THE NASH PROBLEM ON ARC FAMILIES OF SINGULARITIES

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ABSTRACT. Nash proved that every irreducible component of the space of arcs through a singularity corresponds to an exceptional divisor that appears on every resolution. He asked if the converse also holds: does every such exceptional divisor correspond to an arc family? We prove that the converse holds for toric singularities but fails in general.

1. INTRODUCTION

In a 1968 preprint, later published as [21], Nash introduced arc spaces and jet schemes for algebraic and analytic varieties. The problems raised by Nash were studied by Bouvier, Gonzalez-Sprinberg, Hickel, Lejeune-Jalabert, Nobile, Reguera-Lopez and others, see [3, 10, 11, 16, 17, 18, 22, 23].

The study of these spaces was further developed by Kontsevich, Denef and Loeser as the theory of motivic integration, see [15, 7]. Further interesting applications of jet spaces are given by Mustaţă [20].

The main subject of the paper of Nash is the map from the set of irreducible components of the space of arcs through singular points (families of arcs in the original terminology of Nash) to the set of essential components of a resolution of singularities. Roughly speaking, these are the irreducible components of the exceptional set of a given resolution that appear on every possible resolution, see Definition 2.3.

We call this map the *Nash map*, see Theorem 2.15 for a precise definition. The Nash map is always injective and Nash asked if it is always bijective. This problem remained open even for 2-dimensional singularities, though many cases were settled in [23].

In this paper we prove that the Nash map is bijective for toric singularities in any dimension, see Theorem 3.16. On the other hand we also show that the Nash map is not bijective in general. For instance, the 4-dimensional hypersurface singularity

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6 = 0$$

has only 1 irreducible family of arcs but 2 essential divisors over any algebraically closed field of characteristic $\neq 2, 3$. See Example 4.5.

In §2 we define the Nash map and show its injectivity. This is essentially taken from [21] with some scheme theoretic details filled in. The Nash map for toric singularities is studied in §3. Counter examples are given in §4.

In this paper, the ground field k is an algebraically closed field of arbitrary characteristic. A k-scheme is not necessarily of finite type unless we state otherwise. A variety means a separated, irreducible and reduced scheme of finite type over k. Every variety X that we consider is assumed to have a resolution of singularities $f : Y \longrightarrow X$ which is an isomorphism over the smooth locus and whose exceptional set is purely one codimensional. Without this or similar assumptions the definition of essential divisors and components would not make sense. The existence of resolutions is known in characteristic zero and for toric varieties in any characteristic.

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2. The space of arcs and the Nash problem

Definition 2.1. Let X be a variety, $g : X_1 \longrightarrow X$ a proper birational morphism from a normal variety X_1 and $E \subset X_1$ an irreducible exceptional divisor of g. Let $f : X_2 \longrightarrow X$ be another proper birational morphism from a normal variety X_2. The birational map $f^{-1} \circ g : X_1 \dashrightarrow X_2$ is defined on a (nonempty) open subset E^0 of E. The closure of $(f^{-1} \circ g)(E^0)$ is well defined. It is called the *center* of E on X_2.

We say that E appears in f (or in X_2), if the center of E on X_2 is also a divisor. In this case the birational map $f^{-1} \circ g : X_1 \dashrightarrow X_2$ is a local isomorphism at the generic point of E and we denote the birational transform of E on X_2 again by E. For our purposes $E \subset X_1$ is identified with $E \subset X_2$. (Strictly speaking, we should be talking about the corresponding *divisorial valuation* instead.) Such an equivalence class is called an *exceptional divisor over* X.

Definition 2.2. Let X be a variety over k. In this paper, by a *resolution* of the singularities of X we mean a proper, birational morphism $f : Y \longrightarrow X$ with Y non-singular such that $Y \setminus f^{-1}(\operatorname{Sing} X) \longrightarrow X \setminus \operatorname{Sing} X$ is an isomorphism.

A resolution $f : Y \longrightarrow X$ is called a *divisorial resolution* of X if the exceptional set is of pure codimension one.

If X is factorial (or at least $\mathbb{Q}$-factorial) then every resolution is divisorial.

Definition 2.3. An exceptional divisor E over X is called an *essential divisor* over X if for every resolution $f : Y \longrightarrow X$ the center of E on Y is an irreducible component of $f^{-1}(\operatorname{Sing} X)$.

An exceptional divisor E over X is called a *divisorially essential divisor* over X if for every divisorial resolution $f : Y \longrightarrow X$ the center of E on Y is a divisor, (and hence also an irreducible component of $f^{-1}(\operatorname{Sing} X)$).

For a given resolution $f : Y \longrightarrow X$, the set

$$\mathcal{E} = \mathcal{E}_{Y/X} = \left\{ \begin{array}{c} \text{irreducible components of } f^{-1}(\operatorname{Sing} X) \\ \text{which are centers of essential divisors over } X \end{array} \right\}$$

corresponds bijectively to the set of all essential divisors over X.

Therefore we call an element of $\mathcal{E}$ an *essential component* on Y.

Similarly, we can talk about *divisorially essential components* on Y.

It is clear that an essential divisor is also a divisorially essential divisor. We do not know any examples when the two notions are different. In §3 we will see that they coincide for toric singularities.

Example 2.4. Let (X, x) be a normal 2-dimensional singularity. Then the set of the divisorially essential divisors over X coincides with the set of the exceptional curves appearing on the minimal resolution $X' \longrightarrow X$. These are also the essential components on X'.

Example 2.5. Proposition 4 of [1] asserts that if E is an exceptional divisor of a birational morphism $Y \longrightarrow Y'$ with Y' smooth then E is ruled, that is, E is birational to $F \times \mathbb{P}^1$ for some variety F. As noted by Nash, this implies that any nonruled exceptional divisor of a resolution $Y \longrightarrow X$ is essential.

Example 2.6. Let (X, x) be a canonical singularity which admits a crepant divisorial resolution. A quite large group of such singularities is known (see, for example, [6] and the references there). Then the set of the essential divisors over X and also the set of the divisorially essential divisors over X coincide with the set of the crepant exceptional divisors. Indeed, a divisorial essential divisor should be one of the crepant exceptional divisors because of the existence of a divisorial crepant resolution. On the other hand, a crepant exceptional divisor cannot be contracted on a non-singular model of X, because if it could be contracted, the discrepancy of the crepant component would have to be positive. This shows that every crepant divisor is an essential component.

Definition 2.7. Let X be a scheme of finite type over k and $K \supset k$ a field extension. A morphism $\operatorname{Spec} K[t]/(t^{m+1}) \longrightarrow X$ is called an m-jet of X and $\operatorname{Spec} K[[t]] \longrightarrow X$ is called an *arc* of X. We denote the closed point of $\operatorname{Spec} K[[t]]$ by 0 and the generic point by η.

