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SMOOTHING ALGEBRAIC CYCLES BELOW THE MIDDLE DIMENSION

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ABSTRACT. Hironaka proved that the Chow groups $\mathrm{CH}_d(X)$ are generated by smooth subvarieties if $d < \frac{1}{2}\dim X$ and $d \leq 3$. Recently [KV23] extended this to all $d < \frac{1}{2}\dim X$. The aim of this lecture is to explain the methods and sketch the proof.

CONTENTS

Representing homology classes by smooth submanifolds was considered by Hopf for H_2 in [Hop42], and then studied systematically by Thom, who gave a positive answer up to dimension 8, and also established obstructions involving Steenrod powers in [Tho54, Chap.II].

On a smooth algebraic variety over $\mathbb{C}$, homology classes of algebraic subvarieties form a cone. So the right question, as formulated by Borel and Haefliger, is whether homology classes of *smooth* algebraic subvarieties generate the group of algebraic homology classes; see [BH61, 5.17]. For a smooth variety X over an arbitrary field, one should ask whether the classes of *smooth* algebraic subvarieties generate the Chow group $\mathrm{CH}_d(X)$ of d-dimensional cycles.

In this form, the question was considered by Hironaka and Kleiman. [Hir68] gave a positive answer if $d \leq 3$ and $d < \frac{1}{2}\dim X$, and [Kle69, 5.8] showed that the subgroup generated by Chern classes contains $(c-1)!\,\mathrm{CH}^c(X)$ for every c.

The first negative result is in [HRT74], for codimension 2 cycles in Grassmannians. [Deb95] considers codimension 2 cycles on Jacobians; related higher codimension examples are in [BD23]. A large series of counterexamples is given by Benoist, including many satisfying $d = \frac{1}{2}\dim X$; see [Ben22, 4.17]. Thus the following is likely optimal.

Theorem 1. [KV23, 1.2] *Let X be a smooth, projective variety over a field of characteristic* 0. *Then, for every $d < \frac{1}{2}\dim X$, the classes of smooth subvarieties generate* $\mathrm{CH}_d(X)$.

See Corollaries 33–34 for $d \geq \frac{1}{2}\dim X$, and Theorem 39 for positive characteristic. We prove Theorem 1 in Paragraph 32, as a consequence of the following, to be established in Paragraph 26.

Theorem 2. *Let X be a smooth projective variety over a field of characteristic* 0, *and $Z_X \subset X$ an irreducible subvariety. Then there are*

(1) *a smooth, projective variety Y,*
(2) *a flat morphism $g : Y \to X$, and*
(3) *a smooth, complete intersection subvariety $Z_Y \subset Y$,*

such that $g|_{Z_Y} : Z_Y \to Z_X$ is birational.

Complement 3. *The $g : Y \to X$ we construct is birational to $\pi : X^N \times \mathbb{P}^M \to X$ for some (quite large) N, M, where π is a coordinate projection.*

A positive characteristic version is given in Theorem 37.

Although the statement of Theorem 2 is stronger than [KV23, 1.6]—which asserts only that the Chow groups are generated by such images of complete intersections—the proofs are actually the same.

Acknowledgments. All the main results are taken from [KV23]. The positive characteristic variants in Section 6 were also developed in discussions while we worked on [KV23]. Partial financial support was provided by the NSF under grant number DMS-1901855.

1. Outline of the proof of Theorem 2

4 (Hironaka's method for Theorem 1)**.** Let $Z_X^d \subset X^n$ be an irreducible subvariety. As explained in [Hir68, pp.1–2], the method starts with a resolution of singularities $Z'_X \to Z_X$ and an embedding $Z'_X \hookrightarrow \mathbb{P}^N$ for some N. Let $Z \subset X \times \mathbb{P}^N$ be the image of the diagonal embedding of Z'_X, and $\pi : X \times \mathbb{P}^N \to X$ the projection. Then $Z \cong Z'_X$ and $\pi|_Z : Z \to Z_X$ is birational.

Next we try to move Z into 'general position.' Let $|H|$ be a sufficiently ample linear system on $X \times \mathbb{P}^N$, and $Z \cup R$ a general d-dimensional, complete intersection of members of $|H|$ that contain Z. Let S be a general d-dimensional, complete intersection of members of $|H|$. Then Z_X is rationally equivalent to $\pi_*(S) - \pi_*(R)$. S is smooth by Bertini, and so is $\pi_*(S)$ if $d < \frac{1}{2}n$.

Hironaka shows that R and $\pi_*(R)$ are smooth for $d \leq 3$, but always singular for $d \geq 4$; see Example 9. It is impossible to achieve smoothness for $d \geq 4$ by repeating the above linkage trick.

We can resolve the singularities of R, but that does not change its image in $X \times \mathbb{P}^N$. We would need to find an embedded resolution of $Z \cup R$, where the birational transform of R can be moved into general position.

