Convex Reflexive Lattice Polygons and the Number 12

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A Bridge Between Two Lands

A Bridge Between Two Lands

Combinatorics

A Bridge Between Two Lands

Combinatorics

polytope

A Bridge Between Two Lands

Combinatorics

polytope

|

polygon (dim. 2)

A Bridge Between Two Lands

Combinatorics

polytope

|

polygon (dim. 2)

Algebraic Geometry

A Bridge Between Two Lands

Combinatorics

polytope
|
polygon (dim. 2)

Algebraic Geometry

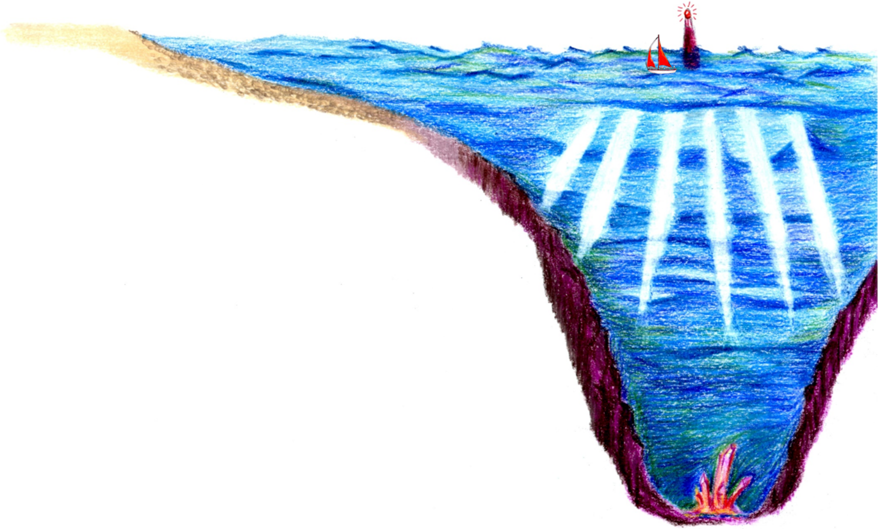
toric varieties

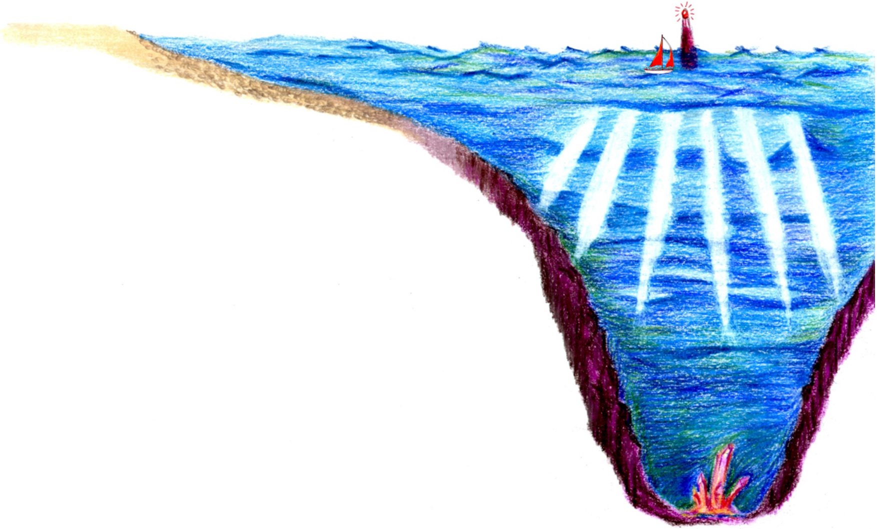
A Bridge Between Two Lands

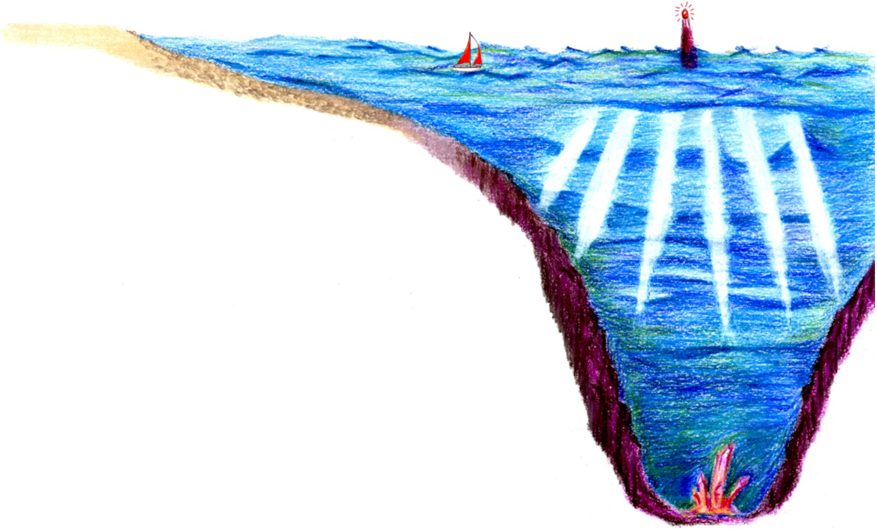
Combinatorics	Algebraic Geometry
polytope	toric varieties
|	|
polygon (dim. 2)	toric surface (dim. 2)

