A STOCHASTIC-LAGRANGIAN APPROACH TO THE NAVIER-STOKES EQUATIONS IN DOMAINS WITH BOUNDARY.

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Abstract. In this paper we derive a probabilistic representation of the deterministic 3-dimensional Navier-Stokes equations in the presence of spatial boundaries. The formulation in the absence of spatial boundaries was done by the authors in CPAM, 2008. While the formulation in the presence of boundaries is similar in spirit, the proof is somewhat different. One notable feature of the formulation in the presence of boundaries is the non-local, implicit influence of the boundary vorticity on the interior fluid velocity.

1. Introduction

The (unforced) incompressible Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla) u - \nu \triangle u + \nabla p = 0 \tag{1.1}$$

$$\nabla \cdot u = 0 \tag{1.2}$$

describe the evolution of the velocity field u of an incompressible fluid with kinematic viscosity $\nu > 0$. Here p is the pressure, which, for incompressible fluids, can be treated as a Lagrange multiplier that ensures incompressibility is preserved. When $\nu = 0$, (1.1)–(1.2) is known as the Euler equations, and describes the evolution of the velocity field of an (ideal) inviscid incompressible fluid. Formally the difference between the Euler and Navier-Stokes equations is only the dissipative Laplacian term. Since the Laplacian is exactly the generator a Brownian motion, one would expect to have an exact stochastic representation of (1.1)–(1.2) which is physically meaningful, i.e. can be thought of as an appropriate average of the inviscid dynamics and Brownian motion.

The difficulty, however, in obtaining such a representation is because of both the nonlinearity, and the nonlocality of equation (1.1)–(1.2). In 2D, an exact stochastic representation of (1.1)–(1.2) dates back to Chorin [12] in 1973, and was obtained using vorticity transport and the Feynman-Kac formula. In three dimensions however, this method fails to provide an exact representation because of the vortex stretching term.

In 3D, a variety of techniques have been used to provide exact stochastic representations of (1.1)–(1.2). One such technique (Le Jan and Sznitman [25]) uses a backward branching process in Fourier space. This approach has been was extensively studied and generalized [2, 3, 29, 32, 33] by many authors. A different,

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and more recent, technique due to Busnello, Flandoli and Romito [5] (see also [4]), uses noisy flow paths and a Girsanov transformation. A related approach in [9] is the stochastic-Lagrangian formulation, exact stochastic representation of solutions to (1.1)–(1.2) which is essentially the averaging of noisy particle trajectories and the inviscid dynamics. Stochastic variational approaches (generalizing Arnold's [1] deterministic variational formulation for the Euler equations) have been used by [11, 14], and a related approach using stochastic differential geometry can be found in [17].

One common setback in all the above methods is the inability to deal with boundary conditions. The main contribution of this paper adapts the stochastic-Lagrangian formulation in [9] (where the authors only considered periodic boundary conditions, or decay at infinity), to the situation with boundaries. The usual probabilistic techniques used to transition to domains with boundary involve stopping the processes at the boundary. This introduces two major problems with the techniques in [9]. First, stopping introduces spatial discontinuities making the proof used in [9] fail, and a different approach is required. Secondly, and more interestingly, is the fact that merely stopping does *not* give the no-slip (0-Dirichlet) boundary condition as one would expect. One needs to also create trajectories at the boundary, which essentially propagate the influence of the vorticity at the boundary to the interior fluid velocity.

1.1. **The stochastic-Lagrangian formulation without boundaries.** We begin by providing a brief description of the stochastic-Lagrangian formulation in the absence of boundaries. For motivation, let us first study a Lagrangian description of the Euler equations (equations (1.1)–(1.2), with $\nu = 0$; we will usually use a superscript of 0 to denote quantities relating to the Euler equations). Let $d = 2, 3$ denote the spatial dimension, and X^0_t be the flow defined by

$$\dot{X}^0_t = u^0_t(X^0_t), \tag{1.3}$$

with initial data $X^0_0(a) = a$, for all $a \in \mathbb{R}^d$. We clarify that we always use a subscript of t to denote the restriction of a function at time t; we use a dot, or ∂_t to denote the derivative in time. One can immediately check (see for instance [6]) that u satisfies the incompressible Euler equations if and only if $\ddot{X}^0$ is a gradient composed with X. By Newton's second law, this admits the physical interpretation that the Euler equations are equivalent to assuming that the force on individual particles is a gradient.

One would naturally expect that solutions to the Navier-Stokes equations can be obtained similarly, by adding noise to particle trajectories, and averaging. However for noisy trajectories, an assumption on $\ddot{X}^0$ will be problematic. In the incompressible case, we can circumvent this difficultly using the Weber formula [34] (equation (1.4) below). Indeed, a direct computation (see for instance [6]) shows that for divergence free u, the assumption that $\ddot{X}^0$ is a gradient is equivalent to

$$u^0_t = \boldsymbol{P}\left[(\nabla^* A^0_t)\,(u^0_0 \circ A^0_t)\right], \tag{1.4}$$

where $\boldsymbol{P}$ denotes the Leray-Hodge projection [8, 13, 26] onto divergence free vector fields, the notation ∇^* denotes the transpose of the Jacobian, and for any $t \geqslant 0$, $A^0_t = (X^0_t)^{-1}$ is the spatial inverse of the map X^0_t (i.e. $A^0_t(X^0_t(a)) = a$ for all $a \in \mathbb{R}^d$ and $X^0_t(A^0_t(x)) = x$ for all $x \in \mathbb{R}^d$).

The above shows that the Euler equations are formally equivalent to equations (1.3) and (1.4). Notice that this formulation no longer involves second (time) derivatives of the flow X^0, and thus does not present any difficulties when we add noise to particle trajectories. Indeed, the authors exploit this fact in [9] and show that adding noise to (1.3), and averaging out the noise in (1.4) gives an equivalent formulation of the Navier-Stokes equations.

Theorem 1.1 (Constantin, Iyer [9]). *Let $d \in \{2,3\}$ be the spatial dimension, $\nu > 0$ represent the kinematic viscosity, and u_0 be a divergence free, periodic, Hölder $2+\alpha$ function, and W be a d-dimensional Wiener process. Consider the system*

$$dX_t = u_t(X_t)\,dt + \sqrt{2\nu}\,dW_t, \tag{1.5}$$

$$X_0(a) = a, \quad \forall a \in \mathbb{R}^d \tag{1.6}$$

$$u_t = E\boldsymbol{P}\left[(\nabla^* A_t)\,(u_0 \circ A_t)\right], \tag{1.7}$$

where as before, for any $t \geqslant 0$, $A_t = X_t^{-1}$ denotes the spatial inverse[1] of X_t. Then u is a classical solution of the Navier-Stokes equations (1.1)–(1.2) *with initial data u_0 and periodic boundary conditions if and only if u is a fixed point of the system* (1.5)–(1.7).

We now explain briefly the idea behind the proof of Theorem 1.1 given in [9], and explain why this methods can not be used in the presence of spatial boundaries. Consider first the solution of the SDE (1.5) with initial data (1.6). We know that any (spatially regular) process θ that is constant along trajectories of X satisfies the SPDE

$$d\theta_t + (u_t \cdot \nabla)\theta_t\,dt - \nu\triangle\theta_t\,dt + \sqrt{2\nu}\nabla\theta_t\,dW_t. \tag{1.8}$$

Since the process A (which as before is defined to be the spatial inverse of X) is constant along trajectories of X, the process θ is defined by

$$\theta_t = \theta_0 \circ A_t \tag{1.9}$$

is constant along trajectories of X. Thus if θ_0 is regular enough (C^2), then θ satisfies SPDE (1.8). If u is deterministic, taking expected values gives a "method of random characteristics" [9, 18, 22, 30], an elegant generalization[2] of the method of characteristics for parabolic equations. Namely $\bar\theta_t = E\theta_0 \circ A_t$ satisfies

$$\partial_t\bar\theta_t + (u_t \cdot \nabla)\bar\theta_t - \nu\triangle\bar\theta_t = 0 \tag{1.10}$$

with initial condition $\bar\theta|_{t=0} = \theta_0$.

Once explicit equations for A, and $u_0 \circ A$ have been established, a direct computation using Itô's formula shows that u given by (1.7) satisfies the Navier-Stokes equations (1.1)–(1.2). This was the proof used in [9]. We remark that this point of view also yields a natural understanding of generalized relative entropies [7, 10, 27, 28]. Eyink's recent work [15] adapted this framework to magnetohydrodynamics and related equations by using the analogous Weber formula [23, 31]. We also mention that Zhang [35] considered a backward analogue, and provided short elegant proofs to classical existence results to (1.1)–(1.2).

[1] It is well known (see for instance Kunita [24]) that the solution to (1.5)–(1.6) gives a stochastic flow of diffeomorphisms, and, in particular guarantee the existence of the spatial inverse of X.

[2] Note that when $\nu = 0$, A is deterministic, so $\bar\theta = \theta$. Further, equation (1.8) reduces to the transport equation, for which the procedure described above is exactly the usual (deterministic) method of characteristics.

2. The formulation for domains with boundary.

In this section we describe how (1.5)–(1.7) can be reformulated in the presence of boundaries. We begin by describing the difficulty in using the techniques from [9] described in Section 1.1.

Let $D \subset \mathbb{R}^d$ be a domain with Lipschitz boundary. Even if we insist $u = 0$ on the boundary of D, we note that the noise in (1.5) is independent of space, and thus insensitive to the presence of the boundary. Consequently, some trajectories of the stochastic flow X will leave the domain D, and for any $t > 0$, the map X_t will (surely) not be spatially invertible. This renders (1.9) meaningless.

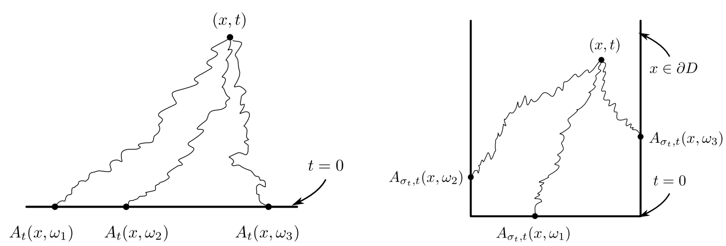

Figure 1. Three sample realizations of A without boundaries (left), and with boundaries (right).

In the absence of spatial boundaries, equation (1.9) dictates that $\bar{\theta}(x,t)$ is determined by averaging the initial temperature of all trajectories of X which reach x at time t. In the presence of boundaries, one must additionally average the boundary value of all trajectories reaching (x,t), starting on ∂D at any intermediate time (Figure 1). As we will see later, this means the analogue of (1.9) in the presence of spatial boundaries is a spatially discontinuous process. This renders (1.8) meaningless, giving a second obstruction to using the methods of [9] in the presence of boundaries.