5 (Change to Theorem 2)**.** More generally, let $Z_Y = D_1 \cap \cdots \cap D_m \subset Y$ be a complete imtersection. Then Z_Y is rationally equivalent to

$$(-1)^m \textstyle\sum_{e_i \in \{0,1\}} (-1)^{\sum e_i} \big((H_1 + e_1 D_1) \cdots (H_m + e_m D_m)\big), \tag{5.1}$$

for any divisors H_i. If the H_i are sufficiently ample, then so are the $H_i + e_i D_i$, and all summands in (5.1) can be moved into 'general position.'

Combining with the first step of Hironaka's method, we see that, after replacing X by $X \times \mathbb{P}^N$, it is enough to prove Theorem 2 for *smooth* subvarieties $Z \subset X$. The singularities of the original Z_X play no role from now on.

6 (Voisin's method)**.** [Voi23] shows that it is enough to prove Theorem 2 after a sequence of blow-ups of smooth, complete intersection subvarieties.

This is conceptually very surprising, since, given a blow-up $X' \to X$ and an equidimensional morphism $Y' \to X'$, there does not seem to be any natural way to construct an equidimensional morphism $Y \to X$ out of them. Nonetheless, the proof, in a series of Lemmas 13–16, is short.

Now the plan is to start with a smooth $Z^d \subset X^n$, choose a general, d-dimensional, complete intersection $Z \cup R \subset X$, and try to find a sequence of blow-ups of smooth, complete intersection subvarieties $X^* \to X$, such that the birational transform $Z^* \cup R^* \subset X^*$ is a smooth, complete intersection.

An easy argument (as in the proof of Corollary 25) shows that Theorem 2 holds for $Z^* \subset X^*$, hence, as we noted above, it also holds for $Z \subset X$.

For $\dim Z \leq 7$ the singularities of $Z \cup R$ are quite simple, and a subtle explicit construction [Voi23, Sec.3] gives the needed blow-ups $X^* \to X$. It is possible that one can push this approach beyond dimension 7, but the singularities of $Z \cup R$ do get more and more complicated as the dimension increases. This leads to the following.

Question 6.1. Can one resolve singularities by blowing up smooth, complete intersection subvarieties only?

The usual embedded resolution methods blow up smooth (hence local complete intersection) subvarieties, but these are almost never global complete intersections. So this may be delicate.

Another twist is that usually $Z^* \cup R^* \subset X^*$ is not a complete intersection, but the zero set of a section of a vector bundle. We call these *complete bundle-sections*—abbreviated as *cbs*—and work with these; see Definition 14 and Lemma 22.

To get the result for every d, we focus on the process of the construction of the complete intersection $Z \cup R$.

7 (Keeping singularities simple)**.** More generally, let $Z \subset Y$ be smooth varieties. Let $Z \subset W \subset Y$ be a smallest dimensional, smooth, complete bundle-section (possibly $W = Y$).

Take a sufficiently ample linear system $|H|$ and a general divisor $H \in |H|$ that contains Z. Then $W \cap H$ has only ordinary quadratic singularities, locally of the form $\left(\sum_{i=1}^{r} x_i x_{r+i} = 0\right)$. In particular, $\operatorname{Sing}(W \cap H) \subsetneq Z$ is smooth; see Lemma 21.

By induction on the dimension of Z, after further blow-ups $\operatorname{Sing}(W \cap H)$ becomes an irreducible component of a complete bundle-section, at which point we can blow it up. We get $Y' \to Y$ and birational transforms $Z' \subset (W \cap H)' \subset Y'$. We check in Lemma 22 that $(W \cap H)' \subset Y'$ is a smooth, complete bundle-section, whose dimension is $\dim W - 1$.

Now we choose a new sufficiently ample linear system $|H'|$ on Y', and repeat the process. Eventually the birational transform of Z becomes an irreducible component of a smooth, complete bundle-section Z^*.

An inconvenient feature is that we have limited control over the other components of Z^*. This is a problem in the induction since having only ordinary quadratic singularities is not preserved by all blow-ups; see Example 19. This issue needs extra considerations in Section 4.

This approach is reminiscent of the observation in [Kol11], that a similar class of singularities was easier to resolve starting with the largest dimensional stratum.

For illustration, let us see how the method works for a point on a surface (which would be simpler using Chern classes as in [Kle69]).

Example 8. Let S be a smooth projective surface and $s \in S$ a point. We construct a smooth 5-fold Y, a flat morphism $\pi : Y \to S$ and 5 divisors $D_i \subset Y$ such that $s_Y := D_1 \cap \cdots \cap D_5$ is a single point (scheme theoretically), and $\pi(s_Y) = s$.

Choose smooth projective curves $C_1, C_2 \subset S$ intersecting transversally, such that $s \in C_1 \cap C_2$. Set

$$Y_1 := B_{C_1 \cap C_2} S \times \mathbb{P}_S\bigl(\mathcal{O}_S(-C_1) + \mathcal{O}_S(-C_2)\bigr),$$

and let $G \subset Y_1$ be the graph of the natural embedding $j : B_{C_1\cap C_2} S \hookrightarrow \mathbb{P}_S\bigl(\mathcal{O}_S(-C_1) + \mathcal{O}_S(-C_2)\bigr)$, induced by the surjection $\mathcal{O}_S(-C_1) + \mathcal{O}_S(-C_2) \twoheadrightarrow I_{C_1\cap C_2}$.