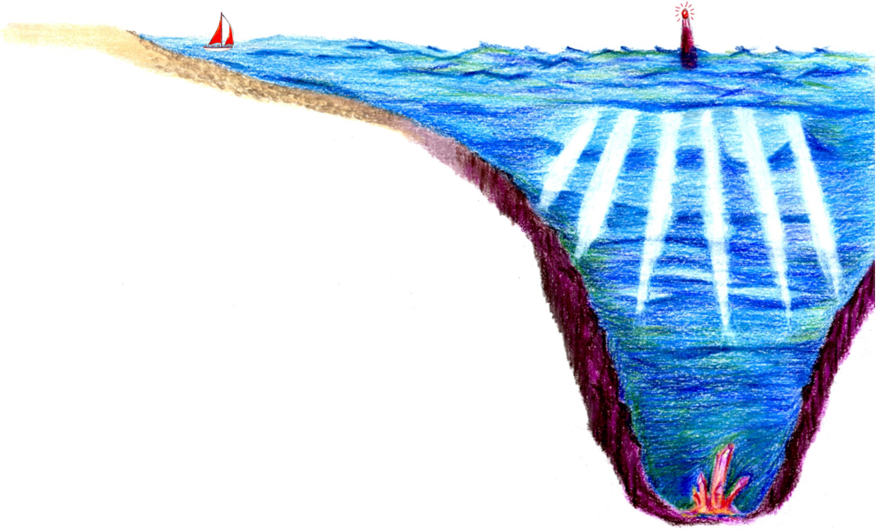
Introduction

First, an overview of our adventure...









Convex Polygons

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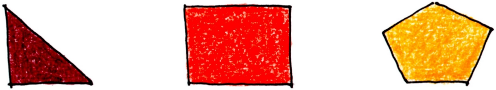
Examples.



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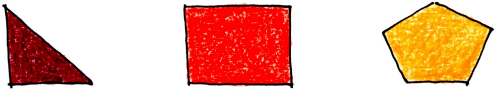

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A lattice polygon is a polygon in $\mathbb{R}^2$ which has vertices in $\mathbb{Z}^2$.

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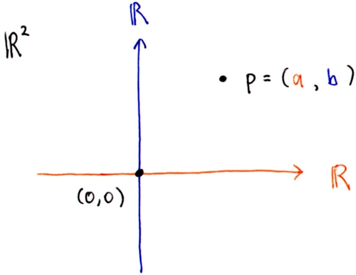

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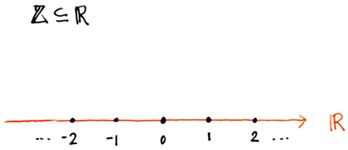

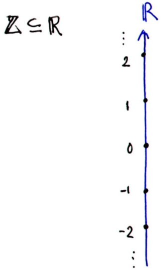

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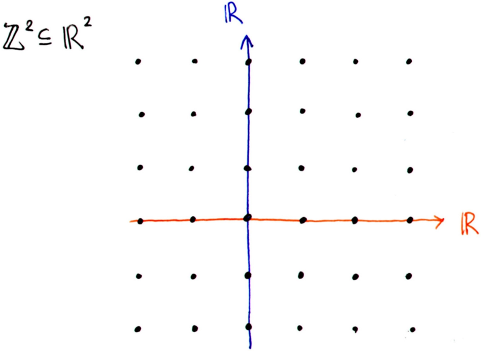

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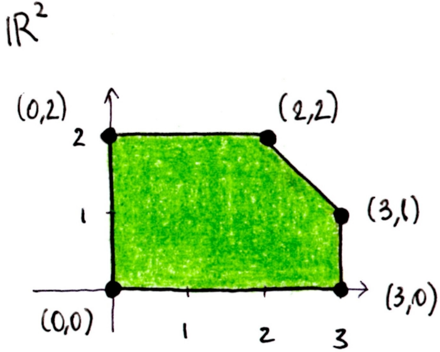

