Advances in Mathematics 270 (2015) 49–59

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

h-Vectors of matroids and logarithmic concavity

June Huh

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

ARTICLE INFO

Article history:
Received 11 November 2012
Accepted 4 November 2014
Available online 13 November 2014
Communicated by Ezra Miller

MSC:
05B35
52C35

Keywords:
Matroid
Hyperplane arrangement
f-Vector
h-Vector
Log-concavity
Characteristic polynomial

ABSTRACT

Let M be a matroid on E, representable over a field of characteristic zero. We show that h-vectors of the following simplicial complexes are log-concave:

1. The matroid complex of independent subsets of E.
2. The broken circuit complex of M relative to an ordering of E.

The first implies a conjecture of Colbourn on the reliability polynomial of a graph, and the second implies a conjecture of Hoggar on the chromatic polynomial of a graph. The proof is based on the geometric formula for the characteristic polynomial of Denham, Garrousian, and Schulze.

1. Introduction and results

A sequence $e_0, e_1, \ldots, e_n$ of integers is said to be *log-concave* if for all $0 < i < n$,

$$e_{i-1}e_{i+1} \leq e_i^2,$$

and is said to have *no internal zeros* if there do not exist $i < j < k$ satisfying

$$e_i \neq 0, \qquad e_j = 0, \qquad e_k \neq 0.$$

E-mail address: junehuh@umich.edu.

http://dx.doi.org/10.1016/j.aim.2014.11.002
0001-8708/

Empirical evidence has suggested that many important enumerative sequences are log-concave, but proving the log-concavity can sometimes be a non-trivial task. See [5,28,29] for a wealth of examples arising from algebra, geometry, and combinatorics. The purpose of this paper is to demonstrate the use of an algebro-geometric tool to the log-concavity problems.

Let X be a complex algebraic variety. A subvariety of X is an irreducible closed algebraic subset of X. If V is a subvariety of X, then the top dimensional homology group $H_{2\dim(V)}(V;\mathbb{Z}) \simeq \mathbb{Z}$ has a canonical generator, and the closed embedding of V in X determines a homomorphism

$$H_{2\dim(V)}(V;\mathbb{Z}) \longrightarrow H_{2\dim(V)}(X;\mathbb{Z}).$$

The image of the generator is called the *fundamental class* of V in X, denoted $[V]$. A homology class in $H_*(X;\mathbb{Z})$ is said to be *representable* if it is the fundamental class of a subvariety.

Hartshorne asks in [14, Question 1.3] which even dimensional homology classes of X are representable by a smooth subvariety. Although the question is exceedingly difficult in general, it has a simple partial answer when X is the product of complex projective spaces $\mathbb{P}^m \times \mathbb{P}^n$. Note in this case that the $2k$-dimensional homology group of X is freely generated by the classes of subvarieties of the form $\mathbb{P}^{k-i} \times \mathbb{P}^i$.

Representable homology classes of $\mathbb{P}^m \times \mathbb{P}^n$ can be characterized numerically as follows [16, Theorem 20].

Theorem 1. *Write $\xi \in H_{2k}(\mathbb{P}^m \times \mathbb{P}^n;\mathbb{Z})$ as the integral linear combination*

$$\xi = \sum_i e_i \big[\mathbb{P}^{k-i} \times \mathbb{P}^i\big].$$

1. *If ξ is an integer multiple of either*

$$\big[\mathbb{P}^m \times \mathbb{P}^n\big], \big[\mathbb{P}^m \times \mathbb{P}^0\big], \big[\mathbb{P}^0 \times \mathbb{P}^n\big], \big[\mathbb{P}^0 \times \mathbb{P}^0\big],$$

   then ξ is representable if and only if the integer is 1.
2. *If otherwise, some positive integer multiple of ξ is representable if and only if the e_i form a nonzero log-concave sequence of nonnegative integers with no internal zeros.*

In short, subvarieties of $\mathbb{P}^m \times \mathbb{P}^n$ correspond to log-concave sequences of nonnegative integers with no internal zeros. Therefore, when trying to prove the log-concavity of a sequence, it is reasonable to look for a subvariety of $\mathbb{P}^m \times \mathbb{P}^n$ which witnesses this property. We demonstrate this method by proving the log-concavity of h-vectors of two simplicial complexes associated to a matroid, when the matroid is representable over a field of characteristic zero. Other illustrations can be found in [16,18,19].

