LEFSCHETZ CLASSES ON PROJECTIVE VARIETIES

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ABSTRACT. The Lefschetz algebra $L^*(X)$ of a smooth complex projective variety X is the subalgebra of the cohomology algebra of X generated by divisor classes. We construct smooth complex projective varieties whose Lefschetz algebras do not satisfy analogues of the hard Lefschetz theorem and Poincaré duality.

1. INTRODUCTION

Let X be a d-dimensional smooth complex projective variety, and let $\mathrm{Alg}^*(X)$ be the commutative graded $\mathbb{Q}$-algebra of algebraic cycles on X modulo homological equivalence

$$\mathrm{Alg}^*(X) = \bigoplus_{k=0}^{d} \mathrm{Alg}^k(X) \subseteq H^{2*}(X, \mathbb{Q}).$$

A hyperplane section of $X \subseteq \mathbb{P}^n$ defines a cohomology class $\omega \in \mathrm{Alg}^1(X)$. Grothendieck's standard conjectures predict that $\mathrm{Alg}^*(X)$ satisfies analogues of the hard Lefschetz theorem and Poincaré duality:

(HL) For every nonnegative integer $k \leq \frac{d}{2}$, the linear map

$$\mathrm{Alg}^k(X) \longrightarrow \mathrm{Alg}^{d-k}(X), \qquad x \longmapsto \omega^{d-2k} x$$

is an isomorphism.

(PD) For every nonnegative integer $k \leq \frac{d}{2}$, the bilinear map

$$\mathrm{Alg}^k(X) \times \mathrm{Alg}^{d-k}(X) \longrightarrow \mathrm{Alg}^d(X) \simeq \mathbb{Q}, \qquad (x_1, x_2) \longmapsto x_1 x_2$$

is nondegenerate.

The properties (HL) and (PD) for $\mathrm{Alg}^*(X)$ are implied by the Hodge conjecture for X.

The *Lefschetz algebra* of X is the graded $\mathbb{Q}$-subalgebra $L^*(X)$ of $\mathrm{Alg}^*(X)$ generated by divisor classes. When X is singular, we may define the Lefschetz algebra to be the graded $\mathbb{Q}$-algebra in the intersection cohomology generated by the Chern classes of line bundles

$$L^*(X) \subseteq IH^{2*}(X, \mathbb{Q}).$$

An application of Lefschetz algebras to the "top-heavy" conjecture in enumerative combinatorics was given in [HW16].

It was asked whether there are smooth projective varieties whose Lefschetz algebras do not satisfy analogues of the hard Lefschetz theorem and Poincaré duality [Kav11]:

(HL) For every nonnegative integer $k \leq \frac{d}{2}$, the linear map

$$L^k(X) \longrightarrow L^{d-k}(X), \qquad x \longmapsto \omega^{d-2k}\, x$$

   is an isomorphism.

(PD) For every nonnegative integer $k \leq \frac{d}{2}$, the bilinear map

$$L^k(X) \times L^{d-k}(X) \longrightarrow L^d(X) \simeq \mathbb{Q}, \qquad (x_1, x_2) \longmapsto x_1 x_2$$

   is nondegenerate.

We show in Proposition 2 that (HL) and (PD) for $L^*(X)$ are equivalent to each other, and to the numerical condition that

$$\dim L^k(X) = \dim L^{d-k}(X) \ \text{ for all } k.$$

Many familiar smooth projective varieties satisfy (HL) and (PD) for $L^*(X)$. In Section 2, we show that this is the case for

(1) toric varieties,

(2) abelian varieties,

(3) Grassmannians and full flag varieties,

(4) wonderful compactifications of arrangement complements,

(5) products of two or more varieties listed above,

(6) complete intersections of ample divisors in the varieties listed above,

(7) all smooth projective varieties with Picard number 1, and

(8) all smooth projective varieties of dimension at most 4.

In Section 3, we construct three varieties whose Lefschetz algebras do not satisfy (HL) and (PD).

Theorem 1. There is a d-dimensional smooth complex projective variety X with the property

$$\dim L^2(X) \neq \dim L^{d-2}(X).$$

The first example has dimension 5 and Picard number 2, and the second example has dimension 6 and Picard number 3. The third example is a partial flag variety; it shows that the assumption made in [Kav11, Theorem 4.1] is not redundant.

