sum_i a_i c_{i\lambda}) \tilde{f}_\lambda.$$

This is zero modulo $(\epsilon F)^{(1)}$, so $\sum_i a_i c_{i\lambda} \in (\epsilon)$ for every λ . Thus $h \in \epsilon F$, $\epsilon F \cong F/mF$ and $\text{emb}(F/\epsilon F) = 0$, so F is flat over A . \square

Relative Cartier divisors also have some unexpected properties over non-reduced base schemes. These do not cause theoretical problems, but it is good to keep them in mind.

EXAMPLE 7.12 (Cartier divisors over $k[\epsilon]$). Let R be an integral domain over a field k . Relative principal ideals in $R[\epsilon]$ over $k[\epsilon]$ are given as $(f + g\epsilon)$ where $f, g \in R$ and $f \neq 0$. We list some properties of such principal ideals that hold for any integral domain R .

$$(7.12.1) \quad (f + g_1\epsilon) = (f + g_2\epsilon) \text{ iff } g_1 - g_2 \in (f),$$

$$(7.12.2) \quad \text{If } u \in R \text{ is a unit then so is } u + g\epsilon \text{ since } (u + g\epsilon)(u^{-1} - u^{-2}g\epsilon) = 1,$$

$$(7.12.3) \quad \text{If } f \text{ is irreducible then so is } f + g\epsilon \text{ for every } g,$$

$$(7.12.4) \quad (f + g\epsilon)(f - g\epsilon) = f^2 \text{ shows that there is no unique factorization.}$$

$$(7.12.5) \quad \text{If } R \text{ is a UFD and the } f_i \text{ are pairwise relatively prime then}$$

$$\prod_i (f_i + g_i\epsilon) = \prod_i (f_i + g'_i\epsilon) \quad \text{iff} \quad (f_i + g_i\epsilon) = (f_i + g'_i\epsilon) \quad \forall i.$$

The following concrete example illustrates several of the above features.

EXAMPLE 7.13 (Picard group of a constant elliptic curve). Let $(0, E)$ be a smooth, projective elliptic curve. Over any base S we have the constant family $\pi : E \times S \rightarrow S$ with the constant section $s_0 : S \cong \{0\} \times S$. Let L be a line bundle on $E \times S$. Then $L \otimes \pi^* s_0^* L^{-1}$ has a canonical trivialization along $\{0\} \times S$, hence it defines a morphism $S \rightarrow \text{Pic}(E)$. So the relative Picard group is computed by the formula

$$\text{Pic}(E \times S/S) \cong \text{Mor}(S, \text{Pic}(E)). \quad (7.13.1)$$

Two consequences are worth mentioning.

Claim 7.13.2. Let (R, m) be a complete local ring. Set $S = \text{Spec } R$ and $S_n = \text{Spec } R/m^n$. Then

$$\text{Pic}(E \times S/S) = \varprojlim \text{Pic}(E \times S_n/S_n). \quad \square$$

Claim 7.13.3. Let $S = \text{Spec } k[t]_{(t)}$ be the local ring of the affine line at the origin and $\hat{S} = \text{Spec } k[[t]]$ its completion. Then

$$\text{Pic}(E \times S/S) \cong \text{Pic}(E) \quad \text{but} \quad \text{Pic}(E \times \hat{S}/\hat{S}) \text{ is infinite dimensional.} \quad \square$$

Next consider the affine elliptic curve $E^\circ = E \setminus \{0\}$ and the constant affine family $E^\circ \times S \rightarrow S$. Note that $\text{Pic}(E^\circ) \cong \text{Pic}^\circ(E)$.

If S is smooth and D° is a Cartier divisor on $E^\circ \times S$ then its closure $D \subset E \times S$ is also Cartier. More generally, this also holds if S is normal, using (4.25). Thus (7.13.1) gives the following.

Claim 7.13.3. If S is normal then

$$\mathrm{Pic}(E^\circ \times S/S) \cong \mathrm{Mor}(S, \mathrm{Pic}^\circ(E)). \quad \square$$

By contrast, (7.9.3) gives the following.

Claim 7.13.4. If $S = \mathrm{Spec} A$ is Artin then

$$\mathrm{Pic}(E^\circ \times S/S) \cong \mathrm{Pic}^\circ(E).$$

So $\mathrm{Pic}(E^\circ \times S/S)$ has dimension 1 but $\dim_k \mathrm{Mor}(S, \mathrm{Pic}^\circ(E)) = \mathrm{length} A$. \square

The following is a good illustration of (7.13.4).

Concrete Example 7.13.5. Start with the plane cubic with equation $Y^2Z = X^3 - Z^3$. In the affine plane $Z = 1$ we get $E^\circ := (y^2 = x^3 - 1)$ (where $x = X/Z, y = Y/Z$) and in the $Y = 1$ plane we get $S := (v = u^3 - v^3)$ (where $u = X/Y, v = Z/Y$). The diagonal in $(y^2 = x^3 - 1) \times (v = u^3 - v^3)$ is a Cartier divisor which is defined by 2 equations $yv = 1$ and $yu = x$. It is a nontrivial element of $\mathrm{Pic}(E^\circ \times S/S)$. Next we check that it is trivial over every Artin subscheme of S supported at $(u = v = 0)$.

At $(u = v = 0)$ the local coordinate is u . Note that u also vanishes at the points where $v^2 + 1 = 0$. If we invert it, then we get that

$$(u^{3r}) = (v^r) \subset k[u, v, (v^2 + 1)^{-1}]/(u^3 - v^3 - v).$$

What is the ideal

$$(yv - 1, yu - x, u^r) \subset k[x, y, u, v, (v^2 + 1)^{-1}]/(y^2 - x^3 + 1, u^3 - v^3 - v).$$

Note that it contains

$$(yv - 1)(y^{r-1}v^{r-1} + \cdots + yv + 1) = y^r v^r - 1 = y^r (v^2 + 1)^{-r} u^{3r} - 1.$$

Thus $1 \in (yv - 1, yu - x, u^r)$ and the ideal is the whole ring.

Relative Mumford divisors.

DEFINITION 7.14. Let S be a scheme and $f : X \rightarrow S$ a morphism of pure relative dimension n . Assume for simplicity that f is flat with S_2 fibers. Two relative Mumford divisors $D_1, D_2 \subset X$ are *linearly equivalent* if $\mathcal{O}_X(-D_1) \cong \mathcal{O}_X(-D_2)$, and *linearly equivalent over S* if $\mathcal{O}_X(-D_1) \cong \mathcal{O}_X(-D_2) \otimes f^*L$ for some line bundle L on S .

(*Asides:* With the weaker assumptions of (4.78), the definition should be $\mathcal{O}_X(-D_1)^H \cong \mathcal{O}_X(-D_2)^H$.)

Note also that, while on a normal variety linear equivalence coincides with rational equivalence, linear equivalence is more restrictive in general. Our definition gives the right generalization of the theory of generalized Jacobians, that was worked out in [Sev47, Ros54, Ser59]; see [Kol18c] for details.)

The linear equivalence classes over S of relative Mumford divisors generate the *relative Mumford class group* $\mathrm{MCl}(X/S)$.

By definition, if D is a Mumford divisor then there is a closed subset $Z \subset X$ such that $\mathcal{O}_X(-D)|_{X \setminus Z}$ is locally free and $\mathrm{codim}_{X_s} Z_s \geq 2$ for every $s \in S$. This gives a natural identification

$$\mathrm{MCl}(X/S) = \lim_Z \mathrm{Pic}((X \setminus Z)/S), \quad (7.14.1)$$

where the limit is over all closed subsets $Z \subset X$ such that $\mathrm{codim}_{X_s} Z_s \geq 2$ for every $s \in S$.

On a normal variety, a Mumford divisor is the same as a Weil divisor and the Mumford class group is the same as the class group. If f has normal fibers, then we recover the traditional notion of the *relative class group* $\text{Cl}(X/S) = \text{MCl}(X/S)$.

As with the Picard group, it may be better to sheafify $\text{MCl}(X/S)$ in the étale topology as in [BLR90, Chap.8], but we will use this notion mostly when S is local, and then this is not important for our current purposes.

For the rest of the section we make some explicit computations about Mumford divisors on schemes that are smooth over an Artin ring. The first steps work equally well for Mumford divisors with Cartier fiber.

PROPOSITION 7.15. *Let (A, k) be a local Artin ring, $k \cong (\epsilon) \subset A$ an ideal and $B = A/(\epsilon)$. Let (R_A, m) be a flat, local, S_2 , A -algebra of dimension 2 and set $X_A := \text{Spec}_A R_A$. Let $f_B \in R_B$ be a non-zerodivisor and set $C_B := (f_B = 0) \subset X_B$. Then the set of all relative Mumford divisors $D_A \subset X_A$ such that $\text{pure}((D_A)|_B) = C_B$ is a torsor under the infinite dimensional k -vector space $H_m^1(C_k, \mathcal{O}_{C_k})$.*

Proof. We can lift f_B to $f_A \in R_A$. Choose $y \in m$ that is not a zerodivisor on C_B and such that D_A is a principal divisor on $X_A \setminus (y = 0)$. After inverting y , we can write the ideal of D_A as

$$(I, y^{-1}) = (f_A + \epsilon y^{-r} g_k) \quad \text{where } g \in R_k, r \in \mathbb{N}. \quad (7.15.1)$$

We can multiply $f_A + \epsilon y^{-r} g_k$ by $1 + \epsilon y^{-s} v$. This changes g_k to $g_k + v y^{r-s} f_A$. Thus the relevant information is carried by the residue class

$$\overline{y^{-r} g_k} \in H^0(C_k^\circ, \mathcal{O}_{C_k^\circ}), \quad (7.15.2)$$

where $C_k^\circ \subset C_k$ denotes the complement of the closed point.

If the residue class is in $H^0(C_k, \mathcal{O}_{C_k})$ then we get a Cartier divisor. Thus the non-Cartier divisors are parametrized by

$$H^0(C_k^\circ, \mathcal{O}_{C_k^\circ})/H^0(C_k, \mathcal{O}_{C_k}) \cong H_m^1(C_k, \mathcal{O}_{C_k}). \quad (7.15.3)$$

We compute in (7.18.2) that different elements of $H_m^1(C_k, \mathcal{O}_{C_k})$ give non-isomorphic divisors. \square

COROLLARY 7.16. *Let (A, k) be a local Artin ring, $k \cong (\epsilon) \subset A$ an ideal and $B = A/(\epsilon)$. Let (R_A, m) be a flat, local, S_2 , A -algebra of dimension 2. Let $f_A \in R_A$ and $g_k \in R_k$ be a non-zerodivisor modulo both f_A and g_k .*

For the divisorial ideal $I := R_A \cap (f_A + \epsilon y^{-r} g_k) R_A[y^{-1}]$ the following are equivalent.

(7.16.1) I is a principal ideal.

(7.16.2) The residue class $\overline{y^{-r} g_k}$ lies in $R_k/(f_k)$.

(7.16.3) $g_k \in (f_k, y^r)$.

Note that we can change $f_A + \epsilon y^{-r} g_k$ to $(f_A + \epsilon h_k) + \epsilon y^{-r} (g_k - y^r h_k)$ for any $h_k \in R_k$, but $g_k \in (f_k, y^r)$ iff $g_k - y^r h_k \in (f_k, y^r)$.

Proof. I is a principal ideal iff it has a generator of the form $f_A + \epsilon h_k$ where $h_k \in R_k$. This holds iff

$$f_A + \epsilon y^{-r} g_k = (1 + \epsilon y^{-s} b_k)(f_A + \epsilon h_k) \quad \text{for some } b_k \in R_A.$$

Equivalently, iff $y^{-r}g_k = h_k + y^{-s}b_k f_k$. If $r > s$ then $g_k = y^r h_k + y^{r-s}b_k f_k$ which is impossible since y is not a zerodivisor modulo g_k . If $r < s$ then $y^{s-r}g_k = y^s h_k + b_k f_k$ which is impossible since y is not a zerodivisor modulo f_k . Thus $r = s$ and then $g_k = y^r h_k + b_k f_k$ is equivalent to $g_k \in (f_k, y^r)$. \square

The next will be crucial in the proof of (7.63).

COROLLARY 7.17. *Let (A, k) be a local Artin ring and (R_A, m) a flat, local, S_2 , A -algebra of dimension 2. Let $f_k \in R_k$ be a non-zerodivisor and $y \in R_A$ a non-zerodivisor modulo f_k . Let $f_A \in R_A[y^{-1}]$ be a lifting of f_k . Then there is an $N > 0$ (depending on f_A) such that the following holds.*

(7.17.1) *Let $f'_A \in R_A[y^{-1}]$ be such that $f_A - f'_A \in y^N R_A$. Then f'_A defines a relative Cartier divisor iff f_A does.*

Proof. Assume first that f_A defines a relative Cartier divisor. Then there is a unit $u_A \in R_A[y^{-1}]$ such that u_k is a unit in R_k and $u_A f_A \in R_A$. Choose N such that $y^N u_A \in R_A$. Then

$$u_A f'_A = u_A (f'_A - f_A) + u_A f_A \in u_A y^N R_A + u_A f_A \in R_A.$$

Conversely, assume that f_A does not define a relative Cartier divisor. By induction on the length of A we may assume that there is an ideal $k \cong (\epsilon) \subset A$ with quotient $B := A/(\epsilon)$, such that, f_B does define a relative Cartier divisor.

There is thus a unit u_A such that $u_A f_A \in R_A + \epsilon R_k[y^{-1}]$. We can thus write $u_A f_A = h_A + \epsilon y^{-r} g_k$ where $h_A \in R_A, g_k \in R_k$ and $r > 0$. Choose N such that $y^N u_A \in y^r R_A$. Computing as above we get that

$$u_A f'_A = u_A (f'_A - f_A) + u_A f_A = (h_A + \phi_A) + \epsilon y^{-r} g_k,$$

where $\phi_A := u_A (f'_A - f_A) \in y^r R_A$. Therefore $(h_k + \phi_k, y^r) = (h_k, y^r)$.

Next note that f_A defines a relative Cartier divisor $\Leftrightarrow u_A f_A$ defines a relative Cartier divisor $\Leftrightarrow g_k \in (h_k, y^r)$, the latter by (7.16). Similarly, f'_A defines a relative Cartier divisor $\Leftrightarrow u_A f'_A$ defines a relative Cartier divisor $\Leftrightarrow g_k \in (h_k + \phi_k, y^r)$.

Thus f_A defines a relative Cartier divisor iff f'_A does. \square

The connection between (7.15) and (7.9) is given by the following.

7.18. Let X be an affine, S_2 scheme and $D := (s = 0) \subset X$ a Cartier divisor. Let $Z \subset D$ be a closed subset that has codimension ≥ 2 in X . Set $X^\circ := X \setminus Z$ and $D^\circ := D \setminus Z$. Restricting the exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

to X° and taking cohomologies we get

$$0 \rightarrow H^0(X^\circ, \mathcal{O}_{X^\circ}) \xrightarrow{s} H^0(X^\circ, \mathcal{O}_{X^\circ}) \rightarrow H^0(D^\circ, \mathcal{O}_{D^\circ}) \xrightarrow{\partial} H^1(X^\circ, \mathcal{O}_{X^\circ}).$$

Note that $H^0(X^\circ, \mathcal{O}_{X^\circ}) = H^0(X, \mathcal{O}_X)$ since X is S_2 and its image in $H^0(D^\circ, \mathcal{O}_{D^\circ})$ is $H^0(D, \mathcal{O}_D)$. Thus ∂ becomes the injection

$$\partial: H^1_Z(D, \mathcal{O}_D) \cong H^0(D^\circ, \mathcal{O}_{D^\circ})/H^0(D, \mathcal{O}_D) \hookrightarrow H^2_Z(X, \mathcal{O}_X). \quad (7.18.1)$$

We are especially interested in the case when (x, X) is local, 2-dimensional and $Z = \{x\}$. In this case (7.18.1) becomes

$$\partial: H^1_x(D, \mathcal{O}_D) \hookrightarrow H^2_x(X, \mathcal{O}_X). \quad (7.18.2)$$

Here the left hand side describes first order deformations of D and the right hand side the Picard group of the first order deformation of X .

We can be especially explicit about first order deformations in the smooth case. Let us start with the description as in (7.15).

7.19 (Mumford divisors in $k[[u, v]][\epsilon]$). Set $X = \text{Spec } k[[u, v]][\epsilon]$ with closed point $x \in X$. By (7.9), the Picard group of the punctured spectrum $X \setminus \{x\}$ is

$$H_x^2(X, \mathcal{O}_X) \cong \bigoplus_{i, j > 0} \frac{1}{x^i y^j} \cdot k.$$

A ideal corresponding to $cx^{-i}y^{-j}$ can be given as

$$I(cx^{-i}y^{-j}) := (x^{2i}, x^i y^j + c\epsilon, x^i \epsilon),$$

a more systematic derivation of this is given in (7.21.1).

This is quite explicit, but we are more interested in the point of view of (7.9).

LEMMA 7.20. *Let $f \in k[[u]][v]$ be a monic polynomial in v of degree n defining a curve $C_k \subset \widehat{\mathbb{A}}_{uv}^2$. Let $D \subset \widehat{\mathbb{A}}_{k[\epsilon]}^2$ be a relative Mumford divisor such that $\text{pure}(D_k) = C_k$. Then the restriction of D to the complement of $(u = 0)$ can be uniquely written as*

$$f + \epsilon \sum_{i=0}^{n-1} v^i \phi_i(u) = 0 \quad \text{where } \phi_i(u) \in u^{-1}k[[u^{-1}]].$$

Thus the set of all such D is naturally isomorphic to the infinite dimensional k -vector space $H_m^1(C_k, \mathcal{O}_{C_k}) \cong \bigoplus_{i=0}^{n-1} u^{-1}k[[u^{-1}]]$.

Note that, by the Weierstrass preparation theorem, almost every curve in $\widehat{\mathbb{A}}_{uv}$ is defined by a monic polynomial in v , so this is a mild restriction.

Proof. Note that $k[[u]][v]/(f) \cong \bigoplus_{i=0}^{n-1} v^i k[[u]]$ as a $k[[u]]$ -module, giving isomorphism

$$H^0(C_k, \mathcal{O}_{C_k}) \cong \bigoplus_{i=0}^{n-1} v^i k[[u]] \quad \text{and} \quad H^0(C_k^\circ, \mathcal{O}_{C_k^\circ}) \cong \bigoplus_{i=0}^{n-1} v^i k((u)). \quad (7.20.1)$$

That is, if $g \in k((u))[v]$ is a polynomial of degree $< n$ in v then $g|_{C_k^\circ}$ extends to a regular function on C iff $g \in k[[u]][v]$. \square

We can also restate (7.20.1) as

$$H_m^1(C_k, \mathcal{O}_{C_k}) \cong \bigoplus_{i=0}^{n-1} v^i k((u))/k[[u]] \cong \bigoplus_{i=0}^{n-1} v^i u^{-1} k[[u^{-1}]]. \quad (7.20.2)$$

EXAMPLE 7.21. Consider next the special case of (7.20) when $f = v$. We can then write the restriction of D as $(v + \phi(u)\epsilon = 0)$ where $\phi \in u^{-1}k[[u^{-1}]]$. Let r denote the pole-order of ϕ and set $q(u) := u^r \phi(u)$.

Claim 7.21.1. The ideal of D is

$$I_D = (v^2, vu^r + q(u)\epsilon, v\epsilon).$$

Thus the fiber over the closed point is $k[[u, v]]/(v^2, vu^r)$. Its torsion submodule is isomorphic to $k[[u, v]]/(v, u^r) \cong k[[u]]/(u^r)$.

Proof. To see this note first that $v^2 = (v + \phi(u)\epsilon)(v - \phi(u)\epsilon)$, $vu^r + q(u) = (v + \phi(u)\epsilon)u^r$ and $v\epsilon = (v + \phi(u)\epsilon)\epsilon$ are elements of I_D . Next note that $q(u)$ is a polynomial with nonzero constant term, hence invertible in $k[[u, v]]$. Therefore

$$k[[u, v]][\epsilon]/(v^2, vu^r + q(u)\epsilon, v\epsilon) \cong k[[u, v]]/(v^2, v^2 u^r q(u)^{-1}) = k[[u, v]]/(v^2)$$

has no embedded points. \square

The ideals of relative Mumford divisors in $k[[u, v]][\epsilon]$ are likely to be more complicated in general. At least the direct generalization of (7.21.1) does not always give the correct generators.

For example, let $f = v^2 - u^3$ and consider the ideal $I \subset k[[u, v]][\epsilon]$ extended from $((v^2 - u^3) + u^{-3}v\epsilon)$. The above procedure gives the elements

$$(v^2 - u^3)^2, u^3(v^2 - u^3) + v\epsilon, (v^2 - u^3)\epsilon \in I.$$

However, $u^3(v^2 - u^3) + v\epsilon = v^2(v^2 - u^3) + v\epsilon$ and we can cancel the v to get that

$$I = ((v^2 - u^3)^2, v(v^2 - u^3) + \epsilon, (v^2 - u^3)\epsilon). \quad (7.21.2)$$

Using the isomorphism $R[\epsilon]/(f^2, fg + \epsilon, f\epsilon) \cong R/(f^2, -f^2g) \cong R/(f^2)$, the above examples can be generalized to the non-smooth case as follows.

Claim 7.21.3. Let (R, m) be a local, S_2 , k -algebra of dimension 2 and $f, g \in m$ a system of parameters. Then $J_{f,g} = (f^2, fg + \epsilon, f\epsilon)$ is (the ideal of) a relative Mumford divisor in $R[\epsilon]$ whose central fiber is $R/(f^2, fg)$, with embedded subsheaf isomorphic to $R/(f, g)$. \square

7.3. Divisorial support

There are at least 3 ways to associate a divisor to a sheaf (7.23) but only one of them—the divisorial support—behaves well in flat families. In this Section we develop this notion and a method to compute it. The latter is especially important for the applications. First we recall the definition of the Fitting ideal sheaf.

7.22 (Fitting ideal). Let R be a noetherian ring, M a finite R -module and

$$R^s \xrightarrow{A} R^r \rightarrow M \rightarrow 0$$

a presentation of M , where A is given by an $s \times r$ -matrix with entries in R . The *Fitting ideal*, or, more precisely, the 0 th *Fitting ideal* of M , denoted by $\text{Fitt}_R(M)$, is the ideal generated by the determinants of $r \times r$ -minors of A . For the following basic properties see [Fit36] or [Eis95, Sec.20.2].

(7.22.1) $\text{Fitt}_R(M)$ is independent of the presentation chosen.

(7.22.2) If R is regular and $M \cong \bigoplus_i R/(g_i^{m_i})$ then $\text{Fitt}_R(M) = (\prod g_i^{m_i})$.

(7.22.3) The Fitting ideal commutes with base change. That is, if S is an R -algebra then $\text{Fitt}_S(M \otimes_R S)$ is generated by $\text{Fitt}_R(M) \otimes_R S$.

The following is a special case of [Lip69, Lem.1].

(7.22.4) Let M be a torsion module. Then $\text{Fitt}_R(M)$ is a principal ideal generated by a non-zerodivisor iff the projective dimension of M is 1.

One direction is easy. If the projective dimension of M is 1, then M has a presentation

$$0 \rightarrow R^s \xrightarrow{A} R^r \rightarrow M \rightarrow 0.$$

Here $r = s$ since M is torsion, thus $\det(A)$ generates $\text{Fitt}_R(M)$.

We prove the converse only in the following special case that we use later, which, however, captures the essence of the general proof.

(7.22.5) Let X be a smooth variety of dimension n and F a coherent sheaf of generic rank 0 on X . Then $\text{Fitt}_X(F)$ is a principal ideal iff F is CM of pure dimension $n - 1$.

Proof. This can be checked after localization and completion. Thus we have a module M over $S := k[[x_1, \dots, x_n]]$, and, after a coordinate change, we may assume that it is finite over $R := k[[x_1, \dots, x_{n-1}]]$ of generic rank say r . Using first (7.22.2) and then (7.22.3) we get that

$$\dim_k M \otimes_R k = \dim_k k[[x_n]] / \text{Fitt}_k(M \otimes_R k) = \dim_k(S / \text{Fitt}_S(M)) \otimes k. \quad (7.22.6)$$

Next note that M is CM $\Leftrightarrow M$ is free over $R \Leftrightarrow \dim M \otimes k = r$. Using (7.22.1) and the previous equivalences for $S / \text{Fitt}_S(M)$ we get that these are equivalent to $S / \text{Fitt}_S(M)$ being CM. This holds iff $\text{Fitt}_S(M)$ is a height 1 unmixed ideal, hence principal. \square

The following explicit formula is quite useful.

Computation 7.22.7. Let S be a smooth R -algebra and $v \in S$ such that $S/(v) \cong R$. (The examples we use are $S = R[v]$ and $S = R[[v]]$.) Let M be an S -module that is free of finite rank as an R -module. Write $M = \bigoplus_{i=1}^r Rm_i$ and $vm_i = \sum_{j=1}^r a_{ij}m_j$. Then $\text{Fitt}_S(M)$ is generated by $\det(v\mathbf{1}_r - (a_{ij}))$.

Proof. A presentation of M as an S -module is given by

$$\bigoplus_{i=1}^r S e_i \xrightarrow{\phi} \bigoplus_{i=1}^r S f_i \xrightarrow{\psi} M \rightarrow 0,$$

where $\psi(f_i) = m_i$ and $\phi(e_i) = v f_i - \sum_{j=1}^r a_{ij} f_j$. Thus $\phi = v\mathbf{1}_r - (a_{ij})$ and so $\det(v\mathbf{1}_r - (a_{ij}))$ generates $\text{Fitt}_S(M)$. \square

DEFINITION 7.23 (Divisorial support I). Let X be a scheme and F a coherent sheaf on X . One usually defines its *support* $\text{Supp } F$ and its *scheme-theoretic support* $\text{SSupp } F := \text{Spec}_X(\mathcal{O}_X / \text{Ann } F)$.

Assume next that $\text{Supp } F$ is nowhere dense and X is regular at every generic point $x_i \in \text{Supp } F$ that has codimension 1 in X . Then there is a unique divisorial sheaf (3.24) associated to the Weil divisor $\sum \text{length}(F_{x_i}) \cdot [x_i]$. We call it the *divisorial support*

of F and denote it by $\text{DSupp } F$. Equivalently,

$$\text{DSupp}(F) = \text{pure}_1(\mathcal{O}_X / \text{Fitt}_X(F)), \quad (7.23.1)$$

where pure_1 denotes the pure codimension 1 part and $\text{Fitt}_X(F)$ is the Fitting ideal sheaf of F (7.22).

If every associated point of F has codimension 1 in X then we have inclusions of subschemes

$$\text{Supp } F \subset \text{SSupp } F \subset \text{DSupp } F. \quad (7.23.2)$$

In general all 3 subschemes are different, though with the same support.

Our aim is to develop a relative version of this notion and some ways of computing it in families. Let $X \rightarrow S$ be a morphism and F a coherent sheaf on X . Informally, we would like the relative divisorial support of F , denoted by $\text{DSupp}_S F$, to be a scheme over S whose fibers are $\text{DSupp}(F_s)$ for all $s \in S$. If S is reduced, this requirement uniquely determines $\text{DSupp}_S F$ but in general there are 2 problems.

- Even in nice situations, this requirement may be impossible to meet.
- For non-reduced base schemes, the fibers do not determine $\text{DSupp}_S F$.

In our main applications X is smooth over some base scheme S that may well have nilpotent elements. As in (7.28), we need to allow embedded subsheaves that ‘come from’ S , but not the others.

DEFINITION 7.24 (Divisorial support II). Let $X \rightarrow S$ be a smooth morphism of pure relative dimension n . Let F be a coherent sheaf on X that is flat over S with CM fibers of pure dimension $n - 1$. We define its divisorial support as

$$\mathrm{DSupp}_S(F) := \mathrm{Fitt}_X(F).$$

LEMMA 7.25. *Under the assumptions of (7.24),*

(7.25.1) $\mathrm{DSupp}_S(F)$ is a relative Cartier divisor, and

(7.25.2) $\mathrm{DSupp}_S(F)$ commutes with base change. That is, let $h : S' \rightarrow S$ be a morphism. By base change we get $g' : X' \rightarrow S'$, $h_X : X' \rightarrow X$, and then $h_X^*(\mathrm{DSupp} F) = \mathrm{DSupp}(h_X^* F)$.

Proof. The first claim can be checked after localization and completion. We may thus assume that $S = \mathrm{Spec} B$ where B is local with residue field k , $X = \mathrm{Spec} B[[x_1, \dots, x_n]]$ and F is the sheafification of M . For a suitable choice of x_1, \dots, x_n we may also assume that M is a finite $R := B[[x_1, \dots, x_{n-1}]]$ -module. Set $R_k = R \otimes_B k \cong k[[x_1, \dots, x_{n-1}]]$. Since M is flat over B , its generic rank over R equals the generic rank of $M \otimes_B k$ over R_k . By assumption $M \otimes_B k$ is CM, hence free over R_k . Thus the generic rank of M over R equals $\dim_k M \otimes_R k$. Thus M is free as an R -module. The rest follows from (7.22.7).

The second claim is immediate from (7.22.3). \square

The following restriction property is also implied by (7.22.3).

LEMMA 7.26. *Continuing with the notation and assumptions of (7.24), let $D \subset X$ be a relative Cartier divisor that is also smooth over S . Assume that D does not contain any generic point of $\mathrm{Supp} F_s$ for any $s \in S$. Then*

$$\mathrm{DSupp}(F|_D) = (\mathrm{DSupp} F)|_D. \quad \square$$

Now we are ready to define the sheaves for which the relative divisorial support makes sense, but first we have to distinguish associated points that come from the base from the other ones.

DEFINITION 7.27. Let $X \rightarrow S$ be a morphism and F a coherent sheaf on X . The *flat locus* of F is the largest open subset $U \subset \mathrm{Supp} F$ such that $F|_U$ is flat over S . We denote it by $\mathrm{Flat}_S(F)$.

It is sometimes more convenient to work with the *flat-CM locus* of F . It is the largest open subset $U \subset \mathrm{Supp} F$ such that $F|_U$ is flat with CM fibers over S . We denote it by $\mathrm{FlatCM}_S(F)$.

These properties are unchanged if we replace X by $\mathrm{SSupp} F$. Thus we may assume that $\mathrm{Supp} F = X$, or even that $\mathrm{Ann}(F) = 0$, whenever it is convenient.

DEFINITION 7.28. Let $g : X \rightarrow S$ be a morphism and F a coherent sheaf on X such that $\mathrm{Supp} F \rightarrow S$ has pure relative dimension d .

We say that F is *vertically pure* if x is a generic point of $\mathrm{Supp}(F_{g(x)})$ for every associated point $x \in \mathrm{Ass}(F)$. (Note, however, that $F_{g(x)}$ is allowed to have embedded points.) Being vertically pure is preserved by flat base change, but usually not preserved by arbitrary base change.

If F is generically flat over S (7.29), then there is a unique largest subsheaf $\mathrm{nv-tors}_S(F) \subset F$ —called the *non-vertical torsion* of F —such that every fiber of the structure map $\mathrm{Supp}(\mathrm{v-tors}_S(F)) \rightarrow S$ has dimension $< d$.

Then $v\text{-pure}(F) := F/\text{nv-tors}_S(F)$ is vertically pure.
All these notions make sense for subschemes of X as well.

DEFINITION 7.29. Let $X \rightarrow S$ be a morphism and F a coherent sheaf on X . We say that F is a *generically flat family of pure sheaves* of dimension d over S if the following hold.

(7.29.1) F is flat at every generic point of F_s for every $s \in S$ and

(7.29.2) $\text{Supp } F \rightarrow S$ has pure relative dimension d .

We usually do not care about the non-vertical torsion, thus we frequently replace F by $v\text{-pure}(F) = F/\text{nv-tors}_S(F)$ and then the following condition is also satisfied.

(7.29.3) F is vertically pure.

We say that $Z \subset X$ is a *generically flat family of pure subschemes* if its structure sheaf \mathcal{O}_Z has this property. (Note that the fibers can have embedded points outside the flat locus.)

The following properties are clear from the definition.

(7.29.4) Conditions (7.29.1–2) are preserved by any base change $S' \rightarrow S$ and (7.29.3) is preserved by flat base change.

(7.29.5) If (7.29.3) holds then the generic fibers F_g are pure of dimension d , but special fibers may have embedded points outside $\text{FlatCM}_S(F)$ (7.27).

DEFINITION–LEMMA 7.30 (Divisorial support III). Let $g : X \rightarrow S$ be a flat morphism of pure relative dimension n and $g^\circ : X^\circ \rightarrow S$ the smooth locus of g .

Let F be a coherent sheaf on X that is generically flat and pure over S of dimension $n-1$. Assume that for every $s \in S$, every generic point of F_s is contained in X° .

Set $Z := \text{Supp } F \setminus (\text{FlatCM}_S(F) \cap X^\circ)$, $U := X \setminus Z$ and $j : U \hookrightarrow X$ the natural injection. We define the *divisorial support* of F over S as

$$\text{DSupp}_S(F) := \overline{\text{DSupp}_S(F|_U)}, \quad (7.30.1)$$

the scheme-theoretic closure of $\text{DSupp}_S(F|_U)$. This makes sense since the latter is already defined by (7.24).

Note that $\text{Supp DSupp}_S(F) = \text{Supp } F$ and $\text{DSupp}_S(F)$ is a generically flat family of pure subschemes of dimension $n-1$ over S , whose restriction to U is relatively Cartier.

It is enough to check the following equalities at codimension 1 points, which follow from (7.25) and (7.22.3).

Claim 7.30.2. Let $g_i : X_i \rightarrow S$ be flat morphisms of pure relative dimension n and $\pi : X_1 \rightarrow X_2$ a finite morphism. Let $D \subset X_1$ be a relative Mumford divisor. Assume that $\text{red } D_s \rightarrow \text{red}(\pi(D_s))$ is birational and π is étale at every generic point of D_s . Then

$$\text{DSupp}_S(\pi_* \mathcal{O}_D) = \pi(D), \quad \text{the scheme-theoretic image of } D. \quad \square$$

Claim 7.30.3. Let $g_i : X_i \rightarrow S$ be flat morphisms of pure relative dimension n and $\pi : X_1 \rightarrow X_2$ a finite morphism. Let F be a coherent sheaf on X_1 that is generically flat and pure over S of dimension $n-1$. Assume that g_1 (resp. g_2) is smooth at every generic point of F_s (resp. $\pi_* F_s$) for every $s \in S$. Then

$$\text{DSupp}_S(\pi_* F) = \text{DSupp}_S(\pi_* \text{DSupp}_S(F)). \quad \square$$

Claim 7.30.4. Let $g_i : X_i \rightarrow S$ be flat morphisms of pure relative dimension n and $\pi_1 : X_1 \rightarrow X_2$, $\pi_2 : X_2 \rightarrow X_3$ finite morphism. Let F be a coherent sheaf on X_1 that is generically flat and pure over S of dimension $n - 1$. Assume that g_1 (resp. g_2, g_3) is smooth at every generic point of F_s (resp. $\pi_{1*}F_s, (\pi_2 \circ \pi_1)_*F_s$) for every $s \in S$. Then

$$\mathrm{DSupp}_S((\pi_2 \circ \pi_1)_*F) = \mathrm{DSupp}_S(\pi_{2*} \mathrm{DSupp}_S(\pi_{1*}F)). \quad \square$$

LEMMA 7.31. *Let $X \rightarrow S$ be a smooth morphism of pure relative dimension n . Let F be a coherent sheaf on X that is generically flat over S with pure fibers of dimension $n - 1$. Assume that*

(7.31.1) *either F is flat over S ,*

(7.31.2) *or S is reduced.*

Then $\mathrm{DSupp}_S F$ is a relative Cartier divisor.

Proof. Assume first that F is flat over S . If $x \in X_s$ is a point of codimension ≤ 2 , then F_s is CM at x , hence $\mathrm{DSupp}_S F$ is a relative Cartier divisor at x by (7.24). Since $X \rightarrow S$ is smooth, $\mathrm{DSupp}_S F$ is a relative Cartier divisor everywhere by (7.9.8).

For the second claim, the above argument gives only that $\mathrm{DSupp}_S F$ is a relative, generically Cartier divisor. By (4.37) it is then enough to check the conclusion after base change $T \rightarrow S$, where T is the spectrum of a DVR. Then X_T is regular, so $\mathrm{DSupp}_T F_T$ is Cartier. \square

Divisorial support commutes with restriction to a divisor, whenever everything makes sense. We just need to make enough assumptions that guarantee that (7.26) applies on a dense set of every fiber.

COROLLARY 7.32. *Continuing with the notation and assumptions of (7.30), let $D \subset X$ be a relative Cartier divisor. Assume that there is an open set $D^\circ \subset D$ such that*

(7.32.1) *$g|_D$ is smooth on D° ,*

(7.32.2) *D_s° is dense in D_s for every $s \in S$,*

(7.32.3) *D does not contain any generic point of $\mathrm{Supp} F_s$ for any $s \in S$, and*

(7.32.4) *$D^\circ \subset \mathrm{FlatCM}_S(F)$.*

Then

$$\mathrm{DSupp}_S(F|_D) = \mathrm{v-pure}((\mathrm{DSupp}_S F)|_D). \quad \square$$

Various Bertini-type theorems show that the above assumptions are quite easy to satisfy, at least locally.

COROLLARY 7.33. *Continuing with the above notation, let $|D|$ be a linear system on X that is base point free in characteristic 0 and very ample in general. Fix $s \in S$ and let $D \in |D|$ be a general member. Then there is an open neighborhood $S^\circ \subset S$ such that*

$$\mathrm{DSupp}_S(F|_D) = (\mathrm{DSupp}_S F)|_D \quad \text{holds over } S^\circ. \quad (7.33.1)$$

Proof. We apply the usual Bertini theorems to X_s . We get that D_s satisfies conditions (7.32.1–4), and then they also hold over some open neighborhood $s \in S^\circ \subset S$. This gives (7.33.1), modulo non-vertical torsion. Finally note that there is no such torsion for general D by (10.9). \square

LEMMA 7.34. *Divisorial support commutes with base change. That is, let $g : X \rightarrow S$ be a flat morphism of pure relative dimension n and F a generically flat family of pure sheaves of dimension $n - 1$ over S . Assume that for every $s \in S$, every generic point of $\text{Supp } F_s$ is contained in the smooth locus of g .*

Let $h : S' \rightarrow S$ be a morphism. By base change we get $g' : X' \rightarrow S'$ and $h_X : X' \rightarrow X$. Then

$$h_X^{[*]}(\text{DSupp}_S F) = \text{DSupp}_{S'}(h_X^* F),$$

where $h_X^{[*]}$ is the divisorial pull-back (4.1.7).

Proof. Set $U := \text{FlatCM}_S(F) \subset X$ with injection $j : U \hookrightarrow X$. Set $U' := h_X^{-1}(U)$ and $h_U : U' \rightarrow U$ the restriction of h_X . Then $h_U^*(\text{DSupp}_S F|_U) = \text{DSupp}_{S'}(h_U^*(F|_U))$ by (7.25).

By (7.29.4) $h_X^{[*]}(\text{DSupp}_S F)$ is a generically flat family of pure divisors and it agrees with $\text{DSupp}_{S'}(h_X^* F)$ over U' . Thus the 2 are equal. \square

DEFINITION 7.35 (Divisorial support of cycles). Let S be a seminormal scheme and Z a well defined family of d -cycles on \mathbb{P}_S^n as in [Kol96, I.3.10].

Let $\rho : \text{Supp } Z \rightarrow \mathbb{P}_S^{d+1}$ be a finite morphism. Then $\rho_* Z$ is a well defined family of d -cycles on \mathbb{P}_S^{d+1} .

If all the residue characteristics are 0, or if Z satisfies the field of definition condition [Kol96, I.4.7], then there is a unique relative Cartier divisor $D \subset \mathbb{P}_S^{d+1}$ whose associated cycle is $\rho_* Z$; see [Kol96, I.3.23.2]. We denote it by $\text{DSupp}_S(\rho_* Z)$. As a practical matter, we usually think of $\rho_* Z$ and $\text{DSupp}_S(\rho_* Z)$ as the same object.

Let F be a coherent sheaf on \mathbb{P}_S^n that is generically flat and pure over S of dimension d . One can associate to it a cycle $Z(F)$ that is a well defined family of d -cycles over S (cf. [Kol96, I.3.15]). Let $\rho : \text{Supp } F \rightarrow \mathbb{P}_S^{d+1}$ be a finite morphism. As in (7.30.3) we get that

$$\text{DSupp}_S(\rho_* F) = \text{DSupp}_S(\rho_* Z(F)). \quad (7.35.1)$$

Thus $\text{DSupp}_S(\rho_* F)$ can be defined in terms of cycles. Note, however, that here the right hand side is defined only for seminormal schemes. One of the main aims defining the divisorial support for sheaves is to be able to work over arbitrary schemes.

7.36 (Proof of (7.4.7)). Assume that we have $f : X \rightarrow (s, S)$ and relative Mumford divisors $D_1, D_2 \subset X$ as in (4.78), where (s, S) is local. Set $Z' = X \setminus U^{\text{cm}}$ where $U^{\text{cm}} \subset U$ is the open subset where f has CM fibers. Let $\pi : X \rightarrow \mathbb{P}_S^n$ be a finite morphism. Set $W_S := \mathbb{P}_S^n \setminus \pi(Z')$ and $U_S := \pi^{-1}(W_S)$. We claim that $\pi : U_S \rightarrow W_S$ is finite and flat. Indeed, $\pi_{s*} \mathcal{O}_{X_s}$ is locally free on W_s since U_s is CM. Choosing local generators and lifting them back to W_S we get a map $\phi : M \rightarrow \pi_* \mathcal{O}_X$ (over some open subset $W_S^c \subset W_S$) where M is free and $\phi \otimes k(s)$ is an isomorphism. Thus ϕ is an isomorphism by [Kol96, I.7.4.1].

If $(f) = D_1 - D_2$ then

$$(\text{norm}_{U_S/W_S}(f)) = \text{DSupp}_S(D_1)|_{W_S} - \text{DSupp}_S(D_2)|_{W_S},$$

since $\pi : U_S \rightarrow W_S$ is finite and flat. Since $\pi(Z')$ has codimension ≥ 2 , this implies that $\text{DSupp}_S(D_1)$ and $\text{DSupp}_S(D_2)$ are linearly equivalent. Thus if one of them is relatively Cartier, then so is the other. \square

7.4. Variants of K-flatness

We introduce 5 versions of K-flatness, which may well be equivalent to each other. From the technical point of view, Cayley-Chow-flatness (or C-flatness) is the easiest to use, but a priori it depends on the choice of a projective embedding. Then most of the work in the next 2 sections goes to proving that a modified version (stable C-flatness) is equivalent to K-flatness, hence independent of the projective embedding.

7.37 (Projections of \mathbb{P}^n). Let S be an affine scheme. Projecting \mathbb{P}_S^n from the section $(a_0 : \cdots : a_n)$ (where $a_i \in \mathcal{O}_S$) to the $(x_n = 0)$ hyperplane is given by

$$\pi : (x_0 : \cdots : x_n) \rightarrow (a_n x_0 - a_0 x_n : \cdots : a_n x_{n-1} - a_{n-1} x_n). \quad (7.37.1)$$

It is convenient to normalize $a_n = 1$ and then we get

$$\pi : (x_0 : \cdots : x_n) \rightarrow (x_0 - a_0 x_n : \cdots : x_{n-1} - a_{n-1} x_n). \quad (7.37.2)$$

Similarly, a Zariski open set of projections of \mathbb{P}_S^n to $L^r = (x_n = \cdots = x_{r+1} = 0)$ is given by

$$\pi : (x_0 : \cdots : x_n) \rightarrow (x_0 - \ell_0(x_{r+1}, \dots, x_n) : \cdots : x_r - \ell_r(x_{r+1}, \dots, x_n)), \quad (7.37.3)$$

where the ℓ_i are linear forms.

Note that in affine coordinates, when we set $x_0 = 1$, the projections become

$$\pi : (x_1, \dots, x_n) \rightarrow \left(\frac{x_1 - \ell_1}{1 - \ell_0}, \dots, \frac{x_r - \ell_r}{1 - \ell_0} \right), \quad (7.37.4)$$

where again the ℓ_i are (homogeneous) linear forms in the x_{r+1}, \dots, x_n . If $\ell_0 \equiv 0$ then we recover the linear projections, but in general the coordinate functions have a non-linear expansion

$$\frac{x_i - \ell_i}{1 - \ell_0} = (x_i - \ell_i)(1 + \ell_0 + \ell_0^2 + \cdots). \quad (7.37.5)$$

Finally, formal projections are given as

$$\pi : (x_1, \dots, x_n) \rightarrow (x_1 - \phi_1(x_1, \dots, x_n), \dots, x_r - \phi_r(x_1, \dots, x_n)), \quad (7.37.6)$$

where ϕ_i are power series such that $\phi_i(x_1, \dots, x_r, 0, \dots, 0) \equiv 0$ for every i .

7.38 (Approximation of formal projections). Let $v_m : \mathbb{P}_S^n \hookrightarrow \mathbb{P}_S^N$ (where $N = \binom{n+m}{n} - 1$) be the m th Veronese embedding. Pulling back the linear coordinates on \mathbb{P}_S^N we get all the monomials of degree m . In affine coordinates x_1, \dots, x_n as above, we get all monomials of degree $\leq m$.

In particular we see that given a formal projection π as in (7.37.6) and $m > 0$, there is a unique linear projection π_m of \mathbb{P}_S^N such that $\pi_m \circ v_m$ is

$$\begin{aligned} (x_1, \dots, x_n) &\rightarrow (x_1 - \psi_1, \dots, x_r - \psi_r) \quad \text{where} \\ \psi_i &\equiv \phi_i \pmod{(x_1, \dots, x_n)^{m+1}} \quad \text{and} \quad \deg \psi_i \leq m \quad \forall i. \end{aligned} \quad (7.38.1)$$

That is, we can approximate formal projections by linear projections composed with a Veronese embedding. Thus it is reasonable to expect that K-flatness is very close to C-flatness for all Veronese images; this leads to the notion of stable C-flatness in (7.40.2).

The uniqueness of the approximation above is not always an advantage. In practice we would like π_m to be in general position away from the chosen point.

This is easy to achieve if we increase m a little. In particular, we get the following obvious result.

Claim 7.38.2. Let (s, S) be a local scheme and $Y \subset \mathbb{P}_S^n$ a closed subset of pure relative dimension d . Let $p \in Y_s$ be a closed point with maximal ideal \mathfrak{m}_p such that $x_0(p) \neq 0$. Fix $m \in \mathbb{N}$ and let $(\widehat{g}_1 : \cdots : \widehat{g}_e) : \widehat{Y}_p \rightarrow \widehat{\mathbb{A}}_S^e$ be a finite morphism.

Then, for every $M \geq m + 1$ there are $g_1, \dots, g_e \in H^0(\mathbb{P}_S^n, \mathcal{O}_{\mathbb{P}_S^n}(M))$ such that

- (a) $\pi : (x_0^M : g_1 : \cdots : g_e) : Y \rightarrow \mathbb{P}_S^e$ is a finite morphism,
- (b) $\pi^{-1}(\pi(p)) \cap Y = \{p\}$ and
- (c) $\widehat{g}_i \equiv g_i/x_0^M \pmod{\mathfrak{m}_p^m}$ for every i . □

Despite having good approximations, the equivalence of K-flatness and stable C-flatness is not clear. The problem is the following.

Assume for simplicity that S is the spectrum of an Artin ring A . For sheaves of dimension d , using the notation of (7.22.7), we can write the equation of $\text{DSupp}(\widehat{\pi}_* \widehat{F})$ in the form

$$\det(M - v\mathbf{1}_n) = 0,$$

where the entries of the matrix M involve rational functions in the power series ϕ_i . The problem is that inverses of power series usually do not have good approximations by rational functions. For example, there is no rational function $g(x_1, x_2)$ such that

$$(x_2 - \sin x_1)^{-1} - g(x_1, x_2) \in k[[x_1, x_2]].$$

The exception is the 1-variable case, where truncations of Laurent series give good approximations. This is what we exploit in (7.63) to prove that K-flatness is equivalent to stable C-flatness for curves.

DEFINITION 7.39. Let E be a vector bundle over a scheme S and $F \subset E$ a vector subbundle. This induces a natural *linear projection* map $\pi : \mathbb{P}_S(E) \dashrightarrow \mathbb{P}_S(F)$. If S is local then E, F are free. After choosing bases, π is given by a matrix of constant rank with entries in \mathcal{O}_S . We call these \mathcal{O}_S -*projections* if we want to emphasize this. If S is over a field k , we can also consider k -*projections*, given by a matrix with entries in k . These, however, only make good sense if we have a canonical trivialization of E ; this rarely happens for us.

We can now formulate various versions of K-flatness and their basic relationships.

DEFINITION 7.40. Let (s, S) first be a local scheme with infinite residue field and F a generically flat family of pure, coherent sheaves of relative dimension d on \mathbb{P}_S^n (7.29), with scheme-theoretic support $Y := \text{SSupp } F$.

(7.40.1) F is *C-flat* over S iff $\text{DSupp}(\pi_* F)$ is Cartier over S for every \mathcal{O}_S -projection $\pi : \mathbb{P}_S^n \dashrightarrow \mathbb{P}_S^{d+1}$ (7.39) that is finite on Y .

(7.40.2) F is *stably C-flat* iff $(v_m)_* F$ is C-flat for every Veronese embedding $v_m : \mathbb{P}_S^n \hookrightarrow \mathbb{P}_S^N$ (where $N = \binom{n+m}{n} - 1$).

(7.40.3) F is *K-flat* over S iff $\text{DSupp}(\rho_* F)$ is Cartier over S for every finite morphism $\rho : Y \rightarrow \mathbb{P}_S^{d+1}$.

(7.40.4) F is *locally K-flat* over S at $y \in Y$ iff $\text{DSupp}(\rho_* F)$ is Cartier over S at $\rho(y)$ for every finite morphism $\rho : Y \rightarrow \mathbb{P}_S^{d+1}$ for which $\{y\} = \text{Supp } \rho^{-1}(\rho(y))$.

(7.40.5) F is *formally K-flat* over S at a closed point $y \in Y$ iff $\text{DSupp}(\rho_*\widehat{F})$ is Cartier over \widehat{S} for every finite morphism $\rho : \widehat{Y} \rightarrow \widehat{\mathbb{A}}_S^{d+1}$, where \widehat{S} (resp. \widehat{Y}) denotes the completion of S at s (resp. Y at y).

Base change properties 7.40.6. We see in (7.53) that being C-flat is preserved by arbitrary base changes and the property descends from faithfully flat base changes. This then implies the same for stable C-flatness. Once we prove that the latter is equivalent to K-flatness, the latter also has the same base change properties. Most likely the same holds for formal K-flatness.

General base schemes 7.40.7. We say that any of the above notions (7.40.1–5) holds for a local base scheme (s, S) (with finite residue field) if it holds after some faithfully flat base change $(s', S') \rightarrow (s, S)$, where $k(s')$ is infinite. Property (7.40.6) assures that this is independent of the choice of S' .

Finally we say that any of the above notions (7.40.1–5) holds for an arbitrary base scheme S if it holds for all its localizations.

Comment on the notation 7.40.8. Here C stands for the initial of either Cayley or Chow and, as before, K stands for the first syllable of Cayley.

VARIANTS 7.41. These definitions each have other versions and relatives. I believe that each of the above 5 are natural and maybe even optimal, though they may not be stated in the cleanest form. Here are some other possibilities and equivalent versions.

(7.41.1) It could have been better to define C-flatness using the Cayley-Chow form; the equivalence is proved in (7.50). The Cayley-Chow form version matches better with the study of Chow varieties; the definition in (7.40.1) emphasizes the similarity with the other 4.

(7.41.2) In (7.40.2) it would have been better to say that F is *stably C-flat* for $L := \mathcal{O}_Y(1)$. However, we see in (7.65) that this notion is independent of the choice of an ample line bundle L , so we can eventually drop L from the name.

(7.41.3) In (7.40.3) we get an equivalent notion if we allow all finite morphisms $\rho : Y \rightarrow W$, where $W \rightarrow S$ is any smooth, projective morphism of pure relative dimension $d + 1$ over S . Indeed, let $\pi : W \rightarrow \mathbb{P}_S^{d+1}$ be a finite morphism. If F is K-flat then $\text{DSupp}((\pi \circ \rho)_*F)$ is a relative Cartier divisor, hence $\text{DSupp}(\rho_*F)$ is K-flat by (7.30.3). Since $W \rightarrow S$ is smooth, $\text{DSupp}(\rho_*F)$ is a relative Cartier divisor by (7.56).

(7.41.4) It would be natural to consider an affine version of C-flatness: We start with a coherent sheaf F on \mathbb{A}_S^n and require that $\text{DSupp}(\pi_*F)$ be Cartier over S for every projection $\pi : \mathbb{A}_S^n \rightarrow \mathbb{A}_S^{d+1}$ that is finite on Y .

The problem is that the relative affine version of Noether's normalization theorem does not hold, thus there may not be any such projections; see (10.67.7). This is why (7.40.4) is stated for projective morphisms only. A more local version is defined in (7.54).

Nonetheless, the notions (7.40.1–4) are étale local on X and most likely the following Henselian version of (7.40.5) does work.

(7.41.5) Assume that $f : (y, Y) \rightarrow (s, S)$ is a local morphism of pure relative dimension d of Henselian local schemes such that $k(y)/k(s)$ is finite. Let F be a coherent sheaf on X that is pure of relative dimension d over S . Then F

is K -flat over S iff $\text{DSupp}(\rho_*F)$ is Cartier over S for every finite morphism $\rho : Y \rightarrow \text{Spec } \mathcal{O}_S\langle x_0, \dots, x_d \rangle$ (where $R\langle \mathbf{x} \rangle$ denotes the Henselization of $R[\mathbf{x}]$).

It is possible that in fact all 5 versions (7.40.1–5) are equivalent to each other, but for now we can prove only 8 of the 10 possible implications. Four of them are easy to see.

PROPOSITION 7.42. *Let F be a generically flat family of pure, coherent sheaves of relative dimension d on \mathbb{P}_S^n . Then*

$$\text{formally } K\text{-flat} \Rightarrow K\text{-flat} \Rightarrow \text{locally } K\text{-flat} \Rightarrow \text{stably } C\text{-flat} \Rightarrow C\text{-flat}.$$

Proof. A divisor D on a scheme X is Cartier iff its completion \widehat{D} is Cartier on \widehat{X} for every $x \in X$ by (7.10). Thus formally K -flat \Rightarrow K -flat.

K -flat \Rightarrow locally K -flat is clear and locally K -flat \Rightarrow stably C -flat follows from (7.55). Finally stably C -flat \Rightarrow C -flat is clear; a stronger version is proved in (7.59). \square

A key technical result of the chapter is the following, to be proved in (7.64).

THEOREM 7.43. *K -flatness is equivalent to stable C -flatness.*

It is quite likely that our methods will prove the following.

CONJECTURE 7.44. *Formal K -flatness is equivalent to K -flatness.*

We prove the special case of relative dimension 1 in (7.63); this is also a key step in the proof of (7.43).

The remaining question is whether C -flat implies stably C -flat. This holds in the examples that I computed, but I have not been able to compute many and I do not have any conceptual argument why these 2 notions should be equivalent.

QUESTION 7.45. Is C -flatness equivalent to stable C -flatness and K -flatness?

Next we show that the above properties are automatic over reduced schemes and can be checked on Artin subschemes. After that we establish the push-forward and additivity properties.

PROPOSITION 7.46. *Let S be a reduced scheme and F a generically flat family of pure, coherent sheaves of relative dimension d on \mathbb{P}_S^n . Then F is K -flat over S .*

Proof. This follows from (7.31). \square

PROPOSITION 7.47. *Let S be a scheme and F a generically flat family of pure, coherent sheaves of relative dimension d on \mathbb{P}_S^n . Then F satisfies one of the properties (7.40.1–5) iff τ^*F satisfies the same property for every Artin subscheme $\tau : A \hookrightarrow S$.*

Proof. Set $Y := \text{SSupp } F$ and let $\pi : Y \rightarrow \mathbb{P}_S^{d+1}$ be a finite morphism. By (7.10) $\text{DSupp}_S(\pi_*F)$ is Cartier iff $\text{DSupp}_A((\pi_A)_*\tau^*F)$ is Cartier for every Artin subscheme $\tau : A \hookrightarrow S$. Thus the Artin versions imply the global ones in all cases.

To check the converse, we may localize at $\tau(A)$. The claim is clear if every finite morphism $\pi_A : Y_A \rightarrow \mathbb{P}_A^{d+1}$ can be extended to $\pi : Y \rightarrow \mathbb{P}_S^{d+1}$. This is obvious for C -flatness, stable C -flatness and formal K -flatness, but it need not hold for K -flatness and local K -flatness.

These cases will be established only after we prove (7.43) in (7.64). Thus we have to be careful not to use this direction in Section 7.5. \square

7.48 (Push-forward, additivity and multiplicativity). First, as a generalization of (7.4.4), let $f : X \rightarrow S$ and $g : Y \rightarrow S$ be projective morphisms of pure relative dimension n and $\tau : X \rightarrow Y$ a finite morphism. Let F be a coherent sheaf on X that is generically flat and pure over S of dimension $n - 1$ such that g is smooth at generic points of f_*F_s for every $s \in S$. Let $\pi : Y \rightarrow \mathbb{P}_S^n$ be any finite morphism. Then

$$\begin{aligned} \text{DSupp}_S((\pi \circ \tau)_*F) &= \text{DSupp}_S(\pi_*(\tau_*F)) \\ &= \text{DSupp}_S(\pi_*\text{DSupp}_S(\tau_*F)), \end{aligned}$$

where the first equality follows from the identity $\pi_*(\tau_*F) = (\pi \circ \tau)_*F$ and for the second we apply (7.30.3) to τ_*F . This proves (7.4.4).

Additivity (7.4.5) is essentially a special case of this. Let $f : X \rightarrow S$ be a projective morphism of pure relative dimension n and $D_1, D_2 \subset X$ K -flat, relative Mumford divisors. Next take 2 copies $X' := X_1 \cup X_2$ of X , mapping to X by the identity map $\tau : X' \rightarrow X$. Let $D' \subset X'$ be the union of the divisors $D_i \subset X_i$. Then $\text{DSupp}_S(\tau_*\mathcal{O}_{D'}) = D_1 + D_2$. Thus if the D_i are K -flat then so is $D_1 + D_2$.

Finally consider (7.4.6). If D is K -flat then so is every mD by additivity, the interesting claim is the converse. Let $\pi : Y \rightarrow \mathbb{P}_S^n$ be any finite morphism. Set $E := \text{DSupp}_S(\pi_*D)$. Then $mE = \text{DSupp}_S(\pi_*(mD))$, thus we need to show that if mE is Cartier then so is E . This was treated in (2.96). \square

7.5. Cayley-Chow flatness

Let $Z \subset \mathbb{P}^n$ be a subvariety of dimension d . Cayley [Cay60, Cay62] associates to it a hypersurface

$$\text{Ch}(Z) := \{L \in \text{Gr}(n-d-1, \mathbb{P}^n) : Z \cap L \neq \emptyset\} \subset \text{Gr}(n-d-1, \mathbb{P}^n),$$

called the Cayley-Chow hypersurface; its equation is called the Cayley-Chow form.

We extend this definition to coherent sheaves on \mathbb{P}_S^n over an arbitrary base scheme. We use 2 variants, but the proof of (7.50) needs 2 other versions as well. All of these are defined in the same way, but $\text{Gr}(n-d-1, \mathbb{P}^n)$ is replaced by other universal varieties.

DEFINITION 7.49 (Cayley-Chow hypersurfaces). Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on \mathbb{P}_S^n (7.29). We define 4 versions of the Cayley-Chow hypersurface associated to F as follows.

In all 4 versions the left hand side map σ is a smooth fiber bundle.

Grassmannian version 7.49.1.

Consider the diagram

$$\begin{array}{ccc} & \text{Flag}_S(\text{point}, n-d-1, \mathbb{P}^n) & \\ \sigma_g \swarrow & & \searrow \tau_g \\ \mathbb{P}_S^n & & \text{Gr}_S(n-d-1, \mathbb{P}^n) \end{array}$$

where the flag variety parametrizes pairs $(\text{point}) \in L^{n-d-1} \subset \mathbb{P}^n$. Set

$$\text{Ch}_g(F) := \text{DSupp}_S((\tau_g)_*\sigma_g^*F).$$

Product version 7.49.2.

Consider the diagram

$$\begin{array}{ccc} & \text{Inc}_S(\text{point}, (\check{\mathbb{P}}^n)^{d+1}) & \\ \sigma_p \swarrow & & \searrow \tau_p \\ \mathbb{P}_S^n & & (\check{\mathbb{P}}^n)_S^{d+1} \end{array}$$

where the incidence variety parametrizes $(d+2)$ -tuples $((\text{point}), H_0, \dots, H_d)$ satisfying $(\text{point}) \in H_i$ for every i . Set

$$\text{Ch}_p(F) := \text{DSupp}_S((\tau_p)_* \sigma_p^* F).$$

Flag version 7.49.3.

Consider the diagram

$$\begin{array}{ccc} & \text{PFlag}_S(0, n-d-2, n-d-1, \mathbb{P}^n) & \\ \sigma_f \swarrow & & \searrow \tau_f \\ \mathbb{P}_S^n & & \text{Flag}_S(n-d-2, n-d-1, \mathbb{P}^n) \end{array}$$

where PFlag parametrizes triples $((\text{point}), L^{n-d-2}, L^{n-d-1})$ such that $(\text{point}) \in L^{n-d-1}$ and $L^{n-d-2} \subset L^{n-d-1}$ (but the point need not lie on L^{n-d-2}). Set

$$\text{Ch}_f(F) := \text{DSupp}_S((\tau_f)_* \sigma_f^* F).$$

Incidence version 7.49.4.

Consider the diagram

$$\begin{array}{ccc} & \text{Inc}_S(\text{point}, L^{n-d-1}, (\check{\mathbb{P}}^n)^{d+1}) & \\ \sigma_i \swarrow & & \searrow \tau_i \\ \mathbb{P}_S^n & & \text{Inc}_S(L^{n-d-1}, (\check{\mathbb{P}}^n)^{d+1}) \end{array}$$

where the incidence variety parametrizes $(d+3)$ -tuples $((\text{point}), L^{n-d-1}, H_0, \dots, H_d)$ satisfying $(\text{point}) \in L^{n-d-1} \subset H_i$ for every i . Set

$$\text{Ch}_i(F) := \text{DSupp}_S((\tau_i)_* \sigma_i^* F).$$

THEOREM 7.50. *Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on \mathbb{P}_S^n . The following are equivalent.*

(7.50.1) $\text{Ch}_p(F) \subset (\check{\mathbb{P}}^n)_S^{d+1}$ is Cartier over S .

(7.50.2) $\text{Ch}_g(F) \subset \text{Gr}_S(n-d-1, \mathbb{P}^n)$ is Cartier over S .

If S is local with infinite residue field then these are also equivalent to

(7.50.3) $\text{DSupp}(\pi_* F)$ is Cartier over S for every \mathcal{O}_S -projection $\pi : \mathbb{P}_S^n \dashrightarrow \mathbb{P}_S^{d+1}$ (7.39) that is finite on $\text{Supp } F$.

(7.50.4) $\text{DSupp}(\pi_* F)$ is Cartier over S for a dense set of \mathcal{O}_S -projections $\pi : \mathbb{P}_S^n \dashrightarrow \mathbb{P}_S^{d+1}$.

Proof. The extreme cases $d = 0$ and $d = n - 1$ are somewhat exceptional, so we deal with them first.

If $d = n - 1$ then $\text{Gr}_S(n-d-1, \mathbb{P}_S^n) = \text{Gr}_S(0, \mathbb{P}_S^n) \cong \mathbb{P}_S^n$ and the only projection is the identity. Furthermore $\text{Ch}_g(F) = \text{DSupp}_S(F)$ by definition, so (7.50.2) and (7.50.3) are equivalent. If these hold then $\text{Ch}_p(F) = \text{Ch}_p(\text{DSupp}_S(F))$ is also flat by (7.24). Conversely, for (7.50.1) \Rightarrow (7.50.2) the argument in (7.51) works.

If $d = 0$ then F is flat over S and (7.50.1–3) hold by (7.31).

We may thus assume from now on that $0 < d < n - 1$. These cases are discussed in (7.51–7.52).

7.51 (Proof of (7.50.1) \Leftrightarrow (7.50.2)). To go between the product and the Grassmannian versions, the basic diagram is the following.

$$\begin{array}{ccc} & \text{Inc}_S(L^{n-d-1}, (\check{\mathbb{P}}^n)^{d+1}) & \\ \swarrow & & \searrow (\mathbb{P}^d)^{d+1}\text{-bundle} \\ (\check{\mathbb{P}}^n)_S^{d+1} & & \text{Gr}_S(n-d-1, \mathbb{P}_S^n) \end{array}$$

The right hand side projection

$$\pi_2 : \text{Inc}_S(L^{n-d-1}, (\check{\mathbb{P}}^n)^{d+1}) \rightarrow \text{Gr}_S(n-d-1, \mathbb{P}_S^n)$$

is a $(\mathbb{P}^d)^{d+1}$ -bundle. Therefore $\text{Ch}_i(F) = \pi_2^* \text{Ch}_g(F)$. Thus $\text{Ch}_g(F)$ is Cartier over S iff $\text{Ch}_i(F)$ is Cartier over S . It remains to compare $\text{Ch}_i(F)$ and $\text{Ch}_p(F)$.

The left hand side projection

$$\pi_1 : \text{Inc}_S(L^{n-d-1}, (\check{\mathbb{P}}^n)^{d+1}) \rightarrow (\check{\mathbb{P}}^n)_S^{d+1}$$

is birational. It is an isomorphism over $(H_0, \dots, H_d) \in (\check{\mathbb{P}}^n)_S^{d+1}$ iff $\dim(H_0 \cap \dots \cap H_d) = n-d-1$, the smallest possible. That is, when the rank of the matrix formed from the equations of the H_i is $d+1$. Thus π_1^{-1} is an isomorphism outside a subset of codimension $n+1-d$ in each fiber of $(\check{\mathbb{P}}^n)_S^{d+1} \rightarrow S$.

Therefore, if $\text{Ch}_i(F)$ is Cartier over S then $\text{Ch}_p(F)$ is Cartier over S , outside a subset of codimension $n+1-d \geq 3$ on each fiber of $(\check{\mathbb{P}}^n)_S^{d+1} \rightarrow S$. Then $\text{Ch}_p(F)$ is Cartier over S everywhere by (7.9.8).

Conversely, let E be the support of the π_1 -exceptional divisor. If $\text{Ch}_p(F)$ is a relative Cartier divisor then so is $\pi_1^* \text{Ch}_p(F)$, which agrees with $\text{Ch}_i(F)$ outside E .

Note that E consists of those tuples $(L^{n-d-1}, H_0, \dots, H_d)$ for which H_0, \dots, H_d are linearly dependent. This is easiest to describe using π_2 , which is a $(\mathbb{P}^d)^{d+1}$ -bundle over $\text{Gr}_S(n-d-1, \mathbb{P}_S^n)$. In a local trivialization, the points in the i th copy of \mathbb{P}^d have coordinates $(a_{i,0} : \dots : a_{i,d})$, and then the equation of E is $\det(a_{i,j}) = 0$. Thus E is irreducible and the restriction of π_2

$$\text{Inc}_S(L^{n-d-1}, (\check{\mathbb{P}}^n)^{d+1}) \setminus E \rightarrow \text{Gr}_S(n-d-1, \mathbb{P}_S^n)$$

is surjective. Since $\text{Ch}_i(F) = \pi_2^* \text{Ch}_g(F)$, this implies that $\text{Ch}_g(F)$ is relative Cartier (4.27). \square

7.52 (Proof of (7.50.2) \Rightarrow (7.50.3) \Rightarrow (7.50.4) \Rightarrow (7.50.2)). To go between the Grassmannian version and the projection versions, the basic diagram is the following.

$$\begin{array}{ccc} & \text{Flag}_S(n-d-2, n-d-1, \mathbb{P}_S^n) & \\ \mathbb{P}^{n-d-1}\text{-bundle } \swarrow & & \searrow \mathbb{P}^{d+1}\text{-bundle} \\ \text{Gr}_S(n-d-1, \mathbb{P}^n) & & \text{Gr}_S(n-d-2, \mathbb{P}^n) \end{array}$$

The left hand side projection

$$\rho_1 : \text{Flag}_S(n-d-2, n-d-1, \mathbb{P}_S^n) \rightarrow \text{Gr}_S(n-d-1, \mathbb{P}_S^n)$$

is a \mathbb{P}^{n-d-1} -bundle and $\text{Ch}_f(X) = \rho_1^* \text{Ch}_g(X)$. Thus $\text{Ch}_g(X)$ is Cartier over S iff $\text{Ch}_f(X)$ is Cartier over S .

The right hand side projection

$$\rho_2 : \text{Flag}_S(n-d-2, n-d-1, \mathbb{P}_S^n) \rightarrow \text{Gr}_S(n-d-2, \mathbb{P}_S^n)$$

is a \mathbb{P}^{d+1} -bundle, but $\text{Ch}_f(X)$ is not the pull-back of something on $\text{Gr}_S(n-d-2, \mathbb{P}_S^n)$.

Let $L \subset \mathbb{P}_S^n$ be a flat family of $(n-d-2)$ -planes; that is, a section of $\mathrm{Gr}(n-d-2, \mathbb{P}_S^n)$ over S . The preimage of $[L]$ is the set of all $n-d-1$ -planes that contain L ; we can identify this with sections of the target of the projection $\pi_L : \mathbb{P}^n \dashrightarrow L^\perp$. Thus the restriction of $\mathrm{Ch}_f(X)$ to the preimage of L is $\mathrm{DSupp}((\pi_L)_*(F))$.

So, if $\mathrm{Ch}_f(F)$ is Cartier over S then $\mathrm{DSupp}((\pi_L)_*(F)) = \mathrm{Ch}_f(F)|_{L^\perp}$ is also Cartier over S . Thus (7.50.2) \Rightarrow (7.50.3) and (7.50.3) \Rightarrow (7.50.4) is obvious.

Conversely, assume that $\mathrm{DSupp}((\pi_L)_*(F))$ is Cartier over S for general L . By (7.9.8) it is enough to show that $\mathrm{Ch}_f(F)$ is flat over S , outside a subset of codimension ≥ 3 .

Let $U_F \subset \mathrm{Gr}_S(n-d-2, \mathbb{P}_S^n)$ be the open subset consisting of those L^{n-d-2} that are disjoint from $\mathrm{DSupp}(F)$. The restriction of the projection π_f to $\mathrm{Supp} \sigma_f^* F$ is finite over $\rho_2^{-1} U_F$, thus $\mathrm{Ch}_f(F) = \mathrm{DSupp}_S((\pi_f)_* \sigma_f^* F)$ is flat over S , outside a codimension ≥ 2 subset of each fiber of $\rho_2^{-1} U_F \rightarrow U_F$ by (7.31). By assumption the non-flat locus is disjoint from the generic fiber, hence the non-flat locus has codimension ≥ 3 over U_F .

It remains to understand what happens over $Z_F := \mathrm{Gr}_S(n-d-2, \mathbb{P}_S^n) \setminus U_F$. Note that $\rho_2^{-1}(Z_F)$ has codimension 2 in $\mathrm{Flag}_S(n-d-2, n-d-1, \mathbb{P}_S^n)$, so it is enough to show that $\mathrm{Ch}_f(F)$ is flat over S at a general point of a general fiber over Z_F .

Thus let L^{n-d-2} be a general point of Z_F . Then $\mathrm{DSupp}(F) \cap L^{n-d-2}$ is a single point p and F is flat over S at p . Furthermore, a general $L^{n-d-1} \supset L^{n-d-2}$ still intersects $\mathrm{DSupp}(F)$ only at p . Thus $\sigma_f^*(F)$ is flat over S at

$$(p, L^{n-d-2}, L^{n-d-1}) \in \mathrm{PFlag}_S(0, n-d-2, n-d-1, \mathbb{P}^n),$$

and $\mathrm{Supp} \sigma_f^* F$ is finite over

$$(L^{n-d-2}, L^{n-d-1}) \in \mathrm{Flag}_S(n-d-2, n-d-1, \mathbb{P}_S^n).$$

Since $\mathrm{Ch}_f(F) = \mathrm{DSupp}_S((\pi_f)_* \sigma_f^* F)$ by (7.49.3), it is flat over S at (L^{n-d-2}, L^{n-d-1}) by (7.31). \square

COROLLARY 7.53. *Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on \mathbb{P}_S^n . Let $h : S' \rightarrow S$ be a morphism. By base change we get $g' : X' \rightarrow S'$ and $F' = \mathrm{v-pure}(h_X^* F)$ (7.28).*

(7.53.1) *If F is C-flat, then so is F' .*

(7.53.2) *If F' is C-flat and h is scheme-theoretically dominant, then F is C-flat.*

Proof. We may assume that S is local with infinite residue field. Being C-flat is exactly (7.50.3) which is equivalent to (7.50.1). $F \mapsto \mathrm{Ch}_p(F)$ commutes with base change by (7.34) and, if h is scheme-theoretically dominant, then, by (4.34), a divisorial sheaf is Cartier iff its divisorial pull-back is. \square

DEFINITION 7.54. Let S be a local scheme with infinite residue field and F a generically flat family of pure, coherent sheaves of dimension d over S (7.29). F is *locally C-flat* over S at $y \in Y := \mathrm{SSupp} F$ iff $\mathrm{DSupp}(\pi_* F)$ is Cartier over S at $\pi(y)$ for every \mathcal{O}_S -projection $\pi : \mathbb{P}_S^n \dashrightarrow \mathbb{P}_S^{d+1}$ that is finite on Y for which $\{y\} = \mathrm{Supp}(\pi^{-1}(\pi(y)) \cap Y)$.

