

LONG TIME DYNAMICS OF FORCED CRITICAL SQG

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ABSTRACT. We prove the existence of a compact global attractor for the dynamics of the forced critical surface quasi-geostrophic equation (SQG) and prove that it has finite fractal (box-counting) dimension. In order to do so we give a new proof of global regularity for critical SQG. The main ingredient is the nonlinear maximum principle in the form of a nonlinear lower bound on the fractional Laplacian, which is used to bootstrap the regularity directly from L^∞ to C^α , without the use of De Giorgi techniques. We prove that for large time, the norm of the solution measured in a sufficiently strong topology becomes bounded with bounds that depend solely on norms of the force, which is assumed to belong merely to $L^\infty \cap H^1$. Using the fact that the solution is bounded independently of the initial data after a transient time, in spaces conferring enough regularity, we prove the existence of a compact absorbing set for the dynamics in H^1 , obtain the compactness of the linearization and the continuous differentiability of the solution map. We then prove exponential decay of high yet finite dimensional volume elements in H^1 along solution trajectories, and use this property to bound the dimension of the global attractor.

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1. INTRODUCTION

Nonlinear forced dissipative partial differential equations can generate physical space patterns which evolve in a temporally complex manner. As parameters are varied, the dynamics may transition from simple to chaotic and ultimately to fully turbulent. Nevertheless, several forced nonlinear dissipative PDE of hydrodynamic origin have been shown to have finite dimensional long time behavior. This has been proved if certain minimal conditions are satisfied. Chief among them is the property that linearizations about time evolving solutions are dominated in a certain sense by the linear dissipative part. This is the case for semi-linear dissipative PDE such as the Navier-Stokes system in 2D, or subcritical quasilinear damped systems. In this paper we study the long time behavior of a critical quasilinear system, the forced SQG system. In this system the nonlinearity has the same differential order as the nonlocal dissipation and the coefficients of the equation are of the same order as the unknown. Even the global regularity of solutions for the unforced equation is nontrivial and has been obtained only in the last years. The existence of a compact, finite

Date: January 10, 2014.

2000 Mathematics Subject Classification. 35Q35.

Key words and phrases. Surface quasi-geostrophic equation (SQG), nonlinear maximum principle, global attractor.

dimensional attractor requires much more than just regularity, it requires an infinite dimensional loss of information. This is what we prove in this paper. In order to do so, we have to invent a new proof of regularity that is both more direct and more explicitly quantitative than all the previous ones. The loss of information is reflected in long time bounds that are independent of initial data. This is entirely nontrivial for a critical quasilinear system, but it is true here because the smoothing effect is controlled by the L^∞ norm, which, in turn, can be shown to be ultimately bounded independently of the initial data. The latter result is based on a new uniform nonlinear Poincaré inequality, which is of independent interest.

We organize the paper in the following manner: the rest of the introduction presents some background material and explains some of the ideas of the proofs. Section 2 deals with preliminaries, including periodic setting, pointwise dissipativity and a statement of the uniform nonlinear Poincaré inequality, whose proof appeared in [CGHV13, Appendix A]. Section 3 is devoted to an example of our method of proof, in the simpler setting of the forced critical Burgers equation. Section 4 is devoted to the proof of global regularity of forced SQG, and Sections 5 and 6 to the existence and finite dimensionality of the attractor. We present some of the technical aspects in appendices, including a proof of backward uniqueness (for marginally smoother forces) and a lemma of general interest about switching the order of derivatives and supremum.

1.1. Physical origins of the model. The forced, critically dissipative surface quasi-geostrophic (SQG) equation is

$$\partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda \theta = f \tag{1.1}$$

$$u = \mathcal{R}^\perp \theta = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) \tag{1.2}$$

$$\theta(\cdot, 0) = \theta_0 \tag{1.3}$$

where $\Lambda = (-\Delta)^{1/2}$, $\mathcal{R}_j = \partial_j \Lambda^{-1}$ is the j th Riesz transform, and the equations are set on the periodic domain $\mathbb{T}^2 = [-\pi, \pi]^2$. Here $\kappa > 0$ is a positive constant, θ_0 is the initial condition, and $f = f(x)$ is a time-independent force. The force is assumed to belong to $L^\infty(\mathbb{T}^2) \cap H^1(\mathbb{T}^2)$, while the initial data is assumed to belong to $H^1(\mathbb{T}^2)$. We consider forcing and initial data of zero average, that is $\int_{\mathbb{T}^2} f(x) dx = \int_{\mathbb{T}^2} \theta_0(x) dx = 0$ which immediately implies that a solution θ of (1.1)–(1.3) obeys $\int_{\mathbb{T}^2} \theta(x, t) dx = 0$ for any $t \geq 0$. Throughout this manuscript we consider mean-zero (zero average) solutions.

The SQG equation describes the evolution of a surface temperature field θ in a rapidly rotating, stably stratified fluid with potential vorticity [CMT94, HPGS95]. From the mathematical point of view, the non-dissipative SQG equations (the system (1.1)–(1.3) with $\kappa = 0$) have properties that are similar to those of the 3D Euler equations in vorticity form [CMT94], and yet one may for instance prove the global existence of finite energy weak solutions [Res95], albeit for completely different reasons than for 2D Euler. Initial numerical simulations [CMT94, OY97, CNS98] have furthermore exhibited highly nonlinear features in the inviscid evolution such as the formation of sharp fronts. The latter issue has been studied analytically [Cor98, CF01, CF02, Cha08, FR11] and also in more recent numerical simulations [CFMR05, DHLY06, OS10, CLS⁺12]. Whether solutions of the inviscid SQG equations can develop singularities in finite time remains an open problem.

If in the three-dimensional quasi-geostrophic equations [Ped82] we take into account damping (Ekman pumping at the boundary) and external sources we arrive at the system the system (1.1)–(1.3). The square root of the Laplacian $\Lambda = (-\Delta)^{1/2}$ naturally arises as the Dirichlet to Neumann map for the surface temperature θ . See e.g. [Con02] and references therein.

Among the dissipative operators Λ^γ with $0 < \gamma \leq 2$, the exponent $\gamma = 1$ appearing in (1.1) is not just physically motivated but also mathematically challenging. In view of the underlying scaling invariance associated with (1.1) (see Remark 2.3 below) and the available conservation laws (the L^∞ maximum principle), it is customary to refer to the dissipation power $\gamma = 1$ as *critical*, although the question of whether a dramatic change in the behavior of the solution occurs for $\gamma < 1$ (the *supercritical* case) remains an open problem.

1.2. Global well-posedness: a brief summary of previous results. From the mathematical point of view, the issue of *global regularity vs finite-time blowup* for the fractionally dissipative SQG equation has been extensively studied over the past two decades. The subcritical case ($\gamma > 1$) has been essentially resolved in [Res95, CW99]. See also [Wu01, SS03, Ju04, Mar08, DL08] and references therein for further qualitative properties of weak and strong solutions in the subcritical case.

The issue of global regularity for the critical SQG equation is more challenging since the balance between the nonlinearity and the dissipative term in (1.1)–(1.3) is the same no matter at which scales one zooms in. This is why treating the equation as a perturbation of the fractional heat equation fails to be useful for large initial data (and nonlinearity). In order to obtain bounds on the solution one has to appeal to finer methods. Two different such sophisticated tools have been introduced independently in [CV10a, KNV07] to show that solutions of (1.1)–(1.3) do not develop singularities in finite time, for arbitrarily large initial data. Before these works, the global regularity was only known for initial data which is small in the L^∞ norm [CCW01] and other scaling critical spaces [CL03, CC04, Wu05, Miu06, Ju07, CMZ07, HK07].

The proof of global regularity in [KNV07] is based on constructing a family of Lipschitz moduli of continuity with the property that if the initial data obeys such a modulus so will the solution of the critical SQG equation for all later times, and so that this family behaves nicely under rescaling. See also [DD08] for applications of the modulus of continuity method to the case of the whole space, [FPV09] for global regularity in the presence of a Lipschitz forcing term, [KN10] in the presence of a linear dispersive force, and [SV12] for regularity of critical linear drift-diffusion equations.

The proof of [CV10a] on the other hand employs the ideas of De Giorgi iteration to the case of a nonlocal parabolic equation, and shows that bounded weak solutions must in fact instantly become Hölder continuous (and hence classical). We also refer to [CV10b, CCV11] and references therein for applications of the De Giorgi ideas to more degenerate nonlocal parabolic equations, and [CW09, Sil10b] for linear fractional advection-diffusion equations (the latter results for linear equations are in fact sharp [SVZ13]).

In an attempt to find a bridge between these two proofs, in [KN09] a completely different proof of global regularity for critical SQG was obtained. The proof of [KN09] relies on keeping track of the action of a dual SQG evolution on a suitable family of test functions (Hardy molecules), in order to control the evolution of a Hölder norm.

In the three proofs of global regularity for (1.1)–(1.2) mentioned above, the quantitative way in which the dissipation is dominating the nonlinear term is not immediately transparent. In order to shed more light on this issue [CV12] introduced *nonlinear* lower bounds for *linear* nonlocal operators such as the fractional Laplacian. The nonlinear nature of these lower bounds appears because of the conservation laws which are available a priori for solutions. The inequalities bound the dissipative term from below and the nonlinear term from above in a way which makes it transparent why the *size* of the dissipation dominates the *size* of the nonlinearity. In [CV12] it was shown that the nonlinear lower bound for Λ is almost sufficient: it implies the global regularity for the critical SQG equation, but only if we additionally know that the solution has the *only small shocks* property (there exists $0 < \delta < 1$ and $L > 0$ such that $|\theta(x, t) - \theta(y, t)| \leq \delta$ whenever $|x - y| \leq L$). It was proved that if the initial data has only small shocks, the size of these shocks can at most double as time evolves. This extra step is however quite technical as it uses once more the nonlinear lower bound for the fractional Laplacian, and it depends strongly on the nature of the nonlinearity.

At this stage we note that to date the global regularity for the supercritical SQG equation with arbitrarily large initial data remains open. The type of known results are: small data global well-posedness (cf. [CCW01, CL03, CC04, Wu05, Miu06, Ju07, CMZ07, HK07, Yu08] and references therein), conditional regularity (cf. [CW08, CW09, DP09a, DP09b]), eventual regularity (cf. [Sil10a, Dab11, Kis11]), or global regularity for dissipative operators which are only logarithmically supercritical [DKV12, XZ12, DKS12]. The nonlinear lower bound for the fractional Laplacian (cf. [CV12] and this manuscript) may be employed to recover most of these results.

1.3. Main results. The main contributions of this paper are as follows. We give a fully transparent, new and final proof of global regularity for the critical SQG equation, which relies *solely* on the nonlinear lower

bound for the fractional Laplacian (Theorem 4.5). The bounds we obtain on the solution incorporate an explicit decay from the initial data in both weak and *strong topologies* (Theorem 5.2). Using this crucial ingredient we prove the existence of a compact global attractor for the forced SQG dynamics (Theorem 5.1), which has finite box-counting and hence Hausdorff dimension (Theorem 6.4). To the best of our knowledge this is the first such result for systems that are both *quasilinear* and *critical*.

1.4. The nonlinear lower bounds imply global regularity. The main ingredient in the proof of the results mentioned above is the nonlinear lower bound for the fractional Laplacian established in [CV12]. One of the main new ideas over [CV12] is to replace the smallness coming from the “only small shocks” property (for which time dependence is difficult to control) with the smallness of the α at the exponent of the Hölder space C^α . This simple idea is also implicitly present in the De Giorgi’s methodology: the size of the Hölder exponent obtained in the bootstrap step depends on the size of the data in L^∞ , the ellipticity constant, and the size of the force. The advantage of the method outlined below is that the propagation of C^α regularity, for some *sufficiently small* α , is achieved directly, without appealing to the complex De Giorgi iteration technique (Theorem 4.3). Once the solution is Hölder continuous, the nonlinear lower bound also may be used to immediately obtain the higher regularity of solutions. To better outline the method introduced here, we first apply it to the critical Burgers equation in Section 3, and then to the critical SQG equation in Section 4.

We note that the proof of propagation of Hölder continuity also works in the presence of a force which lies merely in $L^\infty \cap H^1$ (the modulus of continuity proof given in [FPV09] requires Lipschitz forcing). Moreover, the argument is *dynamic* rather than in the spirit of a maximum principle, in the sense that the size of the Hölder norm of the solution is not just bounded in terms of the initial data and force, but as time evolves its size is estimated solely in terms of the force (Theorem 5.2 and Lemma 5.3). In the unforced case this amounts to proving the decay of the Hölder norm and higher Sobolev norms.

1.5. Long time dynamics: the existence of a finite dimensional global attractor. Armed with a proof of regularity that is dynamic, we can study the long time dynamics of solutions of the forced SQG. The decay in the unforced case has been addressed in [Don10]. The behavior of the long-time averages along solutions of the critical SQG equations via viscous approximations was addressed in [CTV13], where we have obtained the absence of anomalous dissipation. We note that the later result may also be proven using the estimates of this paper, but necessitates higher regularity on the force than in [CTV13].

In the second part of the paper we address the existence of a compact global attractor, and prove that it has finite box-counting dimension, and a fortiori finite Hausdorff dimension. The space-periodic setting is needed for this purpose. The critical SQG equation is *quasilinear*. To the best of our knowledge, until now all proofs of finite dimensionality have been done either for dissipative or damped *semilinear* equations, or *subcritical* quasilinear equations. See e.g. the works [FP67, FM83, FT84, CF85, CFT85, CF85, FST85, Con87, CFT88, FST88, DG91, FT91, JT92, Kuk92, JT93, FK95, CJT97, GT97, Zia97] for the 2D periodic Navier-Stokes equations and related systems, the books [CF88, Hal88, Lad91, BV92, Tem97, FMRT01, Rob01, CV02], and references therein. In the context of the dissipative SQG equation, the global attractor has been addressed previously addressed only for the subcritical regime: [Ju05] proved the existence of compact global attractor and [WT13] showed it has finite dimensionality fractal dimension (see also [Ber02] for weak attractors).

In order to prove finite dimensionality we establish the existence of a compact absorbing set in phase space. We work in the phase space H^1 , which is the largest Hilbert space in which uniqueness of weak solutions of SQG is currently available. Weak solutions are known to exist for initial data in L^2 but their uniqueness is not known [Res95]. It is known that solutions of the unforced SQG with initial data in H^1 exist for short time [Ju07], as a result of weak-strong stability of the equation in $H^{1+\epsilon}$. The time of existence however depends on the initial function and *not only on its norm*. There is no lower bound on the time of existence based solely on the H^1 norm. Nevertheless, the solution is unique, and becomes instantly smooth [Miu06, Ju07, Don10]. The same result can be proved for smooth forced SQG. We need only a

limited amount of smoothness, in particular C^α , for small $\alpha > 0$. We use the nonlinear maximum principle [CV12] (estimate (4.16) below) to show the global persistence of a C^α norm, with α small compared to the L^∞ norm of the solution (Theorem 4.3). We use the L^p Poincaré inequality for the fractional Laplacian (Proposition 2.2 below) to show that solutions become bounded in L^∞ with a bound that depends only on the norm of the force and not on the initial data, after a time that depends on the initial H^1 data. We apply again the new proof of global existence to show that the solution becomes bounded in a C^α space with both α and the solution bound depending only on norms of the forces (Lemma 5.3). Since we now have supercritical information, we use the nonlinear maximum principle again in its C^α variant to show that the solution becomes bounded in $H^{3/2}$. The upshot is that there exists a compact absorbing set \mathcal{B} for the evolution of SQG in H^1 which is a bounded set in $H^{3/2}$ (Theorem 5.2). This means that for any initial data $\theta_0 \in H^1$ there exists a time $t(\theta_0)$ after which the unique solution $S(t)\theta_0$ with initial datum θ_0 belongs to this absorbing set \mathcal{B} , which in turn depends on the forces alone and not on the initial data. We show additional properties of the solution after this transient time: higher regularity, specifically $S(t)\theta_0 \in H^2$ for almost all $t > t(\theta_0)$, continuity in phase space (Proposition 5.5), and backward uniqueness (Proposition 5.7), i.e., the injectivity of $S(t)$ on the absorbing set. These properties are used to show that the set $\mathcal{A} = \bigcap_{t>0} S(t)\mathcal{B}$ is the global attractor for the evolution in H^1 (Theorem 5.1). For any $\theta_0 \in H^1$ the solution tends to the global attractor. The convergence is uniform on bounded sets in $H^{1+\epsilon}$. We also establish compactness of the linearization of the solution map and local uniform approximation results (Proposition 6.2). The finite dimensionality of the attractor (Theorem 6.4) is then obtained by applying classical tools [CF85, CF88].

2. PRELIMINARIES

We abuse notation and denote in the same way spaces of vector functions and scalar functions. We do not write the subindex “per”, to emphasize that we work with \mathbb{T}^2 -periodic functions, i.e. L^2_{per} is simply written as L^2 . We also overload notation to denote by $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}$ the periodic extension to the whole space of a \mathbb{T}^2 periodic function φ .

We use the following convention regarding constants: C shall denote a positive, sufficiently large constant, whose value may change from line to line; C is allowed to depend on the size of the box and other universal constants which are fixed throughout the paper; to emphasize the dependence of a constant on a certain quantity Q we write C_Q or $C(Q)$; c, c_0, c_1, \dots shall denote fixed constants appearing in the estimates, that have to be referred to specifically; again, to emphasize dependence on a certain quantity Q , we write c_Q or $c(Q)$.

The fractional Laplacian Λ^s , with $s \in \mathbb{R}$ may be defined in this context as the Fourier multiplier with symbol $|k|^s$, i.e. $\Lambda^s \varphi(x) = \sum_{k \in \mathbb{Z}_*^2} |k|^s \widehat{\varphi}_k \exp(ik \cdot x)$, where $\varphi(x) = \sum_{k \in \mathbb{Z}_*^2} \widehat{\varphi}_k \exp(ik \cdot x)$. Note that the eigenvalues of $\Lambda = (-\Delta)^{1/2}$ are given by $|k|$, with $k \in \mathbb{Z}_*^2 = \mathbb{Z} \setminus \{0\}$. We label them in increasing order (counting multiplicity) as

$$0 < \lambda_1 \leq \dots \leq \lambda_n \leq \dots$$

and denote the eigenfunction associated to λ_j by e_j . Then $\{e_j\}_{j \geq 1}$ is an orthonormal basis of L^2 , and the sets $\{\lambda_j\}_{j \geq 1}$ and $\{|k|\}_{k \in \mathbb{Z}_*^2}$ are equal. In view of the choice $\mathbb{T}^2 = [-\pi, \pi]^2$, we have that $\lambda_1 = 1$.

As a consequence of the mean-free setting, for $s \in \mathbb{R}$ we may identify the homogenous Sobolev spaces $\dot{H}^s(\mathbb{T}^2)$ and the inhomogenous Sobolev spaces $H^s(\mathbb{T}^2)$, and we simply denote these by H^s (without “dots”). As usual these are the closure of (mean-free) $C^\infty(\mathbb{T}^2)$ under the norm

$$\|\varphi\|_{H^s} = \|\Lambda^s \varphi\|_{L^2}.$$

Moreover, for $p \in [1, \infty]$ we denote by $H^{s,p} = H^{s,p}(\mathbb{T}^2)$ the space of mean-free $L^p(\mathbb{T}^2)$ functions φ , which can be written as $\varphi = \Lambda^{-s} \psi$, with $\psi \in L^p$. This is normed by $\|\varphi\|_{H^{s,p}} = \|\Lambda^s \varphi\|_{L^p}$. Lastly, Hölder spaces are denoted as usual by C^α for $\alpha \in (0, 1)$, with seminorm given by

$$[\varphi]_{C^\alpha} = \sup_{x \neq y \in \mathbb{T}^2} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha}$$

and norm $\|\varphi\|_{C^\alpha} = \|\varphi\|_{L^\infty} + [\varphi]_{C^\alpha}$.

