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Correlation bounds for fields and matroids

Abstract. Let G be a finite connected graph, and let T be a spanning tree of G chosen uniformly at random. The work of Kirchhoff on electrical networks can be used to show that the events $e_1 \in$ T and $e_2 \in$ T are negatively correlated for any distinct edges e_1 and e_2. What can be said for such events when the underlying matroid is not necessarily graphic? We use Hodge theory for matroids to bound the correlation between the events $e \in$ B, where B is a randomly chosen basis of a matroid. As an application, we prove Mason's conjecture that the number of k-element independent sets of a matroid forms an ultra-log-concave sequence in k.

Keywords. Matroid, correlation, Hodge theory

1. Introduction and results

Let G be a finite connected graph, and let $w = (w_e)$ be a set of positive weights on the edges e of G. Randomly pick a spanning tree T of G so that the probability of selecting an individual tree t is proportional to the product of the weights of its edges:

$$\mathbb{P}(\mathrm{T} = t) \propto \prod_{e \in t} w_e.$$

The work of Kirchhoff on electrical networks can be used to show that, for any distinct edges i and j, the events $i \in$ T and $j \in$ T are negatively correlated:

$$\mathbb{P}(\mathrm{T \ contains} \ i \mid \mathrm{T \ contains} \ j) \leq \mathbb{P}(\mathrm{T \ contains} \ i).$$

Equivalently, for any distinct edges i and j, we have

$$\mathbb{P}(i \in \mathrm{T}, j \in \mathrm{T}) \, \mathbb{P}(i \notin \mathrm{T}, j \notin \mathrm{T}) \leq \mathbb{P}(i \in \mathrm{T}, j \notin \mathrm{T}) \, \mathbb{P}(i \notin \mathrm{T}, j \in \mathrm{T}).$$

We refer to [24] and [20, Chapter 4] for modern expositions.

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Let E be a finite set. A *matroid* on E is a nonempty collection of subsets of E, called *bases* of the matroid, that satisfies the exchange property:

- For any bases b_1, b_2 and $e_1 \in b_1 \setminus b_2$, there is $e_2 \in b_2 \setminus b_1$ such that $(b_1 \setminus e_1) \cup e_2$ is a basis.

An *independent set* is a subset of a basis, a *dependent set* is a subset of E that is not independent, a *circuit* is a minimal dependent set, the *rank* of a subset of E is the cardinality of any one of its maximal independent subsets, and a *flat* is a subset of E that is maximal for its rank. The *rank* of a matroid is the cardinality of any one of its bases. For any unexplained matroid terms and facts, we refer to Oxley's book [23]. The collection of spanning trees of a connected graph is the best-known example of a matroid.

Let M be a matroid on E, and fix a set $w = (w_e)$ of positive weights on the elements e of E. Randomly pick a basis B of the matroid so that the probability of selecting an individual basis b is proportional to the product of the weights of its elements:

$$\mathbb{P}(\mathrm{B} = b) \propto \prod_{e \in b} w_e.$$

In this more general setup, for any distinct i and j in E, do we still have the negative correlation

$$\mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) \le \mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B})?$$

The answer is "yes" if the matroid is regular [7], if the matroid is representable over $\mathbb{F}_3$ and $\mathbb{F}_4$ [3], if the cardinality of E is at most 7, or if the rank of M is at most 3 [28]. Examples below show that distinct elements of E can define positively correlated events for more general matroids.

Example 1. Let S be the 2-dimensional skeleton of the 5-dimensional simplex. A spanning tree of S is a maximal subset of the twenty triangles in S that does not contain any 2-cycle over $\mathbb{F}_2$. Choose one such B uniformly at random. Then, for any two disjoint triangles in S, say $i = 123$ and $j = 456$ in Figure 1A, we have

$$\mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) = \frac{11664}{46608} \cdot \frac{11664}{46608} \simeq 0.06263,$$

$$\mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B}) = \frac{11640}{46608} \cdot \frac{11640}{46608} \simeq 0.06237.$$

This example was found by Andrew Newman.

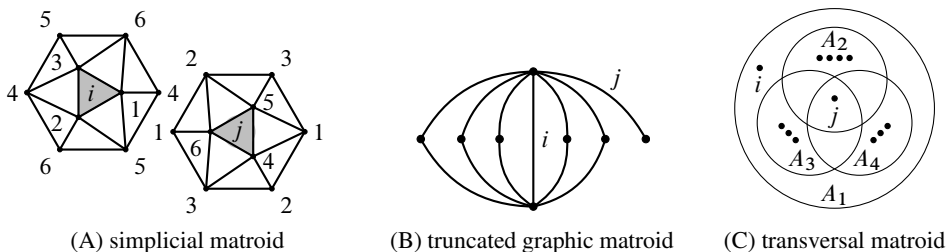(A) simplicial matroid   (B) truncated graphic matroid   (C) transversal matroid

Fig. 1. Positive correlation in matroids.

Example 2. Let G be the graph in Figure 1B. Consider the collection of all forests in G with exactly six edges, and choose one such B uniformly at random. Then, for the edges labelled i and j in Figure 1B, we have

$$\mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B}) \, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) = \frac{80}{384} \cdot \frac{80}{384} \simeq 0.04340,$$

$$\mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B}) \, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B}) = \frac{32}{384} \cdot \frac{192}{384} \simeq 0.04167.$$

This example, attributed to Paul Seymour, Peter Winkler, and Madhu Sudan, is discussed in [7, Section 2].