2. Lefschetz algebras with Poincaré duality

Let $X \subseteq \mathbb{P}^n$ be a d-dimensional smooth complex projective variety, and let $\omega \in L^1(X)$ be the cohomology class of a hyperplane section.

Proposition 2. The following statements are equivalent.

(1) The Lefschetz algebra $L^*(X)$ satisfies the hard Lefschetz theorem, that is, the linear map

$$L^k(X) \longrightarrow L^{d-k}(X), \qquad x \longmapsto \omega^{d-2k} x$$

is an isomorphism for every nonnegative integer $k \leq \frac{d}{2}$.

(2) The Lefschetz algebra $L^*(X)$ satisfies Poincaré duality, that is, the bilinear map

$$L^k(X) \times L^{d-k}(X) \longrightarrow L^d(X) \simeq \mathbb{Q}, \qquad (x_1, x_2) \longmapsto x_1 x_2$$

is nondegenerate for every nonnegative integer $k \leq \frac{d}{2}$.

(3) The Lefschetz algebra $L^*(X)$ has symmetric dimensions, that is, the equality

$$\dim L^k(X) = \dim L^{d-k}(X)$$

holds for every nonnegative integer $k \leq \frac{d}{2}$.

Proof. Clearly, (1) implies (3), and (2) implies (3). The hard Lefschetz theorem for $H^*(X, \mathbb{Q})$ shows that (3) implies (1). We prove that (3) implies (2).

Suppose (3), or equivalently, (1). This implies that, for every nonnegative integer $k \leq \frac{d}{2}$,

$$L^k(X) = \bigoplus_{i=0}^{k} \omega^{k-i} PL^i(X), \ \text{ where } \ PL^i(X) = \ker\Big(\omega^{d-2i+1} : L^i(X) \longrightarrow L^{d-i+1}(X)\Big).$$

In other words, every primitive component appearing in the Lefschetz decomposition of an element $x \in L^k(X)$ is an element of $L^*(X)$. Let us write

$$x = \sum_{i=0}^{k} \omega^{k-i} x_i, \quad x_i \in L^i(X).$$

If x is nonzero, then some summand $\omega^{k-j} x_j$ is nonzero. The Hodge-Riemann relation for the primitive subspace of $H^{j,j}(X)$ shows that

$$(-1)^j \int_X \omega^{d-j-k} x_j x = (-1)^j \int_X \omega^{d-2j} x_j^2 > 0.$$

Thus the product of x with $\omega^{d-j-k} x_j$ is nonzero, and hence $L^*(X)$ satisfies Poincaré duality. $\square$

Many familiar smooth projective varieties have Lefschetz algebras satisfying Poincaré duality. The most obvious examples are the varieties with $L^*(X) = H^{2*}(X, \mathbb{Q})$, such as smooth projective toric varieties, complete flag varieties, wonderful compactifications of hyperplane arrangement complements, etc. We collect more examples in the remainder of this section.

Lemma 3. For every nonnegative integer $k \leq \frac{d}{2}$, the linear map

$$L^k(X) \longrightarrow L^{d-k}(X), \qquad x \longmapsto \omega^{d-2k} x$$

is injective. For $k = 0$ and $k = 1$, the map is bijective.

Proof. We prove the assertion for $k = 1$. By the Lefschetz $(1, 1)$ theorem, we have

$$L^1(X) = H^2(X, \mathbb{Q}) \cap H^{1,1}(X).$$

The hard Lefschetz theorem for $H^*(X, \mathbb{Q})$ implies that the left-hand side is isomorphic to

$$H^{2d-2}(X, \mathbb{Q}) \cap H^{d-1,d-1}(X),$$

which contains $L^{d-1}(X)$ as a subspace. This forces $\dim L^1(X) = \dim L^{d-1}(X)$. $\qquad\square$

Proposition 4. Suppose any one of the following conditions:

(1) X is an abelian variety.

(2) X has Picard number 1.

(3) X has dimension at most 4.

Then $L^*(X)$ satisfies Poincaré duality.

Proof. (1) is proved by Milne [Mil99, Proposition 5.2]. (2) and (3) follow from Proposition 2 and Lemma 3. $\qquad\square$

We may construct Lefschetz algebras with Poincaré duality by taking hyperplane sections. Let ι be the inclusion of a smooth ample hypersurface $D \subseteq X$ with cohomology class $\omega \in L^1(X)$.

