The structure of algebraic varieties

János Kollár

Princeton University

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with the assistance of
Jennifer M. Johnson and Sándor J. Kovács
(Written comments added for clarity that were part of the oral presentation.)

Euler, Abel, Jacobi 1751–1851

Elliptic integrals (multi-valued):

$$\int \frac{dx}{\sqrt{x^3 + ax^2 + bx + c}}.$$

To make it single-valued, look at the algebraic curve

$$C := \{(x, y) : y^2 = x^3 + ax^2 + bx + c\} \subset \mathbb{C}^2.$$

We get the integral

$$\int_\Gamma \frac{dx}{y} \quad \text{for some path } \Gamma \text{ on } C.$$

General case

Let $g(x, y)$ be any polynomial, it determines $y := y(x)$ as a multi-valued function of x.
Let $h(u, v)$ be any function.
Then

$$\int h(x, y(x))\, dx \quad \text{(multi-valued integral)}$$

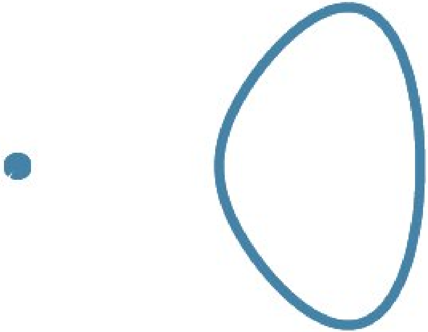
becomes

$$\int_\Gamma h(x, y)\, dx \quad \text{(single-valued integral)}$$

for some path Γ on the algebraic curve

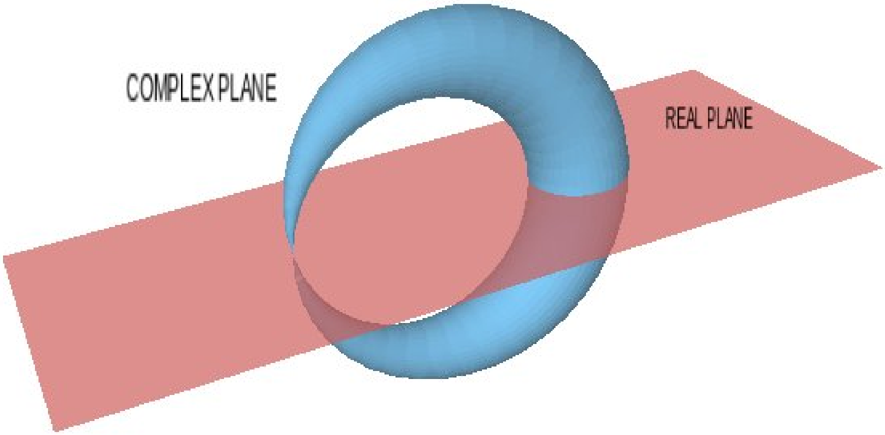
$$C := (g(x, y) = 0) \subset \mathbb{C}^2$$

Example: $C := \left(y^2 = (x+1)^2 x(1-x)\right)$. Real picture:



(Comment: looks like 2 parts, but the real picture can decieve. Complex picture is better.)

Example: $C := \left(y^2 = (x+1)^2 x(1-x)\right)$. Complex picture:



(Comment: the picture is correct in projective space only. The correct picture has 2 missing points at infinity.)

Substitution in integrals

Question (Equivalence)

Given two algebraic curves C *and* D*, when can we transform every integral* $\int_C h \, dx$ *into an integral* $\int_D g \, dx$*?*

Question (Simplest form)

Among all algebraic curves C_i *with equivalent integrals, is there a* ***simplest****?*

Example

For $C := \left(y^2 = (x+1)^2 x(1-x)\right)$
the substitution

$$x = \frac{1}{1-t^2},\ y = \frac{t(2-t^2)}{(1-t^2)^2} \quad \text{with inverse} \quad t = \frac{y}{x(x+1)}$$

transforms $\int_\Gamma h(x,y)dx$ into an integral

$$\int h\left(\frac{1}{1-t^2}, \frac{t(2-t^2)}{(1-t^2)^2}\right) \frac{2t}{(1-t^2)^2}\, dt.$$