2.8. Let X be a scheme of finite type over k. Let $\mathcal{S}ch/k$ be the category of k-schemes and $\mathcal{S}et$ the category of sets. Define a contravariant functor $F_m : \mathcal{S}ch/k \longrightarrow \mathcal{S}et$ by

$$F_m(Y) = \operatorname{Hom}_k(Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X).$$

Then, F_m is representable by a scheme X_m of finite type over k, that is

$$\operatorname{Hom}_k(Y, X_m) \simeq \operatorname{Hom}_k(Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X).$$

This X_m is called the *scheme of m-jets* of X. The canonical surjection $k[t]/(t^{m+1}) \longrightarrow k[t]/(t^m)$ induces a morphism $\phi_m : X_m \longrightarrow X_{m-1}$. Define $\pi_m = \phi_1 \circ \cdots \circ \phi_m : X_m \longrightarrow X$. A point $x \in X_m$ gives an m-jet $\alpha_x : \operatorname{Spec} K[t]/(t^{m+1}) \longrightarrow X$ and $\pi_m(x) = \alpha_x(0)$, where K is the residue field at x.

Let $X_\infty = \varprojlim_m X_m$ and call it the *space of arcs* of X. X_∞ is not of finite type over k but it is a scheme, see [7]. Denote the canonical projection $X_\infty \longrightarrow X_m$ by η_m and the composite $\pi_m \circ \eta_m$ by π. A point $x \in X_\infty$ gives an arc $\alpha_x : \operatorname{Spec} K[[t]] \longrightarrow X$ and $\pi(x) = \alpha_x(0)$, where K is the residue field at x.

Using the representability of F_m we obtain the following universal property of X_∞:

Proposition 2.9. *Let X be a scheme of finite type over k. Then*

$$\operatorname{Hom}_k(Y, X_\infty) \simeq \operatorname{Hom}_k(Y \widehat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]], X)$$

for an arbitrary k-scheme Y, where $Y \widehat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]]$ means the formal completion of $Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[[t]]$ along the subscheme $Y \times_{\operatorname{Spec} k} \{0\}$.

Corollary 2.10. *There is a universal family of arcs*

$$X_\infty \widehat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \longrightarrow X.$$

Definition 2.11. Let X be a k-variety with singular locus $\operatorname{Sing} X \subset X$. Every point x of the inverse image $\pi^{-1}(\operatorname{Sing} X) \subset X_\infty$ corresponds to an arc $\alpha_x : \operatorname{Spec} K[[t]] \longrightarrow X$ such that $\alpha_x(0) \in \operatorname{Sing} X$, where K is the residue field at x. $\pi^{-1}(\operatorname{Sing} X)$ is the space of arcs through $\operatorname{Sing} X$.

Decompose $\pi^{-1}(\operatorname{Sing} X)$ into its irreducible components

$$\pi^{-1}(\operatorname{Sing} X) = (\bigcup_{i \in I} C_i) \cup (\bigcup_{j \in J} C'_j),$$

where the C_i's are the components with a point x corresponding to an arc α_x such that $\alpha_x(\eta) \notin \operatorname{Sing} X$, while the C'_j's are the components without such points. We call the C_i's the *good components* of the space of arcs through $\operatorname{Sing} X$.

The notion of "arc families" in [21] is the same as the above concept of good components.

The next lemma shows that in characteristic zero every irreducible component of $\pi^{-1}(\operatorname{Sing} X) \subset X_\infty$ is good. This can be viewed as a strong form of Kolchin's irreducibility theorem [12, Chap.IV,Prop.10]. See also [9]. It is also interesting to compare this with the results of [20] according to which the jet spaces X_m are usually reducible.

Lemma 2.12. *Let k be a field of characteristic zero and X a k-variety. Then every arc through $\operatorname{Sing} X$ is a specialization of an arc through $\operatorname{Sing} X$ whose generic point maps into $X \setminus \operatorname{Sing} X$.*

Proof. We may assume that X is affine. Pick any arc $\phi : \operatorname{Spec} k'[[s]] \longrightarrow X$ such that $\phi(0) \in \operatorname{Sing} X$. Let Y be the Zariski closure of the image of ϕ. Then $\mathcal{O}_Y$ is an integral domain and ϕ corresponds to an injection $\mathcal{O}_Y \longrightarrow k'[[s]]$, where we can take k' to be algebraically closed. We are done if $Y \not\subset \operatorname{Sing} X$. Otherwise we write ϕ as a specialization in two steps.

First we prove that ϕ is a specialization of an arc $\Phi : \operatorname{Spec} K[[s]] \longrightarrow Y \subset X$ such that $\Phi(0)$ is the generic point of Y.

We have an embedding $k'[[s]] \hookrightarrow k'[[S,T]]$ which sends s to $S+T$. It is easy to check that the composite

$$k'[[s]] \hookrightarrow k'[[S,T]] \longrightarrow k'[[S,T]]/(T) \cong k'[[S]]$$

is an isomorphism. Thus we obtain Φ as the composite

$$\mathcal{O}_Y \xrightarrow{\phi} k'[[s]] \hookrightarrow k'[[S,T]] \hookrightarrow k'((T))[[S]].$$

Set $K = k'((T^{1/n} : n = 1, 2, \ldots))$, the algebraic closure of $k'((T))$. The closed point of $\operatorname{Spec} K[[S]]$ maps to the ideal $\Phi^{-1}(S)$, but the pull back of (S) to $k'[[s]]$ is already the zero ideal. Thus the closed point of $\operatorname{Spec} K[[S]]$ maps to the generic point of Y.

Repeatedly cutting with hypersurfaces containing Y we obtain a subvariety $Y \subset Z \subset X$ such that $\dim Z = \dim Y + 1$ and X is smooth along the generic points of Z. Let $n : \bar{Z} \longrightarrow Z$ be the normalization and $\bar{Y} \subset \bar{Z}$ the preimage of Y with reduced scheme structure. $\bar{Y} \longrightarrow Y$ is finite, surjective, and so generically étale in characteristic zero. Thus the arc $\Phi : \operatorname{Spec} K[[S]] \longrightarrow Y$ can be lifted to $\bar{\Phi} : \operatorname{Spec} K[[S]] \longrightarrow \bar{Y}$. $\bar{Z}$ is normal, so smooth along the generic point of $\bar{Y}$. Thus $\bar{\Phi}$ is the specialization of an arc through $\bar{Y}$ whose generic point maps to the generic point of $\bar{Z}$. Projecting to Z we obtain Φ and hence ϕ as the specialization of an arc through $\operatorname{Sing} X$ whose generic point maps into $X \setminus \operatorname{Sing} X$. $\square$

Example 2.13. Let k have characteristic p and consider the surface $S = (x^p = y^p z) \subset \mathbb{A}^3$ with singular locus $Y = (x = y = 0)$. The normalization is $\bar{S} \cong \mathbb{A}^2$ with $(u, v) \mapsto (uv, v, u^p)$. The preimage of Y is $\bar{Y} = (v = 0)$ and $\bar{Y} \longrightarrow Y$ is purely inseparable. Thus a smooth arc in Y can not be lifted to $\bar{Y}$ and it is also not the specialization of an arc through Y whose generic point maps into $S \setminus \operatorname{Sing} S$. In this case $\pi^{-1}(\operatorname{Sing} S) \subset S_\infty$ has a component which is not good.