While the method of random characteristics has the above inherent difficulties in the presence of spatial boundaries, the well known Feynman-Kac [16, 19] formula, at least for linear equations, has been successfully used in this situation. A certain version of the Feynman-Kac formula, when stated with without making the usual time reversal substitution, is essentially the same as the method of random characteristics. It is this version that will yield the natural generalization of (1.5)–(1.7) in domains with boundary. Before turning to the Navier-Stokes equations, we provide a brief discussion on the relation between the Feynman-Kac formula and the method of random characteristics.

2.1. **Feynman-Kac and the method of random characteristics.** Both the Feynman-Kac formula, and the method of random characteristics have their own advantages and disadvantages: The method of random characteristics only involves forward SDE's and obtains the solution of (1.10) at time t with only the knowledge of the initial data and "X at time t" (or more precisely, the solution, at time t, of the equation (1.5), with initial data specified at time 0). However, this method involves computing the spatial inverse of X, which analytically, and numerically, involves an additional step.

On the other hand, to compute the solution of (1.10) at time t via the Feynman-Kac formula when u is time dependent, involves *backward* SDE's, and further requires the knowledge of the solution to (1.5) with initial conditions specified at all times $s \leqslant t$. However, this does not require computation of spatial inverses, and, more importantly, yields the correct formulation in the presence of spatial boundaries.

Now to see the relation between the method of random characteristics and the Feynman-Kac formula, we rewrite (1.5) in integral form, and keep track of solutions starting at all times $s \geqslant 0$. For any $s \geqslant 0$, we define the process $\{X_{s,t}\}_{t \geqslant s}$ to be the flow defined by

$$X_{s,t}(x) = x + \int_s^t u_r \circ X_{s,r}(x)\, dr + \sqrt{2\nu}\,(W_t - W_s) \tag{2.1}$$

Now, as always, we let $A_{s,t} = X_{s,t}^{-1}$. Then formally composing (2.1) with $A_{s,t}$, and using the semigroup property $X_{s,t} \circ X_{r,s} = X_{r,t}$ gives the self-contained backward equation for $A_{s,t}$

$$A_{s,t}(x) = x - \int_s^t u_r \circ A_{r,t}(x)\, dr - \sqrt{2\nu}\,(W_t - W_s)\,. \tag{2.2}$$

Now (1.9) can be written as

$$\theta_t = \theta_0 \circ A_{0,t}, \tag{2.3}$$

and using the semigroup property $A_{r,s} \circ A_{s,t} = A_{r,t}$ we see that

$$\theta_t = \theta_s \circ A_{s,t}. \tag{2.4}$$

This formal calculation leads to a natural generalization of (1.9) in the presence of boundaries. As before, let $D \subset \mathbb{R}^d$ be a domain with Lipschitz boundary, and assume, for now, that u is a Lipschitz function defined on *all* of $\mathbb{R}^d$. Let $A_{s,t}$ be the flow defined by (2.2), and for $x \in D$, we define the *backward exit time* $\sigma_t(x)$ by

$$\sigma_t(x) = \inf\big\{ s \;\big|\; s \in [0,t] \text{ and } \forall r \in (s,t],\ A_{r,t}(x) \in D \big\} \tag{2.5}$$

Let $g : \partial D \times [0,\infty) \to \mathbb{R}$ and $\theta_0 : D \to \mathbb{R}$ be two given (regular enough) functions, and define the process θ_t by

$$\theta_t(x) = \begin{cases} g_{\sigma_t(x)} \circ A_{\sigma_t(x),t}(x) & \text{if } \sigma_t(x) > 0 \\ \theta_0 \circ A_{0,t}(x) & \text{if } \sigma_t(x) = 0. \end{cases} \tag{2.6}$$

Note that when $\sigma_t(x) > 0$, equation (2.6) is consistent with (2.4). Thus (2.6) is the natural generalization of (1.9) in the presence of spatial boundaries, and we expect $\bar\theta_t = E\theta_t$ satisfies the PDE (1.10) with initial data $\bar\theta_0 = \theta_0$ and boundary conditions $\theta = g$ on $\partial D \times [0,\infty)$. Indeed, this is essentially the Feynman-Kac formula.

Note that the backward exit time σ is usually discontinuous in the spatial variable. Thus, even with smooth g, θ_0, the process θ need not be spatially continuous. As mentioned earlier, equation (1.8) will now become meaningless, and we will not be able to obtain a SPDE for θ. However equation (1.10) describing the evolution of the expected value $\bar\theta = E\theta$ and can be directly derived using the backward Markov property and Itô's formula (see for instance [16]). We will not provide this proof here, but will instead provide a proof for the more complicated analogue for the Navier-Stokes equations. This is described in the next section.

2.2. **Application to the Navier-Stokes equations in domains with boundary.** Note first that if $g = 0$ in (2.6), then the solution to (1.10) with initial data θ_0, and 0-Dirichlet boundary conditions will be given by

$$\bar{\theta}_t = E\chi_{\{\sigma_t=0\}}\theta_0 \circ A_{0,t}. \qquad \big[\text{i.e. } \bar{\theta}_t(x) = E\chi_{\{\sigma_t(x)=0\}}\theta_0 \circ A_{0,t}(x)\big]. \tag{2.7}$$

Recall the no-slip boundary condition for the Navier-Stokes equations is exactly a 0-Dirichlet boundary condition on the velocity field. Let u be a solution to the Navier-Stokes equations in D with initial data u_0, and no slip boundary conditions. Now following (2.7), we would expect that analogous to (1.7), the velocity field u can be recovered from the flow $A_{s,t}$ (equation (2.2)), the backward exit time σ_t (equation (2.5)), and the initial data u_0 by

$$u_t = \boldsymbol{P}E\chi_{\{\sigma_t=0\}}(\nabla^* A_{0,t})\, u_0 \circ A_{0,t} \tag{2.8}$$

This however is *false* and a non-local effect is observed. It turns out that what is missing from (2.8) is exactly the vorticity created at the boundary. What we prove instead is the following result.

Theorem 2.1. *Let $u \in C^1([0,T);C^2(D)) \cap C([0,T];C^1(\bar{D}))$ be a solution of the Navier-Stokes equations* (1.1)–(1.2) *with initial data u_0 and no-slip boundary conditions. Let A be the solution to the backward SDE* (2.2) *and σ be the backward exit time defined by* (2.5)*. There exists a function $\tilde{w} : \partial D \times [0,T] \to \mathbb{R}^3$ such that for*

$$w_t(x) = \begin{cases} (\nabla^* A_{0,t}(x))\, u_0 \circ A_{0,t}(x) & \text{when } \sigma_t = 0, \\ (\nabla^* A_{\sigma_t(x),t}(x))\, \tilde{w}_{\sigma_t} \circ A_{\sigma_t,t}(x) & \text{when } \sigma_t > 0, \end{cases} \tag{2.9}$$

we have

$$u_t = \boldsymbol{P}Ew_t. \tag{2.10}$$

Conversely, given a function $\tilde{w} : \partial D \times [0,T] \to \mathbb{R}^d$ suppose there exists a solution to the system (2.2), (2.9), (2.10)*. If further $u \in C^1([0,T);C^2(D)) \cap C([0,T];C^1(\bar{D}))$, then u satisfies the Navier-Stokes equations* (1.1) *with initial data u_0 and vorticity boundary conditions*

$$\nabla \times u = \nabla \times Ew \quad \text{on } \partial D \times [0,T]. \tag{2.11}$$

Remark 2.2. We remark that by $\nabla^* A_{\sigma_t(x),t}(x)$ in (2.9) we mean $[\nabla^* A_{s,t}(x)]_{s=\sigma_t(x)}$. That is, $\nabla^* A_{\sigma_t(x),t}(x)$ refers to the transpose of the Jacobian of A, evaluated at initial time $\sigma_t(x)$, final time t and position x (see [20, 21, 24] for existence). This is different from the transpose of the Jacobian of the function $A_{\sigma_t(\cdot),t}(\cdot)$, which doesn't exist as the function is certainly not differentiable in space.

Remark 2.3 (Regularity assumptions). In order to simply presentation, the regularity assumptions on u are somewhat generous. Our assumptions on u will immediately guarantee that u has a Lipschitz extension to $\mathbb{R}^d$ Now the process A, defined to be a solution to (2.2) with this Lipschitz extension of u, can be chosen to be a (backward) stochastic flow of diffeomorphisms [24]. Thus ∇A is well defined, and further defining σ by (2.5) is valid. Finally, since the statement of Theorem 2.1 only uses values of $A_{s,t}$ for $s \geqslant \sigma_t$, the choice of the Lipschitz extension of u will not matter.

A weaker assumption on the regularity of u can be made in terms of the domain of the backward generator of the process A (see Lemma 4.1 below). While the formal calculus remains essentially unchanged, there are a couple of technical points that require attention. First, when assumptions on smoothness of u up to the boundary is relaxed (or when ∂D is irregular), a Lipschitz extension of u need not exist. In this case, we can no longer use (2.5) to define σ. Further, we can not regard the process A as a stochastic flow of diffeomorphisms, and some care has to be taken when differentiating it. To avoid unnecessary technicalities, these issues are briefly mentioned in Section 3. Once these issues are sorted out, the proof of Theorem 2.1 remains unchanged.

Remark 2.4. Note that our statement of the converse above *does not* explicitly give any information on the Dirichlet boundary values of u. Of course, the normal component of u must vanish at the boundary of D, since u is the Leray-Hodge projection of a function. But an explicit local relation between $\tilde{w}$ and the boundary values of the tangential component of u cannot be established. We remark, however, that while the vorticity boundary condition (2.11) is somewhat artificial, it is enough to to guarantee uniqueness of solutions to the initial value problem for the Navier-Stokes equations.

Remark 2.5 (Choice of $\tilde{w}$). We explain how $\tilde{w}$ can be chosen to obtain the no-slip boundary conditions. We will show (Lemma 4.1) that for w defined by (2.9), the expected value $\bar{w} \stackrel{\text{def}}{=} Ew$ solves the PDE

$$\partial_t \bar{w}_t + (u_t \cdot \nabla)\bar{w}_t - \nu \triangle \bar{w}_t + (\nabla^* u_t)\bar{w}_t = 0 \tag{2.12}$$

with initial data

$$\bar{w}|_{t=0} = u_0. \tag{2.13}$$

As before, $\nabla^* u_t$ in (2.12) denotes the transpose of the Jacobian of u_t. Now, if $u = \boldsymbol{P}\bar{w}$, then we will have $\nabla \times u = \nabla \times \bar{w}$ in D, and by continuity, on the boundary of D. Thus, to prove existence of the function $\tilde{w}$, we solve the PDE (2.12) with initial conditions (2.13) and *vorticity* boundary conditions

$$\nabla \times \bar{w}_t = \nabla \times u_t \quad \text{on } \partial D. \tag{2.14}$$

We chose $\tilde{w}$ to be the Dirichlet boundary values of this solution.