Theorem (Riemann, 1851)

For every algebraic curve $C \subset \mathbb{C}^2$ *we have*

- S*: a compact Riemann surface and*
- *meromorphic, invertible* $\phi : S \dashrightarrow C$ *establishing an isomorphism between*

– $\mathrm{Merom}(C)$: *meromorphic function theory of* C *and*

– $\mathrm{Merom}(S)$: *meromorphic function theory of* S.

MINIMAL MODEL PROBLEM

Question

X – any algebraic variety.
Is there another algebraic variety X^{m} such that

- $\mathrm{Merom}(X) \cong \mathrm{Merom}(X^{\mathrm{m}})$ *and*
- *the geometry of X^{m} is the simplest possible?*

Answers:

- Curves: Riemann, 1851
- Surfaces: Enriques, 1914; Kodaira, 1966
- Higher dimensions: Mori's program 1981–
  - also called Minimal Model Program
  - many open questions

MODULI PROBLEM

Question

- *What are the simplest families of algebraic varieties?*
- *How to transform any family into a simplest one?*

Answers:

- Curves: Deligne–Mumford, 1969
- Surfaces: Kollár – Shepherd-Barron, 1988; Alexeev, 1996
- Higher dimensions: the KSBA-method works
  but needs many technical details

Algebraic varieties 1

Affine algebraic set: common zero-set of polynomials

$$\begin{aligned} X^{\mathrm{aff}} &= X^{\mathrm{aff}}(f_1, \ldots, f_r) \subset \mathbb{C}^N \\ &= \{(x_1, \ldots, x_N) : f_i(x_1, \ldots, x_N) = 0 \ \forall i\}. \end{aligned}$$

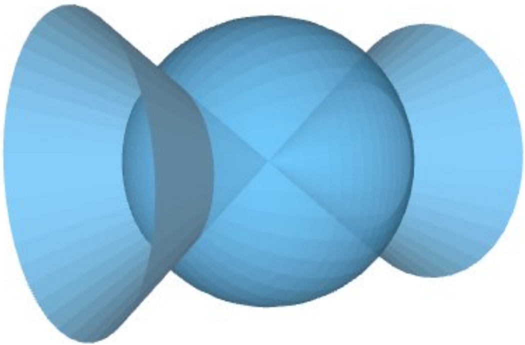
Hypersurfaces: 1 equation, $X(f) \subset \mathbb{C}^N$.

Complex dimension: $\dim \mathbb{C}^N = N$ $(=\frac{1}{2}(\text{topological dim}))$

Curves, surfaces, 3-folds, ...

Algebraic varieties 2

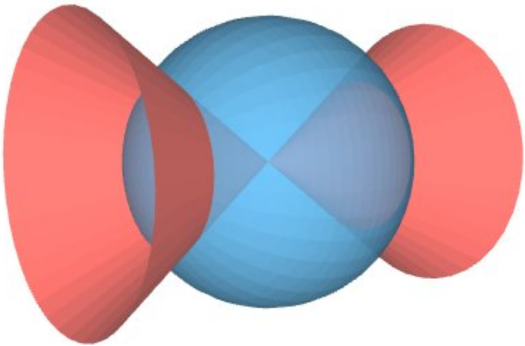
Example: $x^4 - y^4 + z^4 + 2x^2z^2 - x^2 + y^2 - z^2 = 0$.



(Comment: What is going on? Looks like a sphere and a cone together.)

Explanation:

$$x^4 - y^4 + z^4 + 2x^2z^2 - x^2 + y^2 - z^2 =$$
$$(x^2 + y^2 + z^2 - 1)(x^2 - y^2 + z^2)$$



Variety= irreducible algebraic set

Algebraic varieties 3

Projective variety: $X \subset \mathbb{CP}^N$, closure of an affine variety.
Homogeneous coordinates: $[x_0 : \cdots : x_N] = [\lambda x_0 : \cdots : \lambda x_N]$
$\Rightarrow p(x_0, \ldots, x_N)$ makes no sense

Except: If p is homogeneous of degree d then
$p(\lambda x_0, \ldots, \lambda x_N) = \lambda^d p(x_0, \ldots, x_N)$.