In order to fix notations, we recall from [4] some basic definitions on simplicial complexes associated to a matroid. We use Oxley's book as our basic reference on matroid theory [27].

Let Δ be an abstract simplicial complex of dimension r. The *f-vector* of Δ is a sequence of integers $f_0, f_1, \ldots, f_{r+1}$, where

$$f_i = \big(\text{the number of } (i-1)\text{-dimensional faces of } \Delta\big).$$

For example, f_0 is one, f_1 is the number of vertices of Δ, and f_{r+1} is the number of facets of Δ. The *h-vector* of Δ is defined from the f-vector by the polynomial identity

$$\sum_{i=0}^{r+1} f_i(q-1)^{r+1-i} = \sum_{i=0}^{r+1} h_i q^{r+1-i}.$$

Alternatively, the h-vector of Δ can be defined from the Hilbert series

$$\mathrm{HS}(S/I_\Delta; t) = \frac{h_0 + h_1 t + \cdots + h_{r+1}t^{r+1}}{(1-t)^{r+1}},$$

where S/I_Δ is the Stanley–Reisner ring of Δ over any field. See [24, Section 1.1] for Stanley–Reisner basics. When there is a need for clarification, we write the coefficients by $f_i(\Delta)$ and $h_i(\Delta)$ respectively.

Let M be a matroid of rank $r+1$ on an ordered set E of cardinality $n+1$. We are interested in the h-vectors of the following simplicial complexes associated to M:

1. The matroid complex $\mathrm{IN}(M)$, the collection of subsets of E which are independent in M.
2. The broken circuit complex $\mathrm{BC}(M)$, the collection of subsets of E which do not contain any broken circuit of M.

Recall that a *broken circuit* is a subset of E obtained from a circuit of M by deleting the least element relative to the ordering of E. We note that the isomorphism type of the broken circuit complex does depend on the ordering of E. However, the results of this paper will be independent of the ordering of E.

Remark 2. A pure r-dimensional simplicial complex is said to be *shellable* if there is an ordering of its facets such that each facet intersects the complex generated by its predecessors in a pure $(r-1)$-dimensional complex. $\mathrm{IN}(M)$ and $\mathrm{BC}(M)$ are pure of dimension r, and are shellable. As a consequence, the h-vectors of both complexes consist of nonnegative integers [4]. This nonnegativity is recovered in Theorem 3 below.

Dawson conjectured that the h-vector of a matroid complex is a log-concave sequence [11, Conjecture 2.5]. Colbourn repeated this conjecture for graphical matroids in the context of network reliability [10]. Our main result verifies Dawson's conjecture for matroids representable over a field of characteristic zero.

Theorem 3. *Let M be a matroid representable over a field of characteristic zero.*

1. *The h-vector of the matroid complex of M is a log-concave sequence of nonnegative integers with no internal zeros.*
2. *The h-vector of the broken circuit complex of M is a log-concave sequence of nonnegative integers with no internal zeros.*

Indeed, as we explain in the following section, there is a subvariety of a product of projective spaces which witnesses the validity of Theorem 3. I do not know if the log-concavity conjecture for the h-vector of the broken circuit complex of a matroid has ever appeared in the literature, but it is a natural one, since matroid complexes are (up to coning) special cases of broken circuit complexes (Section 2.2).

Lenz shows in [20, Section 4.1] that the log-concavity of the h-vector implies the strict log-concavity of the f-vector:

$$f_{i-1}f_{i+1} < f_i^2, \quad i = 1, 2, \ldots, r.$$

Therefore Theorem 3 implies that the two f-vectors associated to M are strictly log-concave. The strict log-concavity of f-vectors of matroid complexes and broken circuit complexes was conjectured by Mason [23] and Hoggar [15] (for graphical matroids) respectively.