To elaborate on Remark 2.5, we trace through the influence of the vorticity on the boundary on the velocity in the interior. First the vorticity at the boundary influences $\bar{w}$ by entering as a boundary condition on the first derivatives for the PDE (2.12). Now to obtain u, we need to find $\tilde{w}$, the (Dirichlet) boundary values of (2.12), and use this to weight trajectories that start on the boundary of D. The process of finding $\tilde{w}$ is essentially passing from Neumann boundary values of a PDE to the Dirichlet boundary values, which is usually a nonlocal pseudo-differential operator. Thus, while the procedure above is explicit enough, the boundary vorticity influences the interior velocity in a highly implicit, nonlocal manner.

Remark 2.6 (Uniqueness of $\tilde{w}$.). Our choice of $\tilde{w}$ is not unique. Indeed, if $\bar{w}^1$ and $\bar{w}^2$ are two solutions of (2.12)–(2.14), then we must have $\bar{w}^1 - \bar{w}^2 = \nabla q$, where q satisfies the equation

$$\nabla \left(\partial_t q + (u \cdot \nabla) q - \nu \triangle q \right) = 0 \tag{2.15}$$

with initial data $\nabla q_0 = 0$. Since we don't have boundary conditions on q, we can certainly have non-trivial solutions to this equation. Thus our choice of $\tilde{w}$ is only unique up to boundary values of a gradient of a solution to (2.15).

This now raises numerous interesting questions. A fundamental open question in fluid dynamics is about boundary layer separation in the inviscid limit. It is known that in some situations the boundary layer can 'detach' and produce vorticity in the interior [13]. The above probabilistic representation allows a new direction of approach to this problem. One can now quantify boundary separation in terms of the chance that the backward exit time σ_t is strictly positive. Our hope is that this framework will allow the use of probabilistic techniques, especially those with no deterministic analogues (e.g. large deviations), to provide new insight into the study of the inviscid limit.

3. Backward Itô integrals

While the formulation of Theorem 2.1 involves only regular (forward) Itô integrals, the proof requires backward Itô integrals and processes adapted to a two parameter filtration. The need for backward Itô integrals stems from equation (2.2), which, as mentioned earlier, is the evolution of A *backward* in time. This is however obscured because our diffusion coefficient is constant making the martingale term exactly the increment of the Wiener process, and can be explicitly computed without any backward (or even forward) Itô integrals.

To elucidate matters, consider the flow X' given by

$$X'_{s,t}(a) = a + \int_s^t u_r \circ X'_{s,r}(a)\, dr + \int_s^t \sigma_r \circ X'_{s,r}(a)\, dW_r. \tag{3.1}$$

If, as usual, $A'_{s,t} = (X'_{s,t})^{-1}$, then substituting formally[3] $a = A'_{s,t}(x)$ and assuming the semigroup property gives the equation

$$A'_{s,t}(x) = x - \int_s^t u_r \circ A'_{r,t}(x)\, dr - \int_s^t \sigma_r \circ A'_{r,t}(x)\, dW_r. \tag{3.2}$$

for the process $A'_{s,t}$. The need for backward Itô integrals is now evident; the last term above does not make sense as a forward Itô integral since $A'_{r,t}$ is not $\mathcal{F}_r$ measurable. This term however is well defined as a backward Itô integral; an integral with respect to a decreasing filtration where processes are sampled at the right end point. Since forward Itô integrals are more predominant in the literature, we recollect a few standard facts about backward Itô integrals in this section. A more detailed account, with proofs, can be found in [16, 24] for instance.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{W_t\}_{t \geqslant 0}$ be a d-dimensional Wiener process on Ω, and let $\mathcal{F}_{s,t}$ be the σ-algebra generated by the increments $W_{t'} - W_{s'}$ for all $s \leqslant s' \leqslant t' \leqslant t$, augmented so that the filtration $\{\mathcal{F}_{s,t}\}_{0 \leqslant s \leqslant t}$ satisfies the usual conditions.[4] Note that for $s \leqslant s' \leqslant t' \leqslant t$, we have $\mathcal{F}_{s',t'} \subset \mathcal{F}_{s,t}$. Also $W_t - W_s$ is $\mathcal{F}_{s,t}$-measurable, and is independent of both the past $\mathcal{F}_{0,s}$, and the future $\mathcal{F}_{t,\infty}$.

We define a (two parameter) family of random variables $\{\xi_{s,t}\}_{0 \leqslant s \leqslant t}$ to be a (two parameter) process adapted to the (two parameter) filtration $\{\mathcal{F}_{s,t}\}_{0 \leqslant s \leqslant t}$, if for all $0 \leqslant s \leqslant t$, the random variable $\xi_{s,t}$ is $\mathcal{F}_{s,t}$-measurable. For example, $\xi_{s,t} = W_t - W_s$ is an adapted process. More generally, if b and σ are regular enough *deterministic*

[3]The formal substitution does *not* give the correct answer when σ is not spatially constant. This is explained subsequently, and the correct equation is (3.3) below.

[4]By 'usual conditions' in this context, we mean that for all $s \geqslant 0$, $\mathcal{F}_{s,s}$ contains all $\mathcal{F}_{0,\infty}$-null sets. Further, $\mathcal{F}_{s,t}$ is right continuous in t, and left continuous in s. See [19, Definition 2.25] for instance.

functions, then the solution $\{X'_{s,t}\}_{0\leqslant s\leqslant t}$ of the (forward) SDE (3.1) is an adapted process.

Given an adapted (two parameter) process ξ and any $t \geqslant 0$, we define the backward Itô integral $\int_\cdot^t \xi_{r,t}\, dW_r$ by

$$\int_s^t \xi_{r,t}\, dW_r = \lim_{\|P\|\to 0} \sum_i \xi_{t_{i+1},t}(W_{t_{i+1}} - W_{t_i})$$

where $P = (r = t_0 < t_1 \cdots < t_N = t)$, is a partition of $[r,t]$ and $\|P\|$ is the length of the largest subinterval of P. The limit is taken in the L^2 sense, exactly as with forward Itô integrals.

The standard properties (existence, Itô isometry, martingale properties) of the backward Itô integral are of course identical to those of the forward integral. The only difference is in the sign of the Itô correction. Explicitly, consider the process $\{A'_{s,t}\}_{0\leqslant s\leqslant t}$ satisfying the backward Itô differential equation (3.2). If $\{f_{s,t}\}_{0\leqslant s\leqslant t}$ is adapted, C^2 in space, and continuously differentiable with respect to s, then the process $B_{s,t} = f_{s,t} \circ A_{s,t}$ satisfies the backward Itô differential equation

$$B_{t,t} - B_{s,t} = \int_s^t \left[\partial_r f_{r,t} + (u_r \cdot \nabla) f_{r,t} - \frac{1}{2} a_r^{ij} \partial_{ij} f_{r,t}\right] \circ A_{r,t}\, dr + \\ + \int_s^t [\nabla f_{r,t}\, \sigma_r] \circ A_{r,t}\, dW_r.$$

where $a_r^{ij} = \sigma_r^{ik} \sigma_r^{jk}$ with the Einstein sum convention.

Though we only consider solutions to (3.1) for constant diffusion coefficient, we briefly address one issue when σ is not constant. Our motivation for the equation (3.2) was to make the substitution $x = A'_{s,t}(x)$, and formally use the semigroup property. This, however, does not yield the correct equation when σ is not constant, and the equation for $A'_{s,t} = (X'_{s,t})^{-1}$ involves an additional correction term. To see this, we discretize the forward integral in (3.1) (in time), and substitute $a = A'_{s,t}(x)$. This yields a sum sampled at the *left* end point of each time step. While this causes no difficulty for the bounded variation terms, the martingale term is a discrete approximation to a *backward* integral, and hence must be sampled at the right endpoint of each time step. Converting this to sum sampled at the right endpoint via a Taylor expansion of σ is what gives this extra correction. Carrying through this computation (see for instance [24, §4.2]) yields the equation

$$A'_{s,t}(x) = x - \int_s^t u_r \circ A'_{r,t}(x)\, dr - \int_s^t \sigma_r \circ A'_{r,t}(x)\, dW_r + \\ + \int_s^t \left(\partial_j \sigma_r^{i,k} \circ A'_{r,t}(x)\right) \left(\sigma_r^{j,k} \circ A'_{r,t}(x)\right) e_i\, dr \tag{3.3}$$

where $\{e_i\}_{1\leqslant i\leqslant d}$ are the elementary basis vectors, and $\sigma^{i,j}$ denotes the i,j^{th} entry in the $d \times d$ matrix σ.

We recall that the proof of the (forward) Itô formula involves approximating f by it's Taylor polynomial about the left endpoint of the partition intervals. Analogously the backward Itô formula involves approximating f by Taylor polynomial about the *right* endpoint of partition intervals, which accounts for the reversed sign in the Itô correction.

Finally we remark that for any fixed $t \geqslant 0$, the solution $\{A_{s,t}\}_{0\leqslant s\leqslant t}$ of the backward SDE (2.2) is a backward strong Markov process (the same is true for solutions to (3.3)). The backward Markov property states that $r < s < t$ then

$$E_{\mathcal{F}_{s,t}} f \circ A_{r,t}(x) = E_{A_{s,t}(x)} f \circ A_{r,t}(x) = [Ef \circ A_{r,s}(y)]_{y=A_{s,t}(x)}$$

where $E_{\mathcal{F}_{s,t}}$ denotes the conditional expectation with respect to the σ-algebra $\mathcal{F}_{s,t}$, and $E_{A_{s,t}(x)}$ the conditional expectation with respect to the σ-algebra generated by the process $A_{s,t}(x)$.

For the strong Markov property, and we define σ to be a *backward t-stopping time*[5] if almost surely $\sigma \leqslant t$, and for all $s \leqslant t$, the event $\{\sigma \geqslant s\}$ is $\mathcal{F}_{s,t}$-measurable. Now if σ is any backward t-stopping time with $r \leqslant \sigma \leqslant t$ almost surely, the backward strong Markov property states

$$E_{\mathcal{F}_{\sigma,t}} f \circ A_{r,t}(x) = E_{A_{\sigma,t}} f \circ A_{r,t}(x) = [Ef \circ A_{r,s}(y)]_{\substack{s=\sigma,\\ y=A_{\sigma,t}(x)}}$$

The proofs of the backward Markov properties is analogous to the proof of the forward Markov properties, and we refer the reader to [16] for instance.

3.1. **A reformulation without Lipschitz extensions.** We conclude this section by outlining how Theorem 2.1 can be reformulated without using Lipschitz extensions of u. This is mainly of interest when ∇u blows up near ∂D.