Recall cf. [CZ54, Sha64, SW71] that for $\varphi \in C^\infty(\mathbb{T}^2)$ the periodic Riesz transforms \mathcal{R}_j may be defined in terms of their Fourier multiplier symbol $\widehat{\mathcal{R}_j\varphi}_k = ik_j|k|^{-1}\widehat{\varphi}_k$, for all $k \in \mathbb{Z}_*^2$. Alternatively this is defined as the singular integral

$$\mathcal{R}_j\varphi(x) = P.V. \int_{\mathbb{T}^2} \varphi(x+y)R_j^*(y)dy \quad (2.1)$$

where the periodic Riesz transform kernel R_j^* is given by

$$R_j^*(y) = R_j(y) + \sum_{k \in \mathbb{Z}_*^2} (R_j(x+2\pi k) - R_j(2\pi k)) \quad (2.2)$$

for $y \neq 2\pi\mathbb{Z}^2$, and R_j is the whole space Riesz-transform kernel given by

$$R_j(y) = \frac{\Omega_j(y/|y|)}{|y|^2} = \frac{y_j}{2\pi|y|^3} \quad (2.3)$$

for $y \neq 0$. The explicit form of the Ω_j is not important: it is a *smooth* function which has *zero mean on \mathbb{S}^1* . Note that if we extend φ periodically to \mathbb{R}^2 , we may rewrite (2.1) as

$$\mathcal{R}_j\varphi(x) = P.V. \int_{\mathbb{R}^2} \varphi(x+y)R_j(y)dy \quad (2.4)$$

where the principal value is both as $|y| \rightarrow 0$ and $|y| \rightarrow \infty$. See e.g. [CZ54, pp. 256–261], or [SW71, Chapter VII] for a proof.

The functional analytic characterization the fractional Laplacian Λ^α as the Fourier multiplier with symbol $|k|^\alpha$ turns out to be useful for estimates in L^2 -based Sobolev spaces, but not for pointwise in x estimates. For this purpose, we recall the kernel representation of the periodic fractional Laplacian [CC04], see also [DNPV11, RS12]. Note that other very useful characterizations are available, see e.g. [CS07] to obtain monotonicity formulae.

For $\alpha \in (0, 2)$ and $\varphi \in C^\infty(\mathbb{T}^2)$ we have the pointwise definition

$$\Lambda^\alpha\varphi(x) = P.V. \int_{\mathbb{T}^2} (\varphi(x) - \varphi(x+y))K_\alpha(y)dy$$

where kernel K_α is defined on $\mathbb{T}^2 \setminus \{0\}$ as

$$K_\alpha(y) = c_\alpha \sum_{k \in \mathbb{Z}^2} \frac{1}{|y - 2\pi k|^{2+\alpha}} \quad (2.5)$$

and the normalization constant is

$$c_\alpha = \frac{2^\alpha \Gamma(1 + \alpha/2)}{|\Gamma(-\alpha/2)|\pi}. \quad (2.6)$$

Under this normalization one has that $\lim_{\alpha \rightarrow 2^-} \Lambda^\alpha\varphi = -\Delta\varphi$, and $\lim_{\alpha \rightarrow 0^+} \Lambda^\alpha\varphi = \varphi - \bar{\varphi}$, pointwise in $x \in \mathbb{T}^2$, where $\bar{\varphi}$ is the mean of φ over \mathbb{T}^2 . When $\alpha \in (0, 1)$ the above definition is valid for $\varphi \in C^{\alpha+\varepsilon}$, while for $\alpha \in [1, 2)$ we need that $\varphi \in C^{1, \alpha-1+\varepsilon}$, for some $\varepsilon > 0$.

We recall the following two statements, which we use frequently throughout the paper.

Proposition 2.1 (Pointwise identity). *Let $\alpha \in (0, 2)$ and $\varphi \in C^\infty(\mathbb{T}^2)$. Then we have that*

$$2\varphi(x)\Lambda^\alpha\varphi(x) = \Lambda^\alpha(\varphi(x)^2) + D_\alpha[\varphi](x) \quad (2.7)$$

holds, where

$$\begin{aligned} D_\alpha[\varphi](x) &= P.V. \int_{\mathbb{T}^2} (\varphi(x) - \varphi(x+y))^2 K_\alpha(y)dy \\ &= P.V. \int_{\mathbb{R}^2} (\varphi(x) - \varphi(x+y))^2 \frac{c_\alpha}{|y|^{2+\alpha}} dy \end{aligned} \quad (2.8)$$

pointwise for $x \in \mathbb{T}^2$, and we denote by φ the periodic extension of φ to all of \mathbb{R}^2 .

Identity (2.7) was proven in [CC04] (see also [Con06, CV12]). The same identity was used in [Tol00, Lemma 3.1] for the periodic case in one dimension, in the context of Stokes waves. In [CC04] the pointwise estimate $f'(\varphi)\Lambda^\alpha\varphi - \Lambda^\alpha f(\varphi) \geq 0$ was established for functions f which are non-decreasing and convex. The second equality in (2.8) follows from Fubini's theorem, a change of variables and the Dominated Convergence theorem.

Proposition 2.2 (Fractional Laplacian in L^p). *Let $p = 4q$, $q \geq 1$, $0 \leq \alpha \leq 2$, and let $\varphi \in C^\infty$ have zero mean on \mathbb{T}^d . Then*

$$\int_{\mathbb{T}^d} \theta^{p-1}(x)\Lambda^\alpha\theta(x)dx \geq \frac{1}{p}\|\Lambda^{\alpha/2}(\theta^{p/2})\|_{L^2}^2 + \frac{1}{C_{\alpha,d}}\|\theta\|_{L^p}^p \quad (2.9)$$

holds, with an explicit constant $C_{\alpha,d} \geq 1$, which is independent of p .

When the second term on the right of (2.9) is absent, the above statement was proven in [CC04]. Since for $p = 4q$, $q \geq 1$, $\theta^{p/2}$ is not of zero mean, it is not immediately clear that the first term on the right of (2.9) dominates the p th power of the L^p norm. (In contradistinction with the case $p = 2$, which is the classical Poincaré inequality). The proof of Proposition 2.2 to our knowledge was first given in [CGHV13, Appendix A]. For the sake of convenience we include a sketch of the proof in Appendix A below.

Remark 2.3 (Scaling). Choosing to work on the periodic box $\mathbb{T}^2 = [-\pi, \pi]^2$ is just a matter of convenience for the presentation, so that the group of characters is \mathbb{Z}^2 . All the results in this paper may be translated to the case of a general periodic box $\mathbb{T}_L^2 = [-L/2, L/2]^2$ as follows. The critical SQG equation has a natural scaling invariance associated to it. If $\theta(x, t)$ is a solution of (1.1)–(1.3) on $[0, T] \times \mathbb{T}^2$, with force $f(x)$ and initial data $\theta_0(x)$, then

$$\theta_\lambda(x, t) = \theta(x/\lambda, t/\lambda)$$

is also a solution of the equations, but on the space-time domain $[0, \lambda T] \times [-\lambda\pi, \lambda\pi]^2$, with force $f_\lambda(x) = (1/\lambda)f(x/\lambda)$ and initial condition $\theta_{0\lambda}(x) = \theta_0(x/\lambda)$. The value of κ remains unchanged. Thus, in order to work on the box \mathbb{T}_L^2 , one merely has to set $\lambda = L/(2\pi)$ in the below argument. Note also that $\|\theta\|_{L^\infty} = \|\theta_\lambda\|_{L^\infty}$ for any $\lambda > 0$ and that the L^∞ norm is non-increasing along solution paths, which is why it is customary to refer to (1.1)–(1.3) as the *critical* SQG equations.

3. GLOBAL REGULARITY FOR THE FORCED CRITICAL BURGERS EQUATION

In order to present the main idea of the proof of global regularity for the critical forced SQG equation, we first consider the one dimensional critical forced Burgers equation

$$\partial_t\theta + \theta\partial_x\theta + \Lambda\theta = f \quad (3.1)$$

on a periodic domain $\mathbb{T} = [-\pi, \pi]$, with smooth force, and smooth initial data θ_0 . We assume that the data and the force have zero mean on \mathbb{T} , so that the same holds for the solution θ at later times.

First notice that we have a global in time control on L^p norms of the solution. Since

$$\int_{\mathbb{T}} \theta\partial_x\theta\theta^{p-1}dx = \frac{1}{p+1} \int_{\mathbb{T}} \partial_x(\theta^{p+1})dx = 0$$

we multiply (3.1) with θ^{p-1} , integrate over \mathbb{T} , and use the lower bound in Proposition 2.2 to obtain

$$\frac{d}{dt}\|\theta\|_{L^p} + c\|\theta\|_{L^p} \leq \|f\|_{L^p}$$

for all $p \geq 2$ even, where the constant c is independent of p . Integrating in time and then passing $p \rightarrow \infty$ thus yields

$$\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} + \frac{1}{c}\|f\|_{L^\infty} =: B_\infty \quad (3.2)$$

for all $t \geq 0$. See also Proposition 4.1 below for similar estimates for the SQG equation.

Our goal is to give a global in (positive) time bound for the C^α norm of the solution, for some $\alpha \in (0, 1)$. It is well-known that due to the critical power of the dissipation, such a bound would in turn imply that the solution cannot develop singularities in finite time.

Let $\alpha \in (0, \alpha_0]$, where $\alpha_0 \in (0, 1)$ is to be determined later in terms of the initial data and the force. In order to study the propagation of Hölder continuity in (3.1) we study the evolution of

$$v(x, t; h) = \frac{|\delta_h \theta(x)|}{|h|^\alpha}$$

where $\delta_h \theta(x) = \theta(x+h) - \theta(x)$ is the usual finite difference. Note that a bound on $\sup_{x, h \in \mathbb{T}^2} |v(t, x; h)|$ is in fact equivalent to a bound on $[\theta(t)]_{C^\alpha}$. Evaluating (3.1) at $x+h$ and x and taking the difference one obtains

$$(\partial_t + \theta \partial_x + (\delta_h \theta) \partial_h + \Lambda_x) (\delta_h \theta) = (\delta_h f).$$

Multiplying by $|h|^{-2\alpha} (\delta_h \theta)$ and appealing to Proposition 2.1 we thus arrive at the pointwise inequality

$$(\partial_t + \theta \partial_x + (\delta_h \theta) \partial_h + \Lambda_x) v^2 + \frac{D[\delta_h \theta]}{|h|^{2\alpha}} = \frac{4\alpha}{|h|^{2\alpha+1}} (\delta_h \theta)^3 + \frac{2(\delta_h f)}{|h|^\alpha} v \leq \frac{4\alpha}{|h|^{1-\alpha}} v^3 + \frac{4\|f\|_{L^\infty}}{|h|^\alpha} v \quad (3.3)$$

where

$$D[\delta_h \theta] = cP.V. \int (\delta_h \theta(x) - \delta_h \theta(x+y))^2 \frac{1}{|y|^2} dy.$$

The main idea is to combine a nonlinear lower bound for $D[\delta_h \theta]$, with the smallness of α to obtain an ODE for $\|v(t)\|_{L_{t,x}^\infty}$, which has global bounded solutions.

Using the argument in [CV12], see also (4.16) below, we have that

$$D[\delta_h \theta](x) \geq \frac{|\delta_h \theta(x)|^3}{C\|\theta\|_{L^\infty}|h|}$$

for some $C > 0$. Therefore, using (3.2) we arrive at

$$\frac{D[\delta_h \theta]}{|h|^{2\alpha}} \geq \frac{|\delta_h \theta(x)|^3}{C\|\theta\|_{L^\infty}|h|^{1+2\alpha}} = \frac{v^3}{CB_\infty|h|^{1-\alpha}}.$$

Inserting the above bound in (3.3) yields

$$(\partial_t + \theta \partial_x + (\delta_h \theta) \partial_h + \Lambda_x) v^2 + \frac{v^3}{C_0 B_\infty |h|^{1-\alpha}} \leq \frac{4\alpha v^3}{|h|^{1-\alpha}} + \frac{4\|f\|_{L^\infty} v}{|h|^\alpha}. \quad (3.4)$$

Therefore if we choose α_0 such that

$$\alpha_0 \leq \frac{1}{8C_0 B_\infty}$$

the nonlinear term on the right side of (3.4) can be absorbed into the left side of the inequality, and moreover, once we appeal to the ε -Young inequality in order to hide the forcing term in the dissipation, we arrive at

$$(\partial_t + \theta \partial_x + (\delta_h \theta) \partial_h + \Lambda_x) v^2 + \frac{v^3}{4C_0 B_\infty |h|^{1-\alpha}} \leq C_1 B_\infty^{1/2} \|f\|_{L^\infty}^{3/2} |h|^{\frac{1-4\alpha}{2}}. \quad (3.5)$$

for some $C_1 > 0$ which depends only on C_0 . Formally evaluating (3.5) at a point (\bar{x}, \bar{h}) where v^2 attains its maximal value, and noting that such a point must necessarily obey $|\bar{h}| \leq \pi$ (the latter is due to the periodicity in h of $\delta_h \theta$, and the strictly decaying nature of $|h|^{-\alpha}$), we obtain

$$\begin{aligned} (\partial_t v^2)(\bar{x}, \bar{h}) + \frac{v^3(\bar{x}, \bar{h})}{4C_0 B_\infty \pi^{1-\alpha}} &\leq (\partial_t + \theta \partial_x + (\delta_h \theta) \partial_h + \Lambda_x) v^2(\bar{x}, \bar{h}) + \frac{v^3(\bar{x}, \bar{h})}{4C_0 B_\infty \pi^{1-\alpha}} \\ &\leq C_1 B_\infty^{1/2} \|f\|_{L^\infty}^{3/2} \pi^{\frac{1-4\alpha}{2}}, \end{aligned} \quad (3.6)$$

as long as we impose

$$\alpha_0 \leq \frac{1}{4}.$$

In (3.6) we used that at the maximum (in joint x and h) of v^2 we must have $\partial_x v^2 = \partial_h v^2 = 0$, and $\Lambda v^2 \geq 0$. One can then rigorously show that on for almost every t in $[0, T_*)$, the maximal time of existence of a smooth solution to the initial value problem associated to (3.1), we have

$$\frac{d}{dt} \|v(t)\|_{L_{x,h}^\infty}^2 \leq (\partial_t v^2)(\bar{x}, \bar{h})$$

which combined with (3.6) yields that for any $\alpha \in (0, \alpha_0]$, with $\alpha_0 = \min\{1/(8C_0B_\infty), 1/4\}$, we have

$$[\theta(t)]_{C^\alpha} = \|v(t)\|_{L_{x,h}^\infty} \leq \max \left\{ [\theta_0]_{C^\alpha}, (4C_0C_1)^{1/2} \pi^{\frac{1-2\alpha}{2}} B_\infty^{1/2} \|f\|_{L^\infty}^{1/2} \right\}$$

for all $t \in [0, T_*)$, and thus a posteriori for all $t \geq 0$, thereby completing the proof of global regularity for critical Burgers.

Notice that the key ingredients were the nonlinear lower bound on Λ , and the smallness of α . This argument carries over to the SQG case modulo some technical issues having to do with the fact that the velocity depends linearly but in a nonlocal fashion on θ . We give details in Section 4 below, where we also fully justify the arguments presented here only formally for clarity of the exposition.

4. GLOBAL REGULARITY FOR FORCED CRITICAL SQG

In this section we give a new proof of global existence and uniqueness of solutions for (1.1)–(1.3) which has the advantage that the bounds require merely $f \in L^\infty \cap H^1$. The proof is based on the nonlinear lower bound for the fractional Laplacian discovered in [CV12]. We first recall some L^p estimates for solutions of the forced critical SQG equation.

Proposition 4.1 (Absorbing ball in L^p). *Let θ be a smooth solution of (1.1)–(1.3), and let $p \geq 2$ be even. Then we have*

$$\|\theta(\cdot, t)\|_{L^p} \leq \|\theta_0\|_{L^p} e^{-tc_0\kappa} + \frac{1}{c_0\kappa} \|f\|_{L^p} (1 - e^{-tc_0\kappa}) \quad (4.1)$$

for some universal constant c_0 . Moreover,

$$\|\theta(\cdot, t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} e^{-tc_0\kappa} + \frac{1}{c_0\kappa} \|f\|_{L^\infty} (1 - e^{-tc_0\kappa}) \quad (4.2)$$

holds with the same universal constant c_0 .

Proof. Multiplying (1.1) by θ^{p-1} , and using Proposition 2.2 we arrive at

$$\frac{d}{dt} \|\theta\|_{L^p} + c_0\kappa \|\theta\|_{L^p} \leq \|f\|_{L^p}$$

for some $c_0 > 0$ which is independent of p . Therefore we obtain (4.1) for all $t \geq 0$ and $p \geq 2$ even. Moreover, since the constants appearing in (4.1) are independent of p , and we are on a periodic domain $\mathbb{T}^2 = [-\pi, \pi]^2$, we have that

$$\|\theta(\cdot, t)\|_{L^p} \leq (2\pi)^{2/p} \|\theta_0\|_{L^\infty} e^{-tc_0\kappa} + \frac{(2\pi)^{2/p}}{c_0\kappa} \|f\|_{L^\infty} (1 - e^{-tc_0\kappa})$$

which yields (4.2) upon passing $p \rightarrow \infty$. □

In particular, Proposition 4.1 shows that for any $p \in [2, \infty)$ even and $p = \infty$ we have that

$$\|\theta(\cdot, t)\|_{L^p} \leq \|\theta_0\|_{L^p} + \frac{1}{c_0\kappa} \|f\|_{L^p} := M_p(\theta_0, f) = M_p \quad (4.3)$$

for any $t \geq 0$. Next, we recall the following local existence [Miu06, Ju07] and smoothing [Don10] result.

Proposition 4.2 (Local solution). *Assume $\theta_0 \in H^1$ and $f \in H^1 \cap L^\infty$. There exists $T_* = T_*(\theta_0, f) > 0$ and a unique solution θ of the initial value problem (1.1)–(1.3) which obeys the energy inequality and*

$$\theta \in C([0, T_*]; H^1) \cap L^2(0, T_*; H^{3/2}). \quad (4.4)$$

Moreover, for any $\beta \geq 0$, if $f \in H^\beta$ we have that

$$\sup_{0 < t < T_*} t^\beta \|\theta(\cdot, t)\|_{H^{1+\beta}} \leq c_1 \kappa^{-\beta} \|\theta_0\|_{H^1} + c_1 \kappa^{-1} T_*^\beta \|f\|_{H^\beta} \quad (4.5)$$

for some positive universal constant c_1 . Additionally, we have $\lim_{t \rightarrow 0^+} t^\beta \|\theta(\cdot, t)\|_{H^{1+\beta}} = 0$, when $\beta > 0$.

We emphasize that the time of existence T_* doesn't depend in a locally uniform way on $\|\theta_0\|_{H^1}$. The above result was proven in the aforementioned works in the absence of a forcing term, but it is not difficult to verify that estimate (4.5) holds assuming f is sufficiently smooth. We omit these details. Note that the solution may be extended uniquely past T_* assuming that a priori we know e.g. that $\sup_{t \in [T_*/2, T_*]} [\theta(t, \cdot)]_{C^\alpha} < \infty$, for some $\alpha > 0$.

The first result we obtain is the propagation of Hölder continuity for smooth solutions, which may then be applied to a sequence of solutions to a regularized problem, in order to obtain in the limit the corresponding result for weak solutions (see Theorem 4.4 below).

Theorem 4.3 (Propagation of Hölder regularity). *Let θ_0 and f be sufficiently smooth, $T > 0$ be arbitrary, and let $\theta \in C^{1/2}((0, T); C^{1,1/2})$ be the unique classical solution of the initial value problem (1.1)–(1.3). As in (4.3) above, define*

$$M_\infty = M_\infty(\theta_0, f) = \|\theta_0\|_{L^\infty} + (c_0 \kappa)^{-1} \|f\|_{L^\infty}.$$

There exists a sufficiently small universal constant $\varepsilon_0 > 0$ such that for any α with

$$0 < \alpha \leq \alpha_0 = \min \left\{ \frac{\varepsilon_0 \kappa}{M_\infty}, \frac{1}{4} \right\}, \quad (4.6)$$

if $\theta_0 \in C^\alpha$ we have

$$[\theta(t)]_{C^\alpha} \leq M_\alpha(t) \quad (4.7)$$

for all $t \in [0, T]$, where M_α is the solution of the ordinary differential equation (4.36) below. In particular, we have that

$$M_\alpha(t) \leq \max \left\{ [\theta_0]_{C^\alpha}, \frac{M_\infty}{\varepsilon_0} \right\} \quad (4.8)$$

for any $t \in [0, T]$, and also

$$M_\alpha(t) \leq \frac{2M_\infty}{\varepsilon_0} \quad (4.9)$$

for all $t \geq t_\alpha = t_\alpha(M_\infty, [\theta_0]_{C^\alpha})$, which is defined explicitly in (4.37) below.

Proof of Theorem 4.3. For $\alpha > 0$, we look at the evolution of weighted finite differences

$$v(x, t; h) = \frac{|\delta_h \theta(x)|}{|h|^\alpha}.$$

Since $\theta_0 \in C^\alpha$, we have that

$$\|v_0\|_{L_{x,h}^\infty} \leq [\theta_0]_{C^\alpha} \leq M_\alpha(0). \quad (4.10)$$

Our goal is to find an upper bound $M_\alpha(t)$ such that

$$\|v(\cdot, t; \cdot)\|_{L_{x,h}^\infty} \leq M_\alpha(t) \quad (4.11)$$

for any $t \geq 0$. In view of the periodicity in h of $\delta_h \theta(x)$, we would in turn obtain from (4.11) that

$$[v(\cdot, t)]_{C^\alpha} = \sup_{x, h \in \mathbb{T}^2} \frac{|\delta_h \theta(x, t)|}{|h|^\alpha} \leq \|v(\cdot, t; \cdot)\|_{L_{x,h}^\infty} \leq M_\alpha(t) \quad (4.12)$$

which would then conclude the proof of the theorem.