Corollary 4. *Let M be a matroid representable over a field of characteristic zero.*

1. *The f-vector of the matroid complex of M is a strictly log-concave sequence of nonnegative integers with no internal zeros.*
2. *The f-vector of the broken circuit complex of M is a strictly log-concave sequence of nonnegative integers with no internal zeros.*

The main special cases of Theorem 3 and Corollary 4 are treated in the following subsections.

Remark 5. A pure simplicial complex is a matroid complex if and only if every ordering of the vertices induces a shelling [4, Theorem 7.3.4]. In view of this characterization of matroids, one should contrast Theorem 3 with examples of other 'nice' shellable simplicial complexes whose f-vector and h-vector fail to be log-concave. In fact, the unimodality of the f-vector already fails for simplicial polytopes in dimension ≥ 20 [2,3].

These shellable simplicial complexes led to suspect that various log-concavity conjectures on matroids might not be true in general [29,31]. Theorem 3 shows that there is a qualitative difference between the h-vectors of

1. matroid complexes and other shellable simplicial complexes, and/or
2. matroids representable over a field and matroids in general.

The method of the present paper to prove the log-concavity crucially depends on the assumption that the matroid is representable over a field, and the log-concavity conjectures for general matroids remain wide open.

1.1. The reliability polynomial of a graph

The *reliability* of a connected graph G is the probability that the graph remains connected when each edge is independently removed with the same probability $1-p$. If the graph has e edges and v vertices, then the reliability of G is the polynomial

$$\mathrm{Rel}_G(p) = \sum_{i=0}^{e-v+1} f_i\, p^{e-i}(1-p)^i,$$

where f_i is the number of cardinality i sets of edges whose removal does not disconnect G. For example, f_0 is one, f_1 is the number of edges of G that are not isthmuses, and f_{e-v+1} is the number of spanning trees of G. The *h-sequence* of the reliability polynomial is the sequence h_i defined by the expression

$$\mathrm{Rel}_G(p) = p^{v-1}\left(\sum_{i=0}^{e-v+1} h_i(1-p)^i\right).$$

In other words, the h-sequence is the h-vector of the matroid complex of the cocycle matroid of G. Since the cocycle matroid of a graph is representable over every field, Theorem 3 confirms a conjecture of Colbourn that the h-sequence of the reliability polynomial of a graph is log-concave [10].

Corollary 6. *The h-sequence of the reliability polynomial of a connected graph is a log-concave sequence of nonnegative integers with no internal zeros.*

It has been suggested that Corollary 6 has practical applications in combinatorial reliability theory [6].

1.2. The chromatic polynomial of a graph

The *chromatic polynomial* of a graph G is the polynomial defined by

$$\chi_G(q) = (\text{the number of proper colorings of } G \text{ using } q \text{ colors}).$$

The chromatic polynomial depends only on the cycle matroid of the graph, up to a factor of the form q^c. More precisely, the absolute value of the i-th coefficient of the chromatic polynomial is the number of cardinality i sets of edges which contain no broken circuit [32]. Since the cycle matroid of a graph is representable over every field, Corollary 4 confirms a conjecture of Hoggar that the coefficients of the chromatic polynomial of a graph form a strictly log-concave sequence [15].

Corollary 7. *The coefficients of the chromatic polynomial of a graph form a sign-alternating strictly log-concave sequence of integers with no internal zeros.*

Corollary 7 has been previously verified for all graphs with ≤ 11 vertices [21].