Let $D \subset \mathbb{R}^d$ be a domain, not necessarily Lipschitz, $T > 0$, and assume $u : D \times [0,T] \to \mathbb{R}^d$ is bounded and *locally* Lipschitz. Now we can always write $D = \cup_{k\in\mathbb{N}} D^{(k)}$, where the $D^{(k)}$'s are an increasing sequence of locally compact open sets with Lipschitz boundary. Now the function u restricted to $D^{(k)}$ must have a Lipschitz extension to $\mathbb{R}^d$, which we denote by $u^{(k)}$. Let $A^{(k)}$ be the solution to (2.2) with u replaced by $u^{(k)}$. Now we can define $\sigma^{(k)}$ to be the backward exit time of $A^{(k)}$ from the domain $D^{(k)}$. That is, we define $\sigma^{(k)}$ by (2.5), with A replaced with $A^{(k)}$, and D replaced with $D^{(k)}$. Note that by pathwise uniqueness, we must have

$$\sigma_t^{(k+1)}(x) < \sigma_t^{(k)}(x) \quad \text{and} \quad A^{(k)}_{s\vee\sigma_t^{(k)},t}(x) = A^{(k+1)}_{s\vee\sigma_t^{(k)},t}(x)$$

for all $x \in D^{(k)}$, $t \leqslant T$, almost surely. Thus we define

$$\sigma_t(x) = \begin{cases} \lim\limits_{k\to\infty} \sigma_t^{(k)}(x) & \text{for } x \in D, \\ t & \text{for } x \in \partial D. \end{cases} \tag{3.4}$$

For $x \in D$ and any realization where $s > \sigma_t(x)$, the sequence $\big(A^{(k)}_{s\vee\sigma_t^{(k)},t}(x)\big)_k$ is eventually constant. So on the event $\{s \in (\sigma_t(x), t]\}$, we define

$$A_{s,t}(x) = \lim_{k\to\infty} A^{(k)}_{s\vee\sigma_t^{(k)}(x),t}(x) \quad \text{for } s \in (\sigma_t(x), t]. \tag{3.5}$$

Now, A must satisfy (2.2) almost surely on the event $\{s \in (\sigma_t(x), t]\}$. Since u is bounded, equation (2.2) and Lévy's almost sure, modulus of continuity [19, §2.9F] for Brownian motion will guarantee that for almost every realization the function $f(r) = A_{r,t}(x)$ is uniformly continuous on $(\sigma_t(x), t]$. Thus the limit $\lim_{s\to\sigma_t(x)^+} A_{s,t}(x)$ exists almost surely, and we denote it by $A_{\sigma_t(x),t}(x)$.[6] Note that by definition of σ, we must have $A_{\sigma_t(x),t}(x) \in \partial D$. Finally, we remark that the

[5]Our use of the term backward t-stopping time is analogous to s-stopping time in [16, p24].

[6]Though unnecessary for our purposes, we point out that this argument can be made to show that the event where $\lim_{s\to\sigma_t(x)^+} A_{s,t}(x)$ exists is in fact independent of x.

above definition of A is independent of the sequence of domains $D^{(k)}$. This, again, is an immediate consequence of pathwise uniqueness.

Now we can reformulate Theorem 2.1 using σ and A defined by (3.4) and (3.5) respectively. Provided we have enough control on ∇u to guarantee that the limit $\lim_{s\to\sigma_t(x)^+} \nabla A_{s,t}(x)$ exists, Theorem 2.1 and it's proof will remain unchanged in this context.

4. The no-slip boundary condition.

In this section we prove Theorem 2.1. First, we know from [20, 21], that spatial derivatives of A can be interpreted as the limit (in probability) of the usual difference quotient.[7] Further, the Jacobian of A is a process which, almost surely, satisfies the equation

$$\nabla A_{s,t}(x) = x - \int_s^t \nabla u_r\big|_{A_{r,t}(x)} \nabla A_{r,t}(x)\, dr, \tag{4.1}$$

obtained by formally differentiating (2.2) in space. We reiterate that equation (4.1) is an ODE, as the Wiener process is independent of the spatial parameter.

Lemma 4.1. *Let D, u, T be as in Theorem 2.1, σ be the minimal existence time of* (2.2)*, and A be the solution to* (2.2) *with respect to the backward stopping time σ.*

(1) *Let $\bar{w} \in C^1([0,T]; C^2(D)) \cap C([0,T]; C^1(\bar{D}))$ be solution of* (2.12) *with initial data* (2.13)*, and boundary conditions*

$$\bar{w} = \tilde{w} \quad \text{on } \partial D. \tag{4.2}$$

Then, for w defined by (2.9)*, we have $\bar{w} = Ew$.*

(2) *Let w be defined by* (2.9)*, and $\bar{w} = Ew$ as above. If for all $t \in (0,T]$, $\bar{w}_t \in \mathcal{D}(A_{\cdot,t})$, and $\bar{w}$ is C^1 in time, then $\bar{w}$ satisfies*

$$\partial_t \bar{w} + L_t \bar{w} + (\nabla^* u)\bar{w} = 0 \tag{4.3}$$

where L_t is defined by

$$L_t\phi(x) = \lim_{s\to t^-} \frac{\phi(x) - E\phi\left(A_{s\vee\sigma_t(x),t}(x)\right)}{t-s}, \tag{4.4}$$

and $\mathcal{D}(A_{\cdot,t})$ is the set of all ϕ for which the limit on the right exists. Further, $\bar{w}$ has initial data u_0 and boundary conditions (4.2)*.*

Before proceeding any further, we first address the relationship between the two assertions of the lemma. We claim that if $\bar{w} \in C^1((0,T); C^2(D))$, then equation (4.3) reduces to equation (2.12). This follows immediately from the next proposition.

Proposition 4.2. *If $\phi \in C^2(D)$, then $L_t\phi = (u_t \cdot \nabla)\phi - \nu\triangle\phi$.*

Proof. Omitting the spatial variable for notational convenience, the backward Itô formula gives

$$\begin{aligned}\phi - \phi\circ A_{s\vee\sigma_t,t} &= \phi\circ A_{t,t} - \phi\circ A_{s\vee\sigma_t,t}\\ &= \int_{s\vee\sigma_t}^t \left[(u_r\cdot\nabla)\phi\big|_{A_{r,t}} - \nu\, \triangle\phi\big|_{A_{r,t}}\right] dr + \sqrt{2\nu}\int_{s\vee\sigma_t}^t \nabla\phi\big|_{A_{r,t}}\, dW_r\end{aligned}$$

[7] For regular enough velocity fields u (extended to all of $\mathbb{R}^d$), the process A can in fact be chosen to be a flow of diffeomorphisms of $\mathbb{R}^d$ (see for instance [24]), in which case A is surely differentiable in space.

Since $s \vee \sigma_t$ is a backward t-stopping time, the second term above is a martingale. Thus

$$\begin{aligned} L_t\phi &= \lim_{s\to t^-} E\frac{1}{t-s}\int_s^t \chi_{\{r\geqslant\sigma_t\}} \left[(u_r\cdot\nabla)\phi|_{A_{r,t}} - \nu\, \triangle\phi|_{A_{r,t}}\right] dr \\ &= (u_t\cdot\nabla)\phi - \nu\triangle\phi \end{aligned}$$

since the process A has continuous paths, and $\sigma_t < t$ on the interior of D. □

Now we prove each assertion of Lemma 4.1 separately.

Proof of the first assertion in Lemma 4.1. Recall that $\nabla^* A_{s,t}$ is differentiable in s. Differentiating (4.1) in s, and transposing the matrices gives

$$\partial_s \nabla A_{s,t}(x) = \nabla^* A_{s,t}(x)\ \ \nabla^* u_s|_{A_{s,t}(x)}\,. \tag{4.5}$$

Let $t\in(0,T]$, $x\in D$, and σ' be any backward t-stopping time with $\sigma'\geqslant\sigma_t(x)$ almost surely. Omitting the spatial variable for convenience, the backward Itô formula and equations (2.12) and (4.5) give

$$\begin{aligned} \bar w_t - \nabla^* A_{\sigma',t}\,\bar w_{\sigma'}\circ A_{\sigma',t} &= \\ &= \nabla^* A_{t,t}\,\bar w_t\circ A_{t,t} - \nabla^* A_{\sigma',t}\,\bar w_{\sigma'}\circ A_{\sigma',t} \\ &= \int_{\sigma'}^t \partial_r\nabla^* A_{r,t}\,\bar w_r\circ A_{r,t} + \\ &\qquad + \int_{\sigma'}^t \nabla^* A_{r,t}\ (\partial_r\bar w_r + (u_r\cdot\nabla)\bar w_r - \nu\triangle\bar w_r)\circ A_{r,t}\,dr + \\ &\qquad + \sqrt{2\nu}\int_{\sigma'}^t (\nabla^* A_{r,t})(\nabla^*\bar w_r)\circ A_{r,t}\,dW_r \\ &= \int_{\sigma'}^t \nabla^* A_{r,t}\left((\nabla^* u_r)\,\bar w_r + \partial_r\bar w_r + (u_r\cdot\nabla)\bar w_r - \nu\triangle\bar w_r\right)\circ A_{r,t}\,dr + \\ &\qquad + \sqrt{2\nu}\int_{\sigma'}^t (\nabla^* A_{r,t})(\nabla^*\bar w_r)\circ A_{r,t}\,dW_r \\ &= \sqrt{2\nu}\int_{\sigma'}^t (\nabla^* A_{r,t})\,w_r\circ A_{r,t}\,dW_r. \end{aligned}$$

Thus, taking expected values gives

$$\bar w_t(x) = E\nabla^* A_{\sigma',t}(x)\,\bar w_{\sigma'}\circ A_{\sigma',t}(x) \tag{4.6}$$

Recall that when $\sigma_t(x) > 0$, $A_{\sigma_t(x),t}(x)\in\partial D$. Thus choosing $\sigma' = \sigma_t(x)$, and using the boundary conditions (4.2) and initial data (2.13), we have

$$\bar w_{\sigma_t(x)}\circ A_{\sigma_t(x),t} = \begin{cases} \tilde w_{\sigma_t(x)}\circ A_{\sigma_t(x),t} & \text{if } \sigma_t(x) > 0, \\ u_0\circ A_{\sigma_t(x),t} & \text{if } \sigma_t(x) = 0. \end{cases} \tag{4.7}$$

Substitutiung this in (4.6) completes the proof. □

Proof of the second assertion in Lemma 4.1. Let w be defined by (2.9), and $\bar w = Ew$ as in the statement of the second assertion. We will directly deduce (4.6) using the backward strong Markov property.