In order to find $M_\alpha(t)$ for which (4.11) holds, we first write the equation obeyed by $\delta_h \theta$. It follows by taking finite differences in (1.1) that pointwise in x, t, h we have

$$(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_h + \kappa \Lambda) \delta_h \theta = \delta_h f. \quad (4.13)$$

Upon multiplying (4.13) by

$$\frac{\delta_h \theta(x, t)}{|h|^{2\alpha}}$$

and using Proposition 2.1, we obtain that v obeys the equation

$$\begin{aligned} (\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_h + \kappa \Lambda) v^2 + \frac{\kappa D[\delta_h \theta]}{|h|^{2\alpha}} &= 4\alpha(\delta_h u) \cdot \frac{h}{|h|} \frac{v^2}{|h|} + \frac{2(\delta_h f)v}{|h|^\alpha} \\ &\leq 4\alpha|\delta_h u| \frac{v^2}{|h|} + \frac{4\|f\|_{L^\infty} v}{|h|^\alpha} \end{aligned} \quad (4.14)$$

where $\delta_h u = \mathcal{R}_x^\perp(\delta_h \theta)$, and

$$D[\delta_h \theta](x) = \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \left(\delta_h \theta(x) - \delta_h \theta(x+y) \right)^2 \frac{1}{|y|^3} dy \quad (4.15)$$

from (2.8) with $\alpha = 1$, and we have used the explicit formula (2.6) for the normalizing constant. First we give a lower bound on the dissipative term $D[\delta_h \theta]$, and then estimate the velocity increment $|\delta_h u|$ in (4.14).

Throughout the proof we will use a cutoff function $\chi: [0, \infty) \rightarrow [0, \infty)$ which is smooth, non-increasing, identically 1 on $[0, 1]$, vanishes on $[2, \infty)$, and obeys $|\chi'| \leq 2$.

In the spirit of [CV12], we obtain a nonlinear lower bound for the dissipative term in (4.14). Pointwise in x and h it holds that

$$D[\delta_h \theta](x) \geq \frac{|\delta_h \theta(x)|^3}{c_2 \|\theta\|_{L^\infty} |h|} \quad (4.16)$$

for some universal constant $c_2 > 0$. To prove (4.16), we proceed as follows. For $r \geq 4|h|$ to be determined, we have

$$\begin{aligned} D[\delta_h \theta](x) &\geq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(\delta_h \theta(x) - \delta_h \theta(x+y))^2}{|y|^3} \left(1 - \chi\left(\frac{|y|}{r}\right) \right) dy \\ &\geq \frac{1}{2\pi} (\delta_h \theta(x))^2 \int_{|y| \geq 2r} \frac{1}{|y|^3} dy - \frac{1}{\pi} |\delta_h \theta(x)| \left| \int_{\mathbb{R}^2} \frac{\delta_h \theta(x+y)}{|y|^3} \left(1 - \chi\left(\frac{|y|}{r}\right) \right) dy \right| \\ &\geq \frac{1}{2r} |\delta_h \theta(x)|^2 - \frac{1}{\pi} |\delta_h \theta(x)| \int_{\mathbb{R}^2} |\theta(x+y)| \left| \delta_{-h} \left(\frac{1}{|y|^3} - \frac{1}{|y|^3} \chi\left(\frac{|y|}{r}\right) \right) \right| dy \\ &\geq \frac{1}{2r} |\delta_h \theta(x)|^2 - C |\delta_h \theta(x)| \|\theta\|_{L^\infty} |h| \int_{\mathbb{R}^2} \left(\frac{\mathbf{1}_{|y| \geq 3r/4}}{|y|^4} + \frac{\mathbf{1}_{3r/4 \leq |y| \leq 9r/4}}{r^4} \right) dy \\ &\geq \frac{1}{2r} |\delta_h \theta(x)|^2 - C |\delta_h \theta(x)| \|\theta\|_{L^\infty} \frac{|h|}{r^2} \end{aligned} \quad (4.17)$$

for some $C \geq 1$. In the above estimate we have used estimate (4.19) below. Setting

$$r = \frac{4C \|\theta\|_{L^\infty}}{|\delta_h \theta(x)|} |h|$$

in (4.17), which obeys $r \geq 4|h|$ due to $|\delta_h \theta| \leq 2\|\theta\|_{L^\infty}$, completes the proof of (4.16).

In (4.17) above we have used the following. The mean value theorem gives that for a smooth g we have

$$|\delta_h g(y)| = |h \cdot \nabla g((1 - \lambda)y + \lambda(y + h))| \leq |h| \max_{\lambda \in [0,1]} |\nabla g(y + \lambda h)| \quad (4.18)$$

for $h, y \in \mathbb{R}^2$. In particular, for

$$g(y) = \frac{1}{|y|^3} \left(1 - \chi \left(\frac{|y|}{r} \right) \right)$$

we have that

$$\max_{\lambda \in [0,1]} |\nabla g(y + \lambda h)| \leq C \left(\frac{\mathbf{1}_{|y| \geq 3r/4}}{|y|^4} + \frac{\mathbf{1}_{3r/4 \leq |y| \leq 9r/4}}{r^4} \right) \quad (4.19)$$

whenever $|h| \leq r/4$.

Combining estimate (4.16) with the a priori bound (4.3) (with $p = \infty$) we obtain that the positive term on the right side of (4.14) is bounded from below as

$$\frac{\kappa D[\delta_h \theta]}{|h|^{2\alpha}} \geq \frac{\kappa}{|h|^{2\alpha}} \frac{|\delta_h \theta|^3}{c_2 M_\infty |h|} = \frac{\kappa}{c_2 M_\infty} \frac{v^3}{|h|^{1-\alpha}} \quad (4.20)$$

pointwise in x and h .

We now estimate the velocity finite difference $\delta_h u$. For this purpose fix x and h (ignore t dependence) and let $\rho > 0$ be such that

$$\rho \geq 4|h|. \quad (4.21)$$

As before we let χ be a smooth cutoff function, that is 1 on $[0, 1]$, non-increasing, vanishes on $[2, \infty)$, and obeys $|\chi'| \leq 2$. We decompose the singular integral defining $\delta_h u$ as

$$\delta_h u(x) = \mathcal{R}_x^\perp(\delta_h \theta(x)) = \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} \delta_h \theta(x + y) dy = \delta_h u_{in}(x) + \delta_h u_{out}(x),$$

where the inner piece is given by

$$\begin{aligned} \delta_h u_{in}(x) &= \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} \chi \left(\frac{|y|}{\rho} \right) \delta_h \theta(x + y) dy \\ &= \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} \chi \left(\frac{|y|}{\rho} \right) (\delta_h \theta(x + y) - \delta_h \theta(x)) dy \end{aligned}$$

by using that the kernel of \mathcal{R}^\perp has zero average on the unit sphere, and the outer piece is given by

$$\begin{aligned} \delta_h u_{out}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} \left(1 - \chi \left(\frac{|y|}{\rho} \right) \right) \delta_h \theta(x + y) dy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \delta_{-h} \left(\frac{y^\perp}{|y|^3} \left(1 - \chi \left(\frac{|y|}{\rho} \right) \right) \right) \theta(x + y) dy \end{aligned}$$

by using a finite-difference-by-parts.

For the inner piece, by appealing to the Cauchy-Schwartz inequality, we obtain

$$|\delta_h u_{in}| \leq C (\rho D[\delta_h \theta])^{1/2}. \quad (4.22)$$

In order to bound the outer piece, we recall that the mean value theorem gives

$$|\delta_h g(y)| \leq |h| \max_{\lambda \in [0,1]} |\nabla g(y + \lambda h)| \quad (4.23)$$

for smooth functions g . We apply (4.23) with

$$g(y) = \frac{y^\perp}{|y|^3} \left(1 - \chi \left(\frac{|y|}{\rho} \right) \right)$$

which yields

$$|\delta_{-h}g(y)| \leq |h| \max_{\lambda \in [0,1]} |\nabla g(y - \lambda h)| \leq C|h| \frac{\mathbf{1}_{|y| \geq \rho/2}}{|y|^3}$$

when $|h| \leq \rho/4$. This in turn implies that

$$|\delta_h u_{out}| \leq C|h| \int_{|y| \geq \rho/2} \frac{1}{|y|^3} |\theta(x+y)| dy \leq \frac{CM_\infty |h|}{\rho}. \quad (4.24)$$

Combining the inner (4.22) and outer (4.24) velocity estimates, we obtain that

$$|\delta_h u| \leq C \left((\rho D[\delta_h \theta])^{1/2} + \frac{M_\infty |h|}{\rho} \right) \quad (4.25)$$

if ρ is chosen so that $\rho \geq 4|h|$.

Using the Cauchy-Schwartz inequality we obtain from (4.25) that

$$\begin{aligned} 4\alpha |\delta_h u| \frac{v^2}{|h|} &\leq C\alpha (\rho D[\delta_h \theta])^{1/2} \frac{v^2}{|h|} + C\alpha \frac{M_\infty v^2}{\rho} \\ &\leq \frac{\kappa D[\delta_h \theta]}{2|h|^{2\alpha}} + \frac{c_3 \alpha^2 \rho v^4}{\kappa |h|^{2-2\alpha}} + \frac{c_3 \alpha M_\infty v^2}{\rho} \end{aligned}$$

for some sufficiently large $c_3 > 0$. Therefore, letting

$$\rho = \frac{\kappa^{1/2} M_\infty^{1/2} |h|^{1-\alpha}}{\alpha^{1/2} v} = \frac{\kappa^{1/2} M_\infty^{1/2}}{\alpha^{1/2} |\delta_h \theta|} |h| \quad (4.26)$$

we obtain

$$4\alpha |\delta_h u| \frac{v^2}{|h|} \leq \frac{\kappa D[\delta_h \theta]}{2|h|^{2\alpha}} + \frac{2c_3 \alpha^{3/2} M_\infty^{1/2}}{\kappa^{1/2}} \frac{v^3}{|h|^{1-\alpha}}. \quad (4.27)$$

Note that in order to define ρ as in (4.26), we need to ensure $\rho \geq 4|h|$. By the triangle inequality we have $|\delta_h \theta| \leq 2M_\infty$ and hence indeed

$$\rho \geq \left(\frac{\kappa}{2\alpha M_\infty} \right)^{1/2} |h| \geq 4|h|$$

holds, since by assumption $\alpha \in (0, \alpha_0]$ and

$$\alpha_0 \leq \frac{\kappa}{32M_\infty}. \quad (4.28)$$

Combining (4.14), the lower bound (4.20), estimate (4.27), and recalling the definition of M_∞ in (4.3), we arrive at

$$(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_h + \kappa \Lambda) v^2 + \left(\frac{\kappa}{2c_2 M_\infty} - \alpha^{3/2} \frac{2c_3 M_\infty^{1/2}}{\kappa^{1/2}} \right) \frac{v^3}{|h|^{1-\alpha}} \leq \frac{4c_0 \kappa M_\infty v}{|h|^\alpha}. \quad (4.29)$$

Since $\alpha \leq \alpha_0$, with

$$\alpha_0 \leq \frac{\kappa}{(8c_2 c_3)^{2/3} M_\infty} \quad (4.30)$$

we furthermore obtain from (4.29) that

$$(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_h + \kappa \Lambda) v^2 + \frac{\kappa}{4c_2 M_\infty} \frac{v^3}{|h|^{1-\alpha}} \leq \frac{4c_0 \kappa M_\infty v}{|h|^\alpha}. \quad (4.31)$$

pointwise in x and h . Upon using the ε -Young inequality for the right hand side, that

$$(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_h + \kappa \Lambda) v^2 + \frac{\kappa}{6c_2 M_\infty} \frac{v^3}{|h|^{1-\alpha}} \leq c_4 \kappa |h|^{\frac{1-4\alpha}{2}} M_\infty^2 \quad (4.32)$$

for a positive universal constant c_4 which may be computed explicitly ($c_4 = (4c_0)^{3/2}(4c_2)^{1/2}$).

We now proceed as in [CC04, Section 4], and refer to Appendix B below for details. Since θ is sufficiently smooth v^2 is a bounded continuous function in both x and h , which is periodic in x . Moreover, given $x, h \in \mathbb{T}^2$ and $k \in \mathbb{Z}_*^2$ we have that

$$v(x, t; h)^2 \geq v^2(x, t; k + 2\pi k),$$

in view of the periodicity in h of $\delta_h \theta$, and the strict monotonicity of $|h|^{-\alpha}$. Therefore there exists *at least* one point $(\bar{x}, \bar{h}) = (\bar{x}(t), \bar{h}(t)) \in \mathbb{T}^2 \times \mathbb{T}^2$ where the function $v(\cdot, t; \cdot)^2$ attains its maximum. We define

$$g(t) = \sup_{x, h \in \mathbb{T}^2} v(x, t; h)^2 = v(\bar{x}(t), t; \bar{h}(t))^2. \quad (4.33)$$

In Appendix B below we show that g is Lipschitz continuous on $[0, T]$, and that for almost every t there exists $(\bar{x}(t); \bar{h}(t))$ such that

$$g'(t) = (\partial_t v^2)(\bar{x}(t), t; \bar{h}(t))$$

and (4.33) holds. Evaluating (4.32) at the joint x, h -maximum (\bar{x}, \bar{h}) , using that at the maximum we have

$$\nabla_x v(\bar{x}, t; \bar{h})^2 = 0 = \nabla_h v(\bar{x}, t; \bar{h})^2 \quad \text{and} \quad \Delta v(\bar{x}, t; \bar{h})^2 \geq 0,$$

and the fact that $|\bar{h}| \leq 4\pi$ we thus obtain

$$g'(t) + \frac{\kappa}{6c_2 M_\infty (4\pi)^{1-\alpha}} g(t)^{3/2} \leq c_4 \kappa M_\infty^2 (4\pi)^{\frac{1-4\alpha}{2}} \quad (4.34)$$

once we additionally assume that

$$\alpha_0 \leq \frac{1}{4}. \quad (4.35)$$

Since we are now dealing with an ordinary differential equation, by the usual comparison principle for ODEs it follows that

$$\|v(t)\|_{L_{t,x}^\infty}^2 = g(t) \leq M_\alpha(t)^2,$$

where $M_\alpha(t)$ is the solution of the initial value problem

$$\frac{d}{dt} M_\alpha^2 + \frac{\kappa}{c_5 M_\infty} M_\alpha^3 = c_5^2 \kappa M_\infty^2, \quad M_\alpha(0) = [\theta_0]_{C^\alpha}, \quad (4.36)$$

where $c_5 = c_5(c_2, c_4)$, is a fixed deterministic constant which is *independent* of κ, M_∞ , or α .

In particular, there have proven that

$$[\theta(t)]_{C^\alpha} \leq M_\alpha(t) \leq \max\{[\theta_0]_{C^\alpha}, c_5 M_\infty\}$$

for any $t \geq 0$. Moreover there exists $t_\alpha = t_\alpha(M_\infty, [\theta_0]_{C^\alpha}) \geq 0$ defined as

$$t_\alpha = \begin{cases} 0, & \text{if } [\theta_0]_{C^\alpha} \leq 2c_5 M_\infty \\ \frac{1}{7\kappa} \left(\frac{[\theta_0]_{C^\alpha}^2}{4c_5^2 M_\infty^2} - 1 \right), & \text{if } [\theta_0]_{C^\alpha} > 2c_5 M_\infty \end{cases} \quad (4.37)$$

such that

$$[\theta(t)]_{C^\alpha} \leq 2c_5 M_\infty$$

for any $t \geq t_\alpha$. The above bound shows that the solution forgets the initial data even in the C^α norm. \square

Theorem 4.3 implies the propagation of Hölder continuity for weak solutions.

Theorem 4.4 (Hölder propagation for weak solutions). *Assume $f \in L^\infty \cap H^1$, $\theta_0 \in L^\infty \cap H^1$, $T > 0$ is arbitrary, and let $\theta \in L^\infty(0, T; H^1) \cap L^2(0, T; H^{3/2})$ be the unique weak solution of the critical, forced SQG equation (1.1)–(1.3). Let $\alpha_0 = \alpha_0(\|\theta_0\|_{L^\infty}, \|f\|_{L^\infty}) \leq 1/4$ be defined as in (4.6). For any $\alpha \in (0, \alpha_0]$, if $\theta_0 \in C^\alpha$, then $\theta \in L^\infty(0, T; C^\alpha)$. Moreover, we have $[\theta(t)]_{C^\alpha} \leq M_\alpha(t)$ for a function $M_\alpha(t)$ which obeys (4.8) and (4.9).*

Proof of Theorem 4.4. Let J_ε be a standard mollifier operator. For $\varepsilon \in (0, 1]$, we let θ_ε be the solution of

$$\partial_t \theta^\varepsilon + \kappa \Lambda \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon - \varepsilon \Delta \theta^\varepsilon = J_\varepsilon f, \quad u^\varepsilon = \mathcal{R}^\perp \theta^\varepsilon, \quad \theta_0^\varepsilon = \theta_0. \quad (4.38)$$

As in Proposition 4.1 we obtain that

$$\|\theta^\varepsilon(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} + \frac{1}{c_0 \kappa} \|J_\varepsilon f\|_{L^\infty} \leq \|\theta_0\|_{L^\infty} + \frac{1}{c_0 \kappa} \|f\|_{L^\infty} = M_\infty(\theta_0, f). \quad (4.39)$$

Indeed, the addition of the regularizing term $-\varepsilon \Delta \theta^\varepsilon$ in the equation does not change any part of the argument, and J_ε is given by convolving with an L^1 kernel of mean 1. Once we have that $\theta^\varepsilon \in L_{t,x}^\infty$, a supercritical information for the dissipation given by the Laplacian, the existence of a unique global smooth solution of (4.38) follows from classical arguments (see e.g. [CW99] for subcritical SQG). Since $J_\varepsilon f \in C^\infty$, in fact a bootstrap shows that $\theta^\varepsilon \in C^\infty((0, T) \times \mathbb{T}^2)$, but with bounds that depend on ε .

At this stage we compute α_0 as in (4.6), which is independent of ε due to (4.39), and then apply Theorem 4.3 to θ^ε , which we are allowed to since θ^ε is smooth. We emphasize that the presence of the regularizing term $-\varepsilon \Delta$ does not require *any* modification to the proof of Theorem 4.3. The Laplacian (in x) does not affect the finite differences in h , and the negative Laplacian evaluated at the maximum of a function is non-negative. Therefore, for any $\varepsilon \in (0, 1]$, we have

$$[\theta^\varepsilon(t)]_{C^\alpha} \leq M_\alpha(t) \quad (4.40)$$

for all $t \geq 0$, where $M_\alpha(t)$ (which is independent of ε) is given by the solution of (4.36), and obeys the bounds (4.8) for all time, and (4.9) for long enough time.

The sequence of solutions $\{\theta^\varepsilon\}_{\varepsilon \in (0, 1]}$ is thus uniformly bounded in $L^\infty(0, T; C^\alpha)$. In particular, since $|\mathbb{T}^2| < \infty$ this implies that θ^ε is uniformly bounded in $L^\infty(0, T; H^\alpha)$, and from (4.38) we have that $\partial_t \theta^\varepsilon$ is uniformly bounded in $L^\infty(0, T; H^{\alpha-2})$. Since the injection of $H^\alpha(\mathbb{T}^2)$ into $L^2(\mathbb{T}^2)$ is compact, and the injection of $L^2(\mathbb{T}^2)$ in $H^{\alpha-2}(\mathbb{T}^2)$ is continuous, the Aubin-Lions compactness lemma and the uniform in ε estimates obtained earlier, imply that there exists $\bar{\theta} \in L^\infty(0, T; C^\alpha)$, with bounds inherited (e.g. by duality) directly from (4.40), such that

$$\theta^\varepsilon \rightarrow \bar{\theta} \text{ in } L^2(0, T; L^2).$$

The above strong convergence in $L_{t,x}^2$ is enough in order to pass to the limit in the weak formulation of (4.38), and show that $\bar{\theta}$ is a weak solution of the critical forced SQG equation (1.1)–(1.3) on $[0, T)$. This is seen by writing the nonlinear term in divergence form.