LEMMA 7.55. *Let S be a local scheme with infinite residue field and F a generically flat family of pure, coherent sheaves of dimension d on \mathbb{P}_S^n . Then F is C-flat iff it is locally C-flat at every point.*

Proof. It is clear that C-flat implies locally C-flat.

Conversely, assume that F is locally C-flat. Set $Z_s := \text{Supp}(F_s) \setminus \text{FlatCM}_S(F)$ and pick points $\{y_i : i \in I\}$, one in each irreducible component of Z_s . If $\pi : \mathbb{P}_S^n \dashrightarrow \mathbb{P}_S^{d+1}$ is a general \mathcal{O}_S -projection, then $\{y_i\} = \pi^{-1}(\pi(y_i)) \cap Y$ for all $i \in I$.

Note that $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor along $\mathbb{P}_S^{d+1} \setminus \pi(Z_s)$ by (7.24) and it is also relative Cartier at the points $\pi(y_i)$ for $i \in I$ since F is locally C-flat. Thus $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor outside a codimension ≥ 3 subset of \mathbb{P}_S^{d+1} , hence a relative Cartier divisor everywhere by (7.9.8). \square

COROLLARY 7.56. *Let (s, S) be a local scheme and $X \subset \mathbb{P}_S^n$ a closed subscheme that is flat over S of pure relative dimension $d + 1$. Let $D \subset X$ be a relative Mumford divisor. Let $x \in X_s$ be a smooth point. Then \mathcal{O}_D is locally C-flat at x iff D is a relative Cartier divisor at x .*

Proof. We may assume that S has infinite residue field. A general linear projection $\pi : X \rightarrow \mathbb{P}_S^{d+1}$ is étale at x and $X \cap \pi^{-1}(\pi(D)) = \{x\}$. Thus D is a relative Cartier divisor at x iff $\pi(D)$ is a relative Cartier divisor at $\pi(x)$. By (7.30.2) the latter holds iff \mathcal{O}_D is locally C-flat at x . \square

COROLLARY 7.57. *Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d over S . If F is flat at $y \in Y := \text{SSupp } F$ then it is also locally C-flat at y .*

Proof. By (10.8) F_s is CM outside a subset $Z_s \subset Y_s$ of codimension ≥ 3 . Let $W_s \subset Y_s$ be the set of points where F is not flat. Let $\pi : Y \rightarrow \mathbb{P}_S^{d+1}$ be a general linear projection. By (7.24) $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor outside $\pi(Z_s \cup W_s)$, and we may assume that $\pi(y) \notin \pi(W_s)$. Thus, in a neighborhood of $\pi(y)$, $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor outside the codimension ≥ 3 subset $\pi(Z_s)$. Thus $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor at y by (7.9.8). \square

LEMMA 7.58. *Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on \mathbb{P}_S^n . Set $Y := \text{SSupp } F$ and let $\pi : Y \rightarrow \mathbb{P}_S^{d+1}$ be a finite morphism. Let $g_m : Y \hookrightarrow \mathbb{P}_S^N$ be an embedding such that $g_m^* \mathcal{O}_{\mathbb{P}_S^N}(1) \cong \pi^* \mathcal{O}_{\mathbb{P}_S^{d+1}}(m)$. If $(g_m)_*F$ is C-flat then $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor.*

Proof. We may assume that S is local with infinite residue field. Choosing $d + 2$ general sections of $\mathcal{O}_{\mathbb{P}_S^{d+1}}(m)$ gives a morphism $w_m : \mathbb{P}_S^{d+1} \rightarrow \mathbb{P}_S^{d+1}$ and there is a linear projection $\rho : \mathbb{P}_S^N \dashrightarrow \mathbb{P}_S^{d+1}$ such that $w_m \circ \pi = \rho \circ g_m$. By assumption $\text{DSupp}((\rho \circ g_m)_*F)$ is a relative Cartier divisor, hence so is

$$\text{DSupp}((w_m \circ \pi)_*F) = \text{DSupp}((w_m)_* \text{DSupp}(\pi_*F)),$$

where the equality follows from (7.30.3).

Pick a point $x \in \text{DSupp}(\pi_*F)$. A general w_m is étale at x and $\{x\} = w_m^{-1}(w_m(x)) \cap \text{DSupp}(\pi_*F)$. Thus $w_m : \text{DSupp}(\pi_*F) \rightarrow \text{DSupp}((w_m \circ \pi)_*F)$ is étale at x . Thus $\text{DSupp}(\pi_*F)$ is Cartier at x . \square

COROLLARY 7.59. *Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on \mathbb{P}_S^n . Let $v_m : \mathbb{P}_S^n \hookrightarrow \mathbb{P}_S^N$ be the m th Veronese embedding. If $(v_m)_*F$ is C-flat then so is F . \square*

Bertini theorems for C-flatness.

The going down versions are straightforward.

LEMMA 7.60. *Let (s, S) be a local scheme and F a generically flat family of pure, coherent sheaves of dimension $d \geq 1$ on \mathbb{P}_S^n (7.29). Set $Z_s := \text{Supp}(F_s) \setminus \text{FlatCM}_S(F)$. Let $H \subset \mathbb{P}_S^n$ be a hyperplane that does not contain any irreducible component of Z_s .*

If F is C-flat then so is $F|_H$.

Proof. We may assume that the residue field is infinite. Every projection $H \dashrightarrow \mathbb{P}_S^d$ is obtained as the restriction of a projection $\mathbb{P}_S^n \dashrightarrow \mathbb{P}_S^{d+1}$. The rest follows from (7.32). \square

COROLLARY 7.61. *Let (s, S) be a local scheme and F a generically flat family of pure, coherent sheaves of dimension $d \geq 1$ on \mathbb{P}_S^n . Set $Y := \text{SSupp } F$, $Z_s := Y \setminus \text{FlatCM}_S(F)$ and let $D \subset Y$ be a relative Cartier divisor that does not contain any irreducible component of Z_s . If F is stably C-flat then $F|_D$ is also stably C-flat.*

Proof. We may assume that the residue field is infinite. By (7.55) it is sufficient to prove that $F|_D$ is locally C-flat. Pick a point $y \in D$ and let $H \supset D$ be a hypersurface section of Y that does not contain any irreducible component of Z_s and such that H equals D in a neighborhood of y . After a Veronese embedding H becomes a hyperplane section, and then (7.60) implies that $F|_H$ is stably C-flat. Hence $F|_H$ is locally C-flat and so $F|_D$ also locally C-flat at y . \square

The going up version needs a little more care.

LEMMA 7.62. *Let (s, S) be a local Artin scheme with infinite residue field and F a generically flat family of pure, coherent sheaves of dimension $d \geq 2$ on \mathbb{P}_S^n . Then F is C-flat iff $F|_H$ is C-flat for a dense set of hyperplanes $H \subset \mathbb{P}_S^n$.*

Proof. The hyperplanes are parametrized by $H^0(\mathbb{P}_S^n, \mathcal{O}_{\mathbb{P}_S^n}(1)) \cong \mathcal{O}_S^{n+1}$. Since \mathcal{O}_S is Artinian, it makes sense to talk about a dense set of hyperplanes. (This is the only reason why the lemma is stated for Artin schemes.)

One direction follows from (7.60). Conversely, if $F|_H$ is C-flat for a dense set of hyperplanes H then there is a dense set of projections $\pi : \mathbb{P}_S^n \dashrightarrow \mathbb{P}_S^{d+1}$ such that for a dense set of hyperplanes $L \subset \mathbb{P}_S^{d+1}$, the restriction of F to $\pi^{-1}(L)$ is C-flat. Thus $\text{DSupp}(\pi_* F)$ is a relative Cartier divisor in an open neighborhood of such an L by (7.33). Since $d \geq 2$, this implies that $\text{DSupp}(\pi_* F)$ is a relative Cartier divisor everywhere by (7.9.8). Thus F is C-flat by (7.50). \square

Now we come to the key result.

PROPOSITION 7.63. *Let (s, S) be a local scheme and F a generically flat family of pure, coherent sheaves of dimension 1 on \mathbb{P}_S^n . Then F is stably C-flat \Leftrightarrow K-flat \Leftrightarrow formally K-flat.*

Proof. We already proved in (7.42) that formally K-flat \Rightarrow K-flat \Rightarrow stably C-flat.

Thus assume that F is stably C-flat. Set $Y := \text{SSupp } F$ and pick a closed point $p \in Y$. We need to show that F is formally K-flat at p . By the already proved parts of (7.47), it is enough to prove this for Artin base schemes with infinite residue field. We may thus assume that $S = \text{Spec } A$ for a local Artin ring (A, m, k) with k infinite and $p \in Y_s(k)$.

Let $\widehat{\pi} : \widehat{Y} \rightarrow \widehat{\mathbb{A}}_S^2 = \text{Spec } A[[u, v]]$ be a finite morphism. It is convenient to identify u, v with $\widehat{\pi}^*u, \widehat{\pi}^*v$. After a linear coordinate change we may assume that the composite

$$\widehat{\rho} : \widehat{Y} \rightarrow \text{Spec } A[[u, v]] \rightarrow \text{Spec } A[[u]] \quad \text{is also finite.}$$

Thus $\widehat{\rho}_*\widehat{F}$ is a coherent sheaf on $\text{Spec } A[[u]]$; let \widehat{G} denote its restriction to the generic point. Since F is generically flat over A , \widehat{G} is flat over A , hence we can write it as the sheafification of a free $A((u))$ -module $\bigoplus_j A((u))e_j$ of rank r with basis $\{e_j\}$. Then multiplication by v is given by a matrix $M = (m_{ij}(u))$ where the $m_{ij}(u) \in A((u))$. By (7.24) $\text{DSupp}(\widehat{\pi}_*\widehat{F})$ is given by

$$(\det(M - v\mathbf{1}_r) = 0) \subset \widehat{\mathbb{A}}_S^2. \quad (7.63.1)$$

Next we write down an algebraic approximation of $\widehat{\pi}$. To this end fix some $m_0 \in \mathbb{N}$, to be determined later. We may assume that $p = (1:0:\cdots:0)$. By (7.38.2), for $m \gg m_0$ we can choose homogeneous polynomials $g_1, g_2 \in H^0(\mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^n}(m))$ such that

$$\pi : Y \rightarrow \mathbb{P}_S^2 \quad \text{given by} \quad (x_0^m : g_1 : g_2) \quad (7.63.2)$$

is a finite morphism, p is the only point of Y that maps to $(1:0:0)$,

$$g_1/x_0^m \equiv u \pmod{\mathfrak{m}_p^{m_0}} \quad \text{and} \quad g_2/x_0^m \equiv v \pmod{\mathfrak{m}_p^{m_0}}, \quad (7.63.3)$$

where \mathfrak{m}_p is the ideal sheaf of $p \in Y$. Since the relative dimension is 1, $\mathfrak{m}_p^c \subset u\mathcal{O}_{Y_p}$ for some $c \in \mathbb{N}$, thus

$$g_1/x_0^m \equiv u \pmod{u^{m_0/c}\mathcal{O}_{Y_p}} \quad \text{and} \quad g_2/x_0^m \equiv v \pmod{u^{m_0/c}\mathcal{O}_{Y_p}} \quad (7.63.4)$$

holds, whenever $c \mid m_0$.

We now compute $\text{DSupp}(\pi_*F)$ in the affine chart with coordinates $u' := g_1/x_0^m$ and $v' := g_2/x_0^m$. This leads to a matrix $M'(u')$ such that

$$\text{DSupp}(\pi_*F) = (\det(M'(u') - v'\mathbf{1}_{r'}) = 0) \subset \mathbb{A}_S^2. \quad (7.63.5)$$

Choose an open $Y^\circ \subset Y$ such that $\rho : Y^\circ \rightarrow \mathbb{A}_{u'}^1$, the composite of π with the coordinate projection, is finite. Let F° be the restriction of F to Y° and $p = p_0, \dots, p_s \in Y^\circ$ the ρ -preimages of the origin. Then the completion of ρ_*F° at the origin decomposes as

$$\widehat{\rho_*F^\circ} = \bigoplus_{i=0}^s \widehat{\rho}_*\widehat{F}_i,$$

where \widehat{F}_i denotes the completion of F at p_i . Correspondingly, M' becomes a block-diagonal matrix with blocks M'_i . From (7.63.4) we get that

$$M(u) \equiv M'_0(u') \pmod{u^{m_0/c}A[[u]]}. \quad (7.63.6)$$

Let e be the maximal pole order of the $m_{ij}(u)$. Expanding the determinant we get terms whose maximal pole order is $\leq re$. Thus (7.63.6) implies that

$$\det(M - v\mathbf{1}_r) \equiv \det(M'_0 - v'\mathbf{1}_r) \pmod{u^{-re+m_0/c}A[[u, v]]}. \quad (7.63.7)$$

If $m_0 \gg 1$ (depending on c, e, r and M) then, by (7.17), $\det(M - v\mathbf{1}_r)$ defines a Cartier divisor iff $\det(M'_0 - v'\mathbf{1}_r)$ does. Since F is stably C-flat, $\det(M' - v'\mathbf{1}_{r'})$ defines a Cartier divisor, hence so does $\det(M'_0 - v'\mathbf{1}_r)$ and $\det(M - v\mathbf{1}_r)$. Thus F is formally K-flat at p . \square

7.64 (Proof of (7.43) and (7.47)). We already noted in (7.42) that K-flat \Rightarrow stably C-flat.

To see the converse, assume that F is stably C-flat. We aim to prove that it is K-flat. By the already established directions of (7.47), it is enough to prove this over Artin rings. Thus assume that S is the spectrum of an Artin ring and let $\pi : X \rightarrow \mathbb{P}_S^{d+1}$ be a finite projection. Set $L := \pi^* \mathcal{O}_{\mathbb{P}_S^{d+1}}(1)$. By (7.65) F is stably C-flat for L , hence $\text{DSupp}(\pi_* F)$ is a relative Cartier divisor by (7.58). This proves (7.43).

We already proved (7.47) for stable C-flatness. By (7.43) stable C-flatness is equivalent to K-flatness and local C-flatness, hence (7.47) also holds for these. \square

COROLLARY 7.65. *Let (s, S) be a local scheme and F a generically flat family of pure, coherent sheaves of dimension $d \geq 1$ on \mathbb{P}_S^n . Let L, M be relatively ample line bundles on $Y := \text{SSupp } F$. Then F is stably C-flat for L (as in (7.41.2)) iff it is stably C-flat for M .*

Proof. By (7.47), it is enough to prove this when S is Artinian with infinite residue field.

Assume that F is stably C-flat for M . By (7.58) we may assume that L is very ample. Repeatedly using (7.61) we get that, for general $L_i \in |L_i|$, the restriction of F to the complete intersection curve $L_1 \cap \dots \cap L_{d-1} \cap Y$ is stably C-flat for M . Thus the restriction of F to $L_1 \cap \dots \cap L_{d-1} \cap Y$ is formally K-flat by (7.63). Using (7.63) in the other direction for L , we get that the restriction of F to $L_1 \cap \dots \cap L_{d-1} \cap Y$ is stably C-flat for L . Now we can use (7.62) to conclude that F is stably C-flat for L . \square

7.6. Representability Theorems

DEFINITION 7.66. Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on \mathbb{P}_S^n . The *functor of K-flat pull-backs* is

$$KFlat_F(q : T \rightarrow S) = \begin{cases} 1 & \text{if } q_{\mathbb{P}}^{[*]} F \rightarrow T \text{ is K-flat, and} \\ 0 & \text{otherwise,} \end{cases}$$

where $q_{\mathbb{P}} : \mathbb{P}_T^n \rightarrow \mathbb{P}_S^n$ is the induced morphism and $q_{\mathbb{P}}^{[*]} F := \text{v-pure}(q_{\mathbb{P}}^* F)$ is the divisorial pull-back as in (4.1.7) or (7.28). If $Y \subset \mathbb{P}_S^n$ is a generically flat family of pure subschemes of dimension d then we write $KFlat_Y$ instead of $KFlat_{\mathcal{O}_Y}$.

If $KFlat_F$ is representable by a morphism, we denote it by $j_F^{\text{kflat}} : S_F^{\text{kflat}} \rightarrow S$. Note that j_F^{kflat} is necessarily a monomorphism.

One defines analogously the *functor of C-flat pull-backs* $CFlat_F$, and the *functor of stably C-flat pull-backs* $SCFlat_F$. The monomorphisms representing them are denoted by $j_F^{\text{cflat}} : S_F^{\text{cflat}} \rightarrow S$ and $j_F^{\text{scflat}} : S_F^{\text{scflat}} \rightarrow S$.

In our cases several of the monomorphisms are subschemes $S^* \hookrightarrow S$ such that $\text{red } S = \text{red } S^*$. (In particular, $S^* \subset S$ is both open and closed.) We call such a subscheme *full*.

PROPOSITION 7.67. *Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on \mathbb{P}_S^n . Then the functors of C-flat, stably C-flat or K-flat pull-backs of F are represented by full subschemes*

$$S_F^{\text{kflat}} = S_F^{\text{scflat}} \subset S_F^{\text{cflat}} \subset S.$$

Proof. By (7.50), $j_F^{\text{cflat}} : S_F^{\text{cflat}} \rightarrow S$ is the same as $j_{\text{Ch}_p(F)}^{\text{car}} : S_{\text{Ch}_p(F)}^{\text{car}} \rightarrow S$, with the Cayley-Chow hypersurface $\text{Ch}_p(F)$ as defined in (7.49.2). Thus (4.36.2) gives $S_F^{\text{cflat}} \subset S$.

We can apply this to each Veronese embedding $v_m : \mathbb{P}_S^n \hookrightarrow \mathbb{P}_S^N$, to get a full subschemes $S_{v_m(F)}^{\text{cflat}} \subset S$. Their intersection gives $S_F^{\text{scflat}} \subset S$. (A countable intersection of closed subschemes is a subscheme.) Finally $S_{\bar{F}}^{\text{kflat}} = S_F^{\text{scflat}}$ by (7.43). \square

7.68 (Proof of (7.3)). Fix a projective embedding $X \hookrightarrow \mathbf{P}_S$. By (4.86) there is a universal family of C-flat families of Mumford divisors

$$\text{Univ}_d^{\text{mum}} \rightarrow \text{MDiv}(X \subset \mathbf{P}_S),$$

where we follow the notation of (4.86.6).

By (7.67), we get $\text{KDiv}_d(X)$ as full subscheme of $\text{MDiv}(X \subset \mathbf{P}_S)$. \square

7.7. Normal varieties

In the next 3 sections we aim to give explicit descriptions of K-flat deformations of certain varieties. First we show that every K-flat deformation of a normal variety is flat. Then we consider K-flat deformations of planar curves and of seminormal curves. In both cases, we give a complete answer for first order deformations only.

7.69 (K-flat deformations). Let F_0 be a pure, coherent sheaf on \mathbb{P}_k^n . As a first approximation, a K-flat deformation of F_0 over a local scheme (s, S) is a coherent sheaf F on \mathbb{P}_S^n such that

(7.69.1) F is K-flat over S and

(7.69.2) $\text{pure}(F_s) \cong F_0 \otimes_k k(s)$.

As in (7.29), we may replace F by $F/\text{nv-tors}_S(F)$, thus we also assume that

(7.69.3) F is vertically pure.

This seems to be the right notion if $\dim F_0 = 1$, but additional questions appear in codimensions ≥ 2 .

To understand these, note that our definition of divisorial support and K-flatness is insensitive to codimension 2 behavior. For example set $F_0 = (x, y)^r \subset k[x, y]$ and let $(x, y)^r \subset J \subset k[x, y]$ be any ideal. Then $(x, y)^r + \epsilon J \subset k[x, y, \epsilon]$ gives a vertically pure, K-flat deformation of F_0 over $k[\epsilon]$. Its isomorphism class depends on J , but this does not seem to be useful.

In this case one can naturally choose either $J = (x, y)^r$ (and get a flat, trivial deformation) or $J = k[x, y]$ (and get a non-flat deformation). The first of these does not seem possible for arbitrary deformations and the second is also problematic over more complicated base schemes. The only truly canonical choice is to replace F with its hull F^H (9.13). That is, we assume that

(7.69.4) $\text{depth}_p F \geq 2$ for every codimension ≥ 2 point $p \in \text{Supp } F_0$.

This is the right thing to do if F_0 itself is S_2 . That is, if $\text{depth}_p F_0 \geq 2$ for every codimension ≥ 2 point $p \in \text{Supp } F_0$.

However, conditions (7.69.2) and (7.69.4) may be contradictory in general. In the above example, the hull of $(x, y)^r + \epsilon J$ is $k[x, y, \epsilon]$. Also, for divisors we want a K-flat deformation to be a subscheme of the ambient variety; this also may be inconsistent with (7.69.4).

For general sheaves the best choice may be to insist on (7.69.4), but replacing (7.69.2) with

(7.69.2') there are natural embeddings $F_0 \otimes_k k(s) \subset \text{pure}(F_s) \subset F_0^H \otimes_k k(s)$.

This, however, means that the answer depends mostly on F_0^H , very little on F_0 itself.

In the sequel we focus mostly on S_2 sheaves, when (7.69.1–4) give the right definition, and leave the general case undecided.

THEOREM 7.70. *Let $g : Y \rightarrow (s, S)$ be a projective morphism. Assume that*

(7.70.1) \mathcal{O}_Y is vertically pure,

(7.70.2) g is smooth at the generic points of Y_s ,

(7.70.3) g is K -flat and

(7.70.4) $\text{red}(Y_s)$ is normal.

Then g is flat along Y_s .

Proof. Let $Z \subset Y_s$ be a nowhere dense, closed subset such that g is flat along $Y_s \setminus Z$. We show that if $Z \neq \emptyset$ there is a $Z' \subsetneq Z$ such that g is flat along $Y_s \setminus Z'$. Then finish by Noetherian induction.

Let $y \in Z$ be a generic point that has codimension 1 in Y_s . Then $\text{red}(Y_s)$ is smooth at y , hence g is smooth at y by (7.71). This reduces us to the case when the codimension of Z is ≥ 2 . In this case flatness holds even without K -flatness by (10.62). \square

LEMMA 7.71. *Let $g : (y, Y) \rightarrow (s, S)$ be a local morphism of pure relative dimension 1, that is essentially of finite type. Assume that*

(7.71.1) g is smooth along $Y \setminus \{y\}$,

(7.71.2) g is formally K -flat at y and

(7.71.3) $\text{pure}(Y_s)$ is smooth at y .

Then g is smooth at y .

Proof. By (7.47) we may assume that S is Artinian. Then we can reduce it further to the case when Y is complete and $k(y) = k(s) =: k$; see (10.47). Write $Y = \text{Spec } R_A$.

By induction on the length of A we may assume that there is an ideal $A \supset (\epsilon) \cong k$ such that $\text{pure}(R_A/\epsilon R_A) \cong (A/\epsilon)[[\bar{x}]]$.

Let $x \in R_A$ be a lifting of \bar{x} . Set $J := \ker[R_A \rightarrow \text{pure}(R_A/\epsilon R_A)]$. Then J is a rank 1 R_k -module, hence free; let $y \in J$ be a generator. We have $x^r y = \epsilon g_k(x)$, where $g_k \in k[[x]]$ is a unit and $r = \dim_k(J/\epsilon R_A)$. These determine a projection of R_A whose image in $\text{Spec } A[[x, y]]$ is given by the ideal

$$A[[x, y]] \cap (y - \epsilon x^{-r} g_k(x)) A[[x^{-1}, y]].$$

By (7.16) this is a principal ideal iff $g_k(x) \in (y, x^r)$, that is, when $r = 0$. Thus $R_A = A[[x]]$. \square

As the next example shows, the situation is quite different for isolated, non-normal singularities of surfaces.

EXAMPLE 7.72. The deformation theory of the non-normal surface

$$S_r := \text{Spec } k[x^i y^j : i + j \geq r]$$

is quite interesting. For $r \geq 2$ the only singularity is at the origin and the smooth locus is $S_r^\circ \cong \mathbb{A}^2 \setminus \{(0, 0)\}$.

The simplest way to obtain flat deformations of S_r is by deforming the subscheme $\text{Spec } k[x, y]/(x, y)^r$ inside \mathbb{A}^2 . This is the theory of the Hilbert scheme of 0-dimensional subschemes on surfaces; see, for example, [Har10, Sec.9]. We call these the *Hilbert scheme deformations*. These are the deformations of S_r whose coordinate ring can be written as $k + (\text{ideal in } k[x, y])$.

The resulting deformation space is smooth and an open dense subset of it corresponds to distinct points. The number of points equals the length of the subscheme, which is $\binom{r+1}{2}$. Thus we get a deformation space of dimension $r(r+1)$. We should subtract from this 2, corresponding to translations.

First order K-flat deformations of S_r can be studied using the method of Section 6.5. First order flat deformations of the smooth locus S_r° are described by

$$H^1(U, T_U) = \langle x^{-i}y^{-j}\partial_x : i \geq 2, j \geq 1 \rangle \oplus \langle x^{-i}y^{-j}\partial_y : i \geq 1, j \geq 2 \rangle, \quad (7.72.1)$$

where $\partial_x := x \frac{\partial}{\partial x}$ and $\partial_y := y \frac{\partial}{\partial y}$; see (6.57).

Given $D \in H^1(U, T_U)$, let $S_r^\circ(D) \rightarrow \text{Spec } k[\epsilon]$ be the corresponding deformation. By (6.45) a regular function h on S_r° lifts to $S_r^\circ(D)$ iff $D(h) \in H^1(S_r^\circ, \mathcal{O}_{S_r^\circ})$ is 0. Using the conditions (6.57.6.a–b) we get the following.

Claim 7.72.2. In addition to the Hilbert scheme deformations, first order K-flat deformations of S_r are given by

$$\begin{array}{ll} x^{-i}y^{-j}\partial_x & \text{for } i+j \leq r+1, i \geq 2, j \geq 1, \\ x^{-i}y^{-j}\partial_y & \text{for } i+j \leq r+1, i \geq 1, j \geq 2 \quad \text{and} \\ x^{-i}y^{-j}((j-1)\partial_x - (i-1)\partial_y) & \text{for } i+j \leq r+2, i, j \geq 2. \quad \square \end{array}$$

These are the flat deformations of S_r° that can be extended to a K-flat deformation of S_r , but, depending on how we decide between (7.69.2) and (7.69.4), this extension need not be unique.

Remark 7.72.3. It is possible that all flat deformations of S_r admit a simultaneous normalization as in [Kol11b]. For deformations over a reduced base this is proved in [Kol95a, 14.2] and [Kol11b, 22].

7.8. Hypersurface singularities

In this section we give a detailed description of K-flat deformations of hypersurface singularities over $k[\epsilon]$.

7.73 (Non-flat deformations). Let $X \subset \mathbb{A}^n$ be a reduced subscheme of pure dimension d . We aim to describe non-flat deformations of X that are flat outside a subset $W \subset X$.

Choose equations g_1, \dots, g_{n-d} such that

$$(g_1 = \dots = g_{n-d} = 0) = X \cup X',$$

where $Z := X \cap X'$ has dimension $< d$. Let h be an equation of $X' \cup W$ that does not vanish on any irreducible component of X . Thus X is a complete intersection in $\mathbb{A}^n \setminus (h = 0)$ with equation $g_1 = \dots = g_{n-d} = 0$. Its flat deformations over an Artin ring (A, m, k) are then given by

$$g_i(\mathbf{x}) = \Psi_i(\mathbf{x}) \quad \text{where } \Psi_i \in m[x_1, \dots, x_n, h^{-1}]. \quad (7.73.1)$$

Note that we can freely change the Ψ_i by any element of the ideal $(g_1 - \Psi_1, \dots, g_{n-d} - \Psi_{n-d})$. We get especially simple normal forms if $A = k[\epsilon]$, that is, we look at first order deformations. In this case the equations can be written as

$$g_i(\mathbf{x}) = \Phi_i(\mathbf{x})\epsilon \quad \text{where} \quad \Phi_i \in k[x_1, \dots, x_n, h^{-1}]. \quad (7.73.2)$$

Now we can freely change the Φ_i by any element of the ideal (g_1, \dots, g_{n-d}) . Thus the relevant information is carried by $\phi_i := \Phi_i|_X$, and first order generically flat deformations can be given in the form

$$g_i = \phi_i\epsilon \quad \text{where} \quad \phi_i \in H^0(X, \mathcal{O}_X)[h^{-1}]. \quad (7.73.3)$$

Set $X^\circ := X \setminus (Z \cup W)$. By varying h we see that in fact

$$g_i = \phi_i\epsilon \quad \text{where} \quad \phi_i \in H^0(X^\circ, \mathcal{O}_{X^\circ}). \quad (7.73.4)$$

This shows that the choice of h is largely irrelevant.

If the deformation is flat then the equations defining X lift, that is, $\phi_i \in H^0(X, \mathcal{O}_X)$. In some simple cases, for example if X is a complete intersection, this is equivalent to flatness. In the examples that we compute, the most important information is carried by the polar parts

$$\bar{\phi}_i \in H^0(X^\circ, \mathcal{O}_{X^\circ})/H^0(X, \mathcal{O}_X). \quad (7.73.5)$$

We study first order non-flat deformations of hypersurface singularities. Plane curves turn out to be the most interesting ones.

7.74. Consider a hypersurface singularity $X := (f = 0) \subset \mathbb{A}_{\mathbf{x}}^n$ and a generically flat deformation of it

$$\mathbf{X} \subset \mathbb{A}_{\mathbf{x}, z}^{n+1}[\epsilon] \rightarrow \text{Spec } k[\epsilon]. \quad (7.74.1)$$

Aiming to work inductively, we assume that the deformation is flat outside the origin. Choose coordinates such that the x_i do not divide f .

As in (7.73.3) any such deformation can be given as

$$f(\mathbf{x}) = \psi(\mathbf{x})\epsilon \quad \text{and} \quad z = \phi(\mathbf{x})\epsilon, \quad (7.74.2)$$

where $\psi, \phi \in H^0(X, \mathcal{O}_X)[x_n^{-1}]$. Note that the choice of x_n is not intrinsic, so in fact

$$\psi, \phi \in \cap_i H^0(X, \mathcal{O}_X)[x_i^{-1}]. \quad (7.74.3)$$

If $n \geq 3$ then $\cap_i \mathcal{O}_X[x_i^{-1}] = \mathcal{O}_X$ and we get the following special case of (10.64).

Claim 7.74.4. Let $X := (f = 0) \subset \mathbb{A}^n$ be a hypersurface singularity and $\mathbf{X} \subset \mathbb{A}^{n+1}[\epsilon]$ a first order deformation of X that is flat outside the origin. If $n \geq 3$ then \mathbf{X} is flat over $k[\epsilon]$. \square

For $n = 2$ we use the following.

Notation 7.74.5. Let $C = (f(x, y) = 0) \subset \mathbb{A}^2$ be a reduced curve singularity. Set $C^\circ := C \setminus \{(0, 0)\}$. A non-flat deformation \mathbf{C} over $k[\epsilon]$ is written as

$$f(x, y) = \Psi(x, y)\epsilon \quad \text{and} \quad z = \Phi(x, y)\epsilon.$$

As in (7.73), we set $\psi := \Psi|_C$, $\phi := \Phi|_C$ and $\bar{\psi}, \bar{\phi} \in H^0(C^\circ, \mathcal{O}_{C^\circ})/H^0(C, \mathcal{O}_C)$ denote their polar parts.

We say that a (flat, resp. generically flat) deformation over $k[\epsilon]$ *globalizes* if it is induced from a (flat, resp. generically flat) deformation over $k[[t]]$ by base change.

THEOREM 7.75. *Consider a generically flat deformation \mathbf{C} of the plane curve singularity $C := (f = 0) \subset \mathbb{A}_{xy}^2$ given in (7.74.5).*

(7.75.1) If \mathbf{C} is C -flat then $\psi \in H^0(C, \mathcal{O}_C)$.

(7.75.2) If $\psi \in H^0(C, \mathcal{O}_C)$ then the deformation is

(a) flat iff $\phi \in H^0(C, \mathcal{O}_C)$ and

(b) C -flat iff $f_x\phi, f_y\phi \in H^0(C, \mathcal{O}_C)$.

(7.75.3) If C is reduced and $\psi = 0$, then the deformation globalizes iff $\phi \in H^0(\bar{C}, \mathcal{O}_{\bar{C}})$, where $\bar{C} \rightarrow C$ is the normalization.

Remark 7.75.4. Note that Ω_C^1 is generated by $dx|_C, dy|_C$, while ω_C is generated by $f_y^{-1}dx = -f_x^{-1}dy$.

If C is reduced, then Ω_C^1 and ω_C are naturally isomorphic over the smooth locus C° . This gives a natural inclusion $\text{Hom}(\Omega_C^1, \omega_C) \hookrightarrow \mathcal{O}_{C^\circ}$. Then (7.75.2.b) says that C -flat deformations are parametrized by $\text{Hom}(\Omega_C^1, \omega_C)$. We describe this space for monomial curves in (7.76).

Proof. If $\psi, \phi \in H^0(C, \mathcal{O}_C)$ then we can assume that Ψ, Φ are regular, so the deformation is flat. The converse in (7.75.2.a) is clear.

As for (7.75.2.b), we write down the equation of image of the projection

$$(x, y, z) \mapsto (\bar{x}, \bar{y}) = (x - \alpha(x, y, z)z, y - \gamma(x, y, z)z),$$

where α, γ are constants for linear projections and power series that are nonzero at the origin in general. Since $z^2 = \phi^2\epsilon^2 = 0$, Taylor expansion gives that

$$f(\bar{x}, \bar{y}) = f(x, y) - \alpha(x, y, z)f_x(x, y)z - \gamma(x, y, z)f_y(x, y)z.$$

Similarly, for any polynomial $F(x, y)$ we get that $F(\bar{x}, \bar{y}) \equiv F(x, y) \pmod{\epsilon\mathcal{O}_C}$, hence $F(\bar{x}, \bar{y})z = F(x, y)z$ in \mathcal{O}_C since $z\epsilon = 0$. Thus the equation of the projection is

$$f(\bar{x}, \bar{y}) - (\psi(\bar{x}, \bar{y}) - \alpha(\bar{x}, \bar{y}, 0)f_x(\bar{x}, \bar{y})\phi - \gamma(\bar{x}, \bar{y}, 0)f_y(\bar{x}, \bar{y})\phi) \cdot \epsilon = 0. \quad (7.75.5)$$

By (7.16) this defines a relative Cartier divisor for every α, γ iff $\psi, f_x\phi, f_y\phi \in \mathcal{O}_C$, proving (7.75.2.b). (This also shows that linear projections and formal projections give the same restrictions, hence C -flatness implies formal K -flatness in this case.)

We show in (7.77.3) that if \mathbf{C} globalizes then $\phi \in H^0(\bar{C}, \mathcal{O}_{\bar{C}})$. To prove the converse assertion in (7.75.3), we would like to write the global deformation as

$$(f(x, y) = 0, z = \phi(x, y)s) \subset \mathbb{A}_{xyzs}^4.$$

The problem with this is that ϕ has a pole at the origin. Thus we write $\phi = \phi^*h^{-r}$ where ϕ^* is regular at the origin and h is a general linear form in x, y . Then the correct equations are

$$(f(x, y) = 0, zh^r = \phi^*(x, y)s) \subset \mathbb{A}_{xyzs}^4.$$

Note that typically $\phi^*(0, 0) = 0$, hence the 2-plane $(x = y = 0) \subset \mathbb{A}_{xyzs}^4$ appears as an extra irreducible component. We need one more equation to eliminate it.

If $\phi \in H^0(\bar{C}, \mathcal{O}_{\bar{C}})$ then it satisfies an equation

$$\phi^m + \sum_{j=0}^{m-1} r_j \phi^j = 0, \quad \text{where } r_j \in H^0(C, \mathcal{O}_C).$$

Thus $z = \phi s$ satisfies the equation

$$z^m + \sum_{j=0}^{m-1} r_j z^j s^{m-j} = 0.$$

Now the 3 equations

$$f(x, y) = zh^r - \phi^*(x, y)s = z^m + \sum_{j=0}^{m-1} r_j z^j s^{m-j} = 0$$

define the required globalization of the infinitesimal deformation. \square

Complement 7.75.6. One can also consider non-flat deformations in higher dimensional spaces over $k[\epsilon]$. These are then given by equations

$$f(x, y) = \Psi(x, y)\epsilon \quad \text{and} \quad z_i = \Phi_i(x, y)\epsilon.$$

Such a deformation is C-flat iff $f_x\phi_i, f_y\phi_i$ is a regular function on C for every $\phi_i := \Phi_i|_C$. If C is reduced then

$$\{\phi : f_x\phi, f_y\phi \in H^0(C, \mathcal{O}_C)\} \subset H^0(C^\circ, \mathcal{O}_{C^\circ})$$

is a finitely generated $H^0(C, \mathcal{O}_C)$ -module. If $\phi_j = \sum_{i < j} h_i\phi_i$ then $z_j - \sum_{i < j} h_i z_i = 0$, so we can eliminate the variable z_j .

Thus all infinitesimal C-flat deformations of a reduced curve form a finite dimensional vectorspace.

Nonreduced curves 7.75.7. Consider $C = (y^2 = 0)$ with deformations

$$y^2 = (y\psi_1(x) + \psi_0(x))\epsilon \quad \text{and} \quad z = (y\phi_1(x) + \phi_0(x))\epsilon,$$

where $\psi_i, \phi_i \in k[x, x^{-1}]$. If this is C-flat then $\psi_i \in k[x]$ by (7.75.1). Since $f_x \equiv 0$, (7.75.2.b) gives only 1 condition, that $y(y\phi_1(x) + \phi_0(x))$ be regular. Since $y^2 = 0$ we get that $\phi_0 \in k[x]$, but no condition on ϕ_1 . So it can have a pole of arbitrary high order. Note that if ϕ_1 has a pole of order m , then regularizing the second equation we get

$$zx^m = y\epsilon + (\text{other terms}).$$

This suggests that if these deformations lie on a family of surfaces, the total space must have more and more complicated singularity at the origin as $m \rightarrow \infty$.

EXAMPLE 7.76 (Monomial curves). We can be quite explicit if C is the irreducible monomial curve $C := (x^a = y^c) \subset \mathbb{A}^2$ where $(a, c) = 1$. Its miniversal space of flat deformations is given as

$$x^a - y^c + \sum_{i=0}^{a-2} \sum_{j=0}^{c-2} s_{ij} x^i y^j = 0.$$

Its dimension is $(a-1)(c-1)$.

In order to compute C-flat deformations, we parametrize C as $t \mapsto (t^c, t^a)$. Thus $\mathcal{O}_C = k[t^c, t^a]$. Let $E_C = \mathbb{N}a + \mathbb{N}c \subset \mathbb{N}$ denote the semigroup of exponents. Then the condition (7.75.2.b) becomes

$$t^{ac-c}\phi(t), t^{ac-a}\phi(t) \in k[t^a, t^c]. \quad (7.76.1)$$

This needs to be checked one monomial at a time.

For $\phi = t^m$ and $m \geq 0$ the conditions (7.76.1) are automatic, and the deformation is non-flat iff $m \notin E_C$. These give a space of dimension $\frac{1}{2}(a-1)(c-1)$. (This is an integer since one of a, c must be odd.)

For $\phi = t^{-m}$ and $m \geq 0$ we get the conditions $ac-c-m \in E_C$ and $ac-a-m \in E_C$. By (7.76.4) these are equivalent to $ac-a-c-m \in E_C$. The largest value of m satisfying this gives the deformation

$$(x^a - y^c = z - t^{-ac+a+c}\epsilon = 0) \quad \text{over} \quad k[\epsilon]. \quad (7.76.2)$$

Note also that for $0 \leq m \leq ac-a-c$, we have that $ac-a-c-m \in E_C$ iff $m \notin E_C$. These again have $\frac{1}{2}(a-1)(c-1)$ solutions.

Thus we see that the space of C-flat deformations that are non-flat has $(a-1)(c-1)$ extra dimensions; the same as the space of flat deformations. This looks

very promising, but the next example shows that we get different answers for non-monomial curve singularities.

Non-monomial example 7.76.3. Consider the curve singularity $C = (x^4 + y^5 + x^2y^3 = 0)$. Blowing up the origin we get $(x/y)^4 + y + (x/y)^2y = 0$. Thus C is irreducible, it can be parametrized as $x = t^5 + \dots, y = t^4 + \dots$ and it is an equisingular deformation of the monomial curve $(x^4 + y^5 = 0)$.

In the monomial case we have the deformation (7.76.2) where $z - t^{-11}\epsilon = 0$. We claim that C does not have a C -flat deformation $z - \phi\epsilon = 0$ where $\phi = t^{-11} + \dots$. Indeed, such a deformation would satisfy

$$f_x\phi = y(\text{local unit}) \quad \text{and} \quad f_y\phi = x(\text{local unit}).$$

Eliminating ϕ gives that $(xf_x)/(yf_y) = (\text{local unit})$. We can compute the left hand side as

$$\begin{aligned} \frac{4x^4+2x^2y^3}{5y^5+3x^2y^3} &= \frac{-4y^5-4x^2y^3+2x^2y^3}{5y^5+3x^2y^3} = -\frac{4}{5} \cdot \frac{1+(1/2)(x/y)^2}{1+(3/5)(x/y)^2} \\ &= -\frac{4}{5} \left(1 + \left(\frac{1}{2} - \frac{3}{5}\right) \left(\frac{x}{y}\right)^2 + \dots\right). \end{aligned}$$

This is invertible at the origin of the normalization of C , but it is not regular on C since $\frac{x}{y} = t + \dots$. □

The following is left as an exercise.

Lemma 7.76.4. For $(a, c) = 1$ set $E = \mathbb{N}a + \mathbb{N}c \subset \mathbb{N}$. Then

- (a) If $0 \leq m \leq \min\{ac - a, ac - c\}$ then $ac - a - m, ac - c - m \in E$ iff $ac - a - c - m \in E$.
- (b) If $0 \leq m \leq ac - a - c$ then $ac - a - c - m \in E$ iff $m \notin E$. □

7.77 (S_2 hull of a deformation). Let T be the spectrum of a DVR with maximal ideal (t) and residue field k . Let $g : X \rightarrow T$ be a flat morphism of pure relative dimension d and $Z := \text{Supp tors}(X_k)$. Let $j : X \setminus Z \hookrightarrow X$ the natural injection and set $\bar{X} := \text{Spec}_X j_* \mathcal{O}_{X \setminus Z}$. If X is excellent then $\pi : \bar{X} \rightarrow X$ is finite and \bar{X} is S_2 .

By composition we get $\bar{g} : \bar{X} \rightarrow T$. Note that $\pi_k : \bar{X}_k \rightarrow X_k$ is an isomorphism over $X_k \setminus Z$ and \bar{X}_k is S_1 . In particular, if $\text{pure}(X_k)$ is reduced then \bar{X}_k is dominated by the normalization $X_k^{\text{nor}} \rightarrow X_k$.

Note that $t^n \mathcal{O}_X$ usually has some embedded primes contained in Z . The intersection of its height 1 primary ideals (also called the n th symbolic power of $t\mathcal{O}_X$) is

$$(t\mathcal{O}_X)^{(n)} = \mathcal{O}_X \cap t^n \mathcal{O}_{\bar{X}} = \ker[\mathcal{O}_X \rightarrow \text{pure}(\mathcal{O}_X/t^n \mathcal{O}_X)] \tag{7.77.1}$$

Multiplication by t gives injections

$$\text{pure}(\mathcal{O}_{X_k}) = \mathcal{O}_X / (t\mathcal{O}_X)^{(1)} \xrightarrow{t} (t\mathcal{O}_X)^{(1)} / (t\mathcal{O}_X)^{(2)} \xrightarrow{t} \dots \hookrightarrow \tag{7.77.2}$$

Note that

$$(t\mathcal{O}_X)^{(n)} / (t\mathcal{O}_X)^{(n+1)} \hookrightarrow t^n \mathcal{O}_{\bar{X}} / t^{n+1} \mathcal{O}_{\bar{X}} \cong \mathcal{O}_{\bar{X}_k}, \tag{7.77.3}$$

thus the sequence (7.77.2) eventually stabilizes. We can thus view the quotients

$$(t\mathcal{O}_X)^{(n+1)} / t(t\mathcal{O}_X)^{(n)}$$

as graded pieces of two filtrations, one of $\text{tors}(X_k)$ and one of $\mathcal{O}_{\bar{X}_k} / \mathcal{O}_{X_k}$.

To formalize this, let us write $M \preceq N$ to mean that there are filtrations $0 = M_0 \subset \dots \subset M_m = M, 0 = N_0 \subset \dots \subset N_n = N$ and an injection $\sigma : [1, \dots, m] \hookrightarrow [1, \dots, n]$ such that, $M_i / M_{i-1} \cong N_{\sigma(i)} / N_{\sigma(i)-1}$ for every $i = 1, \dots, m$. If M, N are artinian modules over a local ring then this holds iff $\text{length } M \leq \text{length } N$.

We have thus proved the following.

Corollary 7.77.4. Using the notation of (7.77), assume that $\text{pure}(X_k)$ is reduced with normalization $X_k^{\text{nor}} \rightarrow X_k$. Then $\text{tors } \mathcal{O}_{X_k} \leq \mathcal{O}_{X_k^{\text{nor}}}/\mathcal{O}_{X_k}$.

In particular, if $\dim X_k = 1$ then $\text{length}(\text{tors } \mathcal{O}_{X_k}) \leq \text{length}(\mathcal{O}_{X_k^{\text{nor}}}/\mathcal{O}_{X_k})$. \square

A closely related computation is the following.

EXAMPLE 7.78. [Kol99, 4.8] Using (7.37.1) we see that the ideal of Chow equations of the codimension 2 subvariety $(x_{n+1} = f(x_0, \dots, x_n) = 0) \subset \mathbb{P}^{n+1}$ is generated by the forms

$$f(x_0 - a_0x_{n+1} : \dots : x_n - a_nx_{n+1}) \quad \text{for all } a_0, \dots, a_n. \quad (7.78.1)$$

If the characteristic is 0 then Taylor's theorem gives that

$$f(x_0 - a_0x_{n+1} : \dots : x_n - a_nx_{n+1}) = \sum_I \frac{(-1)^I}{I!} a^I \frac{\partial^I f}{\partial x^I} x_{n+1}^{|I|}, \quad (7.78.2)$$

where $I = (i_0, \dots, i_n) \in \mathbb{N}^{n+1}$. The $a^{|I|}$ are linearly independent, hence we get that the ideal of Chow equations is

$$I^{\text{ch}}(f(x_0, \dots, x_n), x_{n+1}) = (f, x_{n+1}D(f), \dots, x_{n+1}^m D^m(f)), \quad (7.78.3)$$

where we can stop at $m = \deg f$. Here we use the usual notation for derivative ideals

$$D(f) := (f, \frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}). \quad (7.78.4)$$

(Note that if we work with the ideal (f) and not just the polynomial f , then we must include f itself in its derivative ideal.)

If we want to work locally at the point $p = (x_1 = \dots = x_n = 0)$, then we can set $x_0 = 1$ to get the local version

$$I^{\text{ch}}(f(1, x_1, \dots, x_n), x_{n+1}) = (f, x_{n+1}D(f), \dots, x_{n+1}^m D^m(f)), \quad (7.78.7)$$

where we can now stop at $m = \text{mult}_p f$. This also holds if f is an analytic function, though this needs to be worked out using the more complicated formulas (7.37.6) that for us become

$$\pi : (x_1, \dots, x_{n+1}) \rightarrow (x_1 - x_{n+1}\psi_1, \dots, x_n - x_{n+1}\psi_n), \quad (7.78.8)$$

where $\psi_i = \psi_i(x_0, \dots, x_{n+1})$ are analytic functions. Expanding as in (7.78.2) we see that

$$f(x_1 - x_{n+1}\psi_1, \dots, x_n - x_{n+1}\psi_n) \in I^{\text{ch}}(f(x_1, \dots, x_n), x_{n+1}). \quad (7.78.9)$$

Thus we get the same ideal if we compute I^{ch} using analytic or formal projections.

7.9. Seminormal curves

Over an algebraically closed field k , every seminormal curve singularity is formally isomorphic to

$$C_n := \text{Spec } k[x_1, \dots, x_n]/(x_i x_j : i \neq j) \subset \mathbb{A}_{\mathbf{x}}^n,$$

formed by the union of the n coordinate axes. In this section we study deformations of C_n over $k[\epsilon]$ that are flat outside the origin.

A normal form is worked out in (7.79.4), which shows that the space of these deformations is infinite dimensional. Then we describe the flat deformations (7.80) and their relationship to smoothings (7.81).

We compute C-flat and K-flat deformations in (7.83); these turn out to be quite close to flat deformations.

The ideal of Chow equations of C_n is computed in (7.87). For $n = 3$ these are close to C-flat deformations, but the difference between the two classes increases rapidly with n .

7.79 (Generically flat deformations of C_n). Let $\mathbf{C}_n \subset \mathbb{A}_{\mathbf{x}}^m[\epsilon]$ be a generically flat deformation of $C_n \subset \mathbb{A}_{\mathbf{x}}^m$ over $k[\epsilon]$.

If \mathbf{C}_n is flat over $k[\epsilon]$ then we can assume that $n = m$, but a priori we only know that $n \leq m$.

Following (7.73), we can describe \mathbf{C}_n as follows.

Along the x_j -axis and away from the origin, the deformation is flat and the x_j -axis is a complete intersection. Thus, in the $(x_j \neq 0)$ open set, \mathbf{C}_n can be given as

$$x_i = \Phi_{ij}(x_1, \dots, x_m)\epsilon \quad \text{where } i \neq j \quad \text{and} \quad \Phi_{ij} \in k[x_1, \dots, x_m, x_j^{-1}]. \quad (7.79.1)$$

Note that $(x_1, \dots, \widehat{x}_j, \dots, x_m, \epsilon)^2$ is identically zero on $\mathbf{C}_n \cap (x_j \neq 0)$, so the terms in this ideal can be ignored. Thus along the x_j -axis we can change (7.79.1) to the simpler form

$$x_i = \phi_{ij}(x_j)\epsilon \quad \text{where } i \neq j \quad \text{and} \quad \phi_{ij} \in k[x_j, x_j^{-1}]. \quad (7.79.2)$$

There is one more simplification that we can make. Write

$$\phi_{ij} = \phi'_{ij} + \gamma_{ij} \quad \text{where } \phi'_{ij} \in k[x_j^{-1}], \gamma_{ij} \in (x_j) \subset k[x_j],$$

and set $x'_i = x_i - \sum_{j \neq i} \gamma_{ij}(x_j)$. Then we get the description

$$x'_i = \phi'_{ij}(x'_j)\epsilon \quad \text{where } i \neq j \quad \text{and} \quad \phi'_{ij} \in k[x'_j, x'_j^{-1}]. \quad (7.79.3)$$

For most of our computations the latter coordinate change is not very important. Thus we write our deformations as

$$\mathbf{C}_n : \{x_i = \phi_{ij}(x_j)\epsilon \quad \text{along the } x_j\text{-axis}\}, \quad (7.79.4)$$

where $\phi_{ij}(x_j) \in k[x_j, x_j^{-1}]$, but we keep in mind that we can choose $\phi_{ij}(x_j) \in k[x_j^{-1}]$ if it is convenient. In order to deal with the cases when $m > n$, we make the following

Convention 7.79.5. We set $\phi_{ij} \equiv 0$ for $j > n$.

Writing \mathbf{C}_n as in (7.79.4) is almost unique; see (7.80.4) for one more coordinate change that leads to a unique normal form.

We get the same result (7.79.4) if we work with the analytic or formal local scheme of C_n : we still end up with $\phi_{ij}(x_j) \in k[x_j^{-1}]$.

PROPOSITION 7.80. *For $n \geq 3$ the generically flat deformation $\mathbf{C}_n \subset \mathbb{A}_{\mathbf{x}}^n[\epsilon]$ as in (7.79.4) is flat iff the $\phi_{ij}(x_j)$ have no poles. (See (7.82.5) for the $n = 2$ case.)*

Proof. If the $\phi_{ij}(x_j)$ are regular then

$$x_i x_j - (x_j \phi_{ij}(x_j) + x_i \phi_{ji}(x_i))\epsilon = 0 \quad (7.80.1)$$

is an equation for \mathbf{C}_n . Thus every equation of C_n lifts to an equation of \mathbf{C}_n , hence \mathbf{C}_n is flat over $k[\epsilon]$ by (7.11).

Conversely, if the deformation is flat then the equations defining C_n lift, so we have a set of defining equations for \mathbf{C}_n of the form

$$x_i x_j = \Psi_{ij}(x_1, \dots, x_n)\epsilon. \quad (7.80.2)$$

As in (7.79.2), this simplifies to

$$x_i x_j = \psi_{ij}(x_j)\epsilon \quad \text{along the } x_j\text{-axis.}$$

Note that $x_i x_j$ vanishes along the other $n - 2$ axes, so we must have $\psi_{ij}(0) = 0$. (Here we use that $n \geq 3$.) Thus $\phi_{ij} := x_j^{-1}\psi_{ij}$ is regular as needed. \square

Remark 7.80.3. Choosing $r \leq n$ of the coordinate axes we get an embedding $\tau_r : C_r \hookrightarrow C_n$, and any generically flat deformation \mathbf{C}_n of C_n induces a generically flat deformation \mathbf{C}_r of C_r .

From (7.80) we conclude that \mathbf{C}_n is flat iff $\tau_3^* \mathbf{C}_n$ is flat for every $\tau_3 : C_3 \hookrightarrow C_n$. Neither direction of this claim seems to follow from general principles. For example, if $\tau_2^* \mathbf{C}_n$ is flat for every $\tau_2 : C_2 \hookrightarrow C_n$ then \mathbf{C}_n need not be flat; see (7.82.5) and (7.83.5).

Remark 7.80.4. Putting (7.79.3) and (7.80) together we get that flat deformations can be given as

$$\mathbf{C}_n : \{x_i = e_{ij}\epsilon \quad \text{along the } x_j\text{-axis, where } e_{ij} \in k\}. \quad (7.80.5)$$

The constants e_{ij} are not yet unique, translations

$$x_i \mapsto x_i - a_i\epsilon \quad \text{change} \quad e_{ij} \mapsto e_{ij} - a_j. \quad (7.80.6)$$

So we get a first order deformation space of dimension $n(n - 1) - n = n(n - 2)$.

We can also think of $\mathcal{O}_{\mathbf{C}_n}$ as a subring of $\bigoplus_j k[X_j, \epsilon_j]$ given by

$$x_i \mapsto (e_{i1}\epsilon_1, \dots, e_{i,i-1}\epsilon_{i-1}, X_i, e_{i,i+1}\epsilon_{i+1}, \dots, e_{in}\epsilon_n).$$

Strangely, (7.80.5) says that every flat first order deformation of C_n is obtained by translating the axes independently of each other. These deformations all globalize in the obvious way, but the globalization is not a flat deformation of C_n unless the translated axes all pass through the same point. If this point is $(a_1\epsilon, \dots, a_n\epsilon)$ then $e_{ij} = a_j$ and applying (7.80.3) we get the trivial deformation.

If $n = 2$ then the universal deformation is $x_1 x_2 + \epsilon = 0$. One may ask why this deformation does not lift to a deformation of C_3 : smooth 2 of the axes to a hyperbola and just move the 3rd axis along. If we use $x_1 x_2 + t = 0$, then the x_3 -axis should move to the line $(x_1 - \sqrt{t} = x_2 - \sqrt{t} = 0)$. This gives the flat deformation given by equations

$$x_1 x_2 + t = x_3(x_1 - \sqrt{t}) = x_3(x_2 - \sqrt{t}) = 0.$$

Of course this only makes sense if t is a square. Thus setting $\epsilon = \sqrt{t} \pmod{t}$ the $t = \epsilon^2 \pmod{t}$ term becomes 0 and we get

$$x_1 x_2 = x_3 x_1 - x_3 \epsilon = x_3 x_2 - x_3 \epsilon = 0,$$

which is of the form given in (7.80.1).

EXAMPLE 7.81 (Smoothing C_n). Rational normal curves $R_n \subset \mathbb{P}^n$ have a moduli space of dimension $(n + 1)(n + 1) - 1 - 3 = n^2 + 2n - 3$. The $C_n \subset \mathbb{P}^n$ have a moduli space of dimension $n + n(n - 1) = n^2$. Thus the smoothings of C_n have a moduli space of dimension $n^2 + 2n - 3 - n^2 = 2n - 3$. We can construct these smoothings explicitly as follows.

Fix distinct $p_1, \dots, p_n \in k$ and consider the map

$$(t, z) \mapsto \left(\frac{t}{z-p_1}, \dots, \frac{t}{z-p_n} \right).$$

Eliminating z gives the equations

$$(p_i - p_j)x_i x_j + (x_i - x_j)t = 0: 1 \leq i \neq j \leq n \quad (7.81.1)$$

for the closure of the image, which is an affine cone over a degree n rational normal curve $R_n \subset \mathbb{P}_{t, \mathbf{x}}^n$. So far this is an $(n-1)$ -dimensional space of smoothings.

Applying the torus action $x_i \mapsto \lambda_i^{-1}x_i$, we get new smoothings given by the equations

$$(p_i - p_j)x_i x_j + (\lambda_j x_i - \lambda_i x_j)t = 0: 1 \leq i \neq j \leq n. \quad 7.81.2$$

Writing it in the form (7.79.4) we get

$$x_i = \frac{\lambda_i}{p_i - p_j} \epsilon \quad \text{along the } x_j\text{-axis.} \quad 7.81.3$$

This looks like a $2n$ -dimensional family, but $\text{Aut}(\mathbb{P}^1)$ acts on it, reducing the dimension to the expected $2n-3$. The action is clear for $z \mapsto \alpha z + \beta$, but $z \mapsto z^{-1}$ also works out using (7.80.6) since

$$\frac{\lambda_i}{p_i^{-1} - p_j^{-1}} = \frac{-\lambda_i p_i^2}{p_i - p_j} + \lambda_i p_i.$$

Claim 7.81.4. For distinct $p_i \in k$ and $\lambda_j \in k^*$, the vectors

$$\left(\frac{\lambda_j}{p_i - p_j} : i \neq j \right) \quad \text{span} \quad \langle e_{ij} \rangle \cong k^{\binom{n}{2}}.$$

So the flat infinitesimal deformations determined in (7.80.5) form the Zariski tangent space of the smoothings.

Proof. Assume that there is a linear relation

$$\sum_{ij} m_{ij} \frac{\lambda_j}{p_i - p_j} = 0.$$

If we let $p_i \rightarrow p_j$ but keep the others fixed, we get that $m_{ij} = 0$. \square

Remark 7.81.5. If $n=3$ then $2n-3 = n(n-2)$ and the Hilbert scheme of degree 3 reduced space curves with $p_a = 0$ is smooth, see [PS85].

EXAMPLE 7.82 (Simple poles). Among non-flat deformations, the simplest ones are given by $\phi_{ij}(x_j) = c_{ij}x_j^{-1} + e_{ij}$. Then we have

$$q_{ij} := x_i x_j - (e_{ij}x_i + e_{ji}x_j)\epsilon = \begin{cases} c_{ij}\epsilon & \text{along the } x_j\text{-axis,} \\ c_{ji}\epsilon & \text{along the } x_i\text{-axis,} \\ 0 & \text{along the other axes.} \end{cases} \quad (7.82.1)$$

Thus we see that $\sum_{ij} \gamma_{ij} q_{ij}$ vanishes on \mathbf{C}_n iff

$$\sum_i \gamma_{ij} c_{ij} \quad \text{is independent of } j. \quad (7.82.2)$$

These impose $n-1$ linear conditions on the γ_{ij} , which are in general independent. Thus we get the following.

Claim 7.82.3. For general c_{ij} , the torsion subsheaf of the central fiber has length $n-1$. \square

In special cases the torsion can be smaller, but if the c_{ij} are not identically 0, then we get at least 1 nontrivial condition. This is in accordance with (7.80).

The $n = 2$ case is exceptional and is worth discussing separately. We get that

$$q_{12} := x_1x_2 - (e_{12}x_1 + e_{21}x_2)\epsilon = \begin{cases} c_{12}\epsilon & \text{along the } x_2\text{-axis,} \\ c_{21}\epsilon & \text{along the } x_1\text{-axis.} \end{cases} \quad (7.82.4)$$

This gives the following.

Claim 7.82.5. For $n = 2$ the deformation as in (7.79.4) is flat iff ϕ_{12}, ϕ_{21} have only simple poles and with the same residue. \square

The main result is the following.

THEOREM 7.83. *For a first order deformation of $C_n \subset \mathbb{A}^m$ specified as in (7.79.4) by*

$$\mathbf{C}_n : \{x_i = \phi_{ij}(x_j)\epsilon \text{ along the } x_j\text{-axis}\} \quad (7.83.1)$$

the following are equivalent.

(7.83.2) \mathbf{C}_n is C -flat.

(7.83.3) \mathbf{C}_n is K -flat.

(7.83.4) The ϕ_{ij} have only simple poles and ϕ_{ij}, ϕ_{ji} have the same residue.

(7.83.5) \mathbf{C}_n induces a flat deformation on any pair of lines $C_2 \hookrightarrow C_n$.

Proof. The proof consist of 2 parts. First we show in (7.84) that (7.83.2) and (7.83.4) are equivalent by explicitly computing the equations of linear projections.

We see in (7.85) that if the ϕ_{ij} have only simple poles then there is only 1 term of the equation of a non-linear projection that could have a pole, and this term is the same for the linearization of the projection. Hence it vanishes iff it vanishes for linear projections. This shows that (7.83.4) \Rightarrow (7.83.3).

Finally (7.83.4) \Leftrightarrow (7.83.5) follows from (7.82.5). \square

Remark 7.83.6. If $j > n$ then $\phi_{ij} \equiv 0$ by (7.79.5), so ϕ_{ji} is regular by (7.83.4). Evaluating them at the origin gives the vector $\mathbf{v}_j \in k^n$. If $\sum_{j>n} \lambda_j \mathbf{v}_j = 0$ then

$$\sum_{j>n} \lambda_j (x_j - \sum_{i=1}^n \phi_{ji} x_i \epsilon)$$

is regular and identically 0 on \mathbf{C}_n . We can thus eliminate some of the x_j for $j > n$ and obtain that every K -flat deformation of \mathbf{C}_n lives in \mathbb{A}^{2n} .

7.84 (Linear projections). Recall that by our convention (7.79.5), $\phi_{ij} \equiv 0$ for $j > n$. Extending this, in the following proof all sums/products involving i go from 1 to m and sums/products involving j go from 1 to n .

With \mathbf{C}_n as in (7.83.1) consider the special projections

$$\pi_{\mathbf{a}} : \mathbb{A}_{\mathbf{x}}^n[\epsilon] \rightarrow \mathbb{A}_{uv}^2[\epsilon] \text{ given by } u = \sum x_i, v = \sum a_i x_i, \quad (7.84.1)$$

where $a_i \in k[\epsilon]$. Write $a_i = \bar{a}_i + a'_i \epsilon$. (One should think that $a'_i = \partial a_i / \partial \epsilon$.)

In order to compute the projection, we follow the method of (7.22.7). Since we compute over $k[u, u^{-1}, \epsilon]$, we may as well work with the $k[u, \epsilon]$ -module $M := \bigoplus_j k[x_j, \epsilon]$ and write $1_j \in k[x_j, \epsilon]$ for the j th unit. Then multiplication by u and v are given by

$$\begin{aligned} u \cdot 1_j &= (\sum_i x_i) 1_j = x_j + \sum_i \phi_{ij} \epsilon \text{ and} \\ v \cdot 1_j &= (\sum_i a_i x_i) 1_j = a_j x_j + \sum_i a_i \phi_{ij} \epsilon. \end{aligned} \quad (7.84.2)$$

Thus

$$v \cdot 1_j = (a_j u + \sum_i (a_i - a_j) \phi_{ij}(u) \epsilon) \cdot 1_j.$$

Thus the v -action on M is given by the diagonal matrix

$$\text{diag}(a_j u + \sum_i (a_i - a_j) \phi_{ij}(u) \epsilon),$$

and by (7.22.7) the equation of the projection is its characteristic polynomial

$$\prod_j (v - a_j u - \sum_i (a_i - a_j) \phi_{ij}(u) \epsilon) = 0. \quad (7.84.3)$$

Expanding it we get an equation of the form

$$\prod_j (v - \bar{a}_j u) - B(u, v, a, \phi) \epsilon = 0, \quad (7.84.4)$$

where

$$B(u, v, a, \phi) = \sum_j (\prod_{i \neq j} (v - \bar{a}_j u)) \cdot (a'_j u + \sum_i (\bar{a}_i - \bar{a}_j) \phi_{ij}(u)). \quad (7.84.5)$$

This is a polynomial of degree $\leq n - 1$ in v , hence by (7.20) its restriction to the curve $(\prod_j (v - \bar{a}_j u) = 0)$ is regular iff $B(u, v, a, \phi)$ is a polynomial in u as well. Let now r be the highest pole order of the ϕ_{ij} and write

$$\phi_{ij}(u) = c_{ij} u^{-r} + (\text{higher terms}).$$

Then the leading part of the coefficient of v^{n-1} in $B(u, v, a, \phi)$ is

$$\sum_j \sum_i (\bar{a}_i - \bar{a}_j) c_{ij} u^{-r} = u^{-r} \sum_i \bar{a}_i (\sum_j (c_{ij} - c_{ji})). \quad (7.84.6)$$

Since the \bar{a}_i are arbitrary, we get that

$$\sum_j (c_{ij} - c_{ji}) = 0 \quad \text{for every } i. \quad (7.84.7)$$

Next we use a linear reparametrization of the lines $x_i = \lambda_i^{-1} y_i$ and then apply a projection $\pi_{\mathbf{a}}$ as in (7.84.1). The equations $x_i = \phi_{ij}(x_j) \epsilon$ become

$$y_i = \lambda_i \phi_{ij}(\lambda_j^{-1} y_j) \epsilon$$

and c_{ij} changes to $\lambda_i \lambda_j^r c_{ij}$. Thus the equations (7.84.7) become

$$\sum_j (\lambda_i \lambda_j^r c_{ij} - \lambda_j \lambda_i^r c_{ji}) = 0 \quad \forall i. \quad (7.84.8)$$

If $r \geq 2$ this implies that $c_{ij} = 0$ and if $r = 1$ then we get that $c_{ij} = c_{ji}$.

This completes the proof of (7.83.2) \Leftrightarrow (7.83.4).

Remark 7.84.9. Note that if we work over \mathbb{F}_2 then necessarily $\lambda_i = 1$, hence (7.84.8) does not exclude the $r \geq 2$ cases.

7.85 (Non-linear projections). Consider a general non-linear projection

$$(x_1, \dots, x_n) \mapsto (\Phi_1(x_1, \dots, x_n), \Phi_2(x_1, \dots, x_n)).$$

After a formal coordinate change we may assume that $\Phi_1 = \sum_i x_i$. Note that the monomials of the form $x_i x_j x_k, x_i^2 x_j^2, x_i x_j \epsilon$ vanish on \mathbf{C}_n , so we can discard these terms from Φ_2 . Thus, in suitable local coordinates a general non-linear projection can be written as

$$u = \sum_i x_i, \quad v = \sum_i \alpha_i(x_i) + \sum_{i \neq j} x_i \beta_{ij}(x_j), \quad (7.85.1)$$

where $\alpha_i(0) = \beta_{ij}(0) = 0$. Note that $\alpha'_i(0) = a_i$ in the notation of (7.84). Now we get that

$$\begin{aligned} u \cdot 1_j &= x_j + \sum_i \phi_{ij}(x_j) \epsilon \quad \text{and} \\ v \cdot 1_j &= \alpha_j(x_j) + \sum_{i \neq j} \alpha_i(\phi_{ij}(x_j) \epsilon) + \sum_{i \neq j} \phi_{ij}(x_j) \beta_{ij}(x_j) \epsilon. \end{aligned} \quad (7.85.2)$$

Note further that $\alpha_i(\phi_{ij}(x_j) \epsilon) = \alpha'_i(0) \phi_{ij}(x_j) \epsilon$ and

$$\alpha_j(x_j) = \alpha_j(u - \sum_i \phi_{ij}(x_j) \epsilon) = \alpha_j(u) - \alpha'_j(u) \sum_i \phi_{ij}(x_j) \epsilon.$$

Thus, as in (7.84.4), the projection is defined by the vanishing of

$$\begin{aligned} & \prod_j \left(v - \alpha_j(u) - \sum_i (\beta_{ij}(u) + \alpha'_i(0) - \alpha'_j(u)) \phi_{ij}(u) \epsilon \right) \\ & =: \prod_j (v - \bar{\alpha}_j(u)) - B(u, v, \alpha, \beta, \phi) \epsilon. \end{aligned} \quad (7.85.3)$$

Let $\bar{\beta}_{ij}, \bar{\alpha}'_j$ denote the residue of β_{ij}, α'_j modulo ϵ and write $\alpha_j(u) = \bar{\alpha}_j(u) + \partial_\epsilon \alpha_j(u) \epsilon$. As in (7.84.5), expanding the product gives that $B(u, v, \alpha, \beta, \phi)$ equals

$$\sum_j \left(\prod_{i \neq j} (v - \bar{\alpha}_i(u)) \right) \cdot (\partial_\epsilon \alpha_j(u) + \sum_i (\bar{\beta}_{ij}(u) + \bar{\alpha}'_i(0) - \bar{\alpha}'_j(u)) \phi_{ij}). \quad (7.85.4)$$

We already know that $\phi_{ij}(u) = c_{ij} u^{-1} + (\text{higher terms})$, hence $B(u, v, \alpha, \beta, \phi)$ has at most simple pole along $(u = 0)$. Computing its residue along $u = 0$ we get

$$v^{n-1} \sum_j \sum_i (\bar{\beta}_{ij}(0) + \bar{\alpha}'_i(0) - \bar{\alpha}'_j(0)) c_{ij} = v^{n-1} \sum_{ij} (\bar{a}_i - \bar{a}_j) c_{ij}. \quad (7.85.5)$$

These are the same as in (7.84.6). Thus $B(u, v, \alpha, \beta, \phi)$ is regular iff it is regular for the linearization of the projection. This completes the proof of (7.83.4) \Rightarrow (7.83.3).

EXAMPLE 7.86. The image of a general linear projection of $C_n \subset \mathbb{A}^n$ to \mathbb{A}^2 is n distinct lines through the origin. Their equation is $g_n(x, y) = 0$ where g_n is homogeneous of degree n with simple roots only. A typical example is $g_n = x^n + y^n$.

A general non-linear projection to \mathbb{A}^2 gives n smooth curve germs with distinct tangent lines through the origin. The equation of the image is $g_n(x, y) + (\text{higher terms}) = 0$ where g_n is homogeneous of degree n with simple roots only.

The miniversal deformation of $(x^n + y^n = 0)$ is

$$(x^n + y^n + \sum_{i,j \leq n-2} t_{ij} x^i y^j = 0) \subset \mathbb{A}_{xy}^2 \times \mathbb{A}_t^{(n-1)^2}. \quad (7.86.1)$$

A general deformation is a smoothing, but deformations that have n smooth branches with the same tangents as $(x^n + y^n = 0)$ form the subfamily

$$(x^n + y^n + \sum_{ij} t_{ij} x^i y^j = 0) \subset \mathbb{A}_{xy}^2 \times \mathbb{A}_t^{\binom{n-3}{2}}, \quad (7.86.2)$$

where summation is over those pairs (i, j) that satisfy $i, j \leq n - 2$ and $n < i + j$. For $n \leq 4$ there is no such pair (i, j) , which gives the following.

Claim 7.86.3. For $n \leq 4$ every analytic projection $\widehat{C}_n \rightarrow \widehat{\mathbb{A}}^2$ is obtained as the composite of an automorphism of \widehat{C}_n , followed by a linear projection and then by an automorphism of $\widehat{\mathbb{A}}^2$. \square

For $n = 5$ we get the deformations

$$(x^5 + y^5 + t x^3 y^3 = 0) \subset \mathbb{A}_{xy}^2 \times \mathbb{A}_t. \quad (7.86.4)$$

For $t \neq 0$ we get curves that are images of \widehat{C}_n by a nonlinear projection, but not as a linear projection pre-composed/composed with automorphisms.

The following strengthens [Kol99, 4.11].

PROPOSITION 7.87. *In characteristic 0, the ideal of Chow equations of C_n is generated by*

(7.87.1) *all degree n monomials, save the x_i^n , if n is even, and*

(7.87.2) *all degree n monomials, save the x_i^n and $x_1 \cdots x_n$, if n is odd.*

These hold both for linear, polynomial and analytic projections.

Note that we can write the even case as $I_{C_n}^{\text{ch}} = I_{C_n} \cap (x_1, \dots, x_n)^n$.