Given $x \in \mathbb{R}^d$, and a $d \times d$ matrix M, define the process $\{B_{s,t}(x,M)\}_{\sigma_t(x) \leqslant s \leqslant t \leqslant T}$ to be the solution of the ODE

$$B_{s,t}(x,M) = M - \int_s^t \nabla u_r|_{A_{r,t}(x)} \, B_{r,t}(x,M)\, dr.$$

If I denotes the $d \times d$ identity matrix, then by (4.1) we have $B_{s,t}(x,I) = \nabla A_{s,t}(x)$ for any $\sigma_t(x) \leqslant s \leqslant t \leqslant T$. Further since the evolution equation for B is linear, we see

$$B_{s,t}(x,M) = B_{s,t}(x,I)M = \nabla A_{s,t}(x)M. \tag{4.8}$$

Note that for any fixed $t \in (0,T]$, the process $\{\nabla A_{s,t}\}_{0 \leqslant s \leqslant t}$ is *not* a backward Markov process. Indeed, the evolution of $\nabla A_{s,t}$ at any time $s \leqslant t$ depends on the time s through the process $A_{s,t}$ appearing on the right in (4.1). However, process $(A_{s,t}, \nabla A_{s,t})$ (or equivalently the process $(A_{s,t}, B_{s,t})$) is a backward Markov process, since the evolution of this system now only depends on the state.

As in the statement of the second assertion, let $\bar{w} = Ew$, where w is defined by (2.9). By our assumption on u and ∂D, the boundary conditions (4.2) and initial data (2.13) are satisfied. For convenience, when $y \in \partial D$, $t > 0$, we define $w_t(y) = \tilde{w}(y)$, and when $t = 0$, $y \in \bar{D}$, we define $w_0(y) = u_0(y)$.

Now, let $x \in D$, $t \in (0,T]$. Since, the point $(A_{\sigma_t(x),t}, t)$ belongs to the parabolic boundary $\partial_p(D \times [0,T]) \stackrel{\text{def}}{=} (\partial D \times [0,T]) \cup (D \times \{0\})$, our boundary conditions and initial data will guarantee (4.6) is satisfied for $\sigma' = \sigma_t(x)$.

To prove (4.6) for arbitrary σ', we choose any backward t-stopping time σ' with $\sigma' \geqslant \sigma_t(x)$ almost surely. We claim

$$\begin{aligned} E_{\mathcal{F}_{\sigma',t}} B^*_{\sigma_t(x),t}(x,I)\, \bar{w}_{\sigma_t(x)} \circ A_{\sigma_t(x),t}(x) \\ = \Big[E B^*_{\sigma_r(y),r}(y,M)\, \bar{w}_{\sigma_r(y)} \circ A_{\sigma_r(y),r}(y) \Big]_{\substack{r=\sigma',\, y=A_{\sigma',t}(x), \\ M=B_{\sigma',t}(x,I),}} \end{aligned} \tag{4.9}$$

holds almost surely. This follows from an appropriate application of the backward strong Markov property. While this is easily believed, checking that the strong Markov property applies in this situation requires a little work, and will distract from the heart of the matter. Thus we postpone the proof of (4.9) momentarily. Now, using the identity (4.9) gives

$$\begin{aligned} \bar{w}_t(x) &= E \nabla^* A_{\sigma_t(x),t}(x)\, \bar{w}_{\sigma_t(x)} \circ A_{\sigma_t(x),t} \\ &= E E_{\mathcal{F}_{\sigma',t}} B^*_{\sigma_t(x),t}(x,I)\, \bar{w}_{\sigma_t(x)} \circ A_{\sigma_t(x),t}(x) \\ &= E \left(\Big[E B^*_{\sigma_r(y),r}(y,M)\, \bar{w}_{\sigma_r(y)} \circ A_{\sigma_r(y),r}(y) \Big]_{\substack{r=\sigma',\, y=A_{\sigma',t}(x), \\ M=B_{\sigma',t}(x,I)}} \right) \\ &= E \left(\Big[M^* E B^*_{\sigma_r(y),r}(y,I)\, \bar{w}_{\sigma_r(y)} \circ A_{\sigma_r(y),r}(y) \Big]_{\substack{r=\sigma',\, y=A_{\sigma',t}(x), \\ M=B_{\sigma',t}(x,I)}} \right) \\ &= E \nabla^* A_{\sigma',t}(x)\, \bar{w}_{\sigma'} \circ A_{\sigma',t}(x), \end{aligned}$$

proving that (4.6) holds.

Now, choose $\sigma' = s \vee \sigma_t(x)$ for $s < t$. Note that for any $x \in D$, we must have $\sigma_t(x) < t$ almost surely. Thus, omitting the spatial coordinate for convenience, we have

$$\begin{aligned}
0 = \lim_{s\to t^-} \frac{\bar w_t - \bar w_t}{t-s} &= \lim_{s\to t^-} \frac{1}{t-s}\Big(\bar w_t - E\nabla^* A_{s\vee\sigma_t,t}\,\bar w_{s\vee\sigma_t}\circ A_{s\vee\sigma_t,t}\Big)\\
&= \lim_{s\to t^-}\Bigg(\frac{1}{t-s}\left[\bar w_t - E\bar w_t\circ A_{s\vee\sigma_t,t}\right]+\\
&\qquad + \frac{1}{t-s}E\left(\bar w_t - \bar w_{s\vee\sigma_t}\right)\circ A_{s\vee\sigma_t,t}+\\
&\qquad + \frac{1}{t-s}E\left(I-\nabla^* A_{s\vee\sigma_t,t}\right)\,\bar w_{s\vee\sigma_t}\circ A_{s\vee\sigma_t,t}\Bigg)\\
&= L_t\bar w_t + \partial_t \bar w_t + (\nabla^* u_t)\bar w_t,
\end{aligned}$$

on the interior of D. This finishes the proof. □

It remains to prove the identity (4.9).

Proof of equation (4.9). Define the stopped processes $A'_{s,t}(x) = A_{\sigma_t(x)\vee s,t}(x)$, and $B'_{s,t}(x,M) = B_{\sigma_t(x)\vee s,t}(x,M)$. Define the process C by

$$C_{s,t}(x,M,\tau) = (A'_{s,t}(x), B'_{s,t}(x,M), \tau + t - \sigma_t(x)\vee s).$$

Note that for any given $s \leqslant t$, we know that $\sigma_t(x)$ need not be $\mathcal{F}_{s,t}$ measurable. However, $\sigma_t(x)\vee s$ *is* an $\mathcal{F}_{s,t}$ measurable backward t-stopping time. Thus $A'_{s,t}$, $B'_{s,t}$, and consequently $C_{s,t}$, are all $\mathcal{F}_{s,t}$ measurable.

Now we claim that almost surely, for $0 \leqslant r \leqslant s \leqslant t \leqslant T$, we have the backward semigroup identity

$$C_{r,s}\circ C_{s,t} = C_{r,t}. \tag{4.10}$$

To prove this, consider first the third component of the left hand side of (4.10)

$$C^{(3)}_{r,s}\circ C_{s,t}(x,M,\tau) = (\tau + t - \sigma_t(x)\vee s) + s - \sigma_s(A'_{s,t}(x))\vee s. \tag{4.11}$$

Consider the event $\{s > \sigma_t(x)\}$. By the semigroup property for A, and strong existence and uniqueness of solutions to (2.2), we have $\sigma_s(A_{s,t}(x)) = \sigma_t(x)$ almost surely. Thus, almost surely on $\{s > \sigma_t(x)\}$, we have

$$\begin{aligned}
C^{(3)}_{r,s}\circ C_{s,t}(x,M,\tau) &= (\tau + t - s) + s - \sigma_t(x)\vee s\\
&= \tau + t - \sigma_t(x)\vee r = C^{(3)}_{r,t}(x,M,\tau).
\end{aligned}$$

Now consider the event $\{s \leqslant \sigma_t\}$. We know $A'_{s,t}(x)\in\partial D$, and so $\sigma_s(A'_{s,t}(x)) = s$. This gives

$$C^{(3)}_{r,s}\circ C_{s,t}(x,M,\tau) = (\tau + t - \sigma_t(x)) + s - s = \tau + t - \sigma_t(x)\vee r = C^{(3)}_{r,t}(x)$$

almost surely on $\{s \leqslant \sigma_t(x)\}$. Therefore we have proved almost sure equality of the third components in equation (4.10).

For the first component $C^{(1)}_{s,t} = A'_{s,t}$, consider as before the case $s > \sigma_t(x)$. In this case $A'_{s,t} = A_{s,t}$, and the semigroup property of A gives equality of the first components in (4.10) almost surely on $\{s > \sigma_t(x)\}$. When $s \leqslant \sigma_t(x)$, as before, $A'_{s,t}\in\partial D$, and $\sigma_s(A'_{s,t}(x)) = s$. Thus

$$A'_{r,s}\circ A'_{s,t}(x) = A_{s,s}\circ A_{\sigma_t(x),t}(x) = A_{\sigma_t(x),t}(x) = A'_{r,t}(x)$$

almost surely on $s \leqslant \sigma_t(x)$. This shows almost sure equality of the first components in equation (4.10). Almost sure equality of the second components follows similarly, completing the proof of (4.10).

Now, for $0 \leqslant r \leqslant s \leqslant t \leqslant T$, the random variable $C_{s,t}$ is $\mathcal{F}_{s,t}$ measurable, and so must be independent of $\mathcal{F}_{r,s}$. This, along with (4.10), will immediately guarantee the Markov property for C. Since the filtration $\mathcal{F}_{\cdot,\cdot}$ satisfies the usual conditions, and for any fixed t, the function $s \mapsto C_{s,t}$ is continuous, C satisfies the strong Markov property (see for instance [16, Theorem 2.4]).

Thus, for any fixed $t \in [0,T]$, and any Borel function φ, the strong Markov property gives

$$\begin{aligned} E_{\mathcal{F}_{\sigma'_t,t}}\varphi(C_{0,t}(x,I,0)) &= [E\varphi(C_{r,t}(y,M,\tau)]_{\substack{r=\sigma'_t \\ (y,M,\tau)=C_{0,\sigma'}(x,I,0)}} \\ &= [E\varphi(C_{r,t}(y,M,\tau)]_{\substack{r=\sigma'_t,\, y=A_{\sigma'_t,t}(x), \\ M=B_{\sigma'_t,t}(x,I),\, \tau=\sigma_r(x),}} \end{aligned}$$

almost surely for any $x \in \mathbb{R}^d$, $M \in \mathbb{R}^{d^2}$, $\tau \geqslant 0$. Choosing $\varphi(x,M,\tau) = M^* \bar{w}_{t-\tau}(x)$ proves (4.9). $\square$

Now a direct computation shows that if $\bar{w}$ satisfies (2.12), then $u = \boldsymbol{P}\bar{w}$ satisfies (1.1), regardless of our choice of $\tilde{w}$. Of course, we will only get the no-slip boundary conditions with the correct choice of $\tilde{w}$. We first obtain the PDE for u.

