Some Identities for Abelian Integrals, II

by

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1. The theta functions associated to Jacobi varieties are known to satisfy a considerable number of special identities, such as the Schottky identity, various nonlinear partial differential equations, the trisecant identity and addition theorems studied by Fay. A preceding paper under the same title was devoted to an examination of various forms of the trisecant identity and some of their implications, while this paper will consider the addition theorems, in both cases as expressed in terms of second-order theta functions and thereby linearized.

The basic second-order theta functions on a Jacobi variety J can be viewed as a single function taking its values in a complex vector space S, and the addition theorem as an expression of the restrictions of this function to the subvarieties $W_r - W_r \subseteq J$ in terms of the standard holomorphic and meromorphic Abelian differentials on the underlying Riemann surface M when $W_r - W_r$ is parametrized by M^{2r}. The main point here is that there are natural filtrations of the vector space S associated to this expression, first according to the index r and then more finely according to the singularities of the Abelian differentials in this expression. The filtration

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can be used to build the general addition theorem by successive extensions, the individual terms of which can be written out quite simply and explicitly, and to simplify the analysis of linear relations between the theta nullwerte correspondingly; by itself it can be viewed as yet another set of invariants associated to Riemann surfaces.

After a brief survey in section 2 to recall the background and establish notation, the general form of the addition theorem and its associated filtration will be derived in sections 3 and 4. This will be examined in rather more detail for small values of r in sections 5 and 6, in part to clarify the general picture but primarily to provide some explicit results that will be needed. The basic quartic formulas as related to the filtration will be examined in section 7. The final section will be devoted to surfaces of genus 3 and 4, to show the extent to which the filtration and the associated quartics are related to other natural invariants of Riemann surfaces. The formulas are numbered separately in separate sections, with the section obviously indicated in references to formulas in another section.

2. Let M be a compact Riemann surface of genus $g > 0$, represented as the quotient $M = \tilde{M}/\Gamma$ of its universal covering surface $\tilde{M}$ by the group Γ of covering transformations, and fix a marking of M, a base point $z_0 \in \tilde{M}$ and a canonical set of generators $A_1, \ldots, A_g$, $B_1, \ldots, B_g$ for Γ. Associated to this marking are a canonical basis $\omega_1, \ldots, \omega_g$ for the Abelian differentials on M, and the corresponding Abelian integrals $w_j(z) = \int_{z_0}^{z} \omega_j$; the latter can be viewed as the coordinate functions of a holomorphic mapping $w : \tilde{M} \longrightarrow \mathbb{C}^g$, and the constant vectors $\omega(T) = w(Tz) - w(z)$ associated to the elements $T \in \Gamma$ are the periods of the Abelian integrals. The periods $\omega(A_j) = \delta_j \in \mathbb{C}^g$ viewed as column vectors are the columns of the $g \times g$ identity matrix $I = \{\delta_1, \ldots, \delta_g\}$, and the periods $\omega(B_j) = \Omega \delta_j \in \mathbb{C}^g$ are the columns of a $g \times g$ symmetric matrix Ω with positive definite imaginary part. The mapping $T \in \Gamma \longrightarrow \omega(T) \in \mathbb{C}^g$ is a group homomorphism $\omega : \Gamma \longrightarrow \mathbb{C}^g$, the image of which is the lattice subgroup $L = \omega(\Gamma) = (I, \Omega)\, \mathbb{Z}^{2g}$ spanned by the columns of the $g \times 2g$ period matrix (I, Ω). The quotient group $J = \mathbb{C}^g/L$ is a compact complex torus of dimension g, the Jacobi variety associated to the Riemann surface M.

Holomorphic line bundles over M can be described by factors of automorphy for the action of the group Γ on $\tilde{M}$. The simplest line bundles are the topologically trivial ones, all of which can be described by factors of automorphy that are merely scalar representations of the group Γ. Indeed to each vector $t \in \mathbb{C}^g$ associate the representation $\rho_t \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ for which $\rho_t(A_j) = 1$ and $\rho_t(B_j) = \exp 2\pi i\, t_j$; any topologically trivial line bundle can be described by one of these factors of automorphy ρ_t, and two such factors of automorphy ρ_t, ρ_s are holomorphically equivalent and hence describe the same line bundle precisely when $t - s \in L$. This leads to a rather concrete identification of the set of topologically trivial line bundles over M with the Jacobi variety J of M, a more detailed description of which can be found in [6]. The simplest line bundle that is not topologically trivial is the point bundle ζ associated to the base point z_0, the bundle of Chern class 1 with a holomorphic section that vanishes to first order at the point z_0 but has no other zeros on M; as in [8] this line bundle can be described by the factor of automorphy

$$(1) \qquad \zeta(A_j, z) = 1 \ , \quad \zeta(B_j, z) = \exp -\frac{2\pi i}{g}\,(w_j(z) + q_j + r_j)$$

for suitable vectors $q \in L \subset \mathbb{C}^g$ and $r \in \mathbb{C}^g$, where r will be called the Riemann point. Any line bundle over M of Chern class n can be described by a factor of automorphy of the form $\rho_t\zeta^n$, where $t \in \mathbb{C}^g$ is uniquely determined up to a vector in the lattice subgroup L; in particular the line bundle associated to a divisor $d = \Sigma_j \nu_j z_j$ of degree $n = \Sigma_j \nu_j$ can be described by the factor of automorphy $\rho_{w(d)}\zeta^n$ where $w(d) = \Sigma_j \nu_j w(z_j) \in \mathbb{C}^g$.

The line bundle ζ_a over M associated to the divisor on M represented by a divisor $1 \cdot a$ on $\tilde{M}$ can thus be described by the factor of automorphy $\zeta_a = \rho_{w(a)}\zeta$; there is then a holomorphic function q_a on $\tilde{M}$ vanishing to the first order at the points $\Gamma a \subset \tilde{M}$ and nonzero otherwise, and satisfying the functional equations

$$(2) \qquad q_a(Tz) = \rho_{w(a)}(T)\ \zeta(T,z)\ q_a(z)$$

for all $T \in \Gamma$. The advantage in viewing the parameter a as varying over $\tilde{M}$ is that it is then possible to find such functions $q_a(z) = q(z,a)$ that are holomorphic in a as well as in z, and that satisfy the skew-symmetry condition $q(z,a) = -q(a,z)$; the resulting function on $\tilde{M} \times \tilde{M}$ is unique up to a constant factor. This function was called the elementary function for the marked Riemann surface M in [6], and is essentially

Klein's prime form [10]. Since the elementary function $q(z,a)$ has a simple zero along the diagonal $z=a$ it follows that $\partial_1 q(z,z) = \partial q(z,a)/\partial z|_{a=z} \neq 0$, where the differentiation is in terms of any local coordinate system near z on $\tilde{M}$; the differential form $\varphi(z) = \partial_1 q(z,z)dz$ is then holomorphic and nowhere vanishing on $\tilde{M}$, so its integral $u(z) = \int_{z_0}^{z}\varphi$ is a well defined holomorphic function on $\tilde{M}$ and can be used as a local coordinate system at each point of $\tilde{M}$. This function is uniquely determined by the choice of elementary function $q(z,a)$, hence is uniquely determined up to a constant factor for a given marked Riemann surface. The resulting local coordinates on $\tilde{M}$ will be called the <u>canonical local coordinates</u> for the marked Riemann surface M , and will be used consistently throughout the remainder of the discussion here. One advantage they have is that the elementary function clearly has a local power series expansion near the diagonal of the form $q(z,a) = (z-a) + (z-a)^2 q_2(z,a)$ in terms of the canonical local coordinates; actually in view of (3) it must be the case that $q_2(z,a) = -q_2(a,z)$, so that $q_2(a,a) = 0$ and this expansion can hence be rewritten

$$(3) \qquad q(z,a) = (z-a) + (z-a)^3 q_3(z,a)$$

where $q_3(z,a)$ is holomorphic near the diagonal and $q_3(z,a) = q_3(a,z)$. Another and more important advantage they have is that the canonical bundle takes a particularly simple form; indeed as shown in [8] when a coordinate transformation $T \in \Gamma$ is expressed in terms of the canonical local coordinates on $\tilde{M}$ then $dT(z)/dz = \rho_{-k}(T)\zeta(T,z)^{2-2g}$ where $k = 2(r+q-\varepsilon)$ and $\varepsilon_j = \omega_{jj}/2$. That means that arbitrary derivatives in terms of the canonical local coordinates of holomorphic sections of line bundles in the standard form $\rho_t\zeta^n$ will again be holomorphic sections of line bundles in the standard form. The canonical bundle itself is just $\varkappa = \rho_k\zeta^{2g-2}$, and k will be called the <u>canonical point</u>.

The canonical meromorphic Abelian differentials and their integrals can be expressed quite simply and usefully in terms of the elementary function. First for a fixed point a in $\tilde{M}$ the function $w_a(z) = \partial \log q(z,a)/\partial a$ is the canonical integral of the second kind with a pole at a : it is a well defined meromorphic function on $\tilde{M}$; its sole singularities are simple poles at the points $\Gamma a \subset \tilde{M}$, indeed in terms of the canonical local coordinate z at $z=a$ it has a Laurent expansion $w_a(z) = -(z-a)^{-1} + \cdots$ near $z=a$; and it satisfies the periodicity properties that $w_a(A_j z) = w_a(z)$ and $w_a(B_j z) = w_a(z) + 2\pi i\, w_j'(a)$. The canonical differential of the second kind is $\omega_a(z) = w_a'(z)\cdot dz = \partial^2 \log q(z,a)/\partial z \partial a \cdot dz$. It is convenient to introduce the associated holomorphic function $Q(z,a) = q(z,a)^2 w_a'(z) = q(z,a)^2\, \partial^2 \log q(z,a)/\partial z \partial a = q(z,a)\, \partial_1 \partial_2 q(z,a) - \partial_1 q(z,a)\, \partial_2 q(z,a)$, which obviously satisfies the symmetry condition $Q(z,a) = Q(a,z)$ and is easily seen to have a Taylor expansion in terms of the canonical local coordinate z at $z=a$ of the form

$$(4) \qquad Q(z,a) = 1 + (z-a)^4\, Q_4(z,a)$$

for some holomorphic function $Q_4(z,a)$ near the diagonal $z=a$; in particular $Q(z,a) = 1$. Next for a pair of fixed points a_+ , a_- in $\tilde{M}$ and a simple arc δ from a_- to a_+ in $\tilde{M}$ the function $w_\delta(z) = \log[q(z,a_+)/q(z,a_-)]$ is a canonical integral of the third kind associated to δ : it is a well defined holomorphic function on $\tilde{M} - \Gamma\delta$ for any choice of branch of the logarithm, hence is determined uniquely up to an additive constant in $2\pi i\,\mathbb{Z}$; it has an analytic continuation through each interior point of the arc δ , with logarithmic branch points at the ends a_+ , a_- ; and it satisfies the periodicity properties that $w_\delta(A_j z) = w_\delta(z)$ and $w_\delta(B_j z) = w_\delta(z) + 2\pi i[w_j(a_+) - w_j(a_-)]$. The canonical differential of the third kind is $\omega_\delta(z) = w'_\delta(z)\cdot dz$, and has simple poles at the points Γa_+ with residue +1 and simple poles at the points Γa_- with residue -1 . It should be noted that the derivative $w'_\delta(z)$ is independent of the choice of a branch of the logarithm or of a particular arc δ from a_- to a_+ ; it thus depends only on the points a_+ , a_- , z , so will be denoted alternatively simply by $w'_{a_+,a_-}(z)$. It is convenient here also to introduce the associated holomorphic function

$$Q(z;a_+,a_-) = q(z,a_+)\ q(z,a_-)\ w'_\delta(z)$$

$$= \det\begin{Bmatrix} \partial_1 q(z,a_+) & q(z,a_+) \\ \partial_1 q(z,a_-) & q(z,a_-) \end{Bmatrix} ,$$

which obviously satisfies the skew-symmetry condition $Q(z;a_+,a_-) = -Q(z;a_-,a_+)$ and is easily seen to have a Taylor expansion in terms of the canonical local coordinate z at $z = a_+$ of the form

$$(5) \qquad Q(z;a_+,a_-) = q(a_+,a_-) + Q_2(z;a_+,a_-)\ (z-a_+)^2$$

for some holomorphic function $Q_2(z;a_+,a_-)$ near the diagonal $z = a_+$; in particular $Q(a_+;a_+,a_-) = q(a_+,a_-)$, and clearly $Q(z;a,a) = 0$.

3. The theta series with characteristic $[\nu|\tau]$ and period matrix Ω has the form

$$\theta[\nu|\tau](w;\Omega) = \sum_{n\in\mathbb{Z}^g} \exp 2\pi i\left[\frac{1}{2}\,{}^t(n+\nu)\Omega(n+\nu) + {}^t(n+\nu)(w+\tau)\right] ,$$

where $\nu, \tau, w \in \mathbb{C}^g$ and $n \in \mathbb{Z}^g$ will be viewed as column vectors or $g \times 1$ matrices with the usual matrix operations. The second-order theta series will be taken here to be of the form

$$\theta_2[\nu|\tau](w;\Omega) = \theta\left[\frac{1}{2}\nu|\tau\right](2w;2\Omega) .$$

It is convenient to view the 2^g second-order theta series that arise as ν varies over $\mathbb{Z}^g/2\mathbb{Z}^g$ as comprising a vector-valued theta function

$$\underline{\underline{\theta}}_2[\tau](w) = \{\theta_2[\nu|\tau](w;\Omega) : \nu \in \mathbb{Z}^g/2\mathbb{Z}^g\} ,$$

and to set $\underline{\underline{\theta}}_2[0](w) = \underline{\underline{\theta}}_2(w)$ so that alternatively $\underline{\underline{\theta}}_2[2\tau](w) = \underline{\underline{\theta}}_2(w+\tau)$. The associated Riemannian theta function $\underline{\underline{\theta}}_2[\tau](w(z))$ is then a holomorphic mapping from $\tilde{M}$ to $\mathbb{C}^G$, where $G = 2^g$, depending on an auxiliary parameter $\tau \in \mathbb{C}^g$. Each entry in this vector-valued function is a holomorphic section of the line bundle $\rho_{k-\tau}\zeta^{2g}$ over M, as demonstrated in [6] among other places; the whole vector-valued function can be viewed as a section of the vector bundle $\rho_{k-\tau}\zeta^{2g} \otimes I_G$, where I_G is the trivial bundle of rank $G = 2^g$, and that will be indicated by writing

(1) $\quad \underline{\underline{\theta}}_2(\tau)[w(z)] \in \Gamma(M, \mathcal{O}(\rho_{k-\tau}\zeta^{2g} \otimes I_G))$.

The first aim here is to express this section suitably in terms of the standard functions and forms over M for suitable parameters τ.

For this purpose fix a divisor $a_1 + \ldots + a_n$ on $\tilde{\tilde{M}}$, where $a_1, \ldots, a_n$ represent n distinct points of the surface M. It follows from the Riemann-Roch theorem that $\underline{\dim}\,\Gamma(M, \mathcal{O}(\kappa\zeta^2_{a_1}\cdots\zeta^2_{a_n})) = g + 2n - 1$; the functions

$$w_j'(z) \cdot \prod_{\nu=1}^{n} q(z,a_\nu)^2 \qquad \text{for } j = 1,\dots,g$$

(2) $$w'_{a_\lambda, a_n}(z) \cdot \prod_{\nu=1}^{n} q(z,a_\nu)^2 \qquad \text{for } \lambda = 1,\dots,n-1$$

$$w'_{a_\mu}(z) \cdot \prod_{\nu=1}^{n} q(z,a_\nu)^2 \qquad \text{for } \mu = 1,\dots,n$$

are evidently linearly independent holomorphic sections of the line bundle $\varkappa\zeta_{a_1}^2 \cdots \zeta_{a_n}^2$, hence form a basis for the vector space of all such sections. As a notational convenience set $u_\lambda'(z) = w'_{a_\lambda, a_n}(z)$ and $v_\mu'(z) = w'_{a_\mu}(z)$, and for nonnegative integers k , ℓ , m for which $k+\ell+m=n$ let $S(m,\ell,k)$ be the subset of the group $S(n)$ of permutations of the indices $1,2,\dots,n$ consisting of those permutations π such that $\pi(1) < \dots < \pi(m)$, $\pi(m+1) < \dots < \pi(m+\ell)$, and $\pi(m+\ell+1) < \dots < \pi(m+\ell+k)$. In these terms there is the following general expansion.

Theorem 1. There are uniquely determined vectors

$$\alpha^{m,\ell,k}_{\mu_1,\dots,\mu_m,\lambda_1,\dots,\lambda_\ell,j_1,\dots,j_k} = \alpha^{m,\ell,k}_{\mu,\lambda,j} \in \mathbb{C}^G$$

that are meromorphic functions of the points $a_1,\dots,a_n \in \tilde{M}$ and nonsingular whenever $a_1,\dots,a_n$ represent distinct points of M , that are symmetric in the m indices μ , in the ℓ indices λ , and in the k indices j , and are such that

$$\underline{\underline{\theta}}_2(w(z_1+\dots+z_n-a_1-\dots-a_n))\Big[\prod_{\mu<\nu} q(z_\mu,z_\nu)\Big]^2\Big[\prod_{\mu<\nu} q(a_\mu,a_\nu)\Big]^2\Big[\prod_{\mu,\nu} q(z_\mu,a_\nu)\Big]^{-2}$$

$$= \sum_{m,\ell,k}\ \sum_{\pi\in S(m,\ell,k)}\ \sum_{\mu,\lambda,j} \alpha^{m,\ell,k}_{\mu,\lambda,j}\, v'_{\mu_1}(z_{\pi(1)})\cdots v'_{\mu_m}(z_{\pi(m)}) \cdot$$

$$\cdot\ \mu'_{\lambda_1}(z_{\pi(m+1)})\cdots\mu'_{\lambda_\ell}(z_{\pi(m+\ell)})\, w'_{j_1}(z_{\pi(m+\ell+1)})\cdots w'_{j_k}(z_{\pi(n)}) \ ,$$

where the first summation is extended over all those indices $k \geqq 0$, $\ell \geqq 0$, $m \geqq 0$ for which $k+\ell+m=n$ and as usual

$$w(z_1+\ldots+z_n - a_1-\ldots-a_n) = w(z_1)+\ldots+w(z_n) - w(a_1)-\ldots-w(a_n) .$$

Proof. First suppose that $a_1,\ldots,a_n$ are fixed points of $\widetilde{M}$ representing distinct points of M. It then follows from (1) that as a function of the variable $z_1 \in \widetilde{M}$ alone

$$\underline{\underline{\theta}}_2(w(z_1+\ldots+z_n - a_1-\ldots-a_n))$$

$$= \underline{\underline{\theta}}_2[2w(z_2+\ldots+z_n - a_1-\ldots-a_n)](w(z_1))$$

$$\in \Gamma(M, \mathcal{O}(\varkappa\zeta_{a_1}^2\cdots\zeta_{a_n}^2\zeta_{z_2}^{-2}\cdots\zeta_{z_n}^{-2} \otimes I_G)) ;$$

since $q(z_1,z_\nu)^2 \in \Gamma(M, \mathcal{O}(\zeta_{z_\nu}^2))$ as a function of z_1 the expression