Proof. Every Chow equation has multiplicity $\geq n$, and we get the same equations modulo $(x_1, \dots, x_n)^{n+1}$, whether we use linear, polynomial and analytic projections (7.37).

In both of our cases, $I_{C_n} \cap (x_1, \dots, x_n)^{n+1} \subset I_{C_n}^{\text{ch}}$, so the ideal of Chow equations coming from linear projections already contains every possible higher order monomial. Thus it is sufficient to prove (7.87.1–2) for linear projections.

The linear projections of C_n to \mathbb{A}_{uv}^2 are given by $u = \sum_i a_i x_i, v = \sum b_i x_i$. The image of the x_j -axis is $(b_j u - a_j v = 0)$. So the pull-back of their product is

$$\prod_j \sum_i (a_i b_j - a_j b_i) x_i. \quad (7.87.3)$$

Since C_n is toric, $I_{C_n}^{\text{ch}}$ is a monomial ideal. Thus we need to understand which degree n monomials in the x_i have a nonzero coefficient in (7.87.3).

First, the coefficient of x_j in $\sum_i (a_i b_j - a_j b_i) x_i$ is 0, so we never get x_j^n . Next consider $x_1 \cdots x_n$. Its coefficient is

$$\sum_{\sigma \in S_n} \prod_i (a_i b_{\sigma(i)} - a_{\sigma(i)} b_i). \quad (7.87.4)$$

Note that the product is 0 if $\sigma(i) = i$ for some i and changes by $(-1)^n$ when σ is replaced by σ^{-1} . Thus if n is odd then (7.87.4) is identically zero. (More generally, the permanent of a skew-symmetric matrix of odd size is 0.) We have thus proved the following.

Claim 7.87.5. If n is odd then the coefficient of $x_1 \cdots x_n$ in (7.87.3) is 0. \square

It remains to show that all other degree n monomials appear in (7.87.3) with nonzero coefficient.

To show this we choose specific values of the a_i, b_i and hope to get enough nonzero terms. Thus fix $1 \leq r \leq n$, choose $a_1 = \cdots = a_r = 1, a_{r+1} = \cdots = a_n = 0$ and $b_1 = \cdots = b_r = 0, b_{r+1} = \cdots = b_n = 1$. Then

$$a_i b_j - a_j b_i = \begin{cases} 1 & \text{if } i \leq r < j, \\ -1 & \text{if } j \leq r < i, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad (7.87.6)$$

Thus (7.87.3) becomes

$$(-1)^r (x_1 + \cdots + x_r)^{n-r} (x_{r+1} + \cdots + x_n)^r \quad (7.87.7)$$

Applying this to various permutations of the x_i and choices of r we get the following.

Claim 7.87.8. Let $M = \prod x_i^{w_i}$ be a degree n monomial. Then $M \in I_{C_n}^{\text{ch}}$ if the following holds.

(a) There is a subset $I \subset \{1, \dots, n\}$ such that $\sum_{i \in I} w_i = n - |I|$. \square

While this is only a sufficient condition, we check next that it applies to every monomial other than x_i^n and $x_1 \cdots x_n$ for n odd.

Lemma 7.87.9. Let $M = \prod x_i^{w_i}$ be a degree n monomial other than x_i^n or $x_1 \cdots x_n$ for odd n . Then there is a subset $I \subset \{1, \dots, n\}$ such that $\sum_{i \in I} w_i = n - |I|$.

Proof. We use induction on n , the case $n = 1$ is empty and $n = 2$ is obvious.

Assume first that $w_{n-1} = w_n = 1$. If $M = x_1^{n-2} x_{n-1} x_n$ then $I = \{1, 2\}$ works. Otherwise, by induction, there is a subset $J \subset \{1, \dots, n-2\}$ such that

$\sum_{i \in J} w_i = n - 2 - |J|$. Set $I = J \cup \{n\}$. Then $\sum_{i \in J} w_i = n - 2 - |J| + 1 = n - |I|$ and we are done.

If the inductive step does not apply, then there is at most one $w_i = 1$, hence at least $\frac{n-1}{2}$ of the $w_i = 0$.

Reorder the x_i such that w_i is a decreasing function and take r such that $w_1 + \cdots + w_{r-1} < n/2$ but $w_1 + \cdots + w_r \geq n/2$. If $w \leq n - r$ then we take $I = \{1, \dots, r, n - s, \dots, n\}$ where $s = n - r - w - 1$. Since $w_i = 0$ for $i \geq \frac{n+1}{2}$,

$$\sum_{i \in J} w_i = w_1 + \cdots + w_r = w \quad \text{and} \quad |I| = r + s + 1 = n - w.$$

What happens if $w > n - r$? Note that then $r \geq 2$ and $w_1 \geq \cdots \geq w_r \geq 2$ so $(r-1)w_r < n/2$ and $2(r-1) < n/2$. On the other hand, $w_1 + \cdots + w_r < n/2 + w_r < n/2 + n/(2r-2)$. One checks that $n/2 + n/(2r-2) > n - r$ and $2(r-1) < n/2$ both hold only for $r = 2$. Furthermore, the only monomial for which the above choice of I does not work is $x_1^{(n-1)/2} x_2^{(n-1)/2} x_3$ for n odd. In this case we can take $I = \{1, 3, n - s, \dots, n\}$ where $s = \frac{n-5}{2}$. \square

This completes the proof of (7.87). \square

Moduli of stable pairs

We bring together the moduli theory of Chapter 6 with K-flatness of Chapter 7 to obtain the moduli theory of stable pairs in full generality. The basic definitions originate in the papers [KSB88, Ale96]; the resulting moduli spaces are usually called KSBA moduli spaces.

In Section 8.1 we discuss a bookkeeping device, called marking: we need to know not only what the boundary divisor Δ is, but also how it is written as a linear combination of \mathbb{Z} -divisors. In the cases considered in Chapter 6 there was always a unique marking; this is why the notion was not introduced before. Simple examples show that, without marking, we get infinite dimensional moduli spaces, already for pointed curves (8.2).

The notion of Kollár–Shepherd-Barron–Alexeev stability is introduced in Section 8.2. The proof that we get good moduli spaces follows the methods of Chapter 6 if the coefficients are rational (8.9), but a few more steps are needed if they are irrational (8.15).

The end result is the following consequence of (8.9) and (8.15).

THEOREM 8.1. *Fix a base scheme S of characteristic 0, a coefficient vector $\mathbf{a} = (a_1, \dots, a_r)$, an integer n and a real number v . Let $\mathcal{MSP}(\mathbf{a}, n, v)$ denote the functor of marked, stable pairs of dimension n and volume v . Then $\mathcal{MSP}(\mathbf{a}, n, v)$ has a coarse moduli space $\text{MSP}(\mathbf{a}, n, v)$, whose irreducible components are projective over S .*

If \mathbf{a} is rational, then $\text{MSP}(\mathbf{a}, n, v)$ is projective over S .

The moduli theory of more general polarized pairs is treated in Section 8.3.

An early difficulty of KSBA theory was that good examples were not easy to write down, and it turned out to be quite hard to fully describe complete moduli spaces. The first notable successes were [Ale02, Hac04]. By now there is a rapidly growing body of fully understood cases.

8.1. Marked stable pairs

So far we have studied slc pairs (X, Δ) but usually did not worry too much about how Δ was written as a sum of divisors. As long as we look at a single variety, we can write Δ uniquely as $\sum a_i D_i$ where the D_i are prime divisors and there is usually not much reason to do anything else. However, the situation changes when we look at families.

8.2 (Is $D = \frac{1}{n}(nD)$?). Assume that we have an slc family over an irreducible base $f : (X, \Delta) \rightarrow S$ with generic point $g \in S$. Then the natural approach is to write $\Delta_g = \sum a_i D_g^i$ where the D_g^i are prime divisors on the generic fiber X_g . For any other point $s \in S$ this gives a decomposition $\Delta_s = \sum a_i D_s^i$, where D_s^i is

the specialization of D_g^i . Note that the D_s^i need not be prime divisors. They can have several irreducible components with different multiplicities and two different D_s^i, D_s^j can have common irreducible components. Thus $\Delta_s = \sum a_i D_s^i$ is not the ‘standard’ way to write Δ_s .

Let us now turn this around. We fix a proper slc pair (X_0, Δ_0) and aim to understand all deformations of it. A first suggestion could be the following:

Naive definition 8.2.1. An slc deformation of (X_0, Δ_0) over a local scheme $(0 \in S)$ is a proper slc morphism $f : (X, \Delta) \rightarrow S$ whose central fiber $(X, \Delta)_0$ is isomorphic to (X_0, Δ_0) .

As an example of this, start with $(\mathbb{P}_{xy}^1, (x=0))$. Pick any $n \geq 1$ and variables t_i . Then

$$\left(\mathbb{P}_{xy}^1 \times \mathbb{A}_{\mathbf{t}}^n, \frac{1}{n}(x^n + t_{n-1}x^{n-1}y + \cdots + t_0y^n = 0) \right) \quad (8.2.2)$$

is a deformation of $(\mathbb{P}_{xy}^1, (x=0))$ over \mathbb{A}^n by the naive definition (8.2.1). We get a deformation space of dimension n ; though it can be reduced to $n-2$ using $\text{Aut}(\mathbb{P}^1, (0:1))$. Letting n vary results in an infinite dimensional deformation space.

The polynomial in (8.2.2) is irreducible over $k(t_0, \dots, t_{n-1})$, thus our recipe above says that we should write $\Delta = \frac{1}{n}D_g$ (where D_g is irreducible) and then the special fiber is written as $(x=0) = \frac{1}{n}(x^n=0)$.

The situation becomes even less clear if we take 2 deformations as in (8.2.2) for 2 different values n, m and glue them together over the origin. The family is locally stable. One side suggests that the fiber over the origin should be $\frac{1}{n}(x^n=0)$, the other side that it should be $\frac{1}{m}(x^m=0)$.

As (8.2) suggests, some bookkeeping is necessary to control the multiplicities of the divisorial part of a pair (X, Δ) in families. This is the role of the marking we introduce next.

At least in characteristic 0, once we control how a given \mathbb{R} -divisor Δ is written as a linear combination of \mathbb{Z} -divisors, we obtain finite dimensional moduli spaces.

DEFINITION 8.3 (Marked pairs). Fix a real vector $\mathbf{a} = (a_1, \dots, a_r)$. (We mostly care about the cases when $a_i \in [0, 1]$.) A *marked pair* with *coefficient vector* \mathbf{a} consists of

(8.3.1) a pair (X, Δ) plus

(8.3.2) a way of writing $\Delta = \sum a_i D_i$, where the D_i are effective \mathbb{Z} -divisors on X .

We also call $\sum a_i D_i$ a *marking* of Δ . We allow the D_i to be empty; this has the advantage that the restriction of a marking to an open subset is again a marking. However in other contexts this is not natural and I will probably sometimes forget about empty divisors.

Observe that $\Delta = \sum a_i D_i$ and $\Delta = \sum (\frac{1}{2}a_i)(2D_i)$ are different as markings. This seems rather pointless for one pair but, as we observed in (8.2), it is a meaningful distinction when we consider deformations of a pair.

Note that, for a given (X, Δ) , markings are combinatorial objects that are not constrained by the geometry of X . If $\Delta = \sum_i b_i B_i$ and the B_i are distinct prime divisors, then the markings correspond to ways of writing the vector (b_1, \dots, b_r) as a positive linear combination of nonnegative integral vectors.

Comments. Working with such markings is a rather natural thing to do. For example, plane curves C of degree d can be studied using the log-CY pair $(\mathbb{P}^2, \Delta_C :=$

$\frac{3}{d}C$) as in [Hac04]. Thus, even if C is reducible, we want to think of the \mathbb{Q} -divisor Δ_C as $\frac{3}{d}C$; hence as a marked divisor with $I := \{1\}$ and $a_1 = \frac{3}{d}$. Similarly, in most cases when we choose the boundary divisor Δ , it has a natural marking.

However, when a part of Δ is forced upon us, for instance coming from the exceptional divisor of a resolution, there is frequently no natural marking, though usually it is possible to choose a marking that works well enough.

If (X, Δ) is slc and $a_i > \frac{1}{2}$ for every i , then the marking is almost determined by Δ . For example, if the a_i are distinct then the obvious marking of $\Delta = \sum a_i D_i$ is the unique one. If all the $a_i = 1$ then the markings of $\sum_{i \in I} D_i$ correspond to partitions of I .

If we allow $a_i = \frac{1}{2}$ then an irreducible divisor D can have 3 different markings: $[D]$, $\frac{1}{2}[2D]$ or $\frac{1}{2}[D] + \frac{1}{2}[D]$. The smaller the a_i , the more markings are possible.

If I is a finite set then a divisor Δ has only finitely many possible markings. More generally, this also holds if I is infinite but the numbers $\{a_i\}$ satisfy the strong descending chain condition (there is no infinite sequence $a_{i_1} \geq a_{i_2} \geq \dots$ where the indices i_1, i_2, \dots are all different) and we ignore empty divisors.

DEFINITION 8.4 (Families of marked pairs). Fix a real vector $\mathbf{a} = (a_1, \dots, a_r)$. A *family of marked pairs* over a scheme S and with coefficient vector \mathbf{a} is a compound object $f : (X, \sum a_i D^i) \rightarrow S$ where

(8.4.1) $f : X \rightarrow S$ is a pure dimensional, flat morphism with geometrically connected, geometrically reduced, S_2 fibers and

(8.4.2) the D^i are relative Mumford \mathbb{Z} -divisors on X .

Note that $(X, \sum a_i D^i)$ is not a variety marked with divisors unless S itself is a variety.

REMARK 8.5. There are some subtle aspects of the notion of marked families of pairs.

First we claim that every locally stable family of pairs can be viewed as a marked family. Indeed, let $f : (X, \Delta) \rightarrow S$ be a locally stable family of pairs. Write $\Delta = \sum_i b_i B_i$ where the B_i are distinct prime divisors. Assume first that the b_i are rational and let N be their smallest common denominator. Then $D = \sum_i (N b_i) B_i$ is a generically Cartier family of divisors over S . Thus $\Delta = \frac{1}{N} D$ gives a marking of (X, Δ) ; we call this the *natural marking*.

If coefficients are real, one can use (11.38), but the resulting marking depends on the choices.

If S is normal then, by (4.25), a marking of (X, Δ) is the same as a marking of the generic fiber (X_g, Δ_g) , hence markings are combinatorial objects, corresponding to ways of writing the coefficient vector (b_1, \dots, b_r) as a positive linear combination of nonnegative integral vectors.

However, if S is not normal, then the geometry of (X, Δ) constrains the allowable markings. The reason for this is that each D^i is generically \mathbb{Q} -Cartier. In particular, if S is connected then $\text{Supp } D^i$ dominates S for every i . For example, consider the family of pairs

$$S := (st = 0) \subset \mathbb{A}_{st}^2, \quad X := \mathbb{P}_{xy}^1 \times S, \quad \Delta := \frac{1}{n} B_1 + \frac{1}{m} B_2,$$

where $B_1 := (s = x^n - ty^n = 0)$ and $B_2 := (t = x^m - sy^m = 0)$. Here $D_1 = B_1, D_2 = B_2$ is not an allowed marking since the B_i are not \mathbb{Q} -Cartier. In fact, the

only possible marking is the natural one

$$\Delta = \frac{(n,m)}{nm} \left(\frac{m}{(n,m)} B_1 + \frac{n}{(n,m)} B_2 \right),$$

and its obvious relatives of the form $\Delta = \sum_j a_j (mB_1 + nB_2)$.

As another example, let C be a nodal curve with normalization (C', p, q) . Fix 4 points a_1, \dots, a_4 on \mathbb{P}^1 . Let $D'_1, D'_2 \subset C' \times \mathbb{P}^1$ be two curves such that $D'_1 + D'_2$ has simple normal crossings only, $D'_i \rightarrow C'$ have degree 2, D'_1 meets \mathbb{P}_p^1 (resp. \mathbb{P}_q^1) in the points a_1, a_2 (resp. a_1, a_3) while D'_2 meets \mathbb{P}_p^1 (resp. \mathbb{P}_q^1) in the points a_3, a_4 (resp. a_2, a_4). We can now glue \mathbb{P}_p^1 to \mathbb{P}_q^1 to get a locally stable family $f : (C \times \mathbb{P}^1, D_1 + D_2) \rightarrow C$. Note that $D_1 + D_2$ is a Cartier divisor, but neither D_1 nor D_2 is \mathbb{Q} -Cartier. Thus the only possible marking is the natural marking $D = D_1 + D_2$ (and its obvious variations).

8.2. Kollár–Shepherd-Barron–Alexeev stability

Now we come to the main theorem of the book, the existence of a good moduli theory for all marked stable pairs (X, Δ) in characteristic 0.

The principle is that, once we have K-flatness to replace flatness in Section 6.2, the rest of the arguments should go through with small changes. This is indeed true for rational coefficients, so we start with that case.

For irrational coefficients it is less clear how to cook up ample line bundles, so the existence of embedded moduli spaces needs more work.

KSBA stability with rational coefficients.

Fix a rational coefficient vector $\mathbf{a} = (a_1, \dots, a_r)$ and let $\text{lcd}(\mathbf{a})$ denote the least common denominator of the a_i .

8.6 (Stable objects). Marked pairs $(X, \Delta = \sum_i a_i D_i)$ with coefficient vector \mathbf{a} such that

$$(8.6.1) \quad (X, \Delta) \text{ is slc,}$$

$$(8.6.2) \quad X \text{ is projective and } K_X + \Delta \text{ is ample.}$$

8.7 (Stable families). A family $f : (X, \Delta = \sum_i a_i D_i) \rightarrow S$ is *KSBA-stable* if the following hold.

$$(8.7.1) \quad f : X \rightarrow S \text{ is flat, finite type, pure dimensional with demi-normal fibers.}$$

$$(8.7.2) \quad \text{The } D_i \text{ are K-flat families of relative Mumford divisors (7.1).}$$

$$(8.7.3) \quad \text{The fibers } (X_s, \Delta_s) \text{ are slc.}$$

$$(8.7.4) \quad \omega_{X/S}^{[m]}(m\Delta) \text{ commutes with base change if } \text{lcd}(\mathbf{a}) \mid m.$$

$$(8.7.5) \quad f \text{ is proper and } K_{X/S} + \Delta \text{ is } f\text{-ample.}$$

The first 4 of these conditions define *locally KSBA-stable* families.

8.8 (Explanation). These conditions are mostly straightforward generalizations of (6.15.1–3). We discussed K-flatness in Chapter 7.

The main question is assumption (8.6.3). For $\omega_{X/S}^{[m]}(m\Delta)$ to make sense, $m\Delta$ must be a \mathbb{Z} -divisor. If the D_i have no multiple or common irreducible components, this holds only if m is a multiple of $\text{lcd}(\mathbf{a})$.

On the other hand, we could ask about the ‘corrected’ forms $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$ as in (6.21.3). As we saw in (2.40), base change can fail for these in 1-parameter families for some values of m , but (2.77) discusses various examples where $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$

does commute with base change for certain values of m . Thus, on a case-by-case basis, a strengthening of assumption (8.6.3) is possible and useful. This was the main theme of Chapter 6.

So one should think of (8.6.3) as the minimal base change assumption, that should be made more stringent whenever possible, without changing the reduced structure of the moduli space.

THEOREM 8.9. *KSBA-stability with rational coefficients, as defined in (8.6–8.7), is a good moduli theory (6.10).*

Proof. We need to check the conditions (6.10.1–5).

Separatedness (6.10.1) follows from (2.48) and valuative-properness (6.10.2) is proved in (2.49) and (7.4.2).

Embedded moduli spaces (6.10.3) are constructed in (8.32). However, the universal family over $\text{C}^m\text{ESP}(\mathbf{a}, r, n, \mathbb{P}_{\mathbb{Q}}^N)$ satisfies (8.7.3) only for a certain value of m for which $\omega_{X/S}^{[m]}(m\Delta)$ is a line bundle. We can then handle the other values as in in the proof of (6.23).

Finally, even strong boundedness (6.8.1) holds by [HMX18]. \square

As in (6.24), we get the following from (8.7.3).

PROPOSITION 8.10. *For KSBA-stable families with rational coefficients as in (8.6–8.7), let m be a multiple of $\text{lcd}(\mathbf{a})$. Then the Euler characteristic $\chi(X, \omega_X^{[m]}(m\Delta))$ and the plurigenus $h^0(X, \omega_X^{[m]}(m\Delta))$ are deformation invariant.* \square

KSBA stability with arbitrary coefficients.

Fix a coefficient vector $\mathbf{a} = (a_1, \dots, a_r)$ where $a_i \in [0, 1]$ are arbitrary real numbers. As we noted in (11.34.1), if $K_X + \sum_i a_i D_i$ is \mathbb{R} -Cartier, then we can get other \mathbb{Q} -Cartier divisors. We start by listing them.

DEFINITION 8.11. Fix a coefficient vector $\mathbf{a} = (a_1, \dots, a_r)$ with linear \mathbb{Q} -envelope $\text{LEnv}_{\mathbb{Q}}(1, \mathbf{a}) \subset \mathbb{Q}^{r+1}$ as in (11.35). For $\Delta = \sum_{i=1}^r a_i D_i$, set

$$\text{LEnv}_{\mathbb{Z}}(K_X + \Delta) := \{m_0 K_X + \sum m_i D_i : (m_0, \dots, m_r) \in \text{LEnv}_{\mathbb{Q}}(1, \mathbf{a}) \cap \mathbb{Z}^{r+1}\}.$$

Let us mention 2 extreme cases.

(8.11.1) If all $a_i \in \mathbb{Q}$ and d is their smallest common denominator, then $\text{LEnv}_{\mathbb{Z}}(K_X + \Delta)$ consists of all \mathbb{Z} -multiples of $d(K_X + \Delta)$.

(8.11.2) If $\{1, a_1, \dots, a_r\}$ are \mathbb{Q} -linearly independent, then

$$\text{LEnv}_{\mathbb{Z}}(K_X + \Delta) := \{m_0 K_X + \sum m_i D_i : m_i \in \mathbb{Z}\},$$

which is the largest possible.

It is very important that, by (11.35) and (11.34.1), if $K_X + \Delta$ is \mathbb{R} -Cartier then all elements of $\text{LEnv}_{\mathbb{Z}}(K_X + \Delta)$ are \mathbb{Q} -Cartier \mathbb{Z} -divisors. (K_X and the D_i may have other linear combinations that are \mathbb{Q} -Cartier \mathbb{Z} -divisors.)

The stable objects are the same as before, but the definition of stable families looks different.

8.12 (Stable objects). We parametrize marked pairs $(X, \Delta = \sum_i a_i D_i)$ with coefficient vector \mathbf{a} such that

(8.12.1) (X, Δ) is slc,

(8.12.2) X is projective and $K_X + \Delta$ is ample.

8.13 (Stable families). A family $f : (X, \Delta = \sum_i a_i D_i) \rightarrow S$ is *KSBA-stable* if the following hold.

- (8.13.1) $f : X \rightarrow S$ is flat, finite type, pure dimensional with demi-normal fibers.
- (8.13.2) The D_i are K-flat families of relative Mumford divisors (7.1).
- (8.13.3) The fibers (X_s, Δ_s) are slc.
- (8.13.4) $\omega_{X/S}^{[m_0]}(\sum m_i D_i)$ commutes with base change if $(m_0, \dots, m_r) \in \text{LEnv}_{\mathbb{Z}}(K_X + \Delta)$.
- (8.13.5) f is proper and $K_{X/S} + \Delta$ is f -ample.

The first 4 of these conditions define *locally KSBA-stable* families.

8.14 (Explanation). These conditions are mostly straightforward generalizations of (8.7), again the main question is assumption (3).

Note first that if the a_i are rational, then, by (8.11.1), $\text{LEnv}_{\mathbb{Z}}(K_X + \Delta)$ consists of the integer multiples of $\text{lcd}(\mathbf{a})(K_X + \Delta)$, so (8.13.3) specializes to (8.7.3).

If $1, a_1, \dots, a_r$ are \mathbb{Q} -linearly independent, then, by (8.11.2), we specialize to (6.38).

For the intermediate cases we follow the philosophy behind KSB stability as in Section 6.2: whenever we can prove that a divisorial sheaf commutes with base change over DVR's, we require this property over all schemes.

Working with all of $\text{LEnv}_{\mathbb{Z}}(K_X + \Delta)$ is (almost) necessary for our proof. We are using several rational perturbations of $K_X + \Delta$ to get enough ample \mathbb{Q} -divisors, and these span $\text{LEnv}_{\mathbb{Z}}(K_X + \Delta)$ (at least with \mathbb{Q} -coefficients).

The sheaves $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$ are not easy to understand. As we already noted in (8.8), they do not always commute with base change.

However, by (11.41), they commute with base change for infinitely many m , depending on the coefficient vector \mathbf{a} . Unfortunately, the method of (11.41) is ineffective, it is not at all clear how to produce such values m .

THEOREM 8.15. *KSBA-stability, as defined in (8.12–8.13), is a good moduli theory (6.10).*

Proof. We need to check the conditions (6.10.1–5).

Separatedness and valuative-properness (6.10.1–2) is as for (8.9).

Embedded moduli spaces (6.10.3) are worked out in (8.16).

For representability (6.10.4), we first use that K-flatness is representable by (7.3). We are left with dealing with condition (8.13.3). This is exactly as in the proof of (6.40).

Finally, here we know only weak boundedness (6.8.2), proved in (4.69). \square

8.16 (Construction of embedded moduli spaces). A way of approximating an \mathbb{R} -Cartier pair with \mathbb{Q} -Cartier pairs is given in (11.38).

Depending on the coefficient vector \mathbf{a} , we have \mathbb{Q} -linear maps $\sigma_j^m : \mathbb{R} \rightarrow \mathbb{Q}$, extended to divisors by $\sigma_j^m(\sum a_i D_i) := \sum \sigma_j^m(a_i) D_i$, with the following properties.

- (8.16.1) If $K_{X/S} + \Delta$ is \mathbb{R} -Cartier then the $K_{X/S} + \sigma_j^m(\Delta)$ are \mathbb{Q} -Cartier.
- (8.16.2) $\lim_{n \rightarrow \infty} \sigma_j^n(\Delta) = \Delta$.
- (8.16.3) If $(X, \Delta) \rightarrow S$ is stable then so are the $(X, \sigma_j^m(\Delta)) \rightarrow S$ for $m \gg 1$.
- (8.16.4) Δ is a convex \mathbb{R} -linear combination of the $\sigma_j^m(\Delta)$ for every fixed m .
- (8.16.5) The $\sigma_j^n(\Delta)$ are convex \mathbb{Q} -linear combination of the $\sigma_j^m(\Delta)$ for $n > m$.

Therefore, if $(X, \sigma_j^m(\Delta))$ are stable for every j then

(8.16.6) (X, Δ) is stable, and

(8.16.7) the $(X, \sigma_j^n(\Delta))$ are stable for every j and every $n > m$.

Now fix m and let $\mathcal{MSP}(\sigma_*^m(\mathbf{a}))$ be the moduli functor of all pairs (X, Δ) for which all the $(X, \sigma_j^m(\Delta))$ are stable. We claim that this is a good moduli theory. Indeed, first $(X, \sigma_0^m(\Delta))$ is a good moduli theory by (8.9). Then we have to add the conditions that the $K_{X/S} + \sigma_j^m(\Delta)$ are \mathbb{Q} -Cartier for $j \neq 0$; these are representable by (4.38). Finally, once the $K_{X/S} + \sigma_j^m(\Delta)$ are \mathbb{Q} -Cartier, ampleness of these is an open condition. Thus we have the moduli spaces $\text{MSP}(\sigma_*^m(\mathbf{a}))$. Using (11.3.4), the properties (6–7) say that $\text{MSP}(\mathbf{a})$ is the union of the infinite increasing chain

$$\dots \subset \text{MSP}(\sigma_*^m(\mathbf{a})) \subset \text{MSP}(\sigma_*^{m+1}(\mathbf{a})) \subset \dots$$

So far we have not specified the dimension and the volume. The dimension can be fixed, but the volume is murkier. Most likely, the volume of (X, Δ) does not determine the volume of $(X, \sigma_0^m(\Delta))$.

However, let us fix a connected component $M \subset \mathcal{MSP}(\mathbf{a})$. Since $K_{X/S} + \sigma_j^m(\Delta)$ is relatively Cartier, its self-intersection number on the fibers is constant. Although the set where $K_{X/S} + \sigma_j^m(\Delta)$ is ample may not be connected, the volume of these is constant, since it agrees with the self-intersection number. Thus there are volumes v_m such that the moduli space of M is contained in the infinite increasing chain

$$\dots \subset \text{MSP}(\sigma_*^m(\mathbf{a}), n, v_m) \subset \text{MSP}(\sigma_*^{m+1}(\mathbf{a}), n, v_{m+1}) \subset \dots \tag{8.16.8}$$

Let us now fix $(X_0, \Delta_0) \in M$ and choose m such that the $(X_0, \sigma_j^m(\Delta_0))$ are stable for every j . By (7), then the $(X_0, \sigma_j^{m'}(\Delta_0))$ are stable for every j and $m' \geq m$. Furthermore, the properties hold for every generalization of (X_0, Δ_0) . Thus the union (8.16.8) stabilizes in a neighborhood of (X_0, Δ_0) . Thus M is proper by (4.69). \square

REMARK 8.17. By (11.39), we can choose the σ_j^m (depending only on \mathbf{a} and the dimension) such that all the $(X, \sigma_j^m(\Delta))$ are slc.

As a consequence of the Strong boundedness theorem (6.8.1), we can choose the σ_j^m (depending only on \mathbf{a} , the dimension and the volume) such that all the $(X, \sigma_j^m(\Delta))$ are stable. However, the volume is really needed here.

To see this, take a series of projective surfaces S_{qn} with smooth curves B_{qn}, D_{qn}, D'_{qn} intersecting transversally as follows.

S_{qn} contains a unique singular point P_{qn} , it is of type $\mathbb{A}^2/\frac{1}{n}(1, q)$. The singular point lies on B_{qn} and the minimal resolution graph as in (3.1)

$$\bar{B}_{qn} \text{ --- } c_1 \text{ --- } \dots \text{ --- } c_d .$$

Assume also that \bar{B}_{qn} is a (-1) -curve, D_{qn} meets B_{qn} transversally at a single point, D'_{qn} is disjoint from B_{qn} and $K_{S_{qn}} + D'_{qn}$ is ample on $S_{qn} \setminus B_{qn}$. We compute that

$$(K_{S_{qn}} \cdot B_{qn}) = -1 + \frac{q+1}{n} .$$

If $\lambda > 1 - \frac{q+1}{n}$ then $K_{S_{qn}} + \lambda D_{qn} + D'_{qn}$ is ample, yet no smaller $\lambda' < \lambda$ has this property.

8.3. Polarized varieties

Assumptions. In this sections we work with arbitrary schemes. Because of functoriality, the situation over $\text{Spec } \mathbb{Z}$ determines everything.

8.18 (Ampleness conditions). Let X be a proper scheme over a field k and L a line bundle on X . The most important positivity notion is *ampleness*, but in connection with projective geometry the notion of *very ampleness* seems more relevant. If L is ample then L^r is very ample for $r \gg 1$ and there are numerous Matsusaka-type theorems that give effective control of the smallest such r [Mat72, LM75, KM83]. In practice, this will not be a major difficulty for us.

A problem with very ampleness is that it is not open in flat families (X_s, L_s) . Thus one needs to consider stronger variants. The two most frequently needed additional conditions are the following.

$$(8.18.1) \quad H^i(X, L) = 0 \text{ for } i > 0.$$

$$(8.18.2) \quad H^0(X, L) \text{ generates the ring } \sum_{r \geq 0} H^0(X, L^r).$$

These are connected by the notion of Castelnuovo-Mumford regularity; see [Laz04, Sec.I.8] for details.

For our purposes the relevant issue is (1). Thus we say that a line bundle L is *strongly ample* if it is very ample and $H^i(X, L^m) = 0$ for $i, m > 0$. By [Laz04, I.8.3], if this holds for all $m \leq \dim X + 1$ then it holds for all m . Thus strong ampleness is an open condition in flat families.

(*Comment.* We mostly need only that $H^i(X, L) = 0$ for $i > 0$. However, it is convenient to have that if L is strongly ample then so is L^m for $m > 1$.)

Let $f : X \rightarrow S$ be a proper, flat morphism and L a line bundle on X . We say that L is *strongly f -ample* or *strongly ample over S* if L is strongly ample on the fibers. Equivalently, if $R^i f_* L^m = 0$ for $i, m > 0$ and L is f -very ample. Thus $f_* L$ is locally free and we get an embedding $X \hookrightarrow \mathbb{P}_S(f_* L)$.

The main case for us is when $f : (X, \Delta) \rightarrow S$ is stable and $L = \omega_{X/S}^{[r]}(r\Delta)$ for some $r > 0$. If $r > 1$ then L is strongly f -ample by (11.33).

DEFINITION 8.19 (Polarization). A *polarized scheme* is a pair (X, L) consisting of a projective scheme X plus an ample line bundle L on X .

In the most basic version of the definition, a *polarized family of schemes* over a scheme S consists of a flat, projective morphism $f : X \rightarrow S$ plus a relatively ample line bundle L on X . (See (8.20) for other variants.)

We are interested only in the relative behavior of L , thus two families (X, L) and (X, L') are considered equivalent if there is a line bundle M on S such that $L \cong L' \otimes f^* M$. There are some quite subtle issues with this in general [Ray70], but if $H^0(X_s, \mathcal{O}_{X_s}) \cong k(s)$ for every $s \in S$, then $L \cong L' \otimes f^* M$ for some M iff $L|_{X_s} \cong L'|_{X_s}$ for every $s \in S$. (If S is reduced, this follows from Grauert's theorem as in [Har77, III.12.9], see (8.33) for the general case.) See also (8.20) for further comments on this.

For technical reasons it is more convenient to deal with the cases when, in addition, L is strongly f -ample (8.18). We call such an L a *strong polarization*. Thus the 'naive' functor of strongly polarized schemes

$$S \mapsto \mathcal{P}^s \text{Sch}(n, N)(S) \tag{8.19.1}$$

associates to a scheme S the equivalence classes of all $f : (X, L) \rightarrow S$ such that

- (8.19.2) f is flat, proper, of pure relative dimension n ,
 (8.19.3) X_s is pure and $H^0(X_s, \mathcal{O}_{X_s}) \cong k(s)$ for every $s \in S$,
 (8.19.4) L is strongly f -ample (8.18) and
 (8.19.5) f_*L is locally free of rank $N + 1$.

Note that, since L is flat over S , strong f -ampleness implies that f_*L is locally free.

We are ultimately interested in the cases when (3) is replaced by one of the following stronger conditions

- (8.19.3') X_s is geometrically connected and reduced for every $s \in S$.
 (8.19.3'') X_s is geometrically connected and demi-normal for every $s \in S$.

Claim 8.19.6. Let $f : X \rightarrow S$ be a flat, proper morphism and L a line bundle on X . Then there is a maximal open subscheme $S^\circ \subset S$ such that $f^\circ : (X^\circ, L^\circ) \rightarrow S^\circ$ satisfies the assumptions (2–5).

Proof. Having pure fibers is an open condition (10.2) and then pure dimensionality is an open condition. The rest is clear. \square

It is frequently more convenient to fix not just $n = \dim X$ and $N = h^0(X, L) - 1$ but the whole Hilbert polynomial $\chi(X, r) := \chi(X, L^r)$. This leads to the functor

$$S \mapsto \mathcal{P}^s \text{Sch}(\chi)(S). \quad (8.19.7)$$

If we also impose condition (3'), we write

$$S \mapsto \mathcal{P}^s \mathcal{V}(\chi)(S). \quad (8.19.8)$$

DEFINITION 8.20 (Pre-polarization). The above definition of polarization is geometrically clear, but it does not have the sheaf property. In analogy with the notion of a presheaf, we could define a *pre-polarization* of a projective morphism $f : X \rightarrow S$ to consist of

- (8.20.1) an open cover $\cup_i U_i \rightarrow S$ and
 (8.20.2) relatively ample line bundles L_i on $X_i := X \times_S U_i$
 such that, for every i, j , the restrictions of L_i and L_j to $X_{ij} := X \times_S U_i \times_S U_j$ are identified as in (8.19). (That is, there are line bundles M_{ij} on $U_i \times_S U_j$ such that $L_i|_{X_{ij}} \cong L_j|_{X_{ij}} \otimes f_{ij}^* M_{ij}$.)

Pre-polarizations form a presheaf and the ‘right’ notion of polarization should be a global section of the corresponding sheaf.

If $\cup_i U_i \rightarrow S$ is a cover by Zariski open subsets, the resulting notion is very similar to what we have in (8.19). The only difference is in property (8.19.5) since f_*L need not exist globally. However, $\mathbb{P}_S(f_*L)$ does exist as a Zariski locally trivial \mathbb{P}^N -bundle over S , and we almost always need to use $\mathbb{P}_S(f_*L)$ anyhow.

If the $U_i \rightarrow S$ are étale, then we still get an object $\mathbb{P}_S(f_*L) \rightarrow S$, but this is a Severi-Brauer scheme, that is, an étale locally trivial \mathbb{P}^N -bundle over S . (See (8.20.3) for an example with $N = 1$.) From the theoretical point of view, it is most natural to use the étale topology for the moduli space of varieties. Thus for the correct functorial notion we need to define the functors

$$S \mapsto \mathcal{P}^s \text{Sch}^{\text{ét}}(n, N)(S) \quad \text{and} \quad S \mapsto \mathcal{P}^s \text{Sch}^{\text{ét}}(\chi)(S), \quad (8.20.3)$$

obtained by sheafification, in the étale topology, of the corresponding functors of polarized schemes (8.19.1) and (8.19.7). (For arbitrary polarized schemes one needs even finer topologies, see [Ray70].)

For the difference between $\mathcal{P}^s\mathcal{S}ch^{et}$ and $\mathcal{P}^s\mathcal{S}ch$, a simple example to keep in mind is the following. Consider

$$X := (x^2 + sy^2 + tz^2 = 0) \subset \mathbb{P}_{xyz}^2 \times (\mathbb{A}_{st}^2 \setminus (st = 0)),$$

with coordinate projection to $S := \mathbb{A}_{st}^2 \setminus (st = 0)$. The fibers are all smooth conics. In the analytic or étale topology there is a pre-polarization whose restriction to each fiber is a degree 1 line bundle but there is no such line bundle on X . However, $\mathcal{O}_{\mathbb{P}^2}(1)$ gives a line bundle on X whose restriction to each fiber has degree 2.

We will, however, stick to the naive versions for several reasons.

(8.20.4) Stable families always come with a preferred polarizing line bundle, $\omega_{X/S}^{[m]}(m\Delta)$ for some $m > 0$.

(8.20.5) $\mathcal{P}^s\mathcal{S}ch^{et}$ and $\mathcal{P}^s\mathcal{S}ch$ have the same coarse moduli spaces (8.38.1).

(8.20.6) A suitable power of any pre-polarization naturally gives an actual polarization using (8.44.6).

So at the end the distinction between the functors $\mathcal{P}^s\mathcal{S}ch^{et}$ and $\mathcal{P}^s\mathcal{S}ch$ does not matter much for us. There is, however, another related notion that does lead to different coarse moduli spaces.

Numerically polarization 8.20.7. It consists of a flat, projective morphism $f : X \rightarrow S$ plus a relatively ample line bundle L on X , but with the difference that two families (X, L) and (X, L') are considered equivalent if $L_s \equiv L'_s$ for every geometric point $s \rightarrow S$. This is the original definition used by [Mat72], and it may be the most natural notion for general polarized pairs. Stable varieties come with an ample divisor, not just with an ample numerical equivalence class, which simplifies our task.

8.21 (Strongly embedded schemes). Fix a projective space $\mathbb{P}_{\mathbb{Z}}^N$. Over the Hilbert scheme there is a universal family, hence we get

$$\text{Univ}(\mathbb{P}_{\mathbb{Z}}^N) \subset \mathbb{P}_{\mathbb{Z}}^N \times \text{Hilb}(\mathbb{P}_{\mathbb{Z}}^N), \quad (8.21.1)$$

and $\mathcal{O}_{\mathbb{P}^N}(1)$ gives a polarization of $\text{Univ}(\mathbb{P}_{\mathbb{Z}}^N) \rightarrow \text{Hilb}(\mathbb{P}_{\mathbb{Z}}^N)$. By (8.19.6) there is a largest open subset

$$\text{Hilb}_n^{\text{str}}(\mathbb{P}_{\mathbb{Z}}^N) \subset \text{Hilb}_n(\mathbb{P}_{\mathbb{Z}}^N) \quad (8.21.2)$$

over which the polarization is strong (8.19.2–5). One should think of this as parametrizing pairs (X, L) that ‘naturally live’ in \mathbb{P}^N . The universal family restricts to

$$\text{Univ}_n^{\text{str}}(\mathbb{P}_{\mathbb{Z}}^N) \rightarrow \text{Hilb}_n^{\text{str}}(\mathbb{P}_{\mathbb{Z}}^N) \quad (8.21.3)$$

The corresponding functor associates to a scheme S the set of all flat families of closed subschemes of pure dimension n of \mathbb{P}_S^N

$$f : (X \subset \mathbb{P}_S^N; \mathcal{O}_X(1)) \rightarrow S, \quad (8.21.4)$$

where $\mathcal{O}_X(1)$ is strongly f -ample. Equivalently, we parametrize objects

$$\left(f : (X; L) \rightarrow S; \phi \in \text{Isom}_S(\mathbb{P}_S(f_*L), \mathbb{P}_S^N) \right) \quad (8.21.5)$$

consisting of a strongly polarized, flat families of purely n -dimensional schemes plus an isomorphism $\phi : \mathbb{P}_S(f_*L) \cong \mathbb{P}_S^N$. We call the latter a *projective framing* of f_*L or of L . We can also fix the Hilbert polynomial χ of X and consider the subschemes

$$\text{Univ}_{\chi}^{\text{str}}(\mathbb{P}_{\mathbb{Z}}^N) \rightarrow \text{Hilb}_{\chi}^{\text{str}}(\mathbb{P}_{\mathbb{Z}}^N) \subset \text{Hilb}_n^{\text{str}}(\mathbb{P}_{\mathbb{Z}}^N), \quad (8.21.6)$$

where $N = \chi(1) - 1$.

By the theory of Hilbert schemes, the spaces $\text{Hilb}_\chi^{\text{str}}(\mathbb{P}_\mathbb{Z}^N)$ are quasi-projective, though usually non-projective, reducible and disconnected; see [Gro62], [Kol96, Chap.I] or [Ser06].

We can summarize these discussions as follows.

PROPOSITION 8.22. *Fix a polynomial $\chi(t)$. Then*

$$\text{Univ}_\chi^{\text{str}}(\mathbb{P}_\mathbb{Z}^N) \rightarrow \text{Hilb}_\chi^{\text{str}}(\mathbb{P}_\mathbb{Z}^N)$$

constructed in (8.21) represents the functor of strongly polarized schemes with Hilbert polynomial χ and a projective framing. That is, for every scheme S , pull-back gives a one-to-one correspondence between

(8.22.1) *$\text{Mor}_\mathbb{Z}(S, \text{Hilb}_\chi^{\text{str}}(\mathbb{P}_\mathbb{Z}^N))$ and*

(8.22.2) *flat, projective families of purely n -dimensional schemes $f : X \rightarrow S$ with a strong polarization L with Hilbert polynomial χ , plus an isomorphism $\mathbb{P}_S(f_*L) \cong \mathbb{P}_S^N$, where $N + 1 = \chi(1)$. \square*

The general correspondence between the moduli of polarized varieties and the moduli of embedded varieties (8.38.1) gives now the following.

COROLLARY 8.23. *Fix a Hilbert polynomial χ with $N + 1 = \chi(1)$. Then the stack $[\text{Hilb}_\chi^{\text{str}}(\mathbb{P}^N)/\text{PGL}_{N+1}]$ represents the functor $\mathcal{P}^s\text{Sch}^{\text{et}}(\chi)$ defined in (8.20.3). \square*

8.24 (Marking points). So far we have studied varieties with marked divisors on them. It is sometimes useful to also mark some points. For curves the points are also divisors and they interact with the log canonical structure. By contrast, in dimension ≥ 2 , the points and the log canonical structure are independent of each other. This makes the resulting notion much less interesting theoretically, but it gives a quick and way to rigidify slc pairs, which was quite useful in Section 5.9.

A flat family of r -pointed schemes is a flat morphism $f : X \rightarrow S$ plus r sections $\sigma_i : S \rightarrow X$. This gives a functor of r -pointed schemes.

Consider the Hilbert scheme with its universal family $\text{Univ}(\mathbb{P}^N) \rightarrow \text{Hilb}(\mathbb{P}^N)$. Then the functor of r -pointed subschemes of \mathbb{P}^N is given by the r -fold fiber product

$$\text{Univ}(\mathbb{P}^N) \times_{\text{Hilb}(\mathbb{P}^N)} \text{Univ}(\mathbb{P}^N) \cdots \times_{\text{Hilb}(\mathbb{P}^N)} \text{Univ}(\mathbb{P}^N).$$

More generally, for any functor that is representable by a flat universal family $\text{Univ}_M \rightarrow M$, its r -pointed version is representable by the r -fold fiber product of Univ_M over M .

In particular, we get MpSP, the moduli of pointed stable pairs.

8.4. Canonically embedded pairs

Assumptions. In this sections we work with \mathbb{Q} -schemes. This is necessary since K -flatness is developed only over \mathbb{Q} . The other parts of the arguments work over \mathbb{Z} . Because of functoriality, the situation over $\text{Spec } \mathbb{Q}$ determines everything.

DEFINITION 8.25. A *strongly polarized family of schemes marked with K -flat divisors* is written as

$$f : (X; D^1, \dots, D^r; L) \rightarrow S, \tag{8.25.1}$$

where

(8.25.2) $f : X \rightarrow S$ satisfies (8.19.2–5),

(8.25.3) the D^i are K-flat families of relative Mumford divisors (7.1), and

(8.25.4) L is a strong polarization (8.19).

If we fix the relative dimension and the rank of f_*L , then, as in (8.19.7–8), we get the functors

$$\mathcal{P}^s\mathcal{MV}(r, n, N) \subset \mathcal{P}^s\mathcal{MSch}(r, n, N). \quad (8.25.5)$$

We write $\mathcal{P}^s\mathcal{MV}(r, \chi)$ and $\mathcal{P}^s\mathcal{MSch}(r, \chi)$ if the Hilbert polynomial $\chi = \chi(X_s, L_s^m)$ of L is also fixed. These can also be sheafified in the étale topology as in (8.20.3).

Warning. Our notation does not indicate K-flatness; but it has enough letters in it already.

The embedded versions are denoted by

$$\mathcal{E}^s\mathcal{MV}(r, n, \mathbb{P}^N) \quad \text{and} \quad \mathcal{E}^s\mathcal{MSch}(r, n, \mathbb{P}^N). \quad (8.25.6)$$

These functors associate to a scheme S the set of all families of closed subschemes of a given \mathbb{P}_S^N (where $N = \chi(1) - 1$) marked with K-flat divisors

$$f : (X \subset \mathbb{P}_S^N; D^1, \dots, D^r; \mathcal{O}_X(1)) \rightarrow S \quad (8.25.7)$$

where $\mathcal{O}_X(1)$ is strongly ample.

Equivalently, we can view $\mathcal{E}^s\mathcal{MSch}(r, n, \mathbb{P}^N)$ as parametrizing objects

$$\left(f : (X; D^1, \dots, D^m; L) \rightarrow S; \phi \in \text{Isom}_S(\mathbb{P}_S(f_*L), \mathbb{P}_S^N) \right) \quad (8.25.8)$$

consisting of a strongly polarized family of varieties marked with K-flat divisors, plus a projective framing $\phi : \mathbb{P}_S(f_*L) \cong \mathbb{P}_S^N$ as in (8.21.5).

8.26 (Universal family of strongly embedded, marked schemes). Fix a projective space $\mathbb{P}_{\mathbb{Q}}^N$ and integers $n \geq 1$ and $r \geq 0$. By (8.21) we have a universal family of strongly embedded schemes

$$\text{Univ}_n^{\text{str}}(\mathbb{P}_{\mathbb{Q}}^N) \rightarrow \text{Hilb}_n^{\text{str}}(\mathbb{P}_{\mathbb{Q}}^N) \quad (8.26.1)$$

satisfying (8.19.2–5). The universal family of K-flat, Mumford divisors on them

$$\text{KDiv}(\text{Univ}_n^{\text{str}}(\mathbb{P}_{\mathbb{Q}}^N) / \text{Hilb}_n^{\text{str}}(\mathbb{P}_{\mathbb{Q}}^N)) \rightarrow \text{Hilb}_n^{\text{str}}(\mathbb{P}_{\mathbb{Q}}^N)$$

was constructed in (7.3). If we need r such divisors, the universal family we want is given by the r -fold fiber product

$$\text{E}^s\text{MSch}(r, n, \mathbb{P}_{\mathbb{Q}}^N) := \times_{\text{Hilb}_n(\mathbb{P}_{\mathbb{Q}}^N)}^r \text{KDiv}(\text{Univ}_n^{\text{str}}(\mathbb{P}_{\mathbb{Q}}^N) / \text{Hilb}_n^{\text{str}}(\mathbb{P}_{\mathbb{Q}}^N)). \quad (8.26.2)$$

Over $\text{E}^s\text{MSch}(r, n, \mathbb{P}_{\mathbb{Q}}^N)$ we have a universal family of strongly polarized schemes marked with K-flat divisors

$$\mathbf{F} : (\mathbf{X}, \mathbf{D}^1, \dots, \mathbf{D}^r; \mathbf{L}) \rightarrow \text{E}^s\text{MSch}(r, n, \mathbb{P}_{\mathbb{Q}}^N), \quad (8.26.3)$$

where we really should have written

$$\left(\mathbf{X}(r, n, \mathbb{P}_{\mathbb{Q}}^N), \mathbf{D}^1(r, n, \mathbb{P}_{\mathbb{Q}}^N), \dots, \mathbf{D}^r(r, n, \mathbb{P}_{\mathbb{Q}}^N); \mathbf{L}(r, n, \mathbb{P}_{\mathbb{Q}}^N) \right)$$

but the latter is rather cumbersome.

It is clear from the construction that the spaces $\text{E}^s\text{MSch}(r, n, \mathbb{P}_{\mathbb{Q}}^N)$ parametrize polarized families of varieties marked with divisors, where the varieties are equipped with an extra framing.

PROPOSITION 8.27. *Fix r, n, N . Then the scheme of embedded, marked varieties $\mathbf{E}^s\mathbf{MSch}(r, n, \mathbb{P}_{\mathbb{Q}}^N)$ constructed in (8.26.3) represents the functor $\mathcal{E}^s\mathcal{MSch}(r, n, \mathbb{P}_{\mathbb{Q}}^N)$, defined in (8.25). That is, for every \mathbb{Q} -scheme S , pulling back the family (8.26.3) gives a one-to-one correspondence between*

(8.27.1) $\text{Mor}_{\mathbb{Q}}(S, \mathbf{E}^s\mathbf{MSch}(r, n, \mathbb{P}_{\mathbb{Q}}^N))$, and

(8.27.2) families $f : (X; D^1, \dots, D^r; L) \rightarrow S$ of n -dimensional schemes, with a strong polarization and marked with K -flat Mumford divisors, plus a projective framing $\mathbb{P}_S(f_*L) \cong \mathbb{P}_S^N$. \square

As in (8.23) and (8.38.1), this implies the following.

COROLLARY 8.28. *Fix n, m, N . Then the stack*

$$[\mathbf{E}^s\mathbf{MSch}(r, n, \mathbb{P}_{\mathbb{Q}}^N)/\text{PGL}_{N+1}]$$

represents the functor $\mathcal{P}^s\mathcal{MSch}(r, n, N)$, defined in (8.25). \square

8.29 (Boundedness conditions). The schemes $\mathbf{E}^s\mathbf{MSch}(r, n, \mathbb{P}^N)$ have infinitely many irreducible components since we have not fixed the degrees of X and of the divisors D^i . Set

$$\text{deg}_L(X; D^1, \dots, D^r) := (\text{deg}_L X, \text{deg}_L D^1, \dots, \text{deg}_L D^r) \in \mathbb{N}^{r+1}. \quad (8.29.1)$$

This multidegree is a locally constant function on $\mathbf{E}^s\mathbf{MSch}(r, n, \mathbb{P}^N)$, hence its level sets give a decomposition

$$\mathbf{E}^s\mathbf{MSch}(r, n, \mathbb{P}^N) = \coprod_{\mathbf{d} \in \mathbb{N}^{r+1}} \mathbf{E}^s\mathbf{MSch}(r, n, \mathbf{d}, \mathbb{P}^N). \quad (8.29.2)$$

The schemes $\mathbf{E}^s\mathbf{MSch}(r, n, \mathbf{d}, \mathbb{P}^N)$ are still not of finite type since the fibers are allowed to be nonreduced. However, the subscheme

$$\mathbf{E}^s\mathbf{MV}(r, n, \mathbf{d}, \mathbb{P}^N) \subset \mathbf{E}^s\mathbf{MSch}(r, n, \mathbf{d}, \mathbb{P}^N), \quad (8.29.3)$$

which parametrizes geometrically reduced fibers is quasi-projective, though usually non-projective, reducible and disconnected.

DEFINITION 8.30. A family of marked pairs $f : (X, \Delta) \rightarrow S$ as in (8.4) is *m -canonically strongly polarized* if

(8.30.1) $\omega_{X/S}$ is locally free outside a codimension ≥ 2 subset of each fiber, and

(8.30.2) $\omega_{X/S}^{[m]}(m\Delta)$ is a line bundle, and

(8.30.3) $\omega_{X/S}^{[m]}(m\Delta)$ is a strong polarization.

If $X \subset \mathbb{P}_S^N$ then $f : (X, \Delta) \rightarrow S$ is *m -canonically strongly embedded* if, in addition,

(8.30.4) $\omega_{X/S}^{[m]}(m\Delta) \cong \mathcal{O}_{\mathbb{P}^N}(1) \otimes f^*M_S$ for some line bundle M_S on S .

These define the functors $\mathcal{C}^m\mathcal{P}^s\mathcal{MSch}$ and $\mathcal{C}^m\mathcal{E}^s\mathcal{MSch}$.

THEOREM 8.31. *Fix $r, m, n, N \in \mathbb{N}$ and a rational coefficient vector $\mathbf{a} = (a_1, \dots, a_r)$. Then the functor $\mathcal{C}^m\mathcal{E}^s\mathcal{MSch}(\mathbf{a}, r, n, \mathbb{P}^N)$ is represented by a monomorphism*

$$\mathcal{C}^m\mathbf{E}^s\mathbf{MSch}(\mathbf{a}, r, n, \mathbb{P}_{\mathbb{Q}}^N) \rightarrow \mathbf{E}^s\mathbf{MSch}(r, n, \mathbb{P}_{\mathbb{Q}}^N)$$

Proof. Start with the universal family

$$\mathbf{F} : (\mathbf{X}, \mathbf{D}^1, \dots, \mathbf{D}^r; \mathbf{L}) \rightarrow \mathbf{E}^s\mathbf{MSch}(r, n, \mathbb{P}_{\mathbb{Q}}^N),$$

as in (8.26.3).

Note that (8.30.1) is an open condition and it holds iff ω_{X_s} is locally free outside a codimension ≥ 2 of X_s for every $s \in S$. Being a line bundle is representable by (4.36) and, once it holds, being a strong polarization is an open condition.

Finally applying (8.34) to $\omega_{X/S}^{[m]}(m\Delta)(-1)$ shows that condition (8.30.4) is representable. \square

By (4.52), if $K_{X/S} + \Delta$ is \mathbb{Q} -Cartier then the stable fibers are parametrized by an open subset. Thus we get the following.

COROLLARY 8.32. *Fix $r, m, n, N \in \mathbb{N}$ and a rational coefficient vector $\mathbf{a} = (a_1, \dots, a_r)$. Then there is open subscheme*

$$\mathrm{C}^m\mathrm{ESP}(\mathbf{a}, r, n, \mathbb{P}_{\mathbb{Q}}^N) \subset \mathrm{C}^m\mathrm{E}^s\mathrm{MSch}(\mathbf{a}, r, n, \mathbb{P}_{\mathbb{Q}}^N)$$

that represents the functor of m -canonically strongly embedded families with stable fibers.

Warning 8.32.1. The reduced subspace of $\mathrm{C}^m\mathrm{ESP}$ is the correct one, but its scheme structure is still a little too large. The reason is that (8.7.3) imposes restrictions on $\omega_{X/S}^{[r]}(r\Delta)$ for various values of r , and we took care only of our chosen m (and its multiples).

We dropped the superscript from E^s since, as we noted in (8.18), an m -canonical polarization is automatically strong.

LEMMA 8.33. *Let $f : X \rightarrow S$ be a proper morphism. Assume that, for every $s \in S$, f is flat at some point of $f^{-1}(s)$ and $H^0(X_s, \mathcal{O}_{X_s}) \cong k(s)$. Then $f_*\mathcal{O}_X = \mathcal{O}_S$.*

Proof. The question is local on S and after a flat base change we may assume that there is a section $S \cong Z \subset X$. Then $\mathcal{O}_S \rightarrow f_*\mathcal{O}_X \rightarrow f_*\mathcal{O}_Z$ shows that $\mathcal{O}_S \rightarrow f_*\mathcal{O}_X$ is a split injection. It is also a surjection, hence an isomorphism. \square

LEMMA 8.34. *Let $f : X \rightarrow S$ be a proper morphism as in (8.33) and G a coherent sheaf on X , flat over S .*

Then there is a largest locally closed subscheme $S' \hookrightarrow S$ with preimage $X' \subset X$, such that $G|_{X'}$ is isomorphic to the pull-back of line bundle from S' .

Proof. By a version of the semicontinuity theorem there is a finite complex of locally free sheaves on S

$$K^\bullet := 0 \rightarrow K^0 \xrightarrow{d_1} K^1 \xrightarrow{d_2} \dots \xrightarrow{d_{n-1}} K^n \rightarrow 0,$$

such that, for every morphism $h : T \rightarrow S$,

$$R^i(f_T)_*h_X^*G \cong H^i(h^*K^\bullet);$$

where we use the notation of (9.66.2). (This form is stated and proved in [Mum70, §5]; [Har77, III.12.2] has a weaker statement but the proof works to give this.)

Locally we can write d_1 as a matrix with entries in \mathcal{O}_S . Then $(\mathrm{rank} d_1 \leq r) \subset S$ is the subscheme defined by the vanishing of the determinants of all $(r+1) \times (r+1)$ -minors. With this definition we see that

$$S_1 := (\mathrm{rank} d_1 \leq \mathrm{rank} K^0 - 1) \setminus (\mathrm{rank} d_1 \leq \mathrm{rank} K^0 - 2)$$

represents the functor of those base changes $h : T \rightarrow S$ for which $(f_T)_*h_X^*G$ is a line bundle. We can thus base change and obtain $f_1 : X_1 \rightarrow S_1$ such that $(f_1)_*G_1$ is

a line bundle. It remains to understand when the natural map $\phi : f_1^*(f_1)_*G_1 \rightarrow G_1$ is an isomorphism. Let $X_2 \subset X_1$ be the set of points $x \in X_1$ such that

$$\phi_s : (f_1^*(f_1)_*G_1)|_{X_s} \rightarrow G|_{X_s}$$

is an isomorphism at x , where $s = f_1(x)$. By [Kol196, I.7.4.1], X_2 is an open subset of X_1 and $\phi|_{X_1}$ is an isomorphism. Thus

$$S' = S_1 \setminus f_1(\text{Supp}(X_1 \setminus X_2)). \quad \square$$

8.5. Moduli spaces as quotients by group actions

NOTATION 8.35. For a scheme S , we use $\text{PGL}_n(S)$ to denote the group scheme PGL_n over S . We will formulate definitions and results for general algebraic group schemes whenever possible, but in the applications we use only PGL_n , which is smooth and reductive over \mathbb{Q} .

Keep in mind that, if k is field, then in the literature $\text{PGL}_n(k)$ usually denotes the k -points of the group scheme PGL_n , not $\text{PGL}_n(\text{Spec } k)$. It is customary to use PGL_n to denote $\text{PGL}_n(\text{Spec } \mathbb{Z})$ if we work with arbitrary schemes, and $\text{PGL}_n(\text{Spec } \mathbb{Q})$ if we work in characteristic 0.

The same conventions apply to all classical named groups.

8.36 (Comment on algebraic spaces). We will consider quotients of schemes by algebraic groups, primarily PGL_n . It turns out that in many cases such quotients are not schemes, but algebraic spaces. For this reason, it is natural to formulate the basic definitions using algebraic spaces.

In our cases, these quotients turn out to be schemes, even projective, but this is not easy to prove.

In any case, this means that the reader can substitute ‘scheme’ for ‘algebraic space’ in the sequel, without effecting the final theorems.

DEFINITION 8.37. An action of an algebraic group scheme G on an algebraic space X is a morphism $\mu : G \times X \rightarrow X$ that satisfies the scheme theoretic version of the condition $g_1(g_2(x)) = (g_1g_2)(x)$. That is, the following diagram commutes.

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{1_G \times \mu} & G \times X \\ \mu \times 1_X \downarrow & & \downarrow \mu \\ G \times X & \xrightarrow{\mu} & X. \end{array}$$

If G acts on X_1, X_2 then $\pi : X_1 \rightarrow X_2$ is a G -morphism if the following diagram commutes.

$$\begin{array}{ccc} G \times X_1 & \xrightarrow{\mu_1} & X_1 \\ 1_G \times \pi \downarrow & & \downarrow \pi \\ G \times X_2 & \xrightarrow{\mu_2} & X_2. \end{array}$$

The *categorical quotient* is a G -morphism $q : X \rightarrow Y$ such that the G -action is trivial on Y , and q is universal among such.

Fix N and consider the functor $\mathcal{P}^s\text{Sch}(N)$ of strongly polarized schemes of embedding dimension N . By (8.22), its embedded version has a moduli space with a universal family $\text{Univ}^{\text{str}}(\mathbb{P}^N) \rightarrow \text{Hilb}^{\text{str}}(\mathbb{P}^N)$. The connection between the 2 versions is the following impressive sounding but quite simple claim.

THEOREM 8.38. *The categorical quotient $\text{Hilb}^{\text{str}}(\mathbb{P}^N)/\text{PGL}_{N+1}$ is also the categorical moduli space $\mathcal{P}^s\text{Sch}(n, N)$.*

Proof. We have a universal family over $\mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N)$, so we get $\mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N) \rightarrow \mathcal{P}^{\mathrm{s}}\mathrm{Sch}(n, N)$ which is PGL_{N+1} -equivariant.

Conversely, let $f : (X, L) \rightarrow S$ be a family in $\mathcal{P}^{\mathrm{s}}\mathrm{Sch}(n, N)$. Then f_*L is locally free on rank $N + 1$ on S , hence S has an open cover $S = \cup S_i$ such that each $f_*L|_{S_i}$ is free. Choosing a trivialization gives embedded families, hence morphisms

$$\phi_i : S_i \rightarrow \mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N).$$

Over $S_i \cap S_j$ we have 2 different trivializations, these differ by a section of $g_{ij} \in H^0(S_i \cap S_j, \mathrm{PGL}_{N+1})$. Thus, composing with the quotient map

$$q : \mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N) \rightarrow \mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N)/\mathrm{PGL}_{N+1}$$

we get that $q \circ (\phi_i|_{S_i \cap S_j}) = q \circ (g_{ij}(\phi_j|_{S_i \cap S_j})) = q \circ (\phi_j|_{S_i \cap S_j})$, since q is PGL_{N+1} -equivariant. Thus the $q \circ \phi_i$ glue to a morphism

$$\phi : S \rightarrow \mathrm{Hilb}^{\mathrm{str}}(\mathbb{P}^N)/\mathrm{PGL}_{N+1}. \quad \square$$

Remark 8.38.1. Since one can glue a morphism form étale charts, we see that $\mathcal{P}^{\mathrm{s}}\mathrm{Sch}^{\mathrm{et}}$ and $\mathcal{P}^{\mathrm{s}}\mathrm{Sch}$ have the same categorical moduli spaces (8.20.5).

Remark 8.38.1. For those conversant with stacks, the above argument proves (8.23) and (8.28).

Existence of quotients.

Let G be an algebraic group acting on an algebraic space X . Under very mild conditions the categorical quotient X/G exists, but it may be very degenerate.

EXAMPLE 8.39. Consider \mathbb{A}_k^n with the scalar \mathbb{G}_m -action $(x_1, \dots, x_n) \mapsto (\lambda x_1, \dots, \lambda x_n)$. Since the origin is in the closure of every orbit, we get that $\mathbb{A}^n/\mathbb{G}_m = \mathrm{Spec} k$.

By contrast, $(\mathbb{A}^n \setminus \{\mathbf{0}\})/\mathbb{G}_m = \mathbb{P}^{n-1}$.

Note that here the stabilizer is \mathbb{G}_m for the origin, but trivial for every other point. This and many other examples suggest that points with infinite stabilizer cause problems.

With PGL_{N+1} acting on the Hilbert scheme, the stabilizer of the point $[X]$ corresponding to a strongly embedded $X \subset \mathbb{P}^N$ is the automorphism group of the polarized scheme $(X, \mathcal{O}_X(1))$. As we saw in Section 1.8, infinite automorphism groups cause many problems.

We get the best results if all automorphism groups are trivial; we discuss these in Section 8.6. For stable pairs the automorphism groups are finite, but we need a scheme-theoretic version of this.

DEFINITION 8.40 (Proper action). Let $m : G \times X \rightarrow X$ be an algebraic group scheme acting on an algebraic space X . Combining m with the coordinate projection to X gives

$$(m, \pi_X) : G \times X \rightarrow X \times X.$$

The action is called *proper* if (m, π_X) is proper and *free* if (m, π_X) is a closed embedding. Note that the preimage of a diagonal point (x, x) is the stabilizer of x . Thus free implies that all stabilizers trivial and, if G is affine (for example PGL) then proper implies that all stabilizers are finite. (The converses are, however, not true; see [Mum65, p.11].)

A quick contemplation shows that for embedded stable pairs, the properness of the PGL_{N+1} -action is equivalent to the uniqueness of stable extensions considered in (2.48) (and called separatedness there). (This clash of terminologies is, unfortunately, well entrenched.)

Now we come to the definition of the right class of quotients.

DEFINITION 8.41. [Mum65, p.4] Let G be an algebraic group scheme acting on an algebraic space X with categorical quotient $q : X \rightarrow X/G$. It is called a *geometric quotient* if

(8.41.1) if K is an algebraically closed field then $q(K) : X(K)/G(K) \rightarrow (X/G)(K)$ is a bijection of sets.

(8.41.2) q is of finite type and universally surjective, and

(8.41.3) $\mathcal{O}_{X/G} = (q_*\mathcal{O}_X)^G$.

The geometric quotient is denoted by $X//G$.

The fundamental theorem for the existence of geometric quotients is the following. Seshadri came close to proving it in [Ses63, Ses72], his ideas were developed in [Kol97] to settle many cases, including PGL that we need. The general case was treated in [KM97]; see [Ols16] for a thorough treatment.

THEOREM 8.42. *Let G be a flat group scheme acting properly on an algebraic space X . Then the geometric quotient $X//G$ exists.* \square

For free actions, the quotient map is especially simple. Over fields, this is proved in [Mum65, Prop.0.9]. The general case follows from [Sta15, Tag 0CQJ].

COMPLEMENT 8.43. *Assume in addition that the G -action is free on X . Then $X \rightarrow X//G$ is a principal G -bundle.* \square

8.6. Descent

Let $q : S' \rightarrow S$ be a morphism of schemes and assume that we have an object over S' . We say that the object *descends* to S if it is the pull-backs of an object on S . Typical examples are

- a (quasi)coherent sheaf F' , in which case we want to get a (quasi)coherent sheaf F on S such that $F' \cong q^*F$, or
- a morphism $X' \rightarrow S'$, in which case we want to get a morphism $X \rightarrow S$ such that $X' \cong X \times_S S'$.

A systematic theory was developed in [Gro62, Lec.1], treating the case when $S' \rightarrow S$ is faithfully flat; see also [Gro71, Chap.VIII], [BLR90, Chap.6] or [Sta15, Tag 03O6] for more detailed treatments. We discuss the basic idea during the proof of (8.51).

Here we discuss the consequences of descent theory for the moduli of stable pairs; the main one is (8.53). We also prove some special cases that are representative of the general theory, yet can be obtained by simpler methods.

8.44 (Functorial polarization). [Kol90] Let \mathcal{F} be a subfunctor of \mathcal{P}^*Sch . A *functorial polarization* (of level r) of \mathcal{F} consists of the following:

(8.44.1) For every $(f : (X, L) \rightarrow S) \in \mathcal{F}(S)$ another $(f : (X, \bar{L}) \rightarrow S) \in \mathcal{F}(S)$ such that \bar{L} is equivalent to L^r ,

(8.44.2) for any morphism $q : S' \rightarrow S$ an isomorphism $\sigma(q) : q_X^*(\bar{L}) \cong \overline{(q_X^*L)}$, such that

(8.44.3) $\sigma(q \circ q') = \sigma(q') \circ (q'_X)^* \sigma(q)$ for every $q' : S'' \rightarrow S'$ and $q : S' \rightarrow S$

Note that in (2) we need to choose an isomorphism, it is not enough to say that the 2 sides are isomorphic.

If the choice of \bar{L} is specified, then we say that \mathcal{F} is *functorially polarized*.

The following are examples of functorial polarizations.

(8.44.4) If $L_s \cong \omega_{X_s}$ for every $s \in S$, then $\bar{L} := \omega_{X/S}$ is a functorial polarization.

(8.44.5) If every family in \mathcal{F} has a natural section $\sigma : S \rightarrow X$, then we can take $\bar{L} := L \otimes f^*(\sigma^*L)^{-1}$. This applies, for instance, to pointed varieties and (depending on our definition) to polarized abelian varieties.

(8.44.6) Assume that $r := \chi(X_s, L_s)$ is constant and positive for every $(X_s, L_s) \in \mathcal{F}$. Then, using the notation of (8.45.1),

$$\bar{L} := L^r \otimes f^*(\det R^*f_*L)^{-1}.$$

is a level r functorial polarization.

An important non-example is the following.

(8.44.7) $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ does not have a functorial polarization of level 1, since that would lead to a nontrivial representation of $\mathrm{PGL}_2 = \mathrm{Aut}(\mathbb{P}^1)$ on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \cong k^2$. On the other hand, $(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{-1})$ gives a functorial polarization of level 2.

Functorial polarizations also give natural line bundles on the base spaces of families. Let \mathcal{F} be a functorially polarized subfunctor of $\mathcal{P}Sch$. For any $(f : (X, \bar{L}) \rightarrow S) \in \mathcal{F}(S)$ we get the line bundle

$$\det R^*f_*(\bar{L}^{\otimes k}). \quad (8.44.8)$$

For $k \gg 1$ it is given by the simpler formula $\det f_*(\bar{L}^{\otimes k})$.

These line bundles are functorial for base changes, thus they give line bundles on the moduli stack of \mathcal{F} .

DEFINITION 8.45. Let $g : U \rightarrow V$ be a proper morphism, F a coherent sheaf on U , flat over V . By [Mum70, §5] there is a finite complex of locally free sheaves

$$K^\bullet := 0 \rightarrow K^0 \rightarrow K^1 \rightarrow \cdots \rightarrow K^n \rightarrow 0,$$

such that $H^i(K^\bullet) \cong R^i g_* F$ (and similarly for any base change). The line bundle

$$\det R^*g_*F := \left(\prod_{\text{even}} \det K^i\right) \otimes \left(\prod_{\text{odd}} \det K^i\right)^{-1} \quad (8.45.1)$$

is independent of the choices made (this is left as an exercise).

The construction is especially simple if $R^i g_* F = 0$ for $i > 0$. Then $g_* F$ is locally free, so $\det R^*g_*F = \det g_*F$. This is the main case that we use.

As we already saw in Section 1.8, varieties with nontrivial automorphisms can have a very complicated moduli theory. The rest of the sections is devoted to illustrating Principle 1.71: pairs without extra automorphisms have fine moduli spaces.

This needs some definitions and general results first.

8.46 (Morphism schemes). For S -schemes X, Y let $\mathrm{Mor}_S(X, Y)$ be the set of morphisms that commute with projections to S . We get the functor of morphisms on S -schemes $T \mapsto \mathrm{Mor}_T(X_T, Y_T)$.

Claim 8.46.1. Assume that $X \rightarrow S$ is flat, proper, and $Y \rightarrow S$ is of finite type. Then the functor of morphisms is representable by a scheme $\mathrm{Mor}_S(X, Y)$.

Proof. We can identify a morphism with its graph in $\mathrm{Hilb}_S(X \times_S Y)$, and a subscheme $Z \subset X \times_S Y$ is the graph of a morphism iff the first projection

$\pi_X : Z \rightarrow X$ is finite and $(\pi_X)_* \mathcal{O}_Z \cong \mathcal{O}_X$. The first of these is always an open condition, for the second we need the flatness of $X \rightarrow S$, cf. [Kol96, I.7.4.1]. \square

As special cases we also get $\text{Isom}_S(X, Y)$ and $\text{Aut}_S(X)$ that represent the functor of isomorphisms and automorphisms. The identity is always in automorphism, thus we have the identity section $S \subset \text{Aut}_S(X)$. We say that X is *rigid* (over S) if $S = \text{Aut}_S(X)$.