To conclude the proof we notice that in fact $\theta = \bar{\theta}$. This follows in the spirit of weak-strong uniqueness: one writes the equation obeyed by $\theta - \bar{\theta}$ and performs an L^2 energy estimate. The equation for the difference has zero initial data and zero force. Using that $\bar{\theta} \in L_t^\infty C_x^\alpha$ we have $\int \mathcal{R}^\perp \bar{\theta} \cdot \nabla(\theta - \bar{\theta})(\theta - \bar{\theta}) dx = 0$, and since $\theta \in L_t^\infty H_x^1 \cap L_t^2 H_x^{3/2}$ we have $|\int \mathcal{R}^\perp(\theta - \bar{\theta}) \cdot \nabla \theta(\theta - \bar{\theta}) dx| \leq \kappa \|\theta - \bar{\theta}\|_{H^{1/2}}^2 + C \|\theta - \bar{\theta}\|_{L^2}^2 \|\theta\|_{H^{3/2}}^2$. The proof of $\theta = \bar{\theta}$ is then concluded via the Grönwall inequality. \square

The results obtained in this section may be summarized as follows.

Theorem 4.5 (Global regularity). *Let $\theta_0 \in H^1(\mathbb{T}^2)$ and $f \in L^\infty(\mathbb{T}^2) \cap H^1(\mathbb{T}^2)$. There exists a unique global solution $\theta \in L^\infty([0, \infty); H^1) \cap L_{loc}^2((0, \infty); H^{3/2})$ of the initial value problem (1.1)–(1.3). For any $t_1 > 0$ we have $\theta \in L^\infty([t_1, \infty); H^{3/2}) \cap L_{loc}^2([t_1, \infty); H^2)$.*

It is clear that if we would furthermore assume $f \in C^\infty(\mathbb{T}^2)$, then $\theta \in C^\infty([t_1, \infty) \times \mathbb{T}^2)$, for any $t_1 > 0$.

Proof of Theorem 4.5. By the local existence result of Proposition 4.2, there exists a time $T_* = T_*(\theta_0, f) > 0$, and a unique solution $\theta \in L_t^\infty H_x^1 \cap L_t^2 H_x^{3/2}$ of (1.1)–(1.3) on $[0, T_*)$. In addition, by the local smoothing estimate (4.5), since $f \in H^1$ we conclude that $\theta(t_1) \in H^2 \supset C^{\alpha_0}$, where $\alpha_0 = \alpha_0(\|\theta_0\|_{L^\infty}, \|f\|_{L^\infty})$ is as in Theorem 4.3, and $t_1 \in (0, T_*)$ is arbitrary. The propagation of Hölder continuity result of Theorem 4.4, with initial data $\theta(t_1)$ then yields

$$\sup_{t \in [t_1, T_* - \tau]} [\theta(t)]_{C^{\alpha_0}} \leq C(\theta_0, f)$$

where the constant $C(\theta_0, f)$ is *independent* of τ . Since the Hölder bound is supercritical for the natural scaling of the equation, we use the C^{α_0} version of the nonlinear lower bound for the fractional Laplacian [CV12, Theorem 2.2] in order to bootstrap in regularity and obtain

$$\sup_{t \in [t_1, T_* - \tau]} \|\theta(t)\|_{H^{3/2}}^2 + \int_{t_1}^{T_* - \tau} \|\theta(t)\|_{H^2}^2 dt \leq C(\theta_0, f)$$

independently of τ . The proof of this bootstrap procedure is given as part the proof of Theorem 5.2 below, and we omit the details here to avoid redundancy. The above estimate in particular shows that the solution θ may be continued uniquely past T_* , since for data in $H^{3/2}$ the time of existence of the $L_t^\infty H_x^1 \cap L_t^2 H_x^{3/2}$ solution depends only on the size of the *norm* in $H^{3/2}$ (we do not have this fact available if the initial data merely lies in H^1). Having extended the solution past T_* , we repeat the above argument and conclude the proof of global regularity. \square

5. EXISTENCE OF A GLOBAL ATTRACTOR

In view of the global existence established in Theorem 4.5, we define a *solution operator* $S(t)$ for the initial value problem (1.1)–(1.3) via

$$S(t): H^1 \rightarrow H^1, \quad S(t)\theta_0 = \theta(\cdot, t), \quad (5.1)$$

for any $t \geq 0$. In this section we establish (cf. Theorem 5.1 below) the existence of a global attractor \mathcal{A} for the long-time dynamics of $S(t)$ on the phase space H^1 .

Theorem 5.1 (Existence of a global attractor). *The solution map $S: [0, \infty) \times H^1 \rightarrow H^1$ associated to (1.1)–(1.2) with $f \in L^\infty \cap H^1$, possesses a global attractor \mathcal{A} , which is a non-empty compact connected invariant set (i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$) such that for every $\theta_0 \in H^1$ we have*

$$\lim_{t \rightarrow \infty} \text{dist}(S(t)\theta_0, \mathcal{A}) = 0.$$

The set \mathcal{A} is maximal in the sense that for any bounded subset $\mathcal{B}_1 \subset H^{1+\delta}$ with $\delta > 0$, which is invariant under $S(t)$, obeys $\mathcal{B}_1 \subset \mathcal{A}$. Moreover, there exists $M_{\mathcal{A}}$ which depends only on $\kappa, \|f\|_{L^\infty \cap H^1}$, and universal constants, such that if $\theta \in \mathcal{A}$, we have that

$$\|\theta\|_{H^{3/2}} \leq M_{\mathcal{A}} \quad (5.2)$$

and

$$\frac{1}{T} \int_t^{t+T} \|S(\tau)\theta\|_{H^2}^2 d\tau \leq M_{\mathcal{A}}^2 \quad (5.3)$$

for any $T > 0$ and $t \in \mathbb{R}$. In particular, for $\theta \in \mathcal{A}$ we have $\|S(t)\theta\|_{H^2} \leq M_{\mathcal{A}}$ for almost every t .

The proof of Theorem 5.1, given at the end of this section, follows closely the steps outlined in [CF88], and relies on the following main ingredients:

- (i) There exists a compact absorbing set \mathcal{B} (which is a ball around the origin in $H^{3/2}$) for the dynamics induced by $S(t)$ on the phase space H^1 (cf. Theorem 5.2 below).
- (ii) For each $\theta_0 \in H^1$ the solution $S(t)\theta_0: [0, \infty) \rightarrow H^1$ is a continuous function of t , and for fixed $t > 0$, we have that $S(t): \mathcal{B} \rightarrow H^1$ is a Lipschitz continuous function of θ_0 (cf. Remark 5.4 and Proposition 5.5 below).

Establishing (i), the existence of a compact absorbing ball, turns out to be the most important step. For this we need to use the global regularity *twice*. From the local existence of solutions we pick up a time when a C^α norm of the solution is finite, with α small. Then we guarantee first that the solution satisfies strong bounds for all time, but the bounds depend of the initial data. However, after long enough time the L^∞ norm of the solution obeys a bound that no longer depends on initial data (its size depends solely on force). At that time, because we have guaranteed that the solution remained smooth enough in the meantime, we apply again the C^α persistence result, but this time the size of θ in L^∞ is given by f , which permits a calculation

with an α that depends only on f . After an additional time, we obtain a bound of this C^α norm that depends only on f . At this stage, because the bounds make the situation subcritical, with constants which depend on f only, we bootstrap in regularity and obtain that the size of the $H^{3/2}$ norm is determined by f alone.

Theorem 5.2 (Absorbing ball in $H^{3/2}$). *Let $\theta_0 \in H^1$ and $f \in L^\infty \cap H^1$. There exists a time $t_{H^{3/2}} = t_{H^{3/2}}(\theta_0, f)$ and an $M_{3/2, f} = M_{3/2, f}(\|f\|_{L^\infty \cap H^1})$ such that for all $t \geq t_{H^{3/2}}$ we have*

$$\|S(t)\theta_0\|_{H^{3/2}} \leq M_{3/2, f}. \quad (5.4)$$

That is,

$$\mathcal{B} = \{\theta \in H^{3/2} : \|\theta\|_{H^{3/2}} \leq M_{3/2, f}\} \quad (5.5)$$

is an absorbing set. Moreover, there exists an $M_{2, f} = M_{2, f}(\|f\|_{L^\infty \cap H^1})$, such that

$$\frac{1}{T} \int_t^{t+T} \|S(\tau)\theta_0\|_{H^2}^2 d\tau \leq M_{2, f}^2 \quad (5.6)$$

for any $t \geq t_{H^{3/2}}$ and any $T > 0$.

As described in the above outline, in order to prove Theorem 5.2, we first need to show that after waiting long enough time, the solution belongs to a Hölder space, with both the Hölder exponent and the Hölder norm, independent of the initial data. We achieve this in the following lemma, by combining the estimates established in Proposition 4.1 and Theorem 4.3 (respectively Theorem 4.4).

Lemma 5.3 (Absorbing ball in C^α). *Let $\theta_0 \in H^1$, $f \in L^\infty \cap H^1$, and define the Hölder exponent*

$$\alpha_* = \alpha_*(\|f\|_{L^\infty}) := \min \left\{ \frac{\varepsilon_1 \kappa^2}{\|f\|_{L^\infty}}, \frac{1}{4} \right\} \quad (5.7)$$

where $\varepsilon_1 > 0$ is a universal constant. There exists a time $t_{\alpha_*} = t_{\alpha_*}(\theta_0, f)$, such that

$$\|S(t)\theta_0\|_{C^{\alpha_*}} \leq M_{\infty, f} := \frac{2\|f\|_{L^\infty}}{\varepsilon_1 \kappa} \quad (5.8)$$

for all $t \geq t_{\alpha_*}$.

Proof of Lemma 5.3. By Proposition 4.2 there exists $t_0 = t_0(\theta_0, f) > 0$ such that $S(t_0)\theta_0 \in H^2$ and moreover by (4.5) we have

$$\|S(t_0)\theta_0\|_{H^2} \leq C\kappa^{-1}t_0^{-1}\|\theta_0\|_{H^1} + C\kappa^{-1}\|f\|_{H^1}. \quad (5.9)$$

By the Sobolev embedding it follows from (5.9) that

$$\begin{aligned} \|S(t_0)\theta_0\|_{C^{1/4}} &\leq C\|S(t_0)\theta_0\|_{H^2} \\ &\leq C\kappa^{-1}t_0^{-1}\|\theta_0\|_{H^1} + C\kappa^{-1}\|f\|_{H^1} = C(\kappa, \theta_0, f). \end{aligned} \quad (5.10)$$

In particular, $\|S(t_0)\theta_0\|_{L^\infty} \leq C(\kappa, \theta_0, f)$ holds.

Throughout this proof the value of $C(\kappa, \theta_0, f)$ may change from line to line, since we just want to emphasize the dependence of this bound solely on data and force.

We now apply Theorem 4.3 with initial data $S(t_0)\theta_0$. Let ε_0 be the constant in Theorem 4.3, and define

$$\alpha_1 = \min \left\{ \frac{\varepsilon_0 \kappa}{M_\infty(S(t_0)\theta_0, f)}, \frac{1}{4} \right\}$$

where

$$M_\infty(S(t_0)\theta_0, f) = \|S(t_0)\theta_0\|_{L^\infty} + \frac{\|f\|_{L^\infty}}{c_0 \kappa}$$

is as defined as in (4.3), with corresponding constant c_0 . Since $S(t_0)\theta_0 \in C^{1/4} \subseteq C^{\alpha_1}$ and (5.10) holds, it follows from estimate (4.8), that

$$[S(t)\theta_0]_{C^{\alpha_1}} \leq C(\kappa, \theta_0, f)$$

for all $t \geq t_0$. Since in fact we know $S(t_0)\theta_0 \in H^{3/2}$, we bootstrap the above global in time estimate for the C^{α_1} norm, which is a subcritical quantity, to obtain

$$\|S(t)\theta_0\|_{H^{3/2}} \leq C(\kappa, \theta_0, f) \quad (5.11)$$

for all $t \geq t_0$. We refer to the Proof of Theorem 5.2 for the main idea in this bootstrap argument.

We now apply Proposition 4.1, with initial data $S(t_0)\theta_0 \in L^\infty$ bounded as in (5.10), to conclude that there exists

$$t_1 = t_1(\theta_0, f) = \frac{1}{c_0\kappa} \log \left(1 + \frac{c_0\kappa \|S(t_0)\theta_0\|_{L^\infty}}{\|f\|_{L^\infty}} \right)$$

such that

$$\|S(t)\theta_0\|_{L^\infty} \leq \frac{2\|f\|_{L^\infty}}{c_0\kappa}$$

for all $t \geq t_1 + t_0$.

Now finally define

$$\alpha_* = \min \left\{ \frac{\varepsilon_0 c_0 \kappa^2}{2\|f\|_{L^\infty}}, \frac{1}{4} \right\}$$

and apply the argument in Theorem 4.3, with initial data taken to be $S(t_0 + t_1)\theta_0 \in H^{3/2} \subseteq C^{\alpha_*}$. We conclude from (4.9) and (5.11) that there exists $t_{\alpha_*} = t_{\alpha_*}(\theta_0, f)$ with $t_1 + t_0 \leq t_{\alpha_*} < \infty$, such that

$$\|S(t)\theta_0\|_{C^{\alpha_*}} \leq \frac{4\|f\|_{L^\infty}}{c_0\varepsilon_0\kappa}$$

holds for all $t \geq t_{\alpha_*}$, which concludes the proof the theorem. \square

The proof of Theorem 5.2 now follows from Lemma 5.3 and a bootstrap procedure, which is based on the sub-criticality of the Hölder norm and the nonlinear lower bound on the fractional Laplacian [CV12].

Proof of Theorem 5.2. Let $\alpha_* = \alpha_*(\|f\|_{L^\infty})$, and $t_{\alpha_*} = t_{\alpha_*}(\theta_0, f)$ be as defined in Lemma 5.3. Using estimate (5.8), we have that

$$\|S(t)\theta_0\|_{C^{\alpha_*}} \leq M_{\infty, f} \left(:= \frac{2\|f\|_{L^\infty}}{\varepsilon_1\kappa} \right) \quad (5.12)$$

for all $t \geq t_{\alpha_*}$, and moreover, by (5.11) we know that

$$\|S(t)\theta_0\|_{H^{3/2}}^2 \leq C(\kappa, \theta_0, f) < \infty \quad (5.13)$$

for all $t \geq t_{\alpha_*}$.

For the rest of the proof, denote

$$\theta_0^* = S(t_{\alpha_*})\theta_0. \quad (5.14)$$

The first step is to obtain a bound on time averages of the $H^{3/2}$ norm of the solution. We apply ∇ to (1.1) and pointwise in x take inner product with $\nabla\theta$, and apply Proposition 2.1 to obtain

$$(\partial_t + u \cdot \nabla + \kappa\Lambda) |\nabla\theta|^2 + \kappa D[\nabla\theta] = -2\partial_k u_j \partial_j \theta \partial_k \theta + 2\nabla f \cdot \nabla\theta \quad (5.15)$$

where

$$D[\nabla\theta](x) = \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} |\nabla\theta(x) - \nabla\theta(x+y)|^2 \frac{1}{|y|^3} dy$$

and we use the same notation for the \mathbb{T}^2 -periodic function $\nabla\theta$, and its extension to all of \mathbb{R}^2 by periodicity. The main observation here is that since we already have from (5.12) a bound for $\sup_{t \geq t_{\alpha_*}} \|S(t)\theta_0\|_{C^{\alpha_*}}$, we have an *improved nonlinear lower bound* on $D[\nabla\theta]$. Indeed, from [CV12, Theorem 2.2] we have

$$D[\nabla\theta](x) \geq \frac{|\nabla\theta(x)|^{\frac{3-\alpha_*}{1-\alpha_*}}}{C[\theta]_{C^{\alpha_*}}^{\frac{1}{1-\alpha_*}}} \quad (5.16)$$

where the constant C depends only on α_* , and is uniformly bounded for $\alpha_* \in (0, 1/2]$. Combining (5.16) with (5.12), we arrive at

$$D[\nabla\theta](x, t) \geq \frac{|\nabla\theta(x, t)|^{\frac{3-\alpha_*}{1-\alpha_*}}}{c_7 M_{\infty, f}^{\frac{1}{1-\alpha_*}}} \quad (5.17)$$

for all $t \geq t_{\alpha_*}$, where c_7 is a universal constant which is independent of α_* , for $\alpha_* \in (0, 1/2]$.

Next, we estimate the nonlinear term on the right side of (5.15). Let χ be a smooth cutoff function, that is 1 on $[0, 1]$, non-increasing, vanishes on $[2, \infty)$, and obeys $|\chi'| \leq 2$. For $\rho > 0$ to be determined, using the same argument which led to (4.25), we obtain

$$\begin{aligned} |\nabla u(x)| &\leq \frac{1}{2\pi} \left| P.V. \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} \chi\left(\frac{|y|}{\rho}\right) \nabla\theta(x+y) dy \right| + \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^3} \left(1 - \chi\left(\frac{|y|}{\rho}\right)\right) \nabla\theta(x+y) dy \right| \\ &\leq C \left(\rho D[\nabla\theta](x) \right)^{1/2} + C \frac{\|\theta\|_{L^\infty}}{\rho} \end{aligned}$$

for some universal constant $C > 0$. Therefore, we have

$$\begin{aligned} 2|\nabla u(x)| |\nabla\theta(x)|^2 &\leq C \left(\rho D[\nabla\theta](x) \right)^{1/2} |\nabla\theta(x)|^2 + C \frac{\|\theta\|_{L^\infty}}{\rho} |\nabla\theta(x)|^2 \\ &\leq \frac{\kappa}{2} D[\nabla\theta](x) + C \left(\frac{\rho}{\kappa} |\nabla\theta(x)|^4 + \frac{\|\theta\|_{L^\infty}}{\rho} |\nabla\theta(x)|^2 \right) \\ &\leq \frac{\kappa}{2} D[\nabla\theta](x) + \frac{C}{\kappa^{1/2}} \|\theta\|_{L^\infty}^{1/2} |\nabla\theta(x)|^3 \end{aligned} \quad (5.18)$$

by letting $\rho = \kappa^{1/2} \|\theta\|_{L^\infty}^{1/2} |\nabla\theta(x)|^{-1}$.

Since $t \geq t_{\alpha_*}$, we combine (5.12) with (5.15), (5.17), and (5.18), to arrive at

$$(\partial_t + u \cdot \nabla + \kappa\Lambda) |\nabla\theta|^2 + \frac{\kappa}{4} D[\nabla\theta] + \frac{\kappa |\nabla\theta(x, t)|^{\frac{3-\alpha_*}{1-\alpha_*}}}{4c_7 M_{\infty, f}^{\frac{1}{1-\alpha_*}}} \leq 2|\nabla f| |\nabla\theta| + \frac{c_8 M_{\infty, f}^{1/2} |\nabla\theta|^3}{\kappa^{1/2}} \quad (5.19)$$

for some universal constant c_8 . Using the ε -Young inequality, since $(3 - \alpha_*)/(1 - \alpha_*) > 3$ we furthermore infer from (5.19) that

$$(\partial_t + u \cdot \nabla + \kappa\Lambda) |\nabla\theta|^2 + \frac{\kappa}{4} D[\nabla\theta] + \frac{\kappa |\nabla\theta(x, t)|^{\frac{3-\alpha_*}{1-\alpha_*}}}{8c_7 M_{\infty, f}^{\frac{1}{1-\alpha_*}}} \leq 2|\nabla f| |\nabla\theta| + \frac{c_8 (8c_7)^{\frac{3-3\alpha_*}{2\alpha_*}}}{\kappa^{\frac{9-7\alpha_*}{4\alpha_*}}} M_{\infty, f}^{\frac{9-\alpha_*}{4\alpha_*}}. \quad (5.20)$$

Next we integrate (5.20) over \mathbb{T}^2 , and use

$$\frac{1}{2} \int_{\mathbb{T}^2} D[\nabla\theta](x) dx = \int_{\mathbb{T}^2} \nabla\theta \cdot \Lambda \nabla\theta dx = \|\theta\|_{H^{3/2}}^2 \geq \|\theta\|_{H^1}^2$$

to obtain

$$\frac{d}{dt} \|\theta\|_{H^1}^2 + \frac{\kappa}{6} \|\theta\|_{H^1}^2 + \frac{\kappa}{6} \|\theta\|_{H^{3/2}}^2 \leq \frac{6}{\kappa} \|f\|_{H^1}^2 + \frac{c_8 (8c_7)^{\frac{3-3\alpha_*}{2\alpha_*}}}{\kappa^{\frac{9-7\alpha_*}{4\alpha_*}}} M_{\infty, f}^{\frac{9-\alpha_*}{4\alpha_*}} \quad (5.21)$$

for times $t \geq t_{\alpha_*}$. Using the Grönwall inequality we obtain from (5.21) that

$$\begin{aligned} \|S(t + t_{\alpha_*})\theta_0\|_{H^1}^2 &= \|S(t)\theta_0^*\|_{H^1}^2 \\ &\leq \|\theta_0^*\|_{H^1}^2 e^{-\frac{t\kappa}{6}} + \left(\frac{36}{\kappa^2} \|f\|_{H^1}^2 + \frac{c_8(8c_7)^{\frac{3-3\alpha_*}{2\alpha_*}}}{6\kappa^{\frac{9-3\alpha_*}{4\alpha_*}}} M_{\infty,f}^{\frac{9-\alpha_*}{4\alpha_*}} \right) (1 - e^{-\frac{t\kappa}{6}}). \end{aligned} \quad (5.22)$$

Recall cf. (5.13) and (5.14) that $\|\theta_0^*\|_{H^1}^2 = \|S(t_{\alpha_*})\theta_0\|_{H^1}^2 \leq C(\kappa, \theta_0, f)$. We conclude from (5.22) that there exists

$$t_{H^1} = t_{H^1}(\theta_0, f) \geq t_{\alpha_*}$$

such that for all $t \geq t_{H^1}$ we have

$$\|S(t)\theta_0\|_{H^1}^2 \leq \frac{72}{\kappa^2} \|f\|_{H^1}^2 + \frac{c_8(8c_7)^{\frac{3-3\alpha_*}{2\alpha_*}}}{3\kappa^{\frac{9-3\alpha_*}{4\alpha_*}}} M_{\infty,f}^{\frac{9-\alpha_*}{4\alpha_*}} =: M_{1,f}^2. \quad (5.23)$$

Note that cf. (5.7) we have $\alpha_* = \alpha_*(\|f\|_{L^\infty})$ and cf. (5.8) we have $M_{\infty,f} = M_{\infty,f}(\|f\|_{L^\infty})$, so that $M_{1,f} = M_{1,f}(\kappa, \|f\|_{L^\infty \cap H^1})$. The dependence on κ and f may be computed explicitly from (5.7)–(5.8) and (5.23).

