Hodge Theory of Matroids

> Editor's Note: *Matt Baker is speaking on this topic in the Current Events Bulletin Lecture at the January 2017 Joint Mathematics Meetings.*

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Introduction

Logarithmic concavity is a property of a sequence of real numbers, occurring throughout algebraic geometry, convex geometry, and combinatorics. A sequence of positive numbers $a_0, \dots, a_d$ is *log-concave* if

$$a_i^2 \;\geq\; a_{i-1}a_{i+1} \quad \text{for all } i.$$

This means that the logarithms, $\log(a_i)$, form a concave sequence. The condition implies unimodality of the sequence (a_i), a property easier to visualize: the sequence is *unimodal* if there is an index i such that

$$a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_d.$$

We will discuss our work on establishing log-concavity of various combinatorial sequences, such as the coefficients of the chromatic polynomial of graphs and the face numbers of matroid complexes. Our method is motivated by complex algebraic geometry, in particular Hodge theory. From a given combinatorial object M (a matroid), we construct a graded commutative algebra over the real numbers

$$A^*(M) = \bigoplus_{q=0}^{d} A^q(M),$$

which satisfies analogues of Poincaré duality, the hard Lefschetz theorem, and the Hodge-Riemann relations for the cohomology of smooth projective varieties. Log-concavity will be deduced from the Hodge-Riemann relations for M. We believe that behind any log-concave sequence that appears in nature there is such a "Hodge structure" responsible for the log-concavity.

Coloring Graphs

Generalizing earlier work of George Birkhoff, in 1932 Hassler Whitney introduced the *chromatic polynomial* of a connected graph G as the function on $\mathbb{N}$ defined by

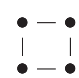
$$\chi_G(q) = |\{\text{proper colorings of } G \text{ using } q \text{ colors}\}|.$$

In other words, $\chi_G(q)$ is the number of ways to color the vertices of G using q colors so that the endpoints of every edge have different colors. Whitney noticed that the chromatic polynomial is indeed a polynomial. In fact, we can write

$$\chi_G(q)/q = a_0(G)q^d - a_1(G)q^{d-1} + \cdots + (-1)^d a_d(G)$$

for some positive integers $a_0(G), \dots, a_d(G)$, where d is one less than the number of vertices of G.

Example 1. The square graph



has the chromatic polynomial $1q^4 - 4q^3 + 6q^2 - 3q$.

The chromatic polynomial was originally devised as a tool for attacking the Four Color Problem, but soon it attracted attention in its own right. Ronald Read conjectured in 1968 that the coefficients of the chromatic polynomial form a unimodal sequence for any graph.

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A few years later, Stuart Hoggar conjectured that the coefficients in fact form a log-concave sequence:

$$a_i(G)^2 \geq a_{i-1}(G)a_{i+1}(G) \quad \text{for any } i \text{ and } G.$$

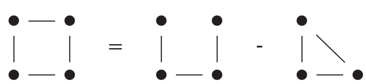
The graph theorist William Tutte quipped, "In compensation for its failure to settle the Four Colour Conjecture, [the chromatic polynomial] offers us the Unimodal Conjecture for our further bafflement."

The chromatic polynomial can be computed using the *deletion-contraction relation*: if $G \backslash e$ is the deletion of an edge e from G and G/e is the contraction of the same edge, then

$$\chi_G(q) \;=\; \chi_{G\backslash e}(q) - \chi_{G/e}(q).$$

The first term counts the proper colorings of G, the second term counts the otherwise-proper colorings of G where the endpoints of e are permitted to have the same color, and the third term counts the otherwise-proper colorings of G where the endpoints of e are mandated to have the same color. Note that, in general, the sum of two log-concave sequences is not a log-concave sequence.

Example 2. To compute the chromatic polynomial of the square graph above, we write



and use

$$\chi_{G\backslash e}(q) = q(q-1)^3, \quad \chi_{G/e}(q) = q(q-1)(q-2).$$

The Hodge-Riemann relations for the algebra $A^*(M)$, where M is the matroid attached to G as in the section "Matroids" below, imply that the coefficients of the chromatic polynomial of G form a log-concave sequence. This is in contrast to a result of Alan Sokal that the set of roots of the chromatic polynomials of all graphs is dense in the complex plane.