Lemma 2.14. *Let $f : Y \longrightarrow X$ be a resolution of the singularities of X and $E_1, \dots, E_r$ the irreducible components of exceptional sets on Y. For a good component C_i, let C_i^o denote the open subset of C_i consisting of arcs $\alpha_x : \operatorname{Spec} K[[t]] \longrightarrow X$ such that $\alpha_x(\eta) \notin \operatorname{Sing} X$. Then, for every $x \in C_i^o$ the arc α_x can be uniquely lifted to an arc $\tilde{\alpha}_x : \operatorname{Spec} K[[t]] \longrightarrow Y$.*

Proof. As f is isomorphic outside of $\operatorname{Sing} X$ and $\alpha_x(\eta) \notin \operatorname{Sing} X$, we obtain the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K((t)) & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ \operatorname{Spec} K[[t]] & \xrightarrow{\alpha_x} & X. \end{array}$$

Since f is proper, there exists a unique morphism $\tilde{\alpha}_x : \operatorname{Spec} K[[t]] \longrightarrow Y$ such that $f \circ \tilde{\alpha}_x = \alpha_x$ by the valuative criterion of properness. $\square$

This $\tilde{\alpha}_x$ is called the lifting of α_x. Now we have a map

$$\varphi : \text{points of } (\bigcup_i C_i^o) \longrightarrow \text{points of } (\bigcup_l E_l)$$

given by $x \mapsto \tilde{\alpha}_x(0)$. We emphasize that this map is not a continuous map of schemes. In fact, the image of an irreducible subset is not necessarily irreducible.

Theorem 2.15 (Nash [21]). *Let X be a k-variety and $f : Y \longrightarrow X$ a resolution of singularities. Let $\{C_i : i \in I\}$ be the good components of the space of arcs through* $\operatorname{Sing} X$ *and let z_i denote the generic point of C_i. Then:*

(i) *$\varphi(z_i)$ is the generic point of an exceptional component $E_{l_i} \subset Y$ for some l_i.*

(ii) *For every $i \in I$, E_{l_i} is an essential component on Y.*

(iii) *The resulting Nash map*

$$\left\{\begin{array}{c}\textit{good components}\\ \textit{of the space of arcs}\\ \textit{through } \operatorname{Sing} X\end{array}\right\} \xrightarrow{\mathcal{N}} \left\{\begin{array}{c}\textit{essential}\\ \textit{components}\\ \textit{on } Y\end{array}\right\} \simeq \left\{\begin{array}{c}\textit{essential}\\ \textit{divisors}\\ \textit{over } X\end{array}\right\}$$

given by $C_i \mapsto E_{l_i}$ is injective. In particular, there are only finitely many good components of the space of arcs through $\operatorname{Sing} X$.

Proof. The resolution $f : Y \longrightarrow X$ induces a morphism $f_\infty : Y_\infty \longrightarrow X_\infty$ of schemes. Let $\pi^Y : Y_\infty \longrightarrow Y$ be the canonical projection. As Y is non-singular, $(\pi^Y)^{-1}(E_l)$ is irreducible for every l. Denote by $(\pi^Y)^{-1}(E_l)^o$ the open subset of $(\pi^Y)^{-1}(E_l)$ consisting of the points y corresponding to arcs $\beta_y : \operatorname{Spec} K[[t]] \longrightarrow Y$ such that $\beta_y(\eta) \notin \bigcup_l E_l$. By restriction f_∞ gives $f'_\infty : \bigcup_{l=1}^r (\pi^Y)^{-1}(E_l)^o \longrightarrow \bigcup_{i\in I} C_i^o$. For a point $x \in C_i^o$, let $\alpha_x : \operatorname{Spec} K[[t]] \longrightarrow X$ be the corresponding arc, where K is the residue field at x. The lifting $\tilde{\alpha}_x : \operatorname{Spec} K[[t]] \longrightarrow Y$ of α_x obtained in Lemma 2.14 determines a K-valued point $\beta : \operatorname{Spec} K \longrightarrow Y_\infty$. Denote the image of β by y, then $f_\infty(y) = x$. Therefore, f'_∞ is surjective. Hence, for each $i \in I$ there is $1 \le l_i \le r$ such that the generic point y_{l_i} of $(\pi^Y)^{-1}(E_{l_i})^o$ is mapped to the generic point z_i of C_i^o. Let $\tilde{\alpha}_{z_i}$ be the lifting of the arc α_{z_i} corresponding to z_i and let $\beta_{y_{l_i}}$ be the arc of Y corresponding to y_{l_i}. Then $\beta_{y_{l_i}} = \tilde{\alpha}_{z_i}$. This is proved as follows: Let L and K be the residue fields at y_{l_i} and z_i, respectively and $g : \operatorname{Spec} L[[t]] \longrightarrow \operatorname{Spec} K[[t]]$ be the canonical morphism induced from the inclusion $K \longrightarrow L$. Then $\beta_{y_{l_i}} = \tilde{\alpha}_{z_i} \circ g$. From this, we have $K = L$ and therefore $\beta_{y_{l_i}} = \tilde{\alpha}_{z_i}$. Note that $\beta_{y_{l_i}}(0) = \pi^Y(y_{l_i})$, which is the generic point of E_{l_i}. To finish the proof of (i), just recall $\varphi(z_i) = \tilde{\alpha}_{z_i}(0) = \beta_{y_{l_i}}(0)$.

Next, we can see that the map $C_i \mapsto E_{l_i}$ is injective. Indeed, if $E_{l_i} = E_{l_j}$ for $i \neq j$, then $z_i = f'_\infty(y_{l_i}) = f'_\infty(y_{l_j}) = z_j$, a contradiction.

To prove that the $\{E_{l_i} : i \in I\}$ are essential components on Y, let $Y' \longrightarrow X$ be another resolution and $\tilde{Y} \longrightarrow X$ a divisorial resolution which factors through both Y and Y'. Let $E'_{l_i} \subset Y'$ and $\tilde{E}_{l_i} \subset \tilde{Y}$ be the irreducible components of the exceptional sets corresponding to C_i. Then, we can see that E_{l_i} and E'_{l_i} are the image of $\tilde{E}_{l_i}$. This shows that $\tilde{E}_{l_i}$ is an essential divisor over X and therefore E_{l_i} is an essential component on Y. □

Nash poses the following problem in his paper [21, p.36].