Example 3. Let A_1, A_2, A_3, A_4 be the four finite sets shown in Figure 1C. A system of distinct representatives is a set $\{x_1, x_2, x_3, x_4\}$ of size four such that $x_k \in A_k$ for all k. Choose one such B uniformly at random. Then, for the elements labelled i and j in Figure 1C, we have

$$\mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B}) \, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) = \frac{33}{309} \cdot \frac{126}{309} \simeq 0.04355,$$

$$\mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B}) \, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B}) = \frac{36}{309} \cdot \frac{114}{309} \simeq 0.04298.$$

This example is from [4, Section 5].[1]

Example 4. Let i and j be distinct elements of a 24-element set E, and let $\mathbb{V}$ be the set of blocks of the Steiner system $S(5, 8, 24)$ that contain exactly one of i and j. Consider the collection of all 6-element subsets of E not contained in any member of $\mathbb{V}$. If we choose one such B uniformly at random, we have

$$\mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B}) \, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) = \frac{7315}{124740} \cdot \frac{72149}{124740} \simeq 0.03391,$$

$$\mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B}) \, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B}) = \frac{22638}{124740} \cdot \frac{22638}{124740} \simeq 0.03293.$$

This example, due to Mark Jerrum, shows that a paving matroid need not have the negatively correlation property [13, Section 4].

In Section 3, we use the Hodge theory for matroids in [1, 12] to bound the correlation between the events $e \in \mathrm{B}$.

Theorem 5. *For any distinct elements* i *and* j *in a matroid* M *of positive rank* d,

$$\mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B}) \, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) \leq 2\left(1 - \frac{1}{d}\right) \mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B}) \, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B}).$$

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[1]The proof of [4, Proposition 5.9] needs a small correction. In the notation of that paper, the numbers should be $L_e = 69$, $L_f = 147$, $L_{ef} = 33$, $L^{ef} = 309$.

Theorem 5 implies the covariance bound

$$\mathrm{Cov}(\mathrm{B}\ \text{contains}\ i, \mathrm{B}\ \text{contains}\ j) < \mathbb{P}(\mathrm{B}\ \text{contains}\ i)\, \mathbb{P}(\mathrm{B}\ \text{contains}\ j).$$

Compare the notion of *approximate independence* in [16, Section 4].

An element e of a rank d matroid M is a *loop* if it is contained in no basis of M, a *coloop* if it is contained in every basis of M, and *free* if it is not a coloop and every circuit of M containing e has cardinality $d + 1$. For example, the elements labelled j in matroids of Examples 2 and 3 are free. In Section 4, we remove the factor 2 in Theorem 5 when both i and j are free.

Theorem 6. *For any distinct free elements i and j in a matroid* M *of positive rank d,*

$$\mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) \le \left(1 - \frac{1}{d}\right) \mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B}).$$

Can we replace the constant 2 in Theorem 5 by a smaller number? To any matroid M, we associate a nonnegative real number $\alpha(\mathrm{M})$ defined by

$$\alpha(\mathrm{M}) = \sup\{\mathbb{P}(i \in \mathrm{B}, j \in \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \notin \mathrm{B}) / \mathbb{P}(i \in \mathrm{B}, j \notin \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B}, j \in \mathrm{B})\},$$

where the supremum is over all distinct non-loop non-coloop elements i and j in M and all sets of positive weights w on the elements of M. When every element of M is either a loop or a coloop, we set $\alpha(\mathrm{M}) = 0$. It is straightforward to check that if $\mathrm{M}^{\perp}$ is the dual matroid of M and M is a minor of another matroid N, then

$$\alpha(\mathrm{M}) = \alpha(\mathrm{M}^{\perp}) \quad \text{and} \quad \alpha(\mathrm{M}) \le \alpha(\mathrm{N}).$$

In addition, if M_1 and M_2 have an element that is neither a loop nor a coloop, then

$$\alpha(\mathrm{M}_1 \oplus \mathrm{M}_2) = \max\{\alpha(\mathrm{M}_1), \alpha(\mathrm{M}_2), 1\}.$$

We define the *correlation constant* $\alpha_{\mathbb{F}}$ of a field $\mathbb{F}$ to be the real number

$$\alpha_{\mathbb{F}} = \sup\{\alpha(\mathrm{M})\},$$

where the supremum is over all matroids M representable over $\mathbb{F}$. The *correlation constant of matroids*, denoted α_{Mat}, is defined in the same way by taking the supremum over all matroids. As we can place any number of new elements in parallel to existing elements in any matroid, the values of $\alpha_{\mathbb{F}}$ and α_{Mat} remain unchanged if we only consider matroids with constant weights.

In Section 5, we construct explicit examples to produce a lower bound of $\alpha_{\mathbb{F}}$ for any field $\mathbb{F}$.

Theorem 7. *The correlation constant of any field* $\mathbb{F}$ *satisfies* $8/7 \le \alpha_{\mathbb{F}} \le \alpha_{\mathrm{Mat}} \le 2$.

What is the correlation constant of $\mathbb{F}_2$? What is the correlation constant of $\mathbb{C}$? Does $\alpha_{\mathbb{F}}$ depend on $\mathbb{F}$? What is the correlation constant α_{Mat}? The first question may be the most tractable one, as the only minor-minimal binary matroid with $\alpha(\mathrm{M})$ larger than 1 is the matroid represented over $\mathbb{F}_2$ by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

This matroid, labelled S_8 in Oxley's list [23, Appendix], was first found by Seymour and Welsh to have positively correlated pair of elements [27]. See [4] for a proof of the assertion on S_8. We conjecture, although without much evidence, that the correlation constant of $\mathbb{F}_2$ is $8/7$. We know no matroid M with $\alpha(\mathrm{M})$ larger than $8/7$.