Counting Independent Subsets

Linear independence is a fundamental notion in algebra and geometry: a collection of vectors is linearly independent if no nontrivial linear combination sums to zero. How many linearly independent collections of i vectors are there in a given configuration of vectors? Write A for a finite subset of a vector space and $f_i(A)$ for the number of independent subsets of A of size i.

Example 3. If A is the set of all nonzero vectors in the three-dimensional vector space over the field with two elements, then

$$f_0 = 1, \quad f_1 = 7, \quad f_2 = 21, \quad f_3 = 28.$$

Examples suggest a pattern leading to a conjecture of Dominic Welsh:

$$f_i(A)^2 \geq f_{i-1}(A)f_{i+1}(A) \quad \text{for any } i \text{ and } A.$$

For any small specific case, the conjecture can be verified by computing the $f_i(A)$'s by the *deletion-contraction relation*: if $A \backslash v$ is the deletion of a nonzero vector v from A and A/v is the projection of A in the direction of v, then

$$f_i(A) \;=\; f_i(A\backslash v) + f_{i-1}(A/v).$$

The first term counts the number of independent subsets of size i, the second term counts the independent subsets of size i not containing v, and the third term counts the independent subsets of size i containing v. As in the case of graphs, we notice the apparent interference between the log-concavity conjecture and the additive nature of $f_i(A)$. The sum of two log-concave sequences is, in general, not log-concave. The conjecture suggests a new description for the numbers $f_i(A)$ and a corresponding structure of A.

Matroids

In the 1930s Hassler Whitney observed that several notions in graph theory and linear algebra fit together in a common framework, that of *matroids*. This observation started a new subject with applications to a wide range of topics such as characteristic classes, optimization, and moduli spaces, to name a few.

Let E be a finite set. A *matroid* M on E is a collection of subsets of E, called *flats* of M, satisfying the following axioms:

(1) If F_1 and F_2 are flats of M, then their intersection is a flat of M.
(2) If F is a flat of M, then any element of $E \backslash F$ is contained in exactly one flat of M covering F.
(3) The ground set E is a flat of M.

Here, a flat of M is said to *cover* F if it is minimal among the flats of M properly containing F. For our purposes, we may assume that M is *loopless*:

(1) The empty subset of E is a flat of M.

Every maximal chain of flats of M has the same length, and this common length is called the *rank* of M. We write $M \backslash e$ for the matroid obtained by deleting e from the flats of M and M/e for the matroid obtained by deleting e from the flats of M containing e. When M_1 is a matroid on E_1, M_2 is a matroid on E_2, and $E_1 \cap E_2$ is empty, the *direct sum* $M_1 \oplus M_2$ is defined to be the matroid on $E_1 \cup E_2$ whose flats are all sets of the form $F_1 \cup F_2$, where F_1 is a flat of M_1 and F_2 is a flat of M_2.

Matroids encode a combinatorial structure common to graphs and vector configurations. If E is the set of edges of a finite graph G, call a subset F of E a flat when there is no edge in $E \backslash F$ whose endpoints are connected by a path in F. This defines a *graphic matroid* on E. If E is a finite subset of a vector space, call a subset F of E a flat when there is no vector in $E \backslash F$ that is contained in the linear span of F. This defines a *linear matroid* on E.

Matroids encode a combinatorial structure common to graphs and vector configurations

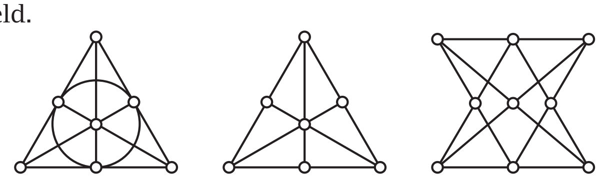
Example 4. Write $E = \{0,1,2,3\}$ for the set of edges of the square graph G in Example 1. The graphic matroid M on E attached to G has flats

$$\emptyset, \{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{0,1,2,3\}.$$

Example 5. Write $E = \{0,1,2,3,4,5,6\}$ for the configuration of vectors A in Example 3. The linear matroid M on E attached to A has flats

$$\emptyset, \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2,3\}, \{3,4,5\}, \{1,5,6\}, \{0,1,4\}, \{0,2,5\}, \{0,3,6\}, \{2,4,6\}, \{0,1,2,3,4,5,6\}.$$

A linear matroid that arises from a subset of a vector space over a field k is said to be *realizable* over k. Not surprisingly, this notion is sensitive to the field k. A matroid may arise from a vector configuration over one field, while no such vector configuration exists over another field. Many matroids are not realizable over any field.