(3) $$\underline{\underline{\theta}}_2(w(z_1+\ldots+z_n - a_1-\ldots-a_n))\cdot\prod_{\mu<\nu}[q(z_\mu,z_\nu)q(a_\mu,a_\nu)]^2$$

belongs to $\Gamma(M, \mathcal{O}(\varkappa\zeta_{a_1}^2\cdots\zeta_{a_n}^2 \otimes I_G))$ when viewed as a function just of z_1. If $f_1,\ldots,f_{g+2n-1}$ is any basis for $\Gamma(M, \mathcal{O}(\varkappa\zeta_{a_1}^2\cdots\zeta_{a_n}^2))$ there are uniquely determined vectors $\alpha_{i_1} \in \mathbb{C}^G$ such that (3) takes the form $\Sigma_{i_1}\alpha_{i_1}f_{i_1}(z_1)$, and these vectors are holomorphic functions of the variables $z_2,\ldots,z_n$. By symmetry (3) has the same properties as a function of z_2 as it has as a function of z_1, so that (3) and therefore each vector α_{i_1} belongs to $\Gamma(M, \mathcal{O}(\varkappa\zeta_{a_1}^2\cdots\zeta_{a_n}^2 \otimes I_G))$ as a function of z_2; hence there are uniquely determined vectors $\alpha_{i_1i_2} \in \mathbb{C}^G$ such that $\alpha_{i_1} = \Sigma_{i_2}\alpha_{i_1i_2}f_{i_2}(z_2)$, and these vectors are holomorphic functions of the variables $z_3,\ldots,z_n$. The argument can be repeated, leading eventually to the result that there are uniquely determined vectors $\alpha_{i_1\ldots i_n} \in \mathbb{C}^G$ such that (3) takes the form

(4) $\Sigma_{i_1\ldots i_n}\alpha_{i_1\ldots i_n} f_{i_1}(z_1)\cdots f_{i_n}(z_n)$,

and since (3) is symmetric in the variables $z_1,\ldots,z_n$ the vectors $\alpha_{i_1\ldots i_n}$ are symmetric in the indices $i_1,\ldots,i_n$. If the index range $I = \{i : 1 \leqq i \leqq g+2n-1\}$ is split into subranges $I' = \{i : 1 \leqq i \leqq g\}$, $I'' = \{i : g+1 \leqq i \leqq g+n-1\}$, and $I''' = \{i : g+n \leqq i \leqq g+2n-1\}$ then for any individual term in the series (4) there will be say k of the indices $i_1,\ldots,i_n$ in the range I' , ℓ in the range I'' , and m in the range I''' where $k+\ell+m=n$. There is a unique rearrangement of the indices $i_1,\ldots,i_n$, grouping together first those m of the i's in the range I''' , then those ℓ in the range I'' , and finally those k in the range I' , but preserving the order of the indices within each subgroup; thus

$$\alpha_{i_1\ldots i_n} f_{i_1}(z_1)\cdots f_{i_n}(z_n) = \alpha_{i_{\pi(1)}\ldots i_{\pi(n)}} f_{i_{\pi(1)}}(z_{\pi(1)})\cdots f_{i_{\pi(n)}}(z_{\pi(n)})$$

where $\pi(1) < \cdots < \pi(m)$, $\pi(m+1) < \cdots < \pi(m+\ell)$, $\pi(m+\ell+1) < \cdots < \pi(n)$, and $i_{\pi(1)}\ldots i_{\pi(m)}$ are in I''' and so on. As the indices $i_1,\ldots,i_n$ vary but k, ℓ, m remain fixed the positions of those indices in I' , in I'' , and in I''' will vary over all the possibilities, hence π will vary over the full subset $S(m,\ell,k)$, while for each π the individual indices $i_{\pi(1)},\ldots,i_{\pi(m)}$ will vary independently over I''' , and so on. The result is precisely the formula in the statement of the theorem, when the basis $f_1,\ldots,f_n$ is taken in the form (2), thus concluding the proof.

The preceding is, of course, a purely formal result, but is a convenient formula on which to base a more detailed analysis. The next step is to observe that the formula can be simplified by introducing a suitable filtration of the vector space $\mathbb{C}^G$, although admittedly some information is lost in the process. The procedure just amounts to applying a suitable projection operator $P : \mathbb{C}^G \longrightarrow \mathbb{C}^G$ to the preceding result, and can be viewed either as ignoring the kernel of P or as considering only those particular second-order theta functions that arise as the image of P.

<u>Theorem 2</u>. If $P : \mathbb{C}^G \longrightarrow \mathbb{C}^G$ is any linear mapping for which

$$P\underline{\underline{\theta}}_2(w(z_1+\cdots+z_{n-1} - a_1-\cdots-a_{n-1})) = 0$$

for all points $z_j, a_j \in \widetilde{M}$ then

$$P\underline{\underline{\theta}}_2(w(z_1+\cdots+z_n-a_1-\cdots-a_n))\Big[\prod_{\mu<\nu} q(z_\mu,z_\nu)\Big]^2\Big[\prod_{\mu<\nu} q(a_\mu,a_\nu)\Big]^2\Big[\prod_{\mu,\nu} q(z_\mu,a_\nu)\Big]^{-2}$$

$$= \sum_{\ell,k}\ \sum_{\pi\in S(0,\ell,k)}\ \sum_{\lambda,j} P\alpha^{0,\ell,k}_{\lambda,j}\ u'_{\lambda_1}(z_{\pi(1)})\cdots u'_{\lambda_\ell}(z_{\pi(\ell)})w'_{j_1}(z_{\pi(\ell+1)})\cdots w'_{j_k}(z_{\pi(n)})$$

where the first summation is extended over all those indices $k \geqq 0$, $\ell \geqq 0$ for which $k+\ell = n$.

<u>Proof</u>. The gist of this result is of course that $P\alpha^{m,\ell,k}_{\mu,\lambda,j} = 0$ whenever $m > 0$, or equivalently that after applying P the expansion of the function (3) in terms of the basis (2) does not actually involve any of the n functions of the third kind in the list (2). Note that each of the $g+n-1$ functions of the first two kinds in (2) vanishes at all of the points $a_1,\ldots,a_n$, while the μ-th function of the third kind in (2) has the value 1 at a_μ and vanishes at the remaining a's. The hypothesis on the projection operator P implies that the function (3) vanishes at all of the points $a_1,\ldots,a_n$ when viewed as a function of any of the variables $z_1,\ldots,z_n$, so the expansion of (3) does only involve the $g+n-1$ functions of the first two kinds in (2) and the proof is thereby concluded.

The problem now is to analyze the coefficients $\alpha^{m,\ell,k}_{\mu,\lambda,j}$ in the expansion just obtained.

Theorem 3. If P is as in the hypothesis of Theorem 2, then

$$P\,\alpha^{0,\ell,k}_{\lambda_1\cdots\lambda_\ell j_1\cdots j_k} = 0 \quad\text{whenever}\quad \lambda_{i_1} = \lambda_{i_2} \quad\text{for}\quad i_1 \neq i_2 .$$

Proof. Multiply the formula of Theorem 2 by $\prod_{\mu,\nu} q(z_\mu, a_\nu)^2$ to make all the terms appearing there holomorphic, apply the differential operator $\partial/\partial z_1$, and set $z_1 = a_1$. The only possibly nontrivial term that can arise on the left-hand side is that in which the differentiation is applied to the term $P\underline{\underline{\theta}}_2(w(z_1+\cdots+z_n-a_1-\cdots-a_n))$, since it has a zero when $z_1 = a_1$ as a consequence of the hypothesis on P; thus if $\partial_j P\underline{\underline{\theta}}_2(w) = \partial P\underline{\underline{\theta}}_2(w)/\partial w_j$ for short then the left-hand side becomes

$$(5)\qquad \Sigma_j w_j'(a_1)\partial_j P\underline{\underline{\theta}}_2(w(z_2+\cdots+z_n-a_2-\cdots-z_n))\cdot$$
$$\cdot\Big[\prod_{1<\nu} q(a_1,z_\nu)\Big]^2\Big[\prod_{1<\mu<\nu} q(z_\mu,z_\nu)\Big]^2\Big[\prod_{\mu<\nu} q(a_\mu,a_\nu)\Big]^2$$

The only possibly nontrivial terms that can arise on the right-hand side are those in which the double zero of the factor $q(z_1,a_1)^2$ at $z_1 = a_1$ is cancelled, and that can only happen when $q(z_1,a_1)^2$ is multiplied by $u_1'(z_1) = w'_{a_1,a_n}(z_1)$ to cancel one of the zeros and the differentiation is applied to the result to cancel the other zero; these are thus the terms for which $\ell \geq 1$, $\pi(1) = 1$, $\lambda_1 = 1$, and in view of (2.5) the result is

$$(6)\qquad \sum_{\substack{\ell,k\\ \ell\geq 1}}\ \sum_{\substack{\pi\in(0,\ell,k)\\ \pi(1)=1}}\ \sum_{\substack{\lambda,j\\ \lambda_1=1}} P\alpha^{0,\ell,k}_{\lambda,j}\, u'_{\lambda_2}(z_{\pi(2)})\cdots w'_{j_k}(z_{\pi(n)})\ \cdot$$
$$\cdot\Big[\prod_{\nu} q(a_1,a_\nu)\Big]^2\Big[\prod_{\substack{\mu,\nu\\ \mu>1}} q(z_\mu\cdot a_\nu)\Big]^2 \quad .$$

Then to the identity (5) = (6) apply the differential operator $\partial/\partial z_2$ and set $z_2 = a_1$. The left-hand side vanishes identically since (5) has the double zero at $z_2 = a_1$ arising from the factor $q(a_1,z_2)^2$. The only possibly nontrivial terms that can arise on the right-hand side are those in which the double zero of the factor $q(z_2,a_1)^2$ at $z_2 = a_1$ is cancelled, and that can only happen when $q(z_2,a_1)^2$ is multiplied by $u_1'(z_2) = w'_{a_1 a_n}(z_2)$ to cancel one of the zeros and the differentiation is applied to the result to cancel the other zero; these are thus the terms for which $\ell \geqq 2$, $\pi(2) = 2$, $\lambda_2 = 1$, and in view of (2.5) the resulting identity is

$$0 = \sum_{\substack{\ell,k \\ \ell \geqq 2}} \quad \sum_{\substack{\pi \in S(0,\ell,k) \\ \pi(1)=1,\pi(2)=2}} \quad \sum_{\substack{\lambda,j \\ \lambda_1=\lambda_2=1}} P\alpha^{0,\ell,k}_{\lambda,j}\, u'_{\lambda_3}(z_{\pi(3)}) \cdots w'_{j_k}(z_{\pi(n)}) \cdot$$

$$\cdot \Big[\prod_{\nu} q(a_1,a_\nu)\Big]^4 \Big[\prod_{\substack{\mu,\nu \\ \mu > 2}} q(z_\mu,a_\nu)\Big]^2$$

The functions $u'_{\lambda_3}(z_{\pi(3)}) \cdots w'_{j_k}(z_{\pi(n)})$ in the different summands appearing here are linearly independent, so all the coefficients $P\alpha^{0,\ell,k}_{\lambda,j}$ appearing here must vanish; thus

$$P\,\alpha^{0,\ell,k}_{11\lambda_3 \ldots \lambda_\ell j_1 \ldots j_k} = 0$$

for all possible choices $\ell \geqq 2$, k, $\lambda_3,\ldots,\lambda_k$. The coefficients are symmetric in the indices $\lambda_1,\ldots,\lambda_\ell$, and the same argument can, of course, be applied similarly when setting $z_1 = z_2 = a_i$ for $i = 1,\ldots,n-1$, and that suffices for the proof.

When considering the expansion given in Theorem 2, it is convenient to take the number ℓ of singular differentials as the <u>order</u> of a term on the right-hand side, and to view the terms of maximal order L actually appearing nontrivially in that expansion as the <u>dominant terms</u>; the dominant order is thus given by

$$(7) \qquad L = \max\{\ell : P\alpha^{0,\ell,k}_{\lambda j} \neq 0 \quad \text{for some} \quad \lambda, j\} ,$$

and depends both on the number n of variable points and the projection operator P chosen. Since $P^{0,\ell,k}_{\lambda,j} = 0$ unless all the λ's are distinct by Theorem 3, while $0 \leqq \lambda \leqq n-1$, it is evident that $P\alpha^{0,n,0}_{\lambda} = 0$ for all indices λ; thus whenever P is as in the hypothesis of Theorem 2, then necessarily $0 \leqq L \leqq n-1$. The coefficients of the dominant terms have a particularly simple form, as follows.

<u>Theorem 4</u>. If P is as in the hypothesis of Theorem 2 and L is the dominant order of the expansion given in that theorem, then there are uniquely determined constant vectors

$$_P\alpha^{i,j}_{k} = {}_P\alpha^{i_1,\ldots,i_L,j_{L+1},\ldots,j_n}_{j_{L+1},\ldots,j_n} \in P\mathbb{C}^G \cong \mathbb{C}^p$$

such that for any indices $1 \leqq \sigma_1 < \cdots < \sigma_L \leqq n-1$ and complementary indices $1 \leqq \tau_{L+1} < \cdots < \tau_n = n$

$$P\alpha^{0,L,n-L}_{\sigma,j} = \sum_k {}_P\alpha^{k(\sigma_1),\ldots,k(\sigma_L),k(\tau_{L+1}),\ldots,k(\tau_n)}_{j_{L+1},\ldots,j_n} \cdot w'_{k_1}(a_1)\cdots w'_{k_n}(a_n) ,$$

where $k(\sigma_1) = k_{\sigma_1}$ and so on. These vectors are symmetric in the indices i, in the indices j, and in the indices k; furthermore

$$_P\alpha^{i,j}_{k} = (-1)^L {}_P\alpha^{i,k}_{j}$$

and $$\sum_{\lambda=L}^{n} \alpha^{i_1,\ldots,i_{L-1},i_\lambda,i_L,\ldots,i_{\lambda-1},i_{\lambda+1},\ldots,i_n}_{k_{L+1},\ldots,k_n} = 0 .$$

Proof. Choose any L indices $1 \leqq \sigma_1 < \cdots < \sigma_L \leqq n$, and let $1 \leqq \tau_{L+1} < \cdots < \tau_n \leqq n$ be the complementary set of indices, so that the σ's and τ's together comprise the indices $1,\ldots,n$. Again multiply the formula of Theorem 2 by $\prod_{\mu,\nu} q(z_\mu, a_\nu)^2$ to make all the terms appearing there holomorphic, but then apply the differential operator $\partial^L/\partial z_{\sigma_1}\cdots\partial z_{\sigma_L}$ and set $z_{\sigma_i} = a_{\sigma_i}$. The only possibly nontrivial terms that can arise on the left-hand side are those in which all the differentiation is applied to $P\underline{\theta}_2(w(z_1+\cdots+z_n-a_1-\cdots-a_n))$, since it is clear from the hypothesis on P that this function vanishes whenever $z_{\sigma_i} = a_{\sigma_i}$ for any i ; thus the left-hand side becomes

$$(8)\quad \Sigma_j w'_{j_1}(a_{\sigma_1})\cdots w'_{j_L}(a_{\sigma_L})\partial_{j_1\cdots j_L} P\underline{\theta}_2(w(z_{\tau_{L+1}}+\cdots+z_{\tau_n}-a_{\tau_{L+1}}-\cdots-a_{\tau_n}))\cdot$$

$$\cdot\Big[\prod_{j<k} q(a_{\sigma_j},a_{\sigma_k})\Big]^4\Big[\prod_{j,k} q(a_{\sigma_j},z_{\tau_k})q(a_{\sigma_j},a_{\tau_k})\Big]^2\cdot$$

$$\cdot\Big[\prod_{j<k} q(z_{\tau_j},z_{\tau_k})q(a_{\tau_j},a_{\tau_k})\Big]^2$$

The only possibly nontrivial terms that can arise on the right-hand side are those in which the double zero of each of the L factors $q(z_{\sigma_j},a_{\sigma_j})^2$ is cancelled, by multiplying by a suitable singular term $u'_{\lambda_j}(z_{\sigma_j})$ to cancel one of the zeros and then differentiating the result to cancel the other zero. All except the dominant terms $\ell = L$ therefore automatically vanish, since the others do not have enough singular terms to allow such cancellation, and among the dominant terms the only ones that can possibly be nontrivial are those for which $\pi(1)=\sigma_1,\ldots,\pi(L)=\sigma_L$, $\pi(L+1)=\tau_{L+1},\ldots,\pi(n)=\tau_n$. There are, however, two separate cases that must then be considered. (i) If $\sigma_L < n$ then $u'_{\lambda_j}(z_{\sigma_j})$ has a pole at

$z_{\sigma_j} = a_{\sigma_j}$ only when $\lambda_j = \sigma_j$ for each j. In view of (2.5), the right-hand side becomes

$$(9) \qquad \Sigma_j \, P\alpha^{0,L,n-L}_{\sigma,j} \, w'_{j_1}(z_{\tau_{L+1}}) \cdots w'_{j_{n-L}}(z_{\tau_n}) \cdot$$

$$\cdot \Big[\prod_{j \neq k} q(a_{\sigma_j}, a_{\sigma_k})\Big]^2 \Big[\prod_{j,k} q(z_{\tau_j}, a_{\tau_k})\Big]^2 \cdot$$

$$\cdot \Big[\prod_{j,k} q(a_{\sigma_j}, a_{\tau_k}) q(a_{\sigma_j}, z_{\tau_k})\Big]^2 \quad .$$

(ii) If $\sigma_L = n$ then for $1 \leqq j \leqq L-1$ as before $u'_{\lambda_j}(z_{\sigma_j})$ has a pole at $z_{\sigma_j} = a_{\sigma_j}$ only when $\lambda_j = \sigma_j$; but $u'_\lambda(z_n) = w'_{a_\lambda a_n}(z_n)$ has a pole at $z_n = a_n$ for any value of λ, although with residue -1, so in view of (2.5) and this change of sign in the residue the right-hand side becomes

$$(10) \qquad -\Sigma_{\lambda,j} \, P\alpha^{0,L,n-L}_{\sigma_1 \cdots \sigma_{L-1} \lambda \, j_1 \cdots j_{n-L}} \, w'_{j_1}(z_{\tau_{L+1}}) \cdots w'_{j_{n-L}}(z_{\tau_n}) \cdot$$

$$\cdot \Big[\prod_{j \neq k} q(a_{\sigma_j}, a_{\sigma_k})\Big]^2 \Big[\prod_{j,k} q(z_{\tau_j}, a_{\tau_k})\Big]^2 \cdot$$

$$\cdot \Big[\prod_{j,k} q(a_{\sigma_j}, a_{\tau_k}) q(a_{\sigma_j}, z_{\tau_k})\Big]^2 \quad .$$

After cancelling out those elementary functions that appear on both sides, the equation (8)=(9) takes the form

$$(11) \qquad \Sigma_j w'_{j_1}(a_{\sigma_1}) \cdots w'_{j_L}(a_{\sigma_L}) \partial_{j_1 \cdots j_L} P\underline{\underline{\theta}}_2(w(z_{\tau_{L+1}} + \cdots + z_{\tau_n} - a_{\tau_{L+1}} - \cdots - a_{\tau_n})) \cdot$$

$$\cdot \prod_{j<k} [q(z_{\tau_j}, z_{\tau_k}) q(a_{\tau_j}, a_{\tau_k})]^2$$

$$= \Sigma_j \, P\alpha^{0,L,n-L}_{\sigma,j} \, w'_{j_1}(z_{\tau_{L+1}}) \cdots w'_{j_{n-L}}(z_{\tau_n}) \cdot \prod_{j,k} q(z_{\tau_j}, a_{\tau_k})^2 \quad .$$