The initial motivation for our paper comes from the work of Mason [22], who offered the following three conjectures of increasing strength. Several other authors studied correlations in matroid theory partly in pursuit of these conjectures [2, 17, 18, 27, 29].

Conjecture 8. For any n-element matroid N and any positive integer k,

(1) $I_k(\mathrm{N})^2 \geq I_{k-1}(\mathrm{N}) I_{k+1}(\mathrm{N})$,

(2) $I_k(\mathrm{N})^2 \geq \frac{k+1}{k} I_{k-1}(\mathrm{N}) I_{k+1}(\mathrm{N})$,

(3) $I_k(\mathrm{N})^2 \geq \frac{k+1}{k} \frac{n-k+1}{n-k} I_{k-1}(\mathrm{N}) I_{k+1}(\mathrm{N})$,

where $I_k(\mathrm{N})$ is the number of k-element independent sets of N.

Conjecture 8(1) was proved in [1]. Conjecture 8(3) is known to hold when n is at most 11 or k is at most 5 [18]. We refer to [6, 9, 11, 19, 21, 25, 30] for other partial results on Conjecture 8.

Conjecture 8(2) follows from the special case of Theorem 6 when the weight w is constant.

Corollary 9. *Conjecture* 8(2) *holds.*

The implication is based on two standard constructions [23, Chapter 7]. First, we use the truncation of N to reduce Conjecture 8(2) to the case $k = d - 1$, where d is the rank of N. Next, we construct the free extension M of N by adding two new free elements i and j. If we pick a basis B of M uniformly at random, then

$$\mathbb{P}(i \in \mathrm{B},\, j \in \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B},\, j \notin \mathrm{B}) = \frac{I_{d-2}(\mathrm{N}) \cdot I_d(\mathrm{N})}{(I_{d-2}(\mathrm{N}) + 2 I_{d-1}(\mathrm{N}) + I_d(\mathrm{N}))^2},$$

$$\mathbb{P}(i \in \mathrm{B},\, j \notin \mathrm{B})\, \mathbb{P}(i \notin \mathrm{B},\, j \in \mathrm{B}) = \frac{I_{d-1}(\mathrm{N}) \cdot I_{d-1}(\mathrm{N})}{(I_{d-2}(\mathrm{N}) + 2 I_{d-1}(\mathrm{N}) + I_d(\mathrm{N}))^2}.$$

Now Conjecture 8(2) for N is Theorem 6 for i and j in M.

Conjecture 8(2) implies an entropy bound that cannot be deduced from Conjecture 8(1). Recall that the *Shannon entropy* $H(\mathrm{X})$ of a discrete random variable X is, by definition,

$$H(\mathrm{X}) = -\sum_k \mathbb{P}(\mathrm{X} = k) \log \mathbb{P}(\mathrm{X} = k),$$

where the logarithm is to base 2 and the sum is over all values of X with nonzero probability. For a rank d matroid M, let I_M be the size of an independent set drawn uniformly at random from the collection of all independent sets of M. For any d, uniform matroids of rank d show that

$$\inf_{\mathrm{rk(M)}=d} H(\mathrm{I}_\mathrm{M}) = 0,$$

where the infimum is over all matroids of rank d. We show that, asymptotically, the entropy of I_M is at most half of the obvious upper bound $\log d$ given by Jensen's inequality.

Corollary 10. *Uniform random independent sets of matroids satisfy*

$$\lim_{d\to\infty}\left(\sup_{\mathrm{rk(M)}=d} H(\mathrm{I}_\mathrm{M})/\log d\right) = \frac{1}{2},$$

where the supremum is over all matroids of rank d.

Corollary 10 is based on a result of Johnson [14, Theorem 2.5], who showed that the Poisson distribution maximizes entropy in the class of ultra log-concave distributions. Recall that a random variable X taking its values in $\mathbb{N}$ is said to have the *Poisson distribution with parameter λ* if

$$\mathbb{P}(\mathrm{X} = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{for all } k \in \mathbb{N}.$$

Combined with Conjecture 8(2), Johnson's result implies that

$$H(\mathrm{I}_\mathrm{M}) \le H(\mathrm{P}(\lambda)),$$

where $\mathrm{P}(\lambda)$ is the Poisson distribution with parameter $\lambda = \mathbb{E}(\mathrm{I}_\mathrm{M})$. Using known bounds for the entropy of Poisson distributions from information theory [5, Theorem 8.6.5], we get

$$H(\mathrm{I}_\mathrm{M}) \le \frac{1}{2}\log\left(2\pi e\left(d + \frac{1}{12}\right)\right).$$

In general, an upper bound of the entropy of a random variable X implies a concentration of X [15, Chapter 22]. The above bound of $H(\mathrm{I}_\mathrm{M})$, for example, gives the following.

Corollary 11. *For any matroid* M *of rank d, there is k such that*

$$\mathbb{P}(\mathrm{I}_\mathrm{M} = k) > \frac{1}{5\sqrt{d}}.$$

Clearly, Corollaries 10 and 11 cannot be deduced from Conjecture 8(1) alone.

2. Hodge theory for matroids

We review the results of [12] and [1] that will be used to prove Theorems 5 and 6. For our purposes, we may assume that matroids do not have any loops. In the rest of this paper, we fix a positive integer n and work with loopless matroids on finite sets

$$E = \{1, \ldots, n\} \quad \text{and} \quad \overline{E} = \{0, 1, \ldots, n\}.$$

We say that two subsets S_1 and S_2 of $\overline{E}$ are *incomparable* if $S_1 \nsubseteq S_2$ and $S_2 \nsubseteq S_1$. Our notations will be consistent with those of [12, Section 2].