Well-defined notions are:

- Zero set of homogeneous p.
- Quotient of homogeneous p, q of the same degree

$$f(x_0, \ldots, x_N) = \frac{p_1(x_0, \ldots, x_N)}{p_2(x_0, \ldots, x_N)}$$

Rational functions on $\mathbb{CP}^N$, and, by restriction, rational functions on $X \subset \mathbb{CP}^N$.

Theorem (Chow, 1949; Serre, 1956)

$M \subset \mathbb{CP}^N$ – *any closed subset that is* *locally* *the common zero set of analytic functions. Then*

- M *is algebraic:* *globally* *given as the common zero set of homogeneous polynomials and*
- *every meromorphic function on* M *is rational:* *globally* *the quotient of two homogeneous polynomials.*

Non-example: $M := (y = \sin x) \subset \mathbb{C}^2 \subset \mathbb{CP}^2$.

(Comment: The closure at infinity is not locally analytic.)

Rational maps = meromorphic maps

Definition

– $X \subset \mathbb{CP}^N$ *algebraic variety*
– $f_0, \ldots, f_M$ *rational functions.*
Map (or rational map) $\mathbf{f} : X \dashrightarrow \mathbb{CP}^M$ *given by*

$$p \mapsto \left[f_0(p):\cdots:f_M(p)\right] \in \mathbb{CP}^M.$$

Where is $\mathbf{f}$ defined?

- away from poles and common zeros, but, as an example,

let $\pi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^1$ be given by $[x:y:z] \mapsto \left[\frac{x}{z} : \frac{y}{z}\right]$.

Note that

$$\left[\frac{x}{z} : \frac{y}{z}\right] = \left[\frac{x}{y} : 1\right] = \left[1 : \frac{y}{x}\right].$$

So π is defined everywhere except $(0:0:1)$.

Isomorphism

Definition

X, Y are isomorphic if there are everywhere defined maps

$$f : X \to Y \text{ and } g : Y \to X$$

that are inverses of each other.

Denoted by $X \cong Y$.

Isomorphic varieties are essentially the same.

Birational equivalence

Unique to algebraic geometry!

Definition

X, Y are *birational*

if there are rational maps

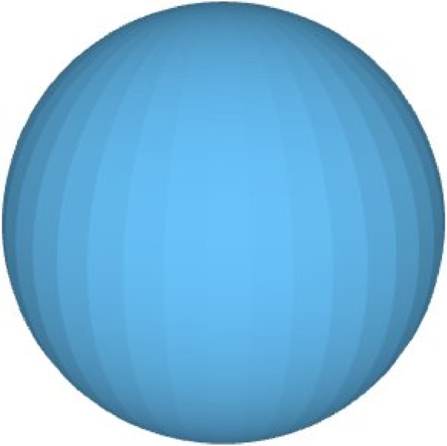
$f : X \dashrightarrow Y$ *and* $g : Y \dashrightarrow X$ *such that*

- $\phi_Y \mapsto \phi_X := \phi_Y \circ f$ *and* $\phi_X \mapsto \phi_Y := \phi_X \circ g$ *give* $\mathrm{Merom}(X) \cong \mathrm{Merom}(Y)$.
- *Equivalent: There are* $Z \subsetneq X$ *and* $W \subsetneq Y$ *such that* $(X \setminus Z) \cong (Y \setminus W)$.

Denoted by $X \overset{\mathrm{bir}}{\sim} Y$.

(Comment: The next 12 slides show that, in topology, one can make a sphere and a torus from a sphere by cutting and pasting. Nothing like this can be done with algebraic varieties. Notice that if we keep the upper cap open and the lower cap closed then the construction is naturally one-to-one on points.)