The definitions of Mor, Isom, Aut also apply to pairs.

With the definition of stable families in place, we get the following consequence of (2.47) about isomorphism schemes.

PROPOSITION 8.47. *Let $f_i : (X_i, \Delta_i) \rightarrow S$ be stable morphisms. Then the structure map $\text{Isom}_S((X_1, \Delta_1), (X_2, \Delta_2)) \rightarrow S$ is finite.*

Proof. Choose m such that $m(K_{X_i/S} + \Delta_i)$ are very ample and set $F_i := (f_i)_* \mathcal{O}_{X_i}(mK_{X_i/S} + m\Delta_i)$. Then

$$\text{Isom}_S((X_1, \Delta_1), (X_2, \Delta_2)) \subset \text{Isom}_S(\mathbb{P}_S(F_1), \mathbb{P}_S(F_2))$$

is closed, hence affine over S .

Let T be the spectrum of a DVR over k with generic point t_g and $\phi_g : t_g \rightarrow \text{Isom}_S((X_1, \Delta_1), (X_2, \Delta_2))$ a morphism. We can view it as an isomorphism of the generic fibers

$$\phi_g : (X_1, \Delta_1) \times_S \{t_g\} \cong (X_2, \Delta_2) \times_S \{t_g\}.$$

By (2.48), ϕ_g extends uniquely to an isomorphism

$$\Phi : (X_1, \Delta_1) \times_S T \cong (X_2, \Delta_2) \times_S T.$$

This is exactly the valuative criterion of properness for $\text{Isom}_S((X_1, \Delta_1), (X_2, \Delta_2))$. Thus $\text{Isom}_S((X_1, \Delta_1), (X_2, \Delta_2))$ is both affine and proper, hence finite. \square

Next we verify (1.72.1) for stable pairs.

COROLLARY 8.48. *Let $f : (X, \Delta) \rightarrow S$ be a stable morphism. Then*

- (8.48.1) *the structure map $\pi : \text{Aut}_S(X, \Delta) \rightarrow S$ is finite,*
- (8.48.2) *the subset $S^\circ \subset S$ of rigid fibers is open,*
- (8.48.3) *$\text{Aut}_S(X, \Delta) = S$ iff $\text{Aut}(X_s, \Delta_s)$ is trivial for every geometric point $s \rightarrow S$.*
- (8.48.4) *if the characteristic is 0, then $\text{Aut}_S(X, \Delta) = S$ iff the automorphism group of (X_s, Δ_s) is trivial for every geometric point $s \rightarrow S$.*

Proof. Finiteness follows from (8.47). The identity section gives that \mathcal{O}_S is a direct summand of $\pi_* \mathcal{O}_{\text{Aut}_S(X, \Delta)}$. Thus S° is the complement of the support of $\pi_* \mathcal{O}_{\text{Aut}_S(X, \Delta)} / \mathcal{O}_S$. The fibers of $\text{Aut}_S(X, \Delta) \rightarrow S$ are the schemes $\text{Aut}(X_s, \Delta_s)$. Group schemes are smooth in characteristic 0, hence then $\text{Aut}(X_s, \Delta_s)$ is the same as the automorphism group. \square

Uniqueness of descent now follows easily.

PROPOSITION 8.49. *Let $S' \rightarrow S$ be a faithfully flat morphism and $X' \rightarrow S'$ a flat, proper morphism such that X' is rigid over S' . Then there is at most 1 scheme $X \rightarrow S$ such that $X' \cong X \times_S S'$.*

Proof. Assume that we have $X_1 \rightarrow S$ and $X_2 \rightarrow S$. Since the $X_i \times_S S' \cong X'$ are flat and proper, so are $X_i \rightarrow S$. We aim to prove that $\text{Isom}_S(X_1, X_2) \cong S$. To see this take any $T \rightarrow S'$ and note that

$$T = \text{Isom}_T(X'_T, X'_T) = \text{Isom}_T(X_1 \times_S T, X_2 \times_S T) = \text{Mor}_S(T, \text{Isom}_S(X_1, X_2)).$$

Thus $\text{Isom}_S(X_1, X_2) \rightarrow S$ is a surjective, proper, monomorphism, hence an isomorphism. \square

The simplest descent result is the following addition to (1.76.5).

LEMMA 8.50. *Let K/k be a finite, separable field extension and (X, L) a rigid, functorially polarized, projective variety defined over K . Then (X, L) descends to k iff $(X, L) \cong (X^\sigma, L^\sigma)$ for every $\sigma \in \text{Gal}(K/k)$.*

Proof. We may assume that K/k is Galois and then only the $\sigma \in \text{Gal}(K/k)$ matter. We get an action of $\text{Gal}(K/k)$ on $H^0(X, L)$ by

$$H^0(X, L) \xrightarrow{\sigma\text{-lin}} H^0(X^\sigma, L^\sigma) \xrightarrow{K\text{-isom}} H^0(X, L).$$

This is well defined since the K -isomorphism is unique, even on L . By the fundamental lemma on quasilinear maps (see [Sha74, Sec.A.3]) there is a unique k -subspace $V(X, L) \subset H^0(X, L)$ such that $V(X, L) \otimes_k K = H^0(X, L)$.

Since $X = \text{Proj}_K \sum H^0(X, L^m)$, we see that $X_k := \text{Proj}_k \sum V(X, L^m)$ defines the descent. \square

THEOREM 8.51. *Let $S' \rightarrow S$ be a faithfully flat morphism and $f' : (X', L') \rightarrow S'$ a flat, functorially polarized projective morphism that is rigid over S' . The following are equivalent.*

(8.51.1) $f' : (X', L') \rightarrow S'$ descends to $f : (X, L) \rightarrow S$.

(8.51.2) For every Artin scheme $\tau : A \rightarrow S$, the pull-back $f'_A : (X'_A, L'_A) \rightarrow A$ is independent of the lifting $\tau' : A \rightarrow S'$.

If S is normal and $S' \rightarrow S$ is smooth, then it is enough to check (2) for spectra of fields.

Proof. We just explain how this fits in the framework of faithfully flat descent, for which we refer to [Sta15, Tag 03O6].

Let $\pi_i : S' \times_S S' \rightarrow S'$ denote the coordinate projections for $i = 1, 2$. Pulling back $f' : (X', L') \rightarrow S'$ to $S' \times_S S'$ by the π_i , we get 2 families

$$f'_i : (X'_i, L'_i) \rightarrow S' \times_S S'.$$

If $f : (X, L) \rightarrow S$ exists then these are both isomorphic to the pull-back of $f : (X, L) \rightarrow S$, hence to each other

$$\sigma_{12} : (X'_1, L'_1) \cong (X'_2, L'_2).$$

The existence of σ_{12} is a necessary condition for descent. The key observation is that it is not sufficient, one also needs certain compatibility conditions over the triple product $S' \times_S S' \times_S S'$. However, if (X', L') is rigid over S' , then σ_{12} is unique and the compatibility conditions are automatic.

To prove that σ_{12} exists, consider

$$\pi : \text{Isom}_{S' \times_S S'}((X'_1, L'_1), (X'_2, L'_2)) \rightarrow S' \times_S S'.$$

Since (X', L') is rigid over S' , π is a monomorphism. Assumption (2) implies that it is scheme-theoretically surjective, hence an isomorphism.

If $S' \rightarrow S$ is smooth then $S' \times_S S' \rightarrow S$ is also smooth, hence $S' \times_S S'$ is normal if S is normal. In that case, surjectivity is a set-theoretic question. \square

COROLLARY 8.52. *Let G be a flat group scheme over S and $S' \rightarrow S$ a principal G -bundle. Let $f' : (X', L') \rightarrow S'$ be a flat, functorially polarized projective morphism that is rigid over S' . Assume that the G actions lifts to (X', L') .*

Then $f' : (X', L') \rightarrow S'$ descends to $f : (X, L) \rightarrow S$.

Proof. We need to check assumption (8.51.2). So fix $\tau : A \rightarrow S$ and liftings $\tau_i : A \rightarrow S'$. Then S'_A is a principal G -bundle with 2 sections τ_i . Thus $\tau_2 = g_{12} \circ \tau_1$ for some section g_{12} of G_A . Since the G -action lifts to (X', L') , the corresponding pull-backs are isomorphic. \square

Now we come to the main theorem.

THEOREM 8.53. *Let $\text{MSP}_{\mathbb{Q}}^{\text{rigid}} \subset \text{MSP}_{\mathbb{Q}}$ be the open subset parametrizing stable pairs without automorphisms. Then there is a universal family over $\text{MSP}_{\mathbb{Q}}^{\text{rigid}}$.*

Proof. First note that $\text{MSP}_{\mathbb{Q}}^{\text{rigid}}$ is indeed open by (8.48.2).

For rigid families the existence is a local question. We may thus fix the dimension n , the number of marked divisors r , the coefficient vector (a_1, \dots, a_r) , the volume v and the intended embedding dimension N .

First consider the case when the a_i are rational, and then also fix $m > 1$, a multiple of $\text{lcd}(a_1, \dots, a_r)$. Set $\mathbf{d} := (n, r, a_1, \dots, a_r, m, v, N)$.

Let $\mathcal{MSP}(\mathbf{d})(S)$ denote the set of marked families $f : (X, \Delta) \rightarrow S$ with these numerical data, for which $m(K_{X/S} + \Delta)$ is a Cartier \mathbb{Z} -divisor and a strong polarization, and such that $f_* \mathcal{O}_X(m(K_{X/S} + \Delta))$ has rank $N + 1$. Similarly, let $\mathcal{EMSP}(\mathbf{d})(S)$ denote the set of these objects together with a strong embedding into \mathbb{P}_S^N .

By (8.32) we have the moduli spaces

$$\text{EMSP}^{\text{rigid}}(\mathbf{d})_{\mathbb{Q}} \subset \text{EMSP}(\mathbf{d})_{\mathbb{Q}},$$

with universal families. By (8.43),

$$\text{EMSP}^{\text{rigid}}(\mathbf{d})_{\mathbb{Q}} \rightarrow \text{MSP}^{\text{rigid}}(\mathbf{d})_{\mathbb{Q}}$$

is a principal PGL_{N+1} -bundle. Therefore the universal family over $\text{EMSP}^{\text{rigid}}(\mathbf{d})_{\mathbb{Q}}$ descends to $\text{MSP}^{\text{rigid}}(\mathbf{d})_{\mathbb{Q}}$ by (8.52).

The case of irrational coefficients is very similar. We need to work with the rational approximations $(X, \sigma_j^m(\Delta)) \rightarrow S$ as in (8.16). \square

Complement 8.53.1. The same proof works for other variants of the moduli of stable pairs, in particular we get universal families over the moduli $\text{MpSP}^{\text{rigid}}$ of rigid, pointed, stable pairs (8.24).

Hulls and Husks

Given a coherent sheaf F over a projective scheme, the quot-scheme—introduced by Grothendieck—parametrizes all quotients $F \twoheadrightarrow Q$ of F . In many applications it is necessary to understand not only surjections $F \twoheadrightarrow Q$ but also ‘almost surjections’ $F \rightarrow G$. The precise notion should depend on the application; for us the most important is to study morphisms $F \rightarrow G$ from F to a pure sheaf G that are surjective at all generic points of $\text{Supp } G$. Such objects are called *quotient husks*. Special cases appeared in [Kol08a, PT09, AK10, Kol11b]. The aim of this chapter is to study quotient husks, prove that they have a fine moduli space $\text{QHusk}(F)$ and then apply this to families of hulls.

In Section 9.1 we recall basic results on S_2 sheaves; the proofs are based on [Gro60, Gro62].

Then we turn to the study of hulls of coherent sheaves. The notion of the *hull* of a coherent sheaf F is the generalization of the concept of reflexive hull of a module over a normal domain. In Section 9.2 we discuss the absolute case, denoted usually by $F^{[**]}$, and in Section 9.3 the relative case, denoted by F^H . For many applications the key is the following.

QUESTION 9.1. Let $f : X \rightarrow S$ be a proper morphism and F a coherent sheaf on X . Do the hulls $F_s^H = F_s^{[**]}$ of the fibers F_s form a coherent sheaf that is flat over S ?

If the answer is yes, the resulting sheaf is called the *universal hull* of F over S . Local criteria for its existence are studied in Section 9.4.

In order to get global criteria, husks and quotient husks are defined in Section 9.5. In Section 9.6, the first main result of the Chapter proves that if $X \rightarrow S$ is projective and F is a coherent sheaf on X then the functor of all quotient husks with a given Hilbert polynomial has a fine moduli space $\text{QHusk}_p(X)$ which is a proper algebraic space over S . The proof closely follows the arguments given in [Kol08a].

This is used in a global study of hulls in Section 9.7. A third answer to our question is given in Section 9.8 in terms of a decomposition of S into locally closed subschemes. This can be viewed as a generalization of the Flattening Decomposition Theorem [Mum66, Lect.8].

Moduli spaces of hulls of powers of the relative dualizing sheaf $\omega_{X/S}^{\otimes m}$ were used to define moduli spaces of stable varieties and pairs.

These results are partially extended to algebraic spaces in Section 9.9.

9.1. S_2 sheaves

In this section we collect some well known results about pushing forward and S_2 sheaves.

Assumptions. In this section we work with arbitrary Noetherian schemes.

DEFINITION 9.2. Let F be a quasi-coherent sheaf on a scheme X . Its *annihilator*, denoted by $\text{Ann}(F)$, is the largest ideal sheaf $I \subset \mathcal{O}_X$ such that $I \cdot F = 0$. The *support* of F is the zero set $Z(I) \subset X$, denoted by $\text{Supp } F$.

The *dimension* of F at a point x , denoted by $\dim_x F$, is the dimension of its support at x . The dimension of F is $\dim F := \dim \text{Supp } F$.

The set of all associated points (or primes) of a quasi-coherent sheaf F is denoted by $\text{Ass}(F)$. An associated point of F is called *embedded* if it is contained in the closure of another associated point of F . Let $\text{emb}(F) \subset F$ denote the largest subsheaf whose associated points are all embedded points of F . Thus $F/\text{emb}(F)$ has no embedded points hence it is S_1 (9.7). Informally speaking, $F \mapsto F/\text{emb } F$ is the best way to associate an S_1 sheaf to an arbitrary coherent sheaf.

If F is coherent then it has only finitely many associated points and $\text{Supp } F$ is the union of their closures.

Let $Z \subset X$ be a closed subscheme. Then $\text{tors}_Z F \subset F$ denotes the subsheaf of all local sections whose support is contained in Z . There is a natural isomorphism $\text{tors}_Z F = \mathcal{H}_Z^0(X, F)$.

Assume that X has a dimension function. Then we use $\text{tors}(F) \subset F$ to denote the subsheaf of all local sections whose support has dimension $< \dim \text{Supp } F$. A coherent sheaf F is called *pure* (of dimension n) if (the closure of) every associated point of F has dimension n . Thus $\text{pure}(F) := F/\text{tors}(F)$ is the maximal *pure quotient* of F . A scheme is pure iff its structure sheaf is.

Let $g : Y \rightarrow X$ be flat and of finite type. Then $\text{tors}(g^*F) = g^*\text{tors}(F)$ and $\text{pure}(g^*F) = g^*\text{pure}(F)$. There are, however, problems with similar claims even if $X = \text{Spec } k, Y = \text{Spec}_k k(y)$; see (2.72.3).

If $\text{Supp } F$ is pure dimensional then $\text{emb}(F) = \text{tors}(F)$.

Let $f : X \rightarrow S$ be of finite type and F a coherent sheaf on X such that F_s is pure for every $s \in S$. Then the same holds after any base change $S' \rightarrow S$.

Warning. If X is pure dimensional, F is coherent and $\dim F = \dim X$, then our terminology agrees with every usage of torsion that we know of. However, in other cases these notions are frequently used with different meaning in the literature. The above distinction between $\text{emb}(F)$ and $\text{tors}(F)$ is not standard.

LEMMA 9.3. Let X be a projective scheme and $q : F \rightarrow G$ a surjective map of coherent sheaves on X . Then $G = \text{pure}(F)$ iff $\chi(X, G(*)) = \chi(X, \text{pure } F(*))$.

Proof. If $\chi(X, G(*)) = \chi(X, \text{pure } F(*))$ then $\chi(X, \ker q(*)) = \chi(X, \text{tors } F(*))$, so $\dim \ker q = \dim \text{tors } F < \dim F$. Thus $\ker q \subset \text{tors } F$ but then $\chi(X, \ker q(*)) = \chi(X, \text{tors } F(*))$ says that they are equal. \square

9.4 (Regular sequences and depth). Let A be a ring and M an A -module. Recall that $x \in A$ is *M -regular* if it is not a zero divisor on M , that is, if $m \in M$ and $xm = 0$ implies that $m = 0$. Equivalently, if x is not contained in any of the associated primes of M .

A sequence $x_1, \dots, x_r \in A$ is an *M -regular sequence* if x_1 is not a zero divisor on M and x_i is not a zero divisor on $M/(x_1, \dots, x_{i-1})M$ for all $i = 2, \dots, r$.

Let $\text{rad } A$ denote the *radical* of A , that is, the intersection of all maximal ideals. Let $I \subset \text{rad } A$ be an ideal. The *depth* of M along I is the maximum length of an M -regular sequence $x_1, \dots, x_r \in I$. It is denoted by $\text{depth}_I M$.

It turns out that if A is Noetherian, M is finite over A and $I \subset \text{rad } A$ then all maximal M -regular sequences $x_1, \dots, x_r \in I$ have the same length; see for instance [Mat86, p.127] or [Eis95, Sec.17]. This is a quite subtle result which makes the depth computable in practice: Pick any $x_1 \in I$ that is not contained in any of the associated primes of M , then $\text{depth}_I M = 1 + \text{depth}_I(M/x_1M)$.

Among other useful consequences we see that $\text{depth}_I M$ depends only on \sqrt{I} and it is the minimum of the depths computed for the localizations A_m . Furthermore, if A is finite over B and $J \subset B$ is an ideal then $\text{depth}_J M = \text{depth}_{JA} M$. (Here the first depth is computed over the ring B and the second over the ring A .)

Warning. The literature is not fully consistent on the depth if $M = 0$ (which is infinite by our definition) or if $I \not\subset \text{rad } A$. While the above definition of depth makes sense for arbitrary rings and ideals, it can give unexpected results. For instance, take $A = k[x, y]$, $I = (x)$ and $M = A/(x - 1)$. Then both $x, 0, 0, \dots$ and $xy, x, 0, 0, \dots$ are maximal M -regular sequences.

DEFINITION 9.5. Let F be a coherent sheaf on X . The *depth of F at x* , denoted by $\text{depth}_x F$, is defined as the depth of its localization F_x along $m_{x,X}$ (as an $\mathcal{O}_{x,X}$ -module). For a closed subscheme $Z \subset X$ we set

$$\text{depth}_Z F := \inf\{\text{depth}_z F : z \in Z\}. \quad (9.5.1)$$

If $X = \text{Spec } A$ is affine, $Z = V(I)$ for some ideal $I \subset \text{rad } A$ and $M = H^0(X, F)$ then $\text{depth}_Z F = \text{depth}_I M$.

Warning. This definition is for coherent sheaves only. See [Gro68, Exp.III] for the correct definition of depth for quasi-coherent sheaves.

A coherent sheaf F is called S_m (or it is said to satisfy *Serre's condition S_m*) if

$$\text{depth}_x F \geq \min\{m, \text{codim}(x, \text{Supp } F)\} \quad \text{for all } x \in \text{Supp } F. \quad (9.5.2)$$

We say that F is S_m at x if the localization F_x is S_m (as an $\mathcal{O}_{x,X}$ -module).

Comments. Frequently condition (9.5.2) is stated for all $x \in X$. For the purposes of the latter version, one should say that the zero module has infinite depth. This, however, messes up other conventions, so we just ignore this problem.

In practice there are two cases that are especially interesting and useful. If $m \geq \dim F$ then S_m is equivalent to CM; see, for instance, [Kol13b, 2.58]. This is pretty much the ideal situation, but if it does not hold, usually one can not do anything about it. The other very useful case is condition S_2 . Not every sheaf is S_2 , but, as we see in (9.15), to any coherent sheaf one can usually associate a coherent S_2 sheaf in a natural way, and this is very helpful in many proofs.

Warning. It is important to note that being S_m at x is *not* the same as $\text{depth}_x F \geq m$; neither implies the other. The difference is clear already for $m = 1$:

- $\text{depth}_x F \geq 1$ iff x is not an associated point of F (cf. (9.7)) and
- F is S_1 at x iff x is not contained in the closure of an embedded associated point of F .

As another example, let $(x \in X)$ be the localization of $k[x_1, \dots, x_4]$ at the origin and $M = k[x_1, \dots, x_4] + k[x_1, \dots, x_4]/(x_3, x_4)$. Then $\text{depth}_{(x_1, \dots, x_4)} M = 2$ but $\text{depth}_{(x_3, x_4)} M = 0$. Thus M is not even S_1 .

By contrast, if F has maximal depth at x , that is, if $\text{depth}_x F = \dim_x F$, then $\text{depth}_z F = \text{codim}(z, \text{Supp } F)$ holds for every point $z \in X$. This is one reason why being CM is much better behaved. (See (10.15) for a discussion of the general case.)

9.6 (Depth and flatness). Let $p : Y \rightarrow X$ be a morphism and G a coherent sheaf on Y that is flat over X . It is easy to see that for any point $y \in Y$ we have

$$\text{depth}_y G = \text{depth}_{p(y)} X + \text{depth}_y G_{p(y)}. \quad (9.6.1)$$

Similarly, assume that $p : Y \rightarrow X$ is flat and let F be a coherent sheaf on X . Then

$$\text{depth}_y p^* F = \text{depth}_{p(y)} F + \text{depth}_y Y_{p(y)}. \quad (9.6.2)$$

In particular, if $p : Y \rightarrow X$ is flat with S_m fibers and F is a quasi-coherent S_m sheaf on X then $p^* F$ is also S_m . The converse also holds if p is faithfully flat.

The assumption on the fibers is necessary and a flat pull-back of an S_m sheaf need not be S_m ; not even for products. Let X_1, X_2 be k -schemes. Then $X_1 \times X_2$ is S_m iff both of the X_i are S_m .

Condition S_1 can be described in terms of embedded points.

LEMMA 9.7. *Let F be a coherent sheaf on a scheme X and $Z \subset X$ a closed subscheme. Then $\text{depth}_Z F \geq 1$ iff none of the associated points of F is contained in Z . In particular, F is S_1 iff it has no embedded associated points.*

Proof. An element $x_1 \in m \subset A$ is not a zero divisor on a module M iff it is not contained in any of the associated primes of M . If M has only finitely many associated primes, then there exists such an x_1 iff m is not an associated prime of M . \square

We will repeatedly use the following lemma which gives several characterizations of S_2 sheaves.

LEMMA 9.8. *Let F be a coherent sheaf and $Z \subset \text{Supp } F$ a nowhere dense subscheme. The following are equivalent.*

$$(9.8.1) \text{ depth}_Z F \geq 2.$$

$$(9.8.2) \text{ depth}_Z F \geq 1 \text{ and } \text{depth}_Z(F|_D) \geq 1 \text{ whenever } D \text{ is a Cartier divisor in an open subset of } X \text{ that does not contain any associated prime of } F.$$

$$(9.8.3) \text{ tors}_Z F = 0 \text{ and } \text{tors}_Z(F|_D) = 0 \text{ whenever } D \text{ is as above.}$$

$$(9.8.4) \text{ An exact sequence } 0 \rightarrow F \rightarrow F' \rightarrow Q \rightarrow 0 \text{ splits whenever } \text{Supp } Q \subset Z.$$

$$(9.8.5) \text{ depth}_Z F \geq 1 \text{ and for any exact sequence } 0 \rightarrow F \rightarrow F' \rightarrow Q \rightarrow 0 \text{ such that } \emptyset \neq \text{Supp } Q \subset Z, F' \text{ has an associated prime contained in } \text{Supp } Q.$$

$$(9.8.6) F = j_*(F|_{X \setminus Z}) \text{ where } j : X \setminus Z \hookrightarrow X \text{ is the natural injection.}$$

$$(9.8.7) \mathcal{H}_Z^0(X, F) = \mathcal{H}_Z^1(X, F) = 0.$$

$$(9.8.8) \text{ Let } z \in Z \text{ be any point. Then } H_z^0(X_z, F_z) = H_z^1(X_z, F_z) = 0.$$

Proof. All but (4) are clearly local conditions on X . By assumption $\text{tors}_Z F = 0$. Thus, if in (4) there is a splitting locally then the unique splitting is given by $\text{tors}_Z F' \subset F'$. Thus (4) is also local, we can thus assume that X is affine.

Conditions (2) and (3) are just restatements of the inductive definition of depth. Assume (1) and consider an extension $0 \rightarrow F \rightarrow F' \rightarrow Q \rightarrow 0$ where $\text{Supp } Q \subset Z$. If $\text{tors}_Z(F') \rightarrow Q$ is surjective then it gives a splitting. If not then after quotienting out by $\text{tors}_Z(F')$ and taking a coherent subsheaf $F'' \subset F'/\text{tors}_Z(F')$ we get an extension $0 \rightarrow F \rightarrow F'' \rightarrow Q'' \rightarrow 0$ where $\text{tors}_Z(F'') = 0$. Pick $s \in I_Z$ that is not a zero divisor on F and F'' but $s \cdot (F''/F) = 0$. Then sF'' is a nonzero submodule of F/sF supported on Z . This proves (1) \Rightarrow (4).

Assuming (4) we claim that $\text{tors}_Z(F) = 0$. After localizing at a generic point of $\text{tors}_Z(F)$, we may assume that $\text{tors}_Z(F)$ is supported at $z \in Z$. Since the

injective hull of $k(z)$ over \mathcal{O}_X has infinite length, there is a non-split extension $j : \text{tors}_Z(F) \hookrightarrow G$. Then the cokernel of $(1, j) : \text{tors}_Z(F) \rightarrow F + G$ gives a non-split extension of F . The rest of (5) is clear.

If $\text{depth}_Z F \geq 1$ then the natural map $F \rightarrow j_*(F|_{X \setminus Z})$ is an injection. The quotient is supported on Z , thus (5) \Rightarrow (6).

Assume (6). Then $F \rightarrow j_*(F|_{X \setminus Z})$ is an injection, so $\text{depth}_Z F \geq 1$. If $\text{depth}_Z F < 2$ then we can pick $s \in I_Z$ such that F/sF has a subsheaf Q supported on Z . Let $F' \subset F$ be the preimage of Q . Then $s^{-1}F' \subset j_*(F|_{X \setminus Z})$ shows that (6) \Rightarrow (1).

We discuss (7) and (8) in (10.28). □

COROLLARY 9.9. *Let F be a coherent, S_2 sheaf and G any coherent sheaf. Then $\mathcal{H}om_X(G, F)$ is also S_2 .*

Proof. It is clear that every irreducible component of $\text{Supp } \mathcal{H}om_X(G, F)$ is also an irreducible component of $\text{Supp } F$.

Let $Z \subset \text{Supp } F$ be a closed subset of codimension ≥ 2 and $j : X \setminus Z \hookrightarrow X$ the injection. Any homomorphism $\phi : G|_{X \setminus Z} \rightarrow F|_{X \setminus Z}$ uniquely extends to

$$j_*\phi : j_*(G|_{X \setminus Z}) \rightarrow j_*(F|_{X \setminus Z}).$$

Since F is S_2 , the target equals F . We have a natural map $G \rightarrow j_*(G|_{X \setminus Z})$ (whose kernel is the subsheaf of sections whose support is in Z). Thus

$$\mathcal{H}om_X(G, F) = j_*(\mathcal{H}om_X(G, F)|_{X \setminus Z}),$$

hence $\mathcal{H}om_X(G, F)$ is S_2 . □

An important property of S_2 sheaves is the following, which can be obtained by combining [Har77, III.7.3] and [Har77, III.12.11].

PROPOSITION 9.10 (Enriques–Severi–Zariski lemma). *Let $f : X \rightarrow S$ be a projective morphism and F a coherent sheaf on X that is flat over S with S_2 fibers of pure dimension ≥ 2 . Then $f_*F(-m) = R^1 f_*F(-m) = 0$ for $m \gg 1$.*

*Therefore, if $H \in |\mathcal{O}_X(m)|$ does not contain any of the associated points of F then the restriction map $f_*F \rightarrow (f|_H)_*(F|_H)$ is an isomorphism.* □

9.2. Hulls of coherent sheaves

Let X be an integral, normal scheme and F a coherent sheaf on X . The reflexive hull of F is the double dual $F^{**} := \mathcal{H}om_X(\mathcal{H}om_X(F, \mathcal{O}_X), \mathcal{O}_X)$. We would like to extend this notion to arbitrary schemes and arbitrary coherent sheaves. For this the key properties of the reflexive hull are the following.

- F^{**} is S_2 and
- F^{**} is the smallest S_2 sheaf containing $F/(\text{torsion})$.

These are the properties that we use to define the hull of a sheaf. Note, however, that for this we need to agree what the torsion subsheaf of a sheaf should be. Two natural candidates are discussed in (9.2):

- $\text{emb } F$, the subsheaf corresponding to embedded points and
- $\text{tors } F$, the largest subsheaf whose support has dimension $< \dim F$.

These are the same if $\text{Supp } F$ is irreducible but quite different in general. For example, if X is a disjoint union of normal schemes of different dimensions, then

$\text{emb } \mathcal{O}_X = 0$ while $\text{tors } \mathcal{O}_X$ is the structure sheaf of the lower dimensional components. A theory of hulls using $\text{emb } F$ is developed in [Kol17].

Here we work with $\text{tors } F$. An advantage is that $F/\text{tors } F$, and hence the hull, are pure dimensional; this is quite important in our applications. A disadvantage is that for this to make sense, one needs the dimension function to be quite well behaved. In all final applications we work with schemes of finite type, so this will not be a problem.

ASSUMPTION 9.11. In this section we consider schemes X such that

(9.11.1) $\dim W$ is finite for every irreducible subscheme $W \subset X$ and

(9.11.2) if $W_1 \subsetneq W_2$ is a maximal (with respect to inclusion) irreducible subscheme of the irreducible $W_2 \subset X$, then $\dim W_1 = \dim W_2 - 1$.

These hold for schemes of finite type over a field (by standard dimension theory) and also for schemes of finite type over a local CM scheme; see [Sta15, Tags 00NM and 02JT]. However, these conditions are not satisfied by many naturally occurring schemes; a typical example is the localization of $k[x, y]$ at $(x, y) \cup (x - 1)$.

(More generally, one could work instead with any scheme that admits a dimension function, see [Kol17].)

A useful property of pure sheaves is the following.

LEMMA 9.12. *Let $p : X \rightarrow Y$ be a finite morphism and F a coherent sheaf on X . Then F is pure and S_m iff p_*F is pure and S_m .*

Proof. The last remark of (9.4) implies that the depth is preserved by push-forward. Thus the only question is whether (co)dimension is preserved or not; this is where (9.11) is used. \square

DEFINITION 9.13 (Hull of a sheaf). Let X be a scheme as in (9.11) and F a coherent sheaf on X . Set $n := \dim F$. The *hull* of F is a coherent sheaf $F^{[**]}$ together with a map $q : F \rightarrow F^{[**]}$ such that

(9.13.1) $\text{Supp}(\ker q)$ has dimension $\leq n - 1$,

(9.13.2) $\text{Supp}(\text{coker } q)$ has dimension $\leq n - 2$ and

(9.13.3) $F^{[**]}$ is pure and S_2 .

We sometimes say *S_2 -hull* or *pure hull* if we want to emphasize these properties. We see below that a hull is unique and it exists if X is excellent.

By definition, $F^{[**]} = (F/\text{tors } F)^{[**]}$, hence it is enough to construct pure hulls of pure, coherent sheaves.

The notation $F^{[**]}$ is chosen to emphasize the close connection between the pure hull and the reflexive hull F^{**} ; see (9.14) for the precise statement. We introduce a relative version, denoted by F^H in (9.18). If $X \rightarrow \text{Spec } k$ is a k -scheme then $F^{[**]} = F^H$.

The following property is clear from the definition.

(9.13.4) Let G be a pure, coherent, S_2 sheaf and $F \subset G$ a subsheaf. Then $G = F^{[**]}$ iff $\dim(G/F) \leq \dim G - 2$.

From (9.12) and (9.6) we obtain the following base change properties of hulls.

(9.13.5) Let $p : X \rightarrow Y$ be a finite morphism. Then $p_*(F^{[**]}) = (p_*F)^{[**]}$.

(9.13.6) $g : Z \rightarrow X$ be flat and pure dimensional with S_2 fibers. Then $g^*(F^{[**]}) = (g^*F)^{[**]}$.

In many cases the pure hull coincides with the usual reflexive hull.

PROPOSITION 9.14. *Let X be an irreducible, normal scheme and F a torsion free coherent sheaf on X . Then*

$$F^{[**]} = F^{**} := \mathcal{H}om_X(\mathcal{H}om_X(F, \mathcal{O}_X), \mathcal{O}_X).$$

Proof. F is locally free outside a codimension ≥ 2 subset $Z \subset X$. Thus the natural map $F \rightarrow F^{**}$ is an isomorphism over $X \setminus Z$. Since F^{**} is S_2 by (9.9), it satisfies the assumptions of (9.13). \square

This can be used to construct the pure hull over finite type schemes. Indeed, we may assume that X is affine and $X = \text{Supp } F$. By Noether normalization there is a finite surjection $p : X \rightarrow \mathbb{A}^n$. Thus, by (9.13.5) and (9.14), $F^{[**]}$ can be identified with $(p_* F)^{**}$. Next we prove that pure hulls exist over excellent schemes; see [Kol17] for a more general result.

PROPOSITION 9.15. *Let X be an excellent scheme and F a pure, coherent sheaf on X .*

(9.15.1) *There is a closed subset $Z \subset \text{Supp } F$ of dimension $\leq \dim F - 2$ such that F is S_2 over $X \setminus Z$.*

(9.15.2) *Let $Z \subset \text{Supp } F$ be any closed subset of dimension $\leq \dim F - 2$ such that F is S_2 over $X \setminus Z$. Then the following hold.*

(9.15.2.a) $F^{[**]} = j_*(F|_{X \setminus Z})$.

(9.15.2.b) *Let $G \supset F$ be any coherent sheaf such that $G|_{X \setminus Z} = F|_{X \setminus Z}$. Then $F \rightarrow F^{[**]}$ uniquely extends to $G \rightarrow F^{[**]}$.*

Proof. The first claim follows from (10.26). To see (2.a), note that $j_*(F|_{X \setminus Z})$ is coherent by (10.25), S_2 over $X \setminus Z$ by assumption and $\text{depth}_Z j_*(F|_{X \setminus Z}) \geq 2$ by (9.8). Thus $j_*(F|_{X \setminus Z})$ is a hull of F .

If G as in (2.b), then we get $G \rightarrow j_*(G|_{X \setminus Z}) = j_*(F|_{X \setminus Z})$.

Let $F^{[**]}$ be any hull of F . Then $F^{[**]}|_{X \setminus Z}$ is a hull of $F|_{X \setminus Z}$; let $W \subset X \setminus Z$ be the support of their quotient. Then $\text{codim}_X W \geq 2$ hence $F^{[**]}|_{X \setminus Z} = F|_{X \setminus Z}$ by (9.8.2). Thus we get a map $F^{[**]} \rightarrow j_*(F|_{X \setminus Z})$. Applying (9.8) again gives that $F^{[**]} = j_*(F|_{X \setminus Z})$. \square

9.16 (Quasi-coherent hulls). The formula (9.15.2.a) suggests that one should define the hull of a quasi-coherent sheaf F as

$$F^{[**]} := \varinjlim (j_Z)_*(F|_{X \setminus Z}),$$

where Z runs through all codimension ≥ 2 closed subsets of $\text{Supp } F$. It is easy to see that $F^{[**]}$ is always S_2 (as defined in [Gro68, Exp.III]) and it agrees with our definition whenever both are defined.

Since j_* is left exact, we obtain that the formation of the hull is also left exact.

COROLLARY 9.17. *Let $0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3$ be an exact sequence of coherent sheaves of the same dimension. Then the hulls also form an exact sequence $0 \rightarrow F_1^{[**]} \rightarrow F_2^{[**]} \rightarrow F_3^{[**]}$.* \square

9.3. Relative hulls

Next we develop a relative version of the notion of hull for coherent sheaves on a scheme X over a base scheme S .

In the absolute case, the hull is an S_2 sheaf that we can associate to any coherent sheaf on X , in particular, the hull does not have embedded points.

In the relative case, assume for simplicity that $f : X \rightarrow S$ is smooth; then \mathcal{O}_X should be its own ‘relative hull.’ Note, however, that the structure sheaf \mathcal{O}_X has no embedded points if and only if the base scheme S has no embedded points. Thus if we want to say that \mathcal{O}_X is its own ‘relative hull’ then we have to distinguish embedded points that are caused by S (these are allowed) from other embedded points (these are forbidden).

The distinction between these two types of embedded points seems to be meaningful only if F is flat over a sufficiently large open subset of $\text{Supp } F$. Further restrictions need to be imposed if we want to allow base changes.

DEFINITION 9.18 (Hull over a base scheme). Let $f : X \rightarrow S$ be a morphism of finite type and F a coherent sheaf on X . Let n denote the maximum fiber dimension of $\text{Supp } F \rightarrow S$.

A *hull* (or *relative hull*) of F over S is a coherent sheaf F^H together with a morphism $q : F \rightarrow F^H$ such that

- (9.18.1) $\text{Supp}(\ker q) \rightarrow S$ has fiber dimension $\leq n - 1$,
- (9.18.2) $\text{Supp}(\text{coker } q) \rightarrow S$ has fiber dimension $\leq n - 2$,
- (9.18.3) there is a closed subset $Z \subset X$ with complement $U := X \setminus Z$ such that
 - (9.18.3.a) $Z \rightarrow S$ has fiber dimension $\leq n - 2$,
 - (9.18.3.b) $(F/\ker q) \rightarrow F^H$ is an isomorphism over U .
 - (9.18.3.c) $F^H|_U$ is flat over S with pure, S_2 fibers and
 - (9.18.3.d) $\text{depth}_Z F^H \geq 2$.

Note that $\text{Supp}(\text{coker } q) \subset Z$ by (3.b) hence in fact (3.a) implies (2). We state the latter separately to emphasize the parallels with the definition of the absolute hull (9.13). If S is the spectrum of a field then clearly $F^H = F^{[**]}$. Note, however, that while the hull is always defined, the relative hull frequently does not exist.

For instance, let $f : X := \mathbb{A}_{st}^2 \rightarrow S := \mathbb{A}_t^1$ be the projection and $F \subset \mathcal{O}_X$ the ideal sheaf of the point $(0,0)$. Then $F^{[**]} = \mathcal{O}_X$ but $F \rightarrow \mathcal{O}_X$ is not a relative hull since $\text{coker}(F \rightarrow \mathcal{O}_X)$ has codimension 1 on the fiber X_0 .

(It would have been more consistent to denote the hull by F^h , but a superscript h is frequently used to denote the Henselization.)

Relative hulls are easy to understand if the base scheme is 1-dimensional and regular.

LEMMA 9.19. *Let $(0, T)$ be the spectrum of a DVR, $f : X \rightarrow T$ a morphism of finite type and $q : F \rightarrow G$ a map between pure, coherent sheaves on X that are flat over T . Then G is a relative hull of F iff the following hold.*

- (9.19.1) G_g is the hull of F_g ,
- (9.19.2) G_0 is S_1 ,
- (9.19.3) $q_0 : F_0 \rightarrow G_0$ is an isomorphism outside a subset $Z_0 \subset \text{Supp } G_0$ of codimension ≥ 2 .

Proof. Assume that $G = F^H$ and let $Z \subset X$ be as in (9.18). By assumption $G|_{X \setminus Z}$ has S_2 fibers thus $G|_{X \setminus Z}$ is S_2 . Hence G is S_2 since $\text{depth}_Z G \geq 2$ and so G_0 is S_1 and $q_0 : F_0 \rightarrow G_0$ is an isomorphism outside $X_0 \cap Z$.

Conversely, if (1–3) hold then G is S_2 by (1–2). By (9.15) there is a closed subset $Z_1 \subset X_0$ of codimension ≥ 2 such that F_0 is S_2 over $X_0 \setminus Z_0$. Thus $q : F \rightarrow G$ satisfies the conditions (9.18.1–3) where Z is the union of 3 closed sets: Z_0, Z_1 and the closure of $\text{Supp}(\text{coker } q_g)$. \square

As a special case, we get the following characterization of relative hulls.

COROLLARY 9.20. *Let $(0, T)$ be the spectrum of a DVR, $f : X \rightarrow T$ a morphism of finite type and F a pure, coherent sheaf on X that is flat over T . Then $F = F^H \Leftrightarrow F$ is $S_2 \Leftrightarrow F_g$ is S_2 and F_0 is S_1 .* \square

COROLLARY 9.21 (Bertini theorem for relative hulls). *Let $(0, T)$ be the spectrum of a DVR, $X \subset \mathbb{P}_T^n$ a quasi-projective scheme and F a coherent sheaf on X with relative hull $q : F \rightarrow F^H$. Then $q|_L : F|_L \rightarrow F^H|_L$ is the relative hull of $F|_L$ for a general hyperplane $L \subset \mathbb{P}_T^n$.*

Proof. We use (10.9) both for the special fiber X_0 and the generic fiber X_g . We get open subsets $U_0 \subset \check{\mathbb{P}}_0^n$ and $U_g \subset \check{\mathbb{P}}_g^n$ such that

$$(9.21.1) \quad F^H|_{L_0} \text{ is } S_1 \text{ for } L_0 \in U_0,$$

$$(9.21.2) \quad (F/\text{tors } F)|_{L_0} = (F|_{L_0})/\text{tors}(F|_{L_0}) \text{ for } L_0 \in U_0,$$

$$(9.21.3) \quad (F|_{L_0})/\text{tors}(F|_{L_0}) \rightarrow G_{L_0} \text{ is an isomorphism outside a subset of codimension } \geq 2 \text{ for } L_0 \in U_0 \text{ and}$$

$$(9.21.4) \quad F^H|_{L_g} \text{ is the hull of } F|_{L_g} \text{ for } L_g \in U_g.$$

Let $W_T \subset \check{\mathbb{P}}_T^n$ denote the closure of $\check{\mathbb{P}}_g^n \setminus U_g$. For dimension reasons, W_T does not contain $\check{\mathbb{P}}_0^n$. Thus any hyperplane corresponding to a section through a point of $U_0 \setminus W_T$ works. (See (9.30) for some examples with non-general hyperplanes.) \square

Next we state the precise conditions needed for the existence of relative hulls. Then we show that a relative hull is unique and does not depend on the choice of $Z \subset X$, generalizing (9.15).

LEMMA 9.22. *Let $f : X \rightarrow S$ be a morphism of finite type and F a coherent sheaf on X . Then F has a relative hull iff*

$$(9.22.1) \quad \text{there is a coherent subsheaf } \text{tors}_S F \subset F \text{ such that } \text{Supp}(\text{tors}_S F) \rightarrow S \text{ has fiber dimension } \leq n - 1 \text{ and}$$

$$(9.22.2) \quad \text{a closed subset } Z \subset X \text{ such that } f|_Z : Z \rightarrow S \text{ has fiber dimension } \leq n - 2 \text{ such that}$$

$$(9.22.3) \quad (F/\text{tors}_S F)|_{X \setminus Z} \text{ is flat over } S \text{ with pure, } S_2 \text{ fibers.}$$

Furthermore, if these conditions are satisfied then

$$(9.22.4) \quad F^H = j_*((F/\text{tors}_S F)|_{X \setminus Z}) \text{ is the unique relative hull of } F \text{ over } S.$$

$$(9.22.5) \quad \text{Let } G \supset F \text{ be any coherent sheaf such that } G|_{X \setminus Z} = F|_{X \setminus Z}. \text{ Then } F \rightarrow F^H \text{ uniquely extends to } G \rightarrow F^H.$$

Proof. If $q : F \rightarrow F^H$ is a relative hull then the conditions (9.22.1–3) are satisfied and $\text{tors}_S(F) = \ker(q)$.

Conversely, assume that the conditions (9.22.1–3) are satisfied. We can harmlessly replace F by $F/\text{tors}_S(F)$. Write $U := X \setminus Z$. Then $j_*(F|_U)$ is coherent by (10.25), $F \rightarrow j_*(F|_U)$ is an isomorphism over U by construction and $\text{depth}_Z j_*(F|_U) \geq 2$ by (9.8).

The last claim follows from the universal property of the push-forward and it implies that F^H is independent of the choice of Z . \square

We see from the above construction that the torsion subsheaf $\text{tors}_S(F)$ plays essentially no role and one can always work with the quotient $F/\text{tors}_S(F)$ instead of F . This leads to the following generalization of (5.46).

DEFINITION 9.23 (Mostly flat families of S_2 sheaves). Let $f : X \rightarrow S$ be a morphism and F a coherent sheaf on X . We say that F is a *mostly flat family of S_2 sheaves* if there is a closed subscheme $Z \subset X$ with complement $U := X \setminus Z$ such that

(9.23.1) $Z \cap X_s$ has codimension ≥ 2 in $\text{Supp } F_s$ for every $s \in S$ and

(9.23.2) $F|_U$ is flat over S with pure, S_2 fibers.

By (9.22), if these hold then F has a hull F^H . Conversely, if F has a hull then $F/\text{tors}_S F$ is a mostly flat family of S_2 sheaves.

The following is a direct analog of (9.13.4).

COROLLARY 9.24. *Let $f : X \rightarrow S$ be a morphism of finite type and G a coherent sheaf on X that is flat over S with pure, S_2 fibers of dimension n . Let $F \subset G$ be a subsheaf. Then $G = F^H$ iff the fiber dimension of $\text{Supp}(G/F) \rightarrow S$ is $\leq n - 2$. \square*

9.4. Universal hulls

For many applications a key question is to understand the behavior of relative hulls under a base change.

NOTATION 9.25. Let $f : X \rightarrow S$ be a morphism of finite type and F a coherent sheaf satisfying the conditions (9.22.1–3). For any $g : T \rightarrow S$ we have a base-change diagram

$$\begin{array}{ccc} X_T & \xrightarrow{g_X} & X \\ f_T \downarrow & & \downarrow f \\ T & \xrightarrow{g} & S. \end{array} \quad (9.25.1)$$

By pull-back we obtain $Z_T := g_X^{-1}(Z)$, $U_T := g_X^{-1}(U)$ and $F_T := g_X^* F$. Note that $f_T : X_T \rightarrow T$, Z_T and F_T again satisfies (9.22.1–3).

In general $g_X^*(F^H)$ is not the relative hull of F_T . Thus we need to distinguish

$$(F^H)_T := g_X^*(F^H) \quad \text{and} \quad (F_T)^H := (g_X^* F)^H. \quad (9.25.2)$$

Since the two sheaves agree over U_T , (9.22.5) implies that there is a natural map

$$r_T^F : (F^H)_T \rightarrow (F_T)^H \quad (9.25.3)$$

We call r_T^F the *restriction map*, especially when T is a subscheme of S . We see in (9.27) that r_T^F is an isomorphism along $g_X^{-1}(x)$ if F is flat with pure, S_2 -fiber at x .

DEFINITION 9.26. Let $f : X \rightarrow S$ be a morphism of finite type and F a coherent sheaf on X satisfying (9.22.1–3).

We say that F^H is a *universal hull* of F at $x \in X$ if the restriction map r_T^F (9.25.3) is an isomorphism along $g_X^{-1}(x)$ for every $g : T \rightarrow S$. We say that F^H is a *universal hull* of F if it is a universal hull at every $x \in X$. That is, F^H is a universal hull iff $g_X^*(F^H)$ is the hull of g_X^*F for every $g : T \rightarrow S$. Equivalently, iff the functor $F \mapsto F^H$ commutes with base change.

We say that $F \mapsto F^H$ is *universally flat* if $(F_T)^H$ is flat over T for every $g : T \rightarrow S$.

The following theorem gives several characterizations of universal hulls.

THEOREM 9.27. *Let $f : X \rightarrow S$ be a morphism of finite type, F a mostly flat family of S_2 sheaves (9.23) and $F \rightarrow F^H$ the relative hull of F over S . The following are equivalent.*

- (9.27.1) F^H is a universal hull of F .
- (9.27.2) $F \mapsto F^H$ is universally flat.
- (9.27.3) F^H is flat over S and has pure, S_2 fibers.
- (9.27.4) F^H is flat over S and has pure, S_2 fibers over closed points of S .
- (9.27.5) For every closed point $s \in S$ the restriction map $r_s^F : F^H \rightarrow (F_s)^H$ is surjective.
- (9.27.6) $(F_A)^H$ is a universal hull of F_A for every Artin scheme $A \rightarrow S$.

Proof. The only obvious implications are (3) \Rightarrow (4) and (1) \Rightarrow (5) but (4) \Rightarrow (3) directly follows from the openness of the S_2 -condition (10.2).

Note that the properties in (3) are preserved by base change, thus $(F^H)_T$ is flat over T and $((F^H)_T)_t$ is S_2 for every point $t \in T$. By (9.24) this implies that $(F^H)_T$ is the relative hull of F_T . Therefore $(F^H)_T = (F_T)^H$, so $F \mapsto F^H$ is universally flat and commutes with base change. That is, (3) \Rightarrow (2) and (3) \Rightarrow (1) both hold.

If (4) holds then $(F^H)_s = (F_s)^H$ by (9.13.4), thus (4) \Rightarrow (5). Applying (10.62) to every localization of S at closed points shows that (5) \Rightarrow (4).

Next we show that (2) \Rightarrow (6). We may assume that $S = \text{Spec } A$ where (A, m) is local, Artinian. Choose the smallest $r \geq 0$ such that $m^{r+1} = 0$; so $m^r \cong \oplus_i A/m$, the sum of certain number of copies of A/m . This gives an injection $j_r : \oplus_i F_s \hookrightarrow F$ which then extends to $j_r^H : \oplus_i (F_s)^H \hookrightarrow F^H$.

Since F^H is flat over A , the image $j_r^H(\oplus_i (F_s)^H)$ is also isomorphic to $(m^r) \otimes_A F^H$ which is the same as $\oplus_i (F^H)_s$. Thus $(F_s)^H = (F^H)_s$ and, by the above arguments, (2) implies the properties (1–5) for local, Artinian base schemes.

In order to see (6) \Rightarrow (5) we may replace S by its completion at s . For $r \in \mathbb{N}$ set $A_r := \text{Spec}_S \mathcal{O}_S/m_s^r$. By base change we get $f_r : X_r \rightarrow A_r$ and $F_r := F|_{X_r}$. By assumption $(F_r)^H$ is flat over A_r and we have proved that $F \mapsto F^H$ commutes with base change over Artin schemes. Set

$$\tilde{F} := \varinjlim (F_r)^H.$$

Then \tilde{F} is flat over S , coherent (cf. [Har77, II.9.3.A]), agrees with F over U and $\tilde{F} \rightarrow F_s^H$ is surjective. Thus $\tilde{F} = F^H$ by (9.24), giving (5). \square

We can restate the characterization (9.27.3) as follows.

COROLLARY 9.28. *Let $f : X \rightarrow S$ be a morphism of finite type, $q : F \rightarrow G$ a map of coherent sheaves on X . Let n denote the maximum fiber dimension of $\text{Supp}(F) \rightarrow S$. Then G is the universal hull of F over S iff the following hold.*

(9.28.1) $q_s : F_s \rightarrow G_s$ is an isomorphism at all n -dimensional points of X_s for every $s \in S$.

(9.28.2) G is flat with purely n -dimensional, S_2 fibers over S and

(9.28.3) $\text{Supp}(\text{coker}(q)) \rightarrow S$ has fiber dimension $\leq n - 2$. □

Combining (9.28) and (10.3) shows that a relative hull is a universal hull over a dense open subset of the base. Thus Noetherian induction gives the following. A much more precise form will be proved in (9.64).

COROLLARY 9.29 (Universal hull decomposition, weak form). *Let $f : X \rightarrow S$ be a proper morphism and F a coherent sheaf on X . Then there is a locally closed decomposition $j : S' \rightarrow S$ such that $j_X^* F$ has a universal hull.* □

The following example illustrates several aspects of (9.27).

EXAMPLE 9.30. Let X be a projective variety with ample line bundle $\mathcal{O}_X(1)$ and $C(X)$ the corresponding affine cone over X with vertex v . Set $C^*(X) := C(X) \setminus \{v\}$ with natural injection $j : C^*(X) \hookrightarrow C(X)$. If F is a coherent sheaf on X then by pull-back we get a coherent sheaf $C(F)$ on $C^*(X)$ and

$$H^0(C^*(X), C(F)) = \sum_{m \in \mathbb{Z}} H^0(X, F(m)). \quad (9.30.1)$$

Next let $g : X \rightarrow S$ be a flat family of projective varieties and F a coherent sheaf on X that is flat over S and of pure relative dimension ≥ 1 . We get the relative cones

$$C_S^*(X) := \text{Spec}_S \sum_{m \in \mathbb{Z}} g_* \mathcal{O}_X(m) \quad \text{and} \quad C_S(X) := \text{Spec}_S \sum_{m \in \mathbb{N}} g_* \mathcal{O}_X(m)$$

with natural injection $j : C_S^*(X) \hookrightarrow C_S(X)$. Note that $C_S^*(X)$ is a \mathbb{G}_m -bundle over X , thus $C_S^*(X)$ is flat over S but $C_S(X)$ need not be flat over S .

Let $C_S(F)$ denote the coherent sheaf on $C_S^*(X)$ corresponding to $\sum_{m \in \mathbb{Z}} g_* F(m)$; then $C_S(F)^H = j_* C_S(F)$ by (9.22).

By (9.27), $C_S(F)^H$ is a universal hull iff the restriction map $r_s : j_* C_S(F) \rightarrow (j_s)_* C(F_s)$ is surjective for every point $s \in S$. Using the grading given by the cone structure this holds iff the restriction map $r_s(m) : g_* F(m) \rightarrow H^0(X_s, F_s(m))$ is surjective for every point $s \in S$ and every $m \in \mathbb{Z}$. Using the Cohomology and base change theorem we conclude that

$$C_S(F)^H \text{ is a universal hull} \iff g_* F(m) \text{ is locally free } \forall m \in \mathbb{Z}. \quad (9.30.2)$$

In order to get some concrete examples, let E_1, E_2 be elliptic curves and $A = E_1 \times E_2$ with projections $\pi_i : A \rightarrow E_i$. Let L_i be line bundles of degree $d \geq 3$ on E_i ; these give a very ample line bundle $\mathcal{O}_A(1) := \pi_1^* L_1 \otimes \pi_2^* L_2$. Let $C \in |\mathcal{O}_A(1)|$ be a smooth curve. Note that $(C^2) = 2g(C) - 2 = 2d^2$.

Set $S := \text{Pic}^{2d}(E_1) \times \text{Pic}^0(E_2)$ and $X := A \times S$ with projection $g : X \rightarrow S$. $H := C \times S$ gives a hyperplane section $C_S(H) \subset C_S(X)$. The universal bundles give $F := \pi_1^* M \otimes \pi_2^* T$. We check that $C_S^*(F)^H$ is not universal along $\text{Pic}^{2d}(E_1) \times \{\mathcal{O}_{E_2}\}$ but if we choose $T = \mathcal{O}_{E_2}$ and M general then the restriction of $C_S(F)$ to the hyperplane section $C_S(H) \subset C_S(X)$ has a universal hull.

By (9.30.2) these claims are equivalent to computing some cohomologies on A and on C . So let M be a line bundle of degree $2d$ on E_1 and T a line bundle of degree 0 on E_2 . One easily computes that $h^0(A, (\pi_1^* M \otimes \pi_2^* T)(m))$ is independent of M, T if $m \neq 0$ but

$$h^0(A, \pi_1^* M \otimes \pi_2^* T) = 2d \cdot h^0(E_2, T)$$

jumps at $T = 0$. Similarly we obtain that $h^0(C, (\pi_1^*M \otimes \pi_2^*T)(m)|_C)$ is independent of M, T if $m \neq 0, -1$ but $h^0(C, (\pi_1^*M \otimes \pi_2^*T)_C)$ jumps iff $h^0(C, (\pi_1^*M \otimes \pi_2^*T)(-1)|_C)$ jumps iff $(\pi_1^*M \otimes \pi_2^*T)|_C \cong \omega_C$.

This shows that the hull $C_S^*(F)^H$ is not universal along $\text{Pic}^{2d}(E_1) \times \{\mathcal{O}_{E_2}\}$ but if we choose $T = \mathcal{O}_{E_2}$ and M general then the restriction of $C_S(F)$ to the hyperplane section $C_S(H) \subset C_X(X)$ has a universal hull.

The following result is a restatement of (10.64), see also [Kol95a, Thm.12].

THEOREM 9.31. *Let $f : X \rightarrow S$ be a morphism and F a mostly flat family of S_2 sheaves (9.23). Assume that, for every $s \in S$, the hull $(F_s)^H$ is coherent and $\text{depth}_{Z_s}(F_s)^H \geq 3$. Then F^H is the universal hull of F over S . \square*

9.5. Husks of coherent sheaves

ASSUMPTION 9.32. In this section we continue with the assumptions (9.11).

DEFINITION 9.33. Let X be a scheme and F a coherent sheaf on X . An n -dimensional *quotient husk* of F is a quasi-coherent sheaf G together with a homomorphism $q : F \rightarrow G$ such that

- (9.33.1) G is pure of dimension n and
- (9.33.2) $q : F \rightarrow G$ is surjective at all generic points of $\text{Supp } G$.

A quotient husk is called a *husk* if $n = \dim F$ and

- (9.33.3) $q : F \rightarrow G$ is an isomorphism at all n -dimensional points of X .

If $h \in \text{Ann}(F)$ then $h \cdot F = 0$, hence $h \cdot G \subset G$ is supported in dimension $< n$, thus it is 0. Therefore G is also an $\mathcal{O}_X/\text{Ann}(F)$ sheaf and so the particular choice of X matters very little.

Any coherent sheaf F has a maximal husk

$$M(F) := \varinjlim (j_Z)_*(F|_{X \setminus Z}),$$

where Z runs through all closed subsets of $\text{Supp } F$ such that $\dim Z < \dim F$. If $\dim F \geq 1$ then $M(F)$ is never coherent, but it is the union of coherent husks. Thus a coherent sheaf has many different coherent husks and there is no maximal coherent husk.

LEMMA 9.34. *Let F be a coherent sheaf on X and $q : F \rightarrow G$ an n -dimensional (quotient) husk of F .*

- (9.34.1) *Let $g : X \rightarrow Z$ be a finite S -morphism. Then g_*G is an n -dimensional (quotient) husk of g_*F .*
- (9.34.2) *Let $h : Y \rightarrow X$ be a flat morphism of pure relative dimension r with S_1 fibers. Then h^*G is an $(n+r)$ -dimensional (quotient) husk of h^*F .*

Proof. If g is a finite morphism and M is a sheaf then the associated primes of g_*M are the images of the associated primes of M . This implies (1). Similarly, if h is flat then the associated primes of h^*M are the preimages of the associated primes of M . Since h^*G is S_1 by (9.6), we get (2). \square

9.35 (Bertini theorem for (quotient) husks). Let F be a coherent sheaf on a quasi-projective variety $X \subset \mathbb{P}^n$ and $q : F \rightarrow G$ a coherent (quotient) husk. Let $H \subset \mathbb{P}^n$ be a general hyperplane. Then $G|_H$ is pure by (10.9). If, in addition, H does not contain any of the associated primes of G/F then $q|_H : F|_H \rightarrow G|_H$ as also a (quotient) husk.

DEFINITION 9.36. Let X be a scheme and F a coherent sheaf on X . Set $n := \dim F$. A husk $q : F \rightarrow G$ is called *tight* if $q : F/\text{tors } F \hookrightarrow G$ is an isomorphism at all $(n-1)$ -dimensional points of X .

Thus the hull $q : F \rightarrow F^{[**]}$ defined in (9.13) is a tight husk of F . We see below that the hull is the maximal tight husk.

LEMMA 9.37. *Let X be a scheme and F a coherent sheaf on X with hull $q : F \rightarrow F^{[**]}$.*

(9.37.1) *Let $r : F \rightarrow G$ be any tight husk. Then q extends uniquely to an injection $q_G : G \hookrightarrow F^{[**]}$.*

(9.37.2) *$F^{[**]}$ is the unique tight husk that is S_2 .*

Proof. After replacing F with $F/\text{tors } F$ we may assume that F is pure. Set $Z := \text{Supp}(G/F) \cup \text{Supp}(F^{[**]}/F)$. Then Z has codimension ≥ 2 and F is S_2 on $X \setminus Z$. Thus, by using (9.15.2) for F we get that

$$G \subset j_*(G|_{X \setminus Z}) = j_*(F|_{X \setminus Z}) = F^{[**]},$$

proving (1). If G is also S_2 , then, (9.15.2) gives that $G = F^{[**]}$. \square

LEMMA 9.38. *Let X be a projective scheme, F a coherent sheaf of pure dimension n and $F \rightarrow G$ a quotient husk. The following are equivalent.*

(9.38.1) $G = F^{[**]}$.

(9.38.2) G is S_2 and $\chi(X, F(t)) - \chi(X, G(t))$ has degree $\leq n-2$.

(9.38.3) $\chi(X, F^{[**]}(t)) \equiv \chi(X, G(t))$.

Proof. The exact sequence $0 \rightarrow K \rightarrow F \rightarrow G \rightarrow Q \rightarrow 0$ defines the sheaves K, Q and

$$\chi(X, F(t)) - \chi(X, G(t)) = \chi(X, K(t)) - \chi(X, Q(t)).$$

Note that K has pure dimension n and Q has dimension $\leq n-1$.

If $G = F^{[**]}$ then $K = 0$ and $\dim Q \leq n-2$ which implies (2) and (1) \Rightarrow (3) is obvious.

Conversely, assume that $\chi(X, F(t)) - \chi(X, G(t))$ has degree $\leq n-2$. Since $\deg \chi(X, Q(t)) \leq n-1$, we see that $\deg \chi(X, K(t)) \leq n-1$. However, K has pure dimension n thus in fact $K = 0$ and so G is a tight husk of F . If G is S_2 then (9.37) implies that $G = F^{[**]}$, hence (2) \Rightarrow (1).

Finally, if (3) holds then $\chi(X, F(t)) - \chi(X, G(t))$ has degree $\leq n-2$, hence, as we proved, G is a tight husk of F . By (9.37.1) G is a subsheaf of $F^{[**]}$. Thus $G = F^{[**]}$ since they have the same Hilbert polynomials. \square

DEFINITION 9.39 (Husks over a base scheme). Let $f : X \rightarrow S$ be a morphism and F a coherent sheaf on X . A *quotient husk* of F over S is a quasi-coherent sheaf G together with a homomorphism $q : F \rightarrow G$ such that

(9.39.1) G is flat and pure over S and

(9.39.2) $q_s : F_s \rightarrow G_s$ is a quotient husk for every $s \in S$.

A quotient husk is called a *husk* if

(9.39.3) $q_s : F_s \rightarrow G_s$ is a husk for every $s \in S$.

As before, G is also an $\mathcal{O}_X/\text{Ann}(F)$ sheaf and so X matters very little.

Warning. The notion of a (quotient) husk over S does depend on $f : X \rightarrow S$. If S is pure, S_2 and $f : F \rightarrow G$ is a (quotient) husk over S then it is a (quotient)

husk as defined in (9.33), but the converse does not hold. The two notions are even more different if the base scheme is not S_2 .

Our main interest is in the relative case; we sometimes omit ‘over S ’ if our choice of S is clear from the context.

Husks and quotient husks are preserved by base change. That is, let $q : F \rightarrow G$ be a (quotient) husk over S and $g : T \rightarrow S$ a morphism. Set $X_T := X \times_S T$ and let $g_X : X_T \rightarrow X$ be the first projection. Then $g_X^*q : g_X^*F \rightarrow g_X^*G$ is a (quotient) husk over T .

9.40 (Openness of husks). Let $\pi : X \rightarrow S$ be a morphism and $q : F \rightarrow G$ a map of coherent sheaves on X . Assume that G is flat and pure over S . By the Nakayama lemma, for a map between sheaves it is an open condition to be surjective and for a surjective map with flat target it is an open condition to be fiber-wise injective (cf. [Mat86, 22.5], see also [Kol96, I.7.4.1]). Thus the set of points

$$\{x \in X : q_{\pi(x)} : F_{\pi(x)} \rightarrow G_{\pi(x)} \text{ is a local isomorphism at } x\}$$

is open in X . In particular, if π is proper then

$$\{s \in S : q_s : F_s \rightarrow G_s \text{ is a (quotient) husk}\}$$

is open in S .

9.6. Moduli space of quotient husks

DEFINITION 9.41. Let $f : X \rightarrow S$ be a proper morphism and F a coherent sheaf on X . Let $\mathcal{QHusk}(F)(*)$ (resp. $\mathcal{Husk}(F)(*)$)

be the functor that to a scheme $g : T \rightarrow S$ associates the set of all coherent quotient husks (resp. husks) of g_X^*F , where $g_X : T \times_S X \rightarrow X$ is the projection.

By (9.40) $\mathcal{Husk}(F)(*)$ is an open subfunctor of $\mathcal{QHusk}(F)(*)$.

If f is projective, H is an f -ample divisor class and $p(t)$ is a polynomial, then $\mathcal{QHusk}_p(F)(*) \subset \mathcal{QHusk}(F)(*)$ (resp. $\mathcal{Husk}_p(F)(*) \subset \mathcal{Husk}(F)(*)$) denotes the subfunctor of all coherent quotient husks (resp. husks) of g_X^*F with Hilbert polynomial $p(t)$. That is, quotient husks $F \rightarrow G$ such that $f_*(G \otimes H^m)$ is locally free of rank $p(m)$ for all $m \gg 1$.

The main existence theorem of this section is the following.

THEOREM 9.42. *Let $f : X \rightarrow S$ be a projective morphism and F a coherent sheaf on X . Let H be an f -ample divisor class and $p(t)$ a polynomial. Then $\mathcal{QHusk}_p(F)$ has a fine moduli space $\text{QHusk}_p(F) \rightarrow S$ which is a proper algebraic space over S .*

Our construction establishes $\text{QHusk}_p(F)$ as an algebraic space. When S is a point, the projectivity of $\text{QHusk}_p(F)$ is proved in [Lin15], see also [Wan15] for earlier results.

As we noted, $\mathcal{Husk}(F)$ is an open subfunctor of $\mathcal{QHusk}(F)$, thus $\mathcal{Husk}_p(F)$ is represented by an open subspace $\text{Husk}_p(F) \subset \text{QHusk}_p(F)$, which is usually not closed. There are, however, many important cases when $\text{Husk}_p(F)$ is also proper over S .

DEFINITION 9.43. Let $f : X \rightarrow S$ be a morphism and F a coherent sheaf on X . Let $n = \max_{s \in S} \dim(F_s)$. We say that F is *generically flat on every fiber* (or,

more precisely, on every fiber of $\text{Supp } F \rightarrow S$) if F is flat at every n -dimensional point of every fiber X_s . If F is coherent, then this is equivalent to the following.

There is a subscheme $Z \subset X$ such that

$$(9.43.1) \quad F|_{X \setminus Z} \text{ is flat over } S, \text{ and}$$

$$(9.43.2) \quad \dim(X_s \cap Z) < n \text{ for every } s \in S.$$

If f is proper and F is coherent, then $\text{Supp } F \rightarrow S$ is also proper. If, in addition, F is generically flat on every fiber then $s \mapsto \dim(X_s \cap \text{Supp } F)$ is locally constant. To simplify notation we always assume that it is actually constant.

COROLLARY 9.44. *Let $f : X \rightarrow S$ be a projective morphism and F a coherent sheaf that is generically flat on every fiber. Let H be an f -ample divisor class and $p(t)$ a polynomial. Then $\mathcal{Husk}_p(F)$ has a fine moduli space $\text{Husk}_p(F) \rightarrow S$ which is a proper algebraic space over S .*

We start the proof of (9.42) by establishing the valuative criteria of properness and separatedness. Then we define certain open subfunctors $\mathcal{QHusk}_p^m(F) \subset \mathcal{QHusk}_p(F)$ and construct their moduli spaces $\text{QHusk}_p^m(F)$ using quot-schemes (9.49). At the end we check that $\mathcal{QHusk}_p(F) = \mathcal{QHusk}_p^m(F)$ for $m \gg 1$. The rest of the section is devoted to the details of these arguments. The implication (9.42) \Rightarrow (9.44) is proved at the end of (9.45).

As a preliminary step, note that the problem is local on S , thus we may assume that S is affine. Then f, X, F are defined over a finitely generated subalgebra of \mathcal{O}_S , hence we may assume in the sequel that S is of finite type.

9.45 (The valuative criteria of separatedness and properness). More generally, we show that $\mathcal{QHusk}(F)$ satisfies the valuative criteria of separatedness and properness whenever f is proper.

Let T be the spectrum of an excellent DVR with closed point $0 \in T$ and generic point $t \in T$. Given $g : T \rightarrow S$ let $g_X : T \times_S X \rightarrow X$ denote the projection. We have the coherent sheaf $g_X^* F$ and, over the generic point, a quotient husk $q_t : g_X^* F_t \rightarrow g_X^* G_t$. We aim to extend it to a quotient husk $\tilde{q} : g_X^* F \rightarrow \tilde{G}$.

Let $K \subset g_X^* F$ be the largest subsheaf that agrees with $\ker q_t$ over the generic fiber. Then $g_X^* F/K$ is a coherent sheaf on X_T and none of its associated primes is contained in X_0 . Thus $g_X^* F/K$ is flat over T . Let $Z_0 \subset X_0$ be the union of the embedded primes of $(g_X^* F/K)_0$.

By construction q_t descends to a morphism $q'_t : (g_X^* F/K)_t \hookrightarrow g_X^* G_t$. Let $Z_t \subset \text{Supp}(g_X^* F/K)_t$ be the closed subset where q'_t is not an isomorphism and $Z_T \subset X_T$ its closure. Finally set $Z = Z_0 \cup (Z_T \cap X_0)$.

The restriction of the sheaf $g_X^* F/K$ to $X_T \setminus (Z_0 \cup Z_T)$ is flat and pure over T and $g_X^* G_t$ is pure on $X_t = X_T \setminus X_0$. Furthermore, when restricted to $X_T \setminus (X_0 \cup Z_T)$, both of these sheaves are naturally isomorphic to $g_X^* F/K$. Thus we can glue them to get a single sheaf G' defined on $X_T \setminus Z$ that is flat and pure over T .

Let $j : X_T \setminus Z \hookrightarrow X_T$ be the injection. By (9.46) $\tilde{G} := j_* G'$ is the unique extension that is flat and pure over T hence

$$\tilde{q} : g_X^* F \rightarrow g_X^* F/K \rightarrow \tilde{G}$$

is the unique quotient husk extending $q_t : F_t \rightarrow G_t$. Thus $\mathcal{QHusk}(F)$ satisfies the valuative criteria of separatedness and properness.

Furthermore, if f is projective then \tilde{G}_0 has the same Hilbert polynomial as G_t .

Finally note that if F is generically flat over S and $q_t : g_X^* F_t \rightarrow g_X^* G_t$ is a husk then $K \subset g_X^* F$ is zero at the generic points of $X_0 \cap \text{Supp } g_X^* F$, thus $\tilde{q} : g_X^* F \rightarrow g_X^* F/K \rightarrow \tilde{G}$ is a husk.

This shows that if F is generically flat over S then $\text{Husk}(F)$ is closed in $\text{QHusk}(F)$ hence (9.44) follows from (9.42).

The following extension result is the key to understanding flat families of coherent sheaves over spectra of discrete valuation rings.

COROLLARY 9.46. *Let T be the spectrum of an excellent DVR with closed point $0 \in T$, generic point $t \in T$ and $g : X \rightarrow T$ a morphism. Let $Z \subset X_0$ be a closed subscheme and $j : X \setminus Z \hookrightarrow X$ the natural injection. Let G be a coherent sheaf on $X \setminus Z$. Assume that*

(9.46.1) G is flat over T ,

(9.46.2) Z does not contain any of the generic points of $(\overline{W})_0$ for any associated prime W of G .

Then $F = j_ G$ is the unique extension of G to a coherent sheaf on X such that*

(9.46.3) F is flat over T and

(9.46.4) F_0 has no associated primes contained in Z .

Note that assumption (2) is automatic if $\dim Z < \dim W_t$ for every associated prime W_t of G_t .

Proof. If F_0 has no associated primes contained in Z then $\text{depth}_{Z_0} F_0 \geq 1$ hence $\text{depth}_{Z_0} F \geq 2$ and so $F = j_* G$ by (9.8)

Conversely, $j_* G$ is coherent by (10.25) and $\text{depth}_{Z_0} F \geq 2$ by (9.8) which implies that $\text{depth}_{Z_0} F_0 \geq 1$. \square

9.47 (Construction of bounded open subfunctors). For a given $m \in \mathbb{N}$ let $\mathcal{QHusk}_p^m(F) \subset \mathcal{QHusk}_p(F)$ be the open subfunctor of all quotient husks $F_s \rightarrow G_s$ such that $G_s(m)$ is generated by global sections and its higher cohomologies vanish.