Let $\overline{M}$ be a loopless matroid on $\overline{E}$ of rank $d + 1$, and let $\overline{\mathcal{L}}$ be the lattice of flats of $\overline{M}$. Introduce variables $x_{\overline{F}}$, one for each nonempty proper flat $\overline{F}$ of $\overline{M}$, and consider the polynomial ring

$$S(\overline{M}) = \mathbb{R}[x_{\overline{F}}]_{\overline{F} \neq \emptyset, \overline{F} \neq \overline{E}, \overline{F} \in \overline{\mathcal{L}}}.$$

The *Chow ring* $A(\overline{M})$ is the quotient of $S(\overline{M})$ by the ideal generated by the linear forms

$$\sum_{e_1 \in \overline{F}} x_{\overline{F}} - \sum_{e_2 \in \overline{F}} x_{\overline{F}},$$

one for each pair of distinct elements e_1 and e_2 of $\overline{E}$, and the quadratic monomials

$$x_{\overline{F}_1} x_{\overline{F}_2},$$

one for each pair of incomparable nonempty proper flats $\overline{F}_1$ and $\overline{F}_2$ of $\overline{M}$. We denote the degree q component of $A(\overline{M})$ by $A^q(\overline{M})$.

Definition 12. A real-valued function c on $2^{\overline{E}}$ is said to be *strictly submodular* if $c_{\emptyset} = 0$, $c_{\overline{E}} = 0$, and, for any pair of incomparable subsets $I_1, I_2 \subseteq \overline{E}$, we have

$$c_{I_1} + c_{I_2} > c_{I_1 \cap I_2} + c_{I_1 \cup I_2}.$$

A strictly submodular function c defines an element $L(c) = \sum_{\overline{F}} c_{\overline{F}} x_{\overline{F}}$ in $A^1(\overline{M})$. We refer to $L(c)$ as the element of $A^1(\overline{M})$ attached to the function c.

Note that strictly submodular functions on $2^{\overline{E}}$ exist. For example, we have the function

$$c_I = (\text{number of elements in } I)(\text{number of elements not in } I).$$

We may now state the hard Lefschetz theorem and the Hodge–Riemann relations for matroids [1, Theorem 1.4]. The function "deg" in Theorem 13 is the isomorphism $A^d(\overline{M}) \simeq \mathbb{R}$ constructed in [1, Section 5.3]. This isomorphism is uniquely determined by its property

$$\deg(x_{\overline{F}_1} x_{\overline{F}_2} \cdots x_{\overline{F}_d}) = 1$$

for any chain of nonempty proper flats $\overline{F}_1 \subsetneq \overline{F}_2 \subsetneq \cdots \subsetneq \overline{F}_d$ in $\overline{M}$.

Theorem 13. *Let* $L \in A^1(\overline{M})$ *be attached to a strictly submodular function on* $2^{\overline{E}}$.

(1) (Hard Lefschetz theorem) *For every nonnegative integer* $q \leq d/2$, *multiplication by* L *defines an isomorphism*

$$A^q(\overline{M}) \to A^{d-q}(\overline{M}), \quad \eta \mapsto L^{d-2q} \eta.$$

(2) (Hodge–Riemann relations) *For every nonnegative integer* $q \leq d/2$, *multiplication by* L *defines a symmetric bilinear form*

$$A^q(\overline{M}) \times A^q(\overline{M}) \to \mathbb{R}, \quad (\eta_1, \eta_2) \mapsto (-1)^q \deg(\eta_1 \eta_2 L^{d-2q}),$$

that is positive definite on the kernel of L^{d-2q+1}.

Theorems 5 and 6, as well as other applications of the Hodge–Riemann relations in combinatorics surveyed in [10], only use the special case $q \leq 1$. It will be interesting to find applications of the Hodge–Riemann relations for $q > 1$.

3. Proof of Theorem 5

Let M be a rank d loopless matroid on E. Let $\overline{M}$ be the matroid on $\overline{E}$ obtained from M by adding 0 as a coloop, the direct sum of M and the rank 1 matroid on $\{0\}$. For every e in E, we define an element

$$y_e = \sum_{0 \in \overline{F}, e \notin \overline{F}} x_{\overline{F}},$$

where the sum is over all flats $\overline{F}$ of $\overline{M}$ that contain 0 and do not contain e. The linear relations in $A(\overline{M})$ show that we may equivalently define y_e by summing over all flats $\overline{F}$ of $\overline{M}$ that contain e and do not contain 0. The quadratic relations in $A(\overline{M})$ show that, for any nonempty proper flat $\overline{F}$ of $\overline{M}$ containing exactly one of e and 0,

$$x_{\overline{F}} \cdot y_e = 0.$$

In what follows, relations of the above kind will be called *xy-relations*. The *xy*-relations imply that, for example, $y_e \cdot y_e$ is zero for any e in E.