Problem 2.16. *Is the Nash map bijective?*

3. The Nash problem for toric singularities

3.1. We use the notation and terminology of [8]. Let M be the free abelian group $\mathbb{Z}^n$ $(n \geq 2)$ and N its dual $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. We denote $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{Z}} \mathbb{R}$ by $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$, respectively. The canonical pairing $\langle\ ,\ \rangle : N \times M \longrightarrow \mathbb{Z}$ extends to $\langle\ ,\ \rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \longrightarrow \mathbb{R}$. For a finite fan Δ in $N_{\mathbb{R}}$, the corresponding toric variety is denoted by $X = X(\Delta)$. For the primitive vector v in a one-dimensional cone $\tau \in \Delta$, denote the invariant divisor $\overline{orb(\tau)}$ in X by D_v.

For a cone $\tau \in \Delta$ denote by U_τ the invariant affine open subset which contains $orb\ \tau$ as the unique closed orbit. A cone τ is called regular or non-singular, if its generators can be extended to a basis of N. A cone is called singular, if it is not regular. Note that a cone τ is regular, if and only if U_τ is non-singular. A cone generated by $v_1, \dots, v_r \in N$ is denoted by $\langle v_1, \dots, v_r \rangle$.

We can write $k[M]$ as $k[x^u]_{u \in M}$, where we use the shorthand $x^u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$ for $u = (u_1, \dots, u_n) \in M$.

Definition 3.2. An exceptional divisor E over a toric variety X is called a *toric divisorially essential divisor* over X if its center is a divisor on every equivariant divisorial resolution of the singularities of X.

The following is obvious by the definition.

Proposition 3.3. *For a toric variety X a divisorially essential divisor over X is a toric divisorially essential divisor over X.*

At this moment the converse of the above proposition is not clear. But later on, as a corollary of our theorem we obtain the converse.

3.4. In what follows, we consider an affine toric variety $X = X(\Delta)$, therefore the fan Δ consists of all faces of a cone σ. Let $\sigma = \langle e_1, \dots, e_s \rangle$, where the right hand side means the cone generated by primitive vectors $e_1, \dots, e_s$. Let T be the open orbit in X. Let W be the singular locus of X, then $W = \bigcup_{\tau:\text{singular}} orb(\tau)$. Let $S = N \cap (\bigcup_{\tau:\text{singular}} \tau^o)$, where ^{o} means the relative interior.

Proposition 3.5. *If D_v is a toric divisorially essential divisor for $v \in N \cap \sigma$, then v belongs to S.*

Proof. If D_v is a toric divisorially essential divisor, then the image of D_v must be in the singular locus $W = \bigcup_{\tau:\text{singular}} orb(\tau)$. Therefore $v \in S$. $\square$

3.6 (Sketch of the proof)**.** We prove that all maps in the following diagram are injective and that composite of all maps is the identity. This shows that all maps are bijective.

$$\begin{array}{ccc}
\{\text{minimal elements in } S\} & \xrightarrow{\mathcal{F}} & \left\{\begin{array}{c}\text{good components of}\\ \text{arcs through } \operatorname{Sing} X\end{array}\right\} \\
 & & \downarrow \mathcal{N} \\
\mathcal{G}\uparrow & & \left\{\begin{array}{c}\text{essential divisors}\\ \text{over } X\end{array}\right\} \\
 & & \cap \\
\left\{\begin{array}{c}\text{toric divisorially}\\ \text{essential divisors}\\ \text{over } X\end{array}\right\} & \supset & \left\{\begin{array}{c}\text{divisorially essential}\\ \text{divisors over } X\end{array}\right\}
\end{array}$$

First we define an order in $N \cap \sigma$.

Definition 3.7. For two elements $v, v' \in N \cap \sigma$ we define $v \leq v'$, if $v' \in v + \sigma$.

For a subset $A \subset N \cap \sigma$, $a \in A$ is called minimal in A, if there is no other element $a' \in A$ such that $a' \leq a$.

Note that $v \leq v'$ if and only if $\langle v, u\rangle \leq \langle v', u\rangle$ for every $u \in M \cap \sigma^\vee$.
It is clear that $\leq$ is a partial order, i.e.,

(1) $v \leq v$,
(2) if $v \leq v'$ and $v' \leq v$, then $v = v'$,
(3) if $v \leq v'$ and $v' \leq v''$, then $v \leq v''$.

Definition 3.8. For an arc $\alpha : \operatorname{Spec} K[[t]] \longrightarrow X$ such that $\alpha(\eta) \in T$, define $v_\alpha \in N \cap \sigma$ as follows:

By the condition of α, we have a commutative diagram of ring homomorphisms:

$$\begin{array}{ccc}
k[M \cap \sigma^\vee] & \xrightarrow{\alpha^*} & K[[t]] \\
\cap & & \cap \\
k[M] & \xrightarrow{\alpha^*} & K((t)).
\end{array}$$

The map $M \longrightarrow \mathbb{Z}$, $u \mapsto \operatorname{ord}(\alpha^* x^u)$ is a group homomorphism, therefore it determines an element $v_\alpha \in N$ such that $\langle v_\alpha, u\rangle = \operatorname{ord}(\alpha^* x^u)$ for every $u \in M$. By the commutative diagram it follows that $v_\alpha|_{M\cap\sigma^\vee} \geq 0$, hence $v_\alpha \in N \cap \sigma$.

Proposition 3.9. (i) *Let α be an arc of X such that $\alpha(\eta) \in T$ and τ a face of σ. Then $\alpha(0) \in orb(\tau)$, if and only if $v_\alpha \in \tau^o$. In particular, $\alpha(0) \in T$, if and only if $v_\alpha = 0$.*

(ii) *Let Σ be a subdivision of the fan Δ and $f : Y \longrightarrow X$ be the toric morphism corresponding to this subdivision. Then, an arc α of X such that $\alpha(\eta) \in T$ is lifted to an arc $\tilde{\alpha}$ of Y. Let $\tau \in \Sigma$. Then, $\tilde{\alpha}(0) \in orb(\tau)$, if and only if $v_\alpha = v_{\tilde{\alpha}} \in \tau^o$.*

Proof. The first statement of (ii) follows immediately from the properness of f and the condition $\alpha(\eta) \in T$. The second statement of (ii) follows from the result (i) with replacing X by U_τ.