Proposition 5. If $L^*(X)$ satisfies Poincaré duality, then $L^*(D)$ satisfies Poincaré duality.

Proof. The Lefschetz hyperplane theorem for $D \subseteq X$ and Poincaré duality for D show that the pullback ι^* in cohomology induces a commutative diagram

$$
\begin{array}{ccc}
L^*(X) & \xrightarrow{\quad\iota^*\quad} & L^*(D). \\
& \searrow \qquad \nearrow_{\bar{\iota}^*} & \\
& L^*(X)/\mathrm{ann}(\omega) &
\end{array}
$$

If $L^*(X)$ satisfies Poincaré duality, then $L^*(X)/\mathrm{ann}(\omega)$ satisfies Poincaré duality:

$$L^k(X)/\mathrm{ann}(\omega) \times L^{d-k-1}(X)/\mathrm{ann}(\omega) \longrightarrow L^{d-1}(X)/\mathrm{ann}(\omega) \simeq L^d(X) \simeq \mathbb{Q} \text{ is nondegenerate.}$$

When $d < 4$, the conclusion follows from Proposition 4 applied to D. Assuming $d \geq 4$, we show that the induced map $\bar{\iota}^*$ is an isomorphism.

When $d \geq 4$, the Lefschetz hyperplane theorem for Picard groups applies to the inclusion $D \subseteq X$, and therefore $\bar{\iota}^*$ is surjective [Laz04, Chapter 3]. To conclude, we deduce from the hard Lefschetz theorem for $H^*(X, \mathbb{Q})$ that

$$L^k(X) \simeq L^k(X)/\mathrm{ann}(\omega) \simeq L^k(D) \text{ for every nonnegative integer } k < d/2.$$

Lemma 3 applied to D shows that $\bar{\iota}^*$ is an injective in the remaining degrees $k \geq \frac{d}{2}$. $\qquad\square$

We may construct Lefschetz algebras with Poincaré duality by taking products. Let X_1 and X_2 be smooth projective varieties, and suppose that $H^1(X_1, \mathbb{Q}) = 0$.

Proposition 6. There is an isomorphism between graded algebras

$$L^*(X_1 \times X_2) \simeq L^*(X_1) \otimes_{\mathbb{Q}} L^*(X_2).$$

Thus $L^*(X_1 \times X_2)$ satisfies Poincaré duality if $L^*(X_1)$ and $L^*(X_2)$ satisfy Poincaré duality.

Proof. By the Künneth formula, there is an isomorphism of Hodge structures

$$H^2(X_1 \times X_2, \mathbb{Q}) \simeq H^2(X_1, \mathbb{Q}) \oplus H^2(X_2, \mathbb{Q}).$$

The above restricts to an isomorphism between subspaces

$$L^1(X_1 \times X_2) \simeq L^1(X_1) \oplus L^1(X_2),$$

which induces an isomorphism of graded $\mathbb{Q}$-algebras $L^*(X_1 \times X_2) \simeq L^*(X) \otimes_{\mathbb{Q}} L^*(X_2)$. $\qquad\square$

Proposition 4, Proposition 5, and Proposition 6 justify that the eight classes of smooth projective varieties listed in the introduction have Lefschetz algebras with Poincaré duality.

3. Lefschetz algebras without Poincaré duality

We construct three smooth projective varieties whose Lefschetz algebras do not satisfy Poincaré duality. Before giving the construction, we recall standard description of the cohomology ring of a blowup.

Let Z be a codimension r smooth subvariety of a d-dimensional smooth projective variety Y. We write $\pi : X \to Y$ for the blowup of Y along Z, and $\iota : Z \to Y$ for the inclusion of Z in Y:

$$X \xrightarrow{\pi} Y \xleftarrow{\iota} Z.$$

The cohomology ring of the blowup X can be described as follows [GH78, Chapter 4].

Proposition 7. There is a decomposition of graded vector spaces

$$(1) \qquad H^*(X, \mathbb{Q}) \simeq \pi^* H^*(Y, \mathbb{Q}) \oplus \left(\sum_{i=1}^{r-1} H^{*-2i}(Z, \mathbb{Q}) \otimes \mathbb{Q} e^i \right),$$

where e is the cohomology class of the exceptional divisor in X. The cup product satisfies

$$\pi^* y \cup (z \otimes e^i) = (\iota^* y \cup z) \otimes e^i \text{ for cohomology classes } y \text{ of } Y \text{ and } z \text{ of } Z,$$

and, writing $N_{Z/Y}$ for the normal bundle of the embedding $\iota : Z \to Y$, we have

$$(-1)^r e^r = \pi^* \iota_*(1) - \sum_{i=1}^{r-1} c_{r-i}(N_{Z/Y}) \otimes (-e)^i.$$

We abuse notation and suppress the symbols ι^* and π^* in computations below.