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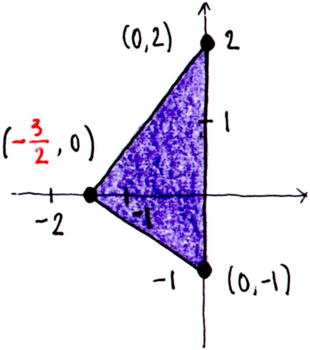

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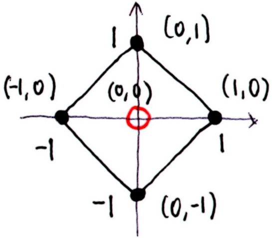

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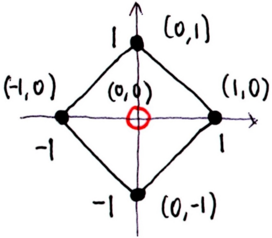

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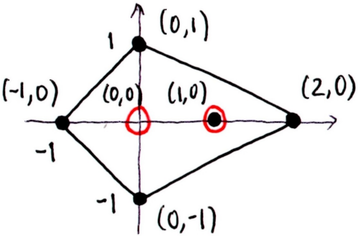

So far we have:

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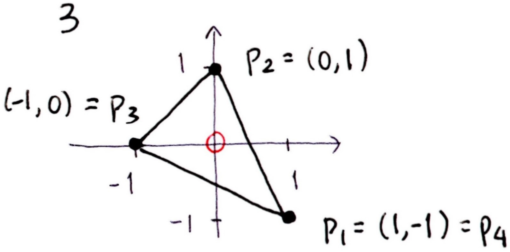

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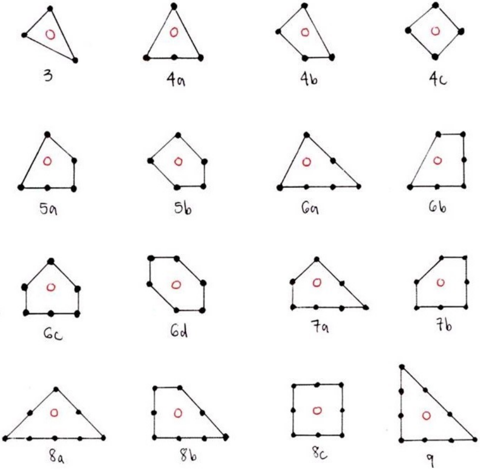

Dual of a Polygon

Given a convex reflexive lattice polygon P with vertices $p_1, p_2, \ldots, p_n$, we define the dual of P to be the polygon with vertices q_i where

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Note: Let $p_{n+1} = p_1$

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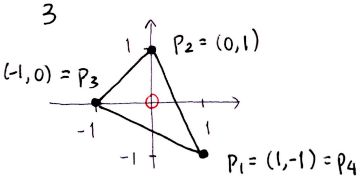

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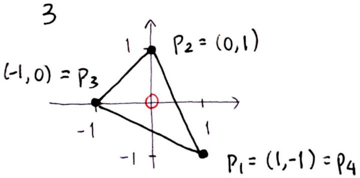

$$q_1 = p_2 - p_1 = (0,1) - (1,-1) = (-1,2)$$
$$q_2 = p_3 - p_2 = (-1,0) - (0,1) = (-1,-1)$$
$$q_2 = p_4 - p_3 = (1,-1) - (-1,0) = (2,-1)$$

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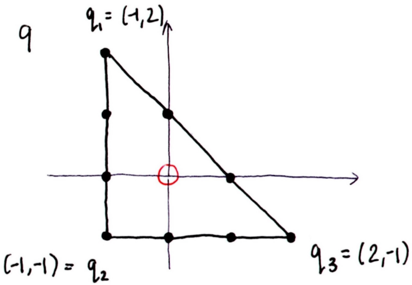

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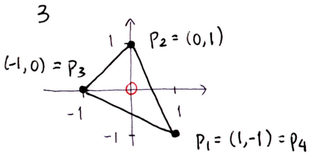
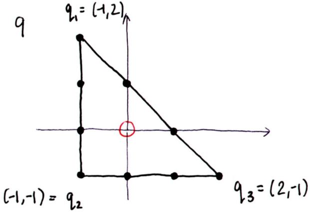
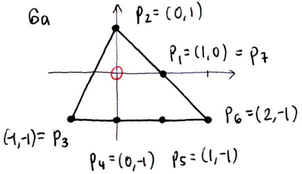




Example 2: Dual of a Polygon



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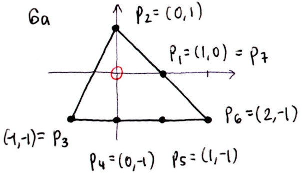