Lemma 4.3. *If $\bar{w}$ satisfies* (2.12)*, and $u = \boldsymbol{P}\bar{w}$, then u satisfies* (1.1)–(1.2)*.*

Proof. By definition of the Leray-Hodge projection, $u = w + \nabla q$ for some function q, and equation (1.2) is automatically satisfied. Thus using equation (2.12) we have

$$\begin{aligned} \partial_t u_t + (u_t \cdot \nabla) u_t - \nu \triangle u_t + (\nabla^* u_t) u_t + \\ + \partial_t \nabla q_t + (u_t \cdot \nabla) \nabla q_t + (\nabla^* u_t) \nabla q_t - \nu \triangle \nabla q_t = 0. \end{aligned} \tag{4.12}$$

Defining p by

$$\nabla p = \nabla \left(\frac{1}{2} |u|^2 + \partial_t q_t + (u_t \cdot \nabla) q_t - \nu \triangle q_t \right)$$

equation (4.12) becomes (1.1). $\square$

Now to address the no-slip boundary condition. The curl of $\bar{w}$ satisfies the vorticity equation, which is how the vorticity enters our boundary condition.

Lemma 4.4. *Let $\bar{w}$ be a solution of* (2.12)*. Then $\xi = \nabla \times \bar{w}$ satisfies the vorticity equation*

$$\partial_t \xi + (u \cdot \nabla)\xi - \nu \triangle \xi = \begin{cases} 0 & \text{if } d = 2, \\ (\xi \cdot \nabla) u & \text{if } d = 3. \end{cases} \tag{4.13}$$

Proof. We only provide the proof for $d = 3$. For this proof we will use subscripts to indicate the component, instead of time as we usually do. If $i, j, k \in \{1, 2, 3\}$ are all distinct, let ε_{ijk} denote the signature of the permutation $(1, 2, 3) \mapsto (i, j, k)$. For convenience we let $\varepsilon_{ijk} = 0$ if i, j, k are not all distinct. Using the Einstein sumation convention, $\xi = \nabla \times \bar{w}$ translates to $\xi_i = \varepsilon_{ijk} \partial_j \bar{w}_k$ on components. Thus, taking the curl of (2.12) gives

$$\partial_t \xi_i + (u \cdot \nabla)\xi_i - \nu \triangle \xi_i + \varepsilon_{ijk} \partial_j u_m \partial_m \bar{w}_k + \varepsilon_{ijk} \partial_k u_m \partial_j \bar{w}_m = 0 \tag{4.14}$$

because $\varepsilon_{ijk}\partial_j\partial_k u_m\,\bar{w}_m = 0$. Making the substitutions $j \mapsto k$ and $k \mapsto j$ in the last sum above we have

$$\begin{aligned}\varepsilon_{ijk}\partial_j u_m \partial_m \bar{w}_k + \varepsilon_{ijk}\partial_k u_m \partial_j \bar{w}_m &= \varepsilon_{ijk}\partial_j u_m \left(\partial_m \bar{w}_k - \partial_k \bar{w}_m\right)\\ &= \varepsilon_{ijk}\partial_j u_m\, \varepsilon_{nmk}\xi_n\\ &= \left(\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn}\right)\partial_j u_m \xi_n\\ &= -\partial_j u_i \xi_j\end{aligned}$$

where δ_{ij} denotes the Kronecker delta function, and the last equality follows because $\partial_j u_j = 0$. Thus (4.14) reduces to (4.13). $\square$

Theorem 2.1 now follows from the above lemmas.

Proof of Theorem 2.1. First suppose u is a solution of the Navier-Stokes equations, as in the statement of the Theorem. We choose $\tilde{w}$ as explained in Remark 2.5. Notice that our assumptions on u and D will guarantee a classical solution to (2.12)–(2.14) exists on the interval $[0,T]$, and thus such a choice is possible.

By Lemma 4.1, we see that for w defined by (2.9), the expected value $\bar{w} = Ew$ satisfies (2.12) with initial data (2.13), and boundary conditions (4.2). By our choice of $\tilde{w}$, and uniqueness to the Dirichlet problem (2.12), (2.13) and (4.2), we must have the vorticity boundary condition (2.14).

Now, let $\xi = \nabla \times \bar{w}$, and $\omega = \nabla \times u$. By Lemma 4.4, we see that ξ satisfies the vorticity equation (4.13). Since u satisfies (1.1)–(1.2), it is well known (see for instance [13,26], or the proof of Lemma 4.4) that ω also satisfies

$$\partial_t \omega_t + (u_t \cdot \nabla)\omega_t - \nu \triangle \omega_t = \begin{cases} 0 & \text{if } d = 2,\\ (\omega_t \cdot \nabla) u_t & \text{if } d = 3.\end{cases} \tag{4.15}$$

From (2.14) we know $\xi = \omega$ on $\partial D \times [0,T]$. By (2.13), we see that $\xi_0 = \nabla \times u_0 = \omega_0$, and hence $\xi = \omega$ on the parabolic boundary $\partial_p(D \times [0,T])$.

The above shows that ω and ξ both satisfy the same PDE (equations (4.13) or (4.15)), with the same initial data, and boundary conditions, and so we must have $\xi = \omega$ on $D \times [0,T]$. Thus $\nabla \times \bar{w} = \nabla \times u$ in $D \times [0,T]$, showing u and $\bar{w}$ differ by a gradient. Since $\nabla \cdot u = 0$, and $u = 0$ on $\partial D \times [0,T]$, we must have $u = \boldsymbol{P}\bar{w}$ proving (2.10).

Conversely, assume we have a solution to the system (2.2), (2.9) and (2.10). As above, Lemma 4.1 shows $\bar{w} = Ew$ satisfies (2.12) with initial data (2.13). By Lemma 4.3, we know u satisfies the equation (1.1)–(1.2) with initial data u_0. Finally, since equation (2.10) shows $\nabla \times u = \nabla \times \bar{w}$ in $D \times [0,T]$, and by continuity, we have the boundary condition (2.11). $\square$

5. Vorticity transport, and ideally conserved quantities.

For the Euler equations, certain conservation laws (e.g. circulation) and exact identities (e.g. vorticity transport) are well known. In the absence of spatial boundaries, inviscid identities usually remain true in expectation. With boundaries, however, we run into regularity issues which, at present, can not always be resolved.

In this section we illustrate the issues involved by considering three inviscid identities. The first identity (vorticity transport) generalizes perfectly to the viscous scenario with boundaries. The second one (Ertel's Theorem) generalizes perfectly to the viscous scenario *without boundaries*, and has a somewhat unsatisfactory generalization in the presence of boundaries. The last one (conservation of circulation), again generalizes perfectly to the viscous scenario without boundaries, but has a completely unsatisfactory generalization to the situation with boundaries.

5.1. **Vorticity transport.** Let u^0 be a solution to the Euler equations with initial data u_0. Let X^0 the inviscid flow map defined by (1.3), and for any $t \geqslant 0$, let $A^0_t = (X^0_t)^{-1}$ be the spatial inverse of the diffeomorphism X^0_t. The vorticity transport (or Cauchy formula) states

$$\omega^0_t = \begin{cases} \omega^0_0 \circ A^0_t & \text{if } d = 2, \\ \left[(\nabla X^0_t)\, \omega^0_0\right] \circ A^0_t & \text{if } d = 3. \end{cases} \tag{5.1}$$

where, we recall that the vorticity ω^0 is defined by $\omega^0 = \nabla \times u^0$, and where $\omega^0_0 = \nabla \times u_0$ is the initial vorticity.

In [9], the authors obtained a natural generalization of (5.1) for the Navier-Stokes equations, in the absence of spatial boundaries. If u solves (1.1)–(1.2) with initial data u_0, and X is the noisy flow map defied by (1.5)–(1.6), then $\omega = \nabla \times u$ is given by

$$\omega_t = \begin{cases} E\omega_0 \circ A_t & \text{if } d = 2, \\ E\left((\nabla X_t)\, \omega_0\right) \circ A_t & \text{if } d = 3. \end{cases} \tag{5.2}$$

We now provide the generalization of this in the presence of boundaries. Note that for any $t \geqslant 0$, $(\nabla X_t) \circ A_t = (\nabla A_t)^{-1}$, so we can rewrite (5.2) completely in terms of the process A. Now, as usual, we replace $A = X^{-1}$ with the solution of (2.2), with respect to the minimal existence time σ. We recall that in Theorem 2.1, in addition to "starting trajectories at the boundary," we had to correct the expression for the velocity by the boundary values of a related quantity (the vorticity). For the vorticity, however, we need no additional correction, and the interior vorticity is completely determined given A, σ and the vorticity on the parabolic boundary[8] $\partial_p(D \times [0,T])$.

Proposition 5.1. *Let u be a solution to (1.1)–(1.2) in D, with initial data u_0, and suppose $\omega = \nabla \times u \in C^1([0,T); C^2(D)) \cap C([0,T] \times \bar{D})$. Let $\tilde{\omega}$ denote the values of ω on the parabolic boundary $\partial_p(D \times [0,T])$. Explicitly, $\tilde{\omega}$ is defined by*

$$\tilde{\omega}(x,t) = \begin{cases} \omega_0(x) & \text{if } x \in D \text{ and } t = 0, \\ \omega_t(x) & \text{if } x \in \partial D. \end{cases}$$

Then,

$$\omega_t(x) = \begin{cases} E\left[\tilde{\omega}_{\sigma_t(x)}\left(A_{\sigma_t(x),t}(x)\right)\right] & \text{if } d = 2, \\ E\left[\left(\nabla A_{\sigma_t(x),t}(x)\right)^{-1} \tilde{\omega}_{\sigma_t(x)}\left(A_{\sigma_t(x),t}(x)\right)\right] & \text{if } d = 3. \end{cases} \tag{5.3}$$

Remark 5.2. More generally, suppose $\tilde{\omega}$ is any function defined on the parabolic boundary of $D \times [0,T]$, and let ω be defined by (5.3). If for all $t \in (0,T]$, $\omega_t \in \mathcal{D}(A_{\cdot,t})$, and ω is C^1 in time, then ω satisfies

$$\partial_t \omega_t + L_t \omega_t = \begin{cases} 0 & \text{if } d = 2, \\ (\omega_t \cdot \nabla) u_t & \text{if } d = 3, \end{cases}$$

[8]Recall, the parabolic boundary $\partial_p(D \times [0,T])$ is defined to be $(D \times \{0\}) \cup (\partial D \times [0,T))$

with $\omega = \tilde{\omega}$ on the parabolic boundary. Here L_t is the generator of $A_{\cdot,t}$, $\mathcal{D}(A_{\cdot,t})$ is the domain of L_t. These are defined in the statement of Lemma 4.1.

Of course, Remark (5.2) along with Proposition 4.2 and uniqueness of (strong) solutions to (4.15) will prove Proposition 5.1. However, direct proofs of both Remark 5.2 and Proposition 5.1 are short and instructive, and we provide independent proofs of each.