For the proof of (i) it is sufficient to prove that $v_\alpha \in \tau$ if and only if $\alpha(0) \in U_\tau$, because $\tau^o = \tau \setminus \bigcup_{\tau'} \tau'$ and $orb(\tau) = U_\tau \setminus \bigcup_{\tau'} U_{\tau'}$, where the unions are over all the proper faces τ' of τ. The condition $v_\alpha \in \tau$ is equivalent to $\langle v_\alpha, u \rangle \geq 0$ for all $u \in M \cap \tau^\vee$. And this holds if and only if the ring homomorphism $\alpha^* : k[M \cap \sigma^\vee] \longrightarrow k[[t]]$ can be extended to $k[M \cap \tau^\vee] \longrightarrow k[[t]]$, which is equivalent to that α factors through U_τ. As U_τ contains T, this is equivalent to that $\alpha(0) \in U_\tau$. □

Proposition 3.10. *For every point $v \in S$, there exists an arc α : $\operatorname{Spec} k[[t]] \longrightarrow X$ such that $\alpha(0) \in W$, $\alpha(\eta) \in T$ and $v = v_\alpha$.*

Proof. Define the ring homomorphism $\alpha^* : k[M] \longrightarrow k((t))$ by $\alpha^*(x^u) = t^{\langle v,u \rangle}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} k[M \cap \sigma^\vee] & \xrightarrow{\alpha^*} & k[[t]] \\ \cap & & \cap \\ k[M] & \xrightarrow{\alpha^*} & k((t)), \end{array}$$

because $\langle v, u \rangle \geq 0$ for every $u \in M \cap \sigma^\vee$. Let $\alpha : \operatorname{Spec} k[[t]] \longrightarrow X$ be the morphism corresponding to α^*, then $v = v_\alpha$ and we obtain $\alpha(\eta) \in T$ by the diagram. On the other hand, as $v \in S$, there is a singular face $\tau < \sigma$ such that $v = v_\alpha \in N \cap \tau^0$. By Proposition 3.9 $\alpha(0) \in orb(\tau) \subset W$. □

Proposition 3.11 (Upper semi-continuity)**.** *Let C be a k-scheme, $\alpha : C \widehat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \longrightarrow X$ a family of arcs on X and $\alpha_c : \operatorname{Spec} k(c)[[t]] \longrightarrow X$ the arc induced from α for each point $c \in Y$. Here $k(c)$ is the residue field at c. Assume $\alpha_c(\eta) \in T$ for every $c \in C$. Then the map $C \longrightarrow N \cap \sigma$, $c \mapsto v_{\alpha_c}$ is upper semi-continuous, i.e., for every $v \in N \cap \sigma$ the subset $U_v := \{c \in C \mid v_{\alpha_c} \leq v\}$ is open in C. In particular, if there is a point $z \in C$ such that v_{α_z} is minimal in S, then there is a non-empty open subset $U \subset C$ such that $v_{\alpha_c} = v_{\alpha_z}$ holds for every $c \in U$.*

Proof. It is sufficient to prove the assertion in the affine case $C = \operatorname{Spec} A$. Let $\alpha^* : k[M \cap \sigma^\vee] \longrightarrow A[[t]]$ be the ring homomorphism corresponding to α. Let $\alpha^*(x^u)$ be $a_0^u + a_1^u t + a_2^u t^2 + \ldots$, where $a_i^u \in A$ for $i \geq 0$. By the definition of U_v, a point $c \in C$ belongs to U_v, if and only if $\langle v_{\alpha_c}, u\rangle \leq \langle v, u\rangle$ for every $u \in M \cap \sigma^\vee$. This is equivalent to that for every element u of generating system of $M \cap \sigma^\vee$ there exists $i \leq \langle v, u\rangle$ such that $a_i^u(c) \neq 0$. Now, we see that U_v is a finite union of the complements of zero locus of functions on C. $\square$

3.12. Let $\{C_i : i \in I\}$ be the good components of the space of arcs through W. For each component $C_i \subset X_\infty$, there exists a corresponding family $\alpha_i : C_i \widehat{\times}_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \longrightarrow X$ of arcs by Corollary 2.10.

Lemma 3.13. *Under the above notation, for a minimal element $v \in S$ there are a good component C_i and a non-empty open subset $U \subset C_i$ such that $v_{\alpha_{ic}} = v$ for every $c \in U$, where $\alpha_{ic} : \operatorname{Spec} k(c)[[t]] \longrightarrow X$ is th arc induced from α_i.*

For a minimal element $v \in S$, take one of these components C_i and define $\mathcal{F}(v) := C_i$. Then the map $\{$minimal elements in $S\} \xrightarrow{\mathcal{F}} \{C_i\}$ is injective.

Proof. For a given minimal element $v \in S$ there is an arc $\alpha : \operatorname{Spec} k[[t]] \longrightarrow X$ such that $\alpha(0) \in W$, $\alpha(\eta) \in T$ and $v_\alpha = v$ by Proposition 3.10. Then, by Definition 2.11, there exist a good component C_i and its k-valued point z such that $\alpha = \alpha_{iz}$. As $\alpha_i(C_i \times_{\operatorname{Spec} k} \{0\}) \subset W$ and $\alpha_{iz}(\eta) \in T$, there exists a non-empty open subset $V \subset C_i$ such that both the conditions $\alpha_i(V \times_{\operatorname{Spec} k} \{0\}) \subset W$ and $\alpha_{ic}(\eta) \in T$ for every $c \in V$ hold. Then, by Proposition 3.11, there exists a non-empty open subset $U \subset V$ such that $v_{\alpha_{ic}} = v$.

The second assertion is obvious from the first statement. $\square$

Lemma 3.14. *Let $\mathcal{N} : \{C_i : i \in I\} \longrightarrow \{$essential divisors$\}$, $C_i \mapsto E_{l_i}$ be the Nash map in Theorem 2.15. Then the composite*

$$\mathcal{N} \circ \mathcal{F} : \{\textit{minimal elements in } S\} \longrightarrow \{\textit{essential divisors}\}$$

satisfies $\mathcal{N} \circ \mathcal{F}(v) = D_v$.

Proof. By Lemma 3.13, the generic point z of $\mathcal{F}(v)$ corresponds to an arc $\alpha : \operatorname{Spec} K[[t]] \longrightarrow X$ such that $v_\alpha = v$. Let $\tilde{\alpha}$ be the lifting of α as an arc of a toric divisorial resolution Y. By the definition of $\mathcal{N}$, $\mathcal{N} \circ \mathcal{F}(v)$ is an exceptional divisor containing $\tilde{\alpha}(0)$ as the generic point. By Proposition 3.9, the exceptional divisor $\overline{orb(\tau)}$ containing $\tilde{\alpha}(0)$ satisfies $v = v_\alpha = v_{\tilde{\alpha}} \in \tau^o$. Therefore this divisor is D_v. $\square$

We prove the following by using the idea of the proof of [4, Théorème 1.10].

Lemma 3.15. *Consider the map*

$$\mathcal{G} : \{\textit{toric divisorially essential divisors over } X\} \longrightarrow S$$

given by $\mathcal{G}(D_v) = v$. *Then, this map is injective and its image is contained in the set of minimal elements of* S. □

Proof. The injectivity is clear by the definition of the map. For the second assertion it is sufficient to prove that if a primitive vector $v \in S$ is not minimal then D_v is not toric divisorially essential. To do this, we construct a regular subdivision Σ of σ such that the map $X(\Sigma) \longrightarrow X$ is a divisorial resolution of X, and in which $v\mathbb{R}_{\geq 0}$ does not appear as a one-dimensional cone.