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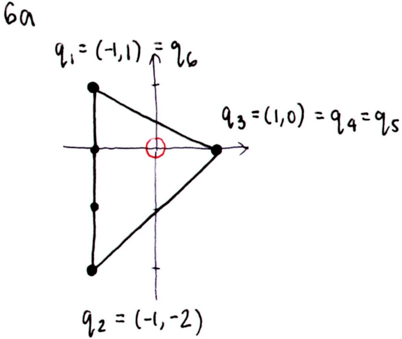

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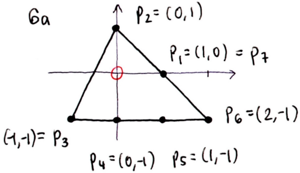

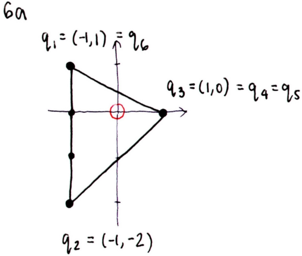

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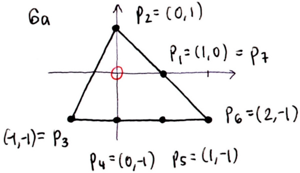

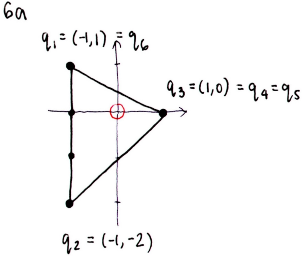

Fact: The dual of a convex reflexive lattice polygon is also a convex reflexive lattice polygon!

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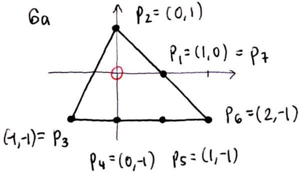

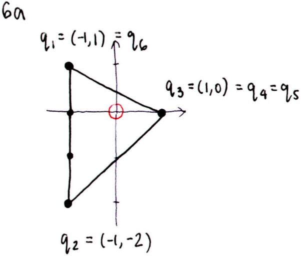

Fact: The dual of a convex reflexive lattice polygon is also a convex reflexive lattice polygon!

Exercise: Try all of them!

Boundary Lattice Points

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Let ∂P denote the number of boundary lattice points of polygon P.

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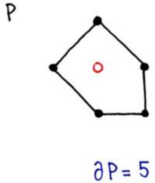

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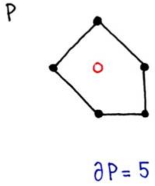

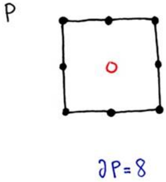

The Number 12

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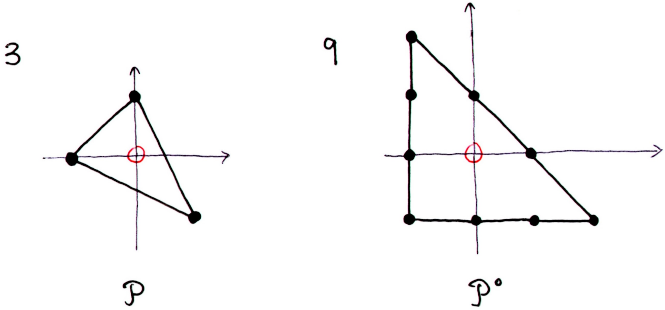
Main Theorem. Let P be a convex reflexive lattice polygon and let P° be its dual. Then

$$\partial P + \partial P^\circ = 12$$

The Number 12

Example.

Example.



$$\partial\mathcal{P} + \partial\mathcal{P}^{\circ} = 3 + 9 = 12$$

Example.

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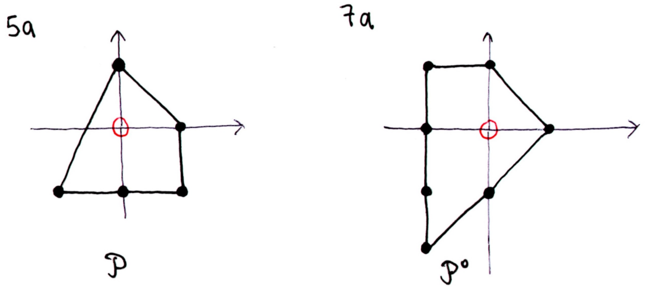