Non-example from topology



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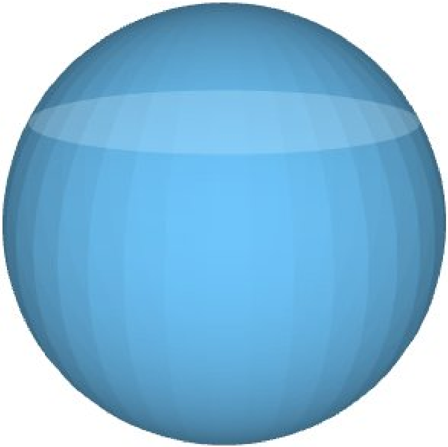

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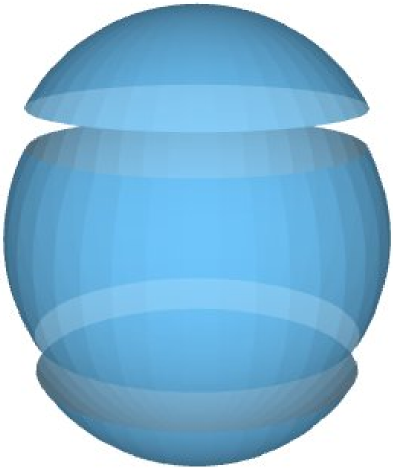
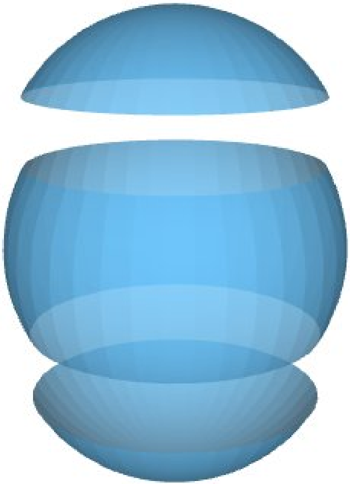


Non-example from topology



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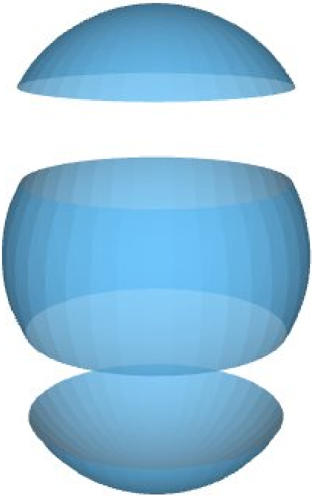

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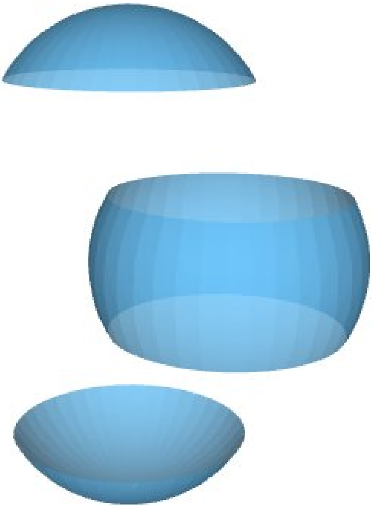

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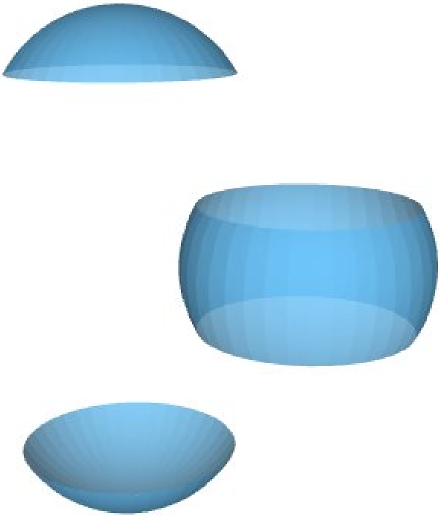