Among the rank 3 loopless matroids pictured above, where rank 1 flats are represented by points and rank 2 flats containing more than two points are represented by lines, the first is realizable over k if and only if the characteristic of k is 2, the second is realizable over k if and only if the characteristic of k is not 2, and the third is not realizable over any field.

The *characteristic polynomial* $\chi_M(q)$ of a matroid M is a generalization of the chromatic polynomial $\chi_G(q)$ of a graph G. It can be recursively defined using the following rules:

(1) If M is the direct sum $M_1 \oplus M_2$, then
$$\chi_M(q) = \chi_{M_1}(q)\,\chi_{M_2}(q).$$
(2) If M is not a direct sum, then, for any e,
$$\chi_M(q) = \chi_{M\setminus e}(q) - \chi_{M/e}(q).$$
(3) If M is the rank 1 matroid on $\{e\}$, then
$$\chi_M(q) = q - 1.$$
(4) If M is the rank 0 matroid on $\{e\}$, then
$$\chi_M(q) = 0.$$

It is a consequence of the Möbius inversion for partially ordered sets that the characteristic polynomial of M is well defined.

We may now state our result in [AHK], which confirms a conjecture of Gian-Carlo Rota and Dominic Welsh.

Theorem 6 ([AHK, Theorem 9.9])**.** The coefficients of the characteristic polynomial form a log-concave sequence for any matroid M.

This implies the log-concavity of the sequence $a_i(G)$ [Huh12] and the log-concavity of the sequence $f_i(A)$ [Len12].

Hodge-Riemann Relations for Matroids

Let X be a mathematical object of "dimension" d. Often it is possible to construct from X in a natural way a graded vector space over the real numbers

$$A^*(X) = \bigoplus_{q=0}^{d} A^q(X),$$

equipped with a graded bilinear pairing

$$P : A^*(X) \times A^{d-*}(X) \longrightarrow \mathbb{R},$$

and a graded linear map

$$L : A^*(X) \longrightarrow A^{*+1}(X), \qquad x \longmapsto \mathrm{L}x$$

("P" is for Poincaré, and "L" is for Lefschetz). For example, $A^*(X)$ may be the cohomology of real (p,p)-forms on a compact Kähler manifold X or the ring of algebraic cycles modulo homological equivalence on a smooth projective variety X or the combinatorial intersection cohomology of a convex polytope X [Kar04] or the Soergel bimodule of an element of a Coxeter group X [EW14] or the Chow ring of a matroid X defined below. We expect that, for every nonnegative integer $q \le \frac{d}{2}$:

(1) the bilinear pairing
$$P : A^q(X) \times A^{d-q}(X) \longrightarrow \mathbb{R}$$
is nondegenerate (Poincaré duality for X),

(2) the composition of linear maps
$$L^{d-2q} : A^q(X) \longrightarrow A^{d-q}(X)$$
is bijective (the hard Lefschetz theorem for X), and

(3) the bilinear form on $A^q(X)$ defined by
$$(x_1, x_2) \longmapsto (-1)^q\, P(x_1, L^{d-2q}x_2)$$
is symmetric and is positive definite on the kernel of
$$L^{d-2q+1} : A^q(X) \longrightarrow A^{d-q+1}(X)$$
(the Hodge-Riemann relations for X).

All three properties are known to hold for the objects listed above except one, which is the subject of Grothendieck's standard conjectures on algebraic cycles.

For a loopless matroid M on E, the vector space $A^*(M)$ has the structure of a graded algebra that can be described explicitly.

Definition 7. We introduce variables x_F, one for each nonempty proper flat F of M, and set

$$S^*(M) = \mathbb{R}[x_F]_{F \neq \emptyset, F \neq E}.$$

The *Chow ring* $A^*(M)$ of M is the quotient of $S^*(M)$ by the ideal generated by the linear forms

$$\sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F,$$

one for each pair of distinct elements i_1 and i_2 of E, and the quadratic monomials

$$x_{F_1} x_{F_2},$$

one for each pair of incomparable nonempty proper flats F_1 and F_2 of M.

The Chow ring of M was introduced by Eva Maria Feichtner and Sergey Yuzvinsky. When M is realizable over a field k, it is the Chow ring of the "wonderful" compactification of the complement of a hyperplane arrangement defined over k as described by Corrado de Concini and Claudio Procesi.

Let d be the integer one less than the rank of M.