$$\partial P + \partial P^\circ = 5 + 7 = 12$$

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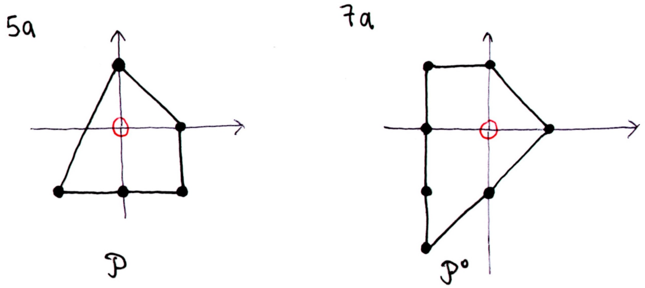

$$\partial \mathcal{P} + \partial \mathcal{P}^\circ = 5 + 7 = 12$$

Exercise: Verify the formula holds for all 16 polygons!

Why?

Why?

Let's turn to algebraic geometry!

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$$\text{convex reflexive lattice polygon } P \quad \leftrightsquigarrow \quad \text{toric surface } X$$

Convex Reflexive Lattice Polygon to Toric Surface

General idea:

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polygon P

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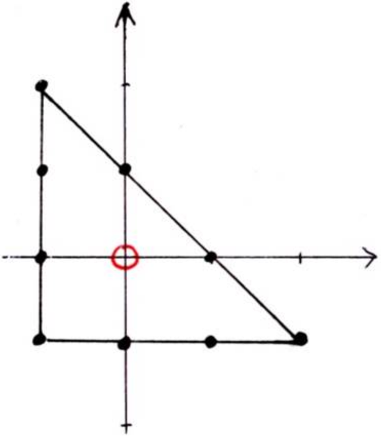

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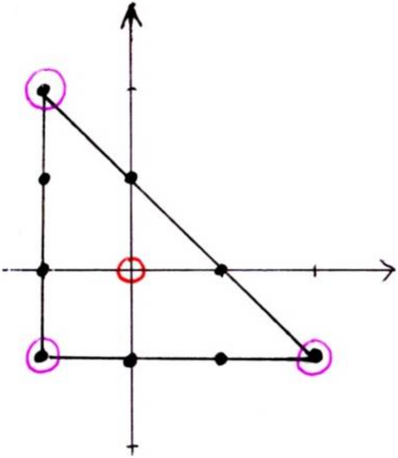

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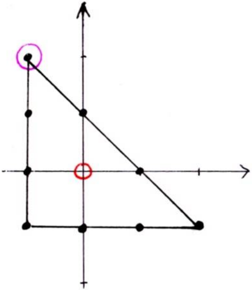

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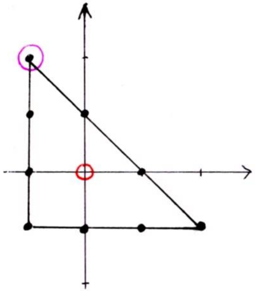

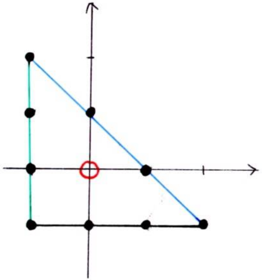

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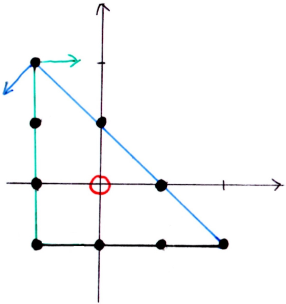

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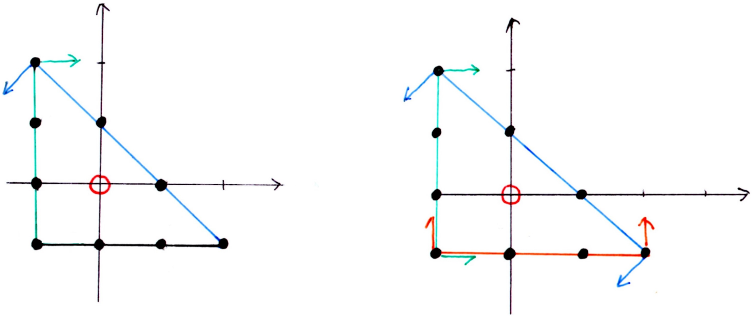

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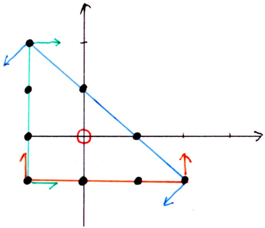

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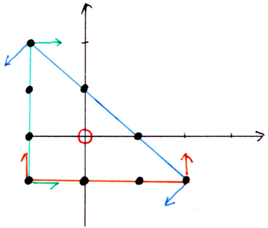