The expression $P\alpha^{0,L,n-L}_{\sigma,j}$ depends on the choice of the points $a_1,\dots,a_n$, and it is clear from (11) that it belongs to $\Gamma(M, \mathcal{O}(\kappa \otimes I_p))$ as a function of any one of the variables $a_{\sigma_1},\dots,a_{\sigma_L}$, where p is the dimension of the image of the projection P ; this expression has the same property as a function of any one of the variables $a_{\tau_{L+1}},\dots,a_{\tau_n}$, although that is perhaps not quite so obvious. For a proof, note that the result of applying the differential operator $\partial^L/\partial w_{j_1}\cdots\partial w_{j_L}$ to the functional equation satisfied by $\underline{\underline{\theta}}_2[\tau](w)$ when the variable w is translated by some lattice vector in L is a rather complicated formula, but one that can be viewed as the assertion that $\partial_{j_1\cdots j_L}\underline{\underline{\theta}}_2[\tau](w)$ satisfies the same functional equations as does $\underline{\underline{\theta}}_2[\tau](w)$ modulo lower-order derivatives; consequently $\partial_{j_1\cdots j_L}\underline{\underline{\theta}}_2[\tau](w(z))$ belongs to the vector space $\Gamma(M, \mathcal{O}(\rho_{k-\tau}\zeta^{2g} \otimes I_G))$ modulo lower-order derivatives, in the obvious sense, and the corresponding result holds upon applying the projection operator P . In particular, in analogy with (3) the expression on the left-hand side of (11) belongs to the vector space $\Gamma(M, \mathcal{O}(\kappa\zeta^2_{z(\tau_{L+1})}\cdots\zeta^2_{z(\tau_n)} \otimes I_p))$ as a function of each variable $a_{\tau_{L+1}},\dots,a_{\tau_n}$, since as already observed at the beginning of the proof of this theorem all the lower-order derivatives vanish at points of the form $w(z_{\tau_{L+1}}+\cdots+z_{\tau_n}-a_{\tau_{L+1}}-\cdots-a_{\tau_n})$; here $z(\tau_i)$ has been used in place of z_{τ_i} for typographical simplicity. It then follows from the equation (11) that the expression $P\alpha^{0,L,n-L}_{\sigma,j}$ viewed as a function of any of the variables $a_{\tau_{L+1}},\dots,a_{\tau_n}$ is a meromorphic function with singularities at most double poles at the points $z_{\tau_{L+1}},\dots,z_{\tau_n}$ and transforms as a section of the bundle $\kappa \otimes I_p$; but since this expression does not depend at all on the points $z_{\tau_{L+1}},\dots,z_{\tau_n}$ it must actually be holomorphic, hence must belong to the vector space $\Gamma(M, \mathcal{O}(\kappa \otimes I_p))$ as desired. Altogether then it is evident that there are constant vectors

$$ {}_p\alpha_{\sigma,j}^{k} = {}_p\alpha_{\sigma_1,\dots,\sigma_L,j_1,\dots,j_{n-L}}^{k_1,\dots\qquad\dots,k_n} \in P\mathbb{C}^G \cong \mathbb{C}^P $$

such that

$$ (12)\qquad P\alpha_{\sigma,j}^{0,L,n-L} = \Sigma_k\ {}_p\alpha_{\sigma,j}^{k} w'_{k_1}(a_1)\cdots w'_{k_n}(a_n) \ . $$

Upon substituting this into (11), replacing j_i by $k_{\sigma_i} = k(\sigma_i)$ on the left-hand side, comparing the coefficients of $w'_{k(\sigma_1)}(a_{\sigma_1})\cdots w'_{k(\sigma_L)}(a_{\sigma_L})$ on the two sides of the resulting equation, and then writing z_i in place of z_{τ_i} and a_i in place of a_{τ_i}, there results the identity

$$ (13)\quad \partial_{k(\sigma_1)\cdots k(\sigma_L)} P\underline{\theta}_2(w(z_{L+1}+\dots+z_n-a_{L+1}-\dots-a_n))\cdot\Big[\prod_{j<k} q(z_j,z_k)q(a_j,a_k)\Big]^2 $$

$$ = \sum_{j,k(\tau_i)} {}_p\alpha_{\sigma,j}^{k} w'_{j_1}(z_{L+1})\cdots w'_{j_{n-L}}(z_n) w'_{k(\tau_{L+1})}(a_{L+1})\cdots w'_{k(\tau_n)}(a_n)\cdot $$

$$ \cdot\prod_{j,k} q(z_j,a_k)^2 \ , $$

which holds for any fixed indices $k_{\sigma_1} = k(\sigma_1),\dots,k_{\sigma_L} = k(\sigma_L)$, for which $\sigma_L < n$. The right-hand side involves both these fixed indices and the indices of summation $k_{\tau_{L+1}} = k(\tau_{L+1}),\dots,k_{\tau_n} = k(\tau_n)$, a notation that might at first seem confusing but on second thought should clearly be a convenient way of indicating which of the superscripts $k_1,\dots,k_n$ in ${}_p\alpha_{\sigma,j}^{k}$ are fixed and which are variable indices of summation; since the extent to which the coefficients ${}_p\alpha_{\sigma,j}^{k}$ are symmetric in k has not yet been determined the exact location of any index must be kept in mind. This formula is particularly simple when $\sigma_i = i$; setting

(14) $$p^{\alpha}{}^{k_1,\dots\dots,k_n}_{j_{L+1},\dots,j_n} = p^{\alpha}{}^{k_1,\dots\dots\dots,k_n}_{1,\dots,L,j_{L+1},\dots,j_n}$$

for short, it becomes the identity

(15) $$\partial_{k_1\cdots k_L} P\underline{\underline{\theta}}_2(w(z_{L+1}+\cdots+z_n-a_{L+1}-\cdots-a_n))\cdot\Big[\prod_{j<k} q(z_j,z_k)q(a_j,a_k)\Big]^2$$

$$= \Sigma\, p^{\alpha}{}^{k_1,\dots,k_L,k_{L+1},\dots,k_n}_{j_{L+1},\dots,j_n}\, w'_{j_{L+1}}(z_{L+1})\cdots w'_{j_n}(z_n) w'_{k_{L+1}}(a_{L+1})\cdots w'_{k_n}(a_n)\cdot$$

$$\cdot \prod_{j,k} q(z_j,a_k)^2 \quad .$$

It is clear from the symmetries of the left-hand side of (15) that the vectors (14) are symmetric in the indices $k_1,\dots,k_L$, in the indices $k_{L+1},\dots,k_n$, and in the indices $j_{L+1},\dots,j_n$; furthermore since $\partial k_1\cdots k_L P\underline{\underline{\theta}}_2(w)$ is an even function of w if L is even and an odd function of w if L is odd , then interchanging the variables z_i with the variables a_i shows that the vector (14) is multiplied by $(-1)^L$ when j_i and k_i are interchanged simultaneously for all subscripts $L+1 \leqq i \leqq n$. Upon comparing (13) and (15) and keeping these symmetries in mind it is also evident that

(16) $$p^{\alpha}{}^{k_1,\dots\dots\dots,k_n}_{\sigma_1,\dots,\sigma_L,j_{L+1},\dots,j_n} = p^{\alpha}{}^{k(\sigma_1),\dots,k(\sigma_L),k(\tau_{L+1}),\dots,k(\tau_n)}_{j_{L+1},\dots\dots,j_n} \quad .$$

Turning back now to case (ii) in which $\sigma_L = n$, and using (12) to rewrite the equation (8)=(10) in the same way that the equation (8)=(9) has just been rewritten, leads to the identity

$$\partial_{k(\sigma_1)\cdots k(\sigma_L)} P\underline{\underline{\theta}}_2(w(z_{L+1}+\cdots+z_n-a_{L+1}-\cdots-a_n))\cdot\prod_{j<k}\Big[q(z_j,z_k)q(a_j,a_k)\Big]^2$$

$$= -\sum_{j,k(\tau_i)\lambda} p^{\alpha}{}^{k_1,\dots\dots\dots\dots,k_n}_{\sigma_1,\dots,\sigma_{L-1},\lambda,j_1,\dots,j_{n-L}}\cdot w'_{j_1}(z_{L+1})\cdots w'_{j_{n-L}}(z_n)\cdot$$

$$\cdot w'_{k(\tau_{L+1})}(a_{L+1})\cdots w'_{k(\tau_n)}(a_n)\cdot\prod_{j,k} q(z_j,a_k)^2 \quad .$$

Here $\sigma_L = n$, and upon taking $\sigma_1 = 1, \ldots, \sigma_{L-1} = L-1$ and hence $\tau_{L+1} = L, \ldots, \tau_n = n-1$, and comparing the result with (15) it follows readily that

$$ {}_p\alpha^{k_1, \ldots, k_{L-1}, k_L, \ldots, k_{n-1}}_{j_1, \ldots, j_{n-L}} = - \sum_{\lambda=L}^{n-1} \alpha^{k_1, \ldots\ldots\ldots, k_n}_{1, \ldots, L-1, \lambda, j_1, \ldots, j_{n-L}} . $$

When rewritten using (14) this clearly leads to the last symmetry relation in the statement of the theorem, and thereby concludes the proof.

A particularly interesting auxiliary result obtained in the course of the proof of the preceding proposition is (15), which can be viewed as a reasonably explicit determination of the constants ${}_p\alpha^{ij}_k$ in terms of derivatives of second-order theta functions, or equivalently as an indication of the independent significance of these constants.

4. The symmetry conditions described in Theorem 4 have some useful formal consequences that will be derived next, in a slight digression. The principal result of interest here can be stated as follows.

Lemma 1. If $\chi_{i,j} = \chi_{i_1,\ldots,i_n;j_1,\ldots,j_n}$ are constants that are symmetric in the indices i and in the indices j separately and that satisfy

(1) $$\Sigma_{\lambda=1}^{n+1} \chi_{i_1,\ldots,i_{n-1},j_\lambda;j_1,\ldots,j_{\lambda-1},j_{\lambda+1},\ldots,j_{n+1}} = 0$$

for all indices $i_1,\ldots,i_{n-1},j_1,\ldots,j_{n+1}$ then

(2) $$\chi_{i;j} = (-1)^n \chi_{j;i} \quad .$$

Proof. To simplify the typography set

$$(1\ 2\cdots n;n+1\cdots 2n) = \chi_{i_1 i_2\cdots i_n;i_{n+1},\ldots,i_{2n}} \quad .$$

It will be demonstrated by induction on m that

(3) $$(1\cdots n\text{-}m\ n\text{-}m\text{+}1\cdots n\ ;\ n\text{+}1\cdots 2n)$$
$$= (-1)^m \sum_J (1\cdots n\text{-}m\ j_1\cdots j_m\ ;\ n\text{-}m\text{+}1\cdots n\ j_{m+1}\cdots j_n) \ ,$$

where the summation is extended over all those permutations $j_1,\ldots,j_n$ of the indices $n+1,\ldots,2n$ such that $j_1 < \cdots < j_m$ and $j_{m+1} < \cdots < j_n$, or equivalently is extended over the $\binom{n}{m}$ distinct ways of splitting the indices $n+1,\ldots,2n$ into a subset $j_1,\ldots,j_m$ of m of them and the complementary subset $j_{m+1},\ldots,j_n$ of the remaining $n-m$ of them. The case $m=0$ is of course trivial, while the case $m=1$ is just the hypothesis (1). For the inductive step, assume that (3) has been established for some $m \geq 1$. By a simple reordering of the indices that formula can be rewritten

$$(1\cdots n\text{-}m\text{-}1\,\lambda\, n\text{-}m\cdots\hat{\lambda}\cdots n\,;\, n{+}1\cdots 2n)$$

$$= (-1)^m \sum_J (1\cdots n\text{-}m\text{-}1\,\lambda\, j_1\cdots j_m\,;\, n\text{-}m\cdots\hat{\lambda}\cdots n\ j_{m+1}\cdots j_n)$$

for $n\text{-}m \leqq \lambda \leqq n$, where $\hat{\lambda}$ signifies that the index λ is to be omitted at that point. Add these equations together for the $m+1$ possible values of λ; the additional symmetry hypothesis implies that the left-hand sides are all identical, while on the right-hand side for any fixed permutation J the case $m=1$ can be applied to the summation over λ and it follows that

$$(m{+}1)(1\cdots n\text{-}m\text{-}1\ n\text{-}m\cdots n\,;\, n{+}1\cdots 2n)$$

$$= (-1)^m \sum_J \sum_{\lambda=n-m}^{m} (1\cdots n\text{-}m\text{-}1\ j_1\cdots j_m\lambda\,;\, n\text{-}m\cdots\hat{\lambda}\cdots n\ j_{m+1}\cdots j_n)$$

$$= (-1)^{m+1} \sum_J \sum_{\nu=m+1}^{n} (1\cdots n\text{-}m\text{-}1\ j_1\cdots j_m j_\nu\,;\, n\text{-}m\cdots n\ j_{m+1}\cdots\hat{j}_\nu\cdots j_n)\ .$$

There are $(n\text{-}m)\binom{n}{m} = (m{+}1)\binom{n}{m+1}$ summands on the right-hand side, corresponding to $m+1$ copies of the summation over the $\binom{n}{m+1}$ ways of splitting the indices $n+1,\ldots,2n$ into a subset of $m+1$ of them and the complementary subset, and that leads to the case $(m+1)$ of (3) and thereby concludes the inductive step. The formula (3) thus holds for all values $0 \leqq m \leqq n$; but the desired result (2) is just the special case $m=n$, and the proof is thereby concluded.

This purely formal observation can then be used to complement the results of Theorem 4 as follows.

Theorem 5. With the hypothesis and notation as in Theorem 4, the dominant order L is subject to the bound

$$2L \leqq n\ .$$

In the extreme case that $2L = n$ the vectors ${}_p\alpha^{i,j}_{k}$ are symmetric under and interchange of the sets i,j,k if L is even and skew-symmetric if L is odd.

Proof. To demonstrate the second assertion of the theorem, if $2L = n$ then the sets of indices i,j,k each contain the same number $L = n-L$ of indices. It follows from Theorem 4 that ${}_p\alpha^{i,j}_{k} = (-1)^L {}_p\alpha^{i,k}_{j}$, and from the other symmetry properties of that theorem and Lemma 1 that ${}_p\alpha^{i,j}_{k} = (-1)^L {}_p\alpha^{j,i}_{k}$; thus these vectors are symmetric or skew-symmetric in the sets i,j,k according to the parity of L as desired. Then for the first assertion of the theorem suppose to the contrary that $2L > n$. Write $i = (i_1,\ldots,i_L) = (i',i'')$ where i' consists of the first $2L-n$ indices and i'' of the last $n-L$ indices. It is clear that the vectors ${}_p\alpha^{i,j}_{k}$ satisfy the hypotheses of Lemma 1 in the indices i'',j for any fixed indices i',k, and it therefore follows from that lemma that ${}_p\alpha^{i,j}_{k} = (-1)^{n-L} {}_p\alpha^{i'j,i''}_{k}$; consequently

$$
\begin{aligned}
&{}_p\alpha^{i_1,\ldots,i_L,j_{L+1},\ldots,j_n}_{k_{L+1},\ldots,k_n} \\
&= (-1)^{n-L}\,{}_p\alpha^{i_1,\ldots,i_{2L-n},j_{L+1},\ldots,j_n\ i_{2L-n+1},\ldots,i_L}_{k_{L+1},\ldots,k_n} \\
&= (-1)^{n-L}\,{}_p\alpha^{i_2,\ldots,i_{2L-n},j_{L+1},\ldots,j_n,i_1,i_{2L-n+1},\ldots,i_L}_{k_{L+1},\ldots,k_n} \\
&= {}_p\alpha^{i_2,\ldots,i_{2L-n},j_{L+1},i_{2L-n+1},\ldots,i_L,j_{L+2},\ldots,j_n,i_1}_{k_{L+1},\ldots,k_n} \\
&= {}_p\alpha^{j_{L+1}i_2,\ldots,i_L,i_1,j_{L+2},\ldots,j_n}_{k_{L+1},\ldots,k_n} .
\end{aligned}
$$

The vectors ${}_p\alpha^{i,j}_{k}$ are thus symmetric in i_1, j_{L+1}, and since they are also symmetric in the indices i and in the indices j separately they must be fully symmetric in the n indices (i,j) . However the last symmetry of Theorem 4 in this case implies that ${}_p\alpha^{i,j}_{k} = 0$ for all indices i,j,k ; but that contradicts the assumption that L is the dominant order, the order of the highest nontrivial terms in the expansion of Theorem 2, so it cannot be the case that $2L > n$ and the proof is thereby concluded.

5. In the simplest case, that in which $n=1$, Theorem 1 just asserts that

$$(1) \qquad \underline{\underline{\theta}}_2(w(z-a))\,q(z,a)^{-2} = \alpha^{1,0,0}\, w'_a(z) + \Sigma_j\, \alpha_j^{0,0,1}\, w'_j(z)$$

for some vectors $\alpha^{1,0,0}$, $\alpha_j^{0,0,1}$ depending holomorphically on the point a. These vectors can be determined quite explicitly by a direct analysis, the result being the following well known and important formula that has been discussed in [3], [2], and [11] among other places.

<u>Corollary 1</u>. For any points $z, a \in \widetilde{M}$,

$$\underline{\underline{\theta}}_2(w(z-a)) = \underline{\underline{\theta}}_2(0)\cdot Q(z,a) + \frac{1}{2}\sum_{j,k} \partial_{jk}\underline{\underline{\theta}}_2(0)\cdot q(z,a)^2 w'_j(z)\, w'_k(a) \quad .$$

<u>Proof</u>. First multiply (1) by $q(z,a)^2$ so that all terms appearing are holomorphic functions of z and a. Setting $z=a$ in the resulting formula leads immediately to the result that $\alpha^{1,0,0} = \underline{\underline{\theta}}_2(0)$, in view of (2.4). Next note from the symmetry of this formula in the variables z and a that $\alpha_j^{0,0,1} \in \Gamma(M, \mathcal{O}(\kappa \otimes I_G))$ as a function of a; hence $\alpha_j^{0,0,1} = \Sigma_k\, \alpha_j^k\, w'_k(a)$ for some vectors α_j^k independent of a. Applying the differential operator $\partial^2/\partial z \partial a$ and setting $z=a$ readily leads to the result that $2\alpha_j^k = \partial^2 \underline{\underline{\theta}}_2(w)/\partial w_j \partial w_k\big|_{w=0} = \partial_{jk}\underline{\underline{\theta}}_2(0)$, and thereby concludes the proof.

Let $S_1 \subseteq \mathbb{C}^G$ be the linear subspace spanned by the values $\underline{\underline{\theta}}_2(w(z-a))$ as z and a vary independently over $\widetilde{M}$. It is evident from Corollary 1 that S_1 can be described equivalently as the subspace of $\mathbb{C}^G$ spanned by the $1+\binom{g+1}{2}$ vectors $\underline{\underline{\theta}}_2(0)$ and $\partial_{jk}\underline{\underline{\theta}}_2(0)$ for $j \leq k$; for the values $\underline{\underline{\theta}}_2(w(z-a))$ clearly lie in the span of these $1+\binom{g+1}{2}$ vectors, and since the coefficients of these vectors are linearly independent functions of z , a the opposite inclusion holds as well. It was demonstrated in [2], for instance, that these $1+\binom{g+1}{2}$ vectors are linearly independent, so they form a basis for the vector space S_1 and that vector space has dimension $1+\binom{g+1}{2}$. Note that $1+\binom{g+1}{2} < 2^g$ whenever $g > 2$, so that S_1 is a proper subspace of $\mathbb{C}^G$ whenever $g > 2$. For the subsequent discussion it is convenient to choose some projection operator $P_1 : \mathbb{C}^G \longrightarrow \mathbb{C}^G$ having S_1 as its kernel; this is then a minimal projection operator for which $P_1\underline{\underline{\theta}}_2(w(z-a)) = 0$ for all points $z, a \in \tilde{M}$.