Let E be any coherent sheaf on X that is flat over S with proper support. If $H^i(X_s, E_s) = 0$ for some $s \in S$ and all $i > 0$, then this vanishing holds in an open neighborhood of $s \in S$. Thus all sections of E_s lift to nearby fibers, hence, if E_s is globally generated then so are the nearby $E_{s'}$. Thus we expect that, for every m , the moduli space $\mathcal{QHusk}_p^m(F)$ is an open subscheme of $\mathcal{QHusk}_p(F)$.

We prove later that $\mathcal{QHusk}_p^m(F) = \mathcal{QHusk}_p(F)$ for $m \gg 1$.

9.48 (Construction of $\mathcal{QHusk}_p^m(F)$). We use the existence and basic properties of quot-schemes (9.49) and hom-schemes (9.50).

By assumption, each $G_s(m)$ can be written as a quotient of $\mathcal{O}_{X_s}^{\oplus p(m)}$. Let

$$Q_{p(t)} := \text{Quot}_{p(t)}^{\circ}(\mathcal{O}_X^{\oplus p(m)}) \subset \text{Quot}(\mathcal{O}_X^{\oplus p(m)})$$

be the universal family of quotients $q_s : \mathcal{O}_{X_s}^{\oplus p(m)} \rightarrow M_s$ that have Hilbert polynomial $p(t)$, are pure, have no higher cohomologies and the induced map

$$q_s : H^0(X_s, \mathcal{O}_{X_s}^{\oplus p(m)}) \rightarrow H^0(X_s, M_s)$$

is an isomorphism. Openness of purity is the $m = 1$ case of (10.3), the other two properties were discussed in (9.47).

Let $\pi : Q_{p(t)} \rightarrow S$ be the structure map, $\pi_X : Q_{p(t)} \times_S X \rightarrow X$ the second projection and M the universal sheaf on $Q_{p(t)} \times_S X$.

By (9.40), the hom-scheme $\underline{\text{Hom}}(\pi_X^* F, M)$ (9.50) has an open subscheme $W_{p(t)}$ parametrizing maps from F to M that are surjective outside a subset of dimension $\leq n - 1$. Let $\sigma : W_{p(t)} \rightarrow Q_{p(t)}$ be the structure map, and $\sigma_X : W_{p(t)} \times_S X \rightarrow Q_{p(t)} \times_S X$ the fiber product.

Note that $W_{p(t)}$ parametrizes triples

$$w := \left[F_w \xrightarrow{r_w} G_w \xleftarrow{q_w} \mathcal{O}_{X_w}(-m)^{\oplus p(m)} \right]$$

where $r_w : F_w \rightarrow G_w$ is a quotient husk with Hilbert polynomial $p(t)$ and $q_w(m) : \mathcal{O}_{X_w}^{\oplus p(m)} \rightarrow G_w(m)$ is a surjection that induces an isomorphism on the spaces of global sections.

Let $w' \in W_{p(t)}$ be another point corresponding to the triple

$$w' := \left[F_{w'} \xrightarrow{r_{w'}} G_{w'} \xleftarrow{q_{w'}} \mathcal{O}_{X_{w'}}(-m)^{\oplus p(m)} \right].$$

such that

$$[F_w \xrightarrow{r_w} G_w] \cong [F_{w'} \xrightarrow{r_{w'}} G_{w'}].$$

The difference between w and w' comes from the different ways that we can write $G_w \cong G_{w'}$ as quotients of $\mathcal{O}_{X_w}(-m)^{\oplus p(m)}$. Since we assume that the induced maps

$$q_w(m), q_{w'}(m) : H^0(X_w, \mathcal{O}_{X_w}^{\oplus p(m)}) \cong H^0(X_w, G_w(m)) = H^0(X_w, G_{w'}(m))$$

are isomorphisms, the different choices of q_w and $q_{w'}$ correspond to different bases in $H^0(X_w, G_w(mH))$. Thus the fiber of $\text{Mor}(*, W_{p(t)}) \rightarrow \mathcal{QHusk}_p(F)(*)$ over $\pi \circ \sigma(w) = \pi \circ \sigma(w') =: s \in S$ is a principal homogeneous space under the algebraic group $\text{GL}(p(m), k(s)) = \text{Aut}(H^0(X_s, G_s(m)))$.

Thus the group scheme $\text{GL}(p(m), S)$ acts on $W_{p(t)}$ and, by (8.38),

$$\text{QHusk}_p^m(F) = W_{p(t)}/\text{GL}(p(m), S).$$

9.49 (Quot-schemes). Let $f : X \rightarrow S$ be a morphism and F a coherent sheaf on X . $\text{Quot}(F)(*)$ denotes the functor that to a scheme $g : T \rightarrow S$ associates the set of all quotients of $g_X^* F$ that are flat over T with proper support, where $g_X : T \times_S X \rightarrow X$ is the projection.

If $F = \mathcal{O}_X$, then a quotient can be identified with a subscheme of X , thus $\text{Quot}(\mathcal{O}_X) = \text{Hilb}(X)$, the Hilbert functor.

If H is an f -ample divisor class and $p(t)$ a polynomial, then $\text{Quot}_p(F)(*)$ denotes those flat quotients that have Hilbert polynomial $p(t)$.

By [Gro62, Lect.IV], $\text{Quot}_p(F)$ is bounded, proper, separated and it has a fine moduli space $\text{Quot}_p(F)$. See [Ser06, Sec.4.4] for a detailed proof. If $F = \mathcal{O}_X$, then $\text{Quot}(\mathcal{O}_X) = \text{Hilb}(X)$, the Hilbert scheme of X .

Note that one can write F as a quotient of $\mathcal{O}_{\mathbb{P}^n}(-m)^r$ for some m, r , thus $\text{Quot}_p(F)$ can be viewed as a subfunctor of $\text{Quot}(\mathcal{O}_{\mathbb{P}^n}^r)$. The theory of $\text{Quot}(\mathcal{O}_{\mathbb{P}^n}^r)$ is essentially the same as the study of the Hilbert functor, discussed in [Mum66] and [Kol96, Sec.I.1].

9.50 (Hom-schemes). Let $f : X \rightarrow S$ be proper morphism and F, G coherent sheaves on X . $\underline{\text{Hom}}_S(F, G)(*)$ denotes the functor that to a scheme $g : T \rightarrow S$ associates the set of all homomorphisms of $g_X^* F$ to $g_X^* G$, where $g_X : T \times_S X \rightarrow X$ is the projection.

As a special case of [Gro60, III.7.7.8–9], if G is flat over S then $\underline{\mathcal{H}om}_S(F, G)(*)$ is represented by an S -scheme $\underline{\mathcal{H}om}_S(F, G)$. That is, for any $g : T \rightarrow S$, there is a natural isomorphism

$$\mathrm{Hom}_T(g_X^* F, g_X^* G) \cong \mathrm{Mor}_S(T, \underline{\mathcal{H}om}_S(F, G)).$$

To see this, note first that there is a natural identification between

(9.50.1) homomorphisms $\phi : F \rightarrow G$, and

(9.50.2) quotients $\Phi : (F + G) \twoheadrightarrow Q$ that induce an isomorphism $\Phi|_G : G \cong Q$.

Next let $\pi : \mathrm{Quot}_S(F + G) \rightarrow S$ denote the quot-scheme parametrizing quotients of $F + G$ with universal quotient $u : \pi_X^*(F + G) \rightarrow Q$, where π_X denotes the induced map $\pi_X : \mathrm{Quot}_S(F + G) \times_S X \rightarrow X$.

Consider now the restriction of u to $u_G : \pi_X^* G \rightarrow Q$. By (9.40) there is an open subset

$$\mathrm{Quot}_S^\circ(F + G) \subset \mathrm{Quot}_S(F + G)$$

that parametrizes those quotients $v : F + G \rightarrow Q$ that induce an isomorphism $v_G : G \cong Q$. Thus $\underline{\mathcal{H}om}_S(F, G) = \mathrm{Quot}_S^\circ(F + G)$. \square

9.51 (Isomorphism-schemes). Let $f : X \rightarrow S$ be proper morphism and F, G coherent sheaves on X . Let $\underline{\mathcal{I}som}_S(F, G)(*)$ denote the functor that to a scheme $g : T \rightarrow S$ associates the set of all isomorphisms $g_X^* F \cong g_X^* G$, where $g_X : T \times_S X \rightarrow X$ is the projection.

Thus $\underline{\mathcal{I}som}_S(F, G)(*) \subset \underline{\mathcal{H}om}_S(F, G)(*)$ and, if G is flat over S , then being an isomorphism is an open condition by (9.40). Thus we conclude that if G is flat over S then $\underline{\mathcal{I}som}_S(F, G)(*)$ is represented by an open subscheme $\underline{\mathcal{I}som}_S(F, G) \subset \underline{\mathcal{H}om}_S(F, G)$. That is, for any $g : T \rightarrow S$, there is a natural isomorphism

$$\mathrm{Isom}_T(g_X^* F, g_X^* G) \cong \mathrm{Mor}_S(T, \underline{\mathcal{I}som}_S(F, G)).$$

9.52 (A boundedness condition). Let $X \subset \mathbb{P}^N$ be a projective scheme over a field. As a temporary convenience, let us say that a coherent sheaf G on X satisfies condition $\mathcal{B}(m)$ if

(9.52.1) $H^i(X, G(r)) = 0$ for $i > 0$ and $r \geq m$,

(9.52.2) $H^0(X, G(r)) \otimes H^0(X, \mathcal{O}_X(1)) \twoheadrightarrow H^0(X, G(r+1))$ for $r \geq m$.

Note that (2) implies that $G(m)$ is generated by global sections. Thus it can be written as a quotient of $\mathcal{O}_X^{p(m)}$ where $p(m) = h^0(X, G(m))$ (and $p(t)$ is the Hilbert polynomial of G). In particular, all sheaves G that satisfy $\mathcal{B}(m)$ and have Hilbert polynomial $p(t)$ form a bounded family by (9.49).

While we care about the latter conclusion, the assumptions (1–2) are better suited for inductive arguments.

Condition $\mathcal{B}(m)$ should be thought of as a crude version of *Castelnuovo-Mumford regularity*; see [Laz04, Sec.I.1.8] for a detailed treatment.

PROPOSITION 9.53. *Let $X \subset \mathbb{P}^N$ be a projective scheme over a field and G a pure, coherent sheaf of dimension ≥ 2 on X with Hilbert polynomial $p(t)$. Let $H \subset X$ be a hyperplane section such that $G_H := G \otimes \mathcal{O}_H$ is also pure. Assume that G_H satisfies condition $\mathcal{B}(m_H)$.*

Then there is an m_X , depending only on m_H and on $p(t)$, such that G satisfies condition $\mathcal{B}(m_X)$.

Proof. Using the cohomology sequence of

$$0 \rightarrow G(r-1) \rightarrow G(r) \rightarrow G_H(r) \rightarrow 0$$

we conclude that $H^i(X, G(r-1)) \cong H^i(X, G(r))$ for $i \geq 2$ and $r \geq m_H + 1$. Thus, by Serre's vanishing, $H^i(X, G(r)) = 0$ for $i \geq 2$ and $r \geq m_H + 1$.

For $i = 1$ we have only an exact sequence

$$H^0(X, G(r)) \xrightarrow{b(r)} H^0(X, G_H(r)) \rightarrow H^1(X, G(r-1)) \xrightarrow{c(r)} H^1(X, G(r)) \rightarrow 0,$$

which shows that $b(r)$ is onto iff $c(r)$ is an isomorphism.

If $b(r)$ is onto for some $r \geq m_H$ then $b(r+1)$ is also onto by (9.54). Thus $c(r)$ is an isomorphism for every $r \geq m_H$. By Serre's vanishing, this again gives that $H^1(X, G(r)) = 0$ for every $r \geq m_H$.

Otherwise $h^1(X, G(r-1)) > h^1(X, G(r))$. In either case we get that

$$H^1(X, G(r)) = 0 \quad \text{for } r \geq m_H + h^1(X, G(m_H)).$$

Since $h^1(X, G(m_H)) = h^0(X, G(m_H)) - p(m_H)$, we are done if we can bound $h^0(X, G(m_H))$ from above. Since G and G_H are pure,

$$h^0(X, G(r)) \leq \sum_{i \geq 0} h^0(X, G_H(r-i)),$$

thus it is enough to bound the sum on the right. Note that $h^0(X, G_H(m_H)) = \chi(X, G_H(m_H)) = p(m_H) - p(m_H - 1)$ and

$$h^0(X, G_H(r)) \neq 0 \Rightarrow h^0(X, G_H(r-1)) < h^0(X, G_H(r)).$$

This bounds $h^0(X, G(m_H))$ and $h^1(X, G(m_H))$ from above. \square

LEMMA 9.54. *Let $X \subset \mathbb{P}^N$ be a projective scheme, $H := (s = 0) \subset X$ a hyperplane section. Let G be a coherent sheaf on X such that s is not a zero divisor on G and set $G_H := G \otimes \mathcal{O}_H$. Assume that*

$$(9.54.1) \quad H^0(X, G) \twoheadrightarrow H^0(H, G_H) \text{ and}$$

$$(9.54.2) \quad H^0(H, G_H) \otimes H^0(X, \mathcal{O}_X(1)) \twoheadrightarrow H^0(H, G_H(1)).$$

Then, for every $m \geq 1$ we have

$$(9.54.3) \quad H^0(X, G(m)) \twoheadrightarrow H^0(H, G_H(m)) \text{ and}$$

$$(9.54.4) \quad H^0(X, G) \otimes H^0(X, \mathcal{O}_X(m)) \twoheadrightarrow H^0(X, G(m)).$$

Proof. By induction it is enough to show this for $m = 1$. Consider the following diagram.

$$\begin{array}{ccc} H^0(X, G) \otimes H^0(X, \mathcal{O}_X) & = & H^0(X, G) \\ \downarrow & & \downarrow \\ H^0(X, G) \otimes H^0(X, \mathcal{O}_X(1)) & \twoheadrightarrow & H^0(X, G(1)) \\ \downarrow & & \downarrow \\ H^0(H, G_H) \otimes H^0(X, \mathcal{O}_X(1)) & \twoheadrightarrow & H^0(H, G_H(1)). \end{array}$$

Here the right vertical sequence is exact and the assumptions say that the lower left vertical and bottom horizontal arrows are surjective. The conclusion follows by an easy diagram chasing. \square

We are now ready to prove the boundedness of $\text{QHusk}_p^m(F)$.

PROPOSITION 9.55. *Let $f : X \rightarrow S$ be a projective morphism, H an f -ample divisor class, $p(t)$ a polynomial and F a coherent sheaf on X . Then there is an m such that for every point $s \rightarrow S$, every quotient husk of F_s with Hilbert polynomial $p(t)$ satisfies the condition $\mathcal{B}(m)$.*

Proof. The proof is by induction on $n := \deg p(t)$ which is also the dimension of the husks. If $n = 0$ then $\dim G_s = 0$ and $\mathcal{B}(m)$ holds for every m .

Next consider the case $n = 1$. Let d be the leading coefficient of $p(t)$. We may assume that $X \subset \mathbb{P}_S^N$, H is the hyperplane class and F is a quotient of a direct sum $\oplus_i \mathcal{O}_{\mathbb{P}^N}(-q)$ for some q . Set $C_s := \text{Spec } \mathcal{O}_{\mathbb{P}^N} / \text{Ann}(G_s)$ and note that $\deg C_s \leq d$. We have a morphism $\oplus_i \mathcal{O}_{C_s}(-q) \rightarrow G_s$ that is surjective at all 1-dimensional points. This gives surjections

$$\oplus_i H^1(C_s, \mathcal{O}_{C_s}(r - q)) \rightarrow H^1(X_s, G_s(r)).$$

Thus $H^1(X_s, G_s(r)) = 0$ for $r \geq q + d - m - 1$ by (9.56).

Finally assume that $n \geq 2$. Let $F_s \rightarrow G_s$ be a quotient husk and $H_s \subset X_s$ a hyperplane section. As long as H_s does not contain any of the associated primes of $F_s, G_s, G_s/F_s$ and $G_s|_{H_s}$ is pure (10.9) we see that $G_s|_{H_s}$ is a quotient husk of $F_s|_{H_s}$ with Hilbert polynomial $p(t) - p(t - 1)$.

If $X \subset \mathbb{P}_S^N$ then the restrictions $F_s|_{H_s}$ are fibers of a coherent sheaf on

$$X \times_S \check{\mathbb{P}}_S^N \rightarrow \check{\mathbb{P}}_S^N$$

where $\check{\mathbb{P}}_S^N$ is the dual projective space bundle parametrizing all hyperplanes in \mathbb{P}_S^N . Therefore, by induction, the $G_s|_{H_s}$ satisfy $\mathcal{B}(m_1)$ for some m_1 by induction. Thus, by (9.53), the G_s satisfy $\mathcal{B}(m)$ for some fixed m . \square

LEMMA 9.56. *Let $X \subset \mathbb{P}^n$ be a subscheme of dimension m and degree d . Then $H^m(X, \mathcal{O}_X(r)) = 0$ for $r \geq d - m - 1$.*

Proof. Choose coordinates on \mathbb{P}^n such that $(x_0 = \dots = x_m = 0)$ is disjoint from X and set $L := (x_{m+1} = \dots = x_n = 0)$ with ideal sheaf I_L . Consider the \mathbb{G}_m -action

$$\rho_t : (x_0 : \dots : x_n) \mapsto (x_0 : \dots : x_m : tx_{m+1} : \dots : tx_n).$$

As $t \rightarrow 0$, the flat limit of the schemes $\rho_t(X)$ is a subscheme X_0 whose support is L . By semicontinuity it is enough to prove that $H^m(X_0, \mathcal{O}_{X_0}(r)) = 0$ for $r \geq d - m - 1$. The top cohomology is unchanged by removing embedded points, thus we may assume that X_0 is pure, in particular \mathcal{O}_{X_0} is a quotient of $\mathcal{O}_{\mathbb{P}^n}/I_L^d$.

Note that $\mathcal{O}_{\mathbb{P}^n}/I_L^d$ is a successive extension of line bundles of the form $\mathcal{O}_L(b)$ where $0 \geq b \geq 1 - d$. Thus $H^m(\mathbb{P}^n, (\mathcal{O}_{\mathbb{P}^n}/I_L^d)(r)) = 0$ for $r + 1 - d \geq -m$. \square

9.7. Hulls and Hilbert polynomials

Let $f : X \rightarrow S$ be a projective morphism with relatively ample line bundle $\mathcal{O}_X(1)$. For a coherent sheaf F on X we aim to understand flatness of F and of its hull F^H in terms of the Hilbert polynomials $\chi(X_s, F_s(t))$ of the fibers F_s . Note that the $\chi(X_s, F_s(t))$ carry no information about the nilpotents in \mathcal{O}_S .

THEOREM 9.57. *Using the above notation, assume that S is reduced. Then $s \mapsto \chi(X_s, F_s^H)$ is an upper semicontinuous function on S and F is flat over S iff this function is locally constant.*

Proof. By generic flatness [Eis95, 14.4], there is a dense open subset $S^\circ \subset S$ such that F is flat over S° . Thus the function $s \mapsto \chi(X_s, F_s(t))$ is locally constant on S° , hence constructible on S by Noetherian induction.

It is thus enough to prove upper semicontinuity when $(0, S)$ is the spectrum of a DVR with generic point g . Let $t_0(F) \subset F$ denote the largest subsheaf supported on X_0 . Then $F/t_0(F)$ is flat over S hence

$$\chi(X_g, F_g(t)) = \chi(X_g, (F/t_0(F))_g(t)) = \chi(X_0, (F/t_0(F))_0(t)).$$

Furthermore, a moment's thought shows that there is an exact sequence

$$0 \rightarrow t_0(F)_0 \rightarrow F_0 \rightarrow (F/t_0(F))_0 \rightarrow 0,$$

hence $\chi(X_0, F_0(*)) \succeq \chi(X_0, (F/t_0(F))_0(*))$ and equality holds iff $t_0(F) = 0$.

The last claim is proved (although not stated) in [Har77, III.9.9]. □

We have similar results for the Hilbert polynomials of hulls.

THEOREM 9.58. *Let $f : X \rightarrow S$ be a projective morphism with relatively ample line bundle $\mathcal{O}_X(1)$ and F a mostly flat family of coherent, S_2 sheaves. Assume that S is reduced. Then $s \mapsto \chi(X_s, F_s^H(*))$ is an upper semicontinuous function and F^H is a universal hull iff this function is locally constant on S .*

Proof. As in the proof of (9.57) we obtain that $s \mapsto \chi(X_s, F_s^{[**]}(t))$ is constructible and it is enough to prove upper semicontinuity when $(0, S)$ is the spectrum of a DVR with generic point g . The argument closely parallels (5.48.3–5).

We may replace F by its hull, hence we may assume that F is S_2 and flat over S . In particular, $\chi(X_0, F_0(t)) = \chi(X_g, F_g(t))$.

Furthermore, F_0 is S_1 , hence the restriction map (9.25) $r_0^F : F_0 \rightarrow F_0^H$ is an injection. The exact sequence

$$0 \rightarrow F_0 \rightarrow F_0^H \rightarrow Q_0 \rightarrow 0$$

defines Q_0 and

$$\chi(X_0, F_0^H(t)) = \chi(X_0, F_0(t)) + \chi(X_0, Q_0(t)).$$

This gives that

$$\chi(X_0, F_0^H(t)) \succeq \chi(X_0, F_0(t)) \equiv \chi(X_g, F_g(t))$$

and equality holds iff $r_0^F : F_0 \rightarrow F_0^H$ is an isomorphism. By (9.27), in this case F^H is a universal hull.

We have thus proved that if $s \mapsto \chi(X_s, F_s^{[**]}(t))$ is locally constant and S is regular and 1-dimensional then F^H is a universal hull of F . We show in (9.65) that this implies the general case. □

PROPOSITION 9.59. *Let $f : X \rightarrow S$ be a projective morphism with relatively ample line bundle $\mathcal{O}_X(1)$ and F a mostly flat family of coherent, S_2 sheaves. Then F^H is a universal hull iff for every local, Artinian ring (A, m_A) with residue field $k = A/m_A$ and every morphism $\text{Spec } A \rightarrow S$ we have*

$$\chi(X_A, (F_A)^H(t)) \equiv \chi(X_k, (F_k)^H(t)) \cdot \text{length } A.$$

Proof. We show that the condition holds iff $(F_A)^H$ is flat over A and then conclude using (9.27.6).

Pick a maximum length filtration of A and lift it to a filtration

$$0 = G_0^U \subset G_1^U \subset \cdots \subset G_r^U = F_A|_{U_A}$$

such that $G_{i+1}^U/G_i^U \cong F_k|_{U_k}$ and $r = \text{length } A$. By pushing it forward to X_A we get a filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_r = (F_A)^H$$

such that $G_{i+1}/G_i \subset (F_k)^H$. Therefore

$$\chi(X_A, (F_A)^H(t)) \leq \chi(X_k, (F_k)^H(t)) \cdot \text{length } A$$

and equality holds iff $G_{i+1}/G_i = (F_k)^H$ for every i , that is, iff F_A^H is flat over A . \square

COROLLARY 9.60. *Let S be a reduced scheme and $g : X \rightarrow S$ a smooth, projective morphism. Let $D \subset X$ be a divisorial subscheme such that $g|_D : D \rightarrow S$ is flat at generic points of the fibers of $g|_D$. Then D is a relative Cartier divisor.*

Proof. We claim that $s \mapsto \chi(X_s, \mathcal{O}_{X_s}(-D_s)^H)$ is locally constant on S . It is enough to prove this after base change to a DVR $T \rightarrow S$. Then X_T is regular hence D_T is a Cartier divisor. Thus $\mathcal{O}_X(-D)$ is its own universal hull by (9.58). So D is a relative Cartier divisor by (4.24). \square

The next example shows a rather typical situation where (9.60) does not hold over nonreduced bases.

EXAMPLE 9.61. Let $f, g \in k[x_1, \dots, x_n]$ be relatively prime, irreducible polynomials. Consider the hypersurface

$$H := (fg = t) \subset \mathbb{A}_x^n \times \mathbb{A}_t^1.$$

Then H is irreducible, flat over \mathbb{A}^1 and a relative Cartier divisor.

For some $r \geq 2$ set $A_r := (t^r = 0)$. By base change we get

$$H_r := (fg = t) \subset \mathbb{A}^n \times A_r.$$

It is the spectrum of

$$k[x_1, \dots, x_n, t]/(fg - t, t^r) \cong k[x_1, \dots, x_n]/(f^r g^r).$$

Thus H_r is reducible with irreducible components

$$H_{r,f} := \text{Spec } k[x_1, \dots, x_n]/(f^r) \quad \text{and} \quad H_{r,g} := \text{Spec } k[x_1, \dots, x_n]/(g^r).$$

Both of these are CM, but the map to A_r is given by fg . Thus the central fiber of $H_{r,f} \rightarrow A_r$ is $\text{Spec } k[x_1, \dots, x_n]/(f^r, fg)$, which has an embedded component along $(f = g = 0)$. Thus $H_{r,f} \rightarrow A_r$ is flat on the open set $(g \neq 0)$ but it is not flat along $(g = 0)$ and $H_{r,f}$ is not a Cartier divisor in $\mathbb{A}^n \times A_r$.

The next result roughly says that flatness of H^0 implies flatness for globally generated sheaves. It is similar to Grauert's theorem on direct images [Har77, III.12.9]; the key difference is that we do not have a flat sheaf to start with.

PROPOSITION 9.62. *Let S be a reduced scheme and $f : X \rightarrow S$ a proper morphism. Let F be a mostly flat family of coherent, S_2 sheaves on X . Assume that*

(9.62.1) $s \mapsto h^0(X_s, F_s^H)$ is a locally constant function on S and

(9.62.2) F_s^H is generated by its global sections for every $s \in S$.

Then F^H is a universal hull and $f_(F^H)$ is locally free.*

Proof. Assume first that S is the spectrum of a DVR. We may replace F by F^H , hence assume that F is flat over S . Then $F_s \hookrightarrow F_s^H$ is an injection and we have inequalities

$$h^0(X_g, F_g) \leq h^0(X_s, F_s) \leq h^0(X_s, F_s^H). \quad (9.62.3)$$

By (1) these are equalities. Since F_s^H is generated by its global sections, this implies that $F_s = F_s^H$. As we explain in (9.65), this implies that F^H is a universal hull for every S . The last claim then follows from Grauert's theorem. \square

9.8. Moduli space of universal hulls

DEFINITION 9.63. Let $f : X \rightarrow S$ be a morphism and F a coherent sheaf on X . For a scheme $g : T \rightarrow S$ set $\mathcal{Hull}(F)(T) = 1$ if $g_X^* F$ has a universal hull and $\mathcal{Hull}(F)(T) = \emptyset$ otherwise, where $g_X : T \times_S X \rightarrow X$ is the projection.

If f is projective and p is a polynomial we set $\mathcal{Hull}_p(F)(T) = 1$ if $g_X^* F$ has a universal hull with Hilbert polynomial p .

The following result is the key to many applications of the theory.

THEOREM 9.64 (Flattening decomposition for universal hulls). *Let $f : X \rightarrow S$ be a projective morphism and F a coherent sheaf on X . Then*

(9.64.1) $\mathcal{Hull}_p(F)$ has a fine moduli space $\text{Hull}_p(F)$.

(9.64.2) The structure map $\text{Hull}_p(F) \rightarrow S$ is a locally closed embedding (10.84).

(9.64.3) The structure map $\text{Hull}(F) = \coprod_p \text{Hull}_p(F) \rightarrow S$ is a locally closed decomposition (10.84).

Proof. Let n be the maximal fiber dimension of $\text{Supp } F \rightarrow S$ and $S_n \subset S$ the closed subscheme parametrizing n -dimensional fibers. We construct $\text{Hull}(F, n)$, the fine moduli space of n -dimensional universal hulls and then repeat the argument for $S \setminus S_n$.

Let $\pi : \text{Husk}(F) \rightarrow S$ be the structure map, $\pi_X : \text{Husk}(F) \times_S X \rightarrow X$ the second projection and $q_{\text{univ}} : \pi_X^* F \rightarrow G_{\text{univ}}$ the universal husk. The set of points $y \in \text{Husk}(F)$ such that $(G_{\text{univ}})_y$ is S_2 and has pure dimension n is open by (10.3). The fiber dimension of

$$\text{Supp coker}[\pi_X^* F \rightarrow G_{\text{univ}}] \rightarrow \text{Husk}(F)$$

is upper semicontinuous. Thus there is a largest open set $W_n \subset \text{Husk}(F)$ parametrizing husks $F_s \rightarrow G_s$ such that G_s is S_2 , has pure dimension n and $\dim \text{Supp } G_s/F_s \leq n - 2$. By (9.28), $\text{Hull}(F, n) = W_n$.

Since hulls are unique (9.22), $\text{Hull}(F) \rightarrow S$ is a monomorphism (10.83). In order to prove that each $\text{Hull}_p(F) \rightarrow S$ is a locally closed embedding, we check the valuative criterion (10.85).

Let $(0, T)$ be the spectrum of a DVR with generic point g and $p : T \rightarrow S$ a morphism such that the hulls of F_g and of F_0 have the same Hilbert polynomials. Let G_g denote the hull of F_g and extend G_g to a husk $F_T \rightarrow G_T$. By assumption and by flatness

$$\chi(X_0, (G_T)_0(t)) = \chi(X_g, (G_T)_g(t)) = \chi(X_g, (F_g)^H(t)) = \chi(X_0, (F_0)^H(t)).$$

Hence $(G_T)_0 = (F_0)^H$ by (9.38) and so G_T is the relative hull of F_T . Thus G_T defines the lifting $T \rightarrow \text{Hull}_p(F)$. \square

9.65 (End of the proof of (9.58) and (9.62)). Let T be the spectrum of a DVR and $p : T \rightarrow S$ a morphism. We have already proved in both cases that $(p^*F)^H$ is a universal hull. Thus $p : T \rightarrow S$ lifts to $\tilde{p} : T \rightarrow \text{Hull}(F)$ and so $\text{Hull}(F) \rightarrow S$ is proper. By (10.83), a proper monomorphism is a closed embedding. Since $\text{Hull}(F) \rightarrow S$ is also surjective, it is an isomorphism if S is reduced. \square

Fiberwise isomorphisms.

DEFINITION 9.66. Let $f : X \rightarrow S$ be a proper morphism. A mostly flat divisorial sheaf L on X is called *relatively trivial* or *f-trivial* if there is a line bundle L_S on S such that $L \cong f^*L_S$. If $f_*\mathcal{O}_X = \mathcal{O}_S$ then this holds iff f_*L is a line bundle on S and the natural map $f^*f_*L \rightarrow L$ is an isomorphism. The functor of *relatively trivial pull-backs* of a mostly flat divisorial sheaf M is defined as

$$\mathcal{RT}_M(T) = \begin{cases} 1 & \text{if } h_X^*M \text{ is relatively trivial, and} \\ 0 & \text{otherwise,} \end{cases} \tag{9.66.1}$$

where we use the usual base change diagram

$$\begin{array}{ccc} X_T & \xrightarrow{h_X} & X \\ f_T \downarrow & & \downarrow f \\ T & \xrightarrow{h} & S. \end{array} \tag{9.66.2}$$

Similarly, the functor of *relatively trivial hulls* of M is defined as

$$\mathcal{RTH}_M(T) = \begin{cases} 1 & \text{if } (h_X^*M)^H \text{ is a relatively trivial universal hull, and} \\ 0 & \text{otherwise.} \end{cases} \tag{9.66.3}$$

More generally, two coherent sheaves F, G on X are called *relatively isomorphic* or *f-isomorphic* if there is a line bundle L_S on S such that $F \cong G \otimes f^*L_S$. One can then define the functor of *relatively isomorphic pull-backs* and also the functor of *relatively isomorphic hulls*. These are denoted by

$$\mathcal{RI}_{F,G}(\ast) \quad \text{and} \quad \mathcal{RTH}_{F,G}(\ast). \tag{9.66.4}$$

In order to compare this with $\underline{\text{Isom}}_S(F, G)(\ast)$ defined in (9.51), note that if T is local then $\mathcal{RI}_{F,G}(T) = 1$ iff $h_X^*F \cong h_X^*G$, but $\underline{\text{Isom}}_S(F, G)(T)$ is the set of all isomorphisms, hence almost always infinite. If $\mathcal{RI}_{F,G}(\ast)$ is represented by a subscheme $\text{RI}_{F,G} \subset S$ then S is the scheme theoretic image of $\underline{\text{Isom}}_S(F, G) \rightarrow S$.

This suggests that in general $\text{RI}_{F,G}$ is only a constructible subset of S . However, if we expect $\underline{\text{Isom}}_S(F, G) \rightarrow S$ to be flat over its image, then $\text{RI}_{F,G} \rightarrow S$ should be a locally closed embedding. This approach seems technically complicated, thus we go around it in the special cases needed for our current applications.

PROPOSITION 9.67. *Let $f : X \rightarrow S$ be a flat, projective morphism with S_2 fibers such that $H^0(X_s, \mathcal{O}_{X_s}) \cong k(s)$ for every $s \in S$. Let L be a mostly flat family of divisorial sheaves on X . Then the functor of relatively trivial hulls is represented by a locally closed embedding $i : \text{RTH}_L \rightarrow S$.*

Proof. We may assume that S is connected. Then $p(\ast) := \chi(X_s, \mathcal{O}_{X_s}(\ast))$ is independent of $s \in S$ and $i : \text{RTH}_L \rightarrow S$ factors through $\text{Hull}_p(S) \rightarrow S$. After replacing S by $\text{Hull}_p(S)$ it remains to prove the special case when L is flat over S . This was treated in (8.34). \square

COROLLARY 9.68. *Let $f : X \rightarrow S$ be a flat, proper morphism with S_2 fibers such that $H^0(X_s, \mathcal{O}_{X_s}) \cong k(s)$ for every $s \in S$. Let L, M be mostly flat families of divisorial sheaves on X . Then the functor of relatively isomorphic hulls $\mathcal{R}\mathcal{I}\mathcal{H}_{F,G}(\ast)$ (9.66.4) is represented by a locally closed embedding $i : \text{RIH}_{L,M} \rightarrow S$.*

Proof. Note that $\text{RIH}_{L,M}$ is isomorphic to RTH_G for $G := \mathcal{H}om_X(L, M)$. \square

Pure quotients.

We get a similar flattening decomposition for pure quotients.

DEFINITION 9.69. Let $f : X \rightarrow S$ be a morphism of finite type and F a coherent sheaf on X . We say that F is *f-pure* or *relatively pure* if F is flat over S and has pure fibers (9.2). We say that $q : F \rightarrow G$ is an *f-pure quotient* or *relatively pure quotient* of F if G is *f-pure* and $G_s = \text{pure}(F_s)$ for every $s \in S$. Note that $\ker q$ is then the largest subsheaf $K \subset F$ such that $\dim(\text{Supp } K_s) < \dim(\text{Supp } F_s)$ for every $s \in S$. In particular, a relatively pure quotient is unique.

For a scheme $g : T \rightarrow S$ set $\mathcal{P}ureq(F)(T) = 1$ if g_X^*F has a relatively pure quotient and $\mathcal{P}ureq(F)(T) = \emptyset$ otherwise; where, as usual, $g_X : T \times_S X \rightarrow X$ denotes the projection.

Given a polynomial p , set $\mathcal{P}ureq_p(F)(T) = 1$ if g_X^*F has a relatively pure quotient with Hilbert polynomial p .

THEOREM 9.70 (Flattening decomposition for relatively pure quotients). *Let $f : X \rightarrow S$ be a projective morphism and F a coherent sheaf on X . Then*

(9.70.1) $\mathcal{P}ureq_p(F)$ has a fine moduli space $\text{Pureq}_p(F)$ for every p ,

(9.70.2) $\text{Pureq}_p(F) \rightarrow S$ is a locally closed embedding (10.84) and

(9.70.3) $\text{Pureq}(F) = \coprod_p \text{Pureq}_p(F) \rightarrow S$ is a locally closed decomposition (10.84).

Proof. Let n be the maximal fiber dimension of $\text{Supp } F \rightarrow S$ and $S_n \subset S$ the closed subscheme parametrizing n -dimensional fibers. We construct $\text{Pureq}(F, n)$, the fine moduli space of n -dimensional relatively pure quotients and then repeat the argument for $S \setminus S_n$.

Let $\pi : \text{Husk}(F) \rightarrow S$ be the structure map, $\pi_X : \text{Husk}(F) \times_S X \rightarrow X$ the second projection and $q_{\text{univ}} : \pi_X^*F \rightarrow G_{\text{univ}}$ the universal husk. The set of points $y \in \text{Husk}(F)$ such that $(G_{\text{univ}})_y$ is has pure dimension n and $F_y \rightarrow (G_{\text{univ}})_y$ is surjective is an open subset $W_n \subset \text{Husk}(F)$ by (10.3) and by the Nakayama lemma. Thus $\text{Pureq}(F, n) = W_n$. At the end we have $\text{Pureq}(F) = \coprod_n \text{Pureq}(F, n)$. (In the above argument we could have used $\text{Quot}(F)$ instead of $\text{Husk}(F)$.)

Since pure quotients are unique (9.69), $\text{Pureq}(F) \rightarrow S$ is a monomorphism (10.83). In order to prove that each $\text{Pureq}_p(F) \rightarrow S$ is a locally closed embedding, we check the valuative criterion (10.85).

Let $(0, T)$ be the spectrum of a DVR with generic point g and $T \rightarrow S$ a morphism such that $\text{pure}(F_g)$ and $\text{pure}(F_0)$ both have Hilbert polynomial p . We need to prove that F_T has a relatively pure quotient.

Let $K \subset F_T$ be the largest subsheaf such that $\dim(\text{Supp } K_g) < \dim(\text{Supp } F_g)$. Then F_T/K is flat over T and $(F_T/K)_g = \text{pure}(F_g)$. Thus we have a surjection $F_0 \twoheadrightarrow (F_T/K)_0$ and $\text{pure}(F_0)$ and $(F_T/K)_0$ both have Hilbert polynomial p . Thus $\text{pure}(F_0) = (F_T/K)_0$ by (9.3) and so $F_T \rightarrow F_T/K$ is the relatively pure quotient. \square

COROLLARY 9.71. *Let S be a reduced scheme, $g : X \rightarrow S$ a projective morphism and F a coherent sheaf on X . Then F has a g -pure quotient $F \twoheadrightarrow G$ iff $s \mapsto \chi(\text{pure } F_s(*))$ is locally constant on S .*

Proof. The condition is clearly necessary. In order to see the converse we may assume that S is connected. Then $\chi(\text{pure } F_s(*))$ is independent of s , call it $p(*)$. Then $\text{Pureq}_p(F) \rightarrow S$ is a locally closed embedding that is also surjective, hence an isomorphism. \square

9.9. Non-projective versions

The proofs in Section 9.8 used in an essential way the projectivity of $X \rightarrow S$. Here we consider similar questions for non-projective morphisms in two cases. If $X \rightarrow S$ is affine then a good theory seems possible only if S is local and complete. Then we study the case when $X \rightarrow S$ is proper.

Local versions.

For affine morphisms we have the following variant of (9.64).

THEOREM 9.72. *Let (S, m_S) be a complete local ring, R a finite type S -algebra and F a finite R -module that is mostly flat with S_2 fibers over S (9.23). Then there is a quotient $S \twoheadrightarrow S^u$ that represents $\text{Hull}(F)$ for local, Artin S -algebras.*

Equivalently, we claim that for every local morphism $h : (S, m_S) \rightarrow (T, m_T)$ the hull $(F_T)^H$ is universal iff there is a factorization $h : S \twoheadrightarrow S^u \rightarrow T$. Compared with (9.64), we only identify the stratum containing the closed point of $\text{Spec } S$.

Since universal hulls commute with completion (9.27.6), (9.72) implies the same statement for complete, local S -algebras.

Proof. We follow the usual method of deformation theory [Art76, Ses75, Har10]. As a first step we construct S^u .

For an ideal $I \subset S$ set $F_I := F \otimes (R/IR)$. First we claim that if $(F_I)^H$ and $(F_J)^H$ are universal hulls then so is $(F_{I \cap J})^H$. To see this, start with the exact sequence

$$0 \rightarrow S/(I \cap J) \rightarrow S/I + S/J \rightarrow S/(I + J) \rightarrow 0. \tag{9.72.1}$$

F is mostly flat over S , thus (9.72.1) stays left exact after tensoring by F and taking the hull. Thus we obtain the exact sequence

$$0 \rightarrow (F_{I \cap J})^H \rightarrow (F_I)^H + (F_J)^H \rightarrow (F_{I+J})^H. \tag{9.72.2}$$

$(F_J)^H \rightarrow (F_{I+J})^H$ is surjective since $(F_J)^H$ is a universal hull, hence (9.72.2) is also right exact.

Set $k := S/m_S$. Since $(F_I)^H$ is a universal hull, $(F_I)^H \otimes k \cong (F_m)^H$, and the same holds for J and $I + J$. Thus tensoring (9.72.2) with k yields an exact sequence

$$(F_{I \cap J})^H \otimes k \rightarrow (F_m)^H + (F_m)^H \xrightarrow{p} (F_m)^H \rightarrow 0. \tag{9.72.3}$$

Since $\ker p \cong (F_m)^H$ we see that $(F_{I \cap J})^H \otimes k \rightarrow (F_m)^H$ is surjective. By (9.27) this implies that $(F_{I \cap J})^H$ is a universal hull.

Let $I^u \subset S$ be the intersection of all those ideals I such that $(F_I)^H$ is a universal hull and $S^u := S/I^u$. By (9.73) and (9.27.6) we obtain that $(F_{S^u})^H$ is a universal hull.

By construction, if $h : S \twoheadrightarrow W := S/I_W$ is a quotient such that $(F_W)^H$ is a universal hull then $I^u \subset I_W$. We still need to prove that if (A, m_A) is a local Artin S -algebra such that $(F_A)^H$ is a universal hull then $h : S \rightarrow A$ factors through S^u .

Let $K := A/m_A$ denote the residue field. Working inductively we may assume that there is an ideal $J_A \subset A$ such that $J_A \cong K$ and $h' : S \rightarrow A' := A/J_A$ factors through S^u . Therefore $h : S \rightarrow A$ factors through $S \rightarrow S/m_S I^u$. Note that $I^u/m_S I^u$ is a finite dimensional k -vector space, call it V_k , and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & V_k & \rightarrow & S/m_S I^u & \rightarrow & S^u & \rightarrow & 0 \\ & & \downarrow \lambda & & \downarrow h & & \downarrow h' & & \\ 0 & \rightarrow & K & \rightarrow & A & \rightarrow & A' & \rightarrow & 0 \end{array} \quad (9.72.4)$$

for some k -linear map $\lambda : V_k \rightarrow K$. If $\lambda = 0$ then h factors through S^u , this is what we want. If $\lambda \neq 0$ then we show that there is an ideal $J^u \subsetneq I^u$ such that F has a universal hull over S/J^u . This contradicts our choice of I^u and proves the theorem.

It is easier to write down the obstruction map in scheme-theoretic language. Thus set $X := \text{Spec}_S R$ and let $i : U \hookrightarrow X$ be the largest open set over which \tilde{F} (the sheaf associated to F) is flat over S . For any $S \rightarrow T$ by base change we get $i : U_T \hookrightarrow X_T$. Let \mathcal{F}_T denote the restriction of \tilde{F}_T to U_T . Then $i_* \mathcal{F}_T$ is the sheaf associated to $(F_T)^H$ and we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & V_k \otimes_k i_* \mathcal{F}_k & \rightarrow & i_* \mathcal{F}_S & \rightarrow & i_* \mathcal{F}_{S^u} \xrightarrow{\delta} V_k \otimes_k R^1 i_* \mathcal{F}_k \\ & & \downarrow \lambda & & \downarrow h & & \downarrow h' \\ 0 & \rightarrow & i_* \mathcal{F}_K & \rightarrow & i_* \mathcal{F}_A & \rightarrow & i_* \mathcal{F}_{A'} \xrightarrow{\Delta} R^1 i_* \mathcal{F}_K. \end{array} \quad (9.72.5)$$

Note that $\Delta = 0$ since $i_* \mathcal{F}_A$ is a universal hull. The right hand square can be factored as

$$\begin{array}{ccc} \delta : i_* \mathcal{F}_{S^u} & \rightarrow & i_* \mathcal{F}_k \xrightarrow{\delta_k} V_k \otimes_k R^1 i_* \mathcal{F}_k \\ \downarrow h' & & \downarrow h_k \quad \downarrow \lambda \otimes \mathbf{1} \\ \Delta : i_* \mathcal{F}_{A'} & \rightarrow & i_* \mathcal{F}_K \xrightarrow{\Delta_K} K \otimes_k R^1 i_* \mathcal{F}_k. \end{array} \quad (9.72.6)$$

By assumption $\Delta_K = 0$. Thus by (9.74) there is a nonzero $\mu : V_k \rightarrow k$ such that $\mu \circ \delta_k = 0$. Set $J^u := m_S I^u + \ker \mu$ and $S' := S/J^u$. Note that $I^u/J^u \cong k$. The extension $I^u/J^u \rightarrow S' \rightarrow S^u$ gives the exact sequence

$$(I^u/J^u) \otimes_k i_* \mathcal{F}_k \hookrightarrow i_* \mathcal{F}_{S'} \rightarrow i_* \mathcal{F}_{S^u} \xrightarrow{\mu \circ \delta} (I^u/J^u) \otimes_k R^1 i_* \mathcal{F}_k. \quad (9.72.7)$$

Since $\mu \circ \delta = 0$ the map $i_* \mathcal{F}_{S'} \rightarrow i_* \mathcal{F}_{S^u}$ is surjective, and so is the composite

$$i_* \mathcal{F}_{S'} \rightarrow i_* \mathcal{F}_{S^u} \rightarrow i_* \mathcal{F}_k.$$

Thus $i_* \mathcal{F}_{S'}$ is a universal hull by (9.27). This contradicts the choice of S^u . \square

The next lemma says that on a complete local ring, all topologies given by m -primary ideals are equivalent. Note that this does not hold for non-complete rings. For example, the intersection of the ideals

$$I_r := (y - \sin x, (x, y)^r) \subset k[x, y]_{(x, y)}$$

is trivial, yet none of them is contained in $(x, y)^2$.

LEMMA 9.73. *Let (S, m) be a complete local ring and $I_1 \supset I_2 \supset \dots$ m -primary ideals such that $\bigcap_i I_i = \{0\}$. Let $J \subset S$ be an m -primary ideal. Then there is a $k = k(J)$ such that $J \supset I_k$.*

Proof. It is enough to prove this for the ideals $J = m^j$. For any j , the $(I_k + m^j)/m^j$ form a descending chain of ideals in the Artin ring S/m^j . Thus the chain stabilizes at some $F_j \subset S/m^j$ and the natural maps $F_{j+1} \rightarrow F_j$ are surjective. If $F_j \neq 0$ for some j then we get an inverse system of elements $r_i \in F_i$ for $i \geq j$. Since S is complete, they have a limit $s \in S$ and $s \in I_k + m^j$ for every k, j . By Krull's intersection theorem $I_k = \bigcap_s (I_k + m^s)$, thus $s \in I_k$ for every k . This is impossible since $\bigcap_k I_k = \{0\}$ by assumption. \square

The following is a simple linear algebra lemma.

LEMMA 9.74. *Let $k \subset K$ be a field extension, M, N (possibly infinite dimensional) k -vector spaces and $\delta_i : M \rightarrow N$ linear maps. The following are equivalent:*

- (9.74.1) *There are $\lambda_i \in K$ (not all 0) such that $\sum_i \lambda_i \delta_i : K \otimes M \rightarrow K \otimes N$ is zero.*
- (9.74.2) *There are $\mu_i \in k$ (not all 0) such that $\sum_i \mu_i \delta_i : M \rightarrow N$ is zero.* \square

One can see that (9.72) does not hold for arbitrary local schemes S , but the following consequence was pointed out by E. Szabó.

COROLLARY 9.75. *The conclusion of (9.72) remains true if S is a Henselian local ring which is the localization of an algebra of finite type over a field or over an excellent DVR.*

Proof. There is a general theorem [Art69, 1.6] about representing functors over Henselian local rings, we check that its conditions are satisfied.

Let \hat{S} denote the completion of S . Let \mathcal{F} be the functor from local S -algebras to sets defined as follows:

$$\mathcal{F}(h : S \rightarrow T) = \begin{cases} 1 & \text{if } (F_T)^H \text{ is a universal hull,} \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to see that if $\mathcal{F}(h : S \rightarrow T) = 1$ then there is a factorization $S \rightarrow T' \rightarrow T$ such that T' is of finite type over S and $\mathcal{F}(h' : S \rightarrow T') = 1$. By definition this means that \mathcal{F} is locally of finite presentation over S (see e.g. [Art69, 1.5]). The universal family over $(\hat{S})^u$ gives an effective versal deformation of the fiber over m_S . The existence of S^u now follows from [Art69, 1.6]. \square

The following is an affine version of (9.70).

THEOREM 9.76. *Let (S, m) be a complete local ring, R a finite type S -algebra and F a finite R -module that is generically flat along R/mR . Then there is a quotient $S \twoheadrightarrow S^u$ that represents $\mathcal{P}ureq(F)$ for local S -algebras.*

Proof. By assumption there is a $Z \subset X := \text{Spec}_S R$ such that $F|_{X \setminus Z}$ is flat over S with pure fibers and $\text{pure}(F_m) = F_m/H_Z^0(F_m)$.

Let $I \subset S$ be an m -primary ideal such that $\text{pure}(F_I)$ is flat over S/I with pure fibers. Consider any ideal $mI \subset J \subset I$ and the exact sequence

$$0 \rightarrow F_I/F_J \rightarrow F_J \rightarrow F_I \rightarrow 0.$$

Note that for each of these modules F_* we have $\text{pure}(F_*) = F_*/H_Z^0(F_*)$. Thus $\text{pure}(F_J)$ is flat over S/J with pure fiber iff $H_Z^0(F_J) \rightarrow H_Z^0(F_I)$ is surjective. The latter sits in the exact sequence

$$0 \rightarrow H_Z^0(F_I/F_J) \rightarrow H_Z^0(F_J) \rightarrow H_Z^0(F_I) \xrightarrow{\partial_J} H_Z^1(F_I/F_J).$$

Thus $\text{pure}(F_J)$ is flat over S with pure fiber iff $\partial_J = 0$. We have a natural surjection $(I/J) \otimes F_m \twoheadrightarrow F_I/F_J$ whose kernel K is supported on Z . Since $H_Z^i(K) = 0$ for $i > 0$ we see that

$$H_Z^1(F_I/F_J) \cong (I/J) \otimes H_Z^1(F_m).$$

Thus ∂_J factors as

$$\partial : H_Z^0(F_I) \xrightarrow{a_I} H_Z^0(F_m) \xrightarrow{c_I} (I/mI) \otimes H_Z^1(F_m) \xrightarrow{q_J} (I/J) \otimes H_Z^1(F_m).$$

If $\text{pure}(F_I)$ is flat over S with pure fiber then a_I is surjective, hence $\partial_J = 0$ iff $c_I(H_Z^0(F_m))$ is killed by q_J . So, there is a unique largest ideal $mI \subset I' \subset I$ such that $\text{pure}(F_{I'})$ is flat over S/I' with pure fibers.

Thus we get a descending chain of ideals $m = I_0 \supset I_1 \supset \dots$ such that $\text{pure}(F_{I_r})$ is flat over S/I_r with pure fibers for every r and if $J \subset S$ is m -primary and $\text{pure} F_J$ is flat over S/J with pure fibers then $J \supset I_r$ for some r . Set $I^u := \bigcap_r I_r$ and $S^u := S/I^u$. Arguing as in (9.72.4–7) shows that $S \twoheadrightarrow S^u$ represents $\mathcal{P}\text{ureq}(F)$ for all local S -algebras. \square

Hulls and husks over algebraic spaces.

Here we present an alternate approach to hulls and husks that does not use projectivity, works for proper algebraic spaces but leaves properness of $\text{Husk}(F)$ unresolved. The proofs were worked out jointly with M. Lieblich.

THEOREM 9.77. *Let S be a Noetherian algebraic space and $p : X \rightarrow S$ a proper morphism of algebraic spaces. Let F be a coherent sheaf on X . Then $\mathcal{Q}\text{Husk}(F)$ is separated and it has a fine moduli space $\text{QHusk}(F)$.*

Proof. Let $f : X \rightarrow S$ be a proper morphism. The functor of flat families of coherent sheaves $\mathcal{F}\text{lat}(X/S)$ is represented by an algebraic stack $\text{Flat}(X/S)$ which is locally of finite type but very non-separated; see [LMB00, 4.6.2.1].

Let $\sigma : \text{Flat}(X/S) \rightarrow S$ be the structure morphism and $U_{X/S}$ the universal family over $\text{Flat}(X/S)$. By (10.3), there is an open substack

$$\text{Flat}^n(X/S) \subset \text{Flat}(X/S)$$

parametrizing pure sheaves of dimension n . Let $U_{X/S}^n$ be the corresponding universal family.

Consider $X \times_S \text{Flat}^n(X/S)$ with coordinate projections π_1, π_2 . The stack

$$\underline{\text{Hom}}(\pi_1^* F, \pi_2^* U_{X/S}^n)$$

parametrizes all maps from the sheaves F_s to pure, n -dimensional sheaves N_s .

We claim that $\mathcal{H}\text{usk}(F)$ is an open substack of $\underline{\text{Hom}}(\pi_1^* F, \pi_2^* U_{X/S}^n)$. Indeed, as in the proof of (9.50), for a map of sheaves $M \rightarrow N$ with N flat over S , it is an open condition to be an isomorphism at the generic points of the support.

As we discussed in (9.45), $\mathcal{H}\text{usk}(F)$ satisfies the valuative criteria of separatedness and properness. Thus the diagonal of $\mathcal{H}\text{usk}(F)$ is a monomorphism. Every algebraic stack with this property is an algebraic space; see [LMB00, Sec.8]. \square

In the projective case, the Hilbert polynomial was used to write $\text{QHusk}(F)$ as a disjoint union of subschemes $\text{QHusk}_p(F)$ that are proper over S . In the proper but non-projective case we do not have Hilbert polynomials, but one could still hope that the connected components of $\text{QHusk}(F)$ are proper over S . This fails even for the quot-scheme but the following weaker variant should be true. (The proof claimed in [Kol08a] is incorrect.)

CONJECTURE 9.78. *Every irreducible component of $\text{QHusk}(F)$ is proper.*

The construction of $\text{Hull}(F)$ given in (9.64) applies to algebraic spaces as well but it does not give boundedness. Nonetheless, we claim that $\text{Hull}(F)$ is of finite type. First, it is locally of finite type since $\text{QHusk}(F)$ is. Second, we claim that $\text{red Hull}(F)$ is dominated by an algebraic space of finite type. In order to see this, consider the (reduced) structure map $\text{red Hull}(F) \rightarrow \text{red } S$. It is an isomorphism at the generic points, hence there is an open dense $S^\circ \subset \text{red } S$ such that S° is isomorphic to an open subspace of $\text{red Hull}(F)$. Repeating this for $\text{red } S \setminus S^\circ$, by Noetherian induction we eventually write $\text{red Hull}(F)$ as a disjoint union of finitely many locally closed subspaces of $\text{red } S$. (We do not claim, however, that every irreducible component of $\text{red Hull}(F)$ is a locally closed subspace of $\text{red } S$.)

These together imply that $\text{Hull}(F)$ is of finite type. (Indeed, if $U \rightarrow V$ is a surjection, U is of finite type and V is locally of finite type then V is of finite type.)

As in (9.22.5), the structure map $\text{Hull}(F) \rightarrow S$ is a monomorphism. However, in the non-projective case, it need not be a locally closed decomposition (9.80). We can summarize these considerations in the following theorem.

THEOREM 9.79 (Flattening decomposition for hulls). *Let $f : X \rightarrow S$ be a proper morphism of algebraic spaces and F a coherent sheaf on X . Then*

(9.79.1) *$\mathcal{Hull}(F)$ is separated and it has a fine moduli space $\text{Hull}(F)$,*

(9.79.2) *$\text{Hull}(F)$ is an algebraic space of finite type over S and*

(9.79.3) *the structure map $\text{Hull}(F) \rightarrow S$ is a surjective monomorphism. \square*

EXAMPLE 9.80. Let C, D be two smooth projective curves. Pick points $p, q \in C$ and $r \in D$. Let X be the surface obtained from the blow-up $B_{(p,r)}(C \times D)$ by identifying $\{q\} \times D$ with the birational transform of $\{p\} \times D$. Note that X is a proper but non-projective scheme and there is a natural proper morphism $\pi : X \rightarrow C'$ where C' is the nodal curve obtained from C by identifying the points p, q .

Then $\text{Hull}(\mathcal{O}_X) = C \setminus \{q\}$ and the natural map $C \setminus \{q\} \rightarrow C'$ is a surjective monomorphism but not a locally closed embedding.

Ancillary results

In this chapter we discuss various results that were used earlier, and for which good references are scarce.

10.1. Flat families of S_m sheaves

Here we consider how the S_m property varies in flat families of coherent sheaves.

DEFINITION 10.1. Recall that a coherent sheaf F on a scheme X satisfies *Serre's condition* S_m if

$$\text{depth}_x F \geq \min\{m, \dim_x F\} \quad \text{for every } x \in X.$$

F is called *Cohen-Macaulay* or *CM* if

$$\text{depth}_x F = \dim_x F \quad \text{for every } x \in X.$$

It is easy to see that if X is CM then $\text{Supp } F$ is locally pure (9.2). In the literature, the definition of CM frequently includes the assumption that $\text{Supp } F$ be pure dimensional; we will most likely lapse into this habit too.

THEOREM 10.2. [Gro60, IV.12.1.6] *Let $\pi : X \rightarrow S$ be a morphism of finite type and F a coherent sheaf on X that is flat over S . Fix $m \in \mathbb{N}$. Then the set of points*

$$\{x \in X : F_{\pi(x)} \text{ is pure and } S_m \text{ at } x\}$$

is open in X .

This immediately implies the following variant for proper morphisms.

COROLLARY 10.3. *Let $\pi : X \rightarrow S$ be a proper morphism and F a coherent sheaf on X that is flat over S . Fix $m \in \mathbb{N}$. Then the set of points*

$$\{s \in S : F_s \text{ is pure and } S_m\}$$

is open in S . □

For non-proper morphisms we get the following.

COROLLARY 10.4. *Let S be an integral scheme, $\pi : X \rightarrow S$ a morphism of finite type and F a coherent sheaf on X . Assume that F is pure and S_m . Then there is a dense open subset $S^\circ \subset S$ such that F_s is pure and S_m for every $s \in S^\circ$.*

Proof. Let $Z \subset X$ denote the set of points $x \in X$ such that either F is not flat at x or $F_{\pi(x)}$ is not pure and S_m at x . Note that Z is closed in X by (10.2) and by generic flatness [Eis95, 14.4].

The local rings of the generic fiber of π are also local rings of X , hence the restriction of F to the generic fiber is pure and S_m . Thus Z is disjoint from the generic fiber of π . Therefore $\pi(Z) \subset S$ is a constructible subset that does not contain the generic point, hence $S \setminus \pi(Z)$ contains a dense open subset $S^\circ \subset S$. □

10.5 (Nagata's openness criterion). In many cases one can check openness of a subset of a scheme using the following easy to prove test, which is sometimes called the *Nagata openness criterion*.

Let X be a Noetherian topological space and $U \subset X$ an arbitrary subset. Then U is open iff the following conditions are satisfied.

(10.5.1) If $x_1 \in \bar{x}_2$ and $x_1 \in U$ then $x_2 \in U$.

(10.5.2) If $x \in U$ then there is a nonempty open $V \subset \bar{x}$ such that $V \subset U$.

Assume now that we want to use this to check openness of a fiber-wise property \mathcal{P} for a morphism $\pi : X \rightarrow S$.

We start with condition (10.5.1). Pick points $x_1, x_2 \in X$ such that $x_1 \in \bar{x}_2$.

Let T be the spectrum of a DVR with closed point $0 \in T$, generic point $t_g \in T$ and $q : T \rightarrow X$ a morphism such that $q(0) = x_1$ and $q(t_g) = x_2$. After base change using $\pi \circ q$ we get $Y \rightarrow T$. Usually one can not guarantee that the residue fields are unchanged under q . However, if property \mathcal{P} is invariant under field extensions, then it is enough to check (10.5.1) for $Y \rightarrow T$. Thus we may assume that S is the spectrum of a DVR.

As for (10.5.2), we can replace S by the closure of $\pi(x)$. Then $\pi(x)$ is the generic point of S and, by passing to an open subset, we may assume that S is regular.

We can summarize these considerations in the following form.

PROPOSITION 10.6 (Openness criterion). *Let \mathcal{P} be a property of coherent sheaves over local rings over fields that is invariant under field extensions. The following are equivalent.*

(10.6.1) *Let $\pi : X \rightarrow S$ be a morphism of finite type and F a coherent sheaf on X that is flat over S . Then the set of points*

$$\{x \in X : F_{\pi(x)} \text{ satisfies property } \mathcal{P} \text{ at } x\} \text{ is open in } X.$$

(10.6.2) *The following 2 special cases of (1) hold, where $\sigma : S \rightarrow X$ denotes a section.*

(10.6.2.a) *S is the spectrum of a DVR with closed point 0 , generic point g and \mathcal{P} holds for $\sigma(0) \in X_0$ then \mathcal{P} holds for $\sigma(g) \in X_g$.*

(10.6.2.b) *S is the spectrum of a regular ring with generic point g and \mathcal{P} holds for $\sigma(g) \in X_g$ then \mathcal{P} holds for all points in a nonempty open subset $U \subset \sigma(S)$. \square*

10.7 (Proof of (10.2)). By (10.6) we may assume that S is affine and regular. We may also assume that π is affine and $X = \text{Supp } F$.

First we check (10.6.2.a) for $m = 1$. (Note that pure and S_1 is equivalent to pure (9.2).) Let $W \subset X$ be an associated prime of F . Then $W \cap X_0$ is an associated prime of F_0 by (10.13). Since F_0 is pure, $W \cap X_0$ is an irreducible component of $\text{Supp } F_0$ hence W is an irreducible component of $\text{Supp } F$. Thus F_g is also pure.

Next we check (10.6.2.a) for $m > 1$. Since S_m implies S_1 , we already know that every fiber of F is pure. By (10.8) there is a subset $Z \subset X$ of relative codimension ≥ 2 such that F is CM over $X \setminus Z$. Let $Z \subset H \subset X$ be a Cartier divisor that does not contain any of the associated primes of F_0 . Then $F|_H$ is flat over S and $(F|_H)_0 = F_0|_H$ is pure and S_{m-1} . Thus, by induction, $F|_H$ is pure and S_{m-1} on the generic fiber, hence F_{s_g} is pure and S_m along H . It is even CM on $X \setminus H$, hence F_{s_g} is pure and S_m .

The proof of (10.6.2.b) follows a similar pattern. We start with $m = 1$. We may assume that F_{s_g} is pure. By Noether normalization, there is a finite surjection $p : X \rightarrow \mathbb{A}_S^n$ for some n . Note that p_*F is flat over S and it is pure on the generic fiber by (9.12), hence torsion free. Using (9.12) in the reverse direction for the other fibers, we are reduced to the case when $X = \mathbb{A}_S^n$ and F is torsion free at $x := \sigma(g)$ on the generic fiber. Thus there is an injection of the localizations $F_x \hookrightarrow \mathcal{O}_{x,X}^m$. By generic flatness [Eis95, 14.4], the quotient $\mathcal{O}_{x,X}^m/F_x$ is flat over an open, dense subset $S^\circ \subset S$. Thus if $s \in S^\circ$ then we have an injection $F|_U \hookrightarrow \mathcal{O}_U^m$. Thus every fiber F_s is torsion free over $U \cap \pi^{-1}(S^\circ)$.

For $m > 1$ we follow the same argument as above using $Z \subset H \subset X$ and induction. □

LEMMA 10.8. *Let $\pi : X \rightarrow S$ be a morphism of finite type and F a coherent sheaf on X that is flat over S . Assume that $\text{Supp } F$ is pure-dimensional over S . Set*

$$\text{cm-locus}(F) := \{x \in \text{Supp } F : F_{\pi(x)} \text{ is CM at } x\}.$$

Then, for every $s \in S$,

(10.8.1) $X_s \cap \text{cm-locus}(F)$ *is dense in* $\text{Supp } F_s$ *and*

(10.8.2) *if* F_s *is pure then* $X_s \setminus \text{cm-locus}(F)$ *has codimension* ≥ 2 *in* $\text{Supp } F_s$.

Proof. We may assume that π is affine and $X = \text{Supp } F$. By Noether normalization, there is a finite surjection $p : X \rightarrow Y := \mathbb{A}_S^n$ for some n .

Since p_*F is flat over S , it is locally free at a point $y \in Y$ iff the restriction of p_*F to the fiber $Y_{p(y)}$ is locally free at y . The latter holds outside a codimension ≥ 1 subset of each fiber Y_s . If F is pure then p_*F is torsion free on each fiber, and then local freeness holds outside a subset of codimension ≥ 2 . □

Let F be a coherent, S_m sheaf on \mathbb{P}^n . If a hyperplane $H \subset \mathbb{P}^n$ does not contain any of the irreducible components of $\text{Supp } F$ then $F|_H$ is S_{m-1} , essentially by definition. The following result says that $F|_H$ is even S_m for general hyperplanes, though we can not be very explicit about the meaning of ‘general.’

COROLLARY 10.9 (Bertini theorem for S_m). *Let F be a coherent, pure, S_m sheaf on a finite type k -scheme and $|V|$ a base point free linear system on X . Then there is a dense open subset $U \subset |V|$ such that $F|_H$ is also pure and S_m for $H \in U$.*

Proof. Let $Y \subset X \times |V|$ be the incidence correspondence (that is, the set of pairs (point $\in H$) with projections π and $\tilde{\pi}$). Note that π is a \mathbb{P}^{n-1} -bundle for $n = \dim |V|$, thus π^*F is also pure and S_m by (9.6).

By (10.4) there is a dense open subset $U \subset |V|$ such that $F|_H$ is also pure and S_m for $H \in U$. For a divisor H , the restriction $F|_H$ is isomorphic to the restriction of π^*F to the fiber of $\tilde{\pi}$ over $H \in |V|$. □

COROLLARY 10.10 (Bertini theorem for hulls). *Let $|V|$ be a base point free linear system on a finite type k -scheme X . Let F be a coherent sheaf on X with hull $q : F \rightarrow F^{[**]}$. Then there is a dense open subset $U \subset |V|$ such that*

$$(F^{[**]})|_H = (F|_H)^{[**]} \quad \text{for } H \in U.$$

Proof. If $H \in |V|$ is general then $\dim(\text{tors } F)|_H = \dim \text{tors } F - 1$ and $\text{pure}(F)|_H$ is S_1 by (10.9). Similarly, $(F^{[**]})|_H$ is S_2 and $\text{pure}(F)|_H \rightarrow (F^{[**]})|_H$ is an isomorphism outside $H \cap \text{coker } q$. □

COROLLARY 10.11 (Bertini theorem for S_m in families). *Let T be the spectrum of a local ring, $X \subset \mathbb{P}_T^n$ a quasi-projective scheme and F a coherent sheaf on X that is flat over T with pure, S_m fibers.*

Assume that either X is projective over T or $\dim T \leq 1$. Then $F|_{H \cap X}$ is also flat over T with pure and S_m fibers for a general hyperplane $H \subset \mathbb{P}_T^n$.

Proof. The hyperplanes correspond to sections of $\check{\mathbb{P}}_T^n \rightarrow T$. If X is projective over T then we use (10.9) for the special fiber X_0 and conclude using (10.3).

If $\dim T = 1$ then we use (10.9) both for the special fiber X_0 and the generic fibers X_{g_i} . We get open subsets $U_0 \subset \check{\mathbb{P}}_0^n$ and $U_{g_i} \subset \check{\mathbb{P}}_{g_i}^n$. Let $W_i \subset \check{\mathbb{P}}_T^n$ denote the closure of $\check{\mathbb{P}}_{g_i}^n \setminus U_{g_i}$. For dimension reasons, W_i does not contain $\check{\mathbb{P}}_0^n$. Thus any hyperplane corresponding to a section through a point of $U_0 \setminus (\cup_i W_i)$ works. \square

EXAMPLE 10.12. If $\dim T \geq 2$ then (10.11) does not hold for non-proper maps. Here is a similar example for the classical Bertini theorem on smoothness. Set

$$X := (x^2 + y^2 + z^2 = s) \setminus (x = y = z = s = 0) \subset \mathbb{A}_{xyz}^3 \times \mathbb{A}_{st}^2$$

with smooth second projection $f : X \rightarrow \mathbb{A}_{st}^2$. Over the origin we start with the hyperplane $H_{00} := (x = 0)$, it is a typical member of the base point free linear system $|ax + by + cz = 0|$.

A general deformation of it is given by $H_{st} := x + b(s, t)y + c(s, t)z = d(s, t)$. It is easy to compute that the intersection $H_{st} \cap X_{st}$ is singular iff $s(1 + b^2 + c^2) = d^2$. This equation describes a curve in \mathbb{A}_{st}^2 that passes through the origin.

10.13 (Associated points of restrictions). Let X be a scheme, $D \subset X$ a Cartier divisor and F a coherent sheaf on X . We aim to compare $\text{Ass}(F)$ and $\text{Ass}(F|_D)$. If D does not contain any of the associated points of a sheaf G then $\text{Tor}^1(G, \mathcal{O}_D) = 0$. Thus if $0 = F_0 \subset \cdots \subset F_r = F$ is a filtration of F by subsheaves and D does not contain any of the associated points of F_i/F_{i-1} then $0 = F_1|_D \subset \cdots \subset F_r|_D = F|_D$ is a filtration of $F|_D$ and $F_i|_D/F_{i-1}|_D \cong (F_i/F_{i-1})|_D$. We can also choose any of the associated points of F to be an associated point of F_1 , proving the following.

Claim 10.13.1. If D does not contain any of the associated points of F then

- (a) $\text{Ass}(F|_D) \subset \cup_i \text{Ass}((F_i/F_{i-1})|_D)$ and
- (b) for every $x \in \text{Ass}(F)$, every generic point of $D \cap \bar{x}$ is in $\text{Ass}(F|_D)$. \square

By (10.18) we can choose the F_i such that $\text{Ass}(F_i/F_{i-1})$ is a single associated point of F for every i . Thus it remains to understand $\text{Ass}(G|_D)$ when G is pure. Let $G^H \supset G$ denote the hull of G and set $Q := G^H/G$. As we noted above, if D does not contain any of the associated points of Q then $G^H|_D \supset G|_D$, thus $\text{Ass}(G^H|_D) = \text{Ass}(G|_D)$. Finally, since G^H is S_2 , the restriction $G^H|_D$ is S_1 hence its associated points are exactly the generic points of $D \cap \text{Supp } G$. We have thus proved the following.

Claim 10.13.2. Let $D \subset X$ be a Cartier divisor that contains neither an associated point of F nor an associated point of $(F_i/F_{i-1})^H/(F_i/F_{i-1})$. Then

- (a) the associated points of $F|_D$ are exactly the generic points of $D \cap \bar{x}$ for all $x \in \text{Ass}(F)$ and
- (b) $(F/\text{emb}(F))|_D \cong (F|_D)/(\text{emb}(F|_D))$. \square

Note that the associated points of $(F_i/F_{i-1})^H/(F_i/F_{i-1})$ depend on the choice of the F_i , they are not determined by F . For the Claim to hold it is enough to take

the intersection of all possible sets. This set is still hard to determine, but in many applications the key point is that, as long as X is excellent, we need D to avoid only a finite set of points.

The next result describes how the associated points of fibers of a flat sheaf fit together. The proof is a refinement of the arguments used in (10.7).

THEOREM 10.14. *Let $f : X \rightarrow S$ be a morphism of finite type and F a coherent sheaf on X . Then the following hold.*

- (10.14.1) *There are finitely many locally closed subschemes $W_i \subset X$ such that for every $s \in S$ the associated points of F_s are exactly the generic points of the $(W_i)_s$.*
- (10.14.2) *If F is flat over S then we can choose the W_i to be closed and such that each $f|_{W_i} : W_i \rightarrow f(W_i)$ is equidimensional.*

Proof. Using Noetherian induction it is enough to prove that (1) holds over a non-empty open subset of $\text{red } S$. We may thus assume that S is integral with generic point $g \in S$.

Assume first that X is integral and F is torsion free. By Noether normalization, after again passing to some non-empty open subset of S there is a finite surjection $p : X \rightarrow \mathbb{A}_S^m$. Then p_*F is torsion free of generic rank say r , hence there is an injection $j : p_*F \hookrightarrow \mathcal{O}_{\mathbb{A}_S^m}^r$. After again passing to some non-empty open subset we may assume that $\text{coker}(j)$ is flat over S , thus

$$j_s : p_*(F_s) = (p_*F)_s \hookrightarrow \mathcal{O}_{\mathbb{A}_S^m}^r$$

is an injection for every $s \in S$. Thus each F_s is torsion free and its associated points are exactly the generic points of the fiber X_s .