Let $\pi_G : Y \to Y_1$ be the blow-up of G with exceptional divisor $E_G \subset Y$.

To get the 5 divisors, let $E_s \subset B_{C_1\cap C_2} S$ be the exceptional curve over s, and $C_1' \subset B_{C_1\cap C_2} S$ the birational transform of C_1. Using the coordinate projection $p_1 : X \to B_{C_1\cap C_2} S$, take

$$D_1 := (p_1 \circ \pi_G)^* E_s, \quad D_2 := (p_1 \circ \pi_G)^* C_1', \quad D_3 := E_G.$$

Finally, choose a sufficiently ample divisor H on Y_1 such that $|\pi_G^* H - E_G|$ restricts to a very ample divisor on E_G, and let $D_4, D_5 \in |\pi_G^* H - E_G|$ be general members.

In general, we need to iterate a similar construction many times. A consequence is that Y in Theorem 2 has much larger dimension than X.

The following is an illustrative example showing why singularities appear in Hironaka's method 4.

Example 9. Start with $Z := (x_1 = x_2 = 0) \subset \mathbb{A}^n$. A general $Z \cup R$ is then given by $g_{11}x_1 - g_{12}x_2 = g_{21}x_1 - g_{22}x_2 = 0$. We see that

$$R := \left(\operatorname{rank}\begin{pmatrix} g_{11} & g_{12} & x_2 \\ g_{21} & g_{22} & x_1 \end{pmatrix} \le 1\right).$$

This is singular at the common zeros of the g_{ij} on Z. For generic g_{ij} this happens iff $\dim Z \ge 4$.

2. Complete intersection images

In this section we work over a field of characteristic 0; see Paragraph 36 for positive characteristic.

Definition 10. Let X be a projective variety over a field of characteristic 0, and $Z_X \subset X$ an irreducible subvariety. We say that Z_X is a *smooth, complete intersection image*, abbreviated as *sci-image,* if there are

(1) a smooth, projective variety Y,
(2) an equidimensional morphism $g : Y \to X$, and
(3) a smooth, complete intersection $Z_Y \subset Y$,

such that $g|_{Z_Y} : Z_Y \to Z_X$ is birational.

Remark 10.4. For our purposes, singular complete intersections $Z_Y \subset Y$ would also work. In characteristic 0 our proofs give smoothness of Z_Y for free. Over finite fields it may be convenient to allow singular, complete intersections; see (36.2).

There are 2 obvious lemmas (11 and 12) and 3 subtle ones (13, 15 and 16).

Lemma 11. *Let $\pi : X' \to X$ be a smooth morphism and $Z' \subset X'$ a sci-image. If $\pi|_{Z'} : Z' \to Z$ is birational, then Z is also a sci-image.* □

Lemma 12. *Let $\pi : P \to X$ be a $\mathbb{P}^n$-bundle and $Z \subset X$ a subvariety that is a sci-image. Then there is a sci-image $Z_P \subset P$ such that $\pi|_{Z_P} : Z_P \to Z$ is birational.*

Sketch of proof. By assumption we have $g : Y \to X$ and Z_Y. Let $H_1, \dots, H_n$ be general members of a very ample $\mathcal{O}_P(1)$ on P. We have $Y \times_X P$ with projections p_1, p_2. Then $p_1^*(Z_Y) \cap p_2^* H_1 \cap \cdots \cap p_2^* H_n$ shows that we can take $Z_P := \pi^*(Z) \cap H_1 \cap \cdots \cap H_n$. □

Lemma 13. [KV23, 3.7] *Let $j : H \hookrightarrow X$ be a smooth hypersurface and $Z \subset H$ a subvariety that is a sci-image. Then $j_* Z \subset X$ is also a sci-image.*

Sketch of proof. Let $G \subset H \times X$ be the graph of j, and $\tau : X' := B_G(H \times X) \to H \times X$ its blow-up with exceptional divisor E. Note that $p_1 : X' \to H \times X \to H$ is smooth and $p_2 : X' \to H \times X \to X$ is equidimensional.

Choose a sufficiently ample divisor A on $H \times X$ such that $|\tau^* A - E|$ restricts to a very ample divisor on E, and take general members $A_1, \dots, A_{n-1} \in |\tau^* A - E|$.

By assumption we have $g : Y_H \to H$ and Z_Y a ci in Y_H. Set $Y := Y_H \times_H X'$ with projections $q_1 : Y \to Y_H$ and $q_2 : Y \to X'$. We can now take

$$q_1^* Z_Y \cap q_2^* E \cap q_2^* A_1 \cap \cdots \cap q_2^* A_{n-1}. \quad \square$$

Remark 13.1. H needs to have codimension 1 in X for $p_2 : X' \to X$ to be equidimensional. For smooth, complete intersections $j : W \hookrightarrow X$ one can use induction, but Lemma 13 is not (yet) proved for inclusions of smooth subvarieties.