Inequality (5.23) not only gives an absorbing ball in H^1 , but combined with (5.21), integrated between t and $t + 1$, it also gives the bound

$$\int_t^{t+1} \|\theta(s)\|_{H^{3/2}}^2 ds \leq \frac{6 + \kappa}{\kappa} M_{1,f}^2 \quad (5.24)$$

for all $t \geq t_{H^1}$.

Estimate (5.24) now directly implies the existence of an absorbing ball for $S(t)$ in $H^{3/2}$. To see this, we take the L^2 inner product of (1.1) with $\Lambda^3 \theta$ and write

$$\begin{aligned} \frac{d}{dt} \|\theta\|_{H^{3/2}}^2 + \kappa \|\theta\|_{H^2}^2 &\leq \frac{1}{\kappa} \|f\|_{H^1}^2 + 2 \left| \int_{\mathbb{T}^2} \left(\Lambda^{3/2}(u \cdot \nabla \theta) - u \cdot \nabla \Lambda^{3/2} \theta \right) \Lambda^{3/2} \theta dx \right| \\ &\leq \frac{1}{\kappa} \|f\|_{H^1}^2 + C \|\theta\|_{H^{3/2}} \|\Lambda^{3/2} \theta\|_{L^4} \|\Lambda \theta\|_{L^4} \\ &\leq \frac{1}{\kappa} \|f\|_{H^1}^2 + \frac{\kappa}{2} \|\theta\|_{H^2}^2 + \frac{c_9}{\kappa} \|\theta\|_{H^{3/2}}^4 \end{aligned} \quad (5.25)$$

for some universal constant $c_9 > 0$. In the above estimate we have appealed to the commutator estimate (A.2) of Lemma A.1, and we used the the Sobolev embedding $H^{1/2} \subset L^4$. We obtain from (5.25) that

$$\frac{d}{dt} \|\theta\|_{H^{3/2}}^2 + \frac{\kappa}{2} \|\theta\|_{H^2}^2 \leq \frac{1}{\kappa} \|f\|_{H^1}^2 + \left(\frac{c_9}{\kappa} \|\theta\|_{H^{3/2}}^2 \right) \|\theta\|_{H^{3/2}}^2 \quad (5.26)$$

for $t \geq 0$.

At this stage we apply the Uniform Grönwall Lemma C.1, with the functions

$$x = \|\theta\|_{H^{3/2}}^2, \quad a(t) = \frac{c_9}{\kappa} \|\theta\|_{H^{3/2}}^2, \quad b = \frac{1}{\kappa} \|f\|_{H^1}^2$$

that by (5.24) obey the bounds

$$\int_t^{t+1} x(s) ds \leq \frac{6 + \kappa}{\kappa} M_{1,f}^2, \quad \int_t^{t+1} a(s) ds \leq \frac{c_9(6 + \kappa)}{\kappa^2} M_{1,f}^2, \quad \int_t^{t+1} b(s) ds = \frac{1}{\kappa} \|f\|_{H^1}^2$$

for any $t \geq t_{H^1}$. We conclude from (C.1) that

$$\|S(t)\theta_0\|_{H^{3/2}}^2 \leq \left(\frac{6 + \kappa}{\kappa} M_{1,f}^2 + \frac{1}{\kappa} \|f\|_{H^1}^2 \right) \exp \left(\frac{c_9(6 + \kappa)}{\kappa^2} M_{1,f}^2 \right) =: M_{3/2,f}^2 \quad (5.27)$$

for any $t \geq t_{H^{3/2}}$, where

$$t_{H^{3/2}} = t_{H^{3/2}}(\theta_0, f) =: t_{H^1} + 1.$$

We also note that since both α^* and $M_{1,f}$ depend only on κ and $\|f\|_{L^\infty \cap H^1}$ (and universal constants), we have $M_{3/2,f} = M_{3/2,f}(\kappa, \|f\|_{L^\infty \cap H^1})$.

Lastly, we notice that $L_t^2 H_x^2$ bounds are also available from the above argument. By combining (5.26) with (5.27) we obtain

$$\frac{1}{T} \int_t^{t+T} \|S(\tau)\theta_0\|_{H^2}^2 d\tau \leq \frac{2}{\kappa^2} \|f\|_{H^1}^2 + \frac{2c_9}{\kappa^2} M_{3/2,f}^4 =: M_{2,f}^2 \quad (5.28)$$

for any $t \geq t_{H^{3/2}}$ and $T > 0$. This concludes the proof of the theorem. \square

Remark 5.4 (Uniform attraction). Theorem 5.2 guarantees that for $\theta_0 \in H^1$, there exists a time $t_{H^{3/2}}$ which depends on θ_0 (and f) so that $S(t)\theta_0 \in \mathcal{B}$ for $t \geq t_{H^{3/2}}$. Note however that $t_{H^{3/2}}(\theta_0, f)$ does not depend solely on $\|\theta_0\|_{H^1}$, and it is not a priori locally uniform with respect to initial data. The sole reason for this is that the time of local existence of the solution arising from initial data $\theta_0 \in H^1$ is not guaranteed to depend only on $\|\theta_0\|_{H^1}$. On the other hand, we would like to emphasize that if $\theta_0 \in H^{1+\delta}$ with $\delta > 0$, it can be shown that $t_{H^{3/2}} = t_{H^{3/2}}(\|\theta_0\|_{H^{1+\delta}}, \|f\|_{L^\infty \cap H^1})$ and the time of entering the absorbing ball is a non-decreasing function of its arguments. *In particular, this implies that given a ball $\mathcal{B}_R = \{\theta \in H^{1+\delta} : \|\theta\|_{H^{1+\delta}} \leq R\}$, there exists a time $t_R = t_R(R, \|f\|_{L^\infty \cap H^1})$ such that $S(t)\mathcal{B}_R \subset \mathcal{B}$ for all $t \geq t_R$.* The reason for this fact is the following. For this smoother initial data $\theta_0 \in H^{1+\delta}$, we find a local time of existence of a unique $L_t^\infty H_x^1 \cap L_t^2 H_x^{3/2}$ solution, that depends only on $\|\theta_0\|_{H^{1+\delta}}$ and norms of f . Then going line-by-line through the proofs of Lemma 5.3 and Theorem 5.2 above shows that by waiting long enough, depending only on the $H^{1+\delta}$ norm of θ_0 and on norms of f , the C^{α^*} norm and then the $H^{3/2}$ norm of $S(t)\theta_0$ are under control. To avoid redundancy we omit further details.

As stated in the outline below Theorem 5.1, besides having a compact absorbing set \mathcal{B} we need continuity properties of $S(t)$ on \mathcal{B} . The next lemma in particular proves the Lipschitz continuity of $S(t) : \mathcal{B} \rightarrow H^1$ for fixed $t > 0$. The proof is given in Appendix C below.

Proposition 5.5 (Continuity). *For fixed $\theta_0 \in H^1$ we have that $S(\cdot)\theta_0 : [0, \infty) \rightarrow H^1$ is continuous. Fix a ball $\mathcal{B}_0 \in H^{3/2}$. We have that for $\theta_0, \tilde{\theta}_0 \in \mathcal{B}_0$ with $\|\theta_0 - \tilde{\theta}_0\|_{H^1} \leq \varepsilon\kappa$, for some universal $0 < \varepsilon \ll 1$, we have*

$$\|S(t)\theta_0 - S(t)\tilde{\theta}_0\|_{H^1} \leq e(t)\|\theta_0 - \tilde{\theta}_0\|_{H^1}$$

for some non-decreasing continuous function of time $e(t)$. In particular, if $\{t_n\}_{n \geq 1}$ is a sequence of times that diverge to ∞ as $n \rightarrow \infty$, and $\{\theta_{0,n}\}_{n \geq 1} \subset \mathcal{B}_0$ are a sequence of initial data such that

$$\|S(t_n)\theta_{0,n} - \theta_0\|_{H^1} \rightarrow 0$$

as $n \rightarrow \infty$ for some $\theta_0 \in H^1$, then for any fixed $t > 0$ we have

$$\|S(t + t_n)\theta_{0,n} - S(t)\theta_0\|_{H^1} = \|S(t)S(t_n)\theta_{0,n} - S(t)\theta_0\|_{H^1} \rightarrow 0$$

as $n \rightarrow \infty$.

The proof of the existence of the global attractor on the phase space H^1 now follows. We only outline the main steps, as the proof follows the same argument given in [CF88, pp. 133–136].

Proof of Theorem 5.1. By Theorem 5.2 we have a compact absorbing set $\mathcal{B} = \{\theta \in H^{3/2} : \|\theta\|_{H^{3/2}} \leq M_{3/2,f}\}$, where $M_{3/2,f}$ can be computed in terms of $\kappa, \|f\|_{L^\infty \cap H^1}$, and universal constants. The idea is that for any bounded $\mathcal{B}_1 \subset H^{1+\delta}$ with $\delta > 0$ by Remark 5.4 we have that the omega limit set obeys $\omega(\mathcal{B}_1) \subset \mathcal{B}$ and thus, by Proposition 5.5 we have

$$S(t)(\omega(\mathcal{B}_1)) = \omega(\mathcal{B}_1)$$

for all $t \geq 0$, where the omega limit sets $\omega(\cdot)$ are taken in the H^1 topology. The solution map is continuous with respect to initial data in \mathcal{B} , and the global attractor is just

$$\mathcal{A} = \omega(\mathcal{B}).$$

This set is clearly, compact, non-empty, and invariant in positive time. That $\lim_{t \rightarrow \infty} \text{dist}(S(t)\theta_0, \mathcal{A}) = 0$ for any $\theta_0 \in H^1$ follows since $S(t)\omega(\theta_0) = \omega(\theta_0) \subset \mathcal{B}$ and the definition of \mathcal{A} . Note that one may in fact show that

$$\mathcal{A} = \bigcap_{t>0} S(t)\mathcal{B}.$$

One inclusion follows by the definition of an omega limit set, while the other one follows by the invariance of $\omega(B)$ in positive time and Remark 5.4. The bounds (5.2)–(5.3) follow by taking $M_{\mathcal{A}} = \max\{M_{3/2,f}, M_{2,f}\}$, where $M_{3/2,f}$ and $M_{2,f}$ are as defined by (5.27) and (5.28) above. \square

Remark 5.6 (Higher regularity). If $f \in C^\infty(\mathbb{T}^2)$, it can be shown that in fact $\mathcal{A} \subset C^\infty(\mathbb{T}^2)$ with bounds that depend only on κ and f . Similar statements hold in the real analytic or Sobolev categories.

We conclude this section by noting that if the force f lies in $H^{1+\varepsilon}$ for some $\varepsilon > 0$, one may show the solution operator $S(t)$ is injective on \mathcal{A} .

Proposition 5.7 (Backwards uniqueness). *Assume that $f \in H^{1+\varepsilon}(\mathbb{T}^2)$ for some $\varepsilon > 0$. Let $\theta_0^{(1)}, \theta_0^{(2)} \in H^{1+\varepsilon}$ be two initial data, and let*

$$\theta^{(i)}(t) = S(t)\theta_0^{(i)} \in C([0, \infty); H^{3/2}) \cap L^2(0, \infty; H^2)$$

be the corresponding solutions of the initial value problem (1.1)–(1.3) for $i \in \{1, 2\}$. If there exists $T > 0$ such that $\theta^{(1)}(T) = \theta^{(2)}(T)$, then $\theta_0^{(1)} = \theta_0^{(2)}$ holds.

The proof uses the classical log-convexity method of Agmon and Nirenberg [AN67], and is given in Appendix C below.

6. FINITE DIMENSIONALITY OF THE ATTRACTOR

In this section we establish a bound on the fractal (and a fortiori Hausdorff) dimension of the global attractor \mathcal{A} for $S(t)$ evolving on H^1 . The physical meaning of this bound is that the long-time behavior of solutions to the forced critical SQG equations can be fully described by a finite number of independent degrees of freedom.

We recall that the fractal dimension $d_f(Z)$ of a compact set Z is given by

$$d_f(Z) = \limsup_{r \rightarrow 0} \frac{\log n_Z(r)}{\log(1/r)}$$

where $n_Z(r)$ is the minimal number of balls in H^1 of radii r needed to cover Z . Note that fractal dimension gives an upper bound (which may be strict) for the Hausdorff dimension of a compact set Z .

The proof closely follows the outline given in [CF85, CF88], where the connection with global Lyapunov exponents and the Kaplan-Yorke formula is established. The main idea is as follows. Assume \mathcal{A} is covered by a finite number of balls of radius r . Let the flow $S(t)$ transport a ball of radius r centered at θ_0 . Then up to an $o(r)$ error the image of the ball is an ellipsoid centered at $S(t)\theta_0$, with semi-axes on the directions given by the eigenvalues of $M(t, \theta_0)$, of lengths given by r multiplied by the eigenvalues of $M(t, \theta_0)$, where $M(t, \theta_0) = (S'(t, \theta_0)^* S'(t, \theta_0))^{1/2}$, and $S'(t, \theta_0)$ is the Fréchet derivative of $S(t)\theta_0$. A control on the volume of this ellipsoid (given in terms of the product of the eigenvalues of $M(t, \theta_0)$) then gives a bound on the number of balls of radius r needed to (re-)cover the ellipsoid. It then turns out that in order to estimate the fractal dimension of \mathcal{A} , it is sufficient to find an integer N with the property that n -dimensional volume elements carried by the flow decay exponentially, for all $n \geq N + 1$. We now make these ideas more precise.

Definition 6.1 (Continuous differentiability of $S(t)$). *The solution map $S(t)$ is continuously differentiable on \mathcal{A} if for every $\theta_0 \in \mathcal{A}$ there exists a linear operator*

$$S'(t, \theta_0): H^1 \rightarrow H^1$$

and a positive function $e(r, t)$, which is continuous with respect to both variables, such that

$$\sup_{\theta_0, \varphi_0 \in \mathcal{A}, 0 < \|\theta_0 - \varphi_0\|_{H^1} \leq r} \frac{\|S(t)\varphi_0 - S(t)\theta_0 - S'(t, \theta_0)[\varphi_0 - \theta_0]\|_{H^1}^2}{\|\varphi_0 - \theta_0\|_{H^1}^2} \leq e(r, t) \quad (6.1)$$

with

$$\lim_{r \rightarrow 0^+} e(r, t) = 0 \quad (6.2)$$

and moreover

$$\sup_{\theta_0 \in \mathcal{A}, \|\xi_0\|_{H^1} = 1} \|S'(t, \theta_0)[\xi_0]\|_{H^1} < \infty \quad (6.3)$$

for every $t \geq 0$.

The solution map $S(t)$ induced by the critical SQG equation is indeed continuously differentiable on \mathcal{A} . For $\theta_0 \in \mathcal{A}$, write $\theta = \theta(t) = S(t)\theta_0$ and for $\xi \in H^1$ let us introduce the elliptic operator

$$A_{\theta_0}(t)[\xi] = A_{\theta}[\xi] = -\kappa \Lambda \xi - \mathcal{R}^\perp \theta \cdot \nabla \xi - \mathcal{R}^\perp \xi \cdot \nabla \theta. \quad (6.4)$$

We express $S'(t, \theta_0)$ using $A_{\theta_0}(t)$.

Proposition 6.2 (Linearization about a trajectory on the attractor). *The solution map $S(t)$ associated to (1.1)–(1.2) is continuously differentiable on \mathcal{A} . Moreover, the linear operator $S'(t, \theta_0)$, when acting on an element $\xi_0 \in H^1$ is given by*

$$S'(t, \theta_0)[\xi_0] = \xi(t)$$

where $\xi(t)$ is the solution of

$$\partial_t \xi = A_{\theta_0}(t)[\xi] := -\kappa \Lambda \xi - \mathcal{R}^\perp \theta \cdot \nabla \xi - \mathcal{R}^\perp \xi \cdot \nabla \theta, \quad \xi(0) = \xi_0. \quad (6.5)$$

Also, for any $t > 0$ the operator $S'(t, \theta_0)$ is compact.

It follows from the proof that the function $e(r, t)$ in Definition 6.1 may be taken $\approx r^{2-a} \exp(Ct)$ for some $a \in (0, 1)$ and some $C > 0$, which depends only on κ and $\|f\|_{L^\infty \cap H^1}$. The proof of Proposition 6.2 is quite technical, and we defer it to Appendix C.

We next show that there is an N such that volume elements which are carried by the flow of $S(t)\theta_0$, with $\theta_0 \in \mathcal{A}$, decay exponentially for dimensions larger than N . Consider $\theta_0 \in \mathcal{A}$, and an initial orthogonal set of infinitesimal displacements $\{\xi_{1,0}, \dots, \xi_{n,0}\}$ for some $n \geq 1$. The volume of the parallelepiped they span is given by

$$V_n(0) = \|\xi_{1,0} \wedge \dots \wedge \xi_{n,0}\|_{H^1}.$$

The reason we have introduced in Proposition 6.2 the linearization $S'(t, \theta_0)$ of the flow near $S(t)\theta_0$ is that these displacements ξ_i evolve exactly under this linearization, that is, we define

$$\xi_i(t) = S'(t, \theta_0)[\xi_{i,0}] \quad \text{for all } i \in \{1, \dots, n\}, \text{ and } t \geq 0,$$

or equivalently the ξ_i obey the equation

$$\partial_t \xi_i = A_{\theta_0}(t)[\xi_i], \quad \xi_i(0) = \xi_{i,0},$$

where $A_{\theta_0}(t)$ is defined in (6.4) above. Then it follows cf. [CF85, CF88] that the volume elements

$$V_n(t) = \|\xi_1(t) \wedge \dots \wedge \xi_n(t)\|_{H^1}$$

satisfy

$$V_n(t) = V_n(0) \exp \left(\int_0^t \text{Tr}(P_n(s) A_{\theta_0}(s)) ds \right)$$

where the orthogonal projection $P_n(s)$ is onto the linear span of $\{\xi_1(s), \dots, \xi_n(s)\}$ in the Hilbert space H^1 , and $\text{Tr}(P_n(s)A_\theta)$ is defined by

$$\text{Tr}(P_n(s)A_\theta) = \sum_{j=1}^n \int_{\mathbb{T}^2} (-\Delta \varphi_j(s)) A_\theta[\varphi_j(s)] dx \quad (6.6)$$

for $n \geq 1$, with $\{\varphi_1(s), \dots, \varphi_n(s)\}$ an orthonormal set spanning the linear span of $\{\xi_1(s), \dots, \xi_n(s)\}$. The value of $\text{Tr}(P_n(s)A_\theta)$ does not depend on the choice of this orthonormalization. Therefore, letting

$$\langle P_n A_{\theta_0} \rangle := \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{Tr}(P_n(t)A_{\theta_0}(t)) dt \quad (6.7)$$

we obtain

$$V_n(t) \leq V_n(0) \exp \left(t \sup_{\theta_0 \in \mathcal{A}} \sup_{P_n(0)} \langle P_n A_{\theta_0} \rangle \right) \quad (6.8)$$

for all $t \geq 0$, where the supremum over $P_n(0)$ is a supremum over all choices of initial n orthogonal set of infinitesimal displacements that we take around θ_0 .