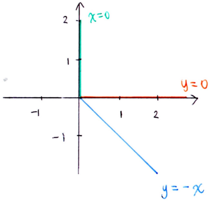

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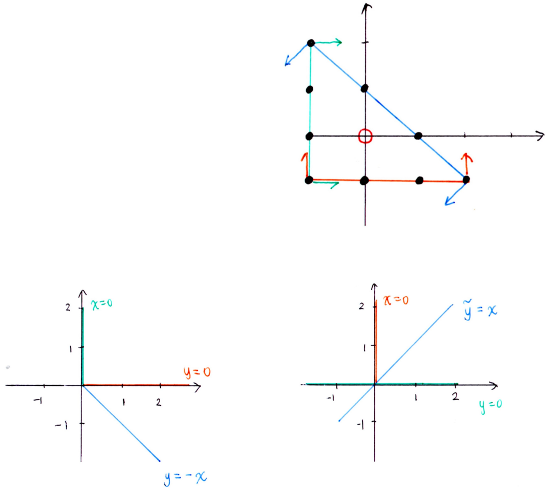

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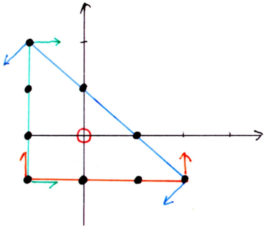

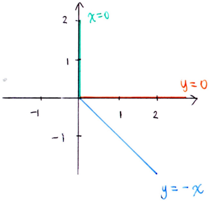

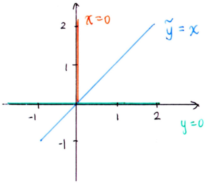

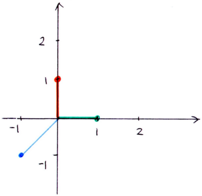

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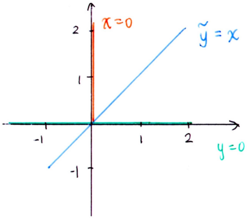

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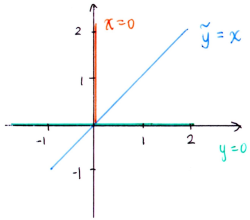

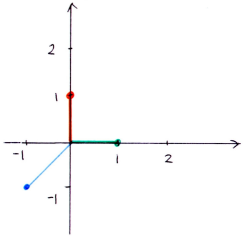

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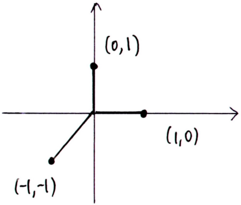

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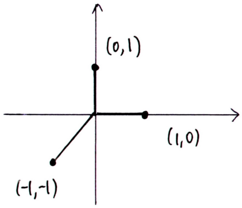

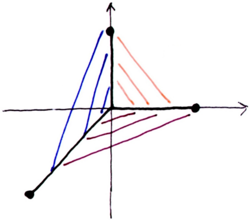

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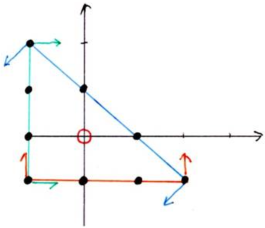

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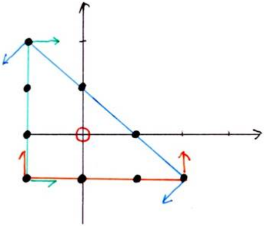

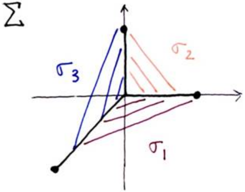

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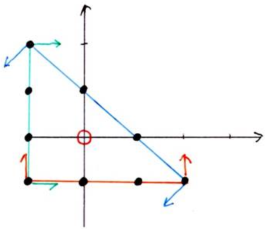
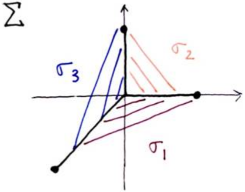
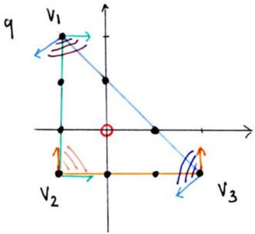
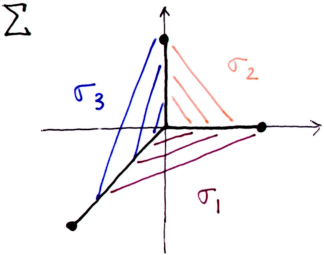