Lemma 14. *For any dependent set* J *of* M, *we have*

$$\prod_{e \in J} y_e = 0.$$

Proof. We may suppose that J is a circuit of M. Choose a maximal independent set I of M in J, an element f in I, and an element g in $J \setminus I$. Since $(I \setminus f) \cup g$ is a basis of J, the set of flats of $\overline{M}$ containing $(I \setminus f) \cup 0$ and not containing f is equal to the set of flats of $\overline{M}$ containing $(I \setminus f) \cup 0$ and not containing g. Therefore, by the *xy*-relations,

$$\prod_{e \in I} y_e = y_f \prod_{e \in I \setminus f} y_e = y_g \prod_{e \in I \setminus f} y_e.$$

Since the square of y_g is zero, this gives $\prod_{e \in J} y_e = \prod_{e \in I} y_e \prod_{e \in J \setminus I} y_e = 0$. ∎

Lemma 15. *For any d-element subset B of E, we have*

$$\deg\left(\prod_{e \in B} y_e\right) = \begin{cases} 1 & \text{if } B \text{ is a basis of } M, \\ 0 & \text{if } B \text{ is not a basis of } M. \end{cases}$$

Proof. Without loss of generality, we may suppose that $B = \{1, \ldots, d\}$. We consider the flats

$$\overline{F}_k = \text{the smallest flat of } \overline{M} \text{ containing } 0, 1, \ldots, k-1, \quad \text{for } k = 1, \ldots, d+1.$$

If B is a basis of M, then $\overline{F}_k$ is the only flat of $\overline{M}$ containing $0, 1, \ldots, k-1$, not containing k, and comparable to $\overline{F}_{k+1}$. Thus the xy-relations imply that

$$y_1 \cdots y_{d-2} y_{d-1} y_d = (y_1 \cdots y_{d-2} y_{d-1}) x_{\overline{F}_d}$$
$$= (y_1 \cdots y_{d-2}) x_{\overline{F}_{d-1}} x_{\overline{F}_d} = \cdots = x_{\overline{F}_1} \cdots x_{\overline{F}_{d-2}} x_{\overline{F}_{d-1}} x_{\overline{F}_d}.$$

If B is not a basis of M, then it contains a dependent set of M, and so $\prod_{e \in B} y_e = 0$ by Lemma 14. $\blacksquare$

Lemma 16. *Let e be an element of E, and let $c(e)$ be the real-valued function on $2^{\overline{E}}$ defined by*

$$c(e)_I = \begin{cases} 1 & \text{if } I \text{ contains } 0 \text{ and does not contain } e, \\ 0 & \text{if } I \text{ contains } e \text{ or does not contain } 0. \end{cases}$$

Then $c(e)_\emptyset = 0$, $c(e)_{\overline{E}} = 0$, and, for any subsets I_1, I_2 of $\overline{E}$,

$$c(e)_{I_1} + c(e)_{I_2} \geq c(e)_{I_1 \cap I_2} + c(e)_{I_1 \cup I_2}.$$

The submodular inequality of Lemma 16 is straightforward to check. In fact,

$$c(e)_{I_1} + c(e)_{I_2} - c(e)_{I_1 \cap I_2} - c(e)_{I_1 \cup I_2}$$
$$= \begin{cases} 1 & \text{if } 0 \text{ is in } I_1 \setminus I_2 \text{ and } e \text{ is in } I_2 \setminus I_1, \\ 1 & \text{if } 0 \text{ is in } I_2 \setminus I_1 \text{ and } e \text{ is in } I_1 \setminus I_2, \\ 0 & \text{otherwise.} \end{cases}$$

We are ready to prove Theorem 5. The equality holds in Theorem 5 when $d = 1$. Suppose from now on that $d \geq 2$. Let $w = (w_e)$ be the given set of positive weights on E. For distinct elements i and j in E, define

$$L_{ij} = L_{ij}(w) = \sum_{e \neq i, e \neq j} w_e y_e,$$

where the sum is over all elements of E other than i and j. Lemma 15 shows that

$$\deg(L_{ij}^d) = d! \left(\sum_{B \in \mathcal{B}^{ij}} \prod_{e \in B} w_e\right),$$

where $\mathcal{B}^{ij}$ is the set of bases of M not containing i and not containing j. Similarly,

$$\deg(y_i L_{ij}^{d-1}) = (d-1)!\Big(\sum_{B \in \mathcal{B}_i^j} \prod_{e \in B} w_e \Big),$$

where $\mathcal{B}_i^j$ is the set of bases of M containing i and not containing j, and

$$\deg(y_i y_j L_{ij}^{d-2}) = (d-2)!\Big(\sum_{B \in \mathcal{B}_{ij}} \prod_{e \in B} w_e \Big).$$

where $\mathcal{B}_{ij}$ is the set of bases of M containing i and containing j. Theorem 5 obviously holds if $\mathcal{B}^{ij}$ or $\mathcal{B}_{ij}$ is empty. We suppose from now on that $\mathcal{B}^{ij}$ and $\mathcal{B}_{ij}$ are nonempty.

Let L be any element of $A^1(\overline{M})$ attached to a strictly submodular function on $2^{\overline{E}}$. By Lemma 16, Theorem 13 applies to the element $L_{ij} + \epsilon L$ for any positive real number ϵ. By the Hodge–Riemann relations for $q \le 1$, any matrix representing the symmetric bilinear form

$$A^1(\overline{M}) \times A^1(\overline{M}) \to \mathbb{R}, \quad (\eta_1, \eta_2) \mapsto \deg\big(\eta_1 \eta_2 (L_{ij} + \epsilon L)^{d-2}\big),$$

must have exactly one positive eigenvalue. Thus, by continuity, any matrix representing the symmetric bilinear form

$$A^1(\overline{M}) \times A^1(\overline{M}) \to \mathbb{R}, \quad (a_1, a_2) \mapsto \deg(\eta_1 \eta_2 L_{ij}^{d-2}),$$

has at most one positive eigenvalue. Now consider the symmetric matrix

$$H_{ij} = \begin{bmatrix} 0 & \deg(y_i y_j L_{ij}^{d-2}) & \deg(y_i L_{ij} L_{ij}^{d-2}) \\ \deg(y_i y_j L_{ij}^{d-2}) & 0 & \deg(y_j L_{ij} L_{ij}^{d-2}) \\ \deg(y_i L_{ij} L_{ij}^{d-2}) & \deg(y_j L_{ij} L_{ij}^{d-2}) & \deg(L_{ij} L_{ij} L_{ij}^{d-2}) \end{bmatrix}.$$