If $v \in S$ is not minimal, then v can be written as $v = n_1 + n_2$, where $n_1 \in S$ and $n_2 \in N \cap \sigma \setminus \{0\}$. Then, we can reduce it into two cases: (1) $n_1, n_2 \in S$, (2) $n_1 \in S$ and n_2 is in a one-dimensional face of σ. Indeed, if $n_2 \notin S$, then n_2 is in a non-singular face τ of σ. Let $\tau = \langle e_1, \dots , e_d\rangle$, then $n_2 = \sum_{i=1}^{d} b_i e_i$ with $b_i \in \mathbb{N} \cup \{0\}$ $(i = 1, \dots , d)$. We may assume that $b_1 \neq 0$. Let γ be the minimal face of σ containing the cone $\langle n_1, \sum_{i=2}^{d} b_i e_i\rangle$, then, since $n_1 \in \gamma$, γ is singular and $n_1 + \sum_{i=2}^{d} b_i e_i \in \gamma^o \subset S$. Here, replace n_1 by $n_1 + \sum_{i=2}^{d} b_i e_i$ and n_2 by $b_1 e_1$, then we can reduce to the case (2).

Next, take the minimal regular subdivision of the 2-dimensional cone $\langle n_1, n_2\rangle$ ([4, Proposition 1.8]) which gives the minimal resolution of the 2-dimensional singularity. Let $\langle v_1, v_2\rangle$ be its 2-dimensional cone containing v, then v is in the relative interior of this cone. We will construct a regular subdivision of σ which contains $\langle v_1, v_2\rangle$ as a cone. We may assume that $v_1 \in S$. First, take the star-shaped subdivision Σ_1 with the center v_1. Then take the star-shaped subdivision Σ_2 of Σ_1 with the center v_2 if v_1, v_2 are in the case (1). If v_1, v_2 are in the case (2), let $\Sigma_2 = \Sigma_1$. Here, we note that the exceptional set for the corresponding equivariant morphism is a divisor. If Σ_2 is not simplicial, let γ be a minimal dimensional cone which is not simplicial. Take $n \in \gamma^o$ and take the star-shaped subdivision of Σ_2 with the center n. Then γ is divided into simplicial cones and the exceptional set for the corresponding equivariant morphism is a divisor. Continuing this procedure, we finally obtain a simplicial subdivision Σ_3. If Σ_3 is not regular, take a cone $\lambda = \langle p_1, \dots , p_t\rangle \in \Sigma_3$ with the maximal multiplicity. The multiplicity is $\operatorname{vol} P_\lambda$, where $P_\lambda = \{\sum_{i=1}^{t} c_i p_i \mid 0 \leq c_i < 1\}$. Since $\operatorname{vol} P_\lambda > 1$, there is a non-zero element $n' \in P_\lambda \cap N$. Take the star-shaped subdivision with the center n'. Then again the exceptional set for the corresponding equivariant morphism is a divisor. Continuing this procedure, we finally obtain a regular subdivision Σ_4. As we did not change the cone

$\langle v_1, v_2\rangle$ in these procedures, Σ_4 contains this cone. Therefore, the exceptional divisor D_v does not appear in $X(\Sigma_4)$. As all regular cones are unchanged, the corresponding equivariant morphism is a resolution which is isomorphic outside the singular locus. It is clear that the resolution is divisorial, as we saw it in each step of subdivisions. $\square$

Theorem 3.16. *Let X be an affine toric variety. Then the Nash map*

$$\mathcal{N} : \{C_i : i \in I\} \longrightarrow \{\textit{essential divisors over } X\}$$

is bijective.

Proof. In the diagram 3.6, we obtain that $\mathcal{F}$ is injective by Lemma 3.13, $\mathcal{N}$ is injective by Nash's theorem 2.15 and $\mathcal{G}$ is injective by Lemma 3.15. We also have that $\mathcal{G}\circ\mathcal{N}\circ\mathcal{F}$ is the identity map on $\{$minimal elements in $S\}$ by Lemma 3.14 and 3.15. Hence, $\mathcal{G}, \mathcal{N}, \mathcal{F}$ are all bijective. $\square$

By the proof of the above theorem, the following are obvious.

Corollary 3.17. *For a toric variety X, E is an essential divisor over X, if and only if E is a toric divisorially essential divisor over X.*

The analogous result for essential divisors is proved in [3], but the definition used there is not quite equivalent to ours.

Corollary 3.18. *For a cone σ in N the number of the minimal elements in $S = N \cap (\bigcup_{\tau:\mathrm{singular}} \tau^o)$ is finite. More precisely this number is the number of essential components and also the number of the good components.*

Corollary 3.19. *For a general point $c \in C_i$ for $(i \in I)$, the corresponding arc α_{ic} satisfies $\alpha_{ic}(\eta) \in T$.*

Example 3.20. Let $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (1,1,e) \in N \simeq \mathbb{Z}^3$ and $\sigma = \langle e_1, e_2, e_3\rangle$. Then all proper faces of σ are regular and σ itself is not regular, therefore the affine toric variety X corresponding to σ has an isolated singularity at the closed orbit. We can also see that $S = N \cap \sigma^o$. By simple calculations we obtain that the minimal elements in S are $(1,1,d)$ $(1 \le d \le e-1)$. Therefore, by our theorem the number of C_i's and the number of the essential components are both $e-1$.

4. Counter examples to the Nash problem

The basic idea of our counter examples to the Nash problem is the following:

Take a singularity $x \in X$ and a partial resolution $p : Y \longrightarrow X$ with exceptional divisor $F \subset Y$. Assume that Y has a singular point $y \in F$

such that every general arc $g : \operatorname{Spec} k[[s]] \longrightarrow (Y, y)$ is contained in an embedded smooth surface germ $G : \operatorname{Spec} k[[s,t]] \longrightarrow (Y, y)$.

Assume that there is an essential divisor E over X whose center on Y is y. The arcs on Y that should correspond to E are all arcs through y. If such an arc is contained in an embedded smooth surface germ $G : \operatorname{Spec} k[[s,t]] \longrightarrow (Y, y)$, then this arc can be moved in Y such that its closed point moves along the curve $G^{-1}(F)$, hence the arcs through E are all limits of arcs through some component of F.

This implies that E does not correspond to an irreducible component of the family of arcs through $x \in X$. If we can also arrange E to be essential, we have a counter example to the Nash problem.