Next, we show that n -dimensional volume elements decay exponentially in time (at a rate that is bounded from below), whenever n is sufficiently large, *independently* of the choice of θ_0 in \mathcal{A} , and *independently* of initial set of orthogonal displacements $\{\xi_{i,0}\}_{i=1}^n$ which define $P_n(0)$. The key is to show that the symmetric part of the operator $A_{\theta_0}(t)$ obeys good quadratic form bounds in the H^1 topology.

Proposition 6.3 (Contractivity of large dimensional volume elements). *There exists $N = N(\kappa, M_{\mathcal{A}})$ such that for any $\theta_0 \in \mathcal{A}$ and any set of initial orthogonal displacements $\{\xi_{i,0}\}_{i=1}^n$, we have*

$$\langle P_n A_{\theta_0} \rangle < 0 \quad (6.9)$$

whenever $n \geq N$. In particular, $V_n(t)$ decays exponentially in t for any $n \geq N$.

Proof of Proposition 6.3. Let $\xi \in H^1$ be arbitrary. By the definition of $A_\theta[\xi]$ in (6.4), and the fact that $\mathcal{R}^\perp \theta$ is divergence-free, we have

$$\begin{aligned} \int_{\mathbb{T}^2} \Lambda^2 \xi A_\theta[\xi] dx &= -\kappa \|\xi\|_{H^{3/2}}^2 + \left| \int_{\mathbb{T}^2} \partial_k \mathcal{R}^\perp \theta \cdot \nabla \xi \partial_k \xi dx \right| + \left| \int_{\mathbb{T}^2} \partial_k (\mathcal{R}^\perp \xi \cdot \nabla \theta) \partial_k \xi dx \right| \\ &\leq -\kappa \|\xi\|_{H^{3/2}}^2 + C \|\theta\|_{H^{3/2}} \|\xi\|_{H^{3/2}} \|\xi\|_{H^1} + C \|\xi\|_{H^{1/2}} \|\theta\|_{H^2} \|\xi\|_{H^{3/2}}. \end{aligned}$$

Here we have appealed to the Sobolev embedding $H^{1/2} \subset L^4$. Using the Poincaré inequality it follows that

$$\int_{\mathbb{T}^2} \Lambda^2 \xi A_\theta[\xi] dx \leq -\frac{\kappa}{2} \|\xi\|_{H^{3/2}}^2 + \frac{c_{10}}{\kappa} \|\theta\|_{H^2}^2 \|\xi\|_{H^1}^2. \quad (6.10)$$

for some universal constant $c_{10} > 0$. Now, for any θ_0 and any $t \geq 0$ the definition (6.6), the inequality (6.10), the normalization of the φ_j 's (recall that $\{\varphi_1(t), \dots, \varphi_n(t)\}$ an orthonormal set spanning the linear span of $\{\xi_1(t), \dots, \xi_n(t)\}$), and estimate (5.3) yield

$$\begin{aligned} \frac{1}{T} \int_0^T \text{Tr}(P_n(t)A_{\theta_0}(t)) dt &= \frac{1}{T} \int_0^T \sum_{j=1}^n \int_{\mathbb{T}^2} (-\Delta \varphi_j(t)) A_\theta[\varphi_j(t)] dx dt \\ &\leq -\frac{\kappa}{4} \frac{1}{T} \int_0^T \sum_{j=1}^n \|\varphi_j(t)\|_{H^{3/2}}^2 dt + \frac{c_{10}}{\kappa} \frac{1}{T} \int_0^T \|\theta(t)\|_{H^2}^2 \sum_{j=1}^n \|\varphi_j(t)\|_{H^1}^2 dt \\ &\leq -\frac{\kappa}{4} \frac{1}{T} \int_0^T \text{Tr}(P_n(t)\Lambda) + n \frac{c_{10}}{\kappa} \frac{1}{T} \int_0^T \|\theta(t)\|_{H^2}^2 dt \\ &\leq -\frac{\kappa}{c_{11}} n^{3/2} + n \frac{c_{10}}{\kappa} M_{\mathcal{A}}^2 \end{aligned} \quad (6.11)$$

where in the last inequality we have used that the eigenvalues $\{\lambda_j\}_{j \geq 1}$ of $\Lambda^{1/2}$ obey

$$\lambda_j \geq \frac{1}{c_{11}} j^{1/2}$$

for a sufficiently large universal constant $c_{11} > 0$ (see e.g. [CF88]). Choosing

$$N = N(\kappa, M_{\mathcal{A}}) = \left\lceil (c_{10} c_{11} \kappa^{-2} M_{\mathcal{A}}^2)^2 \right\rceil \quad (6.12)$$

the lemma now follows directly from (6.11) and the definition (6.7). \square

The upshot of Proposition 6.3 is that N -dimensional volume elements decay exponentially in time. This also implies that the fractal (box-counting) dimension of \mathcal{A} is finite, and is bounded by this N .

Theorem 6.4 (Finite dimensionality of the attractor). *Let $N = N(\kappa^{-1} M_{\mathcal{A}})$ be as defined in (6.12) above. Then the fractal dimension of \mathcal{A} is finite, and we have $\dim_f(\mathcal{A}) \leq N$.*

Proof of Theorem 6.4. We follow precisely the lines of the argument in [CF88, pp. 115–130, and Chapter 14]. The main ingredients are the continuous differentiability of $S(t)$ on \mathcal{A} , the compactness of the linearization, and the exponential decay of large-dimensional volume elements which follows from (6.8) and (6.9). We omit further details and refer to [CF88]. \square

APPENDIX A. FRACTIONAL INEQUALITIES

We recall the following fractional product (Kato-Ponce), commutator (Kenig-Ponce-Vega), and Sobolev estimates, cf. [KP88, KPV91, Tay91, SS03, Ju04] and references therein.

Lemma A.1 (Fractional calculus). *Let $f, g \in C^\infty(\mathbb{T}^2)$, $s > 0$, and $p \in (1, \infty)$. Then we have that*

$$\|\Lambda^s(fg)\|_{L^p} \leq C\|g\|_{L^{p_1}} \|\Lambda^s f\|_{L^{p_2}} + C\|\Lambda^s g\|_{L^{p_3}} \|f\|_{L^{p_4}}, \quad (\text{A.1})$$

where $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$, and $p_2, p_3 \in (1, \infty)$, for a sufficiently large constant C that depends only on s, p, p_i . Moreover,

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{p_2}} + C\|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}} \quad (\text{A.2})$$

where p_i are as above. For $q \in [p, \infty)$ and f of zero mean we also have

$$\|f\|_{L^q} \leq C\|\Lambda^{\frac{2}{p} - \frac{2}{q}} f\|_{L^p} \quad (\text{A.3})$$

for a sufficiently large constant C that depends only on p and q .

We conclude this appendix by giving a sketch of the proof of Proposition 2.2. The detailed proof can be found in [CGHV13], and we give below only the main ideas.

Proof of Proposition 2.2. The case $\alpha = 0$ trivially holds, while in the case $\alpha = 2$ estimate (2.9) follows upon integration by parts. Therefore, henceforth consider $\alpha \in (0, 2)$. For $p = 2$ inequality (2.9) holds due

to the Parseval's identity, and the rest of the proof we let $p \geq 4$ be even. For $0 < \alpha < 2$ we have

$$\begin{aligned}
& \int \theta^{p-1}(x) \Lambda^\alpha \theta(x) dx \\
&= \frac{1}{2} P.V. \iint (\theta^{p-1}(x) - \theta^{p-1}(y)) (\theta(x) - \theta(y)) K_\alpha(x-y) dy dx \\
&= \frac{1}{2p} P.V. \iint \left(p (\theta^{p-1}(x) - \theta^{p-1}(y)) (\theta(x) - \theta(y)) - 2 (\theta^{p/2}(x) - \theta^{p/2}(y))^2 \right) K_\alpha(x-y) dy dx \\
&\quad + \frac{1}{p} P.V. \iint (\theta^{p/2}(x) - \theta^{p/2}(y))^2 K_\alpha(x-y) dy dx \\
&= \frac{1}{2p} P.V. \iint f_p(\theta(x), \theta(y)) K_\alpha(x-y) dy dx + \frac{1}{p} \|\Lambda^{\alpha/2}(\theta^{p/2})\|_{L^2}^2 \\
&=: \frac{1}{2p} \mathcal{T} + \frac{1}{p} \|\Lambda^{\alpha/2}(\theta^{p/2})\|_{L^2}^2
\end{aligned} \tag{A.4}$$

where the double integral is over \mathbb{T}^{2d} , and we have defined

$$f_p(a, b) = p(a^{p-1} - b^{p-1})(a - b) - 2(a^{p/2} - b^{p/2})^2.$$

it can be easily seen that $f_p(a, b) \geq 0$ on \mathbb{R}^2 when p is even, and so the term \mathcal{T} is positive. The main idea is that exactly \mathcal{T} gives the $\|\theta\|_{L^p}^p$ term in the lower bound (2.9).

We next claim that for $p \geq 4$ even, and $a, b \in \mathbb{R}$ we have

$$f_p(a, b) \geq \frac{p-2}{2} (a-b)^2 a^{p-2}. \tag{A.5}$$

This fact may be checked directly using calculus. Using (A.5) we now prove (2.9). Since K_α is positive, letting $e_1 = (1, 0, \dots, 0)$, we have

$$\begin{aligned}
\mathcal{T} &\geq \frac{p-2}{2} P.V. \iint (\theta(x) - \theta(y))^2 \theta(x)^{p-2} K_\alpha(x-y) dy dx \\
&\geq \frac{p-2}{2} c_{d,\alpha} \iint (\theta(x) - \theta(y))^2 \theta(x)^{p-2} \frac{1}{|x-y-2\pi e_1|^{d+\alpha}} dy dx \\
&\geq \frac{(p-2)c_{d,\alpha}}{2(2\pi + |\text{diam}(\mathbb{T}^d)|)^{d+\alpha}} \iint (\theta(x) - \theta(y))^2 \theta(x)^{p-2} dy dx \\
&\geq \frac{(p-2)c_{d,\alpha}}{2(2\pi + |\text{diam}(\mathbb{T}^d)|)^{d+\alpha}} \int_{\mathbb{T}^d} \left(\theta^p(x) |\mathbb{T}^d| - 2\theta^{p-1}(x) \int_{\mathbb{T}^d} \theta(y) dy \right) dx.
\end{aligned} \tag{A.6}$$

At this point we use that θ has zero mean. It then follows from (A.6), that

$$\mathcal{T} \geq \frac{(p-2)c_{d,\alpha} |\mathbb{T}^d|}{2(2\pi + |\text{diam}(\mathbb{T}^d)|)^{d+\alpha}} \|\theta\|_{L^p}^p. \tag{A.7}$$

This proves (2.9) with the constant

$$\frac{(p-2)2^\alpha \Gamma((n+\alpha)/2) |\mathbb{T}^d|}{4p(2\pi + |\text{diam}(\mathbb{T}^d)|)^{d+\alpha} |\Gamma(-\alpha/2)| \pi^{d/2}} \geq \frac{2^\alpha \Gamma((n+\alpha)/2) |\mathbb{T}^d|}{8(2\pi + |\text{diam}(\mathbb{T}^d)|)^{d+\alpha} |\Gamma(-\alpha/2)| \pi^{d/2}} = \frac{1}{C_{d,\alpha}}$$

for any $p \geq 4$. When $d = 2$ and $\alpha = 1$ the above constant $C_{d,\alpha}$ may be taken to equal $2^9 \pi^2$. \square

APPENDIX B. INTERCHANGING THE SPATIAL SUPREMUM WITH THE TIME DERIVATIVE

Lemma B.1 (Switching d/dt and sup). *Let $\mathcal{K} \subset \mathbb{R}^d$ be compact, and let $T > 0$. Consider a function*

$$f: (0, T) \times \mathcal{K} \rightarrow [0, \infty)$$

and assume that for every $\lambda \in \mathcal{K}$ the functions

$$f_\lambda(\cdot) = f(\cdot, \lambda): (0, T) \rightarrow [0, \infty) \quad \text{and} \quad \dot{f}_\lambda(\cdot) = (\partial_t f)(\cdot, \lambda): (0, T) \rightarrow \mathbb{R}$$

are continuous. Additionally, assume that the following properties hold:

- (i) The families $\{f_\lambda\}_{\lambda \in \mathcal{K}}$ and $\{\dot{f}_\lambda\}_{\lambda \in \mathcal{K}}$ are uniformly equicontinuous with respect to t .
- (ii) For every $t \in (0, T)$, the functions $f(t, \cdot): \mathcal{K} \rightarrow [0, \infty)$ and $(\partial_t f)(t, \cdot): \mathcal{K} \rightarrow \mathbb{R}$ are continuous.

Lastly, define

$$F(t) = \sup_{\lambda \in \mathcal{K}} f_\lambda(t)$$

Then, for almost every $t \in (0, T)$ the function F is differentiable at t , and there exists $\lambda_* = \lambda_*(t) \in \mathcal{K}$ such that simultaneously

$$\dot{F}(t) = \dot{f}_{\lambda_*}(t) \quad \text{and} \quad F(t) = f_{\lambda_*}(t) \tag{B.1}$$

hold.

Proof of Lemma B.1. The proof follows along the lines of [CC04, Theorem 4.1] and [KV11, Lemma A.3], but for the sake of completeness we present here the full argument.

The uniform equicontinuity of $\{\dot{f}_\lambda(t)\}_{\lambda \in \mathcal{K}}$ implies that there exists $\delta > 0$ such that

$$\sup_{\lambda \in \mathcal{K}} |\dot{f}_\lambda(t) - \dot{f}_\lambda(s)| \leq 1 \quad \text{whenever} \quad |t - s| < \delta.$$

Since $(\partial_t f)(T/2, \lambda)$ is a continuous function of λ , it attains its maximum over the compact \mathcal{K} at some λ_0 , and we obtain from the above that

$$\sup_{t \in (0, T)} \sup_{\lambda \in \mathcal{K}} |\dot{f}_\lambda(t)| \leq |\dot{f}_{\lambda_0}(T/2)| + \frac{T}{\delta} =: M < \infty$$

Therefore, for $t \in (0, T)$, and $|\Delta t| > 0$ sufficiently small so that $t + \Delta t \in (0, T)$, we have

$$\begin{aligned} |F(t) - F(t + \Delta t)| &= \left| \sup_{\lambda \in \mathcal{K}} f_\lambda(t) - \sup_{\lambda \in \mathcal{K}} f_\lambda(t + \Delta t) \right| \leq \sup_{\lambda \in \mathcal{K}} |f_\lambda(t) - f_\lambda(t + \Delta t)| \\ &= |\Delta t| \sup_{\lambda \in \mathcal{K}} |\dot{f}_\lambda(\tau_\lambda)| \quad (\text{for some } \tau_\lambda \in (t, t + \Delta t)) \\ &\leq |\Delta t| M. \end{aligned}$$

Therefore, F is Lipschitz continuous on $(0, T)$, and Rademacher's theorem implies that F is differentiable almost everywhere.

Fix $t \in (0, T)$ such that F is differentiable at t , and let $\Delta t > 0$. In view of the continuity in λ of $f(t + \Delta t, \lambda)$ and the compactness of \mathcal{K} , there exists $\lambda(t + \Delta t) \in \mathcal{K}$ such that

$$F(t + \Delta t) = f_{\lambda(t + \Delta t)}(t).$$

Since \mathcal{K} is compact, we can find a sequence $\Delta t_n \rightarrow 0^+$ and a point $\lambda_* \in \mathcal{K}$ such that

$$\lambda(t + \Delta t_n) \rightarrow \lambda_* \quad \text{as} \quad \Delta t_n \rightarrow 0^+. \tag{B.2}$$

We now check that

$$F(t) = f_{\lambda_*}(t),$$

i.e., the second statement in (B.1). We write

$$\begin{aligned} |F(t) - f_{\lambda_*}(t)| &\leq \lim_{\Delta t_n \rightarrow 0^+} |F(t) - F(t + \Delta t_n)| \\ &\quad + \lim_{\Delta t_n \rightarrow 0^+} |f_{\lambda(t + \Delta t_n)}(t + \Delta t_n) - f_{\lambda(t + \Delta t_n)}(t)| \\ &\quad + \lim_{\Delta t_n \rightarrow 0^+} |f_{\lambda(t + \Delta t_n)}(t) - f_{\lambda_*}(t)|. \end{aligned} \tag{B.3}$$

Note that

$$|F(t) - F(t + \Delta t_n)| \leq \sup_{\lambda \in \mathcal{K}} |f_\lambda(t) - f_\lambda(t + \Delta t_n)|$$

and

$$|f_{\lambda(t+\Delta t_n)}(t + \Delta t_n) - f_{\lambda(t+\Delta t_n)}(t)| \leq \sup_{\lambda \in \mathcal{K}} |f_\lambda(t) - f_\lambda(t + \Delta t_n)|.$$

In view of the equicontinuity at t of the family $\{f_\lambda\}_{\lambda \in \mathcal{K}}$, we have

$$\lim_{\Delta t_n \rightarrow 0^+} \sup_{\lambda \in \mathcal{K}} |f_\lambda(t) - f_\lambda(t + \Delta t_n)| = 0$$

and thus the first two limits on the right side of (B.3) vanish. The third limit vanishes in view of (B.2) and the assumption of continuity with respect to λ of $f(t, \lambda)$, at any given fixed t . This proves the second part of (B.1).

Let $\varepsilon > 0$ and fix $t \in (0, T)$. In view of the uniform equicontinuity of the family $\{\dot{f}_\lambda\}_{\lambda \in \mathcal{K}}$, we have that there exists $\delta_1 = \delta_1(\varepsilon) > 0$, such that

$$\sup_{\lambda \in \mathcal{K}} |\dot{f}_\lambda(t) - \dot{f}_\lambda(\tau)| < \varepsilon \quad \text{whenever} \quad |t - \tau| < \delta_1. \quad (\text{B.4})$$

Also, in view of the continuity with respect to λ of $(\partial_t f)(t, \lambda)$ at a fixed t , there exists $\delta_2 = \delta_2(\varepsilon, t) > 0$ such that

$$|\dot{f}_\lambda(t) - \dot{f}_{\lambda_*}(t)| < \varepsilon \quad \text{whenever} \quad |\lambda - \lambda_*| < \delta_2. \quad (\text{B.5})$$

Let n be sufficiently large, so that $0 < \Delta t_n < \delta_1$ and $|\lambda(t + \Delta t_n) - \lambda_*| < \delta_2$, which is possible in view of (B.2). Using the fundamental theorem of calculus, (B.4), (B.5), and the fact that $\Delta t_n > 0$, we obtain

$$\begin{aligned} & \frac{F(t + \Delta t_n) - F(t)}{\Delta t_n} \\ &= \frac{f_{\lambda(t+\Delta t_n)}(t + \Delta t_n) - f_{\lambda(t+\Delta t_n)}(t)}{\Delta t_n} + \frac{f_{\lambda(t+\Delta t_n)}(t) - f_{\lambda_*}(t)}{\Delta t_n} \\ &\leq \frac{1}{\Delta t_n} \int_t^{t+\Delta t_n} \dot{f}_{\lambda(t+\Delta t_n)}(s) ds \\ &= \frac{1}{\Delta t_n} \int_t^{t+\Delta t_n} \dot{f}_{\lambda_*}(s) ds + \frac{1}{\Delta t_n} \int_t^{t+\Delta t_n} \left(\dot{f}_{\lambda(t+\Delta t_n)}(s) - \dot{f}_{\lambda_*}(s) \right) ds \\ &\quad + \frac{1}{\Delta t_n} \int_t^{t+\Delta t_n} \left(\dot{f}_{\lambda(t+\Delta t_n)}(s) - \dot{f}_{\lambda(t+\Delta t_n)}(t) \right) + \left(\dot{f}_{\lambda_*}(t) - \dot{f}_{\lambda_*}(s) \right) + \left(\dot{f}_{\lambda(t+\Delta t_n)}(t) - \dot{f}_{\lambda_*}(t) \right) ds \\ &\leq \dot{f}_{\lambda_*}(t) + 4\varepsilon, \end{aligned}$$

which shows that

$$\dot{F}(t) = \lim_{\Delta t_n \rightarrow 0^+} \frac{F(t + \Delta t_n) - F(t)}{\Delta t_n} \leq \dot{f}_{\lambda_*}(t) \quad (\text{B.6})$$

since ε was arbitrary, and we chose t so that $\dot{F}(t)$ exists. Conversely, for $\Delta t_n > 0$ we have

$$\begin{aligned} \frac{F(t + \Delta t_n) - F(t)}{\Delta t_n} &= \frac{f_{\lambda(t+\Delta t_n)}(t + \Delta t_n) - f_{\lambda_*}(t + \Delta t_n)}{\Delta t_n} + \frac{f_{\lambda_*}(t + \Delta t_n) - f_{\lambda_*}(t)}{\Delta t_n} \\ &\geq \frac{f_{\lambda_*}(t + \Delta t_n) - f_{\lambda_*}(t)}{\Delta t_n} \end{aligned}$$

which shows that

$$\dot{F}(t) = \lim_{\Delta t_n \rightarrow 0^+} \frac{F(t + \Delta t_n) - F(t)}{\Delta t_n} \geq \dot{f}_{\lambda_*}(t) \quad (\text{B.7})$$

Estimates (B.6) and (B.7) prove the first part of (B.1), and hence of the lemma. \square

Corollary B.2. *Assume that $\theta \in C^\beta((0, T); C^{1,\beta}(\mathbb{T}^2))$ is a classical solution of (1.1)–(1.2) on $(0, T)$, for some $\beta \in (0, 1)$, with force $f \in C^\beta$. For $0 < \alpha < \beta/2$ we define*

$$v(t, x; h) = \frac{|\delta_h \theta(x, t)|}{|h|^\alpha} = \frac{|\theta(x+h, t) - \theta(x, t)|}{|h|^\alpha} : \mathbb{T}^2 \times (0, T) \times \mathbb{T}^2 \rightarrow \mathbb{R}$$

with the convention that $v(t, x; 0) = 0$. Then for almost every $t \in (0, T)$, there exists a pair

$$(\bar{x}, \bar{h}) = (\bar{x}(t), \bar{h}(t)) \in \mathbb{T}^2 \times \mathbb{T}^2$$

such that

$$v(t, \bar{x}; \bar{h})^2 = \sup_{(x, h) \in \mathbb{T}^2 \times \mathbb{T}^2} v(t, \cdot; \cdot)^2$$

and moreover

$$\frac{d}{dt} \left(\sup_{(x, h) \in \mathbb{T}^2 \times \mathbb{T}^2} v(t, \cdot; \cdot)^2 \right) = (\partial_t v^2)(t, \bar{x}; \bar{h})$$

holds.