Proof of Proposition 5.1. We only provide the proof when $d = 3$. As before, differentiating (4.1) in space, and taking the matrix inverse of both sides gives

$$\partial_r \left(\nabla A_{r,t}(x)\right)^{-1} = -\left(\nabla A_{r,t}(x)\right)^{-1} \nabla u_r|_{A_{r,t}(x)} \tag{5.4}$$

almost surely. Now choose any $x \in D$, $t > 0$ and any backward t-stopping time $\sigma' \geqslant \sigma_t(x)$. Omitting the spatial parameter for notational convenience, the backward Itô formula gives

$$\begin{aligned}
\omega_t - &\left(\nabla A_{\sigma',t}\right)^{-1} \omega_{\sigma'} \circ A_{\sigma',t} = \\
&= \left(\nabla A_{t,t}\right)^{-1} \omega_t \circ A_{t,t} - \left(\nabla A_{\sigma',t}\right)^{-1} \omega_{\sigma'} \circ A_{\sigma',t} \\
&= \int_{\sigma'}^t \partial_r \left(\nabla A_{r,t}\right)^{-1} \omega_r \circ A_{r,t}\, dr + \\
&\qquad + \int_{\sigma'}^t \left(\nabla A_{r,t}\right)^{-1} \left(\partial_r \omega_r + (u_r \cdot \nabla)\omega_r - \nu \triangle \omega_r\right) \circ A_{r,t}\, dr + \\
&\qquad + \sqrt{2\nu} \int_{\sigma'}^t \left(\nabla A_{r,t}\right)^{-1} \left(\nabla \omega_r\right) \circ A_{r,t}\, dW_r \\
&= \int_{\sigma'}^t -\left(\nabla A_{r,t}\right)^{-1} \nabla u_r|_{A_{r,t}}\, \omega_r \circ A_{r,t}\, dr \; + \\
&\qquad + \int_{\sigma'}^t \left(\nabla A_{r,t}\right)^{-1} \left((\omega_r \cdot \nabla) u_r\right) \circ A_{r,t}\, dr + \\
&\qquad + \sqrt{2\nu} \int_{\sigma'}^t \left(\nabla A_{r,t}\right)^{-1} \left(\nabla \omega_r\right) \circ A_{r,t}\, dW_r \\
&= \sqrt{2\nu} \int_{\sigma'}^t \left(\nabla A_{r,t}\right)^{-1} \left(\nabla \omega_r\right) \circ A_{r,t}\, dW_r
\end{aligned}$$

Thus taking expected values gives

$$\omega_t = E\left[(\nabla A_{\sigma',t})^{-1} \omega_{\sigma'} \circ A_{\sigma',t} \right] \tag{5.5}$$

Choosing $\sigma' = \sigma_t(x)$, using the fact that $A_{\sigma_t(x),t}(x)$ always belongs to the parabolic boundary finishes the proof. □

Proof of Remark 5.2. Again, we only consider the case $d = 3$. We will prove (5.5) directly, and then deduce (4.15). Let the process B be as in the proof of the second assertion of Lemma 4.1, and use B^{-1} to denote the process consisting of *matrix* inverses of the process B. Pick $x \in D$, $t \in (0, T]$ and a backward t-stopping time $\sigma' \geqslant \sigma_t(x)$. Using (4.9) we have

$$\begin{aligned}
\omega_t(x) &= E\left[\left(\nabla A_{\sigma_t(x),t}(x)\right)^{-1} \tilde{\omega}_{\sigma_t(x)} \left(A_{\sigma_t(x),t}(x)\right) \right] \\
&= E E_{\mathcal{F}_{\sigma',t}} \left[B^{-1}_{\sigma_t(x),t}(x, I)\, \tilde{\omega}_{\sigma_t(x)} \circ A_{\sigma_t(x),t}(x) \right]
\end{aligned}$$

$$= E\left(\left[EB^{-1}_{\sigma_r(y),r}(y,M)\,\tilde{\omega}_{\sigma_r(y)}\circ A_{\sigma_r(y),r}(y)\right]_{\substack{r=\sigma',y=A_{\sigma',t}(x),\\ M=B_{\sigma',t}(x,I)}}\right)$$
$$= E\left(\left[M^{-1}EB^{-1}_{\sigma_r(y),r}(y,I)\,\tilde{\omega}_{\sigma_r(y)}\circ A_{\sigma_r(y),r}(y)\right]_{\substack{r=\sigma',y=A_{\sigma',t}(x),\\ M=B_{\sigma',t}(x,I)}}\right)$$
$$= E\left[\left(\nabla A_{\sigma',t}(x)\right)^{-1}\omega_{\sigma'}\circ A_{\sigma',t}(x)\right].$$

proving (5.5).

As before, choose $s \leqslant t$ and $\sigma' = \sigma_t(x) \vee s$. Omitting the spatial parameter for notational convenience gives

$$0 = \lim_{s\to t^-}\frac{\omega_t-\omega_t}{t-s} = \lim_{s\to t^-}\frac{1}{t-s}\left[\omega_t - E\left(\nabla A_{\sigma_t\vee s,t}\right)^{-1}\omega_{\sigma_t\vee s}\circ A_{\sigma_t\vee s,t}\right]$$
$$= \lim_{s\to t^-}\Bigg(\frac{1}{t-s}\left[\omega_t - E\omega_t\circ A_{\sigma_t\vee s,t}\right] +$$
$$+\frac{1}{t-s}E\left[\omega_t-\omega_{\sigma_t\vee s}\right]\circ A_{\sigma_t\vee s,t} +$$
$$+\frac{1}{t-s}E\left[I-\left(\nabla A_{\sigma_t\vee s,t}\right)^{-1}\right]\omega_{\sigma_t\vee s}\circ A_{\sigma_t\vee s,t}\Bigg)$$
$$= L_t\omega_t + \partial_t\omega_t - (\nabla u_t)\,\omega_t \qquad \square$$

We remark that the vorticity transport in Proposition 5.1, or Remark 5.2 can be used to provide a stochastic representation of the Navier-Stokes equations. Indeed, since u is divergence free, taking the curl twice gives the negative laplacian. Thus, provided boundary conditions on u are specified, we can obtain u from ω by

$$u_t = (-\triangle)^{-1}\nabla\times\omega_t. \tag{5.6}$$

Therefore, in Theorem 2.1, we can replace (2.10) by (5.3) and (5.6), where $\tilde{\omega}$ is the vorticity on the parabolic boundary, and we impose 0-Dirichlet boundary conditions on (5.6).

5.2. **Ertel's Theorem.** As above, we use a superscript of 0 to denote the appropriate quantities related to the Euler equations. For this section we also assume $d = 3$. Ertel's theorem says that if θ^0 is constant along trajectories of X^0, then so is $(\omega^0\cdot\nabla)\theta^0$. Hence $\phi^0 = (\omega^0\cdot\nabla)\theta^0$ satisfies the PDE

$$\partial_t\phi^0 + (u\cdot\nabla)\phi^0 = 0.$$

For the Navier-Stokes equations, we first consider the situation without boundaries. Let u solve (1.1)–(1.2), X be defined by (1.5), A be the spatial inverse of X, and define ξ by

$$\xi_t(x) = (\nabla A_t(x))^{-1}\omega_0\circ A_t(x).$$

where $\omega_0 = \nabla\times u_0$ is the initial vorticity. From (5.2), we know that $\omega = \nabla\times u = E\xi$. Now we can generalize Ertel's theorem as follows.

Proposition 5.3. *Let θ be a $C^1(\mathbb{R}^d)$ valued process. If θ is constant along trajectories of the (stochastic) flow X, then so is $(\xi\cdot\nabla)\theta$. Hence $\phi = E(\xi\cdot\nabla)\theta$ satisfies the PDE*

$$\partial_t\phi_t + (u_t\cdot\nabla)\phi_t - \nu\triangle\phi_t = 0, \tag{5.7}$$

with initial data $(\omega_0 \cdot \nabla)\theta_0$.

Proof. If θ is constant along trajectories of X, we must have $\theta_t = \theta_0 \circ A_t$ almost surely. Thus,

$$(\xi_t \cdot \nabla)\theta_t = (\nabla\theta_t)\xi_t = \nabla\theta_0\big|_{A_t} (\nabla A_t)(\nabla A_t)^{-1}\omega_0 \circ A_t = (\xi_0 \cdot \nabla\theta_0) \circ A_t,$$

which is certainly constant along trajectories of X. The PDE for ϕ now follows immediately. ☐

Now in the presence of boundaries this further modification. Let A be a solution to (2.2) with minimal existence time σ. The notion of "constant along trajectories" now corresponds to processes θ defined by

$$\theta_t(x) = \tilde{\theta}_{\sigma_t(x)}(A_{\sigma_t(x),t}), \tag{5.8}$$

for some function $\tilde{\theta}$ defined on the parabolic boundary of D.

Unfortunately, irrespective of the regularity of D and $\tilde{\theta}$, the process θ will not be continuous in space, let alone differentiable. The problem arises because while A is regular enough in the spatial variable, the existence time σ_t is not. To work around this, we avoid derivatives on σ in the statement of the theorem.

Proposition 5.4. *Let $\tilde{\theta}$ be a C^1 function defined on the parabolic boundary of $D \times [0,T]$, and let $\tilde{\theta}'$ be any C^1 extension of $\tilde{\theta}$, defined in a neighborhood of the parabolic boundary of $D \times [0,T]$. If θ is defined by* (5.8)*, then*

$$\phi_t(x) = E\big[(\xi_t \cdot \nabla)(\tilde{\theta}'_s \circ A_{s,t})(x)\big]_{s=\sigma_t(x)}$$

satisfies the PDE (5.7) *with initial data $(\omega_0 \cdot \nabla)\tilde{\theta}_0$, and boundary conditions $\phi_t(x) = (\omega_t \cdot \nabla)\tilde{\theta}'(x)$ for $x \in \partial D$.*

Note that when $D = \mathbb{R}^d$ then $\sigma_t \equiv 0$, and hence $\phi_t = E(\xi_t \cdot \nabla)\theta_t$. In this case Proposition 5.4 reduces to Proposition 5.3. The proof of Proposition 5.3 is identical to that of Proposition 5.3, and the same argument obtains

$$\big[(\xi_t \cdot \nabla)(\tilde{\theta}'_s \circ A_{s,t})(x)\big]_{s=\sigma_t(x)} = \big[(\xi_s \cdot \nabla)\tilde{\theta}'_s(y)\big]_{\substack{s=\sigma_t(x),\\ y=A_{\sigma_t(x),t}(x),}}$$

which immediately implies (5.7).

In the scenario with boundaries, it would be interesting to know if one can make sense of $E(\xi_t \cdot \nabla)\theta_t$, and reformulate Proposition 5.4 accordingly, even though θ is not differentiable with respect to space. At present, we are unable to do this.