The case $n=2$ is already somewhat more complicated and illustrates the simplifications that can be effected by using the appropriate projection operators. In this case Theorem 1 asserts that

$$\text{(2)} \qquad \underline{\underline{\theta}}_2(w(z_1+z_2-a_1-a_2))\; q(z_1,z_2)^2\; q(a_1,a_2)^2 \left[\prod_{\mu,\nu} q(z_\mu,a_\nu)\right]^{-2}$$

$$= \sum_{\mu} \alpha^{2,0,0}_{\mu_1\mu_2}\, v'_{\mu_1}(z_1)v'_{\mu_2}(z_2) + \sum_{\pi,\mu} \alpha^{1,1,0}_{\mu}\, v'_{\mu}(z_{\pi(1)})\, u'_1(z_{\pi(2)})$$

$$+ \sum_{\pi,\mu,j} \alpha^{1,0,1}_{\mu,j}\, v'_{\mu}(z_{\pi(1)})\, w'_j(z_{\pi(2)}) + \alpha^{0,2,0}\, u'_1(z_1)u'_1(z_2)$$

$$+ \sum_{\pi,j} \alpha^{0,1,1}_{j}\, u'_1(z_{\pi(1)})\, w'_j(z_{\pi(2)}) + \sum_{j} \alpha^{0,0,2}_{j_1j_2}\, w'_{j_1}(z_1)w'_{j_2}(z_2)$$

for some vectors α that are meromorphic functions of a_1, a_2 with singularities at most at points for which $a_1 = a_2$ on M ; in the summations indicated π ranges over the full symmetric group S_2. At least some of these vectors can be determined explicitly, although something can be said about all of them as follows.

Corollary 2. For any points $z_i, a_i \in \widetilde{M}$,

$$\underline{\underline{\theta}}_2(w(z_1+z_2-a_1-a_2))\ q(z_1,z_2)^2\ q(a_1,a_2)^2 \left[\prod_{\mu,\nu} q(z_\mu,a_\nu)\right]^{-2}$$

$$= \sum_{\pi\in S_2} \underline{\underline{\theta}}_2(0)\cdot[w'_{a_1}(z_{\pi 1})w'_{a_2}(z_{\pi 2}) - w'_{a_2}(a_1)w'_{a_1a_2}(z_{\pi 1})w'_{a_1a_2}(z_{\pi 2})]$$

$$+\frac{1}{2}\sum_{\pi\in S_2}\sum_{j,k}\partial_{jk}\underline{\underline{\theta}}_2(0)\cdot[w'_{a_1}(z_{\pi_1})w'_j(z_{\pi 2})w'_k(a_2)+w'_{a_2}(z_{\pi 1})w'_j(z_{\pi 2})w'_k(a_1)$$

$$- w'_{a_1a_2}(z_{\pi 1})w'_{a_1a_2}(z_{\pi 2})w'_j(a_1)w'_k(a_2)]$$

$$+\sum_{\pi\in S_2}\sum_j \alpha^{0,1,1}_j\ w'_{a_1a_2}(z_{\pi 1})w'_j(z_{\pi 2}) + \sum_j \alpha^{0,0,2}_{j_1j_2}\ w'_{j_1}(z_1)w'_{j_2}(z_2)$$

where $\alpha^{0,1,1}_j$ and $\alpha^{0,0,2}_{j_1j_2}$ are vectors that are meromorphic functions of a_1, a_2 with singularities at most at points for which $a_1 = a_2$ on M. For these vectors

$$(3) \qquad P_1\alpha^{0,1,1}_j = \sum_k \alpha^{k_1k_2}_j\ w'_{k_1}(a_1)w'_{k_2}(a_2)$$

where $\alpha^{k_1k_2}_{k_3} \in P_1\mathbb{C}^G$ are constants that are skew-symmetric in the three indices k_1,k_2,k_3, and if $S'_1 \subseteq \mathbb{C}^G$ is the subspace spanned by S_1 and any representatives of the vectors $\alpha^{k_1k_2}_{k_3}$ and $P'_1 : \mathbb{C}^G \longrightarrow \mathbb{C}^G$ is a projection operator having kernel precisely S'_1 then

$$(4) \qquad P'_1\ \alpha^{0,0,2}_{j_1j_2} = \sum_k \alpha^{k_1k_2}_{j_1j_2}\ w'_{k_1}(a_1)w'_{k_2}(a_2) \quad ,$$

where $\alpha^{k_1k_2}_{j_1j_2} \in P'_1\mathbb{C}^G$ are constants that are symmetric in the indices j_1,j_2, in the indices k_1,k_2, and in the pairs j,k.

Proof. The projection operator P_1 satisfies the hypothesis of Theorem 2 for the case $n=2$, and it follows from Theorems 2 and 3 that

(5) $$P_1\alpha^{2,0,0}_{\mu_1\mu_2} = P_1\alpha^{1,1,0}_{\mu} = P_1\alpha^{1,0,1}_{\mu,j} = P_1\alpha^{0,2,0} = 0 \quad ;$$

these vectors thus all lie in the subspace $S_1 \subseteq \mathbb{C}^G$, so can be expressed in terms of the canonical basis for that subspace. To do so first multiply (2) by $q(z_2,a_2)^2$ and consider the limit in the resulting formula as z_2 tends to a_2. The only possibly nontrivial terms that can arise on the right-hand side are those in which the double zero of $q(z_2,a_2)^2$ at $z_2 = a_2$ is cancelled out by the double pole of the meromorphic function $v_2(z_2)$, and $q(z_2,a_2)^2 v_2(z_2) \longrightarrow 1$ in view of (2.4). The resulting formula is easily seen to have the form

$$\underline{\underline{\theta}}_2(w(z_1-a_1))\; q(z_1,a_1)^{-2}$$
$$= \sum_{\mu} \alpha^{2,0,0}_{\mu 2}\, v'_{\mu}(z_1) + \alpha^{1,1,0}_{2}\, u'_1(z_1) + \sum_{j} \alpha^{1,0,1}_{2,j}\, w'_j(z_1) .$$

A comparison of this with the formula of Corollary 1 and use of the evident symmetry of (2) in the variables z_1 and z_2 lead to the results that

(6) $$\alpha^{2,0,0}_{11} = \alpha^{2,0,0}_{22} = 0 \quad , \quad \alpha^{2,0,0}_{12} = \underline{\underline{\theta}}_2(0) \quad ,$$

$$\alpha^{1,1,0}_{1} = \alpha^{1,1,0}_{2} = 0$$

$$\alpha^{1,0,1}_{1,j} = \frac{1}{2}\sum_k \partial_{jk}\underline{\underline{\theta}}_2(0) w'_k(a_2) \; , \quad \alpha^{1,0,1}_{2,j} = \frac{1}{2}\sum_k \partial_{jk}\underline{\underline{\theta}}_2(0) w'_k(a_1) \; .$$

Next multiply (2) by $q(z_2,a_2)^2$ again, but then apply the differential operator $\partial/\partial z_2$ and consider the limit in the resulting formula as z_2 tends to a_2. The only possibly nontrivial terms that can arise on the right-hand side are those in which at least one of the zeros of $q(z_2,a_2)^2$ at $z_2 = a_2$ is cancelled out by either the double pole of $v'_2(z_2)$ or the single pole of $u'_1(z_2)$; but $\partial/\partial z_2[q(z_2,a_2)^2 v'_2(z_2)] \longrightarrow 0$

as $z_2 \longrightarrow a_2$ by (2.4) while $\partial/\partial z_2[q(z_2,a_2)^2 u_1'(z_2)] \longrightarrow -1$ by (2.3) and (2.5), so only the second case actually leads to something nontrivial. A straightforward calculation thus yields the formula

$$(7) \qquad \sum_j \partial_j \underline{\underline{\theta}}_2(w(z_1-a_1))w_j'(a_2)q(z_1,a_1)^{-2} + 2\,\underline{\underline{\theta}}_2(w(z_1-a_1))w'_{z_1,a_1}(a_2)q(z_1,a_1)^{-2}$$
$$= -\alpha^{0,2,0}\, w'_{a_1,a_2}(z_1) - \sum_j \alpha^{0,1,1}_j\, w_j'(z_1) \quad .$$

Multiplying this by $q(z_1,a_2)$ and taking the limit in the resulting formula as z_1 tends to a_2 readily leads to the identity

$$\alpha^{0,2,0} = -2\; q(a_1,a_2)^{-2}\, \underline{\underline{\theta}}_2(w(a_1-a_2)) \ ,$$

which can be rewritten using Corollary 1 as

$$(8) \qquad \alpha^{0,2,0} = -\underline{\underline{\theta}}_2(0)\cdot 2\, w'_{a_2}(a_1) - \sum_{j,k} \partial_{jk}\underline{\underline{\theta}}_2(0)\cdot w_j'(a_1)w_k'(a_2) \ .$$

Substituting (6) and (8) into (2) yields the first assertion of the corollary. If the projection P_1 is then applied to (2), the terms involving $P\,\alpha^{0,1,1}_j$ will be the dominant ones, in view of (5), and (3) then follows immediately from Theorem 4. Finally, if the projection P_1' is applied to (2), the only terms remaining are $P'\alpha^{0,0,2}_{j_1j_2}$, so they are the dominant ones in this expansion and (4) then follows immediately from Theorem 4 again, to conclude the proof.

Probably the most interesting part of the principal formula of Corollary 2 is that involving the vectors $\alpha^{0,1,1}_j$ and $\alpha^{0,0,2}_j$, which do not lie in the subspace S_1. To describe the singularities of these vectors note first as a simple consequence of the discussion in section 2 that $w'_{a_1a_2}(z) = w'_{a_2}(z)\cdot(a_1-a_2) + O(a_1-a_2)^2$ as a function of a_1

when $a_1 \longrightarrow a_2$, where a_2, z are viewed as fixed and representing distinct points of M . The coefficient of $\underline{\underline{\theta}}_2(0)$ in the formula of Corollary 2 is therefore holomorphic in a_1 near a_2 , indeed tends to zero as $a_1 \longrightarrow a_2$, so the last line must also be holomorphic in a_1 near a_2 . The singularities of $\alpha^{0,1,1}_{j}$, $\alpha^{0,0,2}_{j_1 j_2}$ cannot cancel one another, since the coefficients $w'_{a_1 a_2}(z_{\pi 1})w'_j(z_{\pi 2})$, $w'_{j_1}(z_1)w'_{j_2}(z_2)$ for $j_1 \leqq j_2$, are linearly independent functions of z_1, z_2 , and consequently $\alpha^{0,1,1}_{j}$ <u>has as singularities at most simple poles at points where</u> $a_1 = a_2$ <u>on</u> M <u>while</u> $\alpha^{0,0,2}_{j_1 j_2}$ <u>is holomorphic in</u> a_1, a_2 . If $\rho_j(a_2) = \underline{\lim}_{a_1 \to a_2} (a_1 - a_2)\, \alpha^{0,1,1}_{j}$ is the residue of $\alpha^{0,1,1}_{j}$ viewed as a meromorphic function of a_1 at the point $a_1 = a_2$ then letting $a_1 \longrightarrow a_2$ in the formula of Corollary 2 leads almost immediately to the identity

$$0 = \sum_{\pi} \sum_{j,k} \partial_{jk}\underline{\underline{\theta}}_2(0) \cdot w'_{a_2}(z_{\pi 1}) w'_j(z_{\pi 2}) w'_k(a_2)$$

$$+ \sum_{\pi} \sum_{j} \rho_j(a_2) w'_{a_2}(z_{\pi 1}) w'_j(z_{\pi 2}) + \sum_{j} \alpha^{0,0,2}_{j_1 j_2}(a_2, a_2) w'_{j_1}(z_1) w'_{j_2}(z_2) ,$$

and from this it is clear that

$$(9) \qquad \rho_j(a) = \underline{\lim}_{a_1, a_2 \to a} (a_1 - a_2)\, \alpha^{0,1,1}_{j} = \sum_{k} \partial_{jk}\underline{\underline{\theta}}_2(0) \cdot w'_k(a)$$

$$\alpha^{0,0,2}_{j_1 j_2}(a, a) = 0 \quad .$$

That the singularities of these functions lie in the subspace $S_1 \subseteq \mathbb{C}^G$ in the obvious sense was of course already apparent from (3) and (4). That and the other complications residing in S_1 can be eliminated by applying the projection operator P_1 to the formula of Corollary 2, leading to the simpler formula

$$(10)\qquad P_1\underline{\underline{\theta}}_2(w(z_1+z_2-a_1-a_2))\; q(z_1,z_2)^2 q(a_1,a_2)^2\Big[\prod_{\mu,\nu} q(z_\mu,a_\nu)\Big]^{-2}$$

$$= \sum_{\pi\in S_2}\sum_k \alpha^{k_1k_2}_{k_3}\cdot w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(z_{\pi 1})w'_{a_1a_2}(z_{\pi 2})$$

$$+ \sum_j P_1\alpha^{0,0,2}_{j_1j_2}(a_1,a_2)\cdot w'_{j_1}(z_1)w'_{j_2}(z_2)\ .$$

Applying the projection operator P'_1 to this yields the even simpler formula

$$(11)\qquad P'_1\underline{\underline{\theta}}_2(w(z_1+z_2-a_1-a_2))\; q(z_1,z_2)^2 q(a_1,a_2)^2\Big[\prod_{\mu,\nu} q(z_\mu,a_\nu)\Big]^{-2}$$

$$= \sum_{j,k}\alpha^{k_1k_2}_{j_1j_2}\cdot w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{k_1}(a_1)w'_{k_2}(a_2)\ .$$

The constant vectors appearing in these formulas can be described alternatively by using (3.15), so that

$$(12)\qquad \partial_j P_1\underline{\underline{\theta}}_2(w(z-a)) = \sum_k \alpha^{j\ k_1}_{\ \ k_2}\cdot w'_{k_1}(a)w'_{k_2}(z)\ q(z,a)^2\ ;$$

but the analogue for the other vectors is just (11) itself. These last three formulas were derived in [8] in an alternative manner.

The results obtained in the preceding two corollaries for $n=1$ and $n=2$ can be extended to general n, but with ever increasing complication. Again, though, the complication can be controlled to some extent by applying ever stronger projection operators. For this purpose let $S_n \subseteq \mathbb{C}^G$ be the span of the vectors $\underline{\underline{\theta}}_2(w(z_1+\cdots+z_n-a_1-\cdots-a_n))$ as z_j, a_j vary independently over $\widetilde{M}$, and choose a projection operator P_n having S_n as its kernel; for the case $n=1$ these are the subspace S_1 and projection operator P_1 already introduced. The expansion of Theorem 2 with $P=P_{n-1}$ is then the simplest extension to general n of the version (10) of Corollary 2; the dominant order L satisfies $2L \leqq n$ by Theorem 5, and the dominant terms have the form given in Theorem 4. The remaining terms in the expansion can be made dominant by further strengthening the projection operator, and then they too will have the form given in Theorem 4; this is just the extension to general n of the simplifications (11) and (12) in the case $n=2$. There is not much point in attempting to write out this program in any more detail in general, but it may be helpful to see at least the next two cases $n=3$ and $n=4$. After the sequence of dominant terms and the general form of the expansion are established, it is a fairly mechanical task to write all the terms of the full expansion as the appropriate expressions in terms of this sequence of dominant terms; again, there is not much point at present in attempting to write out the detailed formulas in general, but the cases $n=3$ and $n=4$ will illustrate the procedure and some of the results in these cases will be needed later as well.

<u>Corollary 3</u>. For any points $z_i, a_i \in \tilde{M}$,

$$P_2\underline{\theta}_2(w(z_1+z_2+z_3-a_1-a_2-a_3)) \left[\prod_{\mu<\nu} q(z_\mu,z_\nu)q(a_\mu,a_\nu)\right]^2 \left[\prod_{\mu,\nu} q(z_\mu,a_\nu)\right]^{-2}$$

$$= \sum_{\pi\in S(0,1,2)} \sum_{jk} \alpha^{k_1k_2k_3}_{j_2j_3} \cdot [w'_{a_1a_3}(z_{\pi1})w'_{k_1}(a_1)\, w'_{k_2}(a_2) + w'_{a_2a_3}(z_{\pi1})w'_{k_1}(a_2)w'_{k_2}(a_1)]\cdot$$
$$\cdot w'_{k_3}(a_3)w'_{j_2}(z_{\pi2})w'_{j_3}(z_{\pi3})$$

$$+ \sum_j P_2\alpha^{0,0,3}_{j_1j_2j_3}\, w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{j_3}(z_3)$$

where $\alpha^{k_1k_2k_3}_{j_2j_3} \in P_2\mathbb{E}^G$ are constants that are symmetric in j_2, j_3 and in k_2, k_3 and satisfy

$$\alpha^{k_1k_2k_3}_{j_2j_3} = \alpha^{k_1j_2j_3}_{k_2k_3}\ , \ \alpha^{k_1k_2k_3}_{j_2j_3} + \alpha^{k_2k_1k_3}_{j_2j_3} + \alpha^{k_3k_1k_2}_{j_2j_3} = 0$$

and $P_2\alpha^{0,0,3}_{j_1j_2j_3} \in P_2\mathbb{C}^G$ are holomorphic functions of the points $a_i \in \widetilde{M}$.

The constants $\alpha^{k_1k_2k_3}_{j_2j_3}$ are determined by the formula

$$(13) \qquad \partial_i P_2 \underline{\theta}_2(w(z_1+z_2-a_1-a_2))\ q(z_1,z_2)^2\ q(a_1,a_2)^2 \left[\prod_{\mu,\nu} q(z_\mu,a_\nu)\right]^{-2} = \sum_{j,k} \alpha^{i\,k_1k_2}_{j_1j_2} w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{k_1}(a_1)w'_{k_2}(a_2)\ ,$$

and if $S'_2 \subseteq \mathbb{C}^G$ is the subspace that they together with S_2 span and $P'_2 : \mathbb{C}^G \longrightarrow \mathbb{C}^G$ is a projection operator with kernel S'_2 then

$$(14) \qquad P'_2\alpha^{0,0,3}_{j_1j_2j_3} = \sum_k \alpha^{k_1k_2k_3}_{j_1j_2j_3} w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(a_3)$$

where $\alpha^{k_1k_2k_3}_{j_1j_2j_3} \in P'_2\mathbb{C}^G$ are constants that are symmetric in j_1,j_2,j_3, in k_1,k_2,k_3, and under the interchange of j and k.

Proof. By Theorem 5 the dominant order L satisfies $L \leqq 1$; the dominant terms in the expansion of Theorem 2 have the form given by Theorem 4, and can be rewritten as indicated. The left-hand side and the first term on the right-hand side are holomorphic in a_i, and since the functions of z_i on the right-hand side are linearly independent, the coefficients $P_2\alpha^{0,0,3}_{j}$ must also be holomorphic in a_i. Equation (13) is just the form that (3.15) takes in this case. Finally, after applying the operator P'_2 to the main formula, the only nontrivial terms on the right-hand side are $P'_2\alpha^{0,0,3}_{j}$; they are therefore the dominant terms, so have the form (14) as a consequence of Theorem 4, and that concludes the proof.