Cauchy's eigenvalue interlacing theorem shows that H_{ij} has at most one positive eigenvalue as well. On the other hand, H_{ij} has at least one positive eigenvalue, because its lower-right diagonal entry is positive. A straightforward computation reveals that the determinant of H_{ij} is a positive multiple of

$$2\Big(1 - \frac{1}{d}\Big)\Big(\sum_{B \in \mathcal{B}_i^j} \prod_{e \in B} w_e \Big)\Big(\sum_{B \in \mathcal{B}_j^i} \prod_{e \in B} w_e \Big) - \Big(\sum_{B \in \mathcal{B}^{ij}} \prod_{e \in B} w_e \Big)\Big(\sum_{B \in \mathcal{B}_{ij}} \prod_{e \in B} w_e \Big).$$

The determinant must be nonnegative by the condition on the eigenvalues of H_{ij}, and hence

$$\mathbb{P}(i \in B, j \in B)\, \mathbb{P}(i \notin B, j \notin B) \le 2\Big(1 - \frac{1}{d}\Big)\, \mathbb{P}(i \in B, j \notin B)\, \mathbb{P}(i \notin B, j \in B).$$

This completes the proof of Theorem 5.

4. Proof of Theorem 6

Let i and j be distinct free elements in a rank d matroid Y, and let Z be the deletion of i and j from Y. We prove Theorem 6 for i and j in Y. When $d = 1$, no basis of Y contains both i and j, and equality holds in Theorem 6. Suppose from now on that $d \geq 2$.

Write $\mathcal{B}_{ij}$, $\mathcal{B}_j^i$, $\mathcal{B}_i^j$, $\mathcal{B}^{ij}$ for the sets of bases containing and/or not containing i, j, and $\mathcal{I}_m$ for the collection of m-element independent sets. Since i and j are free, we have natural bijections

$$\mathcal{I}_d(Z) \simeq \mathcal{B}^{ij}(Y), \quad \mathcal{I}_{d-1}(Z) \simeq \mathcal{B}_i^j(Y) \simeq \mathcal{B}_j^i(Y), \quad \mathcal{I}_{d-2}(Z) \simeq \mathcal{B}_{ij}(Y).$$

If the rank of Z is less than d, Theorem 6 clearly holds for Y, as the left hand side of the inequality is zero. If the rank of Z is d, Theorem 6 for Y is equivalent to the following inequality for $Z = M$. The inequality applied to the constant weight w and all the repeated truncations of M implies Corollary 9.

Proposition 17. *For any matroid* M *of rank* $d \geq 2$ *and any set* $w = (w_e)$ *of positive weights,*

$$\left(\sum_{I \in \mathcal{I}_{d-1}} \prod_{e \in I} w_e \right)^2 \geq \frac{d}{d-1} \left(\sum_{I \in \mathcal{I}_{d-2}} \prod_{e \in I} w_e \right) \left(\sum_{I \in \mathcal{I}_d} \prod_{e \in I} w_e \right),$$

where $\mathcal{I}_m = \mathcal{I}_m(M)$ *is the collection of m-element independent sets of* M.

The proof of Proposition 17 is similar to that of Theorem 5. We define an element

$$\alpha = \sum_{0 \in \overline{F}} x_{\overline{F}},$$

where the sum is over all proper flats $\overline{F}$ of $\overline{M}$ containing 0. The linear relations in $A(\overline{M})$ show that we may equivalently define α by summing over all flats $\overline{F}$ of $\overline{M}$ containing e, for any e in E. The main ingredient of the proof is the following extension of Lemma 15.

Lemma 18. *For any m-element subset I of E, we have*

$$deg\left(\alpha^{d-m} \prod_{e \in I} y_e \right) = \begin{cases} 1 & \text{if } I \text{ is independent in } M, \\ 0 & \text{if } I \text{ is dependent in } M. \end{cases}$$

Proof. We use descending induction on m. The case $m = d$ is Lemma 15, and the case of dependent I is Lemma 14. For the induction step, suppose without loss of generality that $\{1, \ldots, d\}$ is a basis of M. It is enough to show that

$$(y_1 \cdots y_{m-1}) y_m \alpha^{d-m} = (y_1 \cdots y_{m-1}) \alpha^{d-m+1}.$$

By the xy-relations, the difference of the right hand side and the left hand side is

$$(y_1 \cdots y_{m-1}) \left(\sum_{\overline{G}} x_{\overline{G}} \right) \alpha^{d-m},$$

where the sum is over all proper flats $\overline{G}$ of $\overline{M}$ containing $0, 1, \ldots, m$. For any such $\overline{G}$, we claim

$$x_{\overline{G}}\, \alpha^{d-m} = 0.$$

To see this, use the linear relations in $A(\overline{M})$ to write

$$x_{\overline{G}}\alpha^{d-m} = x_{\overline{G}}\Big(\sum_{\overline{F}_{m+1}} x_{\overline{F}_{m+1}} \Big) \cdots \Big(\sum_{\overline{F}_d} x_{\overline{F}_d} \Big),$$

where the k-th sum is over all proper flats $\overline{F}_{m+k}$ of $\overline{M}$ containing $m + k$. Since $\{1, \ldots, d\}$ is a basis of M, no proper flat of $\overline{M}$ contains $\{0, 1, \ldots, d\}$, and hence the right hand side is zero by the quadratic relations in $A(\overline{M})$. ∎

We are now ready to prove Proposition 17. Define another element

$$L_0 = L_0(w) = \sum_{e \in E} w_e\, y_e,$$

where the sum is over all elements e in E. By Lemma 18, for any nonnegative integer $m \le d$,

$$\deg(\alpha^{d-m} L_0^m) = m! \Big(\sum_{I \in \mathcal{I}_m} \prod_{e \in I} w_e \Big),$$

where $\mathcal{I}_m$ is the collection of m-element independent sets of M.