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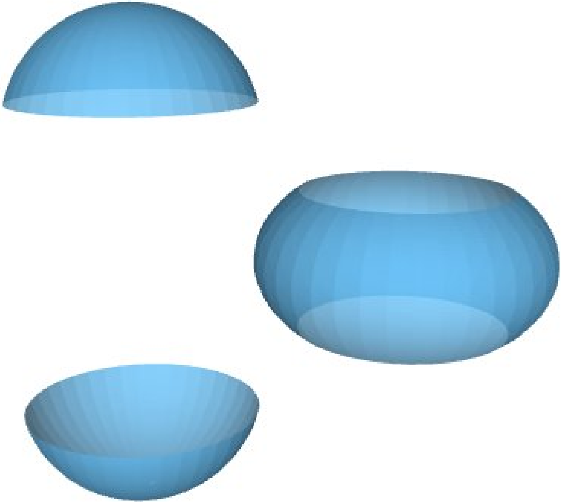
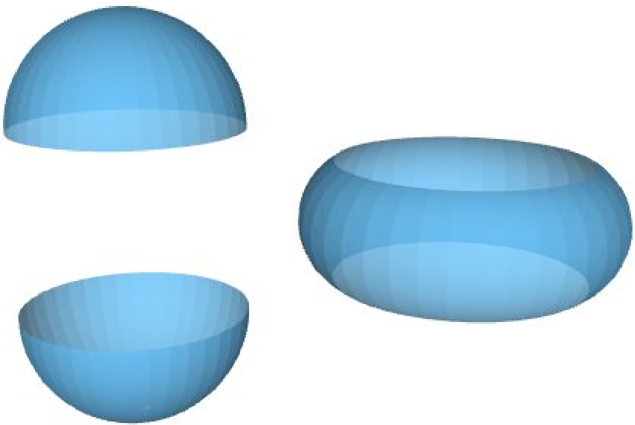
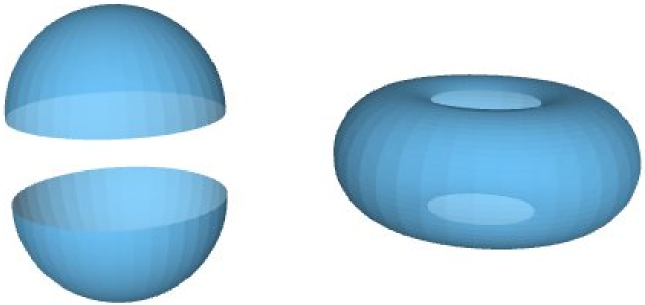


Non-example from topology



Non-example from topology



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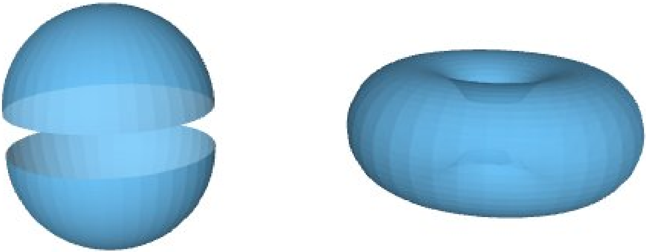

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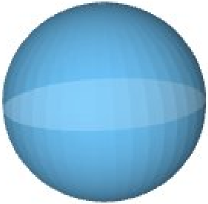

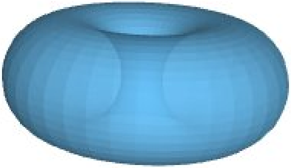

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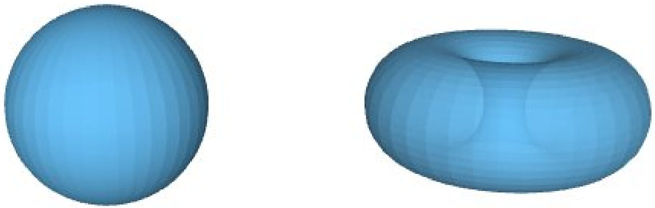


Example of birational equivalence

Affine surface $S := (xy = z^3) \subset \mathbb{C}^3$.
It is birational to $\mathbb{C}^2_{uv}$ as shown by

$$\begin{array}{ccc} \mathbb{C}^3 & \dashleftarrow\dashrightarrow & \mathbb{C}^2 \\ f : (x, y, z) & \longmapsto & (x/z, y/z) \\ (u^2 v, uv^2, uv) & \longleftarrow & (u, v) : g \end{array}$$

f – not defined if $z = 0$
g – defined but maps the coordinate axes to $(0, 0, 0)$.