<u>Corollary 4.</u> For any points $z_i, a_i \in \widetilde{M}$

$$P_2\underline{\theta}_2(w(z_1+\cdots+z_4-a_1-\cdots-a_4))\left[\prod_{\mu<\nu} q(z_\mu,z_\nu)q(a_\mu,a_\nu)\right]^2\left[\prod_{\mu,\nu} q(z_\mu,a_\nu)\right]^{-2}$$

$$= \sum_{\substack{\pi\in S(0,2,2)\\ \rho\in S(0,2,1)}} \sum_{jk} \alpha^{k_1k_2k_3k_4}_{j_3j_4} w'_{a_{\rho1}a_3}(z_{\pi1})\, w'_{a_{\rho2}a_3}(z_{\pi2})\, w'_{j_3}(z_{\pi3})\, w'_{j_4}(z_{\pi4})\cdot$$

$$\cdot w'_{k_1}(a_{\rho1})w'_{k_2}(a_{\rho2})w'_{k_3}(a_{\rho3})w'_{k_4}(a_4)$$

$$+ \sum_{\pi\in S(0,1,3)} \sum_{\lambda j} P_3\alpha^{0,1,3}_{\lambda_1j_2j_3j_4}\, u'_{\lambda_1}(z_{\pi1})w'_{j_2}(z_{\pi2})w'_{j_3}(z_{\pi3})w'_{j_4}(z_{\pi4})$$

$$+ \sum_j P_3\alpha^{0,0,3}_{j_1\cdots j_4}\, w'_{j_1}(z_1)\cdots w'_{j_4}(z_4) ,$$

where $\alpha^{i_1i_2j_1j_2}_{k_1k_2} \in P_3\mathbb{C}^G$ are constants that are symmetric in i_1, i_2, in j_1, j_2, in k_1, k_2, and in the sets i, j, k, and $P_3\alpha^{0,1,3}_{\lambda,j} \in P_3\mathbb{C}^G$, $P_3\alpha^{0,0,3}_{j} \in P_3\mathbb{C}^G$ are holomorphic functions of the points $a_i \in \widetilde{M}$.

<u>Proof.</u> Again by Theorem 5, the dominant order L satisfies $L \leqq 2$, and the expansion of Theorem 2 with the special form for the dominant term given by Theorem 4 has the form indicated. The formula has been simplified by writing out the general form and indicating the permutations necessary to yield the full development; thus ρ runs through the three permutations of the set $(1,2,3)$ for which $\rho1 < \rho2$. The dominant coefficients have further symmetries as in Theorem 4, and they imply the listed symmetries in view of Theorem 5.

More details could be added as in the preceding formula, giving explicit forms for the other terms under suitable further projection and alternative formulas for the various constants appearing; but the pattern should by this point be clear enough without these details being made explicit.

The main formulas of the preceding two corollaries just give the dominant terms of the full expansions of Theorem 1, the most interesting terms in that they involve vectors that do not appear in the expansions for lesser values of n. The other terms can be read off fairly easily from a comparison with the expansions for lesser values of n, much as in the proof of Corollary 2. The full details are somewhat complicated, but can again be simplified by the appropriate use of auxiliary projection operators. For instance, to investigate the vectors of Corollary 2 in more detail, it is useful to consider the result of applying the projection operator P_1 to the full expansions for the cases $n=3$ and $n=4$; the only results that will be needed here are just those in the next two corollaries. As a convenient abbreviation in this discussion set

$$(15)\qquad Q(z_1,\dots,z_n;a_1,\dots,a_n) = \Big[\prod_{\mu<\nu} q(z_\mu,z_\nu)q(a_\mu,a_\nu)\Big]^2 \Big[\prod_{\mu,\nu} q(z_\mu,a_\nu)\Big]^{-2}$$

and note that

$$(16)\qquad \lim_{z_n\to a_n} q(z_n,a_n)^2\, Q(z_1,\dots,z_n;a_1,\dots,a_n)$$

$$= Q(z_1,\dots,z_{n-1};a_1,\dots,a_{n-1}) .$$

<u>Corollary 5</u>. For any points $z_i, a_i \in \widetilde{M}$

$$P_1\underline{\theta}_2(w(z_1+z_2+z_3-a_1-a_2-a_3))\; Q(z_1,z_2,z_3;a_1,a_2,a_3)$$

$$= \Sigma\, P_1\alpha^{1,1,1}_{\mu\lambda j}\, v'_\mu(z_{\pi 1})u'_\lambda(z_{\pi 2})w'_j(z_{\pi 3}) + \Sigma\, P_1\alpha^{1,0,2}_{\mu j_1 j_2}\, v'_\mu(z_{\pi 1})w'_{j_1}(z_{\pi 2})w'_{j_3}(z_{\pi 3})$$

$$+ \Sigma\, P_1\alpha^{0,3,0}_{\lambda_1\lambda_2\lambda_3}\, u'_{\lambda_1}(z_1)u'_{\lambda_2}(z_2)u'_{\lambda_3}(z_3) + \Sigma\, P_1\alpha^{0,2,1}_{\lambda_1\lambda_2 j}\, u'_{\lambda_1}(z_{\pi 1})u'_{\lambda_2}(z_{\pi 2})w'_j(z_{\pi 3})$$

$$+ \Sigma\, P_1\alpha^{0,1,2}_{\lambda j_1 j_2}\, u'_{\lambda_1}(z_{\pi 1})w'_{j_1}(z_{\pi 2})w'_{j_2}(z_{\pi 3}) + \Sigma\, P_1\alpha^{0,0,3}_{j_1 j_2 j_3}\, w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{j_3}(z_3) ,$$

where the summations extend over permutations $\pi \in S(m,\ell,k)$ and indices $\mu = 1,2,3$, $\lambda = 1,2$, $j = 1,\dots,g$. Here

$$P_1\alpha^{1,1,1}_{11j} = P_1\alpha^{1,1,1}_{22j} = 0 \ , \quad P_1\alpha^{1,1,1}_{31j} = -P\,\alpha^{1,1,1}_{32j} = \sum_k \alpha^{k_1k_2}_{j}\ w'_{k_1}(a_1)w'_{k_2}(a_2) \ ,$$

$$P_1\alpha^{1,1,1}_{12j} = \sum_k \alpha^{k_1k_2}_{j}\ w'_{k_1}(a_2)w'_{k_2}(a_3) \ , \quad P_1\alpha^{1,1,1}_{21j} = \sum_k \alpha^{k_1k_2}_{j}\ w'_{k_1}(a_1)w'_{k_2}(a_3) \ ;$$

$$P_1\alpha^{1,0,2}_{1j_1j_2} = P_1\alpha^{0,0,2}_{j_1j_2}(a_2,a_3) \ , \quad P_1\alpha^{1,0,2}_{2j_1j_2} = P_1\alpha^{0,0,2}_{j_1j_2}(a_1,a_3) \ , \ P_1\alpha^{1,0,2}_{3j_1j_2} = P_1\alpha^{0,0,2}_{j_1j_2}(a_1,a_2) \ ;$$

$$P_1\alpha^{0,3,0}_{111} = P_1\alpha^{0,3,0}_{222} = 0 \ ,$$

$$P_1\alpha^{0,3,0}_{112} = -P_1\alpha^{0,3,0}_{122} = -2\sum_k \alpha^{k_1k_2}_{k_3}\ w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(a_3) \ ;$$

$$P_1\alpha^{0,2,1}_{11j} = -2\sum_k \alpha^{k_1k_2}_{j}\ w'_{k_1}(a_2)w'_{k_2}(a_3)w'_{a_2a_3}(a_1) - 2\sum_k P_1\alpha^{0,0,2}_{jk}(a_2,a_3)w'_k(a_1) \ ,$$

$$P_1\alpha^{0,2,1}_{22j} = -2\sum_k \alpha^{k_1k_2}_{j}\ w'_{k_1}(a_1)w'_{k_2}(a_3)w'_{a_1a_3}(a_2) - 2\sum_k P_1\alpha^{0,0,2}_{jk}(a_1,a_3)w'_k(a_2) \ ,$$

$$P_1\alpha^{0,2,1}_{12j} = \sum_k \alpha^{k_1k_2}_{j}\Big[w'_{k_1}(a_1)w'_{k_2}(a_3)w'_{a_1a_3}(a_2) + w'_{k_1}(a_2)w'_{k_2}(a_3)w'_{a_2a_3}(a_1)$$

$$- w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{a_1a_2}(a_3)\Big]$$

$$+ \sum_k \Big[P_1\alpha^{0,0,2}_{jk}(a_1,a_3)w'_k(a_2) + P_1\alpha^{0,0,2}_{jk}(a_2,a_3)w'_k(a_1) - P_1\alpha^{0,0,2}_{jk}(a_1,a_1)w'_k(a_3)\Big] \ .$$

The two coefficients $P_1\alpha^{0,1,2}_{\lambda,j}$, $P_1\alpha^{0,0,3}_{j}$ not discussed explicitly in the preceding corollary do not necessarily lie in the subspace $P_1S_2 \subseteq \mathbb{E}^G$ so cannot be expressed in terms of the vectors $\alpha^{k_1k_2}_{k_3}$, $P_1\alpha^{0,0,2}_{j_1j_2}$ as were all the other coefficients. On the other hand, it is quite natural to view these two coefficients as themselves being primitive vectors, since by Corollary 3 they together with P_1S_2 span the subspace $P_1S_3 \subseteq \mathbb{E}^G$, and their images under P_2 and P_2' have the standard forms as described

<u>Corollary 6</u>. For any points $z_i, a_i \in \widetilde{M}$

$$P_1\underline{\theta}_2(w(z_1+\cdots+z_4 - a_1 - \cdots - a_4))\ Q(z_1,\ldots,z_4\,;\,a_1,\ldots,a_4)$$

$$= \Sigma\, P_1\alpha^{1,3,0}_{\mu_1\lambda_1\lambda_2\lambda_3}\ v'_{\mu_1}(z_{\pi 1})u'_{\lambda_2}(z_{\pi 2})u'_{\lambda_3}(z_{\pi 3})u'_{\lambda_4}(z_{\pi 4})$$

$$+ \Sigma\, P_1\alpha^{0,4,0}_{\lambda_1\lambda_2\lambda_3\lambda_4}\ u'_{\lambda_1}(z_1)u'_{\lambda_2}(z_2)u'_{\lambda_3}(z_3)u'_{\lambda_4}(z_4)$$

$$+ \sum_{\substack{k>0\\ m<3}} \sum_{\pi} \sum_{\mu\lambda j} P_1\alpha^{m,\ell,k}_{\mu\lambda j}\ v'_{\mu_1}(z_{\pi 1})\cdots v'_{\mu_m}(z_{\pi m})\cdots w'_{j_4}(z_4)\ ,$$

where the summations extend over permutations $\pi \in S(m,\ell,k)$ and indices $\mu = 1,2,3,4$, $\lambda = 1,2,3$, $j = 1,\ldots,g$. Here

$$P_1\alpha^{2,2,0}_{\mu_1\mu_2\lambda_1\lambda_2} = 0\ ;$$

$$P_1\alpha^{1,3,0}_{11\lambda_1\lambda_2} = P_1\alpha^{1,3,0}_{1222} = P_1\alpha^{1,3,0}_{1333} = 0\ , \quad \text{all } \lambda_1,\lambda_2\ ,$$

$$P_1\alpha^{1,3,0}_{1223} = -P_1\alpha^{1,3,0}_{1233} = -2\sum_k \alpha^{k_1k_2}_{\ \ k_3}\ w'_{k_1}(a_2)w'_{k_2}(a_3)w'_{k_3}(a_4)\ ,$$

$P_1\alpha^{1,3,0}_{\mu\lambda_1\lambda_2\lambda_3}$ correspondingly for $\mu = 2,3$,

$$P_1\alpha^{1,3,0}_{4111} = P_1\alpha^{1,3,0}_{4222} = P_1\alpha^{1,3,0}_{4333} = P_1\alpha^{1,3,0}_{4123} = 0$$

$$P_1\alpha^{1,3,0}_{4112} = -P_1\alpha^{1,3,0}_{4122} = -P_1\alpha^{1,3,0}_{4113} = P_1\alpha^{1,3,0}_{4223} = P_1\alpha^{1,3,0}_{4133} =$$

$$= -P_1\alpha^{1,3,0}_{4233} = -2\sum_k \alpha^{k_1k_2}_{\ \ k_3}\ w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(a_3)\ ;$$

$$P_1\alpha^{0,4,0}_{111\lambda} = 0 , \quad \text{all} \quad \lambda ,$$

$$P_1\alpha^{0,4,0}_{1122} = 4\sum_k \alpha^{k_1k_2}_{k_3} w'_{k_2}(a_3)w'_{k_3}(a_4)[w'_{k_1}(a_1)w'_{a_3a_4}(a_2) + w'_{k_1}(a_2)w'_{a_3a_4}(a_1)]$$

$$+ 4\sum_k P_1\alpha^{0,0,2}_{k_1k_2}(a_3,a_4)w'_{k_1}(a_1)w'_{k_2}(a_2) ,$$

$$P_1\alpha^{0,4,0}_{1123} = 2\sum_k \alpha^{k_1k_2}_{k_3}[2w'_{k_1}(a_2)w'_{k_2}(a_3)w'_{k_3}(a_4)w'_{a_2a_3}(a_1) + w'_{k_1}(a_2)w'_{k_2}(a_3)w'_{k_3}(a_1)w'_{a_2a_3}(a_4)$$

$$- w'_{k_1}(a_2)w'_{k_2}(a_4)w'_{k_3}(a_1)w'_{a_2a_4}(a_3) - w'_{k_1}(a_3)w'_{k_2}(a_4)w'_{k_3}(a_1)w'_{a_3a_4}(a_2)]$$

$$+ 2\sum_k [P_1\alpha^{0,0,2}_{k_1k_2}(a_2,a_3)w'_{k_1}(a_1)w'_{k_2}(a_4) - P_1\alpha^{0,0,2}_{k_1k_2}(a_2,a_4)w'_{k_1}(a_1)w'_{k_2}(a_3)$$

$$-P_1\alpha^{0,0,2}_{k_1k_2}(a_3,a_4)w'_{k_1}(a_1)w'_{k_2}(a_2)] ,$$

and correspondingly for other indices.

It should be noted that the evident symmetries in the expansion of the preceding corollary can be used to obtain formulas for terms with indices other than those given explicitly in the statement of the corollary; for instance, from the symmetry $P_1\alpha^{0,4,0}_{2213}(a_1,a_2,a_3,a_4) = P_1\alpha^{0,4,0}_{1123}(a_2,a_1,a_3,a_4)$ and the formulas as given in the corollary it is possible to read off an explicit expression for this coefficient. Since the indices λ range only over the values 1,2,3 any term $P_1\alpha^{0,4,0}_{\lambda_1\lambda_2\lambda_3\lambda_4}$ must have at least two equal indices, so all these coefficients can be determined from those listed. It should also be noted that by this calculation the terms $P_1\alpha^{1,3,0}_{\mu\lambda}$ and $P_1\alpha^{0,4,0}_{\lambda}$ take values in the subspace $S_2 \subseteq \mathbb{C}^G$, but that is not the case for all the coefficients; the simplest way of determining just which of these coefficients do take values in S_2 is to apply the projection operator P_2 and analyze the result similarly.

6. The formula of Corollary 1 is reasonably well known, but those of Corollary 2 are possibly not so much so; the latter involve interesting additional phenomena, and are more typical of the general situation, so merit some further discussion. First it is evident from the principal formula of Corollary 1 that the subspace $S_2 \subseteq \mathbb{C}^G$ can be characterized alternatively as the span of the vectors $\underline{\underline{\theta}}_2(0)$, $\partial_{jk}\underline{\underline{\theta}}_2(0)$ for $j \leqq k$, $\alpha^{0,1,1}_{j}(a_1,a_2)$, $\alpha^{0,0,2}_{jk}(a_1,a_2)$ for $j \leqq k$, as the points a_1, a_2 independently over $\tilde{M}$; indeed the values $\underline{\underline{\theta}}_2(w(z_1 + z_2 - a_1 - a_2))$ clearly lie in the subspace of $\mathbb{C}^G$ spanned by these vectors, and since the coefficients of these vectors in the expansion of Corollary 2 are linearly independent functions of z_1, z_2 for any fixed a_1, a_2 the opposite inclusion holds as well. What is particularly interesting is that the extension from S_1 to S_2 is canonically split into two simple stages: first there is the extension S_1'/S_1 spanned by the vectors $\alpha^{k_1k_2}_{j}$, then the extension S_2/S_1' spanned by the vectors $\alpha^{k_1k_2}_{j_1j_2}$, and these vectors and hence the two separate extensions are uniquely determined. Indeed the projection P_1 establishes the isomorphism $S_1'/S_1 \cong P_1S_1'$, and in view of (5.3) the span of the vectors $P_1\alpha^{0,1,1}_{j}(a_1,a_2)$ is just the span of the vectors $\alpha^{k_1k_2}_{j}$; similarly the projection P_1' establishes the isomorphism $S_2/S_1' \cong P_1'S_2$, and in view of (5.4) the span of the vectors $P_1'\alpha^{0,0,2}_{jk}(a_1,a_2)$ is just the span of the vectors $\alpha^{k_1k_2}_{j_1j_2}$. The extension S_2/S_1 can thus be viewed as the span of the vectors $\alpha^{k_1k_2}_{j} \in P_1\mathbb{C}^G$ together with any representatives of the vectors $\alpha^{k_1k_2}_{j_1j_2} \in P_1'\mathbb{C}^G$, although this is no longer a canonical description since some choice of representatives has to be made.

To avoid at least some complications, considerations here will for the most part be limited to the forms (5.10) and (5.11) of Corollary 2;

the interest really lies in properties of the vectors $\alpha^{k_1k_2}_{j}$ and $\alpha^{k_1k_2}_{j_1j_2}$, although it is necessary to consider the holomorphic functions $P_1\alpha^{0,0,2}_{j_1j_2}(a_1,a_2)$ to some extent as well. The symmetries of these vectors in their indices have already been noted, and it is obvious from (5.10) that the functions $P_1\alpha^{0,0,2}_{j_1j_2}(a_1,a_2)$ are symmetric in the variables a_1,a_2 as well. The left-hand side of (5.10) is also symmetric under the interchange of the variables z and a, from which it readily follows that

$$(1)\quad \sum_j \Big[P_1\alpha^{0,0,2}_{j_1j_2}(a_1,a_2)w'_{j_1}(z_1)w'_{j_2}(z_2) - P_1\alpha^{0,0,2}_{j_1j_2}(z_1,z_2)w'_{j_1}(a_1)w'_{j_2}(a_2)\Big]$$

$$= \sum_{\pi,k} \alpha^{k_1k_2}_{k_3}\Big[w'_{k_1}(z_1)w'_{k_2}(z_2)w'_{k_3}(a_{\pi1})w'_{z_1z_2}(a_{\pi2})-w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(z_{\pi1})w'_{a_1a_2}(z_{\pi2})\Big].$$

Next the left-hand side of (5.10) vanishes if either $a_1=a_2$ or $z_1=z_2$, because of the presence of the terms $q(a_1,a_2)^2$ and $q(z_1,z_2)^2$, and from this it follows quite easily that

$$(2)\quad P_1\alpha^{0,0,2}_{j_1j_2}(a,a) = 0$$

and

$$(3)\quad \sum_j P_1\alpha^{0,0,2}_{j_1j_2}(a_1,a_2)w'_{j_1}(z)w'_{j_2}(z)$$

$$= -2\sum_k \alpha^{k_1k_2}_{k_3}\, w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(z)w'_{a_1a_2}(z) \quad ;$$

under the further projection operator P'_1 these last two identities reduce to

$$(4)\quad \sum_k \alpha^{k_1k_2}_{j_1j_2}\, w'_{k_1}(z)w'_{k_2}(z) = 0 \qquad \text{for all indices } j_1,j_2\,,$$

while taking the limit in (3) as z tends to a_2 leads by a simple calculation to the identity

$$(5)\quad \sum_j P_1 \alpha^{0,0,2}_{j_1 j_2}(a,z) w'_{j_1}(z) w'_{j_2}(z)$$

$$= 2 \sum_k \alpha^{k_1 k_2}_{k_3}\, w'_{k_1}(a) w'_{k_2}(z) w''_{k_3}(z) \ .$$

The terms on the right-hand side of (3) are evidently holomorphic quadratic differentials in the variable z, in the sense that they are vectors the components of which lie in $\Gamma(M, \mathcal{O}(\kappa^2))$ as functions of z, and (3) expresses these quadratic differentials as products of abelian differentials; in the same sense the expressions on either side in (5) are holomorphic cubic differentials in the variable z.