Let L be any element of $A^1(\overline{M})$ attached to a strictly submodular function on $2^{\overline{E}}$. By Lemma 16, Theorem 13 applies to the element $L_0 + \epsilon L$ for any positive real number ϵ. By the Hodge–Riemann relations for $q \le 1$, any matrix representing the symmetric bilinear form

$$A^1(\overline{M}) \times A^1(\overline{M}) \to \mathbb{R}, \quad (\eta_1, \eta_2) \mapsto \deg\big(\eta_1 \eta_2 (L_0 + \epsilon L)^{d-2}\big),$$

must have exactly one positive eigenvalue. Thus any matrix representing the symmetric bilinear form

$$A^1(\overline{M}) \times A^1(\overline{M}) \to \mathbb{R}, \quad (a_1, a_2) \mapsto \deg(\eta_1 \eta_2 L_0^{d-2}),$$

has at most one positive eigenvalue. Now consider the symmetric matrix

$$H_0 = \begin{bmatrix} \deg(\alpha\, \alpha L_0^{d-2}) & \deg(\alpha L_0 L_0^{d-2}) \\ \deg(\alpha L_0 L_0^{d-2}) & \deg(L_0 L_0 L_0^{d-2}) \end{bmatrix}.$$

Cauchy's eigenvalue interlacing theorem shows that H_0 has at most one positive eigenvalue. On the other hand, H_0 has at least one positive eigenvalue, because its lower-right diagonal entry is positive. The determinant of H_0 is a positive multiple of

$$\frac{d}{d-1} \Big(\sum_{I \in \mathcal{I}_d} \prod_{e \in I} w_e \Big) \Big(\sum_{I \in \mathcal{I}_{d-2}} \prod_{e \in I} w_e \Big) - \Big(\sum_{I \in \mathcal{I}_{d-1}} \prod_{e \in I} w_e \Big)^2,$$

which must be nonpositive by the condition on the eigenvalues of H_0.

5. Proof of Theorem 7

The upper bound follows from Theorem 5. We construct explicit vector configurations over $\mathbb{F}$ to show the lower bound $8/7$.

Fix a prime number p and an integer $d \geq 2$. Consider the d-dimensional vector space $\mathbb{F}_p^d$ over the field with p elements, and let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the standard basis vectors of $\mathbb{F}_p^d$.

Definition 19. Let M_p^d be the rank d matroid represented by the vectors $\mathbf{e}_1, \mathbf{e}_2 + \cdots + \mathbf{e}_d$, and

$$
\begin{array}{cccc}
1\mathbf{e}_1 + \mathbf{e}_2, & 2\mathbf{e}_1 + \mathbf{e}_2, & \cdots & p\mathbf{e}_1 + \mathbf{e}_2, \\
1\mathbf{e}_1 + \mathbf{e}_3, & 2\mathbf{e}_1 + \mathbf{e}_3, & \cdots & p\mathbf{e}_1 + \mathbf{e}_3, \\
\vdots & \vdots & \ddots & \vdots \\
1\mathbf{e}_1 + \mathbf{e}_d, & 2\mathbf{e}_1 + \mathbf{e}_d, & \cdots & p\mathbf{e}_1 + \mathbf{e}_d.
\end{array}
$$

We write i for the vector $\mathbf{e}_1$ and j for the vector $\mathbf{e}_2 + \cdots + \mathbf{e}_d$.

The matroid M_2^4 is isomorphic to the matroid S_8 mentioned in the introduction. For any d, the matroid M_2^d is the self-dual matroid obtained from the binary spike Z_d in Oxley's list by deleting any element other than the tip [23, Appendix].[2] For any p, the matroid M_p^d has a spike-like structure in that it has a "tip" i and "legs"

$$
\mathrm{L}_m = \{\mathbf{e}_1, 1\mathbf{e}_1 + \mathbf{e}_m, 2\mathbf{e}_1 + \mathbf{e}_m, \ldots, p\mathbf{e}_1 + \mathbf{e}_m\} \quad \text{for } m = 2, \ldots, d.
$$

For general spikes and their role in structural matroid theory, see [23, Chapter 14]. As before, we write $\mathcal{B}_{ij}, \mathcal{B}_j^i, \mathcal{B}_i^j, \mathcal{B}^{ij}$ for the sets of bases of $\mathrm{M} = \mathrm{M}_p^d$ containing and/or not containing i, j.

(1) The contraction $\mathrm{M}/i/j$ is the uniform matroid $\mathrm{U}_{d-2,d-1}$ with each element replaced by p parallel copies. Any basis of the contraction is disjoint from one of the parallel classes and contains exactly one point from each of the remaining parallel classes. Therefore,

$$
|\mathcal{B}_{ij}(\mathrm{M})| = (d-1)\binom{p}{1}^{d-2}.
$$

(2) The deletion $\mathrm{M} \setminus i \setminus j$ is represented by the p-point lines $\mathrm{L}_2 \setminus \mathbf{e}_1, \ldots, \mathrm{L}_d \setminus \mathbf{e}_1$ in $\mathbb{F}_p^d$. Any basis of the deletion must contain exactly two points from one of the lines and one point from each of the remaining lines. Therefore,

$$
|\mathcal{B}^{ij}(\mathrm{M})| = (d-1)\binom{p}{2}\binom{p}{1}^{d-2}.
$$

---

[2]According to Geelen [8], "it all goes wrong for spikes." The spike Z_d was first used by Seymour to demonstrate that an independence oracle algorithm for testing whether a matroid is binary cannot run in polynomial time relative to the size of the ground set [26].