We also need the following generalization of complete intersections.

Definition 14. Let E be a locally free sheaf of rank r on X, and s a global section of E. If $C := (s = 0)$ has pure codimension r, we call it a *complete bundle-section* subscheme, abbreviated as *cbs*. We are mostly interested in cases where C is smooth.

The next 2 lemmas are proved together.

Lemma 15. [KV23, 3.9] *Let $j : C \hookrightarrow X$ be the inclusion of a smooth complete bundle-section, and $Z \subset C$ a sci-image. Then $j_* Z \subset X$ is a sci-image.*

Lemma 16. [KV23, 3.11] *Let $C \subset X$ be a smooth complet bundle-section and $Z' \subset B_C X$ a sci-image. If $\pi|_{Z'} : Z' \to Z$ is birational, then $Z \subset X$ is also a sci-image.*

Sketch of proof. Let C be the zero set of $s : \mathcal{O}_X \to E$. Dually we have a surjection $E^* \to I_C \subset \mathcal{O}_X$, which gives an embedding $i_B : B_C X \hookrightarrow P(E^*) := \operatorname{Proj}_X \operatorname{Sym}(E^*)$. The projections are denoted by $\pi_E : P(E^*) \to X$, $\pi_B : B_C X \to X$ and $\pi_F : F \to C$, where $i_F : F \subset B_C X$ is the π_B-exceptional divisor

Let $\pi_E^* E \to Q$ be the universal quotient bundle. The composite of $\pi_E^* s$ with $\pi_E^* E \to Q$ gives a section s_Q of Q whose zero set is $B_C X$. We use induction on

$$r := \operatorname{codim}(C \subset X) = 1 + \operatorname{codim}\big(B_C X \subset P(E^*)\big).$$

The $r = 1$ case of Lemma 15 is Lemma 13. We have a commutative diagram

$$\begin{array}{ccccc} F & \xrightarrow{i_F} & B_C X & \xrightarrow{i_B} & P(E^*) \\ \pi_C \downarrow & & \pi_B \downarrow & & \downarrow \pi_E \\ C & \xrightarrow{i_C} & X & = & X. \end{array}$$

If $Z \subset B_C X$ is a sci-image, then so is $(i_B)_* Z$ by induction on r, hence also $(\pi_E)_*(i_B)_* Z$ by Lemma 11, proving Lemma 16.

If $Z_C \subset C$ is a sci-image, lift it to $Z_F \subset F$ using Lemma 12. Now $(i_F)_* Z_F$ is a sci-image by Lemma 13, hence so is $j_* Z = (\pi_E)_*(i_B)_*(i_F)_* Z_F$, proving Lemma 15. □

3. Blow-up sequences

In this section we work over an infinite perfect field. All varieties are allowed to be reducible, but assumed pure dimensional.

17 (Blow-up sequences)**.** A *blow-up sequence* is a sequence of morphisms

$$Y_r \xrightarrow{\pi_{r-1}} Y_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_0} Y_0, \tag{17.1}$$

where each $\pi_i : Y_{i+1} \to Y_i$ is the blow up of a subscheme $C_i \subset Y_i$, called the *center* of the blow-up.

Let $W_0 \subset Y_0$ be a subscheme. If the images of the centers C_i are nowhere dense in W_0, then we let $W_i \subset Y_i$ denote the *birational transforms* of W_0. (Also called *proper transform.*)

Here we only deal with blow-up sequences where Y_0 is smooth, and the C_i are smooth. In this case all the Y_i are smooth.

Following Definition 14, we say that (17.1) is a *complete bundle-section blow-up sequence* (abbreviated as *cbs blow-up sequence*), if the $C_i \subset Y_i$ are all complete bundle-sections.

18 (Blow-up lemmas)**.** Let Y be a smooth variety and Z, C reduced, pure dimensional, closed subsets and C smooth. Let $\pi : Y' := B_C Y \to Y$ be the blow-up and $Z' \subset Y'$ the birational transform.

(1) If Z and $Z \cap C$ are smooth, then so is Z'.
(2) If Z is a smooth cbs and $C \subsetneq Z$, then Z' is a smooth cbs.
(3) If Z is a cbs and $\operatorname{codim}_Y C = \operatorname{codim}_Z (Z \cap C)$, then Z' is a cbs.
(4) Let $H \subset Y$ be a hypersurface that has only ordinary double points along some smooth $D \subsetneq H$. If $C = D$ then H' is smooth, and if $C \subsetneq D$, then H' has only ordinary double points, necessarily along D'.

Example 19 (Blow-ups to avoid)**.** Consider the cone $H := (xy + z^2) \subset \mathbb{A}^3$ and blow-up the line $L := (x = y = 0)$. In one chart we get the equation $H' = (x_1^2 y_1 + z^2 = 0)$, so H' does not have ordinary double points.

This leads to the following definition.