Proof of Corollary B.2. The proof follows by applying Lemma B.1 to the function

$$f(t, \lambda) = v(t, x; h)^2$$

with $\lambda = (x; h) \in \mathbb{T}^2 \times \mathbb{T}^2 = \mathcal{K}$, which is clearly compact. It is clear that f is a non-negative function. By assumption, for fixed $\lambda = (x; h)$, the function

$$f_\lambda(t) = f(\lambda, t) = v(t, x; h)^2 = \frac{(\theta(x+h, t) - \theta(x, t))^2}{|h|^{2\alpha}}$$

is continuous with respect to t . In fact, since $2\alpha < \beta < 1$, for $t, s \in [0, T]$ and $\lambda = (x; h)$ we have that

$$\begin{aligned} |f_\lambda(t) - f_\lambda(s)| &= \frac{1}{|h|^{2\alpha}} |\delta_h \theta(x, t) + \delta_h \theta(x, s)| |\delta_h \theta(x, t) - \delta_h \theta(x, s)| \\ &\leq (|\theta(t)|)_{C^{2\alpha}} + |\theta(s)|_{C^{2\alpha}} (|\theta(x+h, t) - \theta(x+h, s)| + |\theta(x, t) - \theta(x, s)|) \\ &\leq 4\|\theta\|_{L^\infty(0, T; C^\beta)} |t - s|^\beta \|\theta\|_{C^\beta(0, T; L^\infty)} \\ &\leq 4|t - s|^\beta \|\theta\|_{C^\beta(0, T; C^\beta)}^2 \end{aligned}$$

which shows that the family $\{f_\lambda\}_{\lambda \in \mathcal{K}}$ is uniformly equicontinuous on $(0, T)$.

Using the equation (1.1) obeyed by θ , we moreover have that

$$\begin{aligned} \dot{f}_\lambda(t) &= (\partial_t f)(t, \lambda) = \frac{\partial}{\partial t} v(t, x; h)^2 \\ &= \frac{2(\theta(x+h, t) - \theta(x, t))}{|h|^{2\alpha}} (\partial_t \theta(x+h, t) - \partial_t \theta(x, t)) \\ &= \frac{2\delta_h \theta(x, t)}{|h|^{2\alpha}} (\delta_h f(x) - \kappa \delta_h \Lambda \theta(x, t) - u(x, t) \cdot \delta_h \nabla \theta(x, t) - \delta_h u(x, t) \cdot \theta(x+h, t)) \end{aligned}$$

is also continuous with respect to t , since by assumption $\nabla \theta, \Lambda \theta \in C^\beta(0, T; C^\beta)$, and $u \in C^\beta(0, T; C^{1,\beta})$. To verify the equicontinuity of the family $\{\dot{f}_\lambda\}_{\lambda \in \mathcal{K}}$, we note that for $t, s \in [0, T]$ and $(x; h) \in \mathbb{T}^2 \times \mathbb{T}^2$ it

holds that

$$\begin{aligned}
& |\dot{f}_\lambda(t) - \dot{f}_\lambda(s)| \\
& \leq C|t-s|^\beta \|\theta\|_{C^\beta(0,T;L^\infty)} \left(\|f\|_{C^{2\alpha}} + \kappa \|\theta\|_{L^\infty(0,T;C^{1,2\alpha})} + \|\theta\|_{L^\infty(0,T;C^{2\alpha})} \|\theta\|_{L^\infty(0,T;C^{1,2\alpha})} \right) \\
& \quad + C|t-s|^\beta \|\theta\|_{L^\infty(0,T;C^{2\alpha})} \left(\kappa \|\theta\|_{C^\beta(0,T;C^{1,\beta})} + \|\theta\|_{C^\beta(0,T;C^{1,\beta})}^2 \right) \\
& \leq C|t-s|^\beta \left(\|f\|_{C^\beta} + \kappa \|\theta\|_{C^\beta(0,T;C^{1,\beta})} + \|\theta\|_{C^\beta(0,T;C^{1,\beta})}^2 \right)
\end{aligned}$$

which by assumption is a finite number times $|t-s|^\beta$.

It is left to check that for fixed $t \in (0, T)$, the quantities $f(t, \lambda)$ and $\partial_t f(t, \lambda)$ vary continuously with respect to $\lambda = (x; h)$. This can be verified similarly to the equicontinuity of f_λ and \dot{f}_λ . Note that there is no problem at $h = 0$ since by assumption $\beta > 2\alpha$. We omit further details. \square

APPENDIX C. TECHNICAL DETAILS ABOUT THE EXISTENCE AND SIZE OF THE ATTRACTOR

In this appendix, we present a number of technical lemmas which are needed in order to establish the existence of the global attractor for the solution map $S(t): H^1 \rightarrow H^1$ associated to the critical SQG equation, and to give an estimate on its fractal dimension.

The following variant of the classical Grönwall lemma is used in the proof of Theorem 5.2, in order to bootstrap information about the time average of the $H^{3/2}$ norm, to information about the pointwise in time behavior of the $H^{3/2}$ norm. The lemma is due to Foias and Prodi [FP67]. See also [CF88, Tem97, Rob01].

Lemma C.1 (Uniform Grönwall Lemma). *Assume $x, a, b: [0, \infty) \rightarrow [0, \infty)$ are functions such that*

$$\frac{dx}{dt} \leq ax + b$$

and in addition assume that there exists $r > 0, t_0 > 0$ such that

$$\int_t^{t+r} x(s) ds \leq X, \quad \int_t^{t+r} a(s) ds \leq A, \quad \int_t^{t+r} b(s) ds \leq B$$

for all $t \geq t_0$. Then we have

$$x(t) \leq (Xr^{-1} + B)e^A \tag{C.1}$$

for all $t \geq t_0 + r$.

Next, we give the proof of the backwards uniqueness property for $S(t)$. The proof uses the classical log-convexity method of Agmon and Nirenberg [AN67], see also [Tem97, Rob01, Kuk07].

Proof of Proposition 5.7. First we note that if $\theta \in L^\infty(0, T; H^{3/2}(\mathbb{T}^2)) \cap L^2(0, T; H^2(\mathbb{T}^2))$ is a solution of

$$\partial_t \theta + \mathcal{R}^\perp \theta \cdot \nabla \theta + \kappa \Lambda \theta = f \tag{C.2}$$

with initial data

$$\theta_0 \in H^{1+\varepsilon}(\mathbb{T}^2)$$

and force

$$f \in H^{1+\varepsilon}(\mathbb{T}^2)$$

for some $\varepsilon \in (0, 1)$. Then in fact

$$\theta \in L^\infty(0, T; H^{3/2+\varepsilon}(\mathbb{T}^2)) \cap L^2(0, T; H^{2+\varepsilon}(\mathbb{T}^2)) \tag{C.3}$$

holds.

To see this, let $s = 3/2 + \varepsilon$. Apply Λ^s to (C.2) and take an L^2 inner product with $\Lambda^s \theta$ to obtain

$$\begin{aligned} \frac{d}{dt} \|\Lambda^s \theta\|_{L^2}^2 + \kappa \|\Lambda^{s+1/2} \theta\|_{L^2}^2 &\leq \frac{C}{\kappa} \|\Lambda^{s-1/2} f\|_{L^2}^2 + C \|\Lambda^s \theta\|_{L^4} \|[\Lambda^s, \mathcal{R}^\perp \cdot \nabla] \theta\|_{L^{4/3}} \\ &\leq \frac{C}{\kappa} \|\Lambda^{s-1/2} f\|_{L^2}^2 + C \|\Lambda^{s+1/2} \theta\|_{L^2} \|\Lambda^s \theta\|_{L^2} \|\nabla \theta\|_{L^4} \\ &\leq \frac{C}{\kappa} \|\Lambda^{s-1/2} f\|_{L^2}^2 + \frac{\kappa}{2} \|\Lambda^{s+1/2} \theta\|_{L^2}^2 + \frac{C}{\kappa} \|\Lambda^s \theta\|_{L^2}^2 \|\theta\|_{H^{3/2}}^2. \end{aligned}$$

The proof of (C.3) is completed using Grönwall's inequality, since $s - 1/2 = 1 + \varepsilon$.

In view of (C.3), it is sufficient to show that if $\theta^{(1)}, \theta^{(2)} \in C(0, T; H^{3/2+\varepsilon}(\mathbb{T}^2)) \cap L^2(0, T; H^{2+\varepsilon}(\mathbb{T}^2))$ are two solutions of (C.2) such that $\theta^{(1)}(T) = \theta^{(2)}(T)$, then we must have that $\theta_0^{(1)} = \theta_0^{(2)}$.

We proceed by contradiction. Without loss of generality we assume that $\theta^{(1)} - \theta^{(2)} \neq 0$ on $[0, T)$, and that the solutions agree at time T . Let $\theta = \theta^{(1)} - \theta^{(2)}$ and $2\bar{\theta} = \theta^{(1)} + \theta^{(2)}$. Similar notation for the velocities. The equation for the difference is

$$\partial_t \theta + \kappa \Lambda \theta + \bar{u} \cdot \nabla \theta + u \cdot \nabla \bar{\theta} = 0,$$

where we know that $\bar{\theta} \in L^\infty(0, T; H^{3/2+\varepsilon}) \cap L^2(0, T; H^{2+\varepsilon})$ for some $\varepsilon > 0$.

We introduce the Hamiltonian

$$H(t) = \|\Lambda^{-1/2} \theta(t)\|_{L^2}^2,$$

the energy

$$E(t) = \|\theta(t)\|_{L^2}^2,$$

the energy dissipation

$$D(t) = \|\Lambda^{1/2} \theta(t)\|_{L^2}^2,$$

the maximum of the Hamiltonian

$$M = 3 \max_{t \in [0, T]} H(t),$$

the Dirichlet quotient

$$Q(t) = \frac{E(t)}{H(t)},$$

and

$$W(t) = \log \frac{M}{H(t)}.$$

By assumption we have that $0 < M < \infty$, $W(t) > 0$ on $[0, T]$, and most importantly

$$\lim_{t \rightarrow T^-} W(t) = +\infty$$

which is what we will show is not possible.

As a preliminary, we compute

$$\frac{1}{2} \frac{d}{dt} H = -\kappa \|\theta\|_{L^2}^2 - \int u \cdot \nabla \bar{\theta} \Lambda^{-1} \theta - \int \bar{u} \cdot \nabla \theta \Lambda^{-1} \theta =: -\kappa E - R \quad (\text{C.4})$$

Here we used the two cancellation property

$$\int u \cdot \nabla \bar{\theta} \Lambda^{-1} \theta = - \int u \cdot \nabla \Lambda^{-1} \theta \bar{\theta} = - \int \nabla^\perp \Lambda^{-1} \theta \cdot \nabla \Lambda^{-1} \theta \bar{\theta} = 0.$$

Also, we shall need

$$\frac{1}{2} \frac{d}{dt} E = -\kappa \|\Lambda^{1/2} \theta\|_{L^2}^2 - \int u \cdot \nabla \bar{\theta} \theta =: -\kappa D - S \quad (\text{C.5})$$

where we used the cancellation property

$$\int \bar{u} \cdot \nabla \theta \theta = 0.$$

We now compute the time evolution of W . Using (C.4) we obtain

$$\frac{1}{2} \frac{d}{dt} W = \frac{-\frac{1}{2} \frac{d}{dt} H}{H} = \frac{\kappa \|\theta\|_{L^2}^2 + R}{H}.$$

From the estimate

$$|R| = \left| \int \bar{u} \cdot \nabla \theta \Lambda^{-1} \theta \right| = \left| \int \bar{u} \cdot \nabla \Lambda^{-1} \theta \theta \right| \leq \|\bar{u}\|_{L^\infty} \|\theta\|_{L^2}^2 = \|\bar{u}\|_{L^\infty} E.$$

we obtain

$$\frac{1}{2} \frac{d}{dt} W \leq \left(\kappa + \|\mathcal{R}^\perp \bar{\theta}\|_{L^\infty} \right) Q \leq \left(\kappa + C \|\bar{\theta}\|_{H^{1+\varepsilon}} \right) Q. \quad (\text{C.6})$$

Next we estimate the evolution of Q . This is the main computation. From (C.4) and (C.5) we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} Q &= \frac{\frac{1}{2} \frac{d}{dt} E}{H} - Q \frac{\frac{1}{2} \frac{d}{dt} H}{H} \\ &= \frac{-\kappa D - S}{H} - Q \frac{(-\kappa E - R)}{H} = -\kappa \frac{D - QE}{H} + \frac{QR - S}{H}. \end{aligned} \quad (\text{C.7})$$

The first observation is that

$$\Omega := D - QE = \|\Lambda^{1/2} \theta\|_{L^2}^2 - \frac{\|\theta\|_{L^2}^4}{\|\Lambda^{-1/2} \theta\|_{L^2}^2} = \|\Lambda^{1/2} \theta - Q \Lambda^{-1/2} \theta\|_{L^2}^2 \geq 0.$$

The second observation is that by appealing to the previous cancellation property we have

$$\begin{aligned} S &= \int u \cdot \nabla \bar{\theta} \theta = \int \Lambda^{-1/2} (u \cdot \nabla \bar{\theta}) \Lambda^{1/2} \theta \\ &= \int \Lambda^{-1/2} (u \cdot \nabla \bar{\theta}) \left(\Lambda^{1/2} \theta - Q \Lambda^{-1/2} \theta \right) + Q \int u \cdot \nabla \bar{\theta} \Lambda^{-1} \theta \\ &= \int \Lambda^{-1/2} (u \cdot \nabla \bar{\theta}) \left(\Lambda^{1/2} \theta - Q \Lambda^{-1/2} \theta \right) \end{aligned}$$

and therefore, using Cauchy-Schwartz and the fractional Sobolev embedding, it follows that

$$\begin{aligned} |S| &\leq \Omega^{1/2} \|\Lambda^{-1/2} (u \cdot \nabla \bar{\theta})\|_{L^2} \\ &\leq \frac{\kappa}{2} \Omega + \frac{C}{\kappa} \|u \cdot \nabla \bar{\theta}\|_{L^{4/3}}^2 \leq \frac{\kappa}{2} \Omega + \frac{C}{\kappa} E \|\bar{\theta}\|_{H^{3/2}}^2. \end{aligned} \quad (\text{C.8})$$

Next, we write R as a commutator

$$R = \int \bar{u} \cdot \nabla \theta \Lambda^{-1} \theta = \int \left(\Lambda^{-1/2} (\bar{u} \cdot \nabla \theta) - \bar{u} \cdot \nabla \Lambda^{-1/2} \theta \right) \Lambda^{-1/2} \theta.$$

The claim is that

$$|R| \leq C \|\bar{\theta}\|_{H^{2+\varepsilon}} H \quad (\text{C.9})$$

for any $\varepsilon > 0$, and some $C > 0$. To prove (C.9) we write R using Fourier series as

$$R = C \sum_{j+k+l=0, j, k, l \in \mathbb{Z}_*^2} \frac{|l| - |k|}{|l|^{1/2} |k|^{1/2} (|l|^{1/2} + |k|^{1/2})} \widehat{\theta}_j^{\perp} \frac{j^\perp \cdot k}{|j|} \widehat{\theta}_k |l|^{-1/2} \widehat{\theta}_l.$$

Taking absolute values in the above we obtain

$$\begin{aligned}
|R| &\leq C \sum_{j+k+l=0, j, k, l \in \mathbb{Z}_*^2} \frac{|j||k|^{1/2}}{|l|(|l|^{1/2} + |k|^{1/2})} |\widehat{\theta}_j| |\widehat{\theta}_k| |\widehat{\theta}_l| \\
&\leq C \sum_{j+k+l=0, j, k, l \in \mathbb{Z}_*^2} \left(1 + \frac{|j|^{1/2}}{|l|^{1/2}}\right) (|j| |\widehat{\theta}_j|) (|k|^{-1/2} |\widehat{\theta}_k|) (|l|^{-1/2} |\widehat{\theta}_l|) \\
&\leq C \| |j| |\widehat{\theta}_j| \|_{\ell^1(\mathbb{Z}^2)} \| |k|^{-1/2} |\widehat{\theta}_k| \|_{\ell^2(\mathbb{Z}_*^2)} \| |l|^{-1/2} |\widehat{\theta}_l| \|_{\ell^2(\mathbb{Z}_*^2)} \\
&\quad + C \| |j|^{3/2} |\widehat{\theta}_j| \|_{\ell^{4/(3+3\delta)}(\mathbb{Z}^2)} \| |k|^{-1/2} |\widehat{\theta}_k| \|_{\ell^2(\mathbb{Z}_*^2)} \| |l|^{-1} |\widehat{\theta}_l| \|_{\ell^{4/(3-3\delta)}(\mathbb{Z}_*^2)} \\
&\leq C \|\bar{\theta}\|_{H^{2+\varepsilon}} \|\Lambda^{-1/2} \theta\|_{L^2}^2
\end{aligned} \tag{C.10}$$

as long as we choose $\delta \in (0, 2\varepsilon/3)$. This proves inequality (C.9).

Inserting (C.8) and (C.9) into (C.7) yields

$$\frac{d}{dt} Q + \kappa \frac{\Omega}{H} \leq \left(C \|\bar{\theta}\|_{H^{2+\varepsilon}} + \frac{C}{\kappa} \|\bar{\theta}\|_{H^{3/2}}^2 \right) Q$$

and by the Grönwall inequality

$$Q(t) \leq Q(0) \exp \left(CT^{1/2} \|\bar{\theta}\|_{L^2(0, T; H^{2+\varepsilon})} + \frac{C}{\kappa} \|\bar{\theta}\|_{L^2(0, T; H^{3/2})}^2 \right) \tag{C.11}$$

for any $t \in [0, T)$.

Estimate (C.11) shows that $Q \in L^\infty(0, T)$, which combined with (C.6) implies that $W \in L^\infty(0, T)$. This contradicts the fact that $H(T) = 0 \Leftrightarrow W(T) = +\infty$, thereby concluding the proof. \square

We now give the proof of the continuity property of the solution map with respect to time and with respect to perturbations in the initial data, in the H^1 topology.

Proof of Proposition 5.5. The continuity in time for fixed initial data was already given by Proposition 4.2, so it remains to check continuity with respect to the initial data which originates from \mathcal{B}_0 .