In general, we use (10.17) for the generic fiber and then extend the resulting filtration to X . Thus, after replacing S by a non-empty open subset if necessary, we may assume that there is a filtration $0 = F^0 \subset \dots \subset F^n = F$ such that each F^{m+1}/F^m is a coherent, torsion free sheaf over some integral subscheme $W_m \subset X$ and $W_{m_1} \not\subset W_{m_2}$ for $m_1 > m_2$. As we proved, we may assume that the associated points of each $(F^{m+1}/F^m)_s$ are exactly the generic points of the fiber $(W_m)_s$. Using generic flatness we may also assume that each F^{m+1}/F^m is flat over S and, after further shrinking S , none of the generic points of $(W_{m_1})_s$ is contained in $(W_{m_2})_s$ for $m_1 > m_2$. Then the associated points of each F_s are exactly the generic points of the fibers $(W_m)_s$ for every m by (10.16). This proves (1).

In order to see (2), consider first the case when the base ($0 \in T$) is the spectrum of a DVR. The filtration given by (10.17) for the generic fiber extends to a filtration $0 = F^0 \subset \dots \subset F^n = F$ over X giving closed integral subschemes $W_m \subset X$. Since T is the spectrum of a DVR, the F^{m+1}/F^m are flat over T , hence the associated points of F_0 are exactly the generic points of the fibers $(W_m)_0$ for every m .

To prove (2) in general, we take the $W_i \subset X$ obtained in (1) and replace them by their closures. A possible problem arises if $f|_{W_i} : W_i \rightarrow f(W_i)$ is not equidimensional. Assume that $W_i \rightarrow f(W_i)$ has generic fiber dimension d and let $(W_i)_s$ be a special fiber. Pick any closed point $x \in (W_i)_s$ and the spectrum of a DVR ($0 \in T$) mapping to W_i such that the special point of T maps to x and the generic point of T to the generic point of W_i . After base change to T we see that F_s has a d -dimensional associated subscheme containing x . Thus $(W_i)_s$ is covered by d -dimensional associated subschemes of F_s . Since F_s is coherent, this is only

possible if $\dim(W_i)_s = d$ and every generic point of the $(W_i)_s$ is an associated point of F_s . \square

10.15 (Semicontinuity and depth). Let X be a scheme and F a coherent sheaf on X . As we noted in (9.5), the function $x \mapsto \text{depth}_x F$ is not lower semicontinuous. This is, however, caused by the non-closed points. A quick way to see this is the following.

Assume that X is regular and let $0 \in X$ be a closed point. By the Auslander–Buchsbaum formula (cf. [Eis95, 19.9]) F_0 has a projective resolution of length $\dim X - \text{depth}_0 F$. Thus there is an open subset $0 \in U \subset X$ such that $F|_U$ has a projective resolution of length $\dim X - \text{depth}_0 F$. This shows that

$$\text{depth}_x F \geq \text{depth}_0 F - \dim \bar{x} \quad \forall x \in U. \quad (10.15.1)$$

That is, $x \mapsto \text{depth}_x F$ is lower semicontinuous for closed points. In general, we have the following analog of (10.2).

Proposition 10.15.2. Let $\pi : X \rightarrow S$ be a morphism of finite type and F a coherent sheaf on X that is flat over S with pure fibers. Let $0 \in X$ be a closed point. Then there is an open subset $0 \in U \subset X$ such that

$$\text{depth}_x F_{\pi(x)} \geq \text{depth}_0 F_{\pi(0)} - \text{tr-deg}_{k(\pi(x))} k(x) \quad \forall x \in U,$$

where $F_{\pi(x)}$ is the restriction of F to the fiber $X_{\pi(x)}$ and tr-deg denotes the transcendence degree. Hence $x \mapsto \text{depth}_x F_{\pi(x)}$ is lower semicontinuous on closed points.

In order to see this, using Noether normalization and (10.8.1) as in (10.7), we can reduce to the case when $X = \mathbb{A}_S^n$ for some n . Next we take a projective resolution of the fiber $F_{\pi(0)}$ and lift it to a suitable neighborhood $0 \in U \subset X$ using the flatness of F . \square

10.2. Dévissage

Dévissage is a method that writes a coherent sheaf as an extension of simpler coherent sheaves and uses these to prove various theorems. There are many ways to do this, different ones are useful in different contexts.

Recall that $\text{Ass}(F)$ denotes the set of associated points of a sheaf F (9.2) and that a sheaf is S_1 iff it has no embedded points (9.7).

If $W := \text{Supp } F$ is integral and F has no embedded points then we also say that F is *torsion free on W* . We write $\text{tors}_Z F \subset F$ for the largest subsheaf whose support is contained in Z .

LEMMA 10.16. *Let X be a Noetherian scheme, F a coherent sheaf on X and $F_1 \subset F_2 \subset F$ subsheaves. Assume that $w \in \text{Ass}(F_2/F_1)$ and $w \notin \text{Supp } F_1$. Then $w \in \text{Ass}(F)$. \square*

LEMMA 10.17. *Let X be a Noetherian scheme, F a coherent sheaf on X and write $\text{Ass}(F) = \{w_i : i = 1, \dots, m\}$ in some fixed order. Assume that $w_j \notin W_i$ for $i < j$. Then F admits a unique filtration $0 = G_0 \subset G_1 \cdots \subset G_m = F$ such that G_i/G_{i-1} is (isomorphic to) a coherent S_1 sheaf that is supported on W_i for $i = 1, \dots, m$. Moreover, the natural map $\text{tors}_{W_i} F \rightarrow G_i/G_{i-1}$ is an isomorphism at w_i for $i = 1, \dots, m$.*

Proof. It is easy to see that we must set $G_1 = \text{tors}_{W_1} F$. Then pass to F/G_1 and use induction on the number of associated points. \square

For an arbitrary ordering of the w_i the filtration still exists but it is not canonical and not even the generic rank of the graded pieces is unique, see (10.19).

LEMMA 10.18. *Let X be a Noetherian scheme, F a coherent sheaf on X and write $\text{Ass}(F) = \{w_i : i = 1, \dots, m\}$ in some fixed order. Then F admits a finite filtration $0 = G_0 \subset G_1 \cdots \subset G_m = F$ such that G_i/G_{i-1} is (isomorphic to) a coherent S_1 sheaf that is supported on W_i for $i = 1, \dots, m$.*

Proof. First set $G' := \text{tors}_{W_1} F$. Then the associated points of F/G' are those w_j that are not contained in W_1 . Let $G'' \subset G'$ be the largest subsheaf whose support is not dense in W_1 . Set $Z = \text{Supp } G''$. A power of I_Z kills G'' , hence, by the Artin–Rees lemma, $G'' \cap I_Z^r G' = 0$ for some r . By Noetherian induction, there is a coherent subsheaf $H \subset (F/I_Z^r G')$ such that none of the w_i are associated points of H and every associated point of $(F/I_Z^r G')/H$ is among the w_i . Finally let $G_1 \subset F$ be the preimage of H . Every associated point of F/G_1 is among the $\{w_i : i \geq 2\}$ by construction. The associated points of G_1 are w_1 and possibly a few others that are distinct from the $\{w_i : i \geq 2\}$. However, since $G_1 \subset F$, its associated points are a subset of $\{w_i : i = 1, \dots, m\}$. Thus w_1 is the only associated point of G_1 .

Next we pass to F/G_1 and finish by induction as before. \square

EXAMPLE 10.19. The graded pieces G_i/G_{i-1} depend on the ordering of the w_i in (10.18), even their generic rank can change. For example, let $X = \text{Spec } k[x, y]$ and F the sheaf corresponding to $k[x, y]/(xy, y^2)$. The associated points are (y) and (x, y) . If we take (x, y) first, we get the most natural filtration

$$0 \rightarrow k \xrightarrow{y} k[x, y]/(xy, y^2) \rightarrow k[x, y]/(y) \rightarrow 0.$$

If we take (y) first then for every $n \geq 1$ we get different possibilities

$$0 \rightarrow k[x, y]/(y) \xrightarrow{x^n} k[x, y]/(xy, y^2) \rightarrow k[x, y]/(xy, x^n, y^2) \rightarrow 0.$$

The above filtration can be further refined.

LEMMA 10.20. *Let W be an irreducible, Noetherian scheme and F a torsion free, coherent sheaf on W . Then F admits a finite filtration $0 = G_0 \subset G_1 \cdots \subset G_m = F$ such that G_i/G_{i-1} is a torsion free, coherent sheaf on $\text{red } W$ of generic rank 1 for $i = 1, \dots, m$.*

Proof. Let $J \subset \mathcal{O}_W$ be the nil-radical. Let $U \subset W$ be a dense, open, affine subset and $s \in H^0(U, F|_U)$ a nonzero section such that $J \cdot s = 0$. We can take G_1 to be the subsheaf of local sections ϕ such that $\phi|_V \subset \mathcal{O}_V \cdot (s|_V)$ for some dense, open subset $V \subset W$. We then pass to F/G_1 and repeat the process. \square

Over a quasi-affine scheme, any global section of G_1 shows the following consequence, which is also easy to prove directly.

COROLLARY 10.21. *Let X be a Noetherian, quasi-affine scheme and F a coherent sheaf on X with associated points $\{w_i : i = 1, \dots, m\}$. For every i there are injections $\mathcal{O}_{W_i} \hookrightarrow F$ where $W_i := \bar{w}_i$.* \square

The following is probably the best known variant of dévissage and it is sufficient for most applications.

COROLLARY 10.22. *The K-group of coherent sheaves on a Noetherian scheme is generated by the structure sheaves of closed, integral subschemes.*

Proof. Using (10.17) and (10.20) we need to deal with the case when F is a torsion free sheaf of generic rank 1 on an integral scheme W . There is a nowhere dense subscheme $Z \subset W$ and an injection $\mathcal{O}_W(-Z) \hookrightarrow F$. Thus

$$[F] = [\mathcal{O}_W(-Z)] + [F/\mathcal{O}_W(-Z)] = [\mathcal{O}_W] - [\mathcal{O}_Z] + [F/\mathcal{O}_W(-Z)],$$

where the bracket denotes the class of a sheaf in the K-group of coherent sheaves. By Noetherian induction the claim holds for $F/\mathcal{O}_W(-Z)$. \square

This is probably as far as one can go on a general Noetherian scheme. On an integral quasi-projective scheme every torsion free, coherent sheaf of generic rank 1 has a subsheaf that is a line bundle. If X is quasi-affine, we can choose the line bundle to be trivial. The quotient has smaller dimensional support but we do not know its associated points. Thus we get the following, where the Z_i need not be closures of associated points of F .

LEMMA 10.23. *Let X be a quasi-projective scheme and F a coherent sheaf on X . Then F admits a finite filtration $0 = G_0 \subset G_1 \cdots \subset G_m = F$ such that G_i/G_{i-1} is isomorphic to a line bundle on a closed, integral subvariety $Z_i \subset X$ for $i = 1, \dots, m$. \square*

LEMMA 10.24. *Let X be a Noetherian, quasi-affine scheme and F a coherent sheaf on X . Then F admits a finite filtration $0 = G_0 \subset G_1 \cdots \subset G_m = F$ such that G_i/G_{i-1} is isomorphic to the structure sheaf of a closed, integral subscheme $Z_i \subset X$ for $i = 1, \dots, m$. \square*

10.3. Cohomology over non-proper schemes

The cohomology theory of coherent sheaves is trivial over affine schemes and well understood over proper schemes. If X is a scheme and $j : U \hookrightarrow X$ is an open subscheme then one can study the cohomology theory of coherent sheaves on U by understanding the cohomology theory of quasi-coherent sheaves on X and the higher direct image functors $R^i j_*$. The key results are (10.25) and (10.29).

We start with the basic coherence result for push-forwards.

PROPOSITION 10.25. [**Gro60**, IV.5.11.1] *Let X be an excellent scheme, $Z \subset X$ a closed subscheme and $U := X \setminus Z$ with injection $j : U \hookrightarrow X$. Let G be a coherent sheaf on U . Then $j_* G$ is coherent iff $\text{codim}_{\bar{W}}(Z \cap \bar{W}) \geq 2$ for every associated point W of G .*

The case of arbitrary Noetherian schemes is discussed in [**Kol17**].

Proof. This is a local question, hence we may assume that X is affine. By (10.18) and (10.20), G has a filtration $0 = G_0 \subset \cdots \subset G_r = G$ such that each G_{m+1}/G_m is isomorphic to a subsheaf of some \mathcal{O}_W where W is an associated prime of G . Since j_* is left exact, it is enough to show that each $j_* \mathcal{O}_W$ is coherent.

Let $\bar{W} \subset X$ denote the closure of W and $p : V \rightarrow W$, $\bar{p} : \bar{V} \rightarrow \bar{W}$ the normalizations. Since X is excellent, p and \bar{p} are finite. $\mathcal{O}_{\bar{V}}$ is S_2 (by Serre's criterion) and so is $\bar{p}_* \mathcal{O}_{\bar{V}}$ by (9.12). Thus

$$j_* \mathcal{O}_W \subset j_* (p_* \mathcal{O}_V) = j_* (\bar{p}_* \mathcal{O}_{\bar{V}}),$$

where the equality follows from (9.8) using $\text{codim}_{\bar{W}}(Z \cap \bar{W}) \geq 2$. Thus $j_*\mathcal{O}_W$ is coherent. \square

It is frequently quite useful to know that coherent sheaves are ‘nice’ over large open subsets. For finite type schemes this was established in (10.8).

PROPOSITION 10.26. *Let X be a Noetherian scheme. Assume that every integral subscheme $W \subset X$ has an open dense subscheme $W^\circ \subset W$ that is regular (or at least CM). Let F be a coherent sheaf on X .*

- (10.26.1) *There is a closed subset $Z_1 \subset \text{Supp } F$ of codimension ≥ 1 such that F is CM on $X \setminus Z_1$.*
- (10.26.2) *If F is S_1 then there is a closed subset $Z_2 \subset \text{Supp } F$ of codimension ≥ 2 such that F is CM on $X \setminus Z_2$.*

Proof. We put the intersections of different irreducible components of $\text{Supp } F$ into Z_1 . Since (1) is a local question, we may thus assume that $\text{Supp } F$ is irreducible. Since an extension of CM sheaves of the same dimensional support is CM (10.27), using (10.18) we may assume that F is torsion free over an integral subscheme $W \subset X$. Then F is locally free over a dense open subset $W^\circ \subset W$ and we can take $Z_1 := W \setminus W^*$, where W^* is the regular locus of W° .

In order to prove (2), we may assume that X is affine. Let $s = 0$ be a local equation of Z_1 . We apply the first part to F/sF to obtain a closed subset $Z_2 \subset \text{Supp}(F/sF)$ of codimension ≥ 1 such that F/sF is CM on $X \setminus Z_2$. Thus F is CM on $X \setminus Z_2$. \square

The next lemma is quite straightforward; see [Kol13b, 2.60] for details.

LEMMA 10.27. *Let X be a scheme and $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ a sequence of coherent sheaves on X that is exact at $x \in X$.*

- (10.27.1) *If $\text{depth}_x F \geq r$ and $\text{depth}_x F'' \geq r - 1$ then $\text{depth}_x F' \geq r$.*
- (10.27.2) *If $\text{depth}_x F \geq r$ and $\text{depth}_x F' \geq r - 1$ then $\text{depth}_x F'' \geq r - 1$.* \square

10.28 (Cohomology over quasi-affine schemes). (See [Gro67] for details.)

Let X be an affine scheme, $Z \subset X$ a closed subscheme and $U := X \setminus Z$. Here our primary interest is in the case when $Z = \{x\}$ is a closed point.

For a quasi-coherent sheaf F on X , let $H_Z^0(X, F)$ denote the space of global sections whose support is in Z . There is a natural exact sequence

$$0 \rightarrow H_Z^0(X, F) \rightarrow H^0(X, F) \rightarrow H^0(U, F|_U).$$

This induces a long exact sequence of the corresponding higher cohomology groups. Since X is affine, $H^i(X, F) = 0$ for $i > 0$, hence the long exact sequence breaks up into a shorter exact sequence

$$0 \rightarrow H_Z^0(X, F) \rightarrow H^0(X, F) \rightarrow H^0(U, F|_U) \rightarrow H_Z^1(X, F) \rightarrow 0 \tag{10.28.1}$$

and a collection of isomorphisms

$$H^i(U, F|_U) \cong H_Z^{i+1}(X, F) \quad \text{for } i \geq 1. \tag{10.28.2}$$

The vanishing of the local cohomology groups is closely related to the depth of the sheaf F . Two instances of this follow from already established results. First, for coherent sheaves (9.7) can be restated as

$$H_Z^0(X, F) = 0 \iff \text{depth}_Z F \geq 1. \tag{10.28.3}$$

Second, (9.8) tells us when the map $H^0(X, F) \rightarrow H^0(U, F|_U)$ in (10.28.1) is an isomorphism. This implies that, for coherent sheaves,

$$H_Z^0(X, F) = H_Z^1(X, F) = 0 \Leftrightarrow \text{depth}_Z F \geq 2. \quad (10.28.4)$$

More generally, Grothendieck's vanishing theorem (see [Gro67, Sec.3] or [BH93, 3.5.7]) says that

$$H_Z^i(X, F) = 0 \quad \text{for } i < \text{depth}_Z F. \quad (10.28.5)$$

Combined with (10.28.2-3) this shows that

$$H^i(U, F|_U) = 0 \quad \text{for } 1 \leq i \leq \text{depth}_Z F - 2. \quad (10.28.6)$$

All the above groups are naturally modules over $H^0(X, \mathcal{O}_X)$ and we need to understand when they are finitely generated.

More generally, let G be a coherent sheaf on U . When is the group $H^i(U, G)$ a finite $H^0(X, \mathcal{O}_X)$ -module? Since X is affine,

$$H^i(U, G) = H^0(X, R^i j_* G),$$

where $j : U \hookrightarrow X$ denotes the natural open embedding. Thus $H^i(U, G)$ is a finite $H^0(X, \mathcal{O}_X)$ -module iff $R^i j_* G$ is a coherent sheaf. For $i \geq 1$, the sheaves $R^i j_* G$ are supported on Z , which implies the following.

Lemma 10.28.7. Notation as above. Assume that $i \geq 1$.

- (a) Every associated prime of $H^i(U, G)$ (viewed as an $H^0(X, \mathcal{O}_X)$ -module) is contained in Z .
- (b) If $Z = \{x\}$ then $H^i(U, G)$ is a finite $H^0(X, \mathcal{O}_X)$ -module iff $H^i(U, G)$ has finite length. \square

The general finiteness condition is stated in (10.29); but first we work out the special cases that we use. We start with $H^0(U, G)$; here we have the following restatement of (10.25).

Lemma 10.28.8. Let X be an excellent, affine scheme, $Z \subset X$ a closed subscheme, $U := X \setminus Z$ and G a coherent sheaf on U . Assume in addition that $Z \cap \bar{W}_i$ has codimension ≥ 2 in \bar{W}_i for every associated prime $W_i \subset U$ of G . Then $H^0(U, G)$ is a finite $H^0(X, \mathcal{O}_X)$ -module. \square

It is considerably harder to understand finiteness for $H^1(U, G)$. The following special case is used in Section 5.8.

Lemma 10.28.9. Let X be an excellent scheme, $Z \subset X$ a closed subscheme, $U := X \setminus Z$ and G a coherent sheaf on U . Assume in addition that

- (a) G is S_2 ,
- (b) there is a coherent CM sheaf F on X and an injection $G \hookrightarrow F|_U$,
- (c) Z has codimension ≥ 3 in $\text{Supp } F$.

Then $R^1 j_* G$ is coherent.

Proof. Set $Q = F|_U/G$. Since G is S_2 , it has no extensions with a sheaf whose support has codimension ≥ 2 by (9.8), thus every associated prime of Q has codimension ≤ 1 in $\text{Supp } F$. Thus Q satisfies the assumptions of (10.25) and so $j_* Q$ is coherent. By (10.28.4) $R^1 j_*(F|_U) = 0$, hence the exact sequence

$$0 \rightarrow j_* G \rightarrow j_*(F|_U) \rightarrow j_* Q \rightarrow R^1 j_* G \rightarrow R^1 j_*(F|_U) = 0$$

shows that $R^1 j_* G$ is coherent. \square

Not every S_2 -sheaf can be realized as a subsheaf of a CM sheaf, but this can be arranged in some important cases.

Lemma 10.28.10. Notation as above. Assume in addition that

- (a) X is embeddable into a regular, affine scheme R as a closed subscheme.
- (b) $\text{Supp } G$ has pure dimension $n \geq 3$ and $Z = \{x\}$ is a closed point.
- (c) G is S_2 .

Then $H^1(U, G)$ has finite length. Thus, if X is of finite type over a field k , then $H^1(U, G)$ is a finite dimensional k -vector space.

Outline of proof. X plays essentially no role. Let $Y \subset R$ be a complete intersection subscheme defined by $\dim R - n$ elements of $\text{Ann } G$. Then Y is Gorenstein, we can view G as a coherent sheaf on $Y \setminus \{x\}$ and $H^i(X \setminus \{x\}, G) = H^i(Y \setminus \{x\}, G)$. Thus it is enough to prove vanishing of the latter for $i = 1$.

By (10.28.11) there is an embedding $G \hookrightarrow \mathcal{O}_{Y \setminus \{x\}}^m$, hence (10.28.9) applies. \square

Lemma 10.28.11. Let U be a quasi-affine scheme of pure dimension n and G a pure, coherent sheaf on U of dimension n . Assume that

- (10.28.1) either U is reduced
- (10.28.2) or U is Gorenstein at its generic points.

Then G is isomorphic to a subsheaf of \mathcal{O}_U^m for some m . \square

Outline of proof. Assume that such an embedding exists at the generic points. Then we have an embedding $G \hookrightarrow \mathcal{O}_U^m$ over some dense open set $U^\circ \subset U$. Pick $s \in \mathcal{O}_U$ invertible at the generic points and vanishing along $U \setminus U^\circ$. Multiplying by s^r for $r \gg 1$ gives the embedding $G \hookrightarrow \mathcal{O}_U^m$.

The remaining question is, what happens at the generic point. The existence of the embedding is clear if U is reduced.

In general, we are reduced to the following algebra question: given an Artin ring A , when is every finite A -module M a submodule of A^m for some m ? Usually the answer is no. However, local duality theory (see, for instance, [Eis95, Secs.21.1–2]) shows that every finite A -module is a submodule of ω_A^m for some m . Finally A is Gorenstein iff $A \cong \omega_A$. \square

Much of the following result can be established by the above methods, but it is easier to prove it using local duality theory; see [Gro68, VIII.2.3] for details.

THEOREM 10.29. *Let X be an excellent scheme, $Z \subset X$ a closed subscheme, $U := X \setminus Z$ and $j : U \hookrightarrow X$ the open embedding. Assume in addition that X is locally embeddable into a regular scheme. For a coherent sheaf G on U and $n \in \mathbb{N}$ the following are equivalent.*

- (10.29.1) $R^i j_* G$ is coherent for $i < n$.
- (10.29.2) $\text{depth}_u G \geq n$ for every point $u \in U$ such that $\text{codim}_{\bar{u}}(Z \cap \bar{u}) = 1$. \square

10.4. Volumes and intersection numbers

We have used several general results that compare intersection numbers and volumes under birational morphisms.

DEFINITION 10.30. [Laz04, Sec.2.2.C] Let X be a proper scheme of dimension n over a field k and D a Mumford \mathbb{R} -divisor on X (4.20.4). Its *volume* is defined as

$$\mathrm{vol}(D) := \lim_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X([mD]))}{m^n/n!}.$$

Algebraically equivalent divisors have the same volume and for $D = \sum d_i D_i$, the volume is a continuous function of the d_i . If D is nef then $\mathrm{vol}(D) = (D^n)$.

PROPOSITION 10.31. *Let $p : Y \rightarrow X$ be a birational morphism of normal, proper varieties of dimension n . Let D_Y be a p -nef \mathbb{R} -Cartier \mathbb{R} -divisor such that $D_X := p_*(D_Y)$ is also \mathbb{R} -Cartier. Then*

$$(10.31.1) \quad \mathrm{vol}(D_X) \geq \mathrm{vol}(D_Y) \text{ and}$$

$$(10.31.2) \quad \text{if } D_X \text{ is ample then equality holds in (1) iff } D_Y \sim_{\mathbb{R}} p^* D_X.$$

Furthermore, let H is an ample divisor on X . Then

$$(10.31.3) \quad I(H, D_X) \succeq I(p^* H, D_Y) \text{ (with } I(*, *) \text{ as in (5.13)), and}$$

$$(10.31.4) \quad \text{equality holds in (3) iff } D_Y \sim_{\mathbb{R}} p^* D_X.$$

The general case when equality holds in (1) is considered in (10.38).

Proof. Write $D_Y = p^* D_X - E$ where E is p -exceptional. By assumption $-E$ is p -nef, hence E is effective by (11.50) Thus $\mathrm{vol}(D_X) = \mathrm{vol}(p^* D_X) \geq \mathrm{vol}(D_Y)$, proving (1). Parts (2) and (4) are a special cases of (10.38), but here is a more direct argument.

Set $r = \dim(p(\mathrm{Supp} E))$. For any \mathbb{R} -Cartier divisors A_i on X the intersection number $(p^* A_1 \cdots p^* A_j \cdot E)$ vanishes whenever $j > r$. Thus, if $j > r$ then

$$(p^* H^j \cdot D_Y^{n-j}) = (p^* H^j \cdot (p^* D_X - E)^{n-j}) = (p^* H^j \cdot p^* D_X^{n-j}) = (H^j \cdot D_X^{n-j})$$

and for $j = r$ we get that

$$(p^* H^r \cdot D_Y^{n-r}) = (H^r \cdot D_X^{n-r}) + (p^* H^r \cdot (-E)^{n-r}).$$

Thus we need to understand $(p^* H^r \cdot (-E)^{n-r})$. We may assume that H is very ample. Intersecting with $p^* H$ is then equivalent to restricting to the preimage of a general member of $|H|$. Using this r -times (and normalizing if necessary), we get a birational morphism $p' : Y' \rightarrow X'$ between normal varieties of dimension $n - r$ and an effective, nonzero, p -exceptional \mathbb{R} -Cartier \mathbb{R} -divisor E' such that $-E'$ is p' -nef and $p'(E'_i)$ is 0-dimensional. Thus, by (10.32), $(p^* H^r \cdot (-E)^{n-r}) = (-E')^{n-r} < 0$ which proves (3–4).

If D_X is ample then we can use this for $H := D_X$. Then $(H^r \cdot D_X^{n-r}) = (D_X^n)$ and we get (2). \square

LEMMA 10.32. *Let $p : Y \rightarrow X$ be a proper, birational morphism of normal schemes. Let E be an effective, nonzero, p -exceptional \mathbb{R} -Cartier \mathbb{R} -divisor such that $p(E)$ is 0-dimensional and $-E$ is p -nef. Set $n = \dim E$.*

$$\text{Then } -(-E)^{n+1} = (-E|_E)^n > 0.$$

Proof. Assume that there is an effective, nonzero, p -exceptional \mathbb{R} -Cartier \mathbb{R} -divisor F such that $p(F) = p(E)$, $-F$ is p -nef and $-(-F)^{n+1} > 0$. Note that E, F have the same support, namely $p^{-1}(p(E))$, thus $E - \epsilon F$ is effective for $0 < \epsilon \ll 1$. Thus $-(-E)^n \geq -(-\epsilon F)^n$ by (10.33) applied to $N_2 = -E, N_1 = -\epsilon F$.

Such a divisor F exists on the normalization of the blow-up $B_{p(E)}X$. Let now $Z \rightarrow X$ be a proper, birational morphism that dominates both Y and $B_{p(E)}X$. We can apply the above observation to the pull-backs of E and F to Z . \square

LEMMA 10.33. *Let N_1, N_2 be \mathbb{R} -Cartier divisors with proper support on an $n + 1$ -dimensional scheme. Assume that there exists an effective divisor with proper support D such that $D \sim_{\mathbb{R}} N_1 - N_2$ and the $N_i|_D$ are both nef. Then $(N_1^{n+1}) \geq (N_2^{n+1})$.*

Proof. $(N_1^{n+1}) - (N_2^{n+1}) = D \cdot \sum_{i=0}^n N_1^i N_2^{n-i} = \sum_{i=0}^n (N_1|_D)^i (N_2|_D)^{n-i}$. \square

The next results compare the volumes of different perturbations of the canonical divisor.

LEMMA 10.34. *Let X be a normal, proper variety of dimension n and D an effective \mathbb{R} -divisor such that $K_X + D$ is \mathbb{R} -Cartier, nef and big. Let Y be a smooth, proper variety birational to X . Then*

$$(10.34.1) \quad \text{vol}(K_Y) \leq (K_X + D)^n \text{ and}$$

$$(10.34.2) \quad \text{equality holds iff } D = 0 \text{ and } X \text{ has canonical singularities.}$$

Proof. Let Z be a normal, proper variety birational to X such that there are morphisms $q : Z \rightarrow Y$ and $p : Z \rightarrow X$. Write

$$K_Z \sim_{\mathbb{R}} q^* K_Y + E \quad \text{and} \quad K_Z \sim_{\mathbb{R}} p^*(K_X + D) - p_*^{-1}D + F, \quad (10.34.3)$$

where E is effective, q -exceptional and F is p -exceptional (not necessarily effective). Thus

$$q^* K_Y \sim_{\mathbb{R}} p^*(K_X + D) - p_*^{-1}D + F - E. \quad (10.34.4)$$

Write $F - E = G^+ - G^-$ where G^+, G^- are effective and without common irreducible components. Note that G^+ is p -exceptional, so

$$H^0(Z, \mathcal{O}_Z(\lfloor mp^*(K_X + D) + mG^+ \rfloor)) = H^0(Z, \mathcal{O}_Z(\lfloor mp^*(K_X + D) \rfloor))$$

and hence also

$$\begin{aligned} H^0(Z, \mathcal{O}_Z(\lfloor mp^*(K_X + D) - p_*^{-1}(mD) + mG^+ - mG^- \rfloor)) \\ = H^0(Z, \mathcal{O}_Z(\lfloor mp^*(K_X + D) - p_*^{-1}(mD) - mG^- \rfloor)). \end{aligned} \quad (10.34.5)$$

Thus

$$\begin{aligned} \text{vol}(K_Y) &= \text{vol}(p^*(K_X + D) - p_*^{-1}D + G^+ - G^-) \\ &= \text{vol}(p^*(K_X + D) - p_*^{-1}D - G^-) \\ &\leq \text{vol}(p^*(K_X + D)) \\ &= \text{vol}(K_X + D) = (K_X + D)^n. \end{aligned}$$

Furthermore, by (10.38) equality holds iff $p_*^{-1}D + G^- = 0$, that is, when $D = 0$ and $G^- = 0$. In such a case (10.34.4) becomes

$$q^* K_Y \sim_{\mathbb{R}} p^* K_X + G^+ \quad \text{and } G^+ \text{ is effective.}$$

Thus $a(E, X) \geq a(E, Y)$ for every divisor E (cf. [Kol13b, 2.5]), hence X has canonical singularities. \square

A similar birational statement does not hold for pairs in general, but a variant holds if Y is a resolution of X . We can also add some other auxiliary divisors; these are needed in our applications.

LEMMA 10.35. *Let X be a normal, proper variety of dimension n and Δ a reduced, effective \mathbb{R} -divisor on X . Let A be an \mathbb{R} -Cartier \mathbb{R} -divisor and D an effective \mathbb{R} -divisor such that $K_X + \Delta + A + D$ is \mathbb{R} -Cartier, nef and big. Let $p : Y \rightarrow X$ be any log resolution of (X, Δ) . Then*

$$(10.35.1) \quad \text{vol}(K_Y + p_*^{-1}\Delta + p^*A) \leq (K_X + \Delta + A + D)^n \text{ and}$$

(10.35.2) *equality holds iff $D = 0$ and (X, Δ) is canonical.*

Proof. As usual, write

$$K_Y + p_*^{-1}\Delta \sim_{\mathbb{R}} p^*(K_X + \Delta + D) - p_*^{-1}D - F_1 + F_2, \quad (10.35.3)$$

where the F_i are p -exceptional, effective and without common irreducible components. As in (10.34.5) we get that

$$\begin{aligned} H^0(Y, \mathcal{O}_Y([mp^*(K_X + \Delta + A + D) - p_*^{-1}(mD) - mF_1 + mF_2])) \\ = H^0(Y, \mathcal{O}_Y([mp^*(K_X + \Delta + A + D) - p_*^{-1}(mD) - mF_1])). \end{aligned}$$

Thus

$$\begin{aligned} \text{vol}(K_Z + p_*^{-1}\Delta + p^*A) &= \text{vol}(p^*(K_X + \Delta + A + D) - p_*^{-1}D + F_2 - F_1) \\ &= \text{vol}(p^*(K_X + \Delta + A + D) - p_*^{-1}D - F_1) \\ &\leq \text{vol}(p^*(K_X + \Delta + A + D)) \\ &= \text{vol}(K_X + \Delta + A + D) = (K_X + \Delta + A + D)^n. \end{aligned}$$

Furthermore, by (10.38) equality holds iff $p_*^{-1}D + F_1 = 0$, that is, when $D = 0$ and $F_1 = 0$. Thus (10.35.3) becomes

$$K_Z + p_*^{-1}\Delta \sim_{\mathbb{R}} p^*(K_X + \Delta) + F_2$$

where F_2 is effective. This says that (X, Δ) is canonical. \square

Essentially the same argument gives the following log canonical version.

LEMMA 10.36. *Let X be a normal, proper variety of dimension n , Δ a reduced, effective \mathbb{R} -divisor on X and A an \mathbb{R} -Cartier \mathbb{R} -divisor on X . Let $q : \bar{X} \rightarrow X$ be a proper birational morphism, \bar{E} the reduced q -exceptional divisor, $\bar{\Delta} := q_*^{-1}\Delta$ and \bar{D} an effective \mathbb{R} -divisor on \bar{X} such that $K_{\bar{X}} + \bar{\Delta} + \bar{E} + D + q^*A$ is \mathbb{R} -Cartier, nef and big. Let $p : Y \rightarrow X$ be any log resolution of singularities with reduced exceptional divisor E . Then*

$$(10.36.1) \quad \text{vol}(K_Y + p_*^{-1}\Delta + E + p^*A) \leq (K_{\bar{X}} + \bar{\Delta} + \bar{E} + \bar{D} + q^*A)^n \text{ and}$$

(10.36.2) *equality holds iff $\bar{D} = 0$ and $(\bar{X}, \bar{\Delta} + \bar{E})$ is log canonical.* \square

We have also used the following elementary estimate.

LEMMA 10.37. *Let $p : Y \rightarrow X$ be a separable, generically finite morphism between smooth, proper varieties. Then $\text{vol}(K_Y) \geq \text{deg}(Y/X) \cdot \text{vol}(K_X)$.*

Proof. This is obvious if $\text{vol}(K_X) = 0$, hence we may assume that K_X is big. Pulling back differential forms gives a natural map $p^*\omega_X \rightarrow \omega_Y$. This gives an injection

$$\omega_X^r \otimes p_*\omega_Y \hookrightarrow p_*(\omega_Y^{r+1}).$$

Since $p_*\omega_Y$ has rank $\text{deg}(Y/X)$ and K_X is big, $H^0(X, \omega_X^r \otimes p_*\omega_Y)$ grows at least as fast as $\text{deg}(Y/X) \cdot H^0(X, \omega_X^r)$. \square

The following result describes the variation of the volume near a nef and big divisor. The assertions are special cases of [FKL16, Thms.A–B].

THEOREM 10.38. *Let X be a proper variety, L a big \mathbb{R} -Cartier divisor and E an effective divisor. The following are equivalent.*

$$(10.38.1) \quad \text{vol}(L - E) = \text{vol}(L) \text{ and}$$

$$(10.38.2) \quad H^0(\mathcal{O}_X(\lfloor mL - mE \rfloor)) = H^0(\mathcal{O}_X(\lfloor mL \rfloor)) \text{ for every } m \geq 0.$$

If L is nef then these are further equivalent to

$$(10.38.3) \quad E = 0.$$

Note that the implications (3) \Rightarrow (2) \Rightarrow (1) are clear but the converse is somewhat surprising. It says that although the volume measures only the asymptotic growth of the Hilbert function, one can not change the Hilbert function without changing the volume. For proofs see the original paper.

10.5. Double points

We used a variety of results about hypersurface double points. For the rest of the section we work with rings R that contain $\frac{1}{2}$. In this case, all the definitions that we have seen are equivalent to the ones given below. If $\frac{1}{2} \notin R$, there are differing conventions, especially if $\text{char } R = 2$.

The following results on normal forms, deformations and resolutions of double points are well known, but not easy to find in one place.

DEFINITION 10.39. A *quadratic form* over a field k is a degree 2 homogeneous polynomial $q(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$. The *rank* of q is defined either as the dimension of the space spanned by the derivatives

$$\left\langle \frac{\partial q}{\partial x_1}, \dots, \frac{\partial q}{\partial x_n} \right\rangle, \quad (10.39.1)$$

or as the rank of the Hessian matrix

$$\text{Hess}(q) := \left(\frac{\partial^2 q}{\partial x_i \partial x_j} \right), \quad (10.39.2)$$

or as the number of variables in any diagonalized form

$$q = a_1 y_1^2 + \dots + a_r y_r^2 \quad \text{where } a_i \in k^\times. \quad (10.39.3)$$

More abstractly, if V is a k -vector space, we can think of q as an element of its symmetric square $S^2 V$. (With this convention, $V = \langle x_1, \dots, x_n \rangle$. It is also natural to think of $W := \text{Spec}_k k[x_1, \dots, x_n]$ to be the basic object, then quadratic forms are elements of $S^2(W^*)$.)

DEFINITION 10.40. Let (S, m) be a regular local ring with residue field k such that $\text{char } k \neq 2$. We can identify m^2/m^3 with $S^2(m/m^2)$. Thus, for any $g \in m^2$, we can view $g + m^3 \in m^2/m^3$ as a quadratic form.

Let Y be a smooth variety over a field of characteristic $\neq 2$ and $X = (g = 0) \subset Y$ a hypersurface. Given a point $p \in X$, we let $\text{rank}_p X$ denote the rank of $g + m_p^3 \in m_p^2/m_p^3$.

We say that $p \in X$ is a *double point* if $\text{rank}_p X \geq 1$, a *cA point* if $\text{rank}_p X \geq 2$ and an *ordinary double point* if $\text{rank}_p X = \dim_p X$. An ordinary double point is also called a *node*, especially if $\dim S = 2$.

If y_1, \dots, y_n are étale coordinates on Y then we can compute the Hessian as

$$\text{Hess}_y(g) = \left(\frac{\partial^2 g}{\partial y_i \partial y_j} \right). \quad (10.40.1)$$

Since the rank is lower semicontinuous, we see that, for every r ,

$$\{p \in \text{Sing } X : \text{rank}_p X \geq r\} \text{ is open in } \text{Sing } X. \quad (10.40.2)$$

For us the most interesting case is $r = 2$. The relative version of (10.40.2) is then the following.

Claim 10.40.3. Let $f : Y \rightarrow S$ be smooth and $X \subset Y$ a relative Cartier divisor. Then $\{p \in X : p \text{ is } cA \text{ (or smooth) on } X_{f(p)}\} \subset X$ is open. \square

This implies that if $X \rightarrow S$ is proper and X_s has only cA -singularities (and smooth points) outside a closed subset $Z_s \subset X_s$ of codimension $\geq m$ for some $s \in S$ then the same holds in an open neighborhood $s \in S^\circ \subset S$.

COROLLARY 10.41. *Let $\pi : X \rightarrow S$ be a flat and pure dimensional morphism. Then the set of points $\{x : X_{\pi(x)} \text{ is demi-normal at } x\}$ is open in X .*

Proof. Being S_2 is an open condition by (10.3). A proper S_1 scheme is geometrically reduced iff it is generically smooth and smoothness is an open condition. Thus being S_2 and geometrically reduced is an open condition.

It remains to show that having only nodes in codimension 1 is also an open condition. If all residue characteristics are $\neq 2$, this follows from (10.40.3) since having only cA -singularities in codimension 1 is an open condition.

See [Kol13b, 1.41] for the right definition and the universal deformation space of a node in characteristic 2. \square

Morse lemma.

Let f be a function on \mathbb{R}^n that has an ordinary critical point at the origin. The Morse lemma says that in suitable local coordinates y_1, \dots, y_n we can write f as $\pm y_1^2 \pm \dots \pm y_n^2$; see [Mil63, p.6] and [AGZV85, Vol.I;Sec.6.2] for the precise differentiable and analytic versions. Algebraically the best is to work with formal power series.

LEMMA 10.42 (Formal Morse lemma). *Let k be a field of characteristic $\neq 2$ and $g \in k[[x_1, \dots, x_n]]$ a power series of multiplicity ≥ 2 such that $\text{rank Hess}(g) = r$. Then there are local coordinates y_1, \dots, y_n such that*

$$g = a_1 y_1^2 + \dots + a_r y_r^2 + h(y_{r+1}, \dots, y_n),$$

where $a_i \in k^\times$ and $\text{mult } h \geq 3$.

We state and prove a more general version of this next.

Let (R, m) be a local ring and $g \in R[[x_1, \dots, x_n]]$ a power series. Reduction modulo m is denoted by \bar{g} . Thus $\bar{g}(x_1, \dots, x_n) \in (R/m)[[x_1, \dots, x_n]]$. We aim to understand those cases when $\text{mult } \bar{g} = 2$. The next result is stated in a form that also works if $\text{char}(R/m) = 2$.

LEMMA 10.43 (Formal Morse lemma with parameters). *Let (R, m) be a complete local ring and $G \in R[[x_1, \dots, x_N]]$ a power series of multiplicity ≥ 2 . Assume that there is a quadratic form $q(x_1, \dots, x_n)$ such that*

$$(10.43.1) \quad \dim \langle \partial \bar{q} / \partial x_1, \dots, \partial \bar{q} / \partial x_n \rangle = n \text{ and}$$

$$(10.43.2) \quad \bar{G}(x_1, \dots, x_n, 0, \dots, 0) - \bar{q}(x_1, \dots, x_n, 0, \dots, 0) \in (x_1, \dots, x_n)^3.$$

Then there are local coordinates $y_1, \dots, y_n, x_{n+1}, \dots, x_N$ such that

$$(10.43.3) \quad y_i \equiv x_i \pmod{(x_1, \dots, x_N)^2 + mR[[x_1, \dots, x_N]]} \text{ and}$$

$$(10.43.4) \quad G = q(y_1, \dots, y_n) + b \text{ for some } b \in (x_{n+1}, \dots, x_N)^3 + mR[[x_{n+1}, \dots, x_N]].$$

Proof. Replacing R by $R[[x_{n+1}, \dots, x_N]]$ reduces everything to the case when $N = n$. Thus we need to find local coordinates y_1, \dots, y_n such that

$$(3') \quad y_i \equiv x_i \pmod{(x_1, \dots, x_n)^2 + mR[[x_1, \dots, x_n]]} \text{ and}$$

$$(4') \quad G = q(y_1, \dots, y_n) + b \text{ for some } b \in m.$$

Let us start with the case when $R = k$ is a field. Set $x_{2,i} := x_i$. Assume inductively (starting with $r = 2$) that there are local coordinate systems $(x_{s,1}, \dots, x_{s,n})$ for $2 \leq s \leq r$ such that

$$\begin{aligned} x_{s,i} &\equiv x_{s-1,i} \pmod{(x_1, \dots, x_n)^{s-1}} \text{ and} \\ G &\equiv q(x_{r,1}, \dots, x_{r,n}) \pmod{(x_1, \dots, x_n)^{r+1}}. \end{aligned}$$

Next we choose $x_{r+1,i} := x_{r,i} + h_{r,i}$ for suitable $h_{r,i} \in (x_1, \dots, x_n)^r$. Note that

$$q(x_{r+1,1}, \dots, x_{r+1,n}) = q(x_{r,1}, \dots, x_{r,n}) + \sum_i h_{r,i} \frac{\partial q}{\partial x_i} \pmod{(x_1, \dots, x_n)^{2r}}.$$

(We use this only modulo $(x_1, \dots, x_n)^{r+2}$.) Since q is nondegenerate,

$$\sum_i \frac{\partial q}{\partial x_i} (x_1, \dots, x_n)^r = (x_1, \dots, x_n)^{r+1}.$$

Thus we can choose the $h_{r,i}$ such that

$$G - q(x_{r+1,1}, \dots, x_{r+1,n}) \in (x_1, \dots, x_n)^{r+2}.$$

In the limit we get $(x_{\infty,1}, \dots, x_{\infty,n})$ as required.

Applying this $k = R/m$ we can assume from now on that

$$G - q(x_1, \dots, x_n) \in mR[[x_1, \dots, x_n]].$$

Working inductively (starting with $r = 1$) assume that there are local coordinate systems $(y_{s,1}, \dots, y_{s,n})$ for $2 \leq s \leq r$ such that

$$\begin{aligned} y_{s,i} &\equiv y_{s-1,i} \pmod{m^{s-1}R[[x_1, \dots, x_n]]} \text{ and} \\ G &\equiv q(y_{r,1}, \dots, y_{r,n}) \pmod{m + m^r R[[x_1, \dots, x_n]]}. \end{aligned}$$

Next we choose $y_{r+1,i} := y_{r,i} + c_{r,i}$ for suitable $c_{r,i} \in m^r R[[x_1, \dots, x_n]]$. Note that

$$q(y_{r+1,1}, \dots, y_{r+1,n}) = q(y_{r,1}, \dots, y_{r,n}) + \sum_i c_{r,i} \frac{\partial q}{\partial x_i} \pmod{m^{2r} R[[x_1, \dots, x_n]]}.$$

(We use this only modulo $m^{r+1} R[[x_1, \dots, x_n]]$.) Since q is nondegenerate,

$$\sum_i \frac{\partial q}{\partial x_i} m^r R[[x_1, \dots, x_n]] = (x_1, \dots, x_n) m^r R[[x_1, \dots, x_n]].$$

Thus we can choose the $c_{r,i}$ such that

$$G - q(y_{r+1,1}, \dots, y_{r+1,n}) \in m + m^{r+1} R[[x_1, \dots, x_n]].$$

In the limit we get $(y_{\infty,1}, \dots, y_{\infty,n})$ as required. \square

In (1.26) we used various results on resolutions of double points of surfaces that contain a pair of lines and double points of 3-folds that contain a pair of planes. The normal forms can be obtained using the method of (10.43) but we did not follow how linear subvarieties transform under the (non-linear) coordinate changes used there. However, in the next examples one can be quite explicit about the coordinate changes and the resolutions.

10.44 (Ordinary double points of surfaces). Let $S := (h(x_1, x_2, x_3) = 0) \subset \mathbb{A}^3$ be a surface with an ordinary double point at the origin that contains the pair of lines $(x_1x_2 = x_3 = 0)$. Then h can be written as

$$h = f(x_1, x_2, x_3)x_1x_2 - g(x_1, x_2, x_3)x_3.$$

If the quadratic part has rank 3 then $f(0, 0, 0) \neq 0$ and we can write $g = x_1g_1 + x_2g_2 + x_3g_3$ for some polynomials g_i . Thus

$$h = f(x_1 - f^{-1}g_1x_3)(x_2 - f^{-1}g_2x_3) - (g_3 + f^{-1}g_1g_2)x_3^2.$$

Here $g_3 + f^{-1}g_1g_2$ is nonzero at $(0, 0, 0)$ and we can set

$$y_1 := x_1 - f^{-1}g_1x_3, \quad y_2 := f(x_2 - f^{-1}g_2x_3)(g_3 + f^{-1}g_1g_2)^{-1} \quad \text{and} \quad y_3 := x_3$$

to bring the equation to the normal form $S = (y_1y_2 - y_3^2 = 0)$. The pair of lines is still $(y_1y_2 = y_3 = 0)$.

Now we consider 3 ways of resolving the singularity of X . First, one can blow up the origin $0 \in \mathbb{A}^3$. We get

$$B_0\mathbb{A}^3 \subset \mathbb{A}_{\mathbf{y}}^3 \times \mathbb{P}_{\mathbf{s}}^2$$

defined by the equations $\{y_i s_j = y_j s_i : 1 \leq i, j \leq 3\}$. Besides these equations, B_0S is defined by $y_1y_2 - y_3^2 = s_1s_2 - s_3^2 = y_1s_2 - y_3s_3 = s_1y_2 - y_3s_3 = 0$.

One can also blow up (y_1, y_3) . We get

$$B_{(y_1, y_3)}\mathbb{A}^3 \subset \mathbb{A}_{\mathbf{y}}^3 \times \mathbb{P}_{u_1 u_3}^1$$

defined by the equation $y_1u_3 = y_3u_1$. Besides this equation, $B_{(y_1, y_3)}S$ is defined by $y_1y_2 - y_3^2 = u_1y_2 - u_3y_3 = 0$.

These two blow-ups are actually isomorphic, as shown by the embedding

$$\mathbb{A}_{\mathbf{y}}^3 \times \mathbb{P}_{u_1 u_3}^1 \hookrightarrow \mathbb{A}_{\mathbf{y}}^3 \times \mathbb{P}_{\mathbf{s}}^2 \quad : \quad ((y_1, y_2, y_3), (u_1 : u_3)) \mapsto ((y_1, y_2, y_3), (u_1^2 : u_3^2 : u_1u_3))$$

restricted to $B_{(y_1, y_3)}S$.

The same things happen if we blow up (y_2, y_3) .

10.45 (Ordinary double points of 3-folds). Let $X := (h(x_1, \dots, x_4) = 0) \subset \mathbb{C}^4$ be a hypersurface with an ordinary double point at the origin that contains the pair of planes $(x_1x_2 = x_3 = 0)$. Then h can be written as

$$h = f(x_1, \dots, x_4)x_1x_2 - g(x_1, \dots, x_4)x_3.$$

The quadratic part has rank 4 iff $f(0, \dots, 0) \neq 0$ and x_4 appears in g with nonzero coefficient. In this case we can set

$$y_i := x_i \quad \text{for } i = 1, 2, 3, \quad \text{and} \quad y_4 := f^{-1}g$$

to bring the equation to the normal form $X = (y_1y_2 - y_3y_4 = 0)$. The original pair of planes is still $(y_1y_2 = y_3 = 0)$.

Now we consider 3 ways of resolving the singularity of X . First, one can blow up the origin $0 \in \mathbb{A}^4$. We get

$$B_0\mathbb{A}^4 \subset \mathbb{A}_{\mathbf{y}}^4 \times \mathbb{P}_{\mathbf{s}}^3$$

defined by the equations $\{y_i s_j = y_j s_i : 1 \leq i, j \leq 4\}$ and $p : B_0X \rightarrow X$ by the additional equations

$$y_1y_2 - y_3y_4 = s_1s_2 - s_3s_4 = y_i s_{3-i} - y_j s_{7-j} = 0 \quad : \quad i \in \{1, 2\}, j \in \{3, 4\}.$$

The exceptional set is the smooth quadric $(s_1s_2 = s_3s_4) \subset \mathbb{P}^3$ lying over the origin $0 \in \mathbb{A}^4$.

One can also blow up (y_1, y_3) . We get

$$B_{(y_1, y_3)}\mathbb{A}^4 \subset \mathbb{A}_{\mathbf{y}}^4 \times \mathbb{P}_{u_1u_3}^1$$

defined by the equation $y_1u_3 = y_3u_1$. Besides this equation, $B_{(y_1, y_3)}X$ is defined by $y_1y_2 - y_3y_4 = u_1y_2 - u_3y_4 = 0$. The exceptional set is the smooth rational curve $E \cong \mathbb{P}_{u_1u_3}^1$ lying over the origin $0 \in \mathbb{A}^4$.

Note furthermore that the birational transform P_{24}^* of the plane $P_{24} := (y_2 = y_4 = 0)$ is the blown-up plane B_0P_{24} , but the birational transform P_{14}^* of the plane $P_{14} := (y_1 = y_4 = 0)$ is the plane $(y_1 = u_1 = 0)$. The latter intersects E at the point $(u_1 = 0) \in E$, thus $(P_{14}^* \cdot E) = 1$. Since $P_{14}^* + P_{24}^*$ is the pull-back of the Cartier divisor $(y_4 = 0)$, it has 0 intersection number with E . Thus $(P_{24}^* \cdot E) = -1$.

We claim that the rational map $p : \mathbb{A}_{\mathbf{y}}^4 \times \mathbb{P}_{\mathbf{s}}^3 \dashrightarrow \mathbb{A}_{\mathbf{y}}^4 \times \mathbb{P}_{\mathbf{u}}^1$ given by

$$p_1 : (y_1, \dots, y_4, s_1 : \dots : s_4) \mapsto (y_1, \dots, y_4, s_1 : s_3)$$

gives a morphism $p_1 : B_0X \rightarrow B_{(y_1, y_3)}X$.

To see this note that the quadric $Q := (s_1s_2 - s_3s_4 = 0)$ is isomorphic to $\mathbb{P}_{\mathbf{u}}^1 \times \mathbb{P}_{\mathbf{v}}^1$, with the isomorphism given as

$$j : ((u_0 : u_1), (v_0 : v_1)) \mapsto (u_0v_0 : u_0v_1 : u_1v_0 : u_1v_1).$$

Thus the map $(s_1 : \dots : s_4) \mapsto (s_1 : s_3)$ is the inverse of j followed by the 1st coordinate projection. Thus p_1 restricts to a morphism on $\mathbb{A}_{\mathbf{y}}^4 \times Q$ and $B_0X \subset \mathbb{A}_{\mathbf{y}}^4 \times Q$.

Similarly, we obtain $p_2 : B_0X \rightarrow B_{(y_2, y_3)}X$. Putting these together, we get an isomorphism

$$p_1 \times p_2 : B_0X \cong B_{(y_1, y_3)}X \times_X B_{(y_2, y_3)}X.$$

(The above considerations show that this is an isomorphism of reduced schemes, and this is all we need. However, by explicit computation, the right hand side is reduced, so we have a scheme theoretic isomorphism.) In particular, this shows that the two maps $p_i : B_{(y_i, y_3)}X \rightarrow X$ are not isomorphic to each other.

Finally, set $S := (y_3 = y_4) \subset X$. By the computations of (10.44), the p_i restrict to isomorphisms $p_i : B_0S \cong B_{(y_i, y_3)}S$. Thus $p^{-1}S = B_0S \cup E$ and B_0S is the graph of the isomorphism $p_2 \circ p_1^{-1} : B_{(y_1, y_3)}S \cong B_{(y_2, y_3)}S$.

10.6. Flatness criteria

Let $f : X \rightarrow S$ be a morphism that we would like to prove to be flat. If f is of finite type then flatness is an open property. Let $U \subset X$ denote the largest open set over which f is flat and set $Z := X \setminus U$. The situation is technically simpler if Z is a single closed point. To achieve this, one can use a Bertini-type theorem (10.46) to pass to a general hyperplane section of X and repeat if necessary. At the end we arrive at a finite type morphism $g : X' \rightarrow S$ that is flat except possibly at a finite set of points. Localizing at any one of them we have a local morphism of local schemes

$$f' : (x', X') \rightarrow (s', S')$$

that is flat over $X' \setminus \{x'\}$ and $k(x')/k(s')$ is finite field extension.

Alternatively, we can localize at a generic point of Z and then use (10.47) to reach the same situation.

If f is not of finite type then we have to be more careful since flatness is not an open property for morphisms of arbitrary Noetherian schemes. A morphism is flat iff it is flat at all points and the latter can be checked after localization. A local, Noetherian scheme is finite dimensional, there is thus a point of largest dimension where f is not flat. Localizing at that point we again get $f' : (x', X') \rightarrow (s', S')$ that is flat over $X' \setminus \{x'\}$.

If $k(x')/k(s')$ is finitely generated then we can again use (10.47) but the situation is more complicated in general. We wrangle with this issue in (10.67).

We can also complete X' and S' , thus we are reduced to the case when we have a local morphism of complete, local, Noetherian schemes. Note, however, that some of our results hold only over base schemes that are normal, seminormal or reduced. These conditions are preserved by completion if S' is excellent but not in general.

PROPOSITION 10.46. *Let $(x, X) \rightarrow (s, S)$ be a local morphism of local schemes and F a coherent sheaf on X . Assume that $r \in m_{x,X}$ is a non-zerodivisor both on F and on F_s . Then F is flat over S iff F/rF is flat over S .*

Proof. By assumption we have an exact sequence

$$0 \rightarrow F \xrightarrow{r} F \rightarrow F/rF \rightarrow 0.$$

Tensoring with $k = k(s)$ gives

$$\mathrm{Tor}^1(k, F) \xrightarrow{r} \mathrm{Tor}^1(k, F) \rightarrow \mathrm{Tor}^1(k, F/rF) \rightarrow F_s \xrightarrow{r} F_s \rightarrow (F/rF)_s \rightarrow 0.$$

By the second assumption $r : F_s \rightarrow F_s$ is injective, hence we get a shorter exact sequence

$$\mathrm{Tor}^1(k, F) \xrightarrow{r} \mathrm{Tor}^1(k, F) \rightarrow \mathrm{Tor}^1(k, F/rF) \rightarrow 0.$$

F is flat over S iff $\mathrm{Tor}^1(k, F) = 0$ by the local criterion of flatness (see [Mat86, Sec.22] or [Eis95, Sec.6.4]) hence $\mathrm{Tor}^1(k, F/rF) = 0$ and so F/rF is flat over S . Conversely, if F/rF is flat over S then $\mathrm{Tor}^1(k, F/rF) = 0$ hence $r : \mathrm{Tor}^1(k, F) \rightarrow \mathrm{Tor}^1(k, F)$ is surjective. Thus $\mathrm{Tor}^1(k, F) = 0$ by the Nakayama lemma and so F is flat over S . \square

10.47 (Flatness and residue field extension). The following simple trick reduces most flatness questions for local morphisms $f : (x, X) \rightarrow (s, S)$ with finitely generated residue field extension $k(x)/k(s)$ to the special case when $k(x)/k(s)$ is purely inseparable. (See 10.67–10.68 for other versions.)

Let $f : X \rightarrow S$ be a morphism, $x \in X$ a point and $s := f(x)$ its image. Choose $k(x) \supset K \supset k(s)$ such that $K/k(s)$ is separable and finitely generated. (We usually also want $k(x)/K$ to be purely inseparable, but this is not needed for the general construction.)

We can thus realize K as the the generic point of a hypersurface $s' \in (h = 0) \subset \mathbb{A}_{k(s)}^n$ where $n = 1 + \mathrm{tr}\text{-deg}_{k(s)} K$. Lift h to a polynomial $H \in \mathcal{O}_S[z_1, \dots, z_n]$ and set $T := (H = 0) \subset \mathbb{A}_S^n$. Let S' be the localization of T at s' and X' the localization of $X \times_S T$ at $x' := (x, s')$. Thus we have a commutative diagram of pointed schemes

$$\begin{array}{ccc} (x' \in X') & \xrightarrow{\pi_{X'}} & (x \in X) \\ f' \downarrow & & \downarrow f \\ (s' \in S') & \xrightarrow{\pi_{S'}} & (s \in S) \end{array}$$

with the following properties.

(10.47.1) π_X, π_S are localizations of smooth morphisms,

(10.47.2) $\pi_X^{-1}(x) = x'$ and $\pi_S^{-1}(s) = s'$ (as sets), and

(10.47.3) $k(x')/k(s') \cong k(x')/K$.

Many local properties of schemes are preserved by smooth morphisms, so the properties of (s, S) are inherited by (s', S') and once we prove a result about (x', X') it descends to (x, X) . In particular, we get the following using (4.27).

Claim 10.47.5. f is flat at x iff f' is flat at x' . □

Claim 10.47.6. π_X^* gives an injection $\text{Pic}^{\text{loc}}(x, X) \hookrightarrow \text{Pic}^{\text{loc}}(x', X')$. □

I do not know a similar method that works for non-finitely generated residue field extensions. A different idea, discussed in (10.67), applies whenever $k(x)/k(s)$ is separable, but non-finitely generated purely inseparable extensions cause numerous problems.

Flatness is usually easy to check if we know all the fibers of a morphism. For projective morphisms there are criteria using the Hilbert function; see [Har77, III.9.9] or (3.20). In the local case we have the following.

LEMMA 10.48. *Let S be a reduced scheme and $f : X \rightarrow S$ a morphism that is essentially of finite type, pure dimensional and its fibers are geometrically reduced. Then f is flat.*

Proof. By (3.22) it is enough to show this when (s, S) is the spectrum of a DVR. In this case f is flat iff none of the associated points of X is contained in X_s . By assumption X_s is reduced, so only generic points of X_s could occur. Then the corresponding irreducible component of X_s is also an irreducible component of X , but we also assumed that f has pure relative dimension. □

In many cases we have some information about the fibers, but we do not fully understand them. Thus we are looking for flatness criteria that do not require complete knowledge of the fibers.

10.49 (Format of flatness criteria). Let $\pi : X \rightarrow S$ be a morphism of Noetherian schemes, (s, S) local. Let F be a coherent sheaf on X and $Z \subset \text{Supp } F_s$ a nowhere dense closed subset such that $F|_{X \setminus Z}$ is flat over S . We aim to prove various flatness theorems with assumptions

(10.49.1) on $F_s/\text{tors}_Z(F_s)$,

(10.49.2) on $\text{depth}_Z F$,

(10.49.3) on $F|_{X \setminus Z}$ and

(10.49.4) on S .

Our main focus is on $F_s/\text{tors}_Z(F_s)$; the assumptions (2–3) are then chosen as needed.

Let G be another coherent sheaf on X such that $G|_{X \setminus Z}$ is flat over S and $X_s \cap \text{Supp } G \subset Z$. Then $F_s/\text{tors}_Z F_s = (F_s + G_s)/\text{tors}_Z(F_s + G_s)$, so assumptions of type (1–2) do not give control over G . Thus we have to make sure that the assumptions of type (2) exclude G .

There are two ways of achieving this. Let $x_G \in \text{Supp } G$ be a generic point and set $s_G := \pi(x_G)$. Then x_G is an associated point of the fiber $(F + G)_{s_G}$ and $X_s \cap \bar{x}_G \subset Z$. Thus the presence of G is excluded by the assumption: $\text{depth}_x(F_{\pi(x)}) \geq 1$ whenever $X_s \cap \bar{x} \subset Z$. A similar argument with extensions suggests that we frequently need the stronger variant

(3') $\text{depth}_x(F_{\pi(x)}) \geq 2$ whenever $X_s \cap \bar{x} \subset Z$.

This is a quite mild restriction and probably the best one can do for Noetherian schemes. It has a geometrically transparent reformulation if there is a ‘good’ dimension function for $\pi : X \rightarrow S$. A precise definition is not important, we mean by this a function $x \mapsto \dim_S x$ that is upper semicontinuous on X , strictly decreasing under specialization and if a coherent sheaf G is flat over $S^\circ \subset S$ then $s \mapsto \text{Supp}\{\dim_S x : x \in \text{Supp}(G_s)\}$ is locally constant on S° .

The prime example for this is the usual dimension function if $X \rightarrow S$ is of finite type. If we have a ‘good’ dimension function and $\text{codim}(Z, X_s) \geq 2$ then we can replace (3') by the more convenient assumption

(3'') the fibers of $F|_{X \setminus Z}$ are pure and S_2 .

In order to simplify notation in (1) we use the following

Notation 10.49.5. Let X be a scheme, $Z \subset X$ a closed subset and F a coherent sheaf on X . We set

$$\text{pure}_Z(F) := F / \text{tors}_Z(F) \quad \text{and} \quad \text{pure}_Z(X) := \text{Spec}_X \text{pure}_Z(\mathcal{O}_X).$$

The key step in the following proofs is to show that certain sheaves associated to F do not have any associated points contained in Z . We start with the case when $\dim X_s = 0$ and $F = \mathcal{O}_X$. With each increase of $\dim X_s$ the results become more general.

Flatness in relative codimension 0.

The basic result is the following, proved in [Gro71, II.2.3].

PROPOSITION 10.50. *Let $f : (x, X) \rightarrow (s, S)$ be a local morphism of local, Noetherian schemes of the same dimension such that $f^{-1}(s) = x$ holds scheme theoretically, that is, $m_{x,X} = m_{s,S}\mathcal{O}_X$. Assume that*

(10.50.1) $k(x) \supset k(s)$ is separable and

(10.50.2) \hat{S} , the completion of S , is normal.

Then f is flat at x .

Note that if S is normal and excellent then \hat{S} is normal.

Proof. We may replace S and X by their completions. As we discuss in (10.67), we can factor f as

$$f : (x, X) \xrightarrow{p} (y, Y) \xrightarrow{q} (x, S)$$

where (y, Y) is also complete, local, Noetherian, $k(x) = k(y)$, $m_{x,X} = m_{s,S}\mathcal{O}_X$ and q is flat.

Thus $f^* : m_{y,Y}/m_{y,Y}^2 \rightarrow m_{x,X}/m_{x,X}^2$ is surjective, hence $p^* : \mathcal{O}_Y \rightarrow \mathcal{O}_X$ is surjective by the Nakayama lemma. Equivalently, $p : X \rightarrow Y$ is a closed embedding. It is thus an isomorphism, provided $\dim X = \dim Y$ and Y is integral.

In order to ensure these properties of Y we need to know more about q . If $k(x)/k(s)$ is finitely generated then q is the localization of a smooth morphism (10.67.3). Thus Y is normal and $\dim Y = \dim S$, as required. The general case is technically harder. We use that q is formally smooth and geometrically regular (10.67.4) to reach the same conclusions as before.

Thus p is an isomorphism, so $f = q$ and f is flat. \square

The next examples show that the assumptions in (10.50), and later in (10.53), are necessary.

EXAMPLE 10.51. 10.51.1. Assume that $\text{char } k \neq 2$ and set $C := (y^2 = ax^2 + x^3)$ where $a \in k$ is not a square. Let $f : \bar{C} \rightarrow C$ denote the normalization. Then the fiber over the origin is the spectrum of $k(\sqrt{a})$, which is a separable extension of k . Here C is not normal and f is not flat.

10.51.2. The extension $\mathbb{C}[x, y] \subset \mathbb{C}[\frac{x}{y}, y]_{(y)}$ is not flat yet $(x, y) \cdot \mathbb{C}[\frac{x}{y}, y]_{(y)}$ is the maximal ideal and the residue field extension is purely transcendental. However, the dimension of the larger ring is 1.

A similar thing happens with the injection $\mathbb{C}[x, y] \hookrightarrow \mathbb{C}[[t]]$ given by $(x, y) \mapsto (t, \sin t)$. The fiber over the origin is the origin with reduced scheme structure.

10.51.3. On $\mathbb{C}[x, y]$ consider the involution $\tau(x) = -x, \tau(y) = -y$. The invariant ring is $\mathbb{C}[x^2, xy, y^2] \subset \mathbb{C}[x, y]$. The fiber over the origin is $\mathbb{C}[x, y]/(x^2, xy, y^2)$; it has length 3 and embedding dimension 2. The fiber over any other point has length 2. Thus the extension is not flat.

10.51.4. As in [Kol95a, 15.2], on $S := k[x_1, x_2, y_1, y_2]$ consider the involution $\tau(x_1, x_2, y_1, y_2) = (x_2, x_1, y_2, y_1)$. The ring of invariants is

$$R := k[x_1 + x_2, x_1x_2, y_1 + y_2, y_1y_2, x_1y_1 + x_2y_2].$$

The resulting extension is not flat along the invariant locus $(x_1 - x_2 = y_1 - y_2 = 0)$.

If $\text{char } k = 2$ then $x_1 - x_2, y_1 - y_2$ are invariants. Set $P := (x_1 - x_2, y_1 - y_2)R$. Then $S/PS = S/(x_1 - x_2, y_1 - y_2)S \cong k[x_1, y_1]$ and $R/P \cong k[x_1^2, y_1^2]$.

Thus $S_P \supset R_P$ is a finite extension whose fiber over P is $k(x_1, y_1) \supset k(x_1^2, y_1^2)$. This is an inseparable field extension, generated by 2 elements.

These examples leave open only one question: what happens with curvilinear fibers.

DEFINITION 10.52 (Curvilinear schemes). Let k be a field and (A, m) a local, artinian k -algebra. We say that $\text{Spec}_k A$ is *curvilinear* if A is cyclic as a $k[t]$ -module for some t . That is, if A can be written as a quotient of $k[t]$. It is easy to see that this holds if

- (10.52.1) either A/m is a finite, separable extension of k and m is a principal ideal,
 (10.52.2) or A is a field extension of k of degree $= \text{char } k$.

Let B be a semilocal artinian k -algebra. Then $\text{Spec}_k B$ is called *curvilinear* if all of its irreducible components are curvilinear. If k is an infinite field, this holds iff B can be written as a quotient of $k[t]$. If K/k is a field extension and $\text{Spec}_k B$ is curvilinear then so is $\text{Spec}_K(B \otimes_k K)$.

Let $\pi : X \rightarrow S$ be a finite type morphism. The embedding dimension of fibers is upper semicontinuous, thus the set $\{x \in X : X_{\pi(x)} \text{ is curvilinear at } x\}$ is open.

THEOREM 10.53. Let $f : X \rightarrow S$ be a finite type morphism with curvilinear fibers such that every associated point of X dominates S . Assume that

- (10.53.1) either S is normal,
 (10.53.2) or there is a closed $W \subset S$ such that $\text{depth}_W S \geq 2$ and f is flat over $S \setminus W$.

Then f is flat.

Proof. We start with the classical case when X, S are complex analytic, S is normal, f is finite and $X \subset S \times \mathbb{C}$. Let $s \in S^{\text{ns}}$ be a smooth point. Then $S \times \mathbb{C}$ is smooth along $\{s\} \times \mathbb{C}$ thus X is a Cartier divisor near X_s . In particular, f is flat over S^{ns} . Set $d := \deg f$. For each $s \in S^{\text{ns}}$ there is a unique monic polynomial $t^d + a_{d-1}(s)t^{d-1} + \cdots + a_0(s)$ of degree d whose zero set is precisely $X_s \subset \mathbb{C}$.

As in the proof of the analytic form of the Weierstrass preparation theorem (see, for instance, [GH94, p.8] or [GR65, Sec.II.B]) we see that the $a_i(s)$ are analytic functions on S^{ns} . By Hartogs's theorem they extend to analytic functions on the whole of S ; we denote these still by $a_i(s)$. Thus

$$X = (t^d + a_{d-1}(s)t^{d-1} + \cdots + a_0(s) = 0) \subset S \times \mathbb{C}$$

is a Cartier divisor and f is flat. This completes the complex analytic case.

In general we argue similarly but replace the polynomial $t^d + a_{d-1}(s)t^{d-1} + \cdots + a_0(s)$ by the point in the Hilbert scheme corresponding to X_s .

Assume first that f is finite. Again set $d := \deg f$ and let $S^\circ \subset S$ denote a dense open subset over which f is flat. Since f is finite, it is (locally) projective, thus we have

$$\begin{array}{ccc} \text{Univ}_d(X/S) & \xrightarrow{p} & X \\ \downarrow & & \downarrow f \\ \text{Hilb}_d(X/S) & \xrightarrow{\pi} & S \end{array}$$

parametrizing length d quotients of the fibers of f . If $s \in S^\circ$ then \mathcal{O}_{X_s} has length d , hence its sole length d quotient is itself. Thus π is an isomorphism over S° .

Let now $s \rightarrow S$ be an arbitrary geometric point. Then

$$X_s \cong \text{Spec } k(s)[t]/(\prod_i (t - a_i)^{m_i})$$

for some $a_i \in k(s)$ and $m_i \in \mathbb{N}$. Thus the fiber of p over s is a finite set corresponding to length d quotients of $k(s)[t]/(\prod_i (t - a_i)^{m_i})$, equivalently, to solutions of the equation $\sum_i m'_i = d$ where $0 \leq m'_i \leq m_i$. We have not yet proved that $\text{Hilb}_d(X/S)$ has no embedded points over $\text{Sing } S$, but we obtain that $\text{pure}(\text{Hilb}_d(X/S)) \rightarrow S$ is finite and birational, hence an isomorphism if S is normal or if $S^\circ \supset S \setminus W$ in case (2) by (9.8.4). Furthermore, the natural map

$$\text{pure}(p) : \text{Univ}_d(X/S) \times_{\text{Hilb}_d(X/S)} \text{pure}(\text{Hilb}_d(X/S)) \rightarrow X$$

is a closed immersion whose image is isomorphic to X over S° . Thus $\text{pure}(p)$ is an isomorphism and so f is flat and $\text{Hilb}_d(X/S) \cong S$.

Finally we reduce the general case to the finite one. (Note that any finite type, quasi-finite morphism can be extended to a finite morphism, but there is no reason to believe that the extension still has curvilinear fibers.)

Flatness is a local question on X , thus pick $x \in X$. Set $s := f(x)$ and use (10.67.6) to get a diagram

$$\begin{array}{ccc} (x', X') & \xrightarrow{\pi} & (x, X) \\ g \downarrow & & \downarrow f \\ (y, Y) & \xrightarrow{h} & (s, S), \end{array}$$

where π is étale, g is finite, $g^{-1}(y) = \{x'\}$ and h is also étale. The latter implies that (y, Y) is also normal. As we noted in (10.52), $h \circ g = f \circ \pi$ has curvilinear fibers since π is étale. Thus g has curvilinear fibers, hence g is flat by the already established finite morphism case. Therefore f is also flat at x . \square

Over a non-normal base there does not seem to be any simple analog of (10.53), but the following is sometimes useful.