Definition 20 (Full intersection property)**.** Let $Z \subset Y$ be schemes. A closed subset $W \subset Y$ has *full intersection* with Z, if $Z \cap W$ is a union of connected components of W. A blow-up sequence $Y_r \to \cdots \to Y_0 = Y$ has *full intersection* with Z if the birational transforms $Z_i \subset Y_i$ are defined, and each blow-up center C_i has full intersection with Z_i.

We also need a Bertini-type theorem for linear systems with basepoints; see [Kol97, Sec.4] for similar results.

Lemma 21. *Let $W \subset Y$ be smooth varieties, H a sufficiently ample linear system on Y, and $H \in |H|$ a general member that contains W. Then*

(1) *H is smooth if* $\dim W < \frac{1}{2} \dim Y$.

(2) *If* $\dim W \geq \frac{1}{2}\dim Y$, *then* $\operatorname{Sing} H$ *is smooth, of dimension* $2\dim W - \dim Y$, *contained in* W, *and* H *has only ordinary quadratic singularities.*

Sketch of proof. These are local questions. If $W \subset Y$ is defined by the equations $g_1 = \cdots = g_r = 0$, then H is defined by an equation $\sum_{i=1}^r f_i g_i = 0$, where the f_i can be chosen general in a linear system whose restriction to W is very ample. The singular set is then $(g_1 = \cdots = g_r = f_1 = \cdots = f_r = 0)$. □

Next we show that blowing up such ordinary quadratic singularities preserves complete bundle-sections. This is the step in the proof where going from complete intersections to complete bundle-sections becomes necessary.

Lemma 22. [KV23, 4.5] *Let* Y *be a smooth variety,* E *a vector bundle on* Y, *and* $W \subset Y$ *a cbs subvariety given by a section* $s \in H^0(Y, E)$. *Assume that* W *has only ordinary double points along some smooth* $D \subsetneq W$.

Let $\pi : Y' := B_D Y \to Y$ *be the blow-up. Then* $W' \subset Y'$, *the birational transform of* W, *is a smooth cbs.*

Sketch of proof. Locally at a point $p \in D$ we can choose coordinates y_i such that

$$W = \big(y_1 = \cdots = y_r = Q(y_{r+1}, \ldots, y_{r+s}) = 0\big),$$

Let F denote the exceptional divisor of π. Then, locally over a neighborhood of p, W' is the complete intersection of the hypersurfaces

$$(y_1 \circ \pi = 0) - F, \ldots, (y_r \circ \pi = 0) - F, \quad \text{and} \quad (Q \circ \pi = 0) - 2F.$$

One needs to check that these local charts give a well defined subsheaf $E' \subset \pi^* E(-F)$, which is locally free.

Note that E' is not a subbundle of $\pi^* E(-F)$, and, even if E is a direct sum of line bundles, usually E' is not. □

Remark 23. The proof suggests that if $W \subset Y$ is an arbitrary cbs subvariety and $D \subsetneq W$ is a smooth subvariety, then the birational transform W' of W on $B_D Y$ is a cbs subvariety if W is normally flat along D [Hir64, p.136], but not in general.

4. Creating complete bundle-sections

In this section we work over an infinite perfect field. As before, all varieties are allowed to be reducible, but assumed pure dimensional.

As we noted in Paragraph 7, our aim is to turn subvarieties into complete intersections, using only complete bundle-section blow-ups. We can not do it in full generality, but the following version is sufficient for the current purposes.

Theorem 24. [KV23, 1.8] *Let* $Z \subset Y$ *be smooth, projective varieties such that* $\dim Z < \frac{1}{4}\dim Y$. *Then there is complete bundle-section blow-up sequence* $Y_r \to \cdots \to Y_0 := Y$ *with centers* $C_i \subset Y_i$, *such that* $\dim C_i < \dim Z$ *for every* i, *and* $Z_r \subset Y_r$ *is a union of irreducible components of a smooth, complete bundle-section* $Z_r^* \subset Y_r$.

We discuss the assumption $\dim Z < \frac{1}{4}\dim Y$ during the proof. We did not try to optimize the constant $\frac{1}{4}$.

Sketch of proof. We start as in Paragraph 7. Let $Z \subset X \subset Y$ be the smallest, smooth, complete bundle-section that contains Z. We are done if $\dim Z = \dim X$.

Otherwise, take a sufficiently ample linear system $|H|$ and a general divisor $H \in |H|$ that contains Z. Since $H \cap X$ is a smaller dimensional, complete bundle-section, it must be singular. However, by Lemma 21, $H \cap X$ has only ordinary quadratic singularities, and $W := \operatorname{Sing}(H \cap X) \subsetneq Z$ is smooth.

By induction on d, after further blow-ups W becomes (a union of irreducible components of) a complete bundle-section, at which point we can blow it up. The birational transform of $H \cap X$ is then smooth by (18.4) and a complete bundle-section by Lemma 22.

The birational transform of Z is now contained in the the birational transform of $H \cap X$, which is a smooth, complete bundle-section, and its dimension is $\dim X - 1$. Repeating this $\dim X - \dim Z$ times, we get the theorem.

So what is the problem?