Fans



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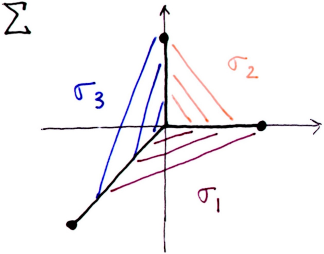

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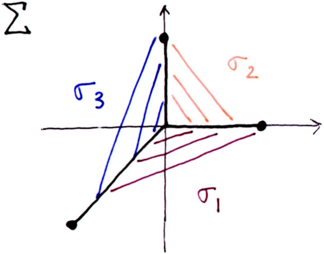


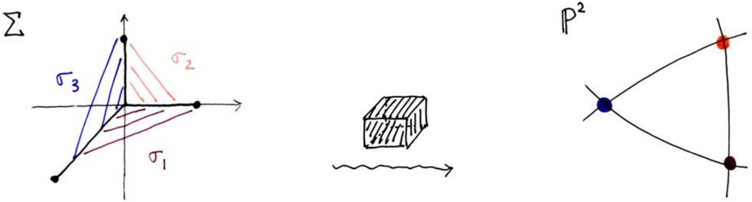
The regions σ_1, σ_2, and σ_3 are two-dimensional cones.

A fan Σ is a union of cones.

Fans

Exercise: Find the associated fan for all 16 polygons!

Example: Fan to Toric Surface



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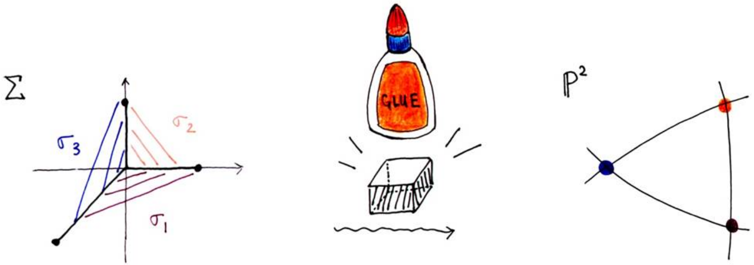

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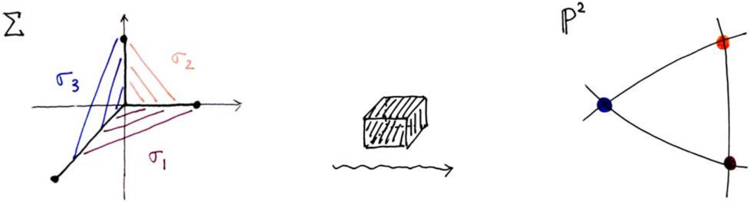

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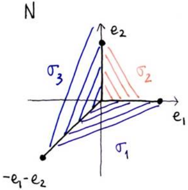

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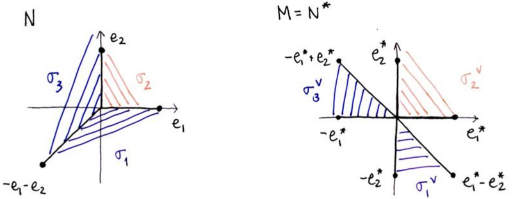

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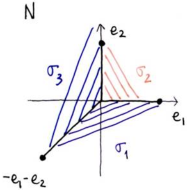

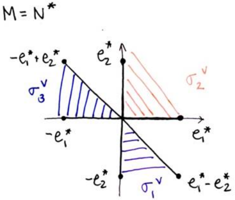

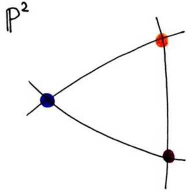

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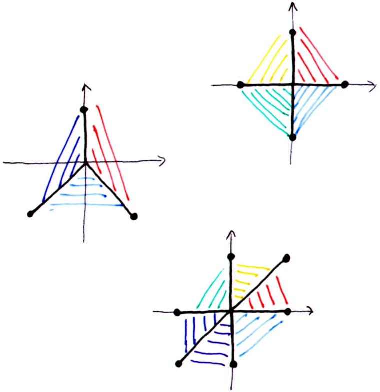

From these fans, we can read what the topological properties of the corresponding toric surface are.

Algebraic Geometry

convex reflexive lattice polygon $P \quad \leftrightsquigarrow \quad$ toric surface X

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It lets us use Noether's Formula!