2. Proof of Theorem 3

We shall assume familiarity with the Möbius function $\mu(x,y)$ of the lattice of flats $\mathscr{L}_M$. For this and more, we refer to [1,34]. An important role will be played by the *characteristic polynomial* $\chi_M(q)$. For a loopless matroid M, the characteristic polynomial is defined from $\mathscr{L}_M$ by the formula

$$\chi_M(q) = \sum_{x \in \mathscr{L}_M} \mu(\varnothing, x) q^{r+1-\mathrm{rank}(x)} = \sum_{i=0}^{r+1} (-1)^i w_i q^{r+1-i}.$$

If M has a loop, then $\chi_M(q)$ is defined to be the zero polynomial. The nonnegative integers w_i are called the *Whitney numbers of the first kind*. The characteristic polynomial is always divisible by $q-1$, defining the *reduced characteristic polynomial*

$$\overline{\chi_M}(q) = \chi_M(q)/(q-1).$$

2.1. Brylawski's theorem I

We need to quote a few results from Brylawski's analysis on the broken circuit complex [7]. The first of these says that the Whitney number w_i is the number of cardinality i subsets of E which contain no broken circuit relative to any fixed ordering of E [7, Theorem 3.3]. This observation goes back to Hassler Whitney, who stated it for graphs [32].

Fix an ordering of E, and let 0 be the smallest element of E. We write $\overline{\mathrm{BC}}(M)$ for the *reduced broken circuit complex* of M, the family of all subsets of $E \setminus \{0\}$ that do not contain any broken circuit of M. Since the broken circuit complex is the cone over $\overline{\mathrm{BC}}(M)$ with apex 0, the above quoted fact says that

$$\overline{\chi_M}(q) = \sum_{i=0}^{r} (-1)^i f_i\big(\overline{\mathrm{BC}}(M)\big) q^{r-i}.$$

In terms of the h-vector, we have

$$\overline{\chi_M}(q+1) = \sum_{i=0}^{r} (-1)^i h_i\big(\overline{\mathrm{BC}}(M)\big) q^{r-i} = \sum_{i=0}^{r} (-1)^i h_i\big(\mathrm{BC}(M)\big) q^{r-i},$$
$$h_{r+1}\big(\mathrm{BC}(M)\big) = 0.$$

Therefore the second assertion of Theorem 3 is equivalent to the statement that the coefficients of $\overline{\chi_M}(q+1)$ form a sign-alternating log-concave sequence with no internal zeros.

2.2. Brylawski's theorem II

We show that the first assertion of Theorem 3 is implied by the second. This follows from the fact that the matroid complex of M is the reduced broken circuit complex of the free dual extension of M [7, Theorem 4.2]. We note that not every reduced broken circuit complex can be realized as a matroid complex [7, Remark 4.3]. The second assertion of Theorem 3 is strictly stronger than the first in this sense.

Recall that the *free dual extension* of M is defined by taking the dual of M, placing a new element p in general position (taking the free extension), and again taking the dual. In symbols,

$$M \times p := \big(M^* + p\big)^*.$$

If M is representable over a field, then $M \times p$ is representable over some finite extension of the same field. Choose an ordering of $E \cup \{p\}$ such that p is smaller than any other element. Then, with respect to the chosen ordering,

$$\mathrm{IN}(M) = \overline{\mathrm{BC}}(M \times p).$$

For more details on the free dual extension, see [7,8,19].

2.3. Reduction to simple matroids

A standard argument shows that it is enough to prove the assertion on $\overline{\chi_M}(q+1)$ when M is simple:

1. If M has a loop, then the reduced characteristic polynomial of M is zero, so there is nothing to show in this case.
2. If M is loopless but has parallel elements, replace M by its *simplification* $\overline{M}$ as defined in [27, Section 1.7]. Then the reduced characteristic polynomials of M and $\overline{M}$ coincide because $\mathscr{L}_M \simeq \mathscr{L}_{\overline{M}}$.

Hereafter M is assumed to be simple of rank $r+1$ with $n+1$ elements, representable over a field of characteristic zero.

2.4. Reduction to complex hyperplane arrangements

We reduce the main assertion to the case of essential arrangements of affine hyperplanes. We use the book of Orlik and Terao as our basic reference in hyperplane arrangements [25].

Note that the condition of representability for matroids of given rank and given number of elements can be expressed in a first-order sentence in the language of fields. Since the theory of algebraically closed fields of characteristic zero is complete [22, Corollary 3.2.3], a matroid representable over a field of characteristic zero is in fact representable over $\mathbb{C}$.