In the above process we would like to blow up certain subvarieties $V \subset W$. By dimension induction we can arrange that V is an irreducible component of a complete bundle-section V^*, but we have to blow up V^*, not V. As in Example 19, we need to make sure that V^* has full intersection (Definition 20) with W. When the codimension of $Z \subset Y$ is small, this is impossible for dimension reasons. However, even if $\dim V^*$ is small, we would need to show that $V^* \setminus V$ is in 'general position' on Y.

So let us see how do we get these other components $V^* \setminus V$. They arise when $Z \subset X$ is a hypersurface. Then we find a general $R \in |H - Z|$. Now $Z \cap R$ is a complete intersection in X, but need not be one in Y. So we need to make some blow-ups. Eventually we can blow up (the birational transform of) $Z \cap R$, and then (the birational transform of) $Z \cup R$ becomes a smooth, complete bundle-section.

The difficulty is that, although we can guarantee that R itself is in 'general position' on X, its birational transform may not be in 'general position,' since the sequence of blow-ups depends on the choice of R.

This is where the assumption $\dim Z < \frac{1}{4}\dim Y$ comes handy. We start with a $Z \subset X \subset Y$ such that $\dim X < \frac{1}{2}\dim Y$. It is now reasonable to hope that we can use the extra $> \frac{1}{2}\dim Y$ dimensions to achieve that all these extra componets $V^* \setminus V$ are disjoint from X. Once this is achieved, they do not cause any further trouble.

In practice we use a double induction on 2 assertions similar to Theorem 24. One is used to push the extra componets $V^* \setminus V$ away from X, the other is basically the argument above. □

Corollary 25. [KV23, 1.9] *Let $Z \subset Y$ be smooth, projective varieties such that $\dim Z < \frac{1}{4}\dim Y$. Then there is complete bundle-section blow-up sequence $\Pi : Y_{r+1} \to Y_r \to \cdots \to Y_0 := Y$, and a complete intersection subvariety $Z_{r+1} \subset Y_{r+1}$ such that $\Pi_*(Z_{r+1}) = Z$.*

Proof. Let $Y_r \to \cdots \to Y_0 := Y$ be as in Theorem 24, and $\pi_r : Y_{r+1} \to Y_r$ the blow-up of Z_r^*. Let E_{r+1} be the π_r-exceptional divisor lying over Z_r.

Choose H_r sufficiently ample on Y_r such that $|\pi_r^* H_r - E_{r+1}|$ restricts to a very ample divisor on E_{r+1}. Then we can choose general members $D^i \in |\pi_r^* H_r - E_{r+1}|$ to obtain a complete intersection subvariety $Z_{r+1} := \big(E_{r+1} \cap D^1 \cap \cdots \cap D^c\big)$, where $c = \dim Y - \dim Z - 1$. □

26 (Proof of Theorem 2)**.** Following Hironaka's method 4, first we choose a smooth subvariety $Z \subset X \times \mathbb{P}^N$ whose first projection is birational onto Z_X. We may choose $N > 3 \dim X$.

Now apply Corollary 25 to $Z \subset Y := X \times \mathbb{P}^N$ to get a complete bundle-section blow-up sequence $\Pi : Y_{r+1} \to Y_r \to \cdots \to Y_0 := Y$, and a complete intersection subvariety $Z_{r+1} \subset Y_{r+1}$ such that $\Pi_* Z_{r+1} = Z$.

Here Z_{r+1} is a sci-image (using the identity map $Y_{r+1} \to Y_{r+1}$), hence $Z \subset Y$ is a sci-image by applying Lemma 16 to each $Y_{i+1} \to Y_i$. Thus $Z_X \subset X$ is also a sci-image by Lemma 11. □

5. Pushing forward complete intersections

In this section we work over infinite fields.

27. Using (5.1) we see that on a smooth projective variety, any complete intersection is rationally equivalent to a linear combination of smooth, complete intersections of ample divisors.

To be precise, below we work with very ample linear systems $|A_i|$ that separate $\dim Y$ points, and the conclusions hold for all complete intersections $W = A_1 \cap \cdots \cap A_m \subset Y$ for a dense open subset of $|A_1| \times \cdots \times |A_m|$.

Lemma 28. *Let $g : Y \to X$ be an equidimensional morphism, and $W \subset Y$ a general, complete intersection of dimension $< \dim X$. Then the set of points $x \in g(W)$ with $\geq s$ preimages in W, has codimension $\geq (s-1)(\dim X - \dim W)$ in $g(W)$.*

Proof. Let $\mathbf{W}$ parametrize all complete intersections, and $(\mathbf{W}, s)$ all s-pointed, complete intersections $\{W, w_1, \dots, w_s : g(w_i) = g(w_j)\}$.