Rather more interesting is the alternative formula (5.12) determining the vectors $\alpha^{k_1 k_2}_{j}$, a formula that can be rewritten as

$$(6)\quad \partial_j P_1 \underline{\theta}_2(w(z-a)) = -q(z,a)^2 \sum_k \alpha^{k_1 k_2}_{j}\, w'_{k_1}(z) w'_{k_2}(a) \ .$$

In one sense the analogue of this formula for the vectors $\alpha^{k_1 k_2}_{j_1 j_2}$ is just (5.11), as already noted; another analogue, although one that does not determine these vectors completely, is the formula

$$(7)\quad \partial_{j_1 j_2} P'_1 \underline{\theta}_2(w(z-a)) = q(z,a)^2 \sum_k \left(\alpha^{j_1 k_1}_{j_2 k_2} + \alpha^{j_1 k_2}_{j_2 k_1} - \alpha^{k_1 k_2}_{j_1 j_2} \right) w'_{k_1}(z) w'_{k_2}(a) \ .$$

To demonstrate this multiply the principal formula of Corollary 5 by $q(z_1,a_1)^2 q(z_2,a_2)^2$, apply the differential operator $\partial^2/\partial z_1 \partial z_2$, and take the limit in the result as $z_1 \longrightarrow a_1$ and $z_2 \longrightarrow a_2$. As before, the only possibly nontrivial terms on the right-hand side are those involving the functions $u'_1(z_1) u'_2(z_2)$, and a straightforward calculation leads to the formula

$$(8)\quad \sum_j \partial_{j_1 j_2} P_1 \underline{\underline{\theta}}_2(w(z_3-a_3))w'_{j_1}(a_1)w'_{j_2}(a_2)q(z_3,a_3)^{-2}$$

$$+ \Sigma_j \partial_j P_1 \underline{\underline{\theta}}_2(w(z_3-a_3))[w'_j(a_1)w'_{z_3,a_3}(a_2) + w'_j(a_2)w'_{z_3,a_3}(a_1)]q(z_3,a_3)^{-2}$$

$$= \Sigma_j \; P_1\alpha^{0,2,1}_{12j}(a_1,a_2,a_3)w'_j(z_3) \quad ;$$

upon applying the further projection operator P'_1 and using the explicit form for the last coefficient as given in Corollary 5 this becomes

$$(9)\quad \sum_j \partial_{j_1 j_2} P'_1 \underline{\underline{\theta}}(w(z_3-a_3))w'_{j_1}(a_1)w'_{j_2}(a_2)q(z_3,a_3)^{-2}$$

$$= \sum_j \Big[P'_1\alpha^{0,0,2}_{j_1 j_2}(a_1,a_3)w'_{j_1}(a_2) + P'_1\alpha^{0,0,2}_{j_1 j_2}(a_2,a_3)w'_{j_1}(a_1) - P'_1\alpha^{0,0,2}_{j_1 j_2}(a_1,a_2)w'_{j_1}(a_3)\Big] w'_{j_2}(z_3),$$

and using the explicit form for the coefficients $P'_1\alpha^{0,0,2}_{j_1 j_2}$ from Corollary 2 leads relatively directly to the desired formula.

7. The vector-valued second-order theta function $\underline{\underline{\theta}}_2$ is an even function, so only even-order terms appear in its Taylor expansion at the origin in $\mathbb{C}^g$. Another interpretation of Corollary 1 is that the values $\underline{\underline{\theta}}_2(w(z-a))$ all lie in the subspace $S_1 \subseteq \mathbb{C}^G$ spanned by the Taylor coefficients of $\underline{\underline{\theta}}_2$ at the origin of orders 0 and 2. That means that the Taylor coefficients of $\underline{\underline{\theta}}_2(w(z-a))$ as a function of z at $z=0$, which are of course just some particular linear combinations of the Taylor coefficients of $\underline{\underline{\theta}}_2$ at the origin, must also lie in this subspace S_1; the formulas expressing this condition are particularly interesting. Perhaps the easiest approach to these formulas is to rewrite Corollary 1 as the identity $P_1\underline{\underline{\theta}}_2(w(z-a)) = 0$, apply a differential operator $\partial^k/\partial z^k$, and set $z=a$. The first non-trivial case is that in which $k=4$, and the resulting formula is easily seen to have the form

$$(1) \qquad \sum_j \partial_{j_1\cdots j_4} P_1\underline{\underline{\theta}}_2(0)\cdot w'_{j_1}(a)w'_{j_2}(a)w'_{j_3}(a)w'_{j_4}(a) = 0 .$$

Of course it is more informative to apply the differential operator directly to the formula of Corollary 1 and then set $z=a$; the result is a rather more complicated formula, expressing the left-hand side of (1) without the projection operator P_1 as a linear combination of the vectors $\underline{\underline{\theta}}_2(0)$ and $\partial_{j_1j_2}\underline{\underline{\theta}}_2(0)$, but reduces to (1) upon applying the projection operator P_1. This is another instance of the simplifications that can be effected by suitable projection. The formula before projection is that given in Corollary 3 in [8] or in equation (1.2) in [2]; it is a quite well known formula, equivalent to the condition that a corresponding expression in terms of first-order theta functions be a solution of the KP equation, as discussed in detail in [2]. The simpler formula is also of interest though, in other respects. For example, since (1) holds for all points $a \in \tilde{M}$ it can be viewed as asserting that the canonical curve associated to M is contained in the locus in $\mathbb{P}^{g-1}$ defined by the quartic equations

$$(2) \qquad \sum_j \partial_{j_1 \cdots j_4} P_1 \underline{\underline{\theta}}_2(0) \cdot x_{j_1} x_{j_2} x_{j_3} x_{j_4} = 0 ,$$

where $(x_1, \ldots, x_g)$ are homogeneous coordinates in $\mathbb{P}^{g-1}$. In many cases the quartic equations (2) describe precisely the canonical curve, as observed by van Geemen and van der Geer [4]; it may be worth digressing briefly here to include a proof of this.

Theorem 6. If the Riemann surface M is neither trigonal nor a plane quintic then the locus (2) is precisely the canonical curve representing M.

Proof. The well known addition theorem for theta functions [10] can be written in the form

$$(3) \qquad {}^t\underline{\underline{\theta}}_2(t) \cdot \underline{\underline{\theta}}_2(w) = \theta(t+w)\theta(t-w) ,$$

where the left-hand side is the indicated matrix product, or equivalently is the scalar product of the two vectors, and $\theta(t) = \theta[0|0](t;\Omega)$ is the standard first-order theta function. Applying the differential operators $\partial^2/\partial w_{j_1} \partial w_{j_2}$ and $\partial^4/\partial w_{j_1} \cdots \partial w_{j_4}$ to (3) and then setting $w = 0$ in (3) and the derived formulas is readily seen to yield the identities

$$(4) \qquad {}^t\underline{\underline{\theta}}_2(t) \cdot \underline{\underline{\theta}}_2(0) = \theta(t)^2$$

$${}^t\underline{\underline{\theta}}_2(t) \cdot \partial_{j_1 j_2} \underline{\underline{\theta}}_2(0) = 2\theta(t)\partial_{j_1 j_2}\theta(t) - 2\partial_{j_1}\theta(t)\partial_{j_2}\theta(t)$$

$${}^t\underline{\underline{\theta}}_2(t) \cdot \partial_{j_1 \cdots j_4} \underline{\underline{\theta}}_2(0) = 2\theta(t)\partial_{j_1 \cdots j_4}\theta(t) - 2[\partial_{j_1}\theta(t)\partial_{j_2 j_3 j_4}\theta(t)]_4 + 2[\partial_{j_1 j_2}\theta(t)\partial_{j_3 j_4}\theta(t)]_3 ,$$

where $[\ ]_n$ indicates the obvious symmetrization of the expression in the brackets, a sum over the n permutations needed to yield a fully symmetric expression. Since S_1 is the span of the vectors $\underline{\underline{\theta}}_2(0)$ and $\partial_{j_1 j_2}\underline{\underline{\theta}}_2(0)$ it is clear from the first two of these identities that ${}^t\underline{\underline{\theta}}_2(t)\cdot S_1 = 0$ precisely when $\theta(t) = \partial_j\theta(t) = 0$ for all j, hence precisely when t represents a point in the singular locus Θ^1 of the subvariety $\Theta = \{t \in J : \theta(t) = 0\}$; and since $\xi - P_1\xi \in S_1$ for any vector $\xi \in \mathbb{C}^G$ it follows that ${}^t\underline{\underline{\theta}}_2(t)\cdot\xi = {}^t\underline{\underline{\theta}}_2(t)\cdot P_1\xi$ for any point $t \in \Theta^1$ and any vector $\xi \in \mathbb{C}^G$. Using this observation and the third identity in (4) leads readily to the result that

$$(5) \qquad {}^t\underline{\underline{\theta}}_2(t)\cdot\sum_j \partial_{j_1\cdots j_4} P_1\underline{\underline{\theta}}_2(0)x_{j_1}x_{j_2}x_{j_3}x_{j_4} = 6\left(\sum_j \partial_{j_1 j_2}\theta(t)x_{j_1}x_{j_2}\right)^2$$

for any point $t \in \Theta^1$. Thus whenever $x \in \mathbb{P}^{G-1}$ is in the locus (2) then $\sum_j \partial_{j_1 j_2}\theta(t)x_{j_1}x_{j_2} = 0$ for all points $t \in \Theta^1$, and it follows from the Enriques-Babbage theorem [1] and the theorem of Green [5] that x lies in the canonical curve, in view of the hypotheses of the present theorem. Any point of the canonical curve is in the locus (2) as a consequence of (1), and the proof is thereby concluded.

The preceding proof actually gives a bit more than was stated in the theorem. Equation (1) of course implies that the canonical curve is contained in the locus (2) for an arbitrary Riemann surface M. Equation (5) implies that the locus (2) is contained in the locus $X = \{t \in J : \sum_j \partial_{j_1 j_2}\theta(t)x_{j_1}x_{j_2} = 0$ for all $t \in \Theta^1\}$ for an arbitrary Riemann surface M, and the theorem of Green asserts that X is the set of zeros of all quadrics through the canonical curve for an arbitrary Riemann surface M. If the vectors $\underline{\underline{\theta}}_2(t)$ for $t \in \Theta^1$ actually span the linear subspace $S_1^{\perp} = \{\xi \in \mathbb{C}^G : {}^t\xi \cdot S_1 = 0\}$ then it follows from (5) that (2) is precisely the locus X.

The result (1) was obtained rather simply from Corollary 1, and as might be expected there are various extensions of this result that can be derived similarly from the other corollaries in section 5. The only such results that will be considered in detail here are the fourth-order equations involving the function $P_1\underline{\underline{\theta}}_2(w)$, and they are as follows.

<u>Theorem 7</u>. For any points $a_i \in \tilde{M}$

(6) $$\sum_j \partial_{j_1\cdots j_4} P_1\underline{\underline{\theta}}_2(0)\cdot w'_{j_1}(a_1)w'_{j_2}(a_1)w'_{j_3}(a_1)w'_{j_4}(a_2)$$
$$= 6\sum_k \alpha^{k_1k_2}_{k_3}\cdot w'_{k_1}(a_1)w''_{k_2}(a_1)w'_{k_3}(a_2) \;,$$

(7) $$\sum_j \partial_{j_1\cdots j_4} P_1\underline{\underline{\theta}}_2(0)\cdot w'_{j_1}(a_1)w'_{j_2}(a_1)w'_{j_3}(a_2)w'_{j_4}(a_2)$$
$$= 4\sum_k P_1\alpha^{0,0,2}_{k_1k_2}(a_1,a_2)\cdot w'_{k_1}(a_1)w'_{k_2}(a_2) \;,$$

(8) $$\sum_j \partial_{j_1\cdots j_4} P_1\underline{\underline{\theta}}_2(0)\cdot w'_{j_1}(a_1)w'_{j_2}(a_2)w'_{j_3}(a_3)w'_{j_4}(a_3)$$
$$= -2\sum_k \alpha^{k_1k_2}_{k_3}\, w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(a_3)w'_{a_1a_2}(a_3)$$
$$+2\sum_k \left[P_1\alpha^{0,0,2}_{k_1k_2}(a_1,a_3)w'_{k_1}(a_2) + P_1\alpha^{0,0,2}_{k_1k_2}(a_2,a_3)w'_{k_1}(a_1)\right]w'_{k_2}(a_3) \;,$$

(9) $$\sum_j \partial_{j_1\ldots j_4} P_1\underline{\theta}_2(0)\cdot w'_{j_1}(a_1)w'_{j_2}(a_2)w'_{j_3}(a_3)w'_{j_4}(a_4)$$

$$= -6 \sum_{\pi\in S(2,1)} \sum_k \alpha^{k_1k_2}_{k_3}\cdot w'_{k_1}(a_{\pi 1})w'_{k_2}(a_{\pi 2})w'_{a_{\pi 1},a_{\pi 2}}(a_{\pi 3})w'_{k_3}(a_4)$$

$$+2 \sum_{\pi\in S(2,1)} \sum_k \Big[2P_1\alpha^{0,0,2}_{k_1k_2}(a_{\pi 3},a_4)\cdot w'_{k_1}(a_{\pi 1})w'_{k_2}(a_{\pi 2})$$

$$-P_1\alpha^{0,0,2}_{k_1k_2}(a_{\pi 1},a_{\pi 2})\cdot w'_{k_1}(a_{\pi 3})w'_{k_2}(a_4)\Big]$$

$$= -\frac{3}{4} \sum_{\pi\in S(4)} \sum_k \alpha^{k_1k_2}_{k_3}\cdot w'_{k_1}(a_{\pi 1})w'_{k_2}(a_{\pi 2})w'_{k_3}(a_{\pi 3})w'_{a_{\pi 1},a_{\pi 2}}(a_{\pi 4})$$

$$+\frac{1}{4} \sum_{\pi\in S(4)} \sum_k P_1\alpha^{0,0,2}_{k_1k_2}(a_{\pi 1},a_{\pi 2})\cdot w'_{k_1}(a_{\pi 3})w'_{k_2}(a_{\pi 4}) \ .$$

Proof. First consider the form (5.10) of Corollary 2, multiply that equation by $\prod_\nu q(z_\nu,a_\nu)^2$, apply the differential operator $\partial^4/\partial z_1^3\partial z_2$, and set $z_i = a_i$. Since the first nontrivial Taylor coefficients of the function $P_1\underline{\theta}_2(w)$ are $P_1\partial_{j_1\ldots j_4}\underline{\theta}_2(0)$ the only possibly nontrivial terms on the left-hand side are those in which all the differentiation is applied to $P_1\underline{\theta}_2(w)$, while the only possibly nontrivial terms on the right-hand side are those involving the function $w'_{a_1a_2}(z_2)$; this leads by a simple calculation to (6). Next multiply (5.10) by $\prod_\nu q(z_\nu,a_\nu)^2$, apply the differential operator $\partial^4/\partial z_1^2\partial z_2^2$, and set $z_i = a_i$. This time the terms on the right-hand side involving the function $w'_{a_1a_2}(z)$ vanish in view of (2.5), so the only possibly nontrivial terms are those involving just the holomorphic Abelian differentials; this leads by another simple calculation to (7). Turn then to Corollary 5, multiply the principal formula by $\prod_\nu q(z_\nu,a_\nu)^2$, apply the differential operator $\partial^4/\partial z_1\partial z_2\partial z_3^2$, and set $z_i = a_i$. As before the only possibly nontrivial terms on the right-hand side are those that involve just the functions $u_1'(z_1)u_2'(z_2)w_j'(z_3)$, so that side becomes $2\,\Sigma_j\, P_1\alpha^{0,2,1}_{12j}(a_1,a_2,a_3)w_j'(a_3)$; using the explicit form for this coefficient as given in Corollary 5 and simplifying the result by applying (6.3) readily leads to (8). Finally, multiply the principal formula of Corollary 6 by $\prod_\nu q(z_\nu,a_\nu)^2$, apply the differential operator $\partial^4/\partial z_1\partial z_2\partial z_3\partial z_4$, and set $z_i = a_i$. The only possibly nontrivial terms on the right-hand side are those that involve just the functions $u_1'(z_1)u_2'(z_2)u_3'(z_3)u_\lambda'(z_4)$ for any value λ , so that side reduces to $-\sum_{\lambda=1}^{3} P_1\alpha^{0,4,0}_{123\lambda}$; using the explicit forms for these coefficients as given in Corollary 6 leads by a straightforward calculation to the first equality in (9). The variable a_4 plays a distinguished role on the right-hand side, while the left-hand side is symmetric in all variables a_i ; that means that the right-hand side must also be fully symmetric, as also follows from (6.3). Either using (6.3) appropriately, or more simply just symmetrizing the right-hand side, yields the second equality in (9), the symmetric version, and thereby concludes the proof.

Although the preceding results are of sufficient interest that it is worth seeing them written out explicitly, it should also be noted that they are all in fact simply consequences of the first identity (9); for by using (6.2), (6.3) and (6.5) it is quite easy to verify that (8) follows from (9) upon setting $a_4 = a_3$, that (6) and (7) follow in turn from (8) upon setting $a_2 = a_3$ and $a_2 = a_1$ respectively, and finally that (1) follows from either (6) or (7) upon setting $a_1 = a_2$. There is a further series of identities that arise upon applying the projection operator P_1' to the results of the preceding theorem; again though all are consequences of a single identity, and in this case such simple consequences that it is really only worth writing out that single identity explicitly.

Theorem 8. With the notation as before,

$$\partial_{j_1\cdots j_4} P_1' \underline{\underline{\theta}}_2(0) = 2\left(\alpha^{j_1 j_2}_{j_3 j_4} + \alpha^{j_1 j_3}_{j_2 j_4} + \alpha^{j_1 j_4}_{j_2 j_3}\right) \quad .$$

Proof. Applying the projection operator P_1' to the first identity (9) and using (5.4) and (6.4) yields the result that

$$\sum_j \partial_{j_1\cdots j_4} P_1' \underline{\underline{\theta}}_2(0)\cdot w'_{j_1}(a_1) w'_{j_2}(a_2) w'_{j_3}(a_3) w'_{j_4}(a_4)$$

$$= 2 \sum_{jk} \alpha^{k_1 k_2}_{j_1 j_2} [\, w'_{j_1}(a_1) w'_{j_2}(a_2) w'_{k_1}(a_3) w'_{k_2}(a_4)$$

$$+ w'_{j_1}(a_1) w'_{j_2}(a_3) w'_{k_1}(a_2) w'_{k_2}(a_4)$$

$$+ w'_{j_1}(a_2) w'_{j_2}(a_3) w'_{k_1}(a_1) w'_{k_2}(a_4)] \quad ,$$

from which the desired result is an immediate consequence.