PROPOSITION 10.54. *Let $f : (x, X) \rightarrow (s, S)$ be a finite, local morphism with curvilinear fibers. Assume that*

(10.54.1) *the pair $(s \in S)$ is weakly normal (10.76),*

(10.54.2) f is flat of constant degree d over $S \setminus \{s\}$,

(10.54.3) $k(x)/k(s)$ is purely inseparable and

(10.54.4) x is not an associated point of X .

Then f is flat.

Proof. As in the proof of (10.53) we consider

$$\begin{array}{ccc} \mathrm{Univ}_d(X/S) & \xrightarrow{p} & X \\ u \downarrow & & \downarrow f \\ \mathrm{Hilb}_d(X/S) & \xrightarrow{\pi} & S. \end{array}$$

By (2) p and π are isomorphisms over $S \setminus \{s\}$. Let $s' \rightarrow s$ be a geometric point. By (3) $X_{s'} \cong \mathrm{Spec} k(s')[t]/(t^r)$ for some r , which has a unique subscheme of length d . Thus π is an injection on geometric points, hence an isomorphism since S is weakly normal. As in the proof of (10.53) we conclude that p is an isomorphism. \square

Example 10.54.5. The normalization of $S = \mathrm{Spec} \mathbb{Q}[x, y]/(x^2 + y^2)$ shows that assumption (3) is necessary.

This result has an interesting consequence for unique factorization in power series rings. The first example of a unique factorization domain A such that $A[[t]]$ is not a UFD was constructed in [Sam61] and the situation was further clarified in [Sto69, Dan70]. An example of a complete, local UFD A such that $A[[t]]$ is not a UFD is given in (10.57). By contrast, the following result of [Ram63, Sam62] shows that many height 1 ideals in $A[[t]]$ are principal; see also [Gro60, IV.21.14.1]. The proof given in (4.25) applies with minor modifications.

THEOREM 10.55 (Principal ideals in power series rings). *Let (R, m) be a normal, complete, local ring and $P \subset R[[x_1, \dots, x_n]]$ a height 1 prime ideal that is not contained in $mR[[x_1, \dots, x_n]]$. Then P is principal.* \square

COROLLARY 10.56 (Unique factorization in power series rings). *Let (R, m) be a normal, complete, local ring and $g \in R[[x_1, \dots, x_n]]$ a power series not contained in $mR[[x_1, \dots, x_n]]$. Then g has a unique factorization as $g = \prod_i p_i$ where each (p_i) is a prime ideal.*

Proof. Let P_i be a height 1 associated prime ideal of (g) . Then (P_i) is not contained in $mR[[x_1, \dots, x_n]]$ thus it is principal by (10.55). \square

REMARK 10.57. A lemma of Gauss says that if R is a UFD then $R[t]$ is also a UFD. More generally, if Y is a normal scheme then $\mathrm{Cl}(Y \times \mathbb{A}^n) \cong \mathrm{Cl}(Y)$. If \mathbb{A}^n is replaced by a smooth variety X then there is an obvious inclusion

$$\mathrm{Cl}(Y) \times \mathrm{Cl}(X) \hookrightarrow \mathrm{Cl}(Y \times X),$$

but, as the example below shows, this map is not surjective, not even if $\mathrm{Cl}(Y) = \mathrm{Cl}(X) = 0$.

Let $E \subset \mathbb{P}^2$ be a cubic defined over \mathbb{Q} such that $\mathrm{Pic}(E)$ is generated by a degree 3 point $P := E \cap L$ for some line $L \subset \mathbb{P}^2$. Let $S \subset \mathbb{A}^3$ be the affine cone over E and $E^\circ := E \setminus P$. Then $\mathrm{Cl}(S) = 0$ and $\mathrm{Cl}(E^\circ) = 0$. However, we claim that $\mathrm{Cl}(S \times E^\circ)$ is infinite.

To see this pick any $\phi \in \mathrm{End}(E)$. (For any m we have multiplication by $3m + 1$ which sends $p \in E(\mathbb{Q})$ to the unique point $\phi(p) \sim (3m + 1)p - mP$.) The lines

$\{\ell_p \times \{\phi(p)\} : p \in E\}$ sweep out a divisor in $S \times E$, where $\ell_p \subset S$ denotes the line over $p \in E$. It is not hard to see that this gives an isomorphism $\text{End}(E) \cong \text{Cl}(S \times E^\circ)$.

As another application, let R denote the complete local ring of S at its vertex. The above considerations also show that R is a UFD but $R[[t]]$ is not.

Flatness in relative codimension 1.

The following result is stated in all dimensions, but we will have even stronger theorems when the codimension is ≥ 2 .

THEOREM 10.58. *Let $f : X \rightarrow S$ be a finite type morphism of Noetherian schemes, $s \in S$ a closed point and $Z \subset X_s$ a nowhere dense closed subset such that f is flat on $X \setminus Z$. Assume that*

- (10.58.1) $\text{pure}_Z(X_s)$ is geometrically regular,
- (10.58.2) the fibers of f over $S \setminus \{s\}$ are pure and S_1 ,
- (10.58.3) $\text{depth}_Z X \geq 1$ and
- (10.58.4) $\{s\}$ is not an associated point of S .

Then f is flat and X_s is geometrically regular.

Proof. By (10.47) we may assume that $Z = \{x\}$ and $k(x) = k(s)$. Now we replace S, X by their completions. Then $\text{pure}_x(\hat{X}_s) \cong \text{Spec}_{k(s)} k(s)[[t_1, \dots, t_n]]$ for some $n \geq 1$.

Lifting the t_i back to sections of $\mathcal{O}_{\hat{X}}$ gives a finite morphism

$$\pi : \hat{X} \rightarrow \hat{\mathbb{A}}_S^n;$$

see (10.58.6) for the notation. We aim to prove that π is an isomorphism. For now we know that it is a local isomorphism along $\hat{X}_s \setminus \{x\}$. Thus there is a closed subscheme $W \subset \hat{\mathbb{A}}_S^n$ such that π is a local isomorphism outside W and $W \cap \hat{\mathbb{A}}_S^n = (0, s)$. In particular, $\pi|_W : W \rightarrow \hat{S}$ is finite.

We show next that π is a local isomorphism outside $(0, s) \in \hat{\mathbb{A}}_S^n$. To see this pick any point $p \in \hat{S} \setminus \{s\}$. Then π restricts to a finite morphism $\pi_p : \hat{X}_p \rightarrow (\hat{\mathbb{A}}_S^n)_p$ whose target is regular by (10.58.6), and which is a local isomorphism outside the nowhere dense set $W \cap (\hat{\mathbb{A}}_S^n)_p$. Thus π_p has a unique irreducible component that maps isomorphically onto $(\hat{\mathbb{A}}_S^n)_p$ and all other associated points of \hat{X}_p are contained in $\pi_p^{-1}(W)$. Such associated points are excluded by (2). Thus π_p is an isomorphism. Since $\hat{X} \rightarrow \hat{S}$ is flat over p , we see that π is a local isomorphism along \hat{X}_p by (10.58.5). Finally,

$$\text{depth}_{(0,s)} \hat{\mathbb{A}}_S^n = n + \text{depth}_y \hat{S} = n + \text{depth}_s \hat{S} \geq 1 + 1 = 2,$$

and \hat{X} has no associated points supported on $\{x\}$ by (3). Hence π is an isomorphism by (9.8). This proves (10.58).

We have used the following easy lemma, cf. [Mat86, 22.5].

Lemma 10.58.5. *Let (s, S) be a local Noetherian scheme and $\pi : X \rightarrow Y$ a finite morphism of Noetherian S -schemes. Assume that X is flat over S . Then π is an isomorphism iff $\pi_s : X_s \rightarrow Y_s$ is an isomorphism. \square*

Notation 10.58.6. Let R be a ring and $Y = \text{Spec } R$. Set

$$\hat{\mathbb{A}}_Y^n := \text{Spec } R[[x_1, \dots, x_n]].$$

If $X \rightarrow Y$ is a finite morphism then $\hat{\mathbb{A}}_X^n \cong X \times_Y \hat{\mathbb{A}}_Y^n$. However, $\hat{\mathbb{A}}_Y^n$ is *not* the product of $\hat{\mathbb{A}}^n$ with Y in any sense.

For example, if R is an integral domain with quotient field K then the generic fiber of $\hat{\mathbb{A}}_Y^n \rightarrow Y$ is the spectrum of the ring of power series over K that have bounded denominators. That is, power series of the form

$$\left\{ \sum_I a_I x^I : a_I \in K \text{ and } \exists r \in R \text{ such that } ra_I \in R \forall I. \right\}$$

This ring is regular (see [Gro60, 0.19.3.5, 0.19.7.1, 0.22.5.8] or [Sta15, Tag 07PM]) but in general more complicated than $K[[x_1, \dots, x_n]]$. For example, it can happen that R is a UFD but $R[[t]]$ is not; see (10.57).

Example 10.58.7. We construct an example $f : X \rightarrow S$ to show that geometric regularity is needed in (10.58), even if X and S are affine varieties.

Let (s, S) be a 1-dimensional reduced scheme and $g : Y \rightarrow (s, S)$ a flat morphism such that Y_s is regular. Let $W \subset Y$ be a closed subset such that $Z := Y_s \cap W$ is not empty and nowhere dense in Y_s . Let $\pi : X \rightarrow Y$ be a finite, birational morphism such that $\text{Supp}(\pi_* \mathcal{O}_X / \mathcal{O}_Y) = W$.

Then $\pi_s : X_s \rightarrow Y_s$ is a finite morphism that is an isomorphism over $Y_s \setminus W$. Since Y_s is regular, we get an isomorphism $\text{pure}_Z X_s \cong Y_s$. Thus the composite $X \rightarrow S$ satisfies (10.58.1–4) but X_s is not regular.

In order to get such an $X \rightarrow S$, let k be a field of characteristic $p > 0$, $c \in k \setminus k^p$ and set $\gamma = c^{1/p}$. Consider the ring extensions

$$k[u, v] / (u^p - cv^p) \subset k[u, v, z, t] / (u^p - cv^p, z^p - c - vt^p).$$

Taking their spectra will give $Y \rightarrow S$.

Over the origin (u, v) the fiber is $k[z, t] / (z^p - c) \cong k(\gamma)[t]$ hence regular. If $v \neq 0$ then we can invert v and the rings become

$$\begin{aligned} k[u, v, v^{-1}] / (u^p - cv^p) &\cong k[u, v, v^{-1}] / ((u/v)^p - c) \cong k(\gamma)[v, v^{-1}] \text{ and} \\ k[u, v, v^{-1}, z, t] / (u^p - cv^p, z^p - c - vt^p) &\cong k(\gamma)[v, v^{-1}, z, t] / ((z - \gamma)^p - vt^p). \end{aligned}$$

The former is regular while the latter is not normal along $(z - \gamma, t)$. Its normalization is obtained by setting $w := (z - \gamma)/t$; it is the ring

$$k(\gamma)[v, v^{-1}, z, t, w] / (w^p - v, z - \gamma - wt) \cong k(\gamma)[w, w^{-1}, t].$$

We obtain X by first gluing $Y \setminus (z - \gamma = t = 0)$ to the normalization of $Y \setminus (u = v = 0)$ and then extending it to a finite morphism $X \rightarrow S$ which has the required properties.

In codimension 1, an slc pair is either smooth or has nodes. Next we show that a close analog of (10.58) holds for nodal fibers if the base scheme is normal; the latter assumption is necessary by (10.61.1).

THEOREM 10.59. *Let (s, S) be a normal, local, excellent scheme, (x, X) a local, S_2 scheme and $f : X \rightarrow S$ a finite type morphism of pure relative dimension 1. Assume that $\text{pure}(X_s)$ has only nodes.*

Then f is flat with reduced fibers that have only nodes.

Proof. By assumption $k(x)/k(s)$ is a finite extension, generated by n elements. Choosing generators gives $s' \in \mathbb{A}_{k(s)}^n$ such that $k(x) \cong k(s')$.

Consider next the trivial lifting $f^{(n)} : \mathbb{A}_X^n \rightarrow \mathbb{A}_S^n$ and the points $s' \in \mathbb{A}_S^n$ projecting to s and $x' = (s', x) \in \mathbb{A}_X^n$ projecting to x . After localization and

completion we get $f' : (x', X') \rightarrow (s', S')$, where, in addition. $k(x') = k(s')$ and the schemes are complete. We change notation and assume these from now on.

By (10.69.4) and (10.43) there is a finite, birational morphism onto a hyper-surface

$$\pi : X \rightarrow H := (q(x_1, x_2) + c = 0) \subset \hat{\mathbb{A}}_S^2$$

where $c \in m_S$. If $c \neq 0$ then H is normal by (10.70), hence π is an isomorphism and we are done. If $c = 0$ then the normalization of H is

$$H^n := \hat{\mathbb{A}}_{S'}^1 \quad \text{where} \quad S' := (q(t, 1) = 0) \subset \hat{\mathbb{A}}_S^1.$$

Note that $\tau : S' \rightarrow S$ is an étale double cover, and we have a natural exact sequence

$$0 \rightarrow \mathcal{O}_S \rightarrow \tau_* \mathcal{O}_{S'} \rightarrow L \rightarrow 0,$$

where L is a line bundle on S . Thus we have an exact sequence

$$0 \rightarrow \mathcal{O}_H \rightarrow \tau_* \mathcal{O}_{H^n} \rightarrow L \rightarrow 0,$$

so there is a coherent subsheaf $L' \subset L$ such that $\tau_* \mathcal{O}_{H^n} / \pi_* \mathcal{O}_X \cong L/L'$.

If $L' = 0$ then $X \cong H^n$ and if $L' = L$ then $X \cong H$; the projection to S is flat in both cases. Otherwise $\text{Supp}(\tau_* \mathcal{O}_{H^n} / \pi_* \mathcal{O}_X) = \text{Supp}(L/L')$ has codimension ≥ 2 in H which is impossible since X is assumed to be S_2 . \square

With different methods, the following generalization of (10.59) is proved in [Kol11b]. The projectivity assumption should not be necessary.

THEOREM 10.60. *Let (s, S) be a normal, local scheme and $f : X \rightarrow S$ a projective morphism of pure relative dimension 1. Assume that X is S_2 and $\text{pure}(X_s)$ is*

- (10.60.1) *either geometrically seminormal*
 (10.60.2) *or has only simple, planar singularities.*

Then f is flat with reduced fibers that are seminormal in case (1) and have only simple, planar singularities in case (2). \square

See [AGZV85, I.p.245] for the conceptual definition of simple, planar singularities. For us it is quickest to note that a plane curve singularity $(f(x, y) = 0)$ is simple iff $(z^2 + f(x, y) = 0)$ is a Du Val surface singularity.

EXAMPLE 10.61. The next examples show that (10.60) does not generalize to non-normal bases or to other curve singularities.

10.61.1 (Deformations of ordinary double points) Let $C \subset \mathbb{P}^2$ be a nodal cubic with normalization $p : \mathbb{P}^1 \rightarrow C$. Over the coordinate axes $S := (xy = 0) \subset \mathbb{A}^2$ consider the family X that is obtained as follows.

Over the x -axis take a smoothing of C , over the y -axis take $\mathbb{P}^1 \times \mathbb{A}_y^1$ and glue them over the origin using $p : \mathbb{P}^1 \rightarrow C$ to get $f : X \rightarrow S$.

Then X is seminormal and S_2 , the central fiber is C with an embedded point, yet f is not flat.

10.61.2 (Deformations of ordinary triple points) Consider the family of plane cubic curves

$$\mathbf{C} := ((x^2 - y^2)(x + t) + t(x^3 + y^3) = 0) \subset \mathbb{A}_{xy}^2 \times \mathbb{A}_t^1.$$

For every t the origin is a singular point, but it has multiplicity 3 for $t = 0$ and multiplicity 2 for $t \neq 0$. Thus blowing up the line $(x = y = 0)$ gives the normalization for $t \neq 0$ but it introduces an extra exceptional curve over $t = 0$. The

normalization of \mathbf{C} is obtained by contracting this extra curve. The fiber over $t = 0$ is then isomorphic to 3 lines through the origin in \mathbb{A}^3 .

10.61.3 (Deformations of ordinary quadruple points) Let $\mathbf{C}_4 \rightarrow \mathbb{P}^{14}$ be the universal family of degree 4 plane curves and $\mathbf{C}_{4,1} \rightarrow S^{12}$ the 12-dimensional subfamily whose general members are elliptic curves with 2 nodes. We normalize both the base and the total space to get

$$\bar{\pi} : \bar{\mathbf{C}}_{4,1} \rightarrow \bar{S}^{12}.$$

We claim that the fiber of $\bar{\pi}$ over the plane quartic with an ordinary quadruple point $C_0 := (x^3y - xy^3 = 0)$ is C_0 with at least 2 embedded points. Most likely, the family is not even flat, but I have not checked this.

We prove this by showing that in different families of curves through $[C_0] \in S^{12}$ we get different flat limits.

To see this, note that the seminormalization C_0^{sn} of C_0 can be thought of as 4 general lines through a point in \mathbb{P}^4 . In suitable affine coordinates we can write it as

$$k[x, y]/(x^3y - xy^3) \hookrightarrow k[u_1, \dots, u_4]/(u_i u_j : i \neq j)$$

using the map $(x, y) \mapsto (u_1 + u_3 + u_4, u_2 + u_3 - u_4)$. Any 3-dimensional linear subspace

$$\langle u_1, \dots, u_4 \rangle \supset W_\lambda \supset \langle u_1 + u_3 + u_4, u_2 + u_3 - u_4 \rangle.$$

corresponds to a projection of C_0^{sn} to \mathbb{P}^3 ; call the image $C_\lambda \subset \mathbb{P}^3$. Then C_λ is 4 general lines through a point in \mathbb{P}^3 ; thus it is a $(2, 2)$ -complete intersection curve of arithmetic genus 1. (Note that the C_λ are isomorphic to each other, but the isomorphism will not commute with the map to C_0 in general.) Every C_λ can be realized as the special fiber in a family $S_\lambda \rightarrow B_\lambda$ of $(2, 2)$ -complete intersection curves in \mathbb{P}^3 whose general fiber is a smooth elliptic curve.

By projecting these families to \mathbb{P}^2 , we get a 1-parameter family $S'_\lambda \rightarrow B_\lambda$ of curves in S^{12} whose special fiber is C_0 .

Let now $\bar{S}'_\lambda \subset \bar{\mathbf{C}}_{4,1}$ be the preimage of this family in the normalization. Then \bar{S}'_λ is dominated by the surface S_λ . There are two possibilities. First, if \bar{S}'_λ is isomorphic to S_λ , then the fiber of $\bar{\mathbf{C}}_{4,1} \rightarrow \bar{S}^{12}$ over $[C_0]$ is C_λ . This, however, depends on λ , a contradiction. Second, if \bar{S}'_λ is not isomorphic to S_λ , then the fiber of $\bar{S}'_\lambda \rightarrow B_\lambda$ over the origin is C_0 with some embedded points. Since C_0 has arithmetic genus 3, we must have at least 2 embedded points.

Flatness in relative codimension ≥ 2 .

Once we know flatness at codimension 1 points of the fibers, the following general result, valid for coherent sheaves, can be used to prove flatness everywhere. We no longer need any restrictions on the base scheme S .

THEOREM 10.62. *Let $f : X \rightarrow S$ be a finite type morphism of Noetherian schemes, (s, S) local, $Z \subset \text{Supp } F_s$ a nowhere dense closed subset and F a coherent sheaf on X . Assume that*

- (10.62.1) $\text{depth}_Z \text{pure}_Z(F_s) \geq 2$,
- (10.62.2) F is flat over S along $X \setminus Z$,
- (10.62.3) the fibers of F over $S \setminus \{s\}$ are pure, and
- (10.62.4) $\text{depth}_Z F \geq 1$.

Then F is flat over S and $\text{tors}_Z(F_s) = 0$.

Proof. Set $X_n := \text{Spec}_X(\mathcal{O}_X/f^*m_{s,S}^n\mathcal{O}_X)$, $F_n := F|_{X_n}$ and $m := m_{s,S}$. We may assume that S is m -adically complete. There are natural complexes

$$0 \rightarrow (m^n/m^{n+1}) \cdot F_0 \rightarrow F_{n+1} \xrightarrow{r_n} F_n \rightarrow 0, \quad (10.62.4)$$

which are exact on $X \setminus Z$ but not (yet) known to be exact along Z , except that r_n is surjective. We also know that

$$(m^n/m^{n+1}) \cdot \text{pure}_Z(F_0) \rightarrow \text{pure}_Z(\ker r_n) \quad (10.62.5)$$

is an isomorphism on $X \setminus Z$. Since $\text{depth}_Z \text{pure}_Z(F_0) \geq 2$, this implies that (10.62.5) is an isomorphism on X by (9.8). Next we show that the induced map

$$r_n : \text{tors}_Z F_{n+1} \rightarrow \text{tors}_Z F_n \quad \text{is surjective.} \quad (10.62.6)$$

Set $K_{n+1} := r_n^{-1}(\text{tors}_Z F_n)$. We have an exact sequence

$$0 \rightarrow \text{pure}_Z(\ker r_n) \rightarrow K_{n+1}/\text{tors}_Z(\ker r_n) \rightarrow \text{tors}_Z F_n \rightarrow 0. \quad (10.62.7)$$

Using that (10.62.5) is an isomorphism, we have $\text{depth}_Z \text{pure}_Z(\ker r_n) \geq 2$, hence the sequence (10.62.7) splits by (9.8). We know that $\varprojlim(\text{tors}_Z F_n)$ is a subsheaf of F , let w be a generic point of its support. Then $w \in \text{Ass}(F_{f(w)})$ by (10.66.3) and $\text{Supp}(\bar{w} \cap X_s) \subset Z$ by construction. Thus $f(w) \neq s$ by (4) and so $F_{f(w)}$ is pure by (3). So $\varprojlim(\text{tors}_Z F_n) = 0$ and $\text{tors}_Z F_n = 0$ for every n by (10.62.6).

Thus (10.62.5) now says that $(m^n/m^{n+1}) \cdot F_0 \cong \ker r_n$. Therefore the sequences (10.62.4) are exact, F is flat and $\text{tors}_Z F_0 = 0$. \square

Putting together the above flatness criteria (10.50), (10.59), (10.60.1) and (10.62) gives the following strengthening of [Hir58].

THEOREM 10.63. *Let (s, S) be a normal, local, excellent scheme, X an S_2 scheme and $f : X \rightarrow S$ a finite type morphism of pure relative dimension n . Assume that $\text{pure}(X_s)$ is*

(10.63.1) *either geometrically normal*

(10.63.2) *or geometrically seminormal and S_2 .*

Then f is flat with reduced fibers that are normal in case (1) and seminormal and S_2 in case (2). \square

Flatness in relative codimension ≥ 3 .

We get an even stronger result in codimension ≥ 3 ; see [Kol95a, Thm.12]. [LN18] pointed out that the purity assumption in (3) is also necessary.

THEOREM 10.64. *Let $f : X \rightarrow S$ be a finite type morphism of Noetherian schemes, (s, S) local, $Z \subset \text{Supp } F_s$ a nowhere dense closed subset and $j : X_s \setminus Z \hookrightarrow X_s$ the natural injection. Assume that*

(10.64.1) *$j_*(F_s|_{X_s \setminus Z})$ is coherent and $\text{depth}_Z(j_*(F_s|_{X_s \setminus Z})) \geq 3$,*

(10.64.2) *$F|_{X \setminus Z}$ is flat over S and $\text{depth}_Z F \geq 2$,*

(10.64.3) *the fibers of F over $S \setminus \{s\}$ are pure and S_2 .*

Then F is flat over S and $F_s = j_(F_s|_{X_s \setminus Z})$.*

Proof. Set $m := m_{s,S}$, $X_n := \text{Spec}_X(\mathcal{O}_X/f^*m^n\mathcal{O}_X)$ and $F_n := F|_{X_n}$. We may assume that \mathcal{O}_S and \mathcal{O}_X are m -adically complete. Set $G_n := F_n|_{X_n \setminus Z}$ and let j denote any of the injections $X_n \setminus Z \hookrightarrow X_n$. By assumption (2) we have exact sequences

$$0 \rightarrow (m^n/m^{n+1}) \cdot G_0 \rightarrow G_{n+1} \rightarrow G_n \rightarrow 0. \quad (10.64.4)$$

Pushing it forward we get the exact sequences

$$\begin{aligned} 0 \rightarrow (m^n/m^{n+1}) \otimes j_*G_0 &\rightarrow j_*G_{n+1} \xrightarrow{r_n} j_*G_n \rightarrow \\ &\rightarrow (m^n/m^{n+1}) \otimes R^1j_*G_0. \end{aligned} \quad (10.64.5)$$

The first part of assumption (1) says that j_*G_0 is coherent and the second part implies (in fact is equivalent to) $R^1j_*G_0 = 0$ by [Gro68, III.3.3, II.6 and I.2.9] or (10.28).

Thus the r_n are surjective. This shows that $G := \varprojlim j_*G_n$ is a coherent sheaf on X that is flat over S and $\text{depth}_x(G_{\pi(x)}) \geq 1$ whenever $X_s \cap \bar{x} \subset Z$ and $x \notin X_s$. Furthermore, the natural map $\rho : F \rightarrow G$ is an isomorphism along $X_s \setminus Z$. Thus (10.65) implies that it is an isomorphism. So $F \cong G$ is flat with central fiber $j_*G_0 = j_*(F_s|_{X_s \setminus Z})$. \square

LEMMA 10.65. *Let $f : X \rightarrow S$ be a finite type morphism of Noetherian schemes, (s, S) local and $Z \subset X_s$ a nowhere dense closed subset. Let F, G be coherent sheaves on X and $\phi : F \rightarrow G$ a morphism. Assume that*

(10.65.1) *G and F are flat over S along $X \setminus Z$,*

(10.65.2) *ϕ is an isomorphism along $X_s \setminus Z$ and*

(10.65.3) *the fibers of G over $S \setminus \{s\}$ are pure and the fibers of F over $S \setminus \{s\}$ are pure and S_2 .*

Then ϕ is an isomorphism.

Proof. Set $W := \text{Supp}(\ker \phi)$ and let $w \in W$ be a generic point. It is also an associated point of $F_{f(w)}$ by (10.66.3). Furthermore, $X_s \cap \bar{w} \subset Z$ hence $\text{depth}_w F_{f(w)} \geq 1$ by (3), a contradiction. Next set $V := \text{Supp}(\text{coker } \phi)$ and let $v \in V$ be a generic point. As before, $X_s \cap \bar{v} \subset Z$, hence $\text{depth}_v F_{f(v)} \geq 2$ by (3). Thus

$$0 \rightarrow F_v \rightarrow G_v \rightarrow \text{coker } \phi_v \rightarrow 0$$

splits by (9.8), so $\text{coker } \phi_v$ is a subsheaf of $G_{f(v)}$ but this contradicts $\text{depth}_v G_{f(v)} \geq 1$ as before. \square

10.66 (Flatness and associated points). Let $f : X \rightarrow S$ be a morphism of Noetherian schemes and F a coherent sheaf on X .

Claim 10.66.1. If F is flat over S then $f(\text{Ass}(F)) \subset \text{Ass}(S)$.

Proof. Let $x \in X$ be an associated point of F and $s := f(x)$. Assume that s is not an associated point of S . Then there is an $r \in m_{s,S}$ such that $r : \mathcal{O}_S \rightarrow \mathcal{O}_S$ is injective near s . Tensoring with F shows that $r : F \rightarrow F$ is injective near X_s . Thus none of the points of X_s is in $\text{Ass}(F)$. \square

Claim 10.66.2. Assume that F is flat over S and $x \in \text{Ass}(F)$. Then every generic point of $\text{Supp}(\bar{x} \cap X_s)$ is an associated point of F_s . In particular, if F is flat with pure fibers then every $x \in \text{Ass}(F)$ is a generic point of $\text{Supp}(F_{f(x)})$.

Proof. Let $G \subset F$ be the largest subsheaf supported on \bar{x} . After localizing at a generic point of $\text{Supp}(\bar{x} \cap X_s)$ we may assume that $\text{Supp}(\bar{x} \cap X_s) = \{w\}$, a single closed point. There is a smallest $n \geq 0$ such that $G \subset m_{s,S}^n F$ but $G \not\subset m_{s,S}^{n+1} F$. Thus $m_{s,S}^n F / m_{s,S}^{n+1} F \cong (m_{s,S}^n / m_{s,S}^{n+1}) \otimes F_s$ has a nonzero subsheaf supported on w . \square

Note that flatness is needed for (10.66.2) as illustrated by the restriction of either of the coordinate projections to the union of the axes ($xy = 0$).

Claim 10.66.3. Assume f is of finite type and $x \in \text{Ass}(F)$. Then every fiber of $\bar{x} \rightarrow f(\bar{x})$ has the same dimension.

Proof. We may assume that $f(x)$ is a minimal associated point of S . Assume that we have $s \in f(\bar{x})$ such that $\dim(X_s \cap \bar{x})$ is larger than the expected dimension d . By restricting to a general relative Cartier divisor $H \subset X$, $F|_H$ is flat along H_s by (10.46) and $H_s \cap \bar{x}$ is a union of associated points of $F|_H$ by (10.13.1). Repeating this $d + 1$ times we get Cartier divisors $H^1, \dots, H^{d+1} \subset X$ and a complete intersection $Z := H^1 \cap \dots \cap H^{d+1}$ such that $F|_Z$ is flat along Z_s , the generic points of $Z \cap \bar{X}$ are associated points of $F|_Z$ yet they do not dominate $f(\bar{x})$. This is impossible by (10.66.1). \square

10.7. Noether normalization

Noether's normalization theorem says that if X is an affine k -variety of dimension m then it admits a finite morphism onto \mathbb{A}_k^m . Equivalently, the structure morphism $X \rightarrow \text{Spec } k$ can be factored as

$$X \xrightarrow{\text{finite}} \mathbb{A}_k^m \rightarrow \text{Spec } k.$$

We aim to generalize this to arbitrary morphisms. That is, we would like to factor an arbitrary morphism $p : X \rightarrow S$ as

$$X \xrightarrow{p_1} Y \xrightarrow{p_2} S,$$

where p_1 has 'finiteness' properties and p_2 has 'smoothness' properties.

In (10.67.7) we give an example of a morphism of pure relative dimension one $p : X \rightarrow S$ from an affine 3-fold X to a smooth, pointed surface $s \in S$ that can not be factored as

$$p : X \xrightarrow{\text{finite}} \mathbb{A}^1 \times S \rightarrow S,$$

not even over a formal neighborhood of s . Such examples are quite typical and, although the projective version of Noether's normalization theorem is easy to generalize to the relative setting, there does not seem to be any sensible global affine analog over base schemes of dimension ≥ 2 . There are, however, very useful local versions.

10.67 (Noether normalization, local version). Let $f : (x, X) \rightarrow (s, S)$ be a morphism of local, Noetherian schemes. We would like to factor f as

$$f : (x, X) \xrightarrow{p} (s', S') \xrightarrow{q} (s, S), \quad (10.67.1)$$

where p has 'finiteness' properties and q has 'smoothness' properties. The most useful version is (10.67.5), but let us start with the case when $k(x) \supset k(s)$ is a finitely generated field extension. Pick any transcendence basis $\bar{y}_1, \dots, \bar{y}_n$ of $k(x)/k(s)$ and lift these back to $y_1, \dots, y_n \in \mathcal{O}_X$. We can then take S' to be the localization of \mathbb{A}_S^n at the generic point of the fiber over $s \in S$. Thus we have proved the following.

Claim 10.67.2. Let $f : (x, X) \rightarrow (s, S)$ be a local morphism of local, Noetherian schemes such that $k(x) \supset k(s)$ is a finitely generated field extension. Then we can factor f as

$$f : (x, X) \xrightarrow{p} (s', S') \xrightarrow{q} (s, S) \quad (10.67.2.a)$$

where $k(x)/k(s')$ is a finite field extension, q has relative dimension 0 and it is the localization of a smooth morphism. \square

Combining this and the method of (10.68) shows that if $k(x) \supset k(s)$ is arbitrary then we get a factorization (10.67.2.a) where $k(x)/k(s')$ is an algebraic field extension and q is formally smooth.

For flatness questions we can freely replace (x, X) and (s, S) by their completions or Henselizations, and in those we can do better.

Pick $\bar{y} \in k(x)$ that is separable over $k(s')$ with separable, monic equation $\bar{g}(\bar{y}) = 0$. If \mathcal{O}_X is Henselian then we can lift \bar{y} to $y \in \mathcal{O}_X$ such that y satisfies a separable, monic equation $g(y) = 0$. We can now replace S' with the Henselization of $\mathcal{O}_{S'}[y]/(g(y))$ at the generic point of the central fiber, and obtain the following.

Claim 10.67.3. Let $f : (x, X) \rightarrow (s, S)$ be a local morphism of local, Henselian, Noetherian schemes such that $k(x)/k(s)$ is a finitely generated field extension. Then we can factor f as

$$f : (x, X) \xrightarrow{p} (s', S') \xrightarrow{q} (s, S) \tag{10.67.3.a}$$

where p is finite, $k(x)/k(s')$ is a purely inseparable field extension, q has relative dimension 0 and it is the localization of a smooth morphism. \square

Combining this with (10.68.3) gives the following.

Claim 10.67.4. Let $f : (x, X) \rightarrow (s, S)$ be a local morphism of local, complete, Noetherian schemes. Then we can factor f as

$$f : (x, X) \xrightarrow{p} (s', S') \xrightarrow{q} (s, S) \tag{10.67.4.a}$$

where $k(x)/k(s')$ is a purely inseparable field extension and q is formally smooth, faithfully flat, regular and of relative dimension 0. \square

For flatness criteria the following form is the most useful.

Claim 10.67.5. Let $f : (x, X) \rightarrow (s, S)$ be a local morphism of local, complete, Noetherian schemes such that $k(x)/k(s)$ is separable. Set $n := \dim X_s$

Then we can factor f as

$$f : (x, X) \xrightarrow{p} ((s', 0), \hat{\mathbb{A}}_{S'}^n) \xrightarrow{\pi} (s', S') \xrightarrow{q} (s, S) \tag{10.67.5.a}$$

such that

- (b) p is finite, $k(x) = k(s', 0) = k(s')$,
- (c) π is the coordinate projection,
- (d) q has relative dimension 0 and
- (e) q is the localization of a smooth morphism if $k(x)/k(s)$ is finitely generated and formally smooth, faithfully flat and regular in general.

Proof. By (10.67.4) we have $q : (s', S') \rightarrow (s, S)$ such that $k(x) = k(s')$. Since \mathcal{O}_{X_s} has dimension n , there are $\bar{t}_1, \dots, \bar{t}_n \in \mathcal{O}_{X_s}$ that generate an ideal that is primary to the maximal ideal. Lift these back to $t_1, \dots, t_n \in \mathcal{O}_X$. These define $p : (x, X) \rightarrow ((s', 0), \hat{\mathbb{A}}_{S'}^n)$. By construction

$$\mathcal{O}_X / (m_S, t_1, \dots, t_n) \cong \mathcal{O}_{X_s} / (\bar{t}_1, \dots, \bar{t}_n)$$

is finite over $k(s')$. Thus p is finite. \square

The following variant is due to [RG71]; see also [Sta15, Tag 052D]. A related factorization theorem is proved in [AFH94].

Claim 10.67.6. Let $f : X \rightarrow S$ be a finite type morphism. Pick $s \in S$, $x \in X_s$ and set $n = \dim_x X_s$. Then there is a commutative diagram

$$\begin{array}{ccc} (x', X') & \xrightarrow{\pi} & (x, X) \\ g \downarrow & & \downarrow f \\ (y, Y) & \xrightarrow{h} & (s, S), \end{array}$$

where π is étale, g is finite, $g^{-1}(y) = \{x'\}$ and h is smooth of relative dimension n . \square

Example 10.67.7. Let S denote the localization (or completion) of \mathbb{A}_{st}^2 at the origin and consider the affine scheme

$$X := ((x^3 + y^3 + 1)(1 + tx) + sy = 0) \subset \mathbb{A}_{xy}^2 \times S.$$

Then $\pi : X \rightarrow S$ is a family of curves. We claim that there is no finite morphism of it onto $\mathbb{A}^1 \times S$.

Assume to the contrary that such a map $g : X \rightarrow \mathbb{A}^1 \times S$ exists. Then g can be extended to a finite morphism $\bar{g} : \bar{X} \rightarrow \mathbb{P}^1 \times S$.

Here $\bar{X}_{(0,0)}$ is a compactification of $X_{(0,0)}$, hence a curve of geometric genus 1.

For $t \neq 0$ the line $(1 + tx = s = 0)$ gives an irreducible component of $\bar{X}_{(0,t)}$ that is a rational curve. As $t \rightarrow 0$, the limit of these rational curves is a union of rational, irreducible, geometric components of $\bar{X}_{(0,0)}$, a contradiction.

10.68 (Residue field extensions). Let (s, S) be a Noetherian, local scheme and $K/k(s)$ a field extension. We would like to find a Noetherian, local scheme (x, X) and a flat morphism $g : (x, X) \rightarrow (s, S)$ such that $g^*m_{s,S} = m_{x,X}$ (that is, the scheme theoretic fiber $g^{-1}(s)$ is the reduced point $\{x\}$) and $k(x) \cong K$. The answer is given in [Gro60, 0_{III}.10.3.1].

Claim 10.68.1. Such a $g : (x, X) \rightarrow (s, S)$ always exists.

Outline of proof. Let us start with extensions with 1 generator $K = k(s)(t)$. If t is transcendental over $k(s)$, we can take X to be the localization of \mathbb{A}_S^1 at the generic point of the fiber over $s \in S$. If t is algebraic, let $\bar{g}(z) \in k(s)[z]$ be a monic minimal polynomial of t . Lift it back to a monic polynomial $g(z) \in \mathcal{O}_S[z]$ and take X to be the localization of $(g(z) = 0) \subset \mathbb{A}_S^1$ at the central fiber. Combining these steps gives a solution for any finitely generated field extension $K/k(s)$. In the separable case the proof gives the following stronger form.

Claim 10.68.2. If $K/k(s)$ is a finitely generated separable extension then we can choose $g : (x, X) \rightarrow (s, S)$ to be the localization of a smooth morphism. In particular, if S is normal then so is X . \square

The general case is proved by iterating the same steps but one also needs a somewhat tricky limit argument to show that the resulting scheme is Noetherian. See [Gro60, 0_{III}.10.3.1] for details. Combining it with [And74] we get the following.

Claim 10.68.3. If $K/k(s)$ is an arbitrary separable extension then we can choose $g : (x, X) \rightarrow (s, S)$ to be formally smooth. If S is complete then g is also regular. In particular, if S is normal then so is X . \square

The following example illustrates some of the subtle aspects.

Example 10.68.4. Let k be a field. Fix a prime p and $a \in k \setminus k^p$. Set $S = \text{Spec } k[x]_{(x-a)}$ with maximal ideal $m = (x-a)S$. Then $\mathcal{O}_S/m \cong k$. Set $K = k(a^{1/p^n} : n = 1, 2, \dots)$. Then $X := \text{Spec } K[t]_{(t-a)}$ is a solution of (10.68.3), but here is a more interesting one.

Start with $k[x^{1/p^n} : n = 1, 2, \dots]$. This is not Noetherian since the ideal $(x^{1/p^n} : n = 1, 2, \dots)$ is not finitely generated. However, $x - a$ is irreducible in $k[x^{1/p^n}]$ for every n (equivalently, $x^{p^n} - a$ is irreducible in $k[x]$) hence also in $k[x^{1/p^n} : n = 1, 2, \dots]$. Thus $k[x^{1/p^n} : n = 1, 2, \dots]_{(x-a)}$ is a DVR with maximal ideal $(x - a)$.

As a concrete example, take $k = \mathbb{Q}$. Then $\mathbb{Q}[t^{1/n} : n = 1, 2, \dots]$ is not Noetherian and the polynomials $t, t-1, t+1$ all have infinitely many divisors. However, it seems that these are the only ones and $\mathbb{Q}[\frac{1}{t^3-t}, t^{1/n} : n = 1, 2, \dots]$ is Noetherian.

Note also that if $\text{char } k = p$ then $K[x^{1/p^n} : n = 1, 2, \dots]_{(x-a)}$ is not Noetherian, as shown by the ideal $(x^{1/p^n} - a^{1/p^n} : n = 1, 2, \dots)$.

Remark on inseparable extensions 10.68.5. Infinite inseparable extensions do cause problems in the above arguments, leading to finite type assumptions in (10.58) and its consequences in positive characteristic. However, I do not know whether these restrictions are actually necessary or not.

From the technical point of view, one difficulty is that infinite inseparable extensions can lead to non-excellent schemes. For example, let k be a perfect field of characteristic $p > 0$ and $K := k(x_1, x_2, \dots)$, a purely transcendental extension in infinitely many variables. Then $K^p = k(x_1^p, x_2^p, \dots)$ is abstractly isomorphic to K . Now start with $K^p[[t]]$ and try to get a residue field extension K/K^p . The above method gives $R := \cup_L L[[t]] \subset K[[t]]$, where L runs through all finite degree subextensions of K/K^p . Note that $R \neq K[[t]]$ since $\sum_i x_i t^i$ is not in R .

It is easy to see that R is a DVR and its completion is $K[[t]]$, which is purely inseparable over R . Thus R is not excellent. It is the source of many counter examples in [Nag62].

10.69 (Noether normalization, birational version). An affine form of Noether's normalization theorem says that every geometrically reduced affine variety admits a finite, birational morphism onto a hypersurface. Over an infinite base field this can be obtained by a general linear projection.

More generally, we have the following relative version.

Claim 10.69.1. Let S be an affine, integral, positive dimensional scheme and $X \rightarrow S$ a finite, dominant morphism. The following are equivalent.

- (a) The generic fiber X_{gen} is curvilinear (10.52) over $k(S)$.
- (b) There is a Cartier divisor $H \subset \mathbb{A}_S^1$ that is finite over S and a morphism $\pi : X \rightarrow H$ such that π is an isomorphism outside a nowhere dense closed subset of H . (If X is integral then this says that π is birational.)

Proof. It is clear that (b) implies (a). To see the converse, let K denote the function field of S and A the semilocal ring of the generic points of X . If (a) holds then, as we noted in (10.52), $A \cong K[t]/(g(t))$ for some monic polynomial $g(t)$ of degree m . Thinking of t as an element of A , there is a non-zero $c \in \mathcal{O}_S$ such that $x := ct \in \mathcal{O}_X$ and $cg(t) \in \mathcal{O}_S[t]$. Thus we can take $H := (c^m g(x/c) = 0) \subset \mathbb{A}_S^1$. \square

A construction as in (10.67.7) shows that the relative version of (10.69.1) fails for morphisms of finite type, but, as in (10.67.3–5), we can generalize (10.69.1) to morphisms of complete local schemes.

Thus let $f : (x, X) \rightarrow (s, S)$ be a local morphism of complete, Noetherian schemes. Assume that there is a finite, birational morphism onto a hypersurface

$$\pi_s : X_s \rightarrow H_s \subset \hat{\mathbb{A}}_{k(s)}^{n+1}. \quad (10.69.2)$$

By lifting the coordinate functions $\pi_s^*(x_i)$ arbitrarily to \mathcal{O}_X , π_s extends to a morphism

$$\pi : X \rightarrow \hat{\mathbb{A}}_S^{n+1}. \quad (10.69.3)$$

Note that π is finite since $\pi^{-1}(s, \mathbf{0}) = \pi_s^{-1}(\mathbf{0})$. Let us denote its image by $H \subset \hat{\mathbb{A}}_S^{n+1}$. The intersection of H with the central fiber $\hat{\mathbb{A}}_S^{n+1}$ is H_s . Thus, if f is flat at the generic points of X_s , then $\pi : X \rightarrow H$ is a local isomorphism at the generic points of H_s and so $H \rightarrow S$ is also flat at the generic points of H_s .

Furthermore, if S is normal then H is a relative Cartier divisor in $\hat{\mathbb{A}}_S^{n+1}$ by (10.55) hence we have proved the following.

Claim 10.69.4. Let $f : (x, X) \rightarrow (s, S)$ be a local morphism of complete, Noetherian schemes. Assume that S is normal and f is flat at the generic points of X_s . Let $\pi_s : X_s \rightarrow H_s \subset \hat{\mathbb{A}}_{k(s)}^{n+1}$ be a finite, birational morphism onto a hypersurface.

Then there is a relative Cartier divisor $H \subset \hat{\mathbb{A}}_S^{n+1}$ such that π_s extends to a finite, birational morphism of S -schemes $\pi : X \rightarrow H$. \square

Informally speaking, normalizations of hypersurfaces describe all deformations over normal base schemes. (Normality of S is necessary by (10.61.1).)

This is a seemingly very useful observation, but in most cases it turns out to be extremely hard to understand the central fiber of the normalization. Next we discuss normalization of double points, where this approach leads to a complete answer. More substantial applications of this method are in [dJvS91].

PROPOSITION 10.70. *Let (R, m) be a complete, normal, local ring, $q(x_1, \dots, x_n)$ a nondegenerate quadratic form over R and $c \in m$. Then $R[[x_1, \dots, x_n]]/(q + c)$ is normal if $n \geq 3$ or $n = 2$ and $c \neq 0$.*

Proof. Set $S := \text{Spec } R$, $X := \text{Spec } R[[x_1, \dots, x_n]]/(q + c)$ with projection $\pi : X \rightarrow S$. Let $W \subset X$ denote the locus where π is not regular. Thus $X \setminus W$ is normal and, by Serre's criterion, X is normal if $\text{depth}_W X \geq 2$.

The fiber of W over $\text{Spec}(R/m)$ has codimension $n - 1$. Thus if $w \in W$ then

$$\text{depth}_w X \geq \text{depth}_{p(w)} S + \text{codim}(w, X_{p(w)}) \geq \text{depth}_{p(w)} S + n - 1.$$

We are done if $n \geq 3$ or if $n = 2$ and $p(w)$ is not a generic point of S . We finish by noting that if $c \neq 0$ then the generic fiber is regular by the Jacobian criterion. \square

LEMMA 10.71. *Let (R, m) be a normal, complete, local ring such that the characteristic of R/m is $\neq 2$. Let $f \in R[[x_1, \dots, x_n]]$ be a power series such that \bar{f} is not identically zero. By (10.56) we can write $f = g^2 h$ where h is square-free. Then $y \mapsto gz$ gives the normalization map*

$$R[[y, x_1, \dots, x_n]]/(y^2 - f) \hookrightarrow R[[z, x_1, \dots, x_n]]/(z^2 - h).$$

Proof. It is clear that the ring extension is finite and birational. Thus we need to show that $R[[z, x_1, \dots, x_n]]/(z^2 - h)$ is normal. Projection to $R[[x_1, \dots, x_n]]$ is étale away from $(h = 0)$, hence $R[[x_1, \dots, x_n]]$ is normal away from $(h = 0)$.

To check normality along $(h = 0)$ we localize at a generic point of $(h = 0)$. Then we have a DVR A with maximal ideal $(h)A$, otherwise we would have a multiple factor of h . The unique maximal ideal of $A[z]/(z^2 - h)$ is (z, h) and it is generated by z , thus $A[z]/(z^2 - h)$ is a DVR, hence normal. \square

10.8. Seminormality and weak normality

(For more details see [Kol96, Sec.I.7.2] and [Kol13b, Sec.10.2].)

Normalization is a very useful operation that can be used to improve a scheme X . However, the normalization $X^n \rightarrow X$ usually creates new points and this makes it harder to relate X and X^n . The notions of semi and weak normalization intend to do as much of the normalization as possible, without creating new points.

For example, the normalization of the higher cusps $C_{2m+1} := (x^2 = y^{2m+1})$ is

$$\pi_{2m+1} : \mathbb{A}_t^1 \rightarrow C_{2m+1} \quad \text{given by} \quad t \mapsto (t^{2m+1}, t^2).$$

The map π_{2m+1} is a homeomorphism, so it is also the seminormalization. By contrast, the normalization of the higher tacnode $C_{2m} := (x^2 = y^{2m})$ is

$$\pi_{2m} : \mathbb{A}_t^1 \times \{\pm 1\} \rightarrow C_{2m} \quad \text{given by} \quad t \mapsto (\pm t^m, t).$$

The map π_{2m} is not a homeomorphism since $(0, 0) \in C_{2m}$ has 2 preimages, $(0, 1)$ and $(0, -1)$. The seminormalization of C_{2m} is

$$\tau_{2m} : C_2 \cong (s^2 = t^2) \rightarrow C_{2m} \quad \text{given by} \quad (s, t) \mapsto (s^m, t).$$

These examples lead to a general definition of seminormalization and seminormal schemes, but they do not adequately show how complicated seminormal schemes are in higher dimensions.

DEFINITION 10.72. A finite morphism of reduced schemes $p : Y \rightarrow X$ is a *partial normalization* if it is surjective and every irreducible component of Y maps birationally onto an irreducible component of X . (See (10.76) for a version where X, Y are non-reduced.)

It is called a *partial seminormalization* (resp. *partial weak normalization*) if, in addition, $k(x) \hookrightarrow k(\text{red } p^{-1}(x))$ is an isomorphism (resp. $k(x) \hookrightarrow k(\text{red } p^{-1}(x))$ is purely inseparable) for every $x \in X$. In particular, the semi and weak versions agree in characteristic 0. (If X is nonreduced, sometimes we refer to a partial normalization of $\text{red } X$ as a partial normalization of X .)

A reduced scheme is *normal* (resp. *seminormal*, or *weakly normal*) if every partial normalization (resp. partial seminormalization or partial weak normalization) is an isomorphism.

If X is excellent then it has a unique partial normalization that is normal. It is called the *normalization* of X . Similarly for *seminormalization* and *weak normalization*. We denote these by X^n (resp. $X^{\text{sn}}, X^{\text{wn}}$). We have morphisms

$$X^n \rightarrow X^{\text{wn}} \rightarrow X^{\text{sn}} \rightarrow X,$$

and $X^{\text{wn}} = X^{\text{sn}}$ in characteristic 0.

EXAMPLE 10.73. Let Y be a reduced scheme with finite normalization $\pi : \bar{Y} \rightarrow Y$. Let $D \subset Y$ be the reduced conductor. Let $D_1 \subset D$ be the union of some of

the irreducible components and $D_2 \subset D$ the union of the others. Then there is a partial normalization

$$\bar{Y} \xrightarrow{\pi_2} Y' \xrightarrow{\pi_1} Y$$

such that $\pi_1: Y' \rightarrow Y$ is an isomorphism over $Y \setminus D_2$ and $\pi_2: \bar{Y} \rightarrow Y'$ is an isomorphism over $Y' \setminus \pi_1^{-1}(D_1)$.

Indeed, $Y \setminus D_2$ and $\bar{Y} \setminus \pi_1^{-1}(D_1)$ naturally glue together to a scheme $p: W \rightarrow Y \setminus (D_1 \cap D_2)$. Let F denote the push-forward of $p_*\mathcal{O}_W$ to Y . Then F is a coherent sheaf of algebras (10.25) and $Y' := \text{Spec}_Y F$ has the required properties.

Seminormality is a quite useful notion, but it is not always easy to use. A major difficulty is that an irreducible component of a seminormal scheme need not be seminormal. In fact, by [Kol13b, 10.12], every reduced and irreducible affine scheme that is smooth in codimension 1 occurs as an irreducible component of a seminormal complete intersection scheme.

The following example illustrates some of the differences between weak and seminormality.

EXAMPLE 10.74. Let $g(t) \in k[t]$ be a polynomial without multiple factors and set $C_g := \text{Spec}_k(k + g \cdot k[t])$. Then C_g is an integral curve whose normalization is \mathbb{A}^1 . We can think of C_g as obtained from \mathbb{A}^1 by identifying all roots of g .

If g is separable then C_g is seminormal and weakly normal. If g is irreducible and purely inseparable then C_g is seminormal but not weakly normal; the weak normalization of C_g is \mathbb{A}^1 .

The problem is that, in the latter case, C_g looks and behaves very much like a (higher) cusp. For example if $\text{char } k = 2$ and $g = t^2 - a$ then $k + g \cdot k[t]$ is generated by $x := t^2 - a$ and $z := t(t^2 - a)$ and they satisfy the equation $z^2 = x^3 + ax^2$. If we adjoin $\alpha = \sqrt{a}$ to k and set $y := z - \alpha x$ then this becomes $y^2 = x^3$.

The following lemma is useful in decomposing a weak normalization into simpler partial weak normalizations.

LEMMA 10.75. *Let (R', m') be a local ring and (R, m) a local subring such that R'/m is artinian. Then there is a sequence of increasing subrings $(R, m) = (R_0, m_0) \subset \cdots \subset (R_n, m_n)$ such that*

$$(10.75.1) \quad m_n = m',$$

$$(10.75.2) \quad R_i/m_i \cong R/m \text{ and } m_i R_{i+1} = m_i \text{ for every } i.$$

Proof. Let $I \subset R$ be the conductor ideal of R'/R . By assumption R'/I is artinian. We are done if $I_0 = m'$; otherwise there is an $r \in m' \setminus I_0$ such that $mr \in I_0$. Set $R_1 := R[r]$. Then $mR_1 = m$ and $R_1/R \cong R/m$ since $I_0 \subset R$. Iterating this procedure eventually ends with $I_{n-1} = m'$ for some n . \square

It is frequently convenient to have other versions of these notions.

DEFINITION 10.76. Let X be a noetherian scheme and $Z \subset X$ a closed, nowhere dense subset. A *finite modification* of X centered at Z is a finite morphism $p: Y \rightarrow X$ such that the restriction $p: Y \setminus p^{-1}(Z) \rightarrow X \setminus Z$ is an isomorphism and none of the associated primes of Y is contained in $p^{-1}(Z)$.

Assume that X is reduced. A finite morphism $p: Y \rightarrow X$ is called a *partial normalization* (resp. seminormalization or weak normalization) if it is a finite modification of X centered at some closed, nowhere dense $Z \subset X$ (resp. and

$k(x) \hookrightarrow k(\text{red } p^{-1}(x))$ is an isomorphism for every $x \in X$, or $k(x) \hookrightarrow k(\text{red } p^{-1}(x))$ is purely inseparable for every $x \in X$).

Fix $Z \subset X$ and let $\mathbf{P}(Z, X)$ denote the set of all finite modification of X centered at Z . We say that the pair $Z \subset X$ is

- (10.76.1) *normal*: every $p \in \mathbf{P}(Z, X)$ is an isomorphism;
- (10.76.2) *seminormal*: every partial seminormalization in $\mathbf{P}(Z, X)$ is an isomorphism;
- (10.76.3) *weakly normal*: every partial weak normalization in $\mathbf{P}(Z, X)$ is an isomorphism.

The following properties are proved in [Kol16c].

PROPOSITION 10.77. *For a noetherian scheme X the following are equivalent.*

- (10.77.1) X is normal (resp. seminormal, weakly normal).
- (10.77.2) $Z \subset X$ is a normal (resp. seminormal, weakly normal) pair for every closed, nowhere dense subset $Z \subset X$.
- (10.77.3) $\{x\} \subset \text{Spec } \mathcal{O}_{x,X}$ is a normal (resp. seminormal, weakly normal) pair for every non-generic point $x \in X$.
- (10.77.4) $\{x\} \subset \text{Spec } \hat{\mathcal{O}}_{x,X}$ is a normal (resp. seminormal, weakly normal) pair for every non-generic point $x \in X$. \square

The last equivalence is surprising since the completion of a normal local ring is not always normal.

All 3 properties ascend for flat morphisms in a strong form. For normality this is classical; see for instance [Mat86, 23.9]. The other cases are proved in [GT80, Man80] for N-1 schemes and the general version is in [Kol16c, 37].

PROPOSITION 10.78. *Let $f : Y \rightarrow X$ be a flat morphism of Noetherian schemes with geometrically reduced fibers. Assume that X and the geometric generic fibers are normal (resp. seminormal, weakly normal). Then Y is also normal (resp. seminormal, weakly normal). \square*

Valuative criteria.

The next results show that many questions about seminormal schemes can be settled using points and specializations only.

DEFINITION 10.79. For a scheme X let $|X|$ denote its underlying point set. Somewhat sloppily, we say that a subset $B \subset |X|$ is a subscheme if there is a reduced subscheme $Z \subset X$ such that $|Z| = B$. Note that such a Z is unique.

Let X, Y be reduced schemes and $\phi : |X| \rightarrow |Y|$ a set-map of the underlying sets. We say that ϕ is a *morphism* if there is a morphism $\Phi : X \rightarrow Y$ inducing ϕ . Note that such a Φ is unique.

Our aim is to find simple conditions that guarantee that a subset is a subscheme or that a set-map is a morphism.

We say that ϕ is a *morphism on points* if the natural inclusion $k(x) \hookrightarrow k(x, \phi(x))$ is an isomorphism for every $x \in X$, where we view $(x, \phi(x))$ as point in $X \times Y$. (This in effect says that there is a natural injection $k(\phi(x)) \hookrightarrow k(x)$.)

We say that ϕ is a *morphism on DVRs* if the composite $\phi \circ h$ is a morphism whenever $h : T \rightarrow X$ is a morphism from the spectrum of a DVR to X .

We say that ϕ is a *morphism on generic DVRs* if the composite $\phi \circ h$ is a morphism whenever $h : T \rightarrow X$ is a morphism from the spectrum of a DVR to X that maps the generic point of T to a generic point of X .

LEMMA 10.80 (Valuative criterion of being a section). *Let $g : X \rightarrow S$ be a separated morphism of finite type and $B \subset |X|$ a subset. Then $B = |Z|$ for a subscheme Z and $g|_Z : Z \rightarrow S$ is a finite, universal homeomorphism iff the following hold.*

(10.80.1) *Every point $s \in S$ has a unique preimage $b_s \in B$ and $k(b_s)/k(s)$ is purely inseparable.*

(10.80.2) *Let T be the spectrum of a DVR and $h : T \rightarrow S$ a morphism mapping generic point to generic point. Then there is a spectrum of a DVR T' , a finite morphism $\pi : T' \rightarrow T$ and a lifting $(h \circ \pi)_X : T' \rightarrow X$ whose image is in B .*

Proof. By assumption (1), $g|_B : B \rightarrow S$ is a universal bijection. Let $s_g \in S$ be a generic point and $b_g \in B$ its preimage. We claim that $\bar{b}_g \subset B$. For any $b_0 \in \bar{b}_g$ there is a DVR T and a morphism $T \rightarrow X$ that maps the generic point to b_g and the closed point to b_0 . We apply (2) to $g \circ \tau$ to conclude that $b_0 \in B$.

Thus Z is the union of all \bar{b}_g hence Zariski closed and $g|_Z : Z \rightarrow S$ is a finite, universal bijection, hence a homeomorphism. \square

LEMMA 10.81 (Valuative criterion of being a morphism). *Let X, Y be schemes of finite type, X seminormal and Y separated. Then a set-map $\phi : |X| \rightarrow |Y|$ is a morphism iff it is a morphism on points and on generic DVRs.*

Proof. Let $Z \subset X \times Y$ be the graph of ϕ and $g : X \times Y \rightarrow X$ the projection. By (10.80) $g|_Z : Z \rightarrow X$ is a finite, universal homeomorphism that is residue field preserving since ϕ is a morphism on points. Thus $g|_Z : Z \rightarrow X$ is an isomorphism since X is seminormal. \square

LEMMA 10.82. *Let $i : S' \rightarrow S$ be a geometrically injective morphism of finite type schemes. Let W be a weakly normal scheme and $g : W \rightarrow S$ a morphism. Then $g : W \rightarrow S$ factors through S' iff $g \circ q : T \rightarrow W \rightarrow S$ factors through S' for generic DVRs.*

Proof. One direction is clear. To see the converse consider $i_W : W \times_S S' \rightarrow W$. It is geometrically injective, proper and surjective and an isomorphism over generic points. Thus i_W is an isomorphism since W is weakly normal. \square

Note that in most cases the Lemma holds even if i is not geometrically injective, but the following example should be kept in mind. Let S be the triangle $(xyz = 0) \subset \mathbb{P}^2$ and $S' \rightarrow S$ a nontrivial étale cover of it. Then every $T \rightarrow S$ lifts (non-uniquely) to $T \rightarrow S'$, but the identity $S \cong S$ does not lift to $S \rightarrow S'$.

Locally closed decompositions.

DEFINITION 10.83. A morphism $p : X \rightarrow Y$ is *geometrically injective* if for every geometric point $\bar{y} \rightarrow Y$ the fiber $X \times_Y \bar{y}$ consists of at most 1 point.

Equivalently, for every point $y \in Y$, its preimage $p^{-1}(y)$ is either empty or a single point and $k(p^{-1}(y))$ is a purely inseparable extension of $k(y)$.

If, furthermore, $k(p^{-1}(y))$ equals $k(y)$ then we say that p is *residue field preserving*. The two notions are equivalent in characteristic 0.

A morphism of schemes $f : X \rightarrow Y$ is a *monomorphism* if for every scheme Z the induced map of sets $Mor(Z, X) \rightarrow Mor(Z, Y)$ is an injection.

A monomorphism is geometrically injective. The normalization of the cusp $\pi : \text{Spec } k[t] \rightarrow \text{Spec } k[t^2, t^3]$ is geometrically injective but not a monomorphism.

The problem is with the fiber over the origin, which is $\text{Spec } k[t]/(t^2) \cong \text{Spec } k[\epsilon]$ (where $\epsilon^2 = 0$). The 2 maps $g_i : \text{Spec } k[\epsilon] \rightarrow \text{Spec } k[t]$ given by $g_0^*(t) = 0$ and $g_1^*(t) = \epsilon$ are different but $\pi \circ g_0 = \pi \circ g_1$. A similar argument shows that a morphism is a monomorphism iff it is geometrically injective and unramified; see [Gro60, IV.17.2.6].

As the above example shows, in order to understand when a map between moduli spaces is a monomorphism, the key is to study the corresponding functors over $\text{Spec } k[\epsilon]$ for all fields k .

See (1.64) for an example that is geometrically injective but, unexpectedly, not a monomorphism.

A closed, open or locally closed embedding is a monomorphism. A typical example of a monomorphism that is not a locally closed embedding is the normalization of the node with a point missing, that is $\mathbb{A}^1 \setminus \{-1\} \rightarrow (y^2 = x^3 + x^2)$ given by $(t \mapsto (t^2 - 1, t^3 - t))$.

The following property is frequently useful.

Claim 10.83.1. A proper monomorphism $f : X \rightarrow Y$ is a closed embedding.

Proof. A proper monomorphism is injective on geometric points, hence finite. Thus it is a closed embedding iff $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is onto. By the Nakayama lemma this is equivalent to $f_y : f^{-1}(y) \rightarrow y$ being an isomorphism for every $y \in f(X)$. By passing to geometric points, we are down to the case when $Y = \text{Spec } k$, k is algebraically closed and $X = \text{Spec } A$ where A is an Artin k -algebra. If $A \neq k$ then there are at least 2 different k maps $A \rightarrow k[\epsilon]$, thus $\text{Spec } A \rightarrow \text{Spec } k$ is not a monomorphism. \square

DEFINITION 10.84. A morphism $g : X \rightarrow Y$ is a *locally closed embedding* if it can be factored as $g : X \rightarrow Y^\circ \hookrightarrow Y$ where $X \rightarrow Y^\circ$ is a closed embedding and $Y^\circ \hookrightarrow Y$ is an open embedding.

A monomorphism $g : X \rightarrow Y$ is called a *locally closed partial decomposition* of Y if the restriction of g to every connected component $X_i \subset X$ is a locally closed embedding.

If g is also surjective, it is called a *locally closed decomposition* of Y . For reduced schemes, the key example is the following.

Claim 10.84.1. Let $h : Y \rightarrow \mathbb{Z}$ be a constructible and upper semicontinuous function and set $Y_i := \{y \in Y : h(y) = i\}$. Then $\coprod_i Y_i \rightarrow Y$ defines a locally closed decomposition. \square

The following direct consequence of (10.83.1) is quite useful.

Claim 10.84.2. A proper, locally closed partial decomposition $g : X \rightarrow Y$ is a closed embedding. If Y is reduced then a proper, locally closed decomposition $g : X \rightarrow Y$ is an isomorphism. \square

PROPOSITION 10.85 (Valuative criterion of locally closed embedding). *For a geometrically injective morphism of finite type $f : X \rightarrow Y$ the following are equivalent.*

(10.85.1) $f(X) \subset Y$ is locally closed and $X \rightarrow f(X)$ is finite.

(10.85.2) Let T be the spectrum of a DVR and $g : T \rightarrow Y$ a morphism such that $g(T) \subset f(X)$. Then there is a spectrum of a DVR T' and a finite morphism $\pi : T' \rightarrow T$ such that $g \circ \pi$ lifts to $g'_X : T' \rightarrow X$.

(10.85.3) *The previous condition holds for those $g : T \rightarrow Y$ that map the generic point of T to a generic point of $f(X)$.*

If f is a monomorphism then these are further equivalent to

(10.85.4) *f is a locally closed embedding.*

Proof. It is clear that (1) \Rightarrow (2) \Rightarrow (3). Next assume (3).

A geometrically injective morphism of finite type is quasi-finite, hence, by Zariski's main theorem, there is a finite morphism $\bar{f} : \bar{X} \rightarrow Y$ extending f . Set $Z := \bar{X} \setminus X$.

If $Z \neq \bar{f}^{-1}\bar{f}(Z)$ then there are points $z \in Z$ and $x \in X$ such that $\bar{f}(z) = \bar{f}(x)$. Let T be the spectrum of a DVR and $h : T \rightarrow \bar{X}$ a morphism which maps the closed point to z and the generic point of T to a generic point of X . Set $g := \bar{f} \circ h$. Then $g(T) \subset f(X)$ and the only lifting of g to $T \rightarrow \bar{X}$ is h , but $h(T) \not\subset X$.

Thus $Z = \bar{f}^{-1}\bar{f}(Z)$ hence $X \rightarrow Y \setminus \bar{f}(Z)$ is proper, proving (1). A proper monomorphism is a closed embedding by (10.83.1), showing the equivalence with (4). \square

Seminormalization of moduli functors.

A major advantage of seminormality over normality is that seminormalization $X \mapsto X^{\text{sn}}$ is a functor from the category of excellent schemes to the category of excellent seminormal schemes. It is thus reasonable to expect that taking the coarse moduli space commutes with seminormalization. This is indeed the case for coarse moduli spaces satisfying the following mild condition.

DEFINITION 10.86. Let $\mathcal{M} : (\text{schemes}) \rightarrow (\text{sets})$ be a functor with coarse moduli space M . We say that \mathcal{M} has *enough 1-parameter families* if the following holds.

Let T be the spectrum of a DVR and $\phi : T \rightarrow M$ a morphism. Then there is a spectrum of a DVR T' , a finite morphism $\pi : T' \rightarrow T$ and $F \in \mathcal{M}(T')$ such that $\phi \circ \pi : T' \rightarrow M$ is the moduli map of F .

PROPOSITION 10.87. *Let $\mathcal{M} : (\text{schemes}) \rightarrow (\text{sets})$ be a functor defined on finite type schemes over a field of characteristic 0. Assume that \mathcal{M} has a finite type coarse moduli space M and enough 1-parameter families.*

Then M^{sn} is the coarse moduli space for its restriction to the category Sch^{sn} of finite type, seminormal schemes

$$\mathcal{M}^{\text{sn}} := \mathcal{M}|_{\text{Sch}^{\text{sn}}} : (\text{seminormal schemes}) \rightarrow (\text{sets}).$$

Proof. Since seminormalization is a functor, every morphism $W \rightarrow M$ lifts to $W^{\text{sn}} \rightarrow M^{\text{sn}}$. Thus we have a natural transformation $\Phi : \mathcal{M}^{\text{sn}} \rightarrow \text{Mor}(*, M^{\text{sn}})$.

Assume that M' is a finite type, seminormal scheme and we have another natural transformation $\Psi : \mathcal{M}^{\text{sn}} \rightarrow \text{Mor}(*, M')$. Thus we get a natural transformation to the product $M^{\text{sn}} \times M'$; let $Z \subset M^{\text{sn}} \times M'$ denote the set-theoretic image. Since M^{sn} is a coarse moduli space, the coordinate projection $Z \rightarrow M^{\text{sn}}$ is geometrically bijective. Since \mathcal{M} has enough 1-parameter families, $Z \rightarrow M^{\text{sn}}$ is a universal homeomorphism by (10.80). Thus $Z \rightarrow M^{\text{sn}}$ is an isomorphism since M^{sn} is seminormal and the characteristic is 0.

Thus we get a morphism $M^{\text{sn}} \rightarrow M'$ and Ψ factors through Φ . \square

The next examples show that the characteristic 0 assumption is likely necessary in (10.87) and that the analogous claim for the underlying reduced subscheme is likely to be false. However, I do not know any natural examples in either case.

EXAMPLE 10.88. Let \mathcal{D} be any diagram of schemes with direct limit $\lim \mathcal{D}$. Since seminormalization is a functor, we get a diagram \mathcal{D}^{sn} and a natural morphism $\lim(\mathcal{D}^{\text{sn}}) \rightarrow (\lim \mathcal{D})^{\text{sn}}$. However, this need not be an isomorphism.

(10.88.1) Let k be an infinite field and consider the diagram of all maps $\phi_a : \text{Spec } k[x] \rightarrow \text{Spec } k[(x-a)^2, (x-a)^3]$ for $a \in k$.

If $\text{char } k = 0$ the direct limit is $\text{Spec } k$. After seminormalization, the maps ϕ_a become isomorphisms $\phi_a^{\text{sn}} : \text{Spec } k[x] \cong \text{Spec } k[x]$ and now the direct limit is $\text{Spec } k[x]$.

(10.88.2) If $\text{char } k = p > 0$ then $x^p - a^p = (x-a)^p \in k[(x-a)^2, (x-a)^3]$ shows that the direct limit is $\text{Spec } k[x^p]$. After seminormalization, the direct limit is again $\text{Spec } k[x]$. In this case $\text{Spec } k[x^p]$ behaves like a coarse moduli space.

(10.88.3) Consider the maps $\sigma_i : k[x] \rightarrow k[x, \epsilon]$ given by $\sigma_0(g(x)) = g(x)$ and $\sigma_1(g(x)) = g(x) + g'(0)\epsilon$. We get a universal push-out diagram

$$\begin{array}{ccc} \text{Spec } k[x, \epsilon] & \xrightarrow{\sigma_0} & \text{Spec } k[x] \\ \sigma_1 \downarrow & & \downarrow \\ \text{Spec } k[x] & \rightarrow & \text{Spec } k[x^2, x^3]. \end{array}$$

If we pass to the underlying reduced subspaces, the push-out becomes $\text{Spec } k[x]$.

Minimal models and their singularities

We review the definitions and results of the minimal model program that we use repeatedly. For more complete treatments, see [KM98] or [Kol13b]. Most of the older literature works with \mathbb{Q} -divisors, but later developments emphasize the role of \mathbb{R} -divisors. We discuss their subtler points in Section 11.3.

11.1. Singularities of stable varieties

We recall the key definitions and results about singularities of stable varieties. These are treated much more thoroughly in [Kol13b]. Here we aim to be concise, discussing all that is necessary for the main results but leaving many details untouched.

Singularities of pairs.

DEFINITION 11.1 (Pairs). We are primarily interested in pairs (X, Δ) where X is a normal variety over a perfect field and $\Delta = \sum a_i D_i$ a formal linear combination of prime divisors with rational or real coefficients. More generally, X can be a pure dimensional, reduced scheme of finite type over a perfect field such that ω_X is locally free outside a subset of codimension ≥ 2 and $\Delta = \sum a_i D_i$ a formal linear combination of prime Mumford divisors (4.20.4), that is, none of the D_i are contained in $\text{Sing } X$. (Even more general scheme cases are discussed in [Kol13b, 2.4].)

DEFINITION 11.2 (Simple normal crossing). A *simple normal crossing* pair—usually abbreviated as *snc* pair—is a pair (X, D) consisting of a smooth variety X and distinct smooth divisors, where all intersections are transversal. That is, at each $p \in X$ we can choose local coordinates x_1, \dots, x_n and an open neighborhood $x \in U \subset X$ such that $D \cap U \subset (x_1 \cdots x_n = 0)$. We also say that D is an *snc divisor*.

D is a *normal crossing divisor* if for every $x \in X$ there is an étale neighborhood $x \in U \rightarrow X$ such that $U \cap D$ is an snc divisor. Thus self-intersections are allowed on a normal crossing divisor.

A variety Y has *simple normal crossing* singularities if every point $y \in Y$ has an open neighborhood $y \in V \subset Y$ that is isomorphic to an snc divisor. One defines *normal crossing* singularities analogously.

DEFINITION 11.3 (Discrepancy of divisors). Let (X, Δ) be a pair as in (11.1). We are looking at cases when the pull-back of $K_X + \Delta$ by birational morphisms makes sense. If Δ is a \mathbb{Q} -divisor, the natural assumption is that $K_X + \Delta$ is \mathbb{Q} -Cartier, that is, $m(K_X + \Delta)$ is Cartier for some $m > 0$.

If Δ is an \mathbb{R} -divisor, we need to assume that $K_X + \Delta$ is \mathbb{R} -Cartier, that is, it can be written as an \mathbb{R} -linear combination of Cartier divisors. We discuss this

notion in detail in Section 11.3. (See (4.55) for the even more general notion of numerically \mathbb{R} -Cartier divisors).