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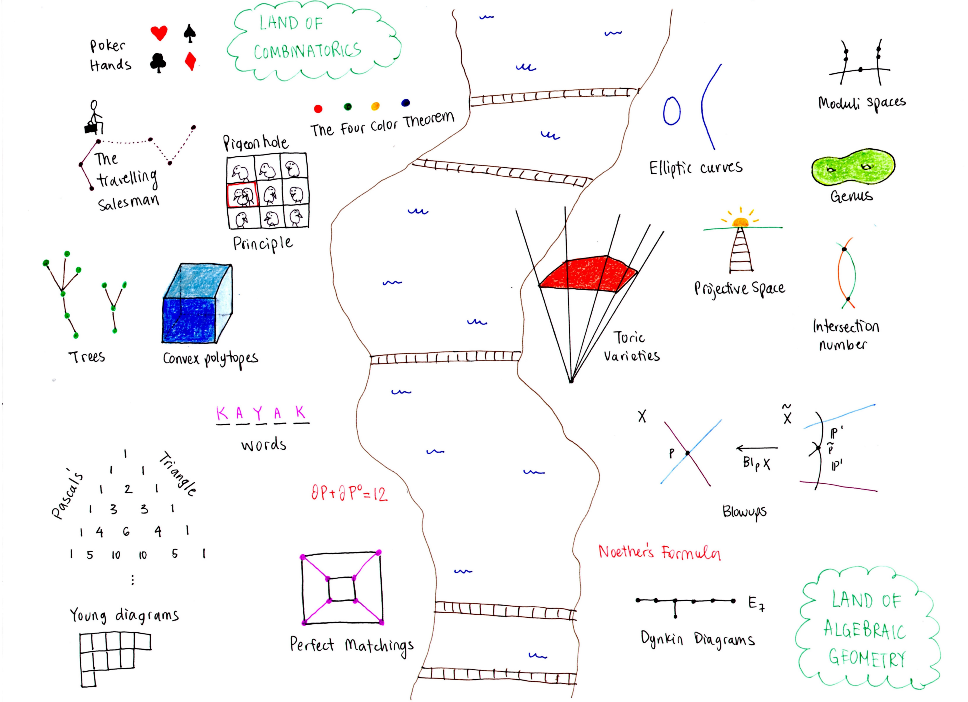
The Main Theorem!

The Big Result

The Main Theorem is another way of stating Noether's Formula for 2-dimensional smooth toric varieties!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The Four Color Theorem
The travelling Salesman
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Pascal's Triangle
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Intersection number
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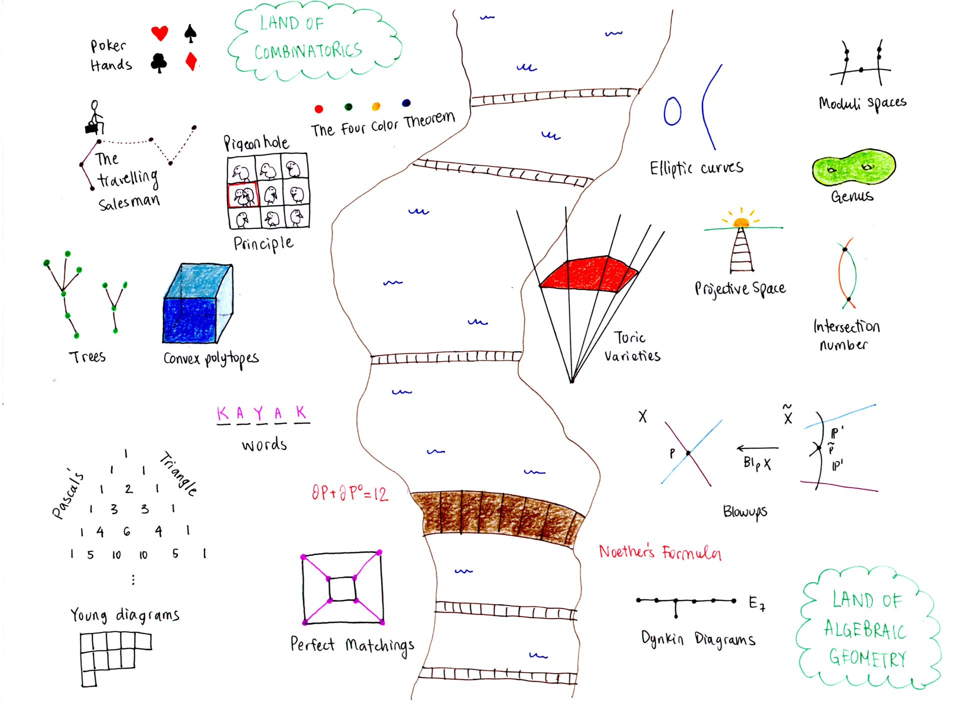
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2) Poonen, B., Rodriguez-Villegas, F., *Lattice Polytopes and the Number 12*, The American Mathematical Monthly, Vol. 107, No. 3 (Mar., 2000), pp. 238-250.