It is perhaps worth noting explicitly here that applying the projection operator P_1' to (6) yields in analogy to (1) the identity

$$(10) \qquad \sum_j \partial_{j_1\cdots j_4} P_1' \underline{\underline{\theta}}_2(0)\cdot w'_{j_1}(a_1) w'_{j_2}(a_1) w'_{j_3}(a_1) w'_{j_4}(a_2) = 0 \;;$$

this also follows immediately from the preceding theorem and (6.4) though.

Let $T_2 \subseteq \mathbb{C}^G$ be the subspace spanned by the vectors $\underline{\underline{\theta}}_2(0)$, $\partial_{j_1 j_2}\underline{\underline{\theta}}_2(0)$, $\partial_{j_1\cdots j_4}\underline{\underline{\theta}}_2(0)$, so that $T_2/S_1 \subseteq \mathbb{C}^G/S_1$ is the subspace spanned by the vectors $\partial_{j_1\cdots j_4} P_1 \underline{\underline{\theta}}_2(0)$. It is clear from (9) that $T_2/S_1 \subseteq S_2/S_1$, although the reversed inclusion is not evident. If the cubic differentials that appear implicitly in (6), those that will be defined explicitly in (8.7) and discussed there for Riemann surfaces of genus $g=4$, are linearly independent, then it follows readily from (9) that $S_1'/S_1 \subseteq T_2/S_1$; these differentials are linearly independent for low enough genus, but cannot be for high enough genus since there are simply too many of them. Theorem 8 indicates the extent to which it is true that $S_2/S_1' \subseteq T_2/S_1'$.

8. The interest of the preceding results can perhaps best be illustrated by examining them more closely for curves of low genus. To begin with a curve M of genus $g=3$, recall that if M is not hyperelliptic the 6 distinct products of pairs of Abelian differentials are a basis for the space of quadratic differentials, the 10 distinct products of triples of Abelian differentials are a basis for the space of cubic differentials, and the 15 distinct products of quadruples of Abelian differentials span the space of quartic differentials but satisfy a single nontrivial linear equation, which yields a quadric equation

$$(1) \qquad f_4(x) = \sum_j c_{j_1 \dots j_4} x_{j_1} \cdots x_{j_4} = 0$$

that defines the canonical curve in $\mathbb{P}^2$; on the other hand if M is hyperelliptic the products of pairs of Abelian differentials only span a 5-dimensional subspace of the space of quadratic differentials, so there is a single nontrivial linear relation among these products, and that yields a quadratic equation

$$(2) \qquad f_2(x) = \sum_j c_{j_1 j_2} x_{j_1} x_{j_2} = 0$$

that defines the canonical curve in $\mathbb{P}^2$, a rational normal curve. In either case the second-order theta function $\underline{\underline{\theta}}_2(w)$ takes values in the space $\mathbb{C}^8$, and the vector $\underline{\underline{\theta}}_2(0)$ together with the 6 vectors $\partial_{j_1 j_2} \underline{\underline{\theta}}_2(0)$ for $j_1 \leqq j_2$ are a basis for the 7-dimensional subspace $S_1 \subseteq \mathbb{C}^8$ spanned by the values $\underline{\underline{\theta}}_2(w(z-a))$. The projection operator P_1 with kernel S_1 can then be viewed as a linear mapping $P_1 : \mathbb{C}^8 \longrightarrow \mathbb{C}^1$, and $P_1 \underline{\underline{\theta}}_2(w)$ is an ordinary complex-valued function, a particular second-order theta function. The subspace $S_2 \subseteq \mathbb{C}^8$ spanned by the values $\underline{\underline{\theta}}_2(w(z_1+z_2-a_1-a_2))$ must be all of $\mathbb{C}^8$, since the points $w(z_1+z_2-a_1-a_2)$ cover the entire Jacobi variety and the 8 functions comprising $\underline{\underline{\theta}}_2(w)$ are linearly independent; the extension S_2/S_1 is consequently one-dimensional, so at least one of the vectors $\alpha^{k_1k_2}_{j}$, $\alpha^{k_1k_2}_{j_1j_2}$ must be nontrivial. Actually, of course, the vectors $\alpha^{k_1k_2}_{j}$ are just complex numbers, since S_2/S_1 is one-dimensional, and since $\alpha^{k_1k_2}_{j}$ is skew-symmetric in its indices and they range over the values 1,2,3 there is really just the single complex constant α^{12}_{3}. If $\alpha^{12}_{3} \neq 0$ it generates the extension S_2/S_1 ; in this case $S_1' = S_2$ and P_1' is the zero mapping, so that $\alpha^{k_1k_2}_{j_1j_2} = 0$. On the other hand, if $\alpha^{12}_{3} = 0$ then $S_1' = S_1$ and S_2/S_1' is one-dimensional; the vectors $\alpha^{k_1k_2}_{j_1j_2}$ lie in this space, so are really just complex numbers, and at least one of them must be nonzero. With this in mind the situation can be summarized as follows.

<u>Theorem 9</u>. If M is a surface of genus $g=3$ then M is hyperelliptic precisely when $\alpha^{12}_{3} = 0$. The equation (7.2) always describes the canonical curve geometrically; if M is not hyperelliptic (7.2) is precisely the defining equation (1) of the canonical curve, while if M is hyperelliptic (7.2) is the square of the defining equation (2) of the canonical curve. The equation $P_1\underline{\underline{\theta}}_2(w) = 0$ always describes geometrically the surface W_1-W_1 in the Jacobi variety; if M is not hyperelliptic $P_1\underline{\underline{\theta}}_2(w)$ is precisely the defining equation of W_1-W_1 , while if M is hyperelliptic $P_1\underline{\underline{\theta}}_2(w)$ is the square of the defining equation of W_1-W_1 .

<u>Proof.</u> First suppose that $\alpha^{12}_{3} = 0$, so that $S_1 = S_1'$ and not all of the constants $\alpha^{k_1k_2}_{j_1j_2}$ vanish. Then (6.4) is a nontrivial linear relation

between products of Abelian differentials, so M must be hyperelliptic, and in view of the known symmetries and the uniqueness of this linear relation it must be the case that $\alpha^{k_1k_2}_{j_1j_2} = \alpha\, c_{j_1j_2} c_{k_1k_2}$ where α is some nonzero complex constant and $c_{j_1j_2}$ are the coefficients of the defining equation (2) for the canonical curve. Furthermore, since $S_1 = S_1'$ then $P_1 = P_1'$, and it follows almost immediately from Theorem 8 that

$$\sum_j \partial_{j_1\cdots j_4} P_1\underline{\theta}_2(0)\cdot x_{j_1} x_{j_2} x_{j_3} x_{j_3} = 6\,\alpha\, f_2(x)^2 ,$$

so that this is the square of the defining equation for the canonical curve.

Next suppose that $\alpha^{12}_{3} \neq 0$, and note that equation (6) of Theorem 7 can be written

$$\text{(3)} \qquad \sum_j \partial_{j_1\cdots j_4} P_1\underline{\theta}_2(0)\cdot w'_{j_1}(a) w'_{j_2}(a) w'_{j_3}(a) w'_{j_4}(b)$$

$$= 6\,\alpha^{12}_{3} \det \begin{Bmatrix} w_1'(a) & w_1''(a) & w_1'(b) \\ w_2'(a) & w_2''(a) & w_2'(b) \\ w_3'(a) & w_3''(a) & w_3'(b) \end{Bmatrix} .$$

For any polynomial $f(x)$ it is, of course, the case that $\partial f(x)/\partial x_i = 0$ for some point x and all indices i precisely when $\Sigma_i\, w_i'(b)\, \partial f(x)/\partial x_i = 0$ for that point x and all points $b \in M$, since the canonical curve can never lie in a proper hyperplane. In particular, for the quartic polynomial $f(x) = \Sigma_j\, \partial_{j_1\cdots j_4} P_1\underline{\theta}_2(0)\cdot x_{j_1} \cdots x_{j_4}$ and a point $x = w(a)$ on the canonical curve it is then an easy consequence of (3) that $\partial f(w(a))/\partial x_i = 0$ for all indices i precisely when the vectors $\{w_i'(a)\}$ and $\{w_i''(a)\}$ are linearly dependent; that is in turn the case precisely when there are two linearly independent Abelian differentials vanishing to the second order at a, hence precisely when M is hyperelliptic and a is a Weierstrass point. Thus if M is not hyperelliptic then $f(x)$ vanishes to the first order on the canonical curve, so that $f(x) = 0$ is precisely the defining equation for the canonical curve. On the other hand, if M is hyperelliptic then since $f(x)$ vanishes on the canonical curve necessarily $f(x) = f_2(x)\, g_2(x)$, where $f_2(x)$ is the defining equation (2) for the canonical curve and $g_2(x)$ is some quadratic polynomial. Thus $f_2(x) = \partial f(x)/\partial x_i = 0$ for all indices i precisely when $f_2(x) = g_2(x) = 0$, hence either for all points of the canonical curve or for at most four points on the canonical curve; but it was just noted that $f_2(x) = \partial f(x)/\partial x_i = 0$ for all indices i precisely at the eight distinct points of the canonical curve that are the images of the Weierstrass points on M, and that is a contradiction. Consequently, if $\alpha^{12}_3 \neq 0$ the curve M cannot be hyperelliptic.

A proof that $P_{1\underline{\theta}2}(w)$ vanishes precisely on the surface $W_1 - W_1$ in the Jacobi variety, indeed vanishes to first order if M is not hyperelliptic and to second order if M is hyperelliptic, was given in [7]; it is an argument of a quite different sort, but does not need to be repeated here. It is worth noting, though, that this result together with (6.6) gives an alternative proof of the result that M is hyperelliptic precisely when $\alpha^{12}_3 = 0$.

Next for a curve of genus $g = 4$ recall that if M is not hyperelliptic the ten distinct products of pairs of Abelian differentials span the 9-dimensional space of quadratic differentials and satisfy a single nontrivial linear equation, which yields a quadratic polynomial

$$(4) \qquad f(x) = \sum_{j} c_{j_1 j_2} x_{j_1} x_{j_2}$$

vanishing on the canonical curve, while the 20 distinct products of triples of Abelian differentials span the 15-dimensional space of cubic differentials and satisfy 5 independent linear equations, which yield 5 linearly independent cubic polynomials vanishing on the canonical curve, the polynomials $x_j f(x)$ for $j = 1,\ldots,4$ together with another cubic polynomial

(5) $$g(x) = \sum_j c_{j_1 j_2 j_3} x_{j_1} x_{j_2} x_{j_3} \quad ;$$

the polynomials f and g define the canonical curve in $\mathbb{P}^3$. On the other hand, if M is hyperelliptic the products of pairs of Abelian differentials only span a 7-dimensional subspace of the space of quadratic differentials, so there are 3 independent linear relations among these 10 products, and that leads to 3 quadratic polynomials

(6) $$f_i(x) = \sum_j c^i_{j_1 j_2} x_{j_1} x_{j_2}$$

that define the canonical curve in $\mathbb{P}^3$.

Some further and possibly less standard results will be needed as well, so will be discussed here before turning to the theta functions themselves. For any indices $1 \leqq i, j \leqq 4$ the expressions

(7) $$\sigma_{ij}(z) = \det \begin{Bmatrix} w_i'(z) & w_i''(z) \\ w_j'(z) & w_j''(z) \end{Bmatrix}$$

are cubic differentials, in the sense that $\sigma_{ij} \in \Gamma(M, \mathcal{O}(\kappa^3))$, and since they are evidently skew-symmetric in the indices i, j there are really 6 of them to be considered.

Lemma 2. If M is a hyperelliptic Riemann surface of genus $g=4$ then $\Gamma(M, \mathcal{O}(\kappa^3))$ is the direct sum of the 5-dimensional linear subspace spanned by the expressions $\sigma_{ij}(z)$ and the 10-dimensional linear subspace spanned by products of triples of the Abelian differentials $w_j'(z)$.

Proof. Since it is quite well known that the subspace of $\Gamma(M, \mathcal{O}(\kappa^3))$ spanned by products of triples of Abelian differentials is 10-dimensional, it is only necessary to show that the expressions $\sigma_{ij}(z)$ span a 5-dimensional subspace of $\Gamma(M, \mathcal{O}(\kappa^3))$ and that no nontrivial element of this subspace can be written as a linear combination of products of triples of Abelian differentials. These assertions are easily seen to be independent of the particular basis chosen for the space of Abelian differentials, and for the proof it is convenient to use another basis. In particular, represent M as the Riemann surface of the function $f(z) = \left[z \prod_{j=1}^{2g+1} (z-a_j) \right]^{1/2}$, where $0, a_1, \ldots, a_{2g+1}$ are distinct complex numbers, and take the basis

$$w_1'(z)\,dz = \frac{dz}{f(z)}, \quad w_2'(z)\,dz = \frac{z\,dz}{f(z)}, \quad w_3'(z)\,dz = \frac{z^2\,dz}{f(z)}, \quad w_4'(z)\,dz = \frac{z^3\,dz}{f(z)}.$$

In terms of the local coordinate $t = cz^{1/2}$ at the point of M lying over the origin $z=0$ the functions $w_j'(t)$ evidently have local power series expansions of the form

$$\begin{aligned}(8)\qquad w_1'(t) &= 1+c_1t+c_2t^2+\cdots, & w_2'(t) &= t^2+c_1t^3+c_2t^4+\cdots, \\ w_3'(t) &= t^4+c_1t^5+c_2t^6+\cdots, & w_4'(t) &= t^6+c_1t^7+c_2t^8+\cdots,\end{aligned}$$

if the constant $c \neq 0$ is chosen appropriately, and using this and the definition (7) it follows easily that the functions $\sigma_{ij}(t)$ have local power series expansions of the form

$$
\begin{aligned}
(9) \qquad \sigma_{12}(t) &= 2t + 4c_1t^2 + \cdots \\
\sigma_{13}(t) &= 4t^3 + 8c_1t^4 + \cdots \\
\sigma_{14}(t) &= 6t^5 + 12c_1t^6 + \cdots \\
\sigma_{23}(t) &= 2t^5 + 4c_1t^6 + \cdots \\
\sigma_{24}(t) &= 4t^7 + 8c_1t^8 + \cdots \\
\sigma_{34}(t) &= 2t^9 + 4c_1t^{10} + \cdots \quad .
\end{aligned}
$$

Inspection of these expansions suggests that $\sigma_{14}(t) = 3\sigma_{23}(t)$, which can readily be verified by observing that $\sigma_{14}(t) = 3\sigma_{23}(t) = 6t^5w_1'(t)^2$, and shows that after deleting $\sigma_{23}(t)$ the remaining 5 functions are linearly independent. Furthermore, any nonzero linear combination of these 5 functions has a power series expansion in t that begins with a term of odd order, while the space spanned by products of triples of the functions $w_j'(t)$ has a basis consisting of the functions $t^{2\nu}\, w_1'(t)^3$ for $\nu = 0,\ldots,9$ and any nonzero element of this space must consequently have a power series expansion in t that begins with a term of even order. These observations suffice to conclude the proof.

For a non-hyperelliptic surface the situation is quite different, since every element of $\Gamma(M, \mathcal{O}(\kappa^3))$ can be written as a linear combination of products of triples of the Abelian differentials $w_j'(z)$; thus there must be some homogeneous cubic polynomials $\tilde{\sigma}_{ij}(x) \in \mathbb{C}[x_1,\ldots,x_4]$ such that

$$(10) \qquad \sigma_{ij}(z) = \tilde{\sigma}_{ij}(w'(z)) \ .$$

These polynomials are of course only determined up to arbitrary elements in the ideal of the canonical curve, the ideal generated by the polynomials f and g ; on the other hand, they must be determined to that extent by the polynomials f and g alone, since the latter define the canonical curve and hence determine the Riemann surface together with the canonical Abelian differentials. To describe these polynomials it is convenient to introduce the polarized forms of the polynomials f and g , namely the multilinear functions

$$(11)\qquad f(x,y) = \sum_j c_{j_1 j_2}\, x_{j_1} y_{j_2}\ , \quad g(x,y,z) = \sum_j c_{j_1 j_2 j_3}\, x_{j_1} y_{j_2} z_{j_3}\ ,$$

and to write

$$(12)\qquad f(x,y) = \sum_j a_j(x) y_j\ , \quad g(x,x,y) = \sum_j b_j(x) y_j\ ,$$

where $a_j(x)$ are linear polynomials and $b_j(x)$ are quadratic polynomials.

<u>Lemma 3</u>. If M is a non-hyperelliptic Riemann surface of genus $g=4$ then the 6 expressions $\sigma_{ij}(z)$ for $i<j$ are linearly independent elements of $\Gamma(M, \mathcal{O}(\kappa^3))$; the polynomials $\tilde{\sigma}_{ij}(x)$ can be taken as

$$\tilde{\sigma}_{12} = c(a_3b_4 - a_4b_3)\ , \quad \tilde{\sigma}_{13} = c(-a_2b_4 + a_4b_2)\ , \quad \tilde{\sigma}_{14} = c(a_2b_3 - a_3b_2)$$
$$\tilde{\sigma}_{23} = c(a_1b_4 - a_4b_1)\ , \quad \tilde{\sigma}_{24} = c(-a_1b_3 + a_3b_1)\ , \quad \tilde{\sigma}_{34} = c(a_1b_2 - a_2b_1)$$

for some nonzero complex constant c .

<u>Proof</u>. The proof of the first assertion of this lemma is much the same as the proof of the corresponding result in the preceding lemma, by examining the orders at a Weierstrass point of an appropriately chosen basis for the Abelian differentials. As shown by T. Kato in [9], any non-hyperelliptic surface of genus 4 has a Weierstrass point for which the first non-gap is 4, hence for which the gap sequence is either (1,2,3,5), (1,2,3,6), or (1,2,3,7). For the natural basis for the space of Abelian differentials

associated to any of these sequences the orders of the expressions σ_{ij} at the Weierstrass point of interest can very readily be calculated directly from the defining equation (7), and the results are as tabulated here:

	(1,2,3,5)	(1,2,3,6)	(1,2,3,7)
order of σ_{12} =	0	0	0
" σ_{13} =	1	1	1
" σ_{14} =	3	4	5
" σ_{23} =	2	2	2
" σ_{24} =	4	5	6
" σ_{34} =	5	6	7

An inspection of this table shows that the 6 functions σ_{ij} are linearly independent in any of these cases. It is perhaps worth noting in passing that this argument does not work at a point other than a Weierstrass point, since the orders of the expressions σ_{ij} are not then distinct (although there are 5 distinct orders so that this can be used alternatively to show that on any Riemann surface these expressions span at least a 5-dimensional space), nor does it work for all Weierstrass points (for when the gap sequence is (1,2,4,5) it again only shows that these expressions span at least a 5-dimensional space.)