3.1. **A 5-dimensional example.** According to Proposition 4, if X is a smooth projective variety whose Lefschetz algebra does not satisfy Poincaré duality, then the dimension of X is at least 5 and the Picard number of X is at least 2. We construct such an example of dimension 5 and Picard number 2.

Let $C \subseteq \mathbb{P}^2$ be a smooth plane cubic curve, let Y be the projective space $\mathbb{P}^5$, and let Z be the product $C \times \mathbb{P}^1$. We write ι for the composition of inclusions

$$Z = C \times \mathbb{P}^1 \longleftrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \longleftrightarrow \mathbb{P}^5 = Y,$$

where the second map is the Segre embedding. Let X be the blowup of Y along Z.

Proposition 8. $\dim L^2(X) = 3$ and $\dim L^3(X) = 4$.

Proof. We have $H^2(Z, \mathbb{Q}) = \mathbb{Q}\, a \oplus \mathbb{Q}\, b$, where a and b are cohomology classes of $C \times$ point and point $\times \mathbb{P}^1$ respectively. Writing c for the cohomology class of a hyperplane in Y, we have

$$
\begin{aligned}
H^0(X, \mathbb{Q}) &= \mathbb{Q}\, 1, \\
H^2(X, \mathbb{Q}) &= \mathbb{Q}\, c^1 \oplus (\mathbb{Q}\, 1)e, \\
H^4(X, \mathbb{Q}) &= \mathbb{Q}\, c^2 \oplus (\mathbb{Q}\, a \oplus \mathbb{Q}\, b)e \oplus (\mathbb{Q}\, 1)e^2, \\
H^6(X, \mathbb{Q}) &= \mathbb{Q}\, c^3 \oplus (\mathbb{Q}\, ab)e \oplus (\mathbb{Q}\, a \oplus \mathbb{Q}\, b)e^2, \\
H^8(X, \mathbb{Q}) &= \mathbb{Q}\, c^4 \oplus (\mathbb{Q}\, ab)e^2, \\
H^{10}(X, \mathbb{Q}) &= \mathbb{Q}\, c^5.
\end{aligned}
$$

where e is the cohomology class of the exceptional divisor in X.

The algebra $L^*(X)$ is generated by c and e. The restriction of c to Z is $a + 3b$, and hence

$$L^2(X) = \mathbb{Q}\, c^2 \oplus \mathbb{Q}\, ce \oplus \mathbb{Q}\, e^2 = \mathbb{Q}\, c^2 \oplus \mathbb{Q}(a + 3b)e \oplus \mathbb{Q}\, e^2.$$

This proves the first assertion.

We next show $L^3(X) = H^6(X, \mathbb{Q})$. It is enough to check that e^3 is not in the subspace

$$V = \mathbb{Q}c^3 \oplus \mathbb{Q}c^2 e \oplus \mathbb{Q}ce^2 = \mathbb{Q}\, c^3 \oplus \mathbb{Q}(ab)e \oplus \mathbb{Q}(a + 3b)e^2 \subseteq H^6(X, \mathbb{Q}).$$

According to Proposition 7, the following relation holds in the cohomology of X:

$$e^3 = -6c^3 - c_2(N_{Z/Y})\, e + c_1(N_{Z/Y})\, e^2 = -6c^3 - c_2(N_{Z/Y})\, e + c_1(T_Y)\, e^2 - c_1(T_Z)\, e^2.$$

Since $c_1(T_Y)\, e^2$ is a multiple of ce^2 and $c_2(N_{Z/Y})\, e$ is a multiple of abe, we have

$$e^3 = -c_1(T_Z)\, e^2 \mod V$$

The tangent bundle of the elliptic curve C is trivial, and therefore $c_1(T_Z)\, e^2$ must be a multiple of ae^2. It follows that e^3 is not contained in V. This proves the second assertion. $\qquad \square$

3.2. **A 6-dimensional example.** A simpler example can be found in dimension 6. Let $Y = \mathbb{P}^3 \times \mathbb{P}^3$, and let $Z = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We write ι for the composition of inclusions

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\;\mathrm{id} \times \delta \times \mathrm{id}\;} \mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{P}^1 = (\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1) \xrightarrow{\;s \times s\;} \mathbb{P}^3 \times \mathbb{P}^3,$$

where id is the identity map, δ is the diagonal embedding, and s is the Segre embedding. Let X be the blowup of Y along Z.