Note that s points in the same fiber of g are parametrized by the s-fold fiber product of $g : Y \to X$ with itself; this has dimension $\dim X + s(\dim Y - \dim X)$. It is $s(\dim Y - \dim W)$ conditions for W to pass through s points. This gives that $\dim(\mathbf{W}, s) - \dim \mathbf{W} = s \dim W - (s-1)\dim X$. □

Corollary 29. *Let $g : Y \to X$ be an equidimensional morphism, and $W \subset Y$ a general, complete intersection of dimension $< \dim X$. Then*

1. *$g|_W : W \to g(W)$ is finite and generically injective,*
2. *$g|_W : W \to g(W)$ is injective if $\dim W < \frac{1}{2} \dim X$, and*
3. *$g(W)$ has finitely many points with 2 preimages if $\dim W = \frac{1}{2} \dim X$.* □

If g is smooth, then $g|_W : W \to X$ is an immersion for $\dim W \leq \frac{1}{2} \dim X$ by [Hir68]. [KV23, 2.1] shows that the same holds for equidimensional morphisms between smooth varieties in characteristic 0. In general, we need a definition.

Definition 30. We call a morphism $\pi : Y \to X$ *piecewise smooth,* if there are finite, locally closed decompositions $X = \cup_i X_i$ and $\pi^{-1}(X_i) = \cup_j Y_{ij}$, such that each $\pi_{ij} := \pi|_{Y_{ij}} : Y_{ij} \to X_i$ is smooth.

In characteristic 0 every morphism is piecewise smooth, so this notion is of interest in positive characteristic only.

Being piecewise smooth is preserved by fiber products and compositions.

Lemma 31. *Let $g : Y \to X$ be an equidimensional and piecewise smooth morphism between smooth, projective varieties.*

Let $W \subset Y$ be a general, complete intersection of dimension $\leq \frac{1}{2}\dim X$. Then $g|_W : W \to X$ is an immersion.

Proof. Let r_{ij} be the rank of T_g on $Y_{ij} \to X_i$ as in Definition 30. Then $\dim X_i \leq r_{ij}$, so $\dim Y_{ij} \leq r_{ij} + (\dim Y - \dim X)$.

If $r_{ij} + \dim W < \dim X$ then W is disjoint from Y_{ij}. Otherwise $r_{ij} \geq \dim W$.

It is $\dim Y - \dim W$ conditions for W to pass through a point $p \in Y_{ij}$, and $r_{ij} - \dim W + 1$ condition for T_pW to have postitive dimensional intersection with the kernel of $T_pY \to T_{g(p)}X$, since the latter has rank r_{ij}. Thus the set of W for which $g|_W$ is not an immersion at some point of Y_{ij} has codimension

$$\begin{aligned} &\geq (\dim Y - \dim W) + (r_{ij} - \dim W + 1) - \dim Y_{ij} \\ &\geq (\dim Y - \dim W) + (r_{ij} - \dim W + 1) - (r_{ij} + \dim Y - \dim X) \\ &= 1 + \dim X - 2\dim W. \end{aligned}$$

This is positive whenever $\dim W \leq \frac{1}{2}\dim X$. □

32 (Proof of Theorem 1)**.** Let $Z_X \subset X$ be a d-dimensional subvariety. By Theorem 2, it is a sci-image of some complete intersection $Z_Y \subset Y$. As we noted in Paragraph 27, Z_Y is rationally equivalent to a linear combination of complete intersections W_j in 'general position.' If $d < \frac{1}{2}\dim X$, then each $g|_{W_j} : W_j \to X$ is injective by Corollary 29.2 and an immersion by Lemma 31. Thus each $g|_{W_j} : W_j \to X$ is an embedding. □

For $d \geq \frac{1}{2}\dim X$, the proof gives the following.

Corollary 33. [KV23, 2.1] *Let X be a smooth, projective variety of dimension $n = 2d$ over a field of characteristic 0. Then $\mathrm{CH}_d(X)$ is generated by immersed subvarieties $Z \subset X$, with finitely many transverse, self-intersection points.* □

Corollary 34. *Let X be a smooth, projective variety over a field of characteristic 0. Then every $\mathrm{CH}_d(X)$ is generated by subvarieties $Z \subset X$ such that the normalization $\tau_Z : Z^{\mathrm{n}} \to Z$ is smooth, and each fiber of τ_Z contains at most $\frac{1}{2}\dim X$ points.* □

6. Positive characteristic

The positive characteristic version of Theorem 2 is not strong enough to imply the positive characteristic version of Theorem 1. We need an improved variant.

Definition 35. Let X be a projective variety over a perfect field, and $Z_X \subset X$ an irreducible subvariety. We say that Z_X is a *complete intersection image* if there are $g : Y \to X$ and a complete intersection $Z_Y \subset Y$ such that $g_*(Z_Y) = Z_X$, where g is piecewise smooth (Definition 30) and (10.1–2) also hold.

36 (Section 2 in positive characteristic)**.** With these definitions, the results of Section 2 hold over any perfect field. There are 2 points that deserve some comments.

(36.1) In the proof of Lemma 13, we need to show that the morphism $p_2 : X' \to H \times X \to X$ is piecewise smooth.