- $S \ncong \mathbb{C}^2$ but
- $S \setminus (z = 0) \cong \mathbb{C}^2 \setminus (uv = 0)$

Basic rule of thumb

Assume $X \overset{\text{bir}}{\sim} Y$, hence $(X \setminus Z) \cong (Y \setminus W)$.

Many questions about X can be answered by

- first studying the same question on Y
- then a similar question involving Z and W.

Aim of the Minimal Model Program:

Exploit this in two steps:

- Given a question and X, find $Y \overset{\text{bir}}{\sim} X$ that is best adapted to the question. This is the Minimal Model Problem.
- Set up dimension induction to deal with Z and W.

When is a variety simple?

- Surfaces: Castelnuovo, Enriques (1898–1914)
- Higher dimensions: There was not even a conjecture until
  - Mori, Reid (1980-82)
  - Kollár–Miyaoka–Mori (1992)

Need: Canonical class or first Chern class

We view it as a map: $\{\text{algebraic curves in } X\} \to \mathbb{Z}$,

it is denoted by: $\int_C c_1(X)$ or $-(K_X \cdot C)$.

(Comment: next few slides give the definition.)

Volume forms

Measure or volume form on $\mathbb{R}^n$:

$$s(x_1, \ldots, x_n) \cdot dx_1 \wedge \cdots \wedge dx_n.$$

Complex volume form: locally written as

$$\omega := h(z_1, \ldots, z_n) \cdot dz_1 \wedge \cdots \wedge dz_n.$$

ω gives a real volume form $\left(\frac{\sqrt{-1}}{2}\right)^n \omega \wedge \bar{\omega}$

(Comment: for the signs note that

$$\begin{aligned} dz \wedge d\bar{z} = (dx + \sqrt{-1}dy) \wedge (dx - \sqrt{-1}dy) \\ = -2\sqrt{-1}dx \wedge dy) \end{aligned}$$

TENSION

Differential geometers want C^∞ volume forms:
$h(z_1, \dots, z_n)$ should be C^∞-functions.

Algebraic/analytic geometers want meromorphic forms:
$h(z_1, \dots, z_n)$ should be meromorphic functions.

Simultaneously possible only for Calabi–Yau varieties.

Connection: Gauss–Bonnet theorem

X – smooth, projective variety,
ω_r – C^∞ volume form,
ω_m – meromorphic volume form,
$C \subset X$ – algebraic curve.

Definition (Chern form or Ricci curvature)

$$\tilde{c}_1(X, \omega_r) := \frac{\sqrt{-1}}{\pi} \sum_{ij} \frac{\partial^2 \log |h_r(\mathbf{z})|}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$

Definition (Algebraic degree)

$$\deg_C \omega_m := \#(\textit{zeros of } \omega_m \textit{ on } C) - \#(\textit{poles of } \omega_m \textit{ on } C),$$

zeros/poles counted with multiplicities.
(assuming ω_m not identically 0 or ∞ on C.)

Theorem (Gauss–Bonnet)

X – smooth, projective variety,
ω_r – C^∞ volume form,
ω_m – meromorphic volume form,
$C \subset X$ – algebraic curve. Then

$$\int_C \tilde{c}_1(X, \omega_r) = -\deg_C \omega_m$$

is independent of ω_r and ω_m .

Denoted by $\int_C c_1(X)$.

(Comment on the minus sign: differential geometers prefer the tangent bundle; volume forms use the cotangent bundle.)