Proposition 9. dim $L^2(X) = 6$ and dim $L^4(X) = 7$.

Proof. Write y_1, y_2 for the cohomology classes

$$\mathrm{cl}(\mathbb{P}^2 \times \mathbb{P}^3), \; \mathrm{cl}(\mathbb{P}^3 \times \mathbb{P}^2) \in H^2(\mathbb{P}^3 \times \mathbb{P}^3, \mathbb{Q}),$$

and write z_1, z_2, z_3 for the cohomology classes

$$\mathrm{cl}(\mathbb{P}^0 \times \mathbb{P}^1 \times \mathbb{P}^1), \; \mathrm{cl}(\mathbb{P}^1 \times \mathbb{P}^0 \times \mathbb{P}^1), \; \mathrm{cl}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^0) \in H^2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q}).$$

Note that $\iota^* y_1 = z_1 + z_2$ and $\iota^* y_2 = z_2 + z_3$. According to Proposition 7,

$$H^4(X, \mathbb{Q}) = \mathbb{Q}\, y_1^2 \oplus \mathbb{Q}\, y_1 y_2 \oplus \mathbb{Q}\, y_2^2 \oplus \big(\mathbb{Q}\, z_1 \oplus \mathbb{Q}\, z_2 \oplus \mathbb{Q}\, z_3\big)e \oplus \big(\mathbb{Q}\, 1\big)e^2 \text{ and}$$

$$H^8(X, \mathbb{Q}) = \mathbb{Q}\, y_1^3 y_2 \oplus \mathbb{Q}\, y_1^2 y_2^2 \oplus \mathbb{Q}\, y_1 y_2^3 \oplus \big(\mathbb{Q}\, z_1 z_2 z_3\big)e \oplus \big(\mathbb{Q}\, z_2 z_3 \oplus \mathbb{Q}\, z_1 z_3 \oplus \mathbb{Q}\, z_1 z_2\big)e^2,$$

where e is the cohomology of the exceptional divisor in X.

The vector space $L^2(X)$ is spanned by the cohomology classes y_1^2, $y_1 y_2$, y_2^2, e^2,

$$y_1 e = z_1 e + z_2 e, \quad \text{and} \quad y_2 e = z_2 e + z_3 e.$$

From the above description of $H^4(X, \mathbb{Q})$, we see that the six elements are linearly independent. This proves the first assertion.

We next check $L^4(X) = H^8(X, \mathbb{Q})$. Note that $L^4(X)$ contains $y_1^3 y_2$, $y_1^2 y_2^2$, $y_1 y_2^3$, and

$$y_1^2 y_2 e = 2 z_1 z_2 z_3 e, \quad y_1^2 e^2 = 2 z_1 z_2 e^2, \quad y_2^2 e^2 = 2 z_2 z_3 e^2, \quad y_1 y_2 e^2 = (z_1 z_2 + z_1 z_3 + z_2 z_3)e^2.$$

From the above description of $H^8(X, \mathbb{Q})$, we see that the seven elements span $H^8(X, \mathbb{Q})$. This proves the second assertion. $\qquad\square$

3.3. **An 8-dimensional example.** Let X be the 8-dimensional partial flag variety

$$X = \left\{ 0 \subseteq V_2 \subseteq V_3 \subseteq \mathbb{C}^5 \mid \dim V_2 = 2, \; \dim V_3 = 3 \right\}.$$

We show that the Lefschetz algebra of X does not satisfy Poincaré duality. This is in contrast with the case of Grassmannians and full flag varieties.

Proposition 10. dim $L^2(X) = 3$ and dim $L^6(X) = 4$.