5.3. **Circulation.** The circulation is the line integral of the velocity field along a closed curve. For the Euler equations, the circulation along a closed curve that is transported by the flow is constant in time. Explicitly, let u^0, X^0, A^0, u_0 be as in the previous subsection. Let Γ be a rectifiable closed curve, then for any $t \geqslant 0$,

$$\oint_{X_t^0(\Gamma)} u_t^0 \cdot dl = \oint_\Gamma u_0^0 \cdot dl. \tag{5.9}$$

For the Navier-Stokes equations, without boundaries, a generalization of (5.9) was considered in [9]. Let u solve (1.1)–(1.2), X be defined by (1.5)–(1.6), and A be the spatial inverse of X. Then

$$\oint_\Gamma u_t \cdot dl = E \oint_{A_t(\Gamma)} u_0 \cdot dl. \tag{5.10}$$

A proof of this (in the absence of boundaries) follows immediately from Theorem 1.1. Indeed,

$$
\begin{aligned}
(5.11)\quad E\oint_{A_t(\Gamma)} u_0 \cdot dl = E\oint_{\Gamma} (\nabla^* A_t)\; u_0 \circ A_t \cdot dl = \\
= E\oint_{\Gamma} \boldsymbol{P}\left[(\nabla^* A_t)\; u_0 \circ A_t\right] \cdot dl = \oint_{\Gamma} u_t \cdot dl,
\end{aligned}
$$

where the first equality follows by definition of line integrals, the second because the line integral of gradients along closed curves is 0, and the last by Fubini and (1.7).

In the presence of boundaries, there are certain obstructions to making this work. Let A, σ, $\tilde{w}$ be as in the statement of Theorem 2.1. We extend $\tilde{w}$ to the parabolic boundary $\partial_p(D \times [0,T])$ by defining $\tilde{w}(x,0) = u_0(x)$ for $x \in D$. Now we would expect the natural generalization of (5.10) to be

$$
(5.12)\qquad \oint_{\Gamma} u_t \cdot dl = E\oint_{A_{\sigma_t,t}(\Gamma)} \tilde{w}_0 \cdot dl.
$$

We remark again, that though $u = 0$ on ∂D, we must have a non-zero contribution from trajectories starting on the side of the cylinder $D\times[0,T]$. However, the integral on the right is not well defined, as the curve $A_{\sigma_t,t}(\Gamma)$ is not necessarily rectifiable!

Now, as with Ertel's theorem, we can try and avoid irregularities from σ when we transport Γ. Indeed, almost tautologically we have

$$
(5.13)\qquad \oint_{\Gamma} u_t \cdot dl = E\oint_{\Gamma} \left[\nabla^* A_{\sigma_t(x),t}(x)\, w_t \circ A_{\sigma_t(x),t}(x)\right] \cdot dl(x).
$$

Further, if $\sigma_t \equiv 0$, the right hand side of equation (5.13) is exactly the right hand side of (5.10).

However (5.13) is essentially a tautological rephrasing of (1.7), and does not capture the essence of (5.10). It would be interesting indeed if one can give meaning to the right hand side of (5.12), and then prove (5.12). At present, we are unable to carry out this construction.

References

[1] V. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier (Grenoble) **16** (1966), no. fasc. 1, 319–361 (French). MR0202082 (34 #1956)

[2] R. N. Bhattacharya, L. Chen, S. Dobson, R. B. Guenther, C. Orum, M. Ossiander, E. Thomann, and E. C. Waymire, *Majorizing kernels and stochastic cascades with applications to incompressible Navier-Stokes equations*, Trans. Amer. Math. Soc. **355** (2003), no. 12, 5003–5040 (electronic), DOI 10.1090/S0002-9947-03-03246-X. MR1997593 (2004k:35294)

[3] R. Bhattacharya, L. Chen, R. B. Guenther, C. Orum, M. Ossiander, E. Thomann, and E. C. Waymire, *Semi-Markov cascade representations of local solutions to 3-D incompressible Navier-Stokes*, Probability and partial differential equations in modern applied mathematics, IMA Vol. Math. Appl., vol. 140, Springer, New York, 2005, pp. 27–40. MR2202031

[4] B. Busnello, *A probabilistic approach to the two-dimensional Navier-Stokes equations*, Ann. Probab. **27** (1999), no. 4, 1750–1780. MR1742887 (2001g:60140)

[5] B. Busnello, F. Flandoli, and M. Romito, *A probabilistic representation for the vorticity of a three-dimensional viscous fluid and for general systems of parabolic equations*, Proc. Edinb. Math. Soc. (2) **48** (2005), no. 2, 295–336. MR2157249 (2006d:35180)

[6] P. Constantin, *An Eulerian-Lagrangian approach for incompressible fluids: local theory*, J. Amer. Math. Soc. **14** (2001), no. 2, 263–278 (electronic). MR1815212 (2002e:76008)

[7] P. Constantin, *Generalized relative entropies and stochastic representation*, Int. Math. Res. Not. (2006), Art. ID 39487, 9. MR2250023 (2008c:76087)

[8] P. Constantin and C. Foias, *Navier-Stokes equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988. MR972259 (90b:35190)
[9] P. Constantin and G. Iyer, *A stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations*, Comm. Pure Appl. Math. **61** (2008), no. 3, 330–345. MR2376844
[10] P. Constantin and G. Iyer, *Stochastic Lagrangian transport and generalized relative entropies*, Commun. Math. Sci. **4** (2006), no. 4, 767–777. MR2264819
[11] F. Cipriano and A. B. Cruzeiro, *Navier-Stokes equation and diffusions on the group of homeomorphisms of the torus*, Comm. Math. Phys. **275** (2007), no. 1, 255–269. MR2335775 (2008j:37170)
[12] A. J. Chorin, *Numerical study of slightly viscous flow*, J. Fluid Mech. **57** (1973), no. 4, 785–796. MR0395483 (52 #16280)
[13] A. J. Chorin and J. E. Marsden, *A mathematical introduction to fluid mechanics*, 3rd ed., Texts in Applied Mathematics, vol. 4, Springer-Verlag, New York, 1993. MR1218879 (94c:76002)
[14] G. L. Eyink, *Stochastic Least-Action Principle for the Incompressible Navier-Stokes Equation* (2008), preprint, available at `arXiv:0810.0817`.
[15] G. L. Eyink, *Stochastic Line-Motion and Stochastic Conservation Laws for Non-Ideal Hydromagnetic Models. I. Incompressible Fluids and Isotropic Transport Coefficients* (2008), preprint, available at `arXiv:0812.0153`.
[16] A. Friedman, *Stochastic differential equations and applications*, Dover Publications Inc., Mineola, NY, 2006. Two volumes bound as one; Reprint of the 1975 and 1976 original published in two volumes. MR2295424 (2007k:60005)
[17] Y. Gliklikh, *Global analysis in mathematical physics*, Applied Mathematical Sciences, vol. 122, Springer-Verlag, New York, 1997. Geometric and stochastic methods; Translated from the 1989 Russian original and with Appendix F by Viktor L. Ginzburg. MR1438545 (98e:58001)
[18] G. Iyer, *A stochastic Lagrangian formulation of the Navier-Stokes and related transport equations*, Ph. D. Thesis, University of Chicago, 2006.
[19] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, 2nd ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991. MR1121940 (92h:60127)
[20] N. V. Krylov, *Quasiderivatives for solutions of Itô's stochastic equations and their applications*, Stochastic analysis and related topics (Oslo, 1992), Stochastics Monogr., vol. 8, Gordon and Breach, Montreux, 1993, pp. 1–44. MR1268004 (95e:60057)
[21] N. V. Krylov, *Quasiderivatives and interior smoothness of harmonic functions associated with degenerate diffusion processes*, Electron. J. Probab. **9** (2004), no. 21, 615–633 (electronic). MR2082053 (2005j:60148)
[22] N. V. Krylov and B. L. Rozovskiĭ, *Stochastic partial differential equations and diffusion processes*, Uspekhi Mat. Nauk **37** (1982), no. 6(228), 75–95 (Russian). MR683274 (84d:60095)
[23] E. A. Kuznetsov and V. P. Ruban, *Hamiltonian dynamics of vortex and magnetic lines in hydrodynamic type systems*, Phys. Rev. E. **61** (2000), 831–841.
[24] H. Kunita, *Stochastic flows and stochastic differential equations*, Cambridge Studies in Advanced Mathematics, vol. 24, Cambridge University Press, Cambridge, 1997. Reprint of the 1990 original. MR1472487 (98e:60096)
[25] Y. Le Jan and A. S. Sznitman, *Stochastic cascades and* 3*-dimensional Navier-Stokes equations*, Probab. Theory Related Fields **109** (1997), no. 3, 343–366. MR1481125 (98j:35144)
[26] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, vol. 27, Cambridge University Press, Cambridge, 2002. MR1867882 (2003a:76002)
[27] P. Michel, S. Mischler, and B. Perthame, *General entropy equations for structured population models and scattering*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 9, 697–702 (English, with English and French summaries). MR2065377 (2005b:35019)
[28] P. Michel, S. Mischler, and B. Perthame, *General relative entropy inequality: an illustration on growth models*, J. Math. Pures Appl. (9) **84** (2005), no. 9, 1235–1260 (English, with English and French summaries). MR2162224 (2006g:35020)
[29] M. Ossiander, *A probabilistic representation of solutions of the incompressible Navier-Stokes equations in* $\mathbf{R}^3$, Probab. Theory Related Fields **133** (2005), no. 2, 267–298. MR2198702 (2007f:76159)

[30] B. L. Rozovskiĭ, *Stochastic evolution systems*, Mathematics and its Applications (Soviet Series), vol. 35, Kluwer Academic Publishers Group, Dordrecht, 1990. Linear theory and applications to nonlinear filtering; Translated from the Russian by A. Yarkho. MR1135324 (92k:60136)

[31] V. P. Ruban, *Motion of magnetic flux lines in magnetohydrodynamics*, JETP **89** (1999), 299–310.

[32] E. Thomann and M. Ossiander, *Stochastic cascades applied to the Navier-Stokes equations*, Probabilistic methods in fluids, World Sci. Publ., River Edge, NJ, 2003, pp. 287–297, DOI 10.1142/9789812703989_0019, (to appear in print). MR2083379 (2005g:76031)

[33] E. C. Waymire, *Multiscale and Multiplicative Processes in Fluid Flows*, Instructional and Research Workshop on Multiplicative Processes and Fluid Flows (MaPhySto, Aarhus University, 2001), Lectures on Multiscale and Multiplicative Processes in Fluid Flows, Vol. 11, 2002, available at `http://www.maphysto.dk/cgi-bin/gp.cgi?publ=407`.

[34] W. Webber, *Über eine Transformation der hydrodynamischen Gleichungen*, J. Reine Angew. Math. **68** (1968), 286-292.

[35] X. Zhang, *A stochastic representation for backward incompressible Navier-Stokes equations*, Probab. Theory Related Fields (2009), to appear, available at `arXiv:0810.4664`.

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