Due to Theorem 5.2, we know there exists an absorbing ball $\mathcal{B} \subset H^{3/2}$ for the dynamics induced by $S(t)$ on H^1 . In particular, by Remark 5.4 there exist a time $t_{H^{3/2}}(\mathcal{B}_0)$ such that $S(t)\mathcal{B}_0 \subset \mathcal{B}$ for all $t \geq t_{H^{3/2}}(\mathcal{B}_0)$. Since the sequence t_n in the statement of the proposition diverges as $n \rightarrow \infty$, we may assume without loss of generality that $\tilde{\theta}_{0,n} = S(t_n)\theta_{0,n} \in \mathcal{B}$ for all $n \geq 1$, and even that $S(t)\tilde{\theta}_{0,n} \in \mathcal{B}$ for all $t \geq 0, n \geq 1$.

Let $\theta_0 \in H^1$ be arbitrary. Fix some $\tilde{\theta}_0 \in \mathcal{B}$ such that $\tilde{\theta}(t) = S(t)\tilde{\theta}_0 \in \mathcal{B}$ for all $t \geq 0$, and such that

$$\|\theta_0 - \tilde{\theta}_0\|_{H^1} \leq \varepsilon \kappa$$

for $\varepsilon > 0$ to be determined later. Denote $\bar{\theta}(t) = \tilde{\theta}(t) - \theta(t) = S(t)\tilde{\theta}_0 - S(t)\theta_0$, for all $t \geq 0$, and let \bar{u} be the corresponding velocity difference. The equation obeyed by $\bar{\theta}$ is

$$\partial_t \bar{\theta} + \kappa \Lambda \bar{\theta} - \bar{u} \cdot \nabla \bar{\theta} + \tilde{u} \cdot \nabla \bar{\theta} + \bar{u} \cdot \nabla \tilde{\theta} = 0.$$

Multiplying the above by $-\Delta \bar{\theta}$, integrating over \mathbb{T}^2 , and integrating by parts, yields

$$\begin{aligned}
\frac{1}{2} \|\bar{\theta}\|_{H^1}^2 + \kappa \|\bar{\theta}\|_{H^{3/2}}^2 &\leq \left| \int (\nabla \bar{u} \cdot \nabla \bar{\theta}) \cdot \nabla \bar{\theta} \right| + \left| \int (\nabla \tilde{u} \cdot \nabla \bar{\theta}) \cdot \nabla \bar{\theta} \right| + \left| \int \Lambda^{1/2} (\bar{u} \cdot \nabla \tilde{\theta}) \Lambda^{3/2} \bar{\theta} \right| \\
&\leq C (\|\nabla \bar{u}\|_{L^4} + \|\nabla \tilde{u}\|_{L^4}) \|\bar{\theta}\|_{H^1} \|\nabla \bar{\theta}\|_{L^4} + C \|\Lambda^{1/2} (\bar{u} \cdot \nabla \tilde{\theta})\|_{L^2} \|\bar{\theta}\|_{H^{3/2}} \\
&\leq C (\|\tilde{\theta}\|_{H^{3/2}} + \|\bar{\theta}\|_{H^{3/2}}) \|\bar{\theta}\|_{H^1} \|\bar{\theta}\|_{H^{3/2}} + C \|\bar{u}\|_{L^\infty} \|\tilde{\theta}\|_{H^{3/2}} \|\bar{\theta}\|_{H^{3/2}}
\end{aligned} \tag{C.12}$$

Here we have also appealed to the fractional Sobolev embedding $H^{1/2} \subset L^4$, and the product estimate from Lemma A.1. We may appeal to Brezis–Gallouët the inequality

$$\|\bar{u}\|_{L^\infty} = \|\mathcal{R}^\perp \bar{\theta}\|_{L^\infty} \leq C \|\bar{\theta}\|_{H^1} \left(1 + \log \frac{\|\bar{\theta}\|_{H^{3/2}}^2}{\|\bar{\theta}\|_{H^1}^2} \right)^{1/2} \quad (\text{C.13})$$

which combined with the inequality

$$a\mu \left(1 + \log \frac{\mu^2}{b^2} \right)^{1/2} \leq \varepsilon \mu^2 + \frac{a^2}{\varepsilon} \log \frac{2a}{\varepsilon b} \quad (\text{C.14})$$

which holds for any $a, \varepsilon > 0$ and $\mu \geq b$ (see [FMT88, Kuk96]), and the estimate (C.12), yields

$$\frac{d}{dt} \|\bar{\theta}\|_{H^1}^2 + \frac{\kappa}{2} \|\bar{\theta}\|_{H^{3/2}}^2 \leq C_0 \|\bar{\theta}\|_{H^1} \|\bar{\theta}\|_{H^{3/2}}^2 + \frac{C_0}{\kappa} \|\bar{\theta}\|_{H^1}^2 \|\tilde{\theta}\|_{H^{3/2}}^2 \left(1 + \log \frac{C_0 \|\tilde{\theta}\|_{H^{3/2}}}{\kappa} \right) \quad (\text{C.15})$$

for some universal constant $C_0 > 0$. Note that the initial data $\bar{\theta}_0$ obeys

$$\|\bar{\theta}_0\|_{H^1} \leq \varepsilon \kappa$$

where ε is chosen so that $4\varepsilon C_0 \leq 1$. Due to continuity in time, we therefore conclude that there exists $T > 0$ such that on $[0, T]$ we have $\|\bar{\theta}(t)\|_{H^1} \leq 2\varepsilon \kappa$, and on this time interval from (C.15) we conclude that

$$\frac{d}{dt} \|\bar{\theta}\|_{H^1}^2 \leq \|\bar{\theta}\|_{H^1}^2 \frac{C_0 M^2}{\kappa} \left(1 + \log \frac{C_0 M}{\kappa} \right). \quad (\text{C.16})$$

Here we used the assumption that $S(t)\tilde{\theta}_0 \in \mathcal{B}$ for all $t \geq 0$ and denoted by M the radius of the ball \mathcal{B} . Thus, a posteriori we conclude that we could have chosen

$$T = \frac{\kappa \log 2}{C_0 M^2 \left(1 + \log \frac{C_0 M}{\kappa} \right)} = T(\kappa, \mathcal{B})$$

so that

$$\|\bar{\theta}(t)\|_{H^1} \leq 2\varepsilon \kappa$$

for $t \in [0, T]$. It is important that this T is independent of ε .

The proof of the proposition may now be concluded. The above estimates shows that as $\|\tilde{\theta}_{0,n} - \theta_0\|_{H^1} \rightarrow 0$, we have

$$\|S(t)\tilde{\theta}_{0,n} - S(t)\theta_0\|_{H^1}^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for all $t \in [0, T(\kappa, \mathcal{B})]$. We then re-iterate this argument for $t \in [iT(\kappa, \mathcal{B}), (i+1)T(\kappa, \mathcal{B})]$ for all $i \geq 1$, which proves the Proposition. \square

We conclude the appendix by giving the proof of continuous differentiability of the solution map around trajectories on the global attractor.

Proof of Proposition 6.2. For $\theta_0, \varphi_0 \in \mathcal{A}$, denote $\xi_0 = \varphi_0 - \theta_0$ and define $\xi(t) = S'(t, \theta_0)[\xi_0]$ via (6.5). We let

$$\eta(t) = \varphi(t) - \theta(t) - \xi(t) = S(t)\varphi_0 - S(t)\theta_0 - S'(t, \theta_0)[\xi_0]$$

and observe that η obeys the equation

$$\partial_t \eta + \kappa \Lambda \eta + \mathcal{R}^\perp \eta \cdot \nabla \theta + \mathcal{R}^\perp \theta \cdot \nabla \eta = -\mathcal{R}^\perp w \cdot \nabla w, \quad \eta(0) = 0. \quad (\text{C.17})$$

where

$$w(t) = \varphi(t) - \theta(t) = S(t)\varphi_0 - S(t)\theta_0 = S(t)\xi_0. \quad (\text{C.18})$$

In order to estimate $\|\eta(t)\|_{H^1}$, take an L^2 inner product of (C.17) with $-\Delta\eta$ and use Lemma A.1 to obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\eta\|_{H^1}^2 + \kappa \|\eta\|_{H^{3/2}}^2 \\
&= \int_{\mathbb{T}^2} \mathcal{R}^\perp \eta \cdot \nabla \theta \Delta \eta dx - \int_{\mathbb{T}^2} \partial_k \mathcal{R}^\perp \theta \cdot \nabla \eta \partial_k \eta dx + \int_{\mathbb{T}^2} \mathcal{R}^\perp w \cdot \nabla w \Delta \eta dx \\
&\leq \|\Lambda^{3/2} \eta\|_{L^2} \|\Lambda^{1/2} (\mathcal{R}^\perp \eta \cdot \nabla \theta)\|_{L^2} + \|\nabla \mathcal{R}^\perp \theta\|_{L^4} \|\nabla \eta\|_{L^4} \|\nabla \eta\|_{L^2} + \|\Lambda^{3/2} \eta\|_{L^2} \|\Lambda^{1/2} (\mathcal{R}^\perp w \cdot \nabla w)\|_{L^2} \\
&\leq C \|\eta\|_{H^{3/2}} (\|\eta\|_{H^1} \|\theta\|_{H^{3/2}} + \|\eta\|_{H^{1/2}} \|\theta\|_{H^2}) + C \|\eta\|_{H^{3/2}} \left(\|w\|_{H^1} \|w\|_{H^{3/2}} + \|\mathcal{R}^\perp w\|_{L^\infty} \|w\|_{H^{3/2}} \right) \\
&\leq \frac{\kappa}{2} \|\eta\|_{H^{3/2}}^2 + \frac{C}{\kappa} \|\eta\|_{H^1}^2 \|\theta\|_{H^2}^2 + \frac{C}{\kappa} \|w\|_{H^1}^2 \|w\|_{H^{3/2}}^2 \left(1 + \log \frac{\|w\|_{H^{3/2}}^2}{\|w\|_{H^1}^2} \right) \tag{C.19}
\end{aligned}$$

for some universal constant $C > 0$. In the last inequality we have also appealed to (C.13).

Next we estimate w , as defined in (C.18). It obeys the equation

$$\partial_t w + \kappa \Lambda w + \mathcal{R}^\perp \varphi \cdot \nabla w + \mathcal{R}^\perp w \cdot \nabla \theta = 0, \quad w(0) = \xi_0. \tag{C.20}$$

We note that in view of Theorem 5.1, we a priori have estimates on the $H^{3/2}$ and even H^2 norms of θ and φ , since they are elements of \mathcal{A} . Multiplying (C.20) with $-\Delta w$ and integrating, similarly to (C.19) we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|w\|_{H^1}^2 + \kappa \|w\|_{H^{3/2}}^2 \\
&\leq \|w\|_{H^{3/2}} \left(\|\nabla \mathcal{R}^\perp \varphi\|_{L^4} \|\nabla w\|_{L^2} + C \|\Lambda^{1/2} \mathcal{R}^\perp w\|_{L^4} \|\nabla \theta\|_{L^4} + C \|\mathcal{R}^\perp w\|_{L^4} \|\Lambda^{1/2} \nabla \theta\|_{L^4} \right) \\
&\leq \frac{\kappa}{2} \|w\|_{H^{3/2}}^2 + \frac{C}{\kappa} \|w\|_{H^1}^2 (\|\varphi\|_{H^{3/2}}^2 + \|\theta\|_{H^2}^2). \tag{C.21}
\end{aligned}$$

The Grönwall inequality and the bounds (5.2)–(5.3) for $\theta, \varphi \in \mathcal{A}$ then yield

$$\|w(t)\|_{H^1}^2 \leq \|\xi_0\|_{H^1}^2 \exp(C\kappa^{-1}tM_{\mathcal{A}}^2) \leq \|\xi_0\|_{H^1}^2 K(t, M_{\mathcal{A}}) \tag{C.22}$$

for all $t \geq 0$. Here and throughout the proof $K(\cdot, \cdot)$ is an increasing continuous function in each variable. This function *may change from line to line*.

Inserting the estimate (C.22) back into (C.21) gives

$$\int_0^t \|w(s)\|_{H^{3/2}}^2 ds \leq \|\xi_0\|_{H^1}^2 C \kappa^{-1} (1 + \kappa^{-1}tM_{\mathcal{A}}^2 \exp(C\kappa^{-1}tM_{\mathcal{A}}^2)) \leq \|\xi_0\|_{H^1}^2 K(t, M_{\mathcal{A}}) \tag{C.23}$$

for $t \geq 0$. Estimate (C.23) can be upgraded to a pointwise in time bound, at the cost of losing the dependence of ξ_0 in H^1 . Since

$$\|w(0)\|_{H^{3/2}} = \|\xi_0\|_{H^{3/2}} = \|\varphi_0 - \theta_0\|_{H^{3/2}} \leq 2M_{\mathcal{A}} \tag{C.24}$$

we are justified to study the time evolution of $\|w(t)\|_{H^{3/2}}$. Taking an L^2 inner product of (C.20) with $\Lambda^3 w$, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|w\|_{H^{3/2}}^2 + \kappa \|w\|_{H^2}^2 \\
&\leq \|\Lambda^{3/2} w\|_{L^2} \|\Lambda^{3/2}, \mathcal{R}^\perp \varphi \cdot \nabla\| w\|_{L^2} + \|\Lambda^2 w\|_{L^2} \|\Lambda (\mathcal{R}^\perp w \cdot \nabla \theta)\|_{L^2} \\
&\leq \|w\|_{H^{3/2}} \left(\|\Lambda^{3/2} \mathcal{R}^\perp \varphi\|_{L^4} \|\nabla w\|_{L^4} + \|\nabla \mathcal{R}^\perp \varphi\|_{L^4} \|\Lambda^{1/2} \nabla w\|_{L^4} \right) \\
&\quad + \|w\|_{H^2} \left(\|\nabla^\perp w\|_{L^4} \|\nabla \theta\|_{L^4} + \|\mathcal{R}^\perp w\|_{L^\infty} \|\Lambda \nabla \theta\|_{L^2} \right) \\
&\leq \frac{\kappa}{2} \|w\|_{H^2}^2 + \|w\|_{H^{3/2}}^2 \left(\|\varphi\|_{H^2} + \frac{C}{\kappa} \|\theta\|_{H^2}^2 \right)
\end{aligned}$$

by using Lemma A.1 and the embedding $H^{1/2} \subset L^4$. It thus follows from the above estimate, the Grönwall inequality, (5.3), and (C.24), that

$$\|w(t)\|_{H^{3/2}}^2 \leq 4M_{\mathcal{A}}^2 \exp(C\kappa^{-1}t(1 + M_{\mathcal{A}}^2)) \leq K(t, M_{\mathcal{A}}) \quad (\text{C.25})$$

for some universal $C > 0$.

We now combine (C.19) with (C.22) and (C.25) to obtain

$$\begin{aligned} \frac{d}{dt} \|\eta\|_{H^1}^2 + \kappa \|\eta\|_{H^{3/2}}^2 &\leq \frac{C}{\kappa} \|\eta\|_{H^1}^2 \|\theta\|_{H^2}^2 + \frac{C}{\kappa} \|w\|_{H^1}^2 \|w\|_{H^{3/2}}^2 \left(1 + \log \frac{\|w\|_{H^{3/2}}^2}{\|w\|_{H^1}^2}\right) \\ &\leq \frac{C}{\kappa} \|\eta\|_{H^1}^2 \|\theta\|_{H^2}^2 + \frac{C}{\kappa} \|w\|_{H^1}^{2-a} \|w\|_{H^{3/2}}^{2+a} \\ &\leq \frac{C}{\kappa} \|\eta\|_{H^1}^2 \|\theta\|_{H^2}^2 + \|\xi_0\|_{H^1}^{2-a} K(t, M_{\mathcal{A}}) \|w\|_{H^{3/2}}^2 \end{aligned} \quad (\text{C.26})$$

where $a \in (0, 1)$ is arbitrary. Using the Grönwall inequality combined with (5.3), (C.23), and the fact that $\eta(0) = 0$ we conclude from (C.26) that

$$\begin{aligned} \|\eta(t)\|_{H^1}^2 &\leq \exp\left(\frac{C}{\kappa} \int_0^t \|\theta(s)\|_{H^2}^2 ds\right) \|\xi_0\|_{H^1}^{2-a} K(t, M_{\mathcal{A}}) \int_0^t \|w(s)\|_{H^{3/2}}^2 ds \\ &\leq \|\xi_0\|_{H^1}^{4-a} K(t, M_{\mathcal{A}}) \end{aligned} \quad (\text{C.27})$$

for a suitable function K as described above, and $a \in (0, 1)$. This proves that

$$\lim_{r \rightarrow 0^+} \left(\sup_{\theta_0, \varphi_0 \in \mathcal{A}, 0 < \|\xi_0\|_{H^1} \leq r} \frac{\|\eta(t)\|_{H^1}^2}{\|\xi_0\|_{H^1}^2} \right) \leq \lim_{r \rightarrow 0^+} r^{2-a} K(t, M_{\mathcal{A}}) = \lim_{r \rightarrow 0^+} e(r, t) = 0$$

with $e(r, t) = r^{2-a} K(t, M_{\mathcal{A}})$, and thus (6.1) holds.

In order to prove (6.3), consider ξ_0 normalized so that $\|\xi_0\|_{H^1} = 1$, and let $\theta_0 \in \mathcal{A}$ be arbitrary. Then, using similar estimates as above we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\xi\|_{H^1}^2 + \kappa \|\xi\|_{H^{3/2}}^2 &\leq C \|\xi\|_{H^{3/2}} \|\xi\|_{H^1} \|\theta\|_{H^{3/2}} + \|\mathcal{R}^\perp \xi\|_{L^\infty} \|\xi\|_{H^1} \|\theta\|_{H^2} \\ &\leq \frac{\kappa}{2} \|\xi\|_{H^{3/2}}^2 + \frac{C}{\kappa} \|\xi\|_{H^1}^2 \|\theta\|_{H^2}^2 \end{aligned} \quad (\text{C.28})$$

which combined with (5.3) yields

$$\|\xi(t)\|_{H^1}^2 \leq \exp(C\kappa^{-1}tM_{\mathcal{A}}^2) \quad (\text{C.29})$$

which indeed proves (6.3).

It remains to prove that for any $t > 0$ and $\theta_0 \in \mathcal{A}$, the operator $S'(t, \theta_0)$ is compact. Without loss of generality we may look at the image under $S'(t, \theta_0)$ of the unit ball in H^1 , and show it is precompact. More precisely, we show that the image of this ball is included in ball in $H^{3/2}$. Combine (C.28) and (C.29) to obtain that

$$\int_0^t \|\xi(s)\|_{H^{3/2}}^2 ds \leq K(t, M_{\mathcal{A}})$$

for any $t > 0$. By the mean value theorem, there exists $\tau \in (0, t/2)$ such that

$$\|\xi(\tau)\|_{H^{3/2}}^2 \leq \frac{1}{t} K(t, M_{\mathcal{A}}). \quad (\text{C.30})$$

Taking the inner product of (6.5) with $\Lambda^3 \xi$, using the usual commutator, Sobolev, and Poincaré inequalities, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\xi\|_{H^{3/2}}^2 + \kappa \|\xi\|_{H^2}^2 &\leq \|[\Lambda^{3/2}, \mathcal{R}^\perp \theta \cdot \nabla] \xi\|_{L^2} \|\Lambda^{3/2} \xi\|_{L^2} + \|\Lambda(\mathcal{R}^\perp \xi \cdot \nabla \theta)\|_{L^2} \|\Lambda^{3/2} \xi\|_{L^2} \\ &\leq C \|\theta\|_{H^2} \|\xi\|_{H^{3/2}}^2 + C \|\theta\|_{H^{3/2}} \|\xi\|_{H^2} \|\xi\|_{H^{3/2}} + C \|\mathcal{R}^\perp \xi\|_{L^\infty} \|\theta\|_{H^2} \|\xi\|_{H^2} \\ &\leq \frac{\kappa}{2} \|\xi\|_{H^2}^2 + \frac{C}{\kappa} \|\xi\|_{H^{3/2}}^2 \|\theta\|_{H^2}^2 \end{aligned} \quad (\text{C.31})$$

at times larger than the τ in (C.30). Integrating (C.31) between τ and t , and using the bound (5.3) and (C.30), we thus obtain

$$\|\xi(t)\|_{H^{3/2}}^2 \leq \|\xi(\tau)\|_{H^{3/2}}^2 \exp\left(\frac{C}{\kappa} \int_\tau^t \|\theta(s)\|_{H^2}^2 ds\right) \leq \frac{1}{t} K(t, M_A)$$

for a suitable function K which is continuous and increasing in all its parameters. This concludes the proof of the Proposition. \square

Acknowledgements.

The authors would like to thank Alexey Cheskidov and Hongjie Dong for very helpful remarks about the first version of the paper. The work of PC was supported in part by NSF grants DMS-1209394, DMS-1265132, and DMS-1240743. The work of AT was supported in part by the NSF GRFP grant. The work of VV was supported in part by the NSF grant DMS-1211828.

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