Let $\widetilde{\mathcal{A}}$ be a central arrangement of $n+1$ distinct hyperplanes in $\mathbb{C}^{r+1}$ representing M. This means that there is a bijective correspondence between E and the set of hyperplanes of $\widetilde{\mathcal{A}}$ which identifies the geometric lattice $\mathscr{L}_M$ with the lattice of flats of $\widetilde{\mathcal{A}}$. Choose any one hyperplane from the projectivization of $\widetilde{\mathcal{A}}$ in $\mathbb{P}^r$. The *decone* of the central arrangement, denoted $\mathcal{A}$, is the essential arrangement of n hyperplanes in $\mathbb{C}^r$ obtained by declaring the chosen hyperplane to be the hyperplane at infinity. If $\chi_{\mathcal{A}}(q)$ is the characteristic polynomial of the decone, then

$$\chi_{\mathcal{A}}(q) = \overline{\chi_M}(q).$$

Therefore it suffices to prove that the coefficients of $\chi_{\mathcal{A}}(q+1)$ form a sign-alternating log-concave sequence of integers with no internal zeros.

2.5. The variety of critical points

Finally, the geometry comes into the scene. We are given an essential arrangement $\mathcal{A}$ of n affine hyperplanes in $\mathbb{C}^r$. Our goal is to find a subvariety of a product of projective spaces, whose fundamental class encodes the coefficients of the translated characteristic polynomial $\chi_{\mathcal{A}}(q+1)$.

The choice of the subvariety is suggested by an observation of Varchenko on the critical points of the master function of an affine hyperplane arrangement [30]. Let $L_1, \dots, L_n$ be the linear functions defining the hyperplanes of $\mathcal{A}$. A *master function* of $\mathcal{A}$ is a nonvanishing holomorphic function defined on the complement $\mathbb{C}^r \setminus \mathcal{A}$ as the product of powers

$$\varphi_{\mathbf{u}} := \prod_{i=1}^{n} L_i^{u_i}, \quad \mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Z}^n.$$

Varchenko's conjecture. *If the exponents u_i are sufficiently general, then all critical points of $\varphi_{\mathbf{u}}$ are nondegenerate, and the number of critical points is equal to $(-1)^r \chi_{\mathcal{A}}(1)$.*

Note that $(-1)^r \chi_{\mathcal{A}}(1)$ is equal to the number of bounded regions in the complement $\mathbb{R}^r \setminus \mathcal{A}$ when $\mathcal{A}$ is defined over the real numbers, and to the signed topological Euler characteristic of the complement $\mathbb{C}^r \setminus \mathcal{A}$. The conjecture is proved by Varchenko in the real case [30], and by Orlik and Terao in general [25].

In order to encode all the coefficients of $\chi_{\mathcal{A}}(q+1)$ in an algebraic variety, we consider the totality of critical points of all possible (multivalued) master functions of $\mathcal{A}$. More precisely, we define the *variety of critical points* $\mathfrak{X}(\mathcal{A})$ as the closure

$$\mathfrak{X}(\mathcal{A}) = \overline{\mathfrak{X}^\circ(\mathcal{A})} \subseteq \mathbb{P}^r \times \mathbb{P}^{n-1},$$
$$\mathfrak{X}^\circ(\mathcal{A}) = \left\{ \sum_{i=1}^n u_i \cdot \operatorname{dlog}(L_i)(x) = 0 \right\} \subseteq \big(\mathbb{C}^r \setminus \mathcal{A}\big) \times \mathbb{P}^{n-1},$$

where $\mathbb{P}^{n-1}$ is the projective space with the homogeneous coordinates $u_1, \ldots, u_n$. The variety of critical points first appeared implicitly in [26], and further studied in [9,13]. See also [17, Section 2].