Building blocks of algebraic varieties

Negatively curved: $\int_C c_1(X) < 0$ for every curve $C \subset X$.
Largest class of the three.

Flat or Calabi–Yau: $\int_C c_1(X) = 0$ for every curve $C \subset X$.
Important role in string theory and mirror symmetry.

Positively curved or Fano: $\int_C c_1(X) > 0$ for every curve.
Few but occur most frequently in applications.

Kähler–Einstein metric : pointwise conditions.
negative/flat: Yau, Aubin, ...
positive: still not settled

Mixed type I

Semi-negatively curved or Kodaira–Iitaka type

$\int_C c_1(X) \leq 0$ for every curve $C \subset X$.

Structural conjecture (Main open problem)

– There is a unique $I_X : X \to I(X)$ such that

$\int_C c_1(X) = 0$ iff $C \subset$ fiber of I_X.

– $I(X)$ is negatively curved in a "suitable sense."

Intermediate case: $0 < \dim I(X) < \dim X$:

family of lower dimensional Calabi–Yau varieties

parametrized by the lower dimensional variety $I(X)$.

(Comment: this is one example why families of varieties are important to study.)

Mixed type II

Positive fiber type

I really would like to tell you that:

– There is a unique $m_X : X \to M(X)$ such that $\int_C c_1(X) > 0$ if $C \subset$ fiber of m_X.

– $M(X)$ is semi–negatively curved.

BUT this is too restrictive.

We fix the definition later.

Main Conjecture

Conjecture (Minimal model conjecture, extended)

Every algebraic variety X is birational to a variety X^{m} that is

- *either semi-negatively curved*
- *or has positive fiber type.*

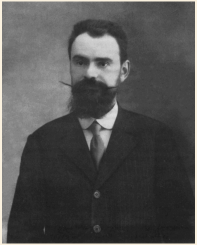
X^{m} is called a minimal model of X (especially in first case)

Caveat. X^{m} may have singularities

(This was a rather difficult point historically.)

Some history

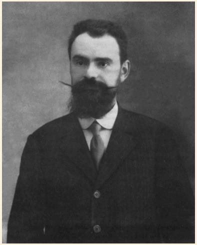
Some history



Enriques

Kodaira

Some history



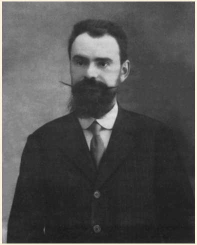
Enriques

Mori

Kodaira

Reid

Some history



Enriques

Mori

Hacon

Kodaira

Reid

McKernan

Rationally connected varieties

Theme: plenty of rational curves $\mathbb{CP}^1 \to X$.

Theorem

X – smooth projective variety. Equivalent:

- *$\forall\, x_1, x_2 \in X$ there is $\mathbb{CP}^1 \to X$ through them.*
- *$\forall\, x_1, \ldots, x_r \in X$ there is a $\mathbb{CP}^1 \to X$ through them.*
- *$\forall\, x_1, \ldots, x_r \in X$ + tangent directions $v_i \in T_{x_i}X$ there is a $\mathbb{CP}^1 \to X$ through them with given directions.*

Definition

X is rationally connected or RC if the above hold.

Properties of rationally connected varieties

- Positively curved $\Rightarrow$ RC
  (Nadel, Campana, Kollár–Miyaoka–Mori, Zhang)
- Birational and smooth deformation invariant
  (Kollár–Miyaoka–Mori)
- Good arithmetic properties:
  p-adic fields (Kollár),
  finite fields (Kollár–Szabó, Esnault)
  $\mathbb{C}(t)$ (Graber–Harris–Starr, de Jong–Starr).
- Loop space of RC is RC (Lempert–Szabó).

Problem

Is RC a symplectic property?

Positive fiber type

Definition

X is of positive fiber type if there is a unique $m_X : X \to M(X)$ such that

- *almost all fibers are rationally connected and*
- *$M(X)$ is semi–negatively curved.*