4.1. Algebraically, a smooth formal curve through $0 \in Y$ is equivalent to a surjection $\phi : \hat{\mathcal{O}}_Y \longrightarrow k[[s]]$, where $\hat{\mathcal{O}}_Y$ denotes the completion of $\mathcal{O}_Y$ at the ideal m_0 of 0. Similarly, a smooth surface germ is equivalent to a surjection $\Phi : \hat{\mathcal{O}}_Y \longrightarrow k[[t,s]]$. The induced maps $m_0/m_0^2 \longrightarrow (s)/(s)^2$ and $m_0/m_0^2 \longrightarrow (s,t)/(s,t)^2$ correspond to a point and a line in the exceptional divisor of the blow up $B_0Y \longrightarrow Y$.

Lemma 4.2. *Let $0 \in Y \subset \mathbb{A}^n$ be a hypersurface singularity of multiplicity m defined by an equation $F = 0$ where $F = F_m + F_{m+1} + \dots$ is the decomposition into homogeneous pieces. Set $Z = (F_m = 0) \subset \mathbb{P}^{n-1}$ and let $z \in Z$ be a point and $z \in L \subset Z$ a line such that Z is smooth along L and $H^1(L, N_{L|Z}) = 0$.*

Let $\phi : \hat{\mathcal{O}}_Y \longrightarrow k[[s]]$ be a smooth formal curve through 0 with tangent direction z. Then ϕ can be extended to a surjection $\Phi : \hat{\mathcal{O}}_Y \longrightarrow k[[t,s]]$ with tangent direction L.

Proof. The line L can be identified with a map $\Phi_1 : k[y_1, \dots, y_n] \longrightarrow k[s,t]$ such that the $\Phi_1(y_i)$ are linear in s,t and $\Phi_1(F) \in (s,t)^{m+1}$. Our aim is to find inductively maps

$$\Phi_r : k[y_1, \dots, y_n] \longrightarrow k[s,t] \quad \text{such that} \quad \Phi_r(F) \in (s,t)^{m+r},$$

Φ_r modulo (t) coincides with ϕ modulo (s^{r+1}) and Φ_r is congruent to Φ_{r+1} modulo $(s,t)^{r+1}$. If this can be done then the inverse limit of the maps

$$k[y_1, \dots, y_n] \xrightarrow{\Phi_r} k[s,t] \longrightarrow k[s,t]/(s,t)^{r+1}$$

gives $\Phi : k[[y_1, \dots, y_n]] \longrightarrow k[[s,t]]$ such that $\Phi(F) = 0$. Thus it descends to $\Phi : \hat{\mathcal{O}}_Y \longrightarrow k[[s,t]]$

A map $g : k[y_1, \dots, y_n] \longrightarrow$ (any ring) can be identified with the vector $(g(y_1), \dots, g(y_n))$. Using this convention, by changing coordinates we may assume that $\phi = (s, 0, \dots, 0)$ and $L = (y_3 = \dots = y_n = 0)$.

The first condition implies that no power of y_1 appears in F and the second means that we can choose $\Phi_1 = (s, t, 0, \dots, 0)$.

Assume that we already have Φ_r which we assume to be of the form

$$\Phi_r = (s, t, tA_{3,r-1}(s,t), \dots, tA_{n,r-1}(s,t))$$

where the $A_{i,r-1}$ are polynomials of degree $\leq r-1$ without constant terms. The vanishing of the constant term comes from extending the map Φ_1 and the divisibility by t comes from the requirement of extending ϕ. We are looking for Φ_{r+1} of the form

$$\Phi_{r+1} = (s, t, tA_{3,r-1}(s,t) + tB_{3,r}(s,t), \dots, tA_{n,r-1}(s,t) + tB_{n,r}(s,t)),$$

where the $B_{i,r}$ are homogeneous of degree r. Let us compute $\Phi_{r+1}(F)$. Using the Taylor expansion, we get that

$$\begin{aligned} \Phi_{r+1}(F) &= \Phi_r(F) + t \cdot \sum_{i=3}^{n} \frac{\partial F_m}{\partial y_i}(s, t, 0, \dots, 0) \cdot B_{i,r}(s,t) \\ &\quad + (\text{terms of multiplicity} \geq m + r + 1). \end{aligned}$$

By the inductive assumption,

$$\Phi_r(F) = t \cdot C_{m+r-1}(s,t) + (\text{terms of multiplicity} \geq m + r + 1),$$

where C_{m+r-1} has degree $m+r-1$. In order to achieve that $\Phi_{r+1}(F) \in (s,t)^{m+r+1}$, we need to find polynomials $B_{i,r}$ such that

$$C_{m+r-1}(s,t) = -\sum_{i=3}^{n} \frac{\partial F_m}{\partial y_i}(s, t, 0, \dots, 0) \cdot B_{i,r}(s,t). \qquad (*)$$

Since we know nothing about C_{m+r-1}, we need to guarantee that the ideal generated by the partials $\partial F_m / \partial y_i(s, t, 0, \dots, 0)$ contains all homogeneous polynomials of degree $m+r-1$ in s, t for every $r \geq 1$. The critical case is $r = 1$.

The normal bundles of L in Z and in $\mathbb{P}^{n-1}$ are related by an exact sequence

$$0 \longrightarrow N_{L|Z} \longrightarrow N_{L|\mathbb{P}^{n-1}} \cong \mathcal{O}(1)^{n-2} \xrightarrow{dF_m} N_{Z|\mathbb{P}^{n-1}}|_L \cong \mathcal{O}(m) \longrightarrow 0,$$

and dF_m is the map $\mathcal{O}(1)^{n-2} \longrightarrow \mathcal{O}(m)$ given by multiplication by the partials $\partial F_m / \partial y_i$ for $i = 3, \dots, n$. We have assumed that $H^1(L, N_{L|Z}) = 0$, thus the induced map

$$\begin{aligned} dF_m : \sum_{i=3}^{n} H^0(L, \mathcal{O}(1)) &\longrightarrow H^0(L, \mathcal{O}(m)), \quad \text{given by} \\ (l_3, \dots, l_n) &\mapsto \sum_{i=3}^{n} l_i \frac{\partial F_m}{\partial y_i}(s, t, 0, \dots, 0) \end{aligned}$$

is surjective. Thus the equation (*) always has a solution. □

Theorem 4.3. *Let $Z \subset \mathbb{P}^{n-1}$ be a smooth hypersurface. Assume that Z is not ruled but through a general point of Z there is a line L such that $H^1(L, N_{L|Z}) = 0$.*

Let $0 \in X$ be any singularity with a partial resolution $p : Y \longrightarrow X$ and $y \in p^{-1}(0)$ a point such that

1. *$y \in Y$ is a hypersurface singularity whose projectivised tangent cone is isomorphic to Z, and*
2. *$p^{-1}(0) \subset Y$ is a Cartier divisor.*

Then the blow up $B_y Y$ gives an essential exceptional divisor $Z \cong E \subset B_y Y$ over $0 \in X$ which does not correspond to an irreducible family of arcs on X.