For the remainder of the proof it is convenient to consider in place of the canonical curve in $\mathbb{P}^3$ the cone over the canonical curve, the holomorphic subvariety $V = \{x \in \mathbb{C}^4 : f(x) = g(x) = 0\} \subseteq \mathbb{C}^4$. This is an irreducible 2-dimensional subvariety of $\mathbb{C}^4$, regular aside from the isolated singularity at the origin, the vertex of the cone, and the germs of the functions f and g generate the local ideal of this subvariety at each of its points. Thus the tangent space to V at a regular point $x \in V$ is the 2-dimensional linear subspace

(13) $$T_x(V) = \{y \in \mathbb{C}^4 : f(x,y) = g(x,x,y) = 0\} \subseteq \mathbb{C}^4 ,$$

with the notation as in (11). On the other hand, the tangent space to V at a point $x = w'(z)$ contains the vector $w'(z)$, since V is a cone, and differentiating with respect to the local coordinate z shows that it also contains the vector $w''(z)$; these 2 vectors are always linearly independent for a non-hyperelliptic Riemann surface, so this tangent space can also be described as

$$T_{w'(z)}(V) = \{y \in \mathbb{C}^4 : y, w'(z), w''(z) \text{ are linearly dependent}\} .$$

The condition that the 3 vectors $y, w'(z), w''(z)$ be linearly dependent just amounts to the vanishing of all 3×3 subdeterminants of the 4×3 matrix with these 3 vectors as columns, so using (7) this tangent space can evidently be described equivalently as

(14) $$T_{w'(z)}(V) = \left\{ y \in \mathbb{C}^4 : \begin{array}{l} y_1\sigma_{23}(z) + y_2\sigma_{31}(z) + y_3\sigma_{12}(z) = 0 \\ y_1\sigma_{24}(z) + y_2\sigma_{41}(z) + y_4\sigma_{12}(z) = 0 \\ y_1\sigma_{34}(z) + y_3\sigma_{41}(z) + y_4\sigma_{13}(z) = 0 \\ y_2\sigma_{34}(z) + y_3\sigma_{42}(z) + y_4\sigma_{23}(z) = 0 \end{array} \right\} .$$

Since the tangent space has dimension 2 the matrix describing this system of linear equations must have rank 2; it is a straightforward matter to verify that this rank condition is equivalently that the functions $\sigma_{ij}(z)$ have no common zeros on M and satisfy the identity

(15) $$\sigma_{12}(z)\sigma_{34}(z) - \sigma_{13}(z)\sigma_{24}(z) + \sigma_{14}(z)\sigma_{23}(z) = 0 .$$

It is also quite easy to see that aside from a common nonzero factor there is a unique system of linear equations of the form appearing in (14) describing a 2-dimensional subspace of $\mathbb{C}^4$. Indeed not all the values $\sigma_{ij}(z)$ are zero, so since the equations are symmetric in the expressions $\sigma_{ij}(z)$ it can be supposed for instance that $\sigma_{12}(z) \neq 0$; the first 2 equations then serve to describe the subspace, so the values $\sigma_{13}(z), \sigma_{14}(z), \sigma_{23}(z), \sigma_{24}(z)$ are determined by $\sigma_{12}(z)$ and that subspace, and $\sigma_{34}(z)$ is then determined by (15). Thus aside from a common factor the values $\sigma_{ij}(z)$ can be determined in terms of the polynomials f and g just by comparing the two descriptions (13) and (14) of the tangent space to V at the point $x = w'(z)$. For instance, using the notation (12) the equations $f(x,y) = g(x,x,y) = 0$ are equivalent to the equations

$$y_1\cdot(a_1b_4-a_4b_1) + y_2\cdot(a_2b_4-a_4b_2) + y_3\cdot(a_3b_4-a_4b_3) = 0$$

$$y_1\cdot(a_3b_1-a_1b_3) + y_2\cdot(a_3b_2-a_2b_3) + y_4\cdot(a_3b_4-a_4b_3) = 0$$

whenever $a_3b_4-a_4b_3 \neq 0$; thus whenever $\sigma_{12}(z) \neq 0$ there is a nonzero value $c(z)$ for which

$$\sigma_{12}(z) = c(z)\cdot[a_3(w'(z))b_4(w'(z)) - a_4(w'(z))b_3(w'(z))] ,$$

and correspondingly for the other expressions $\sigma_{ij}(z)$ with the same value $c(z)$. These equations show that $c(z)$ is a meromorphic function on M, as the quotient of two elements of $\Gamma(M, \mathcal{O}(\kappa^3))$, and is nonvanishing since the functions $\sigma_{ij}(z)$ that appear in the numerators of the various different representations of $c(z)$ have no common zeros; therefore $c(z) = c$ is really a nonzero constant. The cubic polynomial $c[a_3(x)b_4(x) - a_4(x)b_3(x)]$ when restricted to points $x=w'(z)$ on the canonical curve takes the values $\sigma_{12}(z)$, hence can be taken as the polynomial $\tilde{\sigma}_{12}(x)$, and correspondingly for the other terms, to conclude the proof.

To return to the theta functions again, for a Riemann surface of genus $g=4$ the function $\underline{\underline{\theta}}_2(w)$ takes values in the space $\mathbb{C}^{16}$. The vector $\underline{\underline{\theta}}_2(0)$ together with the 10 vectors $\partial_{j_1 j_2}\underline{\underline{\theta}}_2(0)$ for $j_1 \leq j_2$ form a basis for the 11-dimensional subspace $S_1 \subseteq \mathbb{C}^{16}$ spanned by the values $\underline{\underline{\theta}}_2(w(z-a))$. The projection operator P_1 with kernel S_1 can be viewed as a linear mapping $P_1 : \mathbb{C}^{16} \longrightarrow \mathbb{C}^5$, so the function $P_1\underline{\underline{\theta}}_2(w)$ can be viewed as taking values in the space $\mathbb{C}^5$. The subspace $S_2 \subseteq \mathbb{C}^{16}$ spanned by the values $\underline{\underline{\theta}}_2(w(z_1+z_2-a_1-a_2))$ must be all of $\mathbb{C}^{16}$, just as in the case of surfaces of genus $g=3$, so the extension S_2/S_1 is 5-dimensional. In view of the skew-symmetry there are really just 4 of the vectors $\alpha^{k_1k_2}_{j}$ in S_2/S_1, namely α^{12}_{3}, α^{12}_{4}, α^{13}_{4}, α^{23}_{4}, so the extension S_1'/S_1 can be at most 4-dimensional; consequently the extension S_2/S_1' must be at least 1-dimensional.

<u>Theorem 10</u>. If M is a hyperelliptic Riemann surface of genus $g=4$ then $\alpha^{12}_{3} = \alpha^{12}_{4} = \alpha^{13}_{4} = \alpha^{23}_{4} = 0$; the equations (7.2) describe the canonical curve geometrically, but all vanish to at least second order at each point of the canonical curve. If M is a non-hyperelliptic Riemann surface of genus $g=4$ then the vectors α^{12}_{3}, α^{12}_{4}, α^{13}_{4}, α^{23}_{4} are linearly independent; if the rank of the matrix $\{c_{jk}\}$ is 4 the equations (7.2) describe the canonical curve geometrically, while if the rank of that matrix is 3 and P is the point in $\mathbb{P}^3$ it annihilates then the equations (7.2) describe geometrically the union of the canonical curve and the point P .

<u>Proof</u>. First suppose that M is hyperelliptic. It is easy to see that in terms of the expressions (7) equation (7.6) of Theorem 7 can be rewritten as the identity

$$(16)\qquad \sum_j \partial_{j_1 j_2 j_3 k} P_{1}\theta_2(0)\cdot w'_{j_1}(z) w'_{j_2}(z) w'_{j_3}(z) = 6 \sum_{j_1<j_2} \alpha^{j_1 j_2}_{k}\cdot\sigma_{j_1 j_2}(z)$$

for all k, and it follows immediately from this identity and Lemma 2 that

$$(17)\qquad \sum_{j_1<j_2} \alpha^{j_1 j_2}_{k}\cdot\sigma_{j_1 j_2}(z) = 0 \quad \text{for all } k .$$

It is another consequence of Lemma 2 that there is a single nontrivial linear relation $\sum_{j_1<j_2} a_{j_1 j_2}\cdot\sigma_{j_1 j_2}(z) = 0$ between the 6 expressions $\sigma_{j_1 j_2}(z)$, so all of the nontrivial relations (17) must be equivalent to this one relation. Thus if $a_{12} \neq 0$ for instance then all those relations (17) that do not involve $\sigma_{12}(z)$ must be trivial, and therefore $\alpha^{j_1 j_2}_{k} = 0$ whenever $k=1$ or 2; but in view of the skew-symmetry one of the indices in any $\alpha^{j_1 j_2}_{k}$ must be either 1 or 2, and that index can always be taken to be k, so that $\alpha^{j_1 j_2}_{k} = 0$ for all indices. The same argument holds for any nonzero coefficient $a_{j_1 j_2}$, of which there must certainly be some, so that all of the vectors $\alpha^{j_1 j_2}_{k}$ are zero as desired. It follows from Theorem 6 that the quartic equations (7.2) describe the canonical curve geometrically, and from (16) together with the observation that $\alpha^{j_1 j_2}_{k} = 0$ for all indices j_1, j_2, k that the first partial derivatives of all these quartic equations all vanish on the canonical curve also.

Next suppose that M is not hyperelliptic. There is a unique quadratic polynomial (4) vanishing on the canonical curve, so it follows immediately from (6.4) that

$$(18) \qquad \alpha^{k_1k_2}_{j_1j_2} = \alpha\cdot c_{j_1j_2}\, c_{k_1k_2}$$

for some vector $\alpha \in S_2/S_1'$. Since the vectors (18) span the nontrivial vector space S_2/S_1' it must be the case that $\alpha \neq 0$ and $\underline{\dim}\ S_2/S_1' = 1$, and consequently that $\underline{\dim}\ S_1'/S_1 = 4$; the four vectors $\alpha^{12}_{\ 3}$, $\alpha^{12}_{\ 4}$, $\alpha^{13}_{\ 4}$, $\alpha^{23}_{\ 4}$ spanning S_1'/S_1 must therefore be linearly independent as desired. Since the quartic polynomials in (7.2) vanish on the canonical curve while (4) and (5) generate the ideal of the canonical curve there must be unique vectors φ and ψ of quadratic and linear polynomials respectively such that

$$(19) \qquad \sum_j \partial_{j_1\cdots j_4} P_1\underline{\theta}_2(0)\cdot x_{j_1}\cdots x_{j_4} = \varphi(x)\cdot f(x) + \psi(x)\cdot g(x) \ .$$

Applying the further projection operator P_1' to (19) and then using Theorem 8 and equation (18) lead readily to the result that

$$(20) \qquad P_1'\varphi(x)\cdot f(x) + P_1'\psi(x)\cdot g(x) = 6\ P_1'\alpha\cdot f(x)^2 \ ,$$

hence that

$$(21) \qquad P_1'\varphi(x) = 6\ P_1'\alpha\cdot f(x) \ , \quad P_1'\psi(x) = 0 \ .$$

The system of equations (7.2) is therefore precisely equivalent to the system of equations

$$(22) \qquad f(x)^2 = \psi(x)\cdot g(x) = 0 \ ,$$

where $\psi(x)$ is a vector of linear equations with values in the subspace S_1' , hence where

$$(23) \qquad \psi(x) = \psi_1(x)\alpha^{23}_{\ 4} + \psi_2(x)\alpha^{13}_{\ 4} + \psi_3(x)\alpha^{12}_{\ 4} + \psi_4(x)\alpha^{12}_{\ 3}$$

for some uniquely determined ordinary linear polynomials ψ_j. To find these polynomials explicitly note that they are determined completely by their values $\psi_j(x)$ at points x of the canonical curve, since the canonical curve lies in no hyperplane. Further if x is in the canonical curve then it follows from (21) that $P_1'\varphi(x) = 0$, so there is a corresponding expansion

$$(24) \qquad \varphi(x) = \varphi_1(x)\alpha^{23}_{\ 4} + \varphi_2(x)\alpha^{13}_{\ 4} + \varphi_3(x)\alpha^{12}_{\ 4} + \varphi_4(x)\alpha^{12}_{\ 3}$$

for some functions φ_j on the canonical curve. Now for any vector $y \in \mathbb{C}^4$ applying the differential operator $\Sigma\, y_k\partial/\partial x_k$ to (19), considering the resulting equation at a point $x = w'(z)$ on the canonical curve, and applying (16) lead to the result that

$$2\varphi(x)\cdot f(x,y) + 3\psi(x)\cdot g(x,x,y)$$

$$= 4 \sum_{j,k} \partial_{j_1j_2j_3k} P_1\underline{\theta}_2(0)\cdot w'_{j_1}(z)w'_{j_2}(z)w'_{j_3}(z)y_k$$

$$= 24 \sum_{j_1<j_2} \sum_k \alpha^{j_1j_2}_{\ \ k} \cdot \tilde{\sigma}_{j_1j_2}(x) \ ,$$

where $\tilde{\sigma}_{j_1j_2}(x) = \sigma_{j_1j_2}(z)$ as in (10); substituting into this the expansions (23) and (24) and comparing coefficients of the basis vectors $\alpha^{j_1j_2}_{\ \ k}$ then easily yields the system of equations

$$(25) \quad \begin{aligned} 2\varphi_1(x)f(x,y) + 3\psi_1(x)g\ (x,x,y) &= 24[y_2\tilde{\sigma}_{34}(x) - y_3\tilde{\sigma}_{24}(x) + y_4\tilde{\sigma}_{23}(x)] \\ 2\varphi_2(x)f(x,y) + 3\psi_2(x)g\ (x,x,y) &= 24[y_1\tilde{\sigma}_{34}(x) - y_3\tilde{\sigma}_{14}(x) + y_4\tilde{\sigma}_{13}(x)] \\ 2\varphi_3(x)f(x,y) + 3\psi_3(x)g\ (x,x,y) &= 24[y_1\tilde{\sigma}_{24}(x) - y_2\tilde{\sigma}_{14}(x) + y_4\tilde{\sigma}_{12}(x)] \\ 2\varphi_4(x)f(x,y) + 3\psi_4(x)g(x,x,y) &= 24[y_1\tilde{\sigma}_{23}(x) - y_2\tilde{\sigma}_{13}(x) + y_3\tilde{\sigma}_{12}(x)] \end{aligned}$$

that hold for all points x in the canonical curve and all points y in E^4. As observed earlier $f(x,y)$ and $g(x,x,y)$ are linearly independent linear functions of y for each point x on the canonical curve, so this system of equations completely determines the values $\varphi_j(x)$ and $\psi_j(x)$ in terms of the values $\tilde{\sigma}_{j_1j_2}(x)$, and the latter are determined by the values of the coefficients $a_j(x)$ and $b_j(x)$ of the forms $f(x,y)$ and $g(x,x,y)$ as in Lemma 3. For instance, comparing the coefficients of y_1 and y_2 in the first equation of (25) yields the equations

$$2\varphi_1(x)a_1(x) + 3\psi_1(x)b_1(x) = 0$$

$$2\varphi_1(x)a_2(x) + 3\psi_1(x)b_2(x) = 24\,\tilde{\sigma}_{34}(x) \;;$$

if $a_1(x)b_2(x) - a_2(x)b_1(x) \neq 0$ these equations can be solved to yield

$$\psi_1(x)(a_1(x)b_2(x) - a_2(x)b_1(x)) = 8\; a_1(x)\tilde{\sigma}_{34}(x) \;,$$

and it then follows from Lemma 3 that

$$\psi_1(x) = 8\,c\,a_1(x) \;.$$

If $a_1(x)b_2(x) - a_2(x)b_1(x) = 0$ then it is necessary to consider the coefficients of some other pair of variables, but the end result will be the same. A straightforward calculation thus yields the explicit determination

$$\psi_1(x) = 8\,c\,a_1(x) \;, \qquad \psi_2(x) = -8\,c\,a_2(x)$$

$$\psi_3(x) = 8\,c\,a_3(x) \;, \qquad \psi_4(x) = -8\,c\,a_4(x) \;,$$

or equivalently in terms of the coefficients (4)

$$(26) \qquad \psi_j(x) = (-1)^{j+1} 8\,c \sum_k c_{jk}x_k \;.$$

It is a familiar result that the rank of the matrix $\{c_{jk}\}$ is either 4 or 3, according as the subvariety W_3^1 consists of a pair of points or a single point. If the rank is 4 then the vector equation $\psi(x) = 0$ describes just the origin in $\mathbb{E}^4$, and the equations (22) and therefore also the equations (7.2) describe the canonical curve geometrically; however, the ideal generated by these quartics is not the canonical ideal but that generated by $f(x)^2$, $x_1 g(x)$, $x_2 g(x)$, $x_3 g(x)$, and $x_4 g(x)$. If the rank is 3 then the vector equation $\psi(x) = 0$ describes a line in $\mathbb{E}^4$ hence a single point in $\mathbb{P}^3$, the point P annihilated by the matrix $\{c_{jk}\}$; this point of course lies on the locus $f(x) = 0$, so the equations (22) and therefore also the equations (7.2) describe geometrically the union of the canonical curve and this single point. That suffices to conclude the proof.

The last part of the proof of the preceding theorem yields the following rather more precise result as well, extending the geometrical description given in the statement of the theorem; this is an evident consequence of (22) when the vector $\psi(x)$ has the explicit form given by (23) and (26).

Corollary 7. If M is a non-hyperelliptic surface of genus $g = 4$ the ideal generated by the quartic polynomials in (7.2) can be described alternatively as that generated by the quartics $f(x)^2$, $g(x) \sum_j c_{ij} x_j$ for $1 \leqq i \leqq 4$.

In the hyperelliptic case the statement of the preceding theorem can also be made somewhat more precise, but since the canonical curve is just the rational normal curve it is perhaps not worth pursuing this matter too far here, as interesting as it may be. Briefly, though, in this case (6.4) evidently implies that

(27) $$\alpha^{k_1k_2}_{j_1j_2} = \sum_i \alpha^{i_1}_{i_2} c^{i_1}_{k_1k_2} c^{i_2}_{j_1j_2}$$

for some vectors $\alpha^{i_1}_{i_2} \in S_2/S_1'$, and from the known symmetries $\alpha^{i_1}_{i_2} = \alpha^{i_2}_{i_1}$. Furthermore, since $P_1 = P_1'$ in this case, as a consequence of the first part of Theorem 10, it follows from Theorem 8 and (27) that

(28) $$\sum_j \partial_{j_1\cdots j_4} P_{1\underline{=}2}\theta_2(0)\cdot x_{j_1}\cdots x_{j_4} = 6\sum_i \alpha^{i_1}_{i_2} f_{i_1}(x) f_{i_2}(x) \quad .$$

The products $f_{i_1}(x)f_{i_2}(x)$ span a 6-dimensional subspace A of the 35-dimensional space of quartic polynomials; clearly all the polynomials in A vanish to second order on the canonical curve, and this set of polynomials describe the canonical curve geometrically. The vectors $\alpha^{i_1}_{i_2}$ span the 5-dimensional space S_2/S_1, so (28) really describes a 5-dimensional subspace $B \subseteq A$. The condition that this subset of the polynomials in A also describe the canonical curve geometrically is a nontrivial condition on the vectors $\alpha^{i_1}_{i_2}$, indeed is just the condition that $\sum_i \alpha^{i_1}_{i_2} y_{i_1} y_{i_2} = 0$ only when $y = 0$.

The use of this approach in investigating further the various linear relations among the vectors $\partial_{j_1\cdots j_4} P_{1\underline{=}2}\theta_2(0)$ requires an analysis of the quadratic differentials (6.2), somewhat along the lines of the analysis of the cubic differentials in Lemmas 2 and 3 but a bit more complicated, and will be taken up elsewhere. The same analysis seems required in using this approach to the subvarieties of the Jacobi variety defined by equations such as $P_{1\underline{=}2}\theta_2(w) = 0$; a different approach has been used with notable success by Welters [12].

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