The variety of critical points is irreducible because $\mathfrak{X}^\circ(\mathcal{A})$ is a projective space bundle over the complement $\mathbb{C}^r \setminus \mathcal{A}$. The cardinality of a general fiber of the second projection

$$\mathrm{pr}_2 : \mathfrak{X}(\mathcal{A}) \longrightarrow \mathbb{P}^{n-1}$$

is equal to $(-1)^r \chi_{\mathcal{A}}(1)$, as stated in Varchenko's conjecture. More generally, we have

$$\big[\mathfrak{X}(\mathcal{A})\big] = \sum_{i=0}^r v_i \big[\mathbb{P}^{r-i} \times \mathbb{P}^{n-1-r+i}\big] \in H_{2n-2}\big(\mathbb{P}^r \times \mathbb{P}^{n-1}; \mathbb{Z}\big),$$

where v_i are the coefficients of the characteristic polynomial

$$\chi_{\mathcal{A}}(q+1) = \sum_{i=0}^r (-1)^i v_i \, q^{r-i}.$$

The previous statement is [17, Corollary 3.11], which is essentially the geometric formula for the characteristic polynomial of Denham, Garrousian and Schulze [13, Theorem 1.1], modulo a minor technical difference pointed out in [17, Remark 2.2]. A conceptual proof of the geometric formula can be summarized as follows [17, Section 3]:

1. Applying a logarithmic version of the Poincaré–Hopf theorem to a compactification of the complement $\mathbb{C}^r \setminus \mathcal{A}$, one shows that the fundamental class of the variety of critical points captures the characteristic class of $\mathbb{C}^r \setminus \mathcal{A}$.
2. The characteristic class of $\mathbb{C}^r \setminus \mathcal{A}$ agrees with the characteristic polynomial $\chi_{\mathcal{A}}(q+1)$, because the two are equal at $q = 0$ and satisfy the same inclusion–exclusion formula.

See [13, Section 3] for a more geometric approach.

The proof of Theorem 3 is completed by applying Theorem 1 to the fundamental class of the variety of critical points of $\mathcal{A}$.

Simple examples show that equalities may hold throughout in the inequalities of Theorem 3. For example, if M is the uniform matroid of rank $r+1$ with $r+2$ elements, then

$$h_i\big(\mathrm{IN}(M)\big) = h_i\big(\mathrm{BC}(M)\big) = 1, \quad i = 1, \ldots, r.$$

However, a glance at the list of h-vectors of small matroid complexes generated in [12] suggests that there are stronger conditions on the h-vectors than those that are known or conjectured. The answer to the interrogative title of [33] seems to be out of reach at the moment.

Acknowledgments

The author thanks the referee for detailed and helpful comments. The author was partially supported by NSF grant DMS-0943832.

References

[1] M. Aigner, Whitney numbers, in: Combinatorial Geometries, in: Encyclopedia Math. Appl., vol. 29, Cambridge University Press, Cambridge, 1987, pp. 139–160.
[2] L. Billera, C. Lee, A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes, J. Combin. Theory Ser. A 31 (1981) 237–255.
[3] A. Björner, The unimodality conjecture for convex polytopes, Bull. Amer. Math. Soc. 4 (1981) 187–188.
[4] A. Björner, The homology and shellability of matroids and geometric lattices, in: Matroid Applications, in: Encyclopedia Math. Appl., vol. 40, Cambridge University Press, Cambridge, 1992, pp. 226–283.
[5] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, in: Jerusalem Combinatorics '93, in: Contemp. Math., vol. 178, American Mathematical Society, Providence, 1994, pp. 71–89.
[6] J. Brown, C. Colbourn, On the log concavity of reliability and matroidal sequences, Adv. in Appl. Math. 15 (1994) 114–127.
[7] T. Brylawski, The broken-circuit complex, Trans. Amer. Math. Soc. 234 (1977) 417–433.
[8] T. Brylawski, Constructions, in: Theory of Matroids, in: Encyclopedia Math. Appl., vol. 26, Cambridge University Press, Cambridge, 1986, pp. 127–223.
[9] D. Cohen, G. Denham, M. Falk, A. Varchenko, Critical points and resonance of hyperplane arrangements, Canad. J. Math. 63 (2011) 1038–1057.
[10] C. Colbourn, The Combinatorics of Network Reliability, Int. Ser. Monogr. Comput. Sci., The Clarendon Press, Oxford University Press, New York, 1987.
[11] J. Dawson, A collection of sets related to the Tutte polynomial of a matroid, in: Graph Theory, Singapore, 1983, in: Lecture Notes in Math., vol. 1073, Springer, Berlin, 1984, pp. 193–204.
[12] J. De Loera, Y. Kemper, S. Klee, h-Vectors of small matroid complexes, Electron. J. Combin. 19 (2012), Paper 14, 11 pp.
[13] G. Denham, M. Garrousian, M. Schulze, A geometric deletion–restriction formula, Adv. Math. 230 (2012) 1979–1994.
[14] R. Hartshorne, Varieties of small codimension in projective space, Bull. Amer. Math. Soc. 80 (1974) 1017–1032.
[15] S. Hoggar, Chromatic polynomials and logarithmic concavity, J. Combin. Theory Ser. B 16 (1974) 248–254.