Indeed, on the exceptional divisor E, the restriction of p_2 is a $\mathbb{P}^{n-1}$-bundle over H, hence smooth. On $X' \setminus E \cong (H \times X) \setminus G$, it is the coordinate projection, which is smooth since H is smooth. (Note that [KV23, 3.7] is about finite morphisms $H \to X$. Then $p_2 : X' \to H \times X \to X$ is piecewise smooth iff $H \to X$ is piecewise smooth.)

(36.2) In the proofs of Lemmas 12–13, we choose general divisors. If the field is infinite, this is not a problem. Over finite fields, elementary arguments show that we can choose Z_Y to be irreducible. Most likely, the methods of [CP16] can be used to show that, even over finite fields, we can choose Z_Y to be smooth.

If we try to follow the proof of Theorem 2 given in Paragraph 26, everything works, except at the very first step we need to have a resolution $Z'_X \to Z_X$. Thus we get the following.

Theorem 37. *Let X be a smooth projective variety over a prefect field k, and $Z_X \subset X$ an irreducible subvariety. Assume that there is a projective resolution $Z'_X \to Z_X$. Then Z_X is a complete intersection image.*

If k is infinite, then Z_X is a smooth complete intersection image. □

When resolution is not known, there are alterations $p_i : Z'_i \to Z_X$ such that the greatest common divisor of their degrees is a power of $\operatorname{char} k$; see [dJ96, ILO14]. Let c denote the codimension. By [Kle69], $(c-1)!Z_X$ is rationally equivalent to a linear combination of subvarieties that are Chern classes, equivalently, Segre classes. By their definition [Ful98, Sec.3.1], the later are sci-images. So, as in Paragraph 27, $(c-1)!Z_X$ is rationally equivalent to a linear combination of subvarieties that are images of smooth varieties W_j.

We can now choose disjoint embeddings $Z'_i \hookrightarrow \mathbb{P}^N$ and $W_j \hookrightarrow \mathbb{P}^N$. If $\operatorname{char} k > c-1$, then there is a suitable linear combination Z of the diagonal images, such that the projection of Z is rationally equivalent to Z_X.

Thus we proved the following variant of Theorem 2.

Theorem 38. *Let X be a smooth, projective variety over a perfect field k. If $\operatorname{char} k \geq c$ then $\mathrm{CH}^c(X)$ is generated by complete intersection images.* □

We obtain the following version of Theorem 1. Again, the methods of [CP16] should settle the finite field cases.

Theorem 39. *Let X be a smooth projective variety over an infinite, prefect field k. Fix $d < \frac{1}{2}\dim X$. Then the classes of smooth subvarieties generate $\mathrm{CH}_d(X)$ if either $d \leq 3$ or $\operatorname{char} k \geq n-d$.* □

Remark 40. If $p = \operatorname{char} k < n-d$, then the proof shows that the classes of smooth subvarieties generate $p^m \mathrm{CH}_d(X)$, where p^m is the largest p-power dividing $(n-d-1)!$.

7. Open problems

Question 41. [Voi23] In Theorem 26, can one choose $g : Y \to X$ smooth?

Homogeneous spaces are discussed in [Voi23, Sec.4]; see also [KV23, Sec.5].

Question 42. What can one say for $d > \frac{1}{2}\dim X$?

Corollary 34 gives generators Z with smooth normalization $Z^{\mathrm{n}} \to Z$. However, the local structure of $Z^{\mathrm{n}} \to Z$ is not clear from the proof; see [Laz04, 7.2.17–20] and [BE10] for closely related results. Note that, in the topological setting, an unbounded set of singularities must appear by [GSz13].

Question 43. What is the subgring of $\mathrm{CH}(X)$ generated by Chern classes of algebraic vector bundles?

We get all if $\dim X \leq 2$, but not for $\dim X = 3$, as the next example shows.

Example 44. [KV23, 3.5] Let X be a smooth, proper 3-fold and E a vector bundle of rank r with Chern classes c_i. From Riemann-Roch we get that

$$\chi\big(X, E\big) - \chi\big(X, \mathcal{O}_X^{r-1} \oplus \det E\big) = \tfrac{1}{2}(K_X - c_1)c_2 + \tfrac{1}{2}c_3.$$

So, if $(K_X - c_1)c_2$ is even, then so is c_3. To get such example, let $S \subset \mathbb{P}^3$ be a very general surface of even degree $\neq 2$, and $X \subset S \times S$ an ample divisor. Then every curve-divisor intersection number is even.

Question 45. When is $\mathrm{CH}_n(X^{2n})$ generated by smooth subvarieties?

This holds for $n = 1$, and also for $n = 2$ by [Kle69]. [Ben22] gives counterexamples whenever $a(n+1) \geq 3$, where $a(m)$ is the number of 1s in the binary expansion of m.

Question 46. Let X be a smooth, projective variety. Is $\mathrm{CH}_d(X)_{\mathbb{Q}}$ generated by classes of smooth subvarieties for every d?

This is almost completely open for $d \geq \frac{1}{2}\dim X$. [Kle69] shows that $\mathrm{CH}_d(X)_{\mathbb{Q}}$ is generated by classes of subvarieties that have determinantal singularities only.

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