Let $f: Y \rightarrow X$ be a proper, birational morphism from a demi-normal variety Y , $\text{Ex}(f) \subset Y$ the exceptional locus of f and $E_i \subset \text{Ex}(f)$ the irreducible exceptional divisors. Assume that $\text{Ex}(f) \cap \text{Sing } Y \subset Y$ and $f(\text{Ex}(f)) \subset X$ have codimension ≥ 2 ; these are automatic if X, Y are normal. Let $f_*^{-1}\Delta := \sum a_i f_*^{-1}D_i$ denote the birational transform of Δ . Fix a canonical divisor $K^Y \in |K_Y|$ and set $K^X := f_*(K^Y)$.

Assume next that $K_X + \Delta$ is \mathbb{R} -Cartier. Then

$$K^Y + f_*^{-1}\Delta - f^*(K^X + \Delta)$$

makes sense and it is exceptional, hence we can write

$$K^Y + f_*^{-1}\Delta = f^*(K^X + \Delta) + \sum_i a(E_i, X, \Delta)E_i. \tag{11.3.1}$$

The $a(E_i, X, \Delta) \in \mathbb{R}$ are independent of the choice of K^Y . This defines $a(E, X, \Delta)$ for exceptional divisors. Set $a(E, X, \Delta) := -\text{coeff}_E \Delta$ for non-exceptional divisors $E \subset X$.

The real number $a(E, X, \Delta)$ is called the *discrepancy* of E with respect to (X, Δ) ; it depends only on the valuation defined by E , not on the choice of f . (See [KM98, 2.22] for a more canonical definition for \mathbb{Q} -Cartier pairs.)

Warning about terminology. For most cases of interest to us, $a(E, X, \Delta) \geq -1$. For this reason, some authors use *log discrepancies*, defined as

$$a_\ell(E, X, \Delta) := 1 + a(E, X, \Delta). \tag{11.3.2}$$

Most unfortunately, recently some people started to use $a(E, X, \Delta)$ to denote the log discrepancy, creating ample opportunity for confusion.

The discrepancies have the following obvious monotonicity and linearity properties; cf. [KM98, 2.27].

Claim 11.3.3. Let Δ' be an effective, \mathbb{R} -Cartier divisor. Then $a(E, X, \Delta + \Delta') \leq a(E, X, \Delta)$ for every divisor E over X .

Claim 11.3.4. Assume that $K_X + \Delta_i$ are \mathbb{R} -Cartier. Fix $\sum \lambda_i = 1$ and set $\Delta := \sum \lambda_i \Delta_i$. Then $K_X + \Delta$ is \mathbb{R} -Cartier and $a(E, X, \Delta) = \sum \lambda_i a(E, X, \Delta_i)$ for every divisor E over X .

In particular, if the (X, Δ_i) are lc (resp. dlt, klt, canonical, terminal) then so is (X, Δ) . □

DEFINITION 11.4. Let X be a normal variety of dimension ≥ 2 and $\Delta = \sum a_i D_i$ an \mathbb{R} -divisor such that $K_X + \Delta$ is \mathbb{R} -Cartier.

We say that (X, Δ) is

$$\left. \begin{array}{l} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{plt} \\ \textit{dlt} \\ \textit{lc} \end{array} \right\} \text{ if } a(E, X, \Delta) \text{ is } \left\{ \begin{array}{ll} > 0 & \text{for every exceptional } E, \\ \geq 0 & \text{for every exceptional } E, \\ > -1 & \text{for every } E, \\ > -1 & \text{for every exceptional } E, \\ > -1 & \text{if } \text{center}_X E \subset \text{non-snc}(X, \Delta), \\ \geq -1 & \text{for every } E. \end{array} \right.$$

Here klt is short for *Kawamata log terminal*, plt for *purely log terminal* dlt for *divisorial log terminal*, lc for *log canonical* and $\text{non-snc}(X, \Delta)$ denotes the set of points where (X, Δ) is not a simple normal crossing pair [Kol13b, 1.7].

The following observations are useful.

Claim 11.4.1. If (X, Δ) is in any of these 6 classes, $0 \leq \Delta' \leq \Delta$ and $K_X + \Delta'$ is \mathbb{R} -Cartier, then (X, Δ') is also in the same class.

If $K_X + \Delta'$ is not assumed \mathbb{R} -Cartier, then we say that (X, Δ') is *potentially* terminal, canonical, etc.

Claim 11.4.2. Assume that (X, Δ) is terminal (resp. klt) and Θ is an effective \mathbb{R} -Cartier divisor. Then $(X, \Delta + \epsilon\Theta)$ is also terminal (resp. klt) for $0 \leq \epsilon \ll 1$. (See (11.23.6) for the other cases.)

We already gave some examples in (1.33) and (1.40); see also Section 2.2 for such surfaces, (2.35) for cones and [Kol13b] for a detailed treatment.

CM properties.

Many of the divisorial sheaves on an lc pair are Cohen-Macaulay (CM for short). [Elk81] proved that canonical singularities are rational. This was generalized by several authors, the following variant is due to [KM98, 5.25] and [Fuj17, 4.14]; see also [Kol13b, 2.88].

THEOREM 11.5. *Let (X, Δ) be a dlt pair over a field of characteristic 0, L a \mathbb{Q} -Cartier \mathbb{Z} -divisor and $D \leq \lfloor \Delta \rfloor$ a reduced \mathbb{Z} -divisor. Then*

- (11.5.1) \mathcal{O}_X is CM,
- (11.5.2) $\mathcal{O}_X(-D - L)$ is CM,
- (11.5.3) $\omega_X(D + L)$ is CM and
- (11.5.4) if $D + L$ is effective then \mathcal{O}_{D+L} is CM. □

We will also need the following generalization; see [Kol11a] or [Kol13b, 7.31].

THEOREM 11.6. *Let (X, Δ) be dlt, D a (not necessarily effective) \mathbb{Z} -divisor and $\Delta' \leq \Delta$ an effective \mathbb{R} -divisor on X such that $D \sim_{\mathbb{Q}} \Delta'$. Then $\mathcal{O}_X(-D)$ is CM.*

If (X, Δ) is lc then frequently \mathcal{O}_X is not CM. The following variant of the above theorems, while much weaker, is quite useful. In increasing generality it was proved by [Ale08, Kol11a, Fuj17]; see also [Kol13b, 7.20] and [Fuj17, Sec.7.1]. (Even stronger results are proved in [AH12].) We state it for semi-log-canonical pairs—to be defined in (11.11)—using the notion of log canonical centers (11.23).

THEOREM 11.7. *Let (X, Δ) be slc and $x \in X$ a point that is not an lc center (11.23). Let D be a Mumford \mathbb{Z} -divisor. Assume that there is an effective \mathbb{R} -divisor $\Delta' \leq \Delta$ such that $D \sim_{\mathbb{R}} \Delta'$. Then*

- (11.7.1) $\text{depth}_x \mathcal{O}_X(-D) \geq \min\{3, \text{codim}_X x\}$ and
- (11.7.2) $\text{depth}_x \omega_X(D) \geq \min\{3, \text{codim}_X x\}$.

Proof. The first claim is proved in [Kol13b, 7.20]. To get the second, note that, working locally, $K_X + \Delta \sim_{\mathbb{Q}} 0$, thus $-(K_X + D) \sim_{\mathbb{Q}} \Delta - \Delta'$ and $\Delta - \Delta' \leq \Delta$ is effective. Thus, by the first part, $\omega_X(D) \cong \mathcal{O}_X(-(-(K_X + D)))$ has $\text{depth} \geq \min\{3, \text{codim}_X x\}$. □

Taking $D = 0$ gives the following important special case, due to [Ale08].

COROLLARY 11.8. *Let (X, Δ) be slc and $x \in X$ a point of codimension ≥ 3 that is not an lc center. Then $\text{depth}_x \mathcal{O}_X \geq 3$ and $\text{depth}_x \omega_X \geq 3$. □*

Semi-log-canonical pairs.

DEFINITION 11.9. Let (R, m) be a local k -algebra and $\text{char } k \neq 2$. We say that $\text{Spec } R$ has a *node* if $\hat{R} \cong (R/m)[[x, y]]/(x^2 - ay^2)$ for some unit $a \in \hat{R}$. (See [Kol13b, 1.41] for the definition of nodes in characteristic 2.)

As a very simple special case of (2.27) or of (10.43), all deformations of a node can be obtained by pull-back from the diagram

$$\begin{array}{ccccc} (x^2 - ay^2 = 0) & \subset & (x^2 - ay^2 + t = 0) & \subset & \mathbb{A}_{xy}^2 \times \mathbb{A}_t^1 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \in & \mathbb{A}_t^1 & = & \mathbb{A}_t^1. \end{array} \quad (11.9.1)$$

If the characteristic is 0 then all non-trivial deformations over $\hat{\mathbb{A}}_t^1$ are of the form

$$\begin{array}{ccccc} (x^2 - ay^2 = 0) & \subset & (x^2 - ay^2 + t^n = 0) & \subset & \hat{\mathbb{A}}_{xy}^2 \times \hat{\mathbb{A}}_t^1 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \in & \hat{\mathbb{A}}_t^1 & = & \hat{\mathbb{A}}_t^1. \end{array} \quad (11.9.2)$$

Thus the total space has canonical singularities; more precisely, Du Val singularities of type A (2.18).

DEFINITION 11.10. Recall that, by Serre's criterion, a scheme X is normal iff it is S_2 and regular at all codimension 1 points. As a weakening of normality, a scheme is called *demi-normal* if it is S_2 and its codimension 1 points are either regular points or nodes.

A 1-dimensional demi-normal variety is a curve C with nodes. It can be thought of as a smooth curve \bar{C} (the normalization of C) together with pairs of points $p_i, p'_i \in \bar{C}$, obtained as the preimages of the nodes. Equivalently, we have the nodal divisor $\bar{D} = \sum_i (p_i + p'_i)$ on \bar{C} plus a fixed point free involution on \bar{D} given by $\tau: p_i \leftrightarrow p'_i$.

We aim to get a similar description for any demi-normal scheme X . Let $\pi: \bar{X} \rightarrow X$ denote the normalization and $D \subset X$ the divisor obtained as the closure of the nodes of X . Set $\bar{D} := \pi^{-1}(D)$ with reduced structure. Then D, \bar{D} are the conductors of π and the induced map $\bar{D} \rightarrow D$ has degree 2 over the generic points. This gives a rational involution on \bar{D} which becomes a regular involution on the normalization

$$\tau: \bar{D}^n \rightarrow \bar{D}^n, \quad (11.10.1)$$

which is not the identity on any irreducible component. We always assume this condition from now on.

It is easy to see [Kol13b, 5.3] that a demi-normal scheme X is uniquely determined by the triple

$$(\bar{X}, \bar{D}, \tau). \quad (11.10.2)$$

However, it is surprising difficult to understand which triples (\bar{X}, \bar{D}, τ) correspond to demi-normal schemes. The solution of this problem in the log canonical case, given in (11.21), is a key result for us.

Let X be a scheme and $j: X^\circ \hookrightarrow X$ the largest open set that is demi-normal. If the normalization $\pi: X^n \rightarrow X$ is finite (for example, X is excellent) then $j_*\mathcal{O}_{X^\circ} \cap \pi_*\mathcal{O}_{X^n}$ is a coherent sheaf of algebras on X . Its spectrum over X is the *demi-normalization* of X , frequently denoted by X^{dn} . Thus we have a factorization

$$\pi: X^n \rightarrow X^{\text{dn}} \xrightarrow{\tau} X, \quad (11.10.3)$$

X^{dn} is demi-normal and τ is an isomorphism over X° .

Roughly speaking, the concept of semi-log-canonical is obtained by replacing ‘normal’ with ‘demi-normal’ in the definition of log canonical (11.4).

DEFINITION 11.11. Let X be a demi-normal scheme with normalization $\pi: \bar{X} \rightarrow X$ and conductors $D \subset X$ and $\bar{D} \subset \bar{X}$. Let Δ be an effective \mathbb{R} -divisor whose support does not contain any irreducible component of D and $\bar{\Delta}$ the divisorial part of $\pi^{-1}(\Delta)$,

The pair (X, Δ) is called *semi-log-canonical* or *slc* if

(11.11.1) $K_X + \Delta$ is \mathbb{R} -Cartier, and

(11.11.2) one of the following equivalent conditions holds

(11.11.2.a) $(\bar{X}, \bar{D} + \bar{\Delta})$ is lc, or

(11.11.2.b) $a(E, X, \Delta) \geq -1$ for every exceptional divisor E over X .

Note that (2.b) is the exact analog of the definition of log canonical given in (11.4). The equivalence of the conditions (2.a) and (2.b) is proved in [Kol13b, 5.10]. More precisely, one can define $a(E, X, \Delta)$ using semi-resolutions and then $a(E, X, \Delta) = a(E, \bar{X}, \bar{D} + \bar{\Delta})$ for every exceptional divisor. This formula suggests that if $D_i \subset D$ is an irreducible component, then we should declare that $a(D_i, X, \Delta) = -1$.

The discrepancy $a(E, X, \Delta)$ is not defined if $K_X + \Delta$ is not \mathbb{R} -Cartier, thus (11.11.2.b) does not make sense unless (11.11.1) holds. By contrast, (11.11.2.a) makes sense if $K_{\bar{X}} + \bar{D} + \bar{\Delta}$ is \mathbb{R} -Cartier, even if $K_X + \Delta$ is not.

Reid’s covering lemma.

This is a method to compare properties of a scheme with properties of its finite ramified covers.

11.12 (Hurwitz formula). The main example is when $\pi: Y \rightarrow X$ is a finite, separable morphism between normal varieties of the same dimension, but we also need the case when $\pi: Y \rightarrow X$ is a finite, separable morphism between demi-normal varieties such that π is étale over the nodes of X . Then

$$K_Y \sim R + \pi^* K_X, \tag{11.12.1}$$

where R is the *ramification divisor* of π . If none of the ramification indices is divisible by the characteristic, then $R = \sum_D (e(D) - 1)D$ where $e(D)$ denotes the ramification index of π along the divisor $D \subset Y$.

Note that if π is quasi-étale, that is, étale outside a subset of codimension ≥ 2 , then $R = 0$, hence $K_Y \sim \pi^* K_X$.

11.13. Let $\pi: Y \rightarrow X$ be a finite, separable morphism as in (11.12) and Δ_X an \mathbb{R} -divisor on X . Set

$$\Delta_Y := -R + \pi^* \Delta_X. \tag{11.13.1}$$

With this choice, (11.12.1) gives that

$$K_Y + \Delta_Y \sim_{\mathbb{Q}} \pi^* (K_X + \Delta_X). \tag{11.13.2}$$

Reid’s covering lemma compares the discrepancies of divisors over X and Y . For precise forms see [Rei80], [KM98, 5.20] or [Kol13b, 2.42-43]. We need the following special cases.

Claim 11.13.3. Using the above notation, assume that Δ_X and Δ_Y are both effective and one of the following holds.

- (a) The characteristic is 0,

- (b) π is Galois and $\deg \pi$ is not divisible by the residue characteristics, or
- (c) $\deg \pi$ is less than the residue characteristics.

Then (X, Δ_X) is klt (resp. lc or slc) iff (Y, Δ_Y) is klt (resp. lc or slc). \square

There are 2 cases when (11.13.3) especially simple.

Special case 11.13.4. If π is quasi-étale then $\Delta_Y = \pi^* \Delta_X$, thus we compare (X, Δ_X) and $(Y, \pi^* \Delta_X)$.

Special case 11.13.5. Let D_X be a reduced divisor on X such that π is étale over $X \setminus D_X$. Set $D_Y := \text{red } \pi^*(D_X)$. Then $D_Y + R = \pi^*(D_X)$, thus we compare $(X, D_X + \Delta_X)$ and $(Y, D_Y + \pi^* \Delta_X)$.

We frequently use cyclic covers.

11.14 (Cyclic covers). See [KM98, 2.49–52] or [Kol13b, Sec.2.3] for details.

Let X be an S_2 -scheme, L a rank 1 sheaf that is locally free in codimension 1 and s a section of $L^{[n]}$, where, as usual, the bracket denotes that we take the double dual of the usual tensor power. These data define a *cyclic cover* or μ_n -*cover* $\pi: Y \rightarrow X$ such that

$$\pi_* \mathcal{O}_Y = \sum_{i=0}^{m-1} L^{[-i]}$$

and

$$\pi_* \omega_{Y/C} \cong \text{Hom}_X(\pi_* \mathcal{O}_Y, \omega_{X/C}) = \sum_{i=0}^{m-1} L^{[i]} \hat{\otimes} \omega_{X/C},$$

where $\hat{\otimes}$ denotes the double dual of the usual tensor product. The morphism π is étale over $x \in X$ iff L is locally free at x , $s(x) \neq 0$ and $\text{char } k(x) \nmid m$. Thus π is quasi-étale iff s is a nowhere zero section, hence $L^{[n]} \cong \mathcal{O}_X$.

These allow us to reduce many questions about \mathbb{Q} -Cartier divisors to Cartier divisors.

PROPOSITION 11.15. *Let (x, X) be a local scheme over a field of characteristic 0 and $\{D_i : i \in I\}$ a finite set of \mathbb{Q} -Cartier divisors that are Cartier in codimension 1. (This is automatic if X is normal.) Then there is a finite, abelian, quasi-étale cover $\pi: \tilde{X} \rightarrow X$ such that the $\pi^* D_i$ are Cartier.*

Furthermore, if (X, Δ) is klt (resp. lc or slc) for some \mathbb{R} -divisor Δ then $(\tilde{X}, \tilde{\Delta} := \pi^ \Delta)$ is also klt (resp. lc or slc). \square*

Adjunction and the different.

Adjunction is a classical method that allows induction on the dimension by lifting information from divisors to the ambient variety.

DEFINITION 11.16 (Poincaré residue map). Let X be a (pure dimensional) CM scheme and $S \subset X$ a subscheme of pure codimension 1. By applying $\mathcal{H}om(\cdot, \omega_X)$ to the exact sequence

$$0 \rightarrow \mathcal{O}_X(-S) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_S \rightarrow 0$$

we get the short exact sequence

$$0 \rightarrow \omega_X \rightarrow \omega_X(S) \xrightarrow{\mathcal{R}_S} \omega_S \rightarrow 0. \quad (11.16.1)$$

The map $\mathcal{R}_S: \omega_X(S) \rightarrow \omega_S$ is called the *Poincaré residue map*.

By taking tensor powers, we get maps

$$\mathcal{R}_S^{\otimes m} : (\omega_X(S))^{\otimes m} \rightarrow \omega_S^{\otimes m},$$

but, if $m(K_X + S)$ and mK_S are Cartier for some $m > 0$ then we really would like to get a corresponding map between the locally free sheaves

$$\omega_X^{[m]}(mS)|_S \xrightarrow{???} \omega_S^{[m]}. \quad (11.16.2)$$

There is no such map in general and one needs a correction term.

DEFINITION 11.17 (Different). Let X be a demi-normal variety over a perfect field, S a reduced divisor on X and Δ an \mathbb{R} -divisor on X . We assume that there are no coincidences, that is, the irreducible components of $\text{Supp } S, \text{Supp } \Delta$ and $\text{Sing } X$ are all different from each other. Let $\pi : \bar{S} \rightarrow S$ denote the normalization.

Then there is a closed subscheme $Z \subset S$ of codimension 1 such that $S \setminus Z$ and $X \setminus Z$ are both smooth along $S \setminus Z$, the restriction $\pi : (\bar{S} \setminus \pi^{-1}Z) \rightarrow (S \setminus Z)$ is an isomorphism and $\text{Supp } \Delta \cap S \subset Z$.

Assume first that Δ is a \mathbb{Q} -divisor and choose $m > 0$ such that $m(K_X + S + \Delta)$ is Cartier then the Poincaré residue map (11.16) gives an isomorphism

$$\mathcal{R}_{S \setminus Z}^m : \pi^* \omega_X^{[m]}(mS + m\Delta)|_{(\bar{S} \setminus \pi^{-1}Z)} \cong \omega_{\bar{S}}^{[m]}|_{(\bar{S} \setminus \pi^{-1}Z)}.$$

Hence there is a unique (not necessarily effective) divisor $\Delta_{\bar{S}}$ on \bar{S} supported on $\pi^{-1}Z$ such that $\mathcal{R}_{S \setminus Z}^m$ extends to an isomorphism

$$\mathcal{R}_{\bar{S}}^m : \pi^* \omega_X^{[m]}(mS + m\Delta)|_{\bar{S}} \cong \omega_{\bar{S}}^{[m]}(\Delta_{\bar{S}}). \quad (11.17.1)$$

We formally divide by m and define the *different* of Δ on \bar{S} as the \mathbb{Q} -divisor

$$\text{Diff}_{\bar{S}}(\Delta) := \frac{1}{m} \Delta_{\bar{S}}. \quad (11.17.2)$$

We can write (11.17.1) in terms of \mathbb{Q} -divisors as

$$(K_X + S + \Delta)|_{\bar{S}} \sim_{\mathbb{Q}} K_{\bar{S}} + \text{Diff}_{\bar{S}}(\Delta). \quad (11.17.3)$$

Note that (11.17.3) has the disadvantage that it indicates only that the two sides are \mathbb{Q} -linearly equivalent, whereas (11.17.1) is a *canonical* isomorphism.

If Δ is an \mathbb{R} -divisor then by (11.34.4) we can write it as $\Delta' + \Delta''$ where where $K_X + S + \Delta$ is \mathbb{Q} -Cartier and Δ'' is \mathbb{R} -Cartier. Then we set

$$\text{Diff}_{\bar{S}}(\Delta) := \text{Diff}_{\bar{S}}(\Delta') + \pi^* \Delta''. \quad (11.17.4)$$

If X, S are smooth then $K_S = (K_X + S)|_S$, hence in this case $\text{Diff}_{\bar{S}}(\Delta) = \pi^* \Delta$.

Let $f : Y \rightarrow X$ be a proper birational morphism, $S_Y := f_*^{-1}S$ and write $K_Y + S_Y + \Delta_Y \sim_{\mathbb{R}} f^*(K_X + S + \Delta)$. Then

$$\text{Diff}_{\bar{S}}(\Delta) = (f|_{S_Y})_* \text{Diff}_{S_Y}(\Delta_Y) \quad (11.17.5)$$

For simplicity, the above definition is stated only for the cases that we mainly use. We will occasionally need that if $(X, S + \Delta)$ is lc (or slc), then the obvious modification of the definition gives $\text{Diff}_S \Delta$, and the two versions are related by the expected formula

$$\text{Diff}_{\bar{S}}(\Delta) + K_{\bar{S}/S} = \pi^* \text{Diff}_S(\Delta). \quad (11.17.5)$$

See [Kol13b, 4.2] for this result and for the most general setting where the different can be defined. The following basic properties of the different are proved in [Kol13b, 4.4–8].

PROPOSITION 11.18. *Using the notation of (11.17) write $\text{Diff}_{\bar{S}}(\Delta) = \sum d_i V_i$ where $V_i \subset \bar{S}$ are prime divisors. Then the following hold.*

- (11.18.1) *If $(X, S + \Delta)$ is lc (or slc) then $(\bar{S}, \text{Diff}_{\bar{S}}(\Delta))$ is lc.*
 (11.18.2) *If $\text{coeff}_D \Delta \in \{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$ for every prime divisor D then the same holds for $\text{Diff}_{\bar{S}}(\Delta)$.*
 (11.18.3) *If S is Cartier outside a codimension 3 subset then $\text{Diff}_{\bar{S}}(\Delta) = \pi^* \Delta$.*
 (11.18.4) *If $K_X + S$ and D are both Cartier outside a codimension 3 subset then $\text{Diff}_{\bar{S}} D$ is a \mathbb{Z} -divisor and $(K_X + S + D)|_{\bar{S}} \sim K_{\bar{S}} + \text{Diff}_{\bar{S}} D$.*

The following facts about codimension 1 behavior of the different can be proved by elementary but somewhat lengthy computations; see [Kol13b, 2.31, 2.36].

LEMMA 11.19. *Let S be a normal surface, $E \subset S$ a reduced curve and $\Delta = \sum d_i D_i$ an effective \mathbb{R} -divisor. Assume that $0 \leq d_i \leq 1$ and $D_i \not\subset \text{Supp } E$ for every i . Let $\pi: F \rightarrow E$ denote the normalization and let $x \in F$ be a point.*

- (11.19.1) *If E is singular at $\pi(x)$ then $\text{coeff}_x \text{Diff}_F(\Delta) \geq 1$ and equality holds iff E has a node at $\pi(x)$ and $\pi(x) \notin \text{Supp } \Delta$.*
 (11.19.2) *If $\pi(x) \in D_i$ then $\text{coeff}_x \text{Diff}_F(\Delta) \geq d_i$. □*

The next Theorem—whose first part is proved in [Kol92b, 17.4] and second part in [Kaw07]—is frequently referred to as *adjunction* if we assume something about X and obtain conclusions about S , or *inversion of adjunction* if we assume something about S and obtain conclusions about X . See [Kol13b, 4.9] for a proof of a more precise version. The last statement uses the notion of minimal log discrepancy, to be discussed in (11.22).

THEOREM 11.20. *Let X be a normal variety over a field of characteristic 0 and S a reduced divisor on X with normalization $\pi_S: \bar{S} \rightarrow S$. Let Δ be an effective \mathbb{R} -divisor that has no irreducible components in common with S and such that $K_X + S + \Delta$ is \mathbb{R} -Cartier. Then*

- (11.20.1) *$(\bar{S}, \text{Diff}_{\bar{S}}(\Delta))$ is klt iff $(X, S + \Delta)$ is plt in a neighborhood of S and*
 (11.20.2) *$(\bar{S}, \text{Diff}_{\bar{S}}(\Delta))$ is lc iff $(X, S + \Delta)$ is lc in a neighborhood of S .*
 (11.20.3) *For any irreducible subset $Z \subset \bar{S}$, the minimal log discrepancy (11.22) satisfies*

$$\text{mld}(Z, \bar{S}, \text{Diff}_{\bar{S}}(\Delta)) \leq \text{mld}(\pi_S(Z), X, S + \Delta),$$

provided the latter is ≤ 1 . □

Characterization of slc pairs.

Let (X, Δ) be an slc pair. Let $\pi: \bar{X} \rightarrow X$ be the normalization, $\bar{D} \subset \bar{X}$ the conductor, $\bar{\Delta}$ the divisorial part of $\pi^{-1}(\Delta)$ and τ the involution on \bar{D}^n constructed in (11.10). Thus we obtain a map

$$(X, \Delta) \mapsto (\bar{X}, \bar{D} + \bar{\Delta}, \tau)$$

from slc pairs to lc pairs with the extra involution on \bar{D}^n . As we noted in (11.10.2), this map is an injection. That is, $(\bar{X}, \bar{D} + \bar{\Delta}, \tau)$ uniquely determines (X, Δ) . The following theorem, proved in [Kol16b] and [Kol13b, 5.13], describes the image. (See also (6.2) for an instructive special case.)

THEOREM 11.21. *Over a field of characteristic 0, normalization gives a one-to-one correspondence:*

$$\left\{ \begin{array}{l} \text{Proper slc pairs} \\ (X, \Delta) \text{ such that} \\ K_X + \Delta \text{ is ample.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Proper lc pairs } (\bar{X}, \bar{D} + \bar{\Delta}) \text{ plus} \\ \text{an involution } \tau \text{ of } (\bar{D}^n, \text{Diff}_{\bar{D}^n} \bar{\Delta}) \\ \text{such that } K_{\bar{X}} + \bar{D} + \bar{\Delta} \text{ is ample.} \end{array} \right\}$$

(As in (11.10.1), we assume that τ is not the identity on any irreducible component of \bar{D} .) \square

Minimal log discrepancy and log centers.

DEFINITION 11.22. Let (X, Δ) be an slc pair and $W \subset X$ an irreducible subset. The *minimal log discrepancy* of W is defined as the infimum of the numbers $1 + a(E, X, \Delta)$ where E runs through all divisors over X such that $\text{center}_X(E) = W$. It is denoted by

$$\text{mld}(W, X, \Delta) \quad \text{or by} \quad \text{mld}(W) \tag{11.22.1}$$

if the choice of (X, Δ) is clear. Note that if W is an irreducible divisor on X and $W \not\subset \text{Sing } X$ then

$$\text{mld}(W, X, \Delta) = 1 - \text{coeff}_W \Delta. \tag{11.22.2}$$

If $W \subset X$ is a closed subset with irreducible components W_i then we set

$$\text{mld}(W, X, \Delta) = \max_i \text{mld}(W_i, X, \Delta). \tag{11.22.3}$$

If (X, Δ) is slc then, by definition, $\text{mld}(W, X, \Delta) \geq 0$ for every W . The subvarieties with $\text{mld}(W, X, \Delta) = 0$ play a key role in understanding (X, Δ) .

DEFINITION 11.23. Let (X, Δ) be an slc pair. An irreducible subvariety $W \subset X$ is a *log canonical center* or *lc center* of (X, Δ) if $\text{mld}(W, X, \Delta) = 0$. Equivalently, if there is a divisor E over X such that $a(E, X, \Delta) = -1$ and $\text{center}_X E = W$. Log canonical centers have many useful properties.

- (11.23.1) There are only finitely many lc centers.
- (11.23.2) Any union of lc centers is seminormal and Du Bois; see (11.25.1–2).
- (11.23.3) Any intersection of lc centers is also a union of lc centers; see [Amb03, Fuj17, Amb11] or (11.25.4).
- (11.23.4) If (X, Δ) is snc then the lc centers of (X, Δ) are exactly the *strata* of $\Delta_{=1}$, that is, the irreducible components of the various intersections $D_{i_1} \cap \cdots \cap D_{i_s}$ where the D_{i_k} appear in Δ with coefficient 1, see [Kol13b, 2.11]. More generally, this also holds if (X, Δ) is dlt; see [Fuj07, Sec.3.9] or [Kol13b, 4.16].
- (11.23.5) At codimension 2 normal points, the union of lc centers is either smooth or has a node; see [Kol13b, 2.31].
- (11.23.6) Assume that (X, Δ) is slc and Θ is an effective \mathbb{Q} -Cartier divisor. Then $(X, \Delta + \epsilon\Theta)$ is slc for $0 < \epsilon \ll 1$ iff $\text{Supp } \Theta$ does not contain any log canonical center of (X, Δ) .
- (11.23.7) Assume that (X, Δ) is slc and $\epsilon\Theta \leq \Delta$ is an effective \mathbb{Q} -Cartier divisor. Then $\text{Supp } \Theta$ does not contain any log canonical center of $(X, \Delta - \epsilon\Theta)$, cf. [KM98, 2.27].

DEFINITION 11.24. Let (X, Δ) be an slc pair. An irreducible subvariety $W \subset X$ is a *log center* of (X, Δ) if $\text{mld}(W, X, \Delta) < 1$.

Building on earlier results of [Amb03, Fuj17, Amb11], part 1 of the following theorem is proved in [KK10] and the rest in [Kol14]; see also [Kol13b, Chap.7].

THEOREM 11.25. *Let (X, Δ) be an slc pair and $Z, W \subset X$ closed, reduced subsets.*

(11.25.1) *If $\text{mld}(Z, X, \Delta) = 0$ then Z is Du Bois (cf. (2.62) or [Kol13b, 6.32]).*

(11.25.2) *If $\text{mld}(Z, X, \Delta) < \frac{1}{6}$ then Z is seminormal (10.72).*

(11.25.3) *If $\text{mld}(Z, X, \Delta) + \text{mld}(W, X, \Delta) < \frac{1}{2}$ then $Z \cap W$ is reduced.*

(11.25.4) $\text{mld}(Z \cap W, X, \Delta) \leq \text{mld}(Z, X, \Delta) + \text{mld}(W, X, \Delta).$ □

11.2. Relative canonical models and modifications

We used many times the existence of canonical models in the relative setting.

DEFINITION 11.26. Let (Y, Δ_Y) be an lc pair and $p_Y : Y \rightarrow S$ a proper morphism. We say that (Y, Δ_Y) is a *canonical model over S* if $K_Y + \Delta_Y$ is p_Y -ample.

Let (X, Δ) be an lc pair and $p : X \rightarrow S$ a proper morphism. We say that (X^c, Δ^c) is a *canonical model* of (X, Δ) over S if there is a birational map

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X^c \\ p \searrow & & \swarrow p^c \\ & S & \end{array} \quad (11.26.1)$$

such that

(11.26.2) ϕ is a birational contraction, that is ϕ^{-1} has no exceptional divisors,

(11.26.3) $\Delta^c = \phi_* \Delta$,

(11.26.4) (X^c, Δ^c) is a canonical model over S , and

(11.26.5) $\phi_* \mathcal{O}_X(mK_{X/S} + \lfloor m\Delta \rfloor) = \mathcal{O}_{X^c}(mK_{X^c/S} + \lfloor m\Delta^c \rfloor)$ for every $m \geq 0$.

For \mathbb{Q} -divisors we have the following direct generalization of (1.38).

PROPOSITION 11.27. *Let (X, Δ) be an lc pair and $p : X \rightarrow S$ a proper morphism. Assume that X is irreducible and Δ is a \mathbb{Q} -divisor. Then (X, Δ) has a canonical model over S iff*

(11.27.1) *the generic fiber is of general type, and*

(11.27.2) *the relative canonical algebra $\bigoplus_{m \geq 0} p_* \mathcal{O}_X(mK_{X/S} + \lfloor m\Delta \rfloor)$ is finitely generated*

If these hold then the canonical model is

$$X_S^c := \text{Proj}_S \bigoplus_{m \geq 0} p_* \mathcal{O}_X(mK_{X/S} + \lfloor m\Delta \rfloor). \quad (11.27.3)$$

The main conjecture on canonical models says that the relative canonical models always exist. The following known cases are the most important for us.

THEOREM 11.28. *Let (X, Δ) be an lc pair over a field of characteristic 0 and $p : X \rightarrow S$ a proper morphism, S irreducible. The relative canonical model exists in the following cases.*

(11.28.1) [BCHM10] (X, Δ) is klt and the generic fiber is of general type.

(11.28.2) [HX13, HX16] (X, Δ) is dlt, the relative canonical model exists over an open $S^\circ \subset S$, and every lc center intersects $p^{-1}(S^\circ)$. □

DEFINITION 11.29. (cf. [Kol13b, 1.32]) Let X be a normal variety and Δ an effective \mathbb{R} -divisor on X . An *lc modification* of (X, Δ) is a proper, birational morphism $\pi : (X^c, \Delta^c + E^c) \rightarrow (X, \Delta)$ where $\Delta^c := \pi_*^{-1}\Delta$, E^c is the reduced π -exceptional divisor, $(X^c, \Delta^c + E^c)$ is lc and $K_{X^c} + \Delta^c + E^c$ is π -ample.

An lc modification is unique. As for its existence, we clearly need to assume that $\text{coeff } \Delta \subset [0, 1]$. Conjecturally, this is the only necessary condition, but this is known only in some cases.

PROPOSITION 11.30. [OX12] *Let (X, Δ) be a normal pair of finite type over a field of characteristic 0. Assume that $\text{coeff } \Delta \subset [0, 1]$ and $K_X + \Delta$ is \mathbb{R} -Cartier. Then the lc modification of (X, Δ) exists. \square*

PROPOSITION 11.31. [Kol18a, Prop.19] *Let (X, Δ) be a potentially lc pair of finite type over a field of characteristic 0. Then*

- (11.31.1) *it has a projective, small, lc modification $\pi : (X^c, \Delta^c) \rightarrow (X, \Delta)$,*
- (11.31.2) *π is a local isomorphism at every lc center of (X^c, Δ^c) and*
- (11.31.3) *π is a local isomorphism over $x \in X$ iff $K_X + \Delta$ is \mathbb{R} -Cartier at x . \square*

One of the difficulties in dealing with slc pairs is that analogous small modifications need not exist for them; see [Kol13b, 1.40].

Vanishing theorems.

We use various generalizations of Kodaira's vanishing theorem. For most applications the Kawamata-Viehweg variants are sufficient, see [KM98, Secs.2.4–5] for an introductory treatment. A few times we need the stronger Ambro-Fujino version, proved in [Amb03] and [Fuj14, 1.10]. See also [Fuj17, Sec.5.7] and [Fuj17, 6.3.5], where it is called a Reid-Fukuda-type vanishing theorem.

DEFINITION 11.32. Let (X, Δ) be an slc pair, $f : X \rightarrow S$ a proper morphism and L an \mathbb{R} -Cartier, f -nef divisor on X . Then L is called *log f -big* if $L|_W$ is big on the generic fiber of $f|_W : W \rightarrow f(W)$ for every slc center W of (X, Δ) , and also for every irreducible component $W \subset X$.

THEOREM 11.33. *Let (X, Δ) be an slc pair and D a Mumford \mathbb{Z} -divisor on X . Let $f : X \rightarrow S$ be a proper morphism. Assume that $D \sim_{\mathbb{R}} K_X + L + \Delta$, where L is \mathbb{R} -Cartier, f -nef and log f -big. Then*

$$R^i f_* \mathcal{O}_X(D) = 0 \quad \text{for } i > 0. \quad \square$$

11.3. \mathbb{R} -divisors

It is easy to see that, on a \mathbb{Q} -factorial variety, \mathbb{R} -divisors behave very much like \mathbb{Q} -divisors. The same holds in general, but it needs a little more work. The basics are discussed in [Laz04, Sec.1.3], but most other facts are scattered in the literature; see for example [Kol13b, 2.21] or [Bir17, FM21].

11.34 (\mathbb{R} -divisors). Let X be a reduced, S_2 scheme and $\Delta = \sum a_i D_i$ a Mumford \mathbb{R} -divisor. There is a unique such way of writing Δ if the D_i are irreducible, distinct and the $a_i \neq 0$. The \mathbb{Q} -vector space spanned by coefficients is denoted by $\text{CoSp}(\Delta) = \sum_i \mathbb{Q} \cdot a_i \subset \mathbb{R}$.

We say that Δ is \mathbb{R} -Cartier if it can be written as a linear combination of Cartier divisors. Two \mathbb{R} -divisors are \mathbb{R} -linearly equivalent, denoted by $\Delta_1 \sim_{\mathbb{R}} \Delta_2$,

if $\Delta_1 - \Delta_2$ is a linear combination of principal divisors. (11.34.2.d) shows that for \mathbb{Q} -divisors we do not get anything new.

Let $\sigma : \mathbb{R} \rightarrow \mathbb{Q}$ be a \mathbb{Q} -linear map. It extends to a \mathbb{Q} -linear map from \mathbb{R} -divisors to \mathbb{Q} -divisors as $\sigma(\sum d_i D_i) := \sum \sigma(d_i) D_i$.

Claim 11.34.1. Let $\sigma : \mathbb{R} \rightarrow \mathbb{Q}$ be a \mathbb{Q} -linear map. Then

- (a) $\text{Supp}(\sigma(D)) \subset \text{Supp}(D)$,
- (b) if $D_1 \sim_{\mathbb{R}} D_2$ then $\sigma(D_1) \sim_{\mathbb{Q}} \sigma(D_2)$, and
- (c) if D is \mathbb{R} -Cartier then $\sigma(D)$ is \mathbb{Q} -Cartier.

Proof. The first claim is clear from the definition. If $D_1 - D_2 = \sum c_i(f_i)$ then $\sigma(D_1) - \sigma(D_2) = \sum \sigma(c_i)(f_i)$, showing (b), which in turn implies (c). \square

Let $D = \sum_{i \in I} b_i B_i$ be an \mathbb{R} -divisor with B_i irreducible. Choosing a \mathbb{Q} -basis of $\langle b_i : i \in I \rangle_{\mathbb{Q}} \subset \mathbb{R}$, we can write $D = \sum d_j D_j$ where the D_j are \mathbb{Q} -divisors (usually not irreducible) and $d_j \in \mathbb{R}$ linearly independent over \mathbb{Q} . Note that $\text{Supp } D_j \subset \text{Supp } D$, but the D_j depend on the choice of the basis. Nonetheless, they inherit many properties of D .

Claim 11.34.2. Let D_i be \mathbb{Q} -divisors and $d_i \in \mathbb{R}$ linearly independent over \mathbb{Q} . Then

- (a) $\sum d_i D_i$ is \mathbb{R} -Cartier iff each D_i is \mathbb{Q} -Cartier.
- (b) $\sum d_i D_i \sim_{\mathbb{R}} 0$ iff $D_i \sim_{\mathbb{Q}} 0$ for every i .
- (c) If X is proper then $\sum d_i D_i \equiv 0$ iff $D_i \equiv 0$ for every i .
- (d) A \mathbb{Q} -divisor D_i is \mathbb{R} -Cartier iff it is \mathbb{Q} -Cartier.
- (e) $D_1 \sim_{\mathbb{R}} D_2$ iff $D_1 \sim_{\mathbb{Q}} D_2$.

Proof. If the $d_i \in \mathbb{R}$ linearly independent then we can choose σ_i such that $\sigma_i(d_i) = 1$ and $\sigma_i(d_j) = 0$ for $i \neq j$. Then $\sigma_i(D) = D_i$, thus (11.34.1) shows (a) and (b).

For (c) assume that $\sum d_i D_i \equiv 0$ and let $C \subset X$ be a curve. Then $\sum d_i (D_i \cdot C) = 0$. Since $(D_i \cdot C) \in \mathbb{Q}$ and the d_i are linearly independent, we get that $(D_i \cdot C) = 0$ for every i . Applying (a) to D gives (d). Applying (b) $D_1 - D_2$ gives (e). \square

Corollary 11.34.3. Let Θ be a Mumford \mathbb{R} -divisor and $\{d_i\}$ a \mathbb{Q} -basis of $\text{CoSp}(\Delta)$. Then we get a unique representation $\Theta = \sum d_i D_i$ where the D_i are \mathbb{Q} -divisors. If Θ is \mathbb{R} -Cartier, then the D_i are \mathbb{Q} -Cartier. \square

Corollary 11.34.4. Let Δ be a Mumford \mathbb{R} -divisor and $\{d'_i\}$ a \mathbb{Q} -basis of $\mathbb{Q} + \text{CoSp}(\Delta)$ such that $\sum d'_i = 1$. Then we get a unique representation $\Delta = \sum d'_i D_i$ where the D_i are \mathbb{Q} -divisors. If $K_X + \Delta$ is \mathbb{R} -Cartier, then $K_X + D_i$ are \mathbb{Q} -Cartier.

Proof. Note that $K_X + \Delta = \sum d'_i (K_X + D_i)$, so the last assertion follows from (11.34.2). \square

Next we show that \mathbb{R} -divisors can be approximated by \mathbb{Q} -divisors in a way that many properties are preserved. We start with some general comments on vector spaces and field extensions. At the end we care only about $\mathbb{R} \supset \mathbb{Q}$.

DEFINITION–LEMMA 11.35. Let K/k be a field extension, V a k -vector space and $w \in V \otimes_k K$. The *linear k -envelope* of w , denoted by $\text{LEnv}_k(w) \subset V$, is the smallest vector subspace such that $w \in \text{LEnv}_k(w) \otimes_k K$. Let $\sigma : K \rightarrow k$ denote a k -linear map. Then $\text{LEnv}_k(w)$ is spanned by any of the following 3 sets.

(11.35.1) All $(1_V \otimes \sigma)(w)$.

(11.35.2) All $\sum \sigma(c_i)v_i$, where $v_i \in V$ is a basis and $w = \sum c_i v_i$.

(11.35.3) All $\sum_i a_{ij}v_i$, where $e_j \in K$ is a k -basis and $w = \sum_{ij} a_{ij}e_j v_i$.

The *affine k -envelope* of w , denoted by $\text{AEnv}_k(w) \subset V$, is the smallest affine-linear subspace such that $w \in \text{AEnv}_k(w) \otimes_k K$. Let $\sigma : K \rightarrow k$ denote a k -linear map such that $\sigma(1) = 1$. Then $\text{AEnv}_k(w)$ is spanned by any of the following 3 sets.

(11.35.4) All $(1_V \otimes \sigma)(w)$.

(11.35.5) All $\sum \sigma(c_i)v_i$, where $v_i \in V$ is a basis and $w = \sum c_i v_i$.

(11.35.6) All $\sum_i a_{ij}v_i$, where $e_j \in K$ is a k -basis such that $e_1 = 1$ and $w = \sum_{ij} a_{ij}e_j v_i$.

11.36 (Approximating by rational simplices). Fix real numbers d_1, \dots, d_m and consider the \mathbb{Q} -vector space W with basis $\mathfrak{d}_1, \dots, \mathfrak{d}_m$. Set $\mathfrak{d} := \sum d_i \mathfrak{d}_i \in W_{\mathbb{R}}$ and $V := \text{AEnv}_{\mathbb{Q}}(\mathfrak{d})$. We construct a sequence of simplices

$$V \supset S_1 \supset S_2 \supset \dots \quad \text{such that} \quad \cap_n S_n = \{\mathfrak{d}\}.$$

Set $S_0 := V$. For each $n \in \mathbb{N}$ the cubes of the lattice $\frac{1}{n}\mathbb{Z}^m$ give a chamber decomposition of $W_{\mathbb{R}}$. There is a smallest chamber C_n that contains \mathfrak{d} , and then \mathfrak{d} is an interior point of $C_n \cap S_{n-1}$. Thus \mathfrak{d} can be written as a convex linear combination of $\dim V + 1$ vertices of $V_{\mathbb{R}} \cap C_n$; denote them by \mathfrak{d}_j^n . These span S_n . By (11.35), there are \mathbb{Q} -linear maps $\sigma_j^n : \mathbb{R} \rightarrow \mathbb{Q}$ such that $\mathfrak{d}_j^n = \sigma_j^n(\mathfrak{d})$. We can thus write

(11.36.1) $\mathfrak{d} = \sum_j \lambda_j^n \mathfrak{d}_j^n$, where

(11.36.2) $\mathfrak{d}_j^n = \sum_i \sigma_j^n(d_i) \mathfrak{d}_i$,

(11.36.3) $\sum_j \lambda_j^n = 1$ and $\sum_j \lambda_j^n \sigma_j^n(d_i) = d_i \quad \forall i$,

(11.36.4) $\lim_{n \rightarrow \infty} \mathfrak{d}_j^n = \mathfrak{d} \quad \forall j$, and

(11.36.5) for fixed n , the λ_j^n are linearly independent over \mathbb{Q} . (To see this, note that 1 and the d_i are \mathbb{Q} -linear combinations of the λ_j^n for fixed n .)

Remark 11.36.6. The choice of the vertices is not unique, but once we choose them, the linear maps σ_j^n and the constants λ_j^n are unique. Thus, from now on, we view σ_j^n and λ_j^n as depending only on $j, n \in \mathbb{N}$ and $d_1, \dots, d_m \in \mathbb{R}$. Note that these are not continuous functions of the d_i , even the number of the j -indices varies discontinuously with d_1, \dots, d_m .

Also, we only care about the restriction of the σ_j^n to $\mathbb{Q} + \sum \mathbb{Q}d_i$, so there are no axiom of choice issues.

PROPOSITION 11.37 (Convex approximation of \mathbb{R} -divisors I). *Let X be a reduced, S_2 scheme and $\Theta = \sum_i d_i D_i$ a Mumford \mathbb{R} -divisor, where the D_i are \mathbb{Q} -divisors. Let σ_j^n and λ_j^n be as in (11.36) and set $\Theta_j^n := \sum \sigma_j^n(d_i) D_i$. Then*

(11.37.1) $\Theta = \sum_j \lambda_j^n \Theta_j^n$.

(11.37.2) *Let $E \subset X$ be a prime divisor. Then*

(11.37.2.a) $\text{coeff}_E \Theta_j^n = \text{coeff}_E \Theta$ if $\text{coeff}_E \Theta \in \mathbb{Q}$, and

(11.37.2.b) $\lim_{n \rightarrow \infty} \text{coeff}_E \Theta_j^n = \text{coeff}_E \Theta$ in general.

(11.37.3) Θ is effective iff the Θ_j^n are effective for every j for $n \gg 1$ (and then they have the same support).

Assume next that Θ is \mathbb{R} -Cartier. Then

(11.37.4) *The Θ_j^n are \mathbb{Q} -Cartier.*

(11.37.5) *Let E be a prime divisor over X . Then*

- (11.37.5.a) $\text{coeff}_E \Theta_j^n = \text{coeff}_E \Theta$ if $\text{coeff}_E \Theta \in \mathbb{Q}$, and
 (11.37.5.b) $\lim_{n \rightarrow \infty} \text{coeff}_E \Theta_j^n = \text{coeff}_E \Theta$ in general.
 (11.37.6) Let C be a curve on X . Then
 (11.37.6.a) $(C \cdot \Theta_j^n) = (C \cdot \Theta)$ if $(C \cdot \Theta) \in \mathbb{Q}$, and
 (11.37.6.b) $\lim_{n \rightarrow \infty} (C \cdot \Theta_j^n) = (C \cdot \Theta)$ in general.
 (11.37.7) Θ is ample iff the Θ_j^n are ample for every j for $n \gg 1$.

Proof. (1) is a formal consequence of (11.36.2), while (2.b) follows from (11.36.3).

For (2.a), assume that $\text{coeff}_E \Theta =: c \in \mathbb{Q}$. Note that $\sum_i x_i \text{coeff}_E D_i = c$ defines a rational hyperplane in W (as in (11.36)). It contains \mathfrak{d} , hence also V and the other \mathfrak{d}_j^n . The Θ_j^n are the images of the \mathfrak{d}_j^n .

By (11.36.4) the λ_j^n are linearly independent over \mathbb{Q} . Thus, if Θ is \mathbb{R} -Cartier then the Θ_j^n are \mathbb{Q} -Cartier by (11.34.2.a), proving (4). Also, in this case $\text{coeff}_E \Theta$ makes sense for divisors over X and same for the intersection numbers $(C \cdot \Theta)$. The proofs of (5–7) are now the same as for (2). \square

PROPOSITION 11.38 (Convex approximation of \mathbb{R} -divisors II). *Let X be a demi-normal variety and $\Delta = \sum d_i D_i$ a Mumford \mathbb{R} -divisor, where the D_i are \mathbb{Q} -divisors. Assume that $K_X + \Delta$ is \mathbb{R} -Cartier. Let σ_j^n and λ_j^n be as in (11.36) and set $\Delta_j^n := \sum \sigma_j^n(d_i) D_i$.*

- (11.38.1) $\Delta = \sum_j \lambda_j^n \Delta_j^n$ and $K_X + \Delta = \sum_j \lambda_j^n (K_X + \Delta_j^n)$.
 (11.38.2) Δ is effective iff the Δ_j^n are effective for every j for $n \gg 1$ (and then they have the same support).
 (11.38.3) $K_X + \Delta_j^n$ are \mathbb{Q} -Cartier.
 (11.38.4) $K_X + \Delta$ is ample iff the $K_X + \Delta_j^n$ are ample for every j for $n \gg 1$.
 (11.38.5) Let E be a prime divisor over X . Then
 (11.38.5.a) $a(E, X, \Delta_j^n) = a(E, X, \Delta)$ if $a(E, X, \Delta) \in \mathbb{Q}$, and
 (11.38.5.b) $\lim_{n \rightarrow \infty} a(E, X, \Delta_j^n) = a(E, X, \Delta)$ in general.
 (11.38.6) Let C be a curve on X . Then
 (11.38.6.a) $(C \cdot (K_X + \Delta_j^n)) = (C \cdot (K_X + \Delta))$ if $(C \cdot (K_X + \Delta)) \in \mathbb{Q}$, and
 (11.38.6.b) $\lim_{n \rightarrow \infty} (C \cdot (K_X + \Delta_j^n)) = (C \cdot (K_X + \Delta))$ in general.
 Assume next that (X, Δ) has a log resolution.
 (11.38.6) (X, Δ) is lc (resp. dlt or klt) iff (X, Δ_j^n) is lc (resp. dlt or klt) for every j for $n \gg 1$.

Proof. (1–2) follow directly from (11.37) and (3) follows from (1) and (11.37.4). Since ampleness is an open condition, (3) implies (4).

The proofs of (5) and (6) are the same as the proof of (11.37.2.a). If (X, Δ) has a log resolution then being lc (resp. dlt or klt) can be read off from the discrepancies of finitely many divisors on it, hence (5) implies (7). \square

In the slc case, we have the following remarkable sharpening, which we state without proof.

COMPLEMENT 11.39. [HLS20, 5.6] *In (11.38) assume in addition that $(X, \Delta = \sum d_i D_i)$ is slc. Then we can choose the σ_j^n and λ_j^n to depend only on (d_1, \dots, d_r) and the dimension. \square*

Pluricanonical bundles for \mathbb{R} -divisors.

LEMMA 11.40. *Let $(X, \Delta = \sum d_i D_i)$ be an lc pair. Then there are infinitely many $m \in \mathbb{N}$, depending only on (d_1, \dots, d_r) and the dimension, and lc pairs (X, Θ_m) such that $\{m\Delta\} \leq \Theta_m$, and (X, Θ_m) have the same lc centers as (X, Δ) .*

Proof. As in (11.38) write $\Delta = \sum \lambda_i \Delta_i$ where $\sum \lambda_i = 1$. By (11.39) the coefficients of the Δ_i can be chosen depending only on (d_1, \dots, d_r) and the dimension. Let D be a common denominator for the Δ_i . For every $\delta > 0$ there are $m \in \mathbb{N}$ such that, for every i ,

$$m\lambda_i = \eta_{im} + r_{im} \quad \text{where } r_{im} \in D\mathbb{Z} \text{ and } \sum_i |\eta_{im}| \leq \delta.$$

Then

$$\{(m+1)\Delta\} = \Delta + \sum \eta_{im} \Delta_i = \sum (\lambda_i + \eta_{im}) \Delta_i.$$

If $c := \sum_i \eta_{im} \leq 0$ then

$$\sum (\lambda_i + \eta_{im}) \Delta_i \leq \sum \frac{1}{1+c} (\lambda_i + \eta_{im}) \Delta_i =: \Theta_{m+1} \quad \text{and} \quad \sum_i \frac{1}{1+c} (\lambda_i + \eta_{im}) = 1.$$

Furthermore, by explicit computation,

$$\left| \frac{1}{1+c} (\lambda_i + \eta_{im}) - \lambda_i \right| \leq \epsilon \lambda_i \quad \text{if } \delta \leq \epsilon \cdot \min_i \{(1 - \lambda_i - \epsilon \lambda_i)^{-1}\}.$$

Thus (X, Θ_{m+1}) is lc and $\{(m+1)\Delta\} \leq \Theta_{m+1}$.

If $\sum \eta_{im} > 0$ then find $m' < M \in \mathbb{N}$ such that $\sum |\eta_{iM}| < -\sum \eta_{im'}$. Then $\eta_{i(M-m')} = \eta_{iM} - \eta_{im'}$ and so $\sum \eta_{i(M-m')} < 0$. Thus $m = 1 + M - m'$ works as before. \square

COROLLARY 11.41. *Let $(X, \Delta = \sum d_i D_i)$ be an lc pair. Then there are infinitely many $m \in \mathbb{N}$, depending only on (d_1, \dots, d_r) and the dimension, such that*

$$\text{depth}_x \omega_X^{[m]}(\lfloor m\Delta \rfloor) \geq \min\{3, \text{codim}_X x\}, \quad (11.41.1)$$

whenever x is not an lc center of (X, Δ) .

Complement 11.41.2. If $K_X + \Delta$ is \mathbb{Q} -Cartier, then those $m \in \mathbb{N}$ for which (11.41.1) holds form a periodic subset, thus it has positive density. The proof shows that we always get a set of positive density, but usually it is not periodic. In fact, it does not even contain any periodic subset, see (11.41.3).

Proof. Note that $mK_X + \lfloor m\Delta \rfloor + \{m\Delta\} \sim_{\mathbb{R}} 0$. Take any $m \in \mathbb{N}$ for which (11.40) holds. Then $\{m\Delta\} \leq \Theta_m$ and (X, Θ_m) has the same lc centers as (X, Δ) . We can thus apply (11.7) to $-D = mK_X + \lfloor m\Delta \rfloor$ to get our claim. \square

Example 11.41.3. Let $X \subset \mathbb{A}^4$ be the quadric cone and $|A|, |B|$ the 2 families of planes on X . Fix $r \in \mathbb{N}$ and for $0 < c \leq 1/r$ consider the pair

$$(X, \Delta_c := B + cA_1 + \dots + cA_r + (1 - rc)A_0).$$

Then (X, Δ_c) is canonical, and we compute that

$$r\{mc\} + \{m(1 - rc)\} = \begin{cases} 1 & \text{if } \{mc\} \leq 1/r, \text{ and} \\ \geq 2 & \text{otherwise.} \end{cases}$$

Therefore

$$\mathcal{O}_X(\lfloor m\Delta_c \rfloor) \cong \begin{cases} \mathcal{O}_X(-A) & \text{if } \{mc\} \leq 1/r, \text{ and} \\ \mathcal{O}_X(-dA) & \text{for some } d \geq 2 \text{ otherwise.} \end{cases}$$

An easy computation as in [Kol13b, 3.15.2] shows that $\mathcal{O}_X(\lfloor m\Delta_c \rfloor)$ is CM iff $\{mc\} \leq 1/r$. If c is irrational, then the set $\{m : \{mc\} \leq 1/r\}$ has no periodic subsets.

Ampleness criteria.

THEOREM 11.42 (Asymptotic Riemann-Roch). *Let X be a normal, proper algebraic space of dimension n and D a nef, \mathbb{R} -divisor. Then*

$$\begin{aligned} h^0(X, \mathcal{O}_X(\lfloor mD \rfloor)) &= \frac{m^n}{n!}(D^n) + O(m^{n-1}), \quad \text{and} \\ h^0(X, \mathcal{O}_X(\lceil mD \rceil)) &= \frac{m^n}{n!}(D^n) + O(m^{n-1}). \end{aligned} \quad (11.42.1)$$

Proof. By Chow's lemma and (11.49) we may assume that X is projective. Write $D = \sum a_i A_i$ where the A_i are effective, ample \mathbb{Z} -divisors and $a_i \in \mathbb{R}$. Note that

$$\sum \lfloor ma_i \rfloor A_i \leq \lfloor mD \rfloor \leq mD \leq \lceil mD \rceil \leq H + \sum \lceil ma_i \rceil A_i, \quad (11.42.2)$$

for any H ample and effective. It is thus enough to prove that (11.42.1) holds for the 2 divisors on the sides of (11.42.2) for suitable H . Note that

$$\sum \lceil ma_i \rceil A_i \sim_{\mathbb{R}} \sum (\lceil ma_i \rceil - ma_i) A_i + mD.$$

Now choose H_i nef such that $H_i + A_i$ is nef, then $\sum H_i + \sum \lceil ma_i \rceil A_i$ is nef for every $m \geq 0$. Next choose L such that (11.42.4) holds (with $F = \mathcal{O}_X$), and $L + \sum H_i + \sum A_i$ is linearly equivalent to an irreducible divisor B . Set $H = L + \sum H_i$. Then, by Riemann-Roch,

$$h^0(X, \mathcal{O}_X(H + \sum \lceil ma_i \rceil A_i)) = \chi(X, \mathcal{O}_X(H + \sum \lceil ma_i \rceil A_i)) = \frac{m^n}{n!}(D^n) + O(m^{n-1}).$$

Restricting $\mathcal{O}_X(H + \sum \lceil ma_i \rceil A_i)$ to B , the kernel is

$$\mathcal{O}_X(\sum \lceil ma_i \rceil A_i - \sum A_i) \subset \mathcal{O}_X(\sum \lfloor ma_i \rfloor A_i),$$

(the 2 are equal iff none of the ma_i are integers). Thus

$$h^0(X, \mathcal{O}_X(H + \sum \lceil ma_i \rceil A_i)) - h^0(X, \mathcal{O}_X(\sum \lfloor ma_i \rfloor A_i))$$

is at most $h^0(B, \mathcal{O}_B(H|_B + \sum \lfloor ma_i \rfloor A_i|_B))$, and the latter is bounded by $O(m^{n-1})$ using (11.42.3). \square

Matsusaka inequality 11.42.3. Let X be a proper variety of dimension n , L a nef and big \mathbb{Z} -divisor and D a Weil \mathbb{Z} -divisor giving a dominant map $|D| : X \dashrightarrow Z$. Then

$$h^0(X, \mathcal{O}_X(D)) \leq \frac{(D \cdot L^{n-1})^{\dim Z}}{(L^n)^{\dim Z - 1}} + \dim Z.$$

See [Mat72] or [Kol96, VI.2.15] for proofs.

Fujita vanishing 11.42.4. Let X be a projective scheme and F a coherent sheaf on X . Then there is an ample line bundle L such that

$$H^i(X, F \otimes L \otimes M) = 0 \quad \forall i > 0, \quad \forall \text{ nef line bundle } M.$$

See [Fuj83] (or [Laz04, I.4.35] for the characteristic 0 case).

COROLLARY 11.43 (Kodaira lemma). *Let X be a normal, proper, irreducible algebraic space of dimension n and D a nef, \mathbb{R} -divisor. The following are equivalent.*

(11.43.1) $(D^n) > 0$.

(11.43.2) $D = cB + E$, where B is a big \mathbb{Z} -divisor, $c > 0$, and E is an effective \mathbb{R} -divisor.

If X is projective, then these are also equivalent to:

(11.43.3) $D = cA + E$, where A is an ample \mathbb{Z} -divisor, $c > 0$, and E is an effective \mathbb{R} -divisor.

Proof. With (11.42) in place, the arguments in [KM98, 2.61] or [Laz04, 2.2.6] work. See also [Sho96, 6.17] (for characteristic 0) and [Bir17, 1.5] for the original proofs, or [FM21, 2.3]. \square

The proof of the Nakai-Moishezon criterion for \mathbb{R} -divisors uses induction on all proper schemes, so first we need some basic results about them.

11.44 (\mathbb{R} -Cartier divisor classes). [FM21] On an arbitrary scheme one can define \mathbb{R} -line bundles or \mathbb{R} -Cartier divisor classes as elements of $\text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. It is better to think of these as coming from line bundles, but writing divisors keeps the additive notation.

Lemma 11.44.1. Let X be a proper algebraic space with normalization $p : Y \rightarrow X$, and Θ an \mathbb{R} -Cartier divisor class on X . Then Θ is ample iff $p^*\Theta$ is ample.

Proof. For Cartier divisors this is [Har77, Ex.III.5.7], which implies the \mathbb{Q} -Cartier case. Next we reduce the \mathbb{R} -Cartier case to it.

By assumption we can write $\Theta \sim_{\mathbb{R}} \sum_i d_i D_i$ where the D_i are \mathbb{Q} -Cartier divisors. By (11.37) there are $c_{ij} \in \mathbb{Q}$ and $0 < \lambda_j \in \mathbb{R}$ such that the $\Theta_j^Y := \sum_i c_{ij} p^* D_i$ are ample and $d_i = \sum_k \lambda_j c_{kj}$ for every i . In particular, $p^*\Theta \sim_{\mathbb{R}} \sum_j \lambda_j \Theta_j^Y$.

Set $\Theta_j := \sum_i c_{ij} D_i$. Then $\Theta \sim_{\mathbb{R}} \sum_j \lambda_j \Theta_j$ and $p^*\Theta_j = \Theta_j^Y$. The Θ_j are \mathbb{Q} -Cartier, hence ample, and so is Θ . \square

Corollary 11.44.3. Let $g : X \rightarrow S$ be a proper morphism of algebraic spaces and Θ an \mathbb{R} -Cartier divisor class on X . Then

$$S^{\text{amp}} := \{s \in S : \Theta_s \text{ is ample on } X_s\} \subset S \text{ is open.}$$

Proof. Write $\Theta = \sum d_i D_i$ and apply (11.37) to its restriction to X_s . Thus we get \mathbb{Q} -Cartier divisors $\Theta_j := \Theta_j^n$ (for $n \gg 1$) such that $\Theta = \sum_j \lambda_j \Theta_j$ and each $\Theta_j|_{X_s}$ is ample. Thus the Θ_j are ample over some open $s \in S^\circ \subset S$, and so is Θ . \square

THEOREM 11.45. [FM21] Let X be a proper algebraic space and D an \mathbb{R} -Cartier divisor class on X . Then D is ample iff $(D^{\dim Z} \cdot Z) > 0$ for every integral subscheme $Z \subset X$.

Proof. By (11.44.1) we may assume that X is normal. Assume that $(D \cdot C) \geq 0$ for every integral curve $C \subset X$ and $(D^{\dim X}) > 0$. By (11.42) we may assume that D is an effective \mathbb{R} -divisor. By (11.37) we can write $D = \sum \lambda_i D_i$ where the D_i are effective, \mathbb{Q} -Cartier, and $D - D_i$ can be chosen arbitrarily small.

Let $p : Y \rightarrow X$ be the normalization of $\text{Supp } D$. By dimension induction, p^*D is ample, and so are the p^*D_i if the $D - D_i$ are small enough.

Thus the $D_i|_{\text{Supp } D}$ are ample, hence the D_i are semiample by (11.45.1). By assumption $\text{Supp } D$ is not disjoint from any curve, hence the same holds for $\text{Supp } D_i = \text{Supp } D$. So the D_i are ample, and the converse is clear. \square

Claim 11.45.1 Let X be a proper algebraic space and D an effective \mathbb{Q} -Cartier divisor such that $D|_{\text{Supp } D}$ is ample. Then D is semiample. Thus if D is not disjoint from any curve, then D is ample. (See [Laz04, p.35] for a proof.)

The usual proof of the Seshadri criterion (see [Laz04, 1.4.13]) now gives the following.

COROLLARY 11.46 (Seshadri criterion). *Let X be a proper algebraic space and D an \mathbb{R} -Cartier divisor on X . Then D is ample iff there is an $\epsilon > 0$ such that $(D \cdot C) \geq \epsilon \operatorname{mult}_p C$ for every pointed, integral curve $p \in C \subset X$. \square*

Pulling-back divisors.

11.47 (Intersection theory on normal surfaces). [Mum61] Let S be a normal, 2-dimensional scheme and $p : S' \rightarrow S$ a resolution with exceptional curves E_i . The intersection matrix $(E_i \cdot E_j)$ is negative definite by the Hodge index theorem (see [Kol13b, 10.1]). Let D be a \mathbb{K} -divisor on S , where \mathbb{K} denotes \mathbb{Q} or \mathbb{R} . Then there is a unique p -exceptional \mathbb{K} -divisor E_D such that

$$(E_i \cdot (p_*^{-1}D + E_D)) = 0 \quad \text{for every } i. \quad (11.47.1)$$

We call $p^*D := p_*^{-1}D + E_D$ the *numerical pull-back* of D . If D is \mathbb{K} -Cartier then this agrees with the usual pull-back.

If D is effective, then $(E_i \cdot E_D) \leq 0$ for every i , hence E_D is effective by the Hodge index theorem (cf. [Kol13b, 10.3.3]).

More generally, the numerical pull-back is also defined if S' is only normal: we first pull-back to a resolution of S' and then push forward to S .

If D_1, D_2 are \mathbb{K} -divisors and one of them has proper support, then one can define their intersection cycle as

$$(D_1 \cdot D_2) = p_*(p_*^{-1}D_1 \cdot p^*D_2) = p_*(p_*^{-1}D_2 \cdot p^*D_1). \quad (11.47.2)$$

If S is proper, we get the usual properties of intersection theory, except that, even if the D_i are \mathbb{Z} -divisors, their intersection numbers can be rational.

The following connects the numerical push-forward/pull-back with the sheaf-theoretic versions.

Claim 11.47.3. Let $p : T \rightarrow S$ be a proper, birational morphism between normal surfaces. Let B be an \mathbb{R} -divisor on T such that $-B$ is p -nef. Then $g_*\mathcal{O}_T([B]) = \mathcal{O}_S([p_*B])$.

Proof. Write $B = B_h + B_v$ as a sum of its horizontal and vertical parts. We can harmlessly replace B_h with its round down, so we assume that B_h is a \mathbb{Z} -divisor. Let ϕ be a local section of $\mathcal{O}_S(p_*B_h)$. Then $\phi \circ p$ is a rational section of $\mathcal{O}_T(B_h)$, with possible poles along the exceptional curves. There is thus a smallest exceptional \mathbb{Z} -divisor F such that $\phi \circ p$ is a section of $\mathcal{O}_T(B_h + F)$. In particular, $(E_i \cdot (B_h + F)) \geq 0$ for every i . Thus

$$(E_i \cdot (B_h + F - B)) = (E_i \cdot (F - B_v)) \geq 0 \quad \forall i.$$

By the Hodge index theorem (cf. [Kol13b, 10.3.3]) this implies that $B_v - F$ is effective, thus $B_h + F \leq [B]$. \square

Corollary 11.47.4. Let $g : T \rightarrow S$ be a proper, birational morphism between normal surfaces and D an \mathbb{R} -divisor on S . Then $p_*\mathcal{O}_T([p^*D]) = \mathcal{O}_S([D])$. \square

11.48 (Numerical pull-back). Let $g : Y \rightarrow X$ be a projective, birational morphism of normal schemes and H a g -ample Cartier divisor. We define the *H -numerical pull-back* of \mathbb{K} -divisors

$$g_H^{(*)} : \operatorname{WDiv}_{\mathbb{K}}(X) \rightarrow \operatorname{WDiv}_{\mathbb{K}}(Y)$$

as follows. Let $D \subset X$ be a \mathbb{K} -divisor. We inductively define

$$g_H^{(*)}(D) = g_*^{-1}D + \sum_{i \geq 2} F_i(D), \quad (11.48.1)$$

where $F_i(D)$ consists of those g -exceptional divisor $E_{i\ell}$ for which $g(E_{i\ell}) \subset X$ has codimension i .

Assume that we already defined the $F_i(D)$ for $i < j$. Let $x \in X$ be a point of codimension j . After localizing at x we have $g_x : Y_x \rightarrow X_x$. Let F_x be the unique divisor supported in $g_x^{-1}(x)$ such that

$$(E_{j\ell} \cdot (g_*^{-1}D + \sum_{i < j} F_i(D) + F_x) \cdot H^{j-2}) = 0 \quad \forall \ell. \quad (11.48.2)$$

To make sense of this, we may assume that H is very ample, and then let S be a general complete intersection of $j-2$ members of $|H|$. Then S is a normal surface, so we are working with intersection numbers as in (11.47). Also, if S is general then the $g_x|_S$ -exceptional curves are in one-to-one correspondence with the divisors $E_{j\ell}$, so any linear combination of $g_x|_S$ -exceptional curves corresponds to a linear combination of the divisors $E_{j\ell}$.

If we have proper but non-projective $Y \rightarrow X$, we can apply our definition to a projective modification $Y' \rightarrow Y \rightarrow X$ and then push forward to Y . This defines $g_H^{(*)}$ in general.

Already in basic examples, for example cones over cubic surfaces, the divisors $g_H^{(*)}(D)$ do depend on H . However, the notion has several good properties and it is quite convenient in several situations. See, for example, (11.42) or [FKL16, 3.3].

THEOREM 11.49. *Let $g : Y \rightarrow X$ be a projective, birational morphism of normal schemes and H a g -ample Cartier divisor. Let \mathbb{K} be \mathbb{Q} or \mathbb{R} . Then*

- (11.49.1) $g_H^{(*)}$ is \mathbb{K} -linear,
- (11.49.2) $g_* \circ g_H^{(*)}$ is the identity,
- (11.49.3) if D is \mathbb{K} -Cartier then $g_H^{(*)}(D) = g^*(D)$,
- (11.49.4) if D is effective then so is $g_H^{(*)}(D)$,
- (11.49.5) $g_H^{(*)}$ respects \mathbb{K} -linear equivalence, and
- (11.49.6) $g_* \mathcal{O}_Y(\lfloor g_H^{(*)}(B) \rfloor) = \mathcal{O}_X(\lfloor B \rfloor)$.

Proof. (1–3) are clear from the definition and (4) follows from its surface case, which we noted after (11.47.1). If $D_1 \sim_{\mathbb{R}} D_2$ then, using first (1) and then (3) we get that

$$g_H^{(*)}(D_1) = g_H^{(*)}(D_2) + g_H^{(*)}(D_1 - D_2) = g_H^{(*)}(D_2) + g^*(D_1 - D_2),$$

giving (5). Finally (6) is a local question. We may thus assume that (6) holds outside a closed point $x \in X$. Assume to the contrary that $\mathcal{O}_Y(\lfloor g_H^{(*)}(B) \rfloor)$ has a rational section that has poles along $g^{-1}(x)$. After restricting to a general complete intersection surface $S \subset Y$ as in (11.48), we would get a contradiction to (11.47.3). \square

Negativity lemmas.

LEMMA 11.50. [KM98, 3.39] *Let $h : Z \rightarrow Y$ be a proper birational morphism between normal varieties. Let $-B$ be an h -nef \mathbb{R} -Cartier divisor on Z . Then*

- (11.50.1) B is effective iff h_*B is.

(11.50.2) *Assume that B is effective. Then for every $y \in Y$, either $h^{-1}(y) \subset \text{Supp } B$ or $h^{-1}(y) \cap \text{Supp } B = \emptyset$.*

LEMMA 11.51. [**Kol18a**] *Let $\pi : Y \rightarrow X$ be a proper, birational morphism of normal schemes. Let N, B be \mathbb{R} -divisors such that N is π -nef and B is effective and horizontal. Then*

$$\pi_* \mathcal{O}_Y([-N - B]) = \mathcal{O}_X([\pi_*(-N - B)]). \quad (11.51.1)$$

Proof. If $\dim Y = 2$ then B is also π -nef, so the claim follows from (11.47.3). In general, we may assume that π is projective, and reduce to the surface case as in the proof of (11.49.6). \square

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