Proof. We use standard facts and notations in Schubert calculus [EH16]. Let Y be the Grassmannian variety parametrizing 2-dimensional subspaces of $\mathbb{C}^5$. The Schubert classes form a basis of the cohomology of Y:

$$H^*(Y,\mathbb{Q}) = \mathbb{Q}\,\ydiagram{3,3} \oplus \mathbb{Q}\,\ydiagram{3,2} \oplus \mathbb{Q}\,\ydiagram{3,1} \oplus \mathbb{Q}\,\ydiagram{2,2} \oplus \mathbb{Q}\,\ydiagram{3} \oplus \mathbb{Q}\,\ydiagram{2,1} \oplus \mathbb{Q}\,\ydiagram{1,1} \oplus \mathbb{Q}\,\ydiagram{2} \oplus \mathbb{Q}\,\ydiagram{1} \oplus \mathbb{Q}\,1.$$

We write $\mathcal{Q}$ for the universal quotient bundle on Y. Since X is the projectivization $\mathbb{P}\mathcal{Q}$,

$$H^4(X,\mathbb{Q}) = \mathbb{Q}\,\ydiagram{1,1} \oplus \mathbb{Q}\,\ydiagram{2} \oplus \mathbb{Q}\,\square\,\zeta \oplus \mathbb{Q}\,\zeta^2 \text{ and}$$

$$H^{12}(X,\mathbb{Q}) = \mathbb{Q}\,\ydiagram{3,3} \oplus \mathbb{Q}\,\ydiagram{3,2}\,\zeta \oplus \mathbb{Q}\,\ydiagram{3,1}\,\zeta^2 \oplus \mathbb{Q}\,\ydiagram{2,2}\,\zeta^2,$$

where ζ is the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{Q}}(1)$.

The Lefschetz algebra of X is generated by $\square$ and ζ, and therefore

$$L^2(X) = \mathbb{Q}\,\square^2 \oplus \mathbb{Q}\,\square\,\zeta \oplus \mathbb{Q}\,\zeta^2 = \mathbb{Q}\left(\ydiagram{2}+\ydiagram{1,1}\right) \oplus \mathbb{Q}\,\square\,\zeta \oplus \mathbb{Q}\,\zeta^2.$$

This proves the first assertion.

We now show that $L^6(X) = H^{12}(X,\mathbb{Q})$. For this we use four elements

$$\square^6,\ \square^5\,\zeta,\ \square^4\,\zeta^2,\ \square^2\,\zeta^4 \in L^6(X).$$

It is enough to prove that the last element $\square^2\,\zeta^4$ is not contained in the subspace

$$V = \mathbb{Q}\,\square^6 \oplus \mathbb{Q}\,\square^5\,\zeta \oplus \mathbb{Q}\,\square^4\,\zeta^2 = \mathbb{Q}\,\ydiagram{3,3} \oplus \mathbb{Q}\,\ydiagram{3,2}\,\zeta \oplus \mathbb{Q}\left(2\,\ydiagram{2,2} + 3\,\ydiagram{3,1}\right)\zeta^2.$$

According to [EH16, Chapter 5], the Chern classes of $\mathcal{Q}$ are

$$c_0(\mathcal{Q}) = 1,\ \ c_1(\mathcal{Q}) = \square,\ \ c_2(\mathcal{Q}) = \ydiagram{2},\ \ c_3(\mathcal{Q}) = \ydiagram{3}\ \in\ H^*(Y,\mathbb{Q}).$$

In other words, $\zeta^3 + \square\,\zeta^2 + \ydiagram{2}\,\zeta + \ydiagram{3} = 0$ in the cohomology of $X = \mathbb{P}\mathcal{Q}$. It follows that

$$\zeta^4 = -\ydiagram{3}\,\zeta - \ydiagram{2}\,\zeta^2 - \square\,\zeta^3$$
$$= -\ydiagram{3}\,\zeta - \ydiagram{2}\,\zeta^2 - \square\left(-\ydiagram{3} - \ydiagram{2}\,\zeta - \square\,\zeta^2\right) = \ydiagram{3,1} + \ydiagram{2,1}\,\zeta + \ydiagram{1,1}\,\zeta^2,$$

and therefore

$$\square^2\,\zeta^4 = \square^2\left(\ydiagram{3,1} + \ydiagram{2,1}\,\zeta + \ydiagram{1,1}\,\zeta^2\right) = \ydiagram{3,3} + 2\,\ydiagram{3,2}\,\zeta + \left(\ydiagram{2,2} + \ydiagram{3,1}\right)\zeta^2 \notin V.$$

This proves the second assertion. $\hfill\square$

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