[16] J. Huh, Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, J. Amer. Math. Soc. 25 (2012) 907–927.
[17] J. Huh, The maximum likelihood degree of a very affine variety, Compos. Math. 149 (2013) 1245–1266.
[18] J. Huh, E. Katz, Log-concavity of characteristic polynomials and the Bergman fan of matroids, Math. Ann. 354 (3) (2012) 1103–1116.
[19] M. Lenz, The f-vector of a representable-matroid complex is log-concave, Adv. in Appl. Math. 51 (2013) 543–545.
[20] M. Lenz, Matroids and log-concavity, arXiv:1106.2944v4, 2013.
[21] P.H. Lundow, K. Markström, Broken-cycle-free subgraphs and the log-concavity conjecture for chromatic polynomials, Experiment. Math. 15 (2006) 343–353.
[22] D. Marker, Model Theory. An Introduction, Grad. Texts in Math., vol. 217, Springer-Verlag, New York, 2002.
[23] J. Mason, Matroids: unimodal conjectures and Motzkin's theorem, in: Combinatorics, Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972, Inst. Math. Appl., Southend-on-Sea, 1972, pp. 207–220.
[24] E. Miller, B. Sturmfels, Combinatorial Commutative Algebra, Grad. Texts in Math., vol. 227, Springer-Verlag, New York, 2005.
[25] P. Orlik, H. Terao, Arrangements of Hyperplanes, Grundlehren Math. Wiss., vol. 300, Springer-Verlag, Berlin, 1992.
[26] P. Orlik, H. Terao, The number of critical points of a product of powers of linear functions, Invent. Math. 120 (1995) 1–14.
[27] J. Oxley, Matroid Theory, second ed., Oxf. Grad. Texts Math., vol. 21, Oxford University Press, Oxford, 2011.
[28] R. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, in: Graph Theory and Its Applications: East and West, Jinan, 1986, Ann. New York Acad. Sci. 576 (1989) 500–535.
[29] R. Stanley, Positivity problems and conjectures in algebraic combinatorics, in: Mathematics: Frontiers and Perspectives, American Mathematical Society, Providence, 2000, pp. 295–319.
[30] A. Varchenko, Critical points of the product of powers of linear functions and families of bases of singular vectors, Compos. Math. 97 (1995) 385–401.
[31] D. Wagner, Negatively correlated random variables and Mason's conjecture for independent sets in matroids, Ann. Comb. 12 (2008) 211–239.
[32] H. Whitney, A logical expansion in mathematics, Bull. Amer. Math. Soc. 38 (1932) 572–579.
[33] H. Wilf, Which polynomials are chromatic?, in: Colloquio Internazionale sulle Teorie Combinatorie, Tomo I, Roma, 1973, in: Atti Conv. Lincei, vol. 17, Accademia Nazionale dei Lincei, Rome, 1976, pp. 247–256.
[34] T. Zaslavsky, The Möbius function and the characteristic polynomial, in: Combinatorial Geometries, in: Encyclopedia Math. Appl., vol. 29, Cambridge University Press, Cambridge, 1987, pp. 114–138.