Proof. E is an essential divisor by Example 2.5.

In order to prove that E does not correspond to an irreducible family of arcs on X, consider the family W of arcs in $B_y Y$ through E. These correspond to a subset W_y of arcs on Y through y and to a subset W_x of arcs in X through x. We claim that W_x is not an irreducible component of the family of all arcs on X through x.

In order to see this, it is enough to show that a general arc in W_y is a limit of arcs in Y through $p^{-1}(0)$ but not passing through y.

By assumption, the pull back of a general local equation of y contains E with multiplicity 1. A general arc in W is transversal to E, so the general member of W_y is an arc on Y wich has multiplicity 1 intersection with a general local equation of y. Hence the general member of W_y is a smooth arc on Y with general tangent direction. Therefore, by our assumption on Z and by (4.2), a general arc in W_y is contained in a smooth surface germ. Thus it is a limit of arcs through $p^{-1}(0)$ which do not pass through y. Hence W_x is not an irreducible component of the space of arcs on X through x. $\square$

Remark 4.4. In characteristic 0, a smooth hypersurface $Z \subset \mathbb{P}^{n-1}$ is covered by lines if and only if $\deg Z \leq n-2$ (cf. [14, V.4.6]). A general line then has $H^1(L, N_{L|Z}) = 0$ by [14, II.3.11]. Thus the key condition is to check that Z is not birationally ruled. This can not happen if $n \leq 4$. In higher dimensions there are two known sets of examples:

(1) $Z \subset \mathbb{P}^4$ is a smooth cubic. Then Z is not birational to $\mathbb{P}^3$. This was proved by [5] over $\mathbb{C}$ and by [19] in characteristic $\neq 2$. This implies that Z is not ruled. Indeed, assume that Z is birational to $S \times \mathbb{P}^1$. There is a degree 2 map $\mathbb{P}^3 \dashrightarrow Z$ (this goes back to M. Noether, cf. [14, V.5.18.3]), so in characteristic $\neq 2$ we get a dominant separable map $\mathbb{P}^3 \dashrightarrow S \times \mathbb{P}^1 \longrightarrow S$. Thus S is rational by Castelnuovo's theorem. Therefore Z is birational to $\mathbb{P}^2 \times \mathbb{P}^1$ and so rational, a contradiction.

Every line on a smooth cubic satisfies $H^1(L, N_{L|Z}) = 0$ by [14, V.4.4.1] in any characteristic.

(2) $Z \subset \mathbb{P}^{n-1}$ is a very general hypersurface with $n \geq \deg Z \geq \frac{2n}{3}+2$. These are nonruled in characteristic zero by [13].

Example 4.5. The 4-dimensional hypersurface singularity over an algebraically closed field of characteristic $\neq 2, 3$

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6 = 0$$

has only 1 irreducible family of arcs but 2 essential exceptional components.

Proof. Apply Theorem 4.3 to $X = (x_1^3+x_2^3+x_3^3+x_4^3+x_5^6 = 0)$. Blowing up the origin produces Y. The exceptional divisor $F \subset Y$ is Cartier and Y has a unique singular point which is the cone over the cubic 3–fold $Z := (x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0)$. Z is not birationally ruled by (4.4.1).

Blowing up the unique singular point of Y we get a resolution of X with 2 exceptional divisors. One is $E \cong Z$ and the other is F', the birational transform of F.

F' is birationally ruled, but it is still essential. Indeed, the family of arcs on X has to correspond to some exceptional divisor, and F' is the only possibility. Thus F' has to be essential. Another way to see this is to note that X is terminal and F' has minimal discrepancy, namely 1. $\square$

References

1. S. Abhyankar, *On the valuations centered in a local domain.* Amer. J. Math. 78 (1956) 321–348.
2. M. Artin, *On the solutions of analytic equations*, Invent. Math. 5 1968 277–291.
3. C. Bouvier, *Diviseurs essentiels, composantes essentielles des variétés toriques singulières*, Duke Math. J. 91 (1998) 609–620
4. C. Bouvier and G. Gonzalez-Sprinberg, *Système générateur minimal, diviseurs essentiels et G-désingularisations de variétés toriques*, Tohoku Math. J. **47**, (1995) 125–149.
5. C.H. Clemens and P. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. 95 (1972), 281–356
6. D. I. Dais, C. Haase and G. M. Ziegler, *All toric local complete intersection singularities admit projective crepant resolutions*, Tohoku Math. J. (2) **53**, (2001) 95–107.
7. J. Denef and F. Loeser, *Germs of arcs on singular varieties and motivic integration,* Invent. Math. **135**, (1999) 201–232.
8. W. Fulton, *Introduction to Toric Varieties*, Annals of Math. St. **131**, (1993) Princeton University Press.
9. H. Gillet, *Differential Algebra - a scheme theory approach*, preprint, http://www.math.uic.edu/ henri

10. G. Gonzalez-Sprinberg and M. Lejeune-Jalabert, *Families of smooth curves on surface singularities and wedges*, Annales Polonici Mathematici, LXVII.2, (1997) 179–190.
11. M. Hickel, *Fonction de Artin et germes de courbes tracées sur un germe d'espace analytique*, Amer. J. Math. **115**, (1993) 1299–1334.
12. E. R. Kolchin *Differential algebra and algebraic groups*, Pure and Applied Mathematics, Vol. 54. Academic Press, New York-London, 1973.
13. J. Kollár, *Nonrational hypersurfaces*, J. Amer. Math. Soc. 8 (1995) 241–249
14. J. Kollár, *Rational Curves on Algebraic Varieties*, Springer, 1996
15. M. Kontsevich, *Lecture at Orsay* (December 7, 1995)
16. M. Lejeune-Jalabert, *Arcs analytiques et résolution minimale des surfaces quasihomogènes.* in: Lecture Notes in Math. **777**, (1980) 303–336.
17. M. Lejeune-Jalabert, *Courbes tracées sur un germe d'hypersurface.* Amer. J. Math. **112**, (1990) 525–568.
18. M. Lejeune-Jalabert and A. J. Reguera-Lopez, *Arcs and wedges on sandwiched surface singularities*, Amer. J. Math. **121**, (1999) 1191–1213.
19. J. P. Murre, *Reduction of the proof of the non-rationality of a non-singular cubic threefold to a result of Mumford*, Compositio Math. 27 (1973), 63–82
20. M. Mustaţă, *Singularities of Pairs via Jet Schemes*, J. Amer. Math. Soc. 15 (2002), 599-615.
21. J. F. Nash, *Arc structure of singularities*, Duke Math. J. **81**, (1995) 31–38.
22. A. Nobile, *On Nash theory of arc structure of singularities.* Ann. Mat. Pura Appl. 160 (1991), 129–146
23. A. J. Reguera-Lopez, *Families of arcs on rational surface singularities*, Manuscr. Math. **88**, (1995) 321–333.

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