(3) The contraction-deletion $M / i \setminus j$ is the boolean matroid $U_{d-1,d-1}$ with each element replaced by p parallel copies. Any basis of the contraction-deletion contains exactly one element from each parallel class. Therefore,

$$|\mathscr{B}_i^j(M)| = \binom{p}{1}^{d-1}.$$

It remains to compute the number of bases of M not containing i and containing j. There are two types of such bases, corresponding to the two terms in the right hand side of

$$|\mathscr{B}_j^i(M)| = (p^{d-1} - p^{d-2}) + (d-1)(d-2)\binom{p}{2}\binom{p}{1}^{d-3}.$$

A basis of the first type contains exactly one point from each of the p-point lines $L_2 \setminus e_1, \ldots, L_d \setminus e_1$. The determinant formula

$$\det \begin{bmatrix} 0 & k_2 & k_3 & \cdots & k_d \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix} = -k_2 - k_3 - \cdots - k_d$$

shows that there are exactly $p^{d-1} - p^{d-2}$ such bases. A basis of the second type contains exactly two points from one of the lines, no point from another, and one point from each of the remaining lines. It is clear that any basis in $\mathscr{B}_j^i$ must be of one of the two types.

Combining the four numbers, we obtain a ratio that depends only on d and not on p:

$$\frac{|\mathscr{B}_{ij}(M)|\,|\mathscr{B}^{ij}(M)|}{|\mathscr{B}_j^i(M)|\,|\mathscr{B}_j^i(M)|} = \frac{d^2 - 2d + 1}{d^2 - 3d + 4}.$$

The maximum of the ratio is $8/7$, achieved when $d = 5$. This proves Theorem 7 when the field $\mathbb{F}$ has characteristic p.

For fields of characteristic zero, let i and j be distinct elements of a finite set A_1. Let $A_2, \ldots, A_d$ be a family of $(m+1)$-element subsets of $A_1 \setminus i$ whose union is $A_1 \setminus i$ and whose pairwise intersection is $\{j\}$. We extend the transversal matroid construction in Example 3 as follows.

Definition 20. The matroid N_m^d is the transversal matroid of the family $A_1, \ldots, A_d$.

The matroid N_2^6 is isomorphic to the truncated graphic matroid in Example 2. By definition, bases of $N = N_m^d$ are the systems of distinct representatives of $A_1, \ldots, A_d$. For $m = p$, the matroids M_p^d and N_m^d share three of the four minors obtained by deleting and/or contracting i, j.

For any m, we have

$$|\mathcal{B}_{ij}(N)| = (d-1)\binom{m}{1}^{d-2}, \quad |\mathcal{B}^{ij}(N)| = (d-1)\binom{m}{2}\binom{m}{1}^{d-2},$$

$$|\mathcal{B}_i^j(N)| = \binom{m}{1}^{d-1}.$$

There are two types of bases of N not containing i and containing j, corresponding to the two terms in the right hand side of

$$|\mathcal{B}_j^i(N)| = m^{d-1} + (d-1)(d-2)\binom{m}{2}\binom{m}{1}^{d-3}.$$

A basis of the first type contains exactly one element from each of the sets $A_2 \setminus j, \ldots,$ $A_d \setminus j$. A basis of the second type contains exactly two points from one of the sets $A_k \setminus j$, no point from another $A_k \setminus j$, and one point from each of the remaining $A_k \setminus j$.

Combining the four numbers and taking the limit $m \to \infty$, we obtain the same ratio as before:

$$\lim_{m \to \infty} \frac{|\mathcal{B}_{ij}(N)| \, |\mathcal{B}^{ij}(N)|}{|\mathcal{B}_j^i(N)| \, |\mathcal{B}_j^i(N)|} = \frac{d^2 - 2d + 1}{d^2 - 3d + 4}.$$

Since transversal matroids are representable over any infinite field [23, Chapter 11], this proves Theorem 7 when $\mathbb{F}$ has characteristic 0. In fact, for any positive integer $p = m$, the set of vectors in Definition 19 viewed as elements of $\mathbb{Q}^d$ represents N_m^d.

6. Proofs of Corollaries 10 and 11

Let I_M be the size of an independent set drawn uniformly at random from the collection of all independent sets of a rank d matroid M. As discussed in the introduction, Corollary 9 and [14, Theorem 2.5] together imply

$$H(I_M) \leq \frac{1}{2}\log\left(2\pi e\left(\mathbb{E}(I_M) + \frac{1}{12}\right)\right) \leq \frac{1}{2}\log\left(2\pi e\left(d + \frac{1}{12}\right)\right).$$

Corollary 11 follows from the upper bound of $H(I_M)$ and the easy implication

$$H(X) \leq \log t \implies \max_k \mathbb{P}(X = k) \geq 1/t.$$

Corollary 10 follows from the upper bound of $H(I_M)$ and the estimate

$$\frac{1}{2}\log\left(\frac{\pi}{2}d\right) \leq \log\frac{2^d}{\binom{d}{d/2}} \leq \sum_{k=0}^d \frac{\binom{d}{k}}{2^d}\log\frac{2^d}{\binom{d}{k}} \leq \sup_{\text{rk}(M)=d} H(I_M).$$

The first inequality follows from Stirling's approximation, the second from $\binom{d}{k} \leq \binom{d}{d/2}$, and the third is witnessed by the rank d boolean matroid.

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