Theorem 8 ([AHK, Proposition 5.10]). There is a linear bijection

$$\deg\colon A^d(M) \longrightarrow \mathbb{R}$$

uniquely determined by the property that

$$\deg(x_{F_1}x_{F_2}\cdots x_{F_d}) = 1$$

for every maximal chain of nonempty proper flats

$$F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_d.$$

In addition, the bilinear pairing

$$P : A^q(M) \times A^{d-q}(M) \to \mathbb{R}, \quad (x, y) \mapsto \deg(xy)$$

is nondegenerate for every nonnegative $q \le d$.

What should be the linear operator L for M? We collect all valid choices of L in a nonempty open convex cone. The cone is an analogue of the Kähler cone in complex geometry.

Definition 9. A real-valued function c on 2^E is said to be *strictly submodular* if

$$c_\varnothing = 0, \quad c_E = 0,$$

and, for any two incomparable subsets $I_1, I_2 \subseteq E$,

$$c_{I_1} + c_{I_2} > c_{I_1 \cap I_2} + c_{I_1 \cup I_2}.$$

A strictly submodular function c defines an element

$$L(c) = \sum_F c_F x_F \in A^1(M)$$

that acts as a linear operator by multiplication

$$A^*(M) \longrightarrow A^{*+1}(M), \quad x \longmapsto L(c)x.$$

The set of all such elements is a convex cone in $A^1(M)$.

The main result of [AHK] states that the triple $(A^*(M), P, L(c))$ satisfies the hard Lefschetz theorem and the Hodge-Riemann relations for every strictly submodular function c:

Theorem 10. Let q be a nonnegative integer less than $\frac{d}{2}$.

(1) The multiplication by $L(c)$ defines an isomorphism

$$A^q(M) \longrightarrow A^{d-q}(M), \quad x \longmapsto L(c)^{d-2q}\, x.$$

(2) The symmetric bilinear form on $A^q(M)$ defined by

$$(x_1, x_2) \longmapsto (-1)^q\, P(x_1, L(c)^{d-2q}x_2)$$

is positive definite on the kernel of $L(c)^{d-2q+1}$.

The known proofs of the hard Lefschetz theorem and the Hodge-Riemann relations for the different types of objects listed above have certain structural similarities, but there is no known way of deducing one from the others.

Sketch of Proof of Log-Concavity

We now explain why the Hodge-Riemann relations for M imply log-concavity for $\chi_M(q)$. The Hodge-Riemann relations for M, in fact, imply that the sequence (m_i) in the expression

$$\chi_M(q)/(q-1) = m_0q^d - m_1q^{d-1} + \cdots + (-1)^d m_d$$

is log-concave, which is stronger.

We define two elements of $A^1(M)$: for any $j \in E$, set

$$\alpha_M = \sum_{j \in F} x_F, \quad \beta_M = \sum_{j \notin F} x_F.$$

The two elements do not depend on the choice of j, and they are limits of elements of the form $L(c)$ for strictly submodular c. A short combinatorial argument shows that m_i is a mixed degree of α_M and β_M:

$$m_i = \deg(\alpha_M^i\, \beta_M^{d-i}).$$

Thus, it is enough to prove for every i that

$$\deg(\alpha_M^{d-i+1}\beta_M^{i-1})\deg(\alpha_M^{d-i-1}\beta_M^{i+1}) \le \deg(\alpha_M^{d-i}\beta_M^{i})^2.$$

This is an analogue of the *Teisser-Khovanskii inequality* for intersection numbers in algebraic geometry and the *Alexandrov-Fenchel inequality* for mixed volumes in convex geometry. The main case is when $i = d-1$.

By a continuity argument, we may replace β_M by $L = L(c)$ sufficiently close to β_M. The desired inequality in the main case then becomes

$$\deg(\alpha_M^2 L^{d-2})\deg(L^d) \le \deg(\alpha_M L^{d-1})^2.$$

This follows from the fact that the signature of the bilinear form

$$A^1(M) \times A^1(M) \to \mathbb{R}, \quad (x_1, x_2) \mapsto \deg(x_1 L^{d-2} x_2)$$

restricted to the span of α_M and L is semi-indefinite, which, in turn, is a consequence of the Hodge-Riemann relations for M in the cases $q = 0, 1$.

This application uses only a small piece of the Hodge-Riemann relations for M. The general Hodge-Riemann relations for M may be used to extract other interesting combinatorial information about M.

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