Farfield perturbations of vortex patches

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ABSTRACT. We investigate the dynamics of vortex patches in the Yudovitch phase space. We derive an approximation for the evolution of the vorticity in the case of nested vortex patches with distant boundaries, and study its long time behavior.
AMS Subject Classification Number: 35Q31, 76D03
Keywords: Euler equations, vortex patches, long time behavior, instability.

1. Introduction

The long time behavior of solutions of two dimensional incompressible Euler equations is an interesting and highly nontrivial subject. It is well-known that smooth and localized initial data lead to a global in time well-posed evolution in spaces of smooth functions. Beyond this, little is known about the long time dynamics. In this paper we consider the evolution of non-smooth solutions, in a well-known phase space of functions with a limited degree of non-smoothness. The equations of ideal incompressible fluids in two dimensions can be described in terms of a single scalar field, the vorticity ω, which is a function of space and time, $\omega = \omega(x,t)$, with $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$. The vorticity is transported by a flow it creates: it is an active scalar. The transport

$$\partial_t \omega + u \cdot \nabla \omega = 0 \tag{1}$$

is done by an incompressible velocity field $u(x,t)$ whose curl is the vorticity, $\omega = \partial_1 u_2 - \partial_2 u_1$. This linear relation can be inverted by writing $u = \nabla^{\perp}\psi$ and seeking ψ whose gradient decays at infinity and solves $\Delta\psi = \omega$. The global existence and uniqueness of solutions of (1) for vorticity in the class $Y = L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is a classical result of Yudovitch ([**8**]). The evolution (1) results in a rearrangement of the vorticity distribution by a volume-preserving transformation with quasi-Lipschitz classical trajectories. If the initial datum $\omega(x,0)$ is a step function, then it remains a step function, with only the plane domains of constant value evolving in time. An equation of evolution for the boundary of such a domain, termed "contour dynamics" was derived and studied numerically by Zabusky and co-authors ([**9**]). If the initial vorticity equals a constant Ω in a simply connected bounded domain with smooth boundary (a vortex patch), then evolution of vorticity is reduced to a nonlocal evolution equation for a complex valued function $z(\alpha,t)$ representing the boundary of the vortex patch at time t, parameterized by a parameter $\alpha \in [0, 2\pi]$,

$$\partial_t(z(\alpha,t)) = \frac{\Omega}{2\pi}\int_0^{2\pi} \log|z(\alpha,t) - z(\beta,t)|\, \frac{\partial z(\beta,t)}{\partial\beta} d\beta. \tag{2}$$

The derivative $z'(\alpha,t) = \frac{\partial z(\alpha,t)}{\partial\alpha}$ obeys

$$\partial_t z'(\alpha,t) = \frac{\Omega}{2\pi}\int_0^{2\pi} Re\left(\frac{z'(\alpha,t)}{z(\alpha,t) - z(\beta,t)}\right) z'(\beta,t) d\beta. \tag{3}$$

This equation resembles very much the simple equation $\partial_t\omega = \omega H\omega$ ([**3**]) where H is the Hilbert transform, an equation that served as a one-dimensional scalar model for the three dimensional vectorial vortex stretching equation. The simple equation blows up in finite time. Motivated by this, it was conjectured ([**6**]) that the vortex patch equation develops singularities in z'. It turned out ([**2**]), ([**1**]) that the boundaries of vortex patches remain smooth, if they were initially so. If the initial patch is an ellipse, then it remains an ellipse for all time, and the evolution consists of a rigid rotation with constant angular velocity, around a fixed center, the symmetry center of vorticity. The stability of these Kirkhhoff ellipses under strain or local perturbations was investigated ([**7**], [**5**]). In this paper we study the effect of far-field perturbations. We derive equations for a couple of contours which approximate the Eulerian evolution when one contour is far from the other. The system becomes almost uncoupled: the outer curve has a self-determined evolution influenced by the inner curve only via a constant coefficient computed from the area of the region surrounded by the inner curve. That area is conserved under the evolution. The effect that the evolving inner curve has on the outer curve is one of pure rotation around the conserved vorticity field center. The rotation however is not rigid: its angular velocity depends on radius, is constant at fixed radius, but decreases with increased radius. The evolution of the inner curve is influenced by the outer curve via a time dependent complex coefficient $\zeta(t)$. Remarkably, if the inner curve is an ellipse, it stays an ellipse. The nonlinear stability of this ellipse is determined by the long time correlation of $\zeta(t)$ with a geometric quantity representing the inner ellipse. If the outer curve is initially an ellipse, it does not stay one, except in the case it was a circle. If the outer curve is initially an ellipse of small eccentricity, then its evolution can be approximated for long time by that of an ellipse, and in that case ζ can be computed explicitly. The resulting system can be investigated in detail and instability can be proved. The instability is strong, in the sense that the perturbed ellipse's aspect ratio degenerates, while keeping constant area. The proof of this instability is done by studying the dynamics of a complex quantity that represents the aspect ratio of the inner ellipse and the angle it makes with a coordinate system. Degenerate ellipses are represented by the boundary of the unit disk, and the dynamics is such that there can be a stable fixed point on the boundary of the unit disk which attracts trajectories from inside the circle. This means that nondegenerate ellipses degenerate in infinite time.

2. Vortex Patches

We consider the evolution of a two dimensional incompressible inviscid fluid. We describe first the vorticity distribution. We take N smooth, disjoint, oriented, closed plane curves, $\Gamma_j \subset \mathbb{R}^2$, $j = 1, \dots, N$. The complement of their union is an open set $D = \mathbb{R}^2 \backslash \cup_{j=1}^N \Gamma_j$. We denote by D_j the connected components of D, $D = \cup_{j=1}^{N+1} D_j$. We denote by D_{N+1} the unbounded connected component. Each curve Γ_j divides $\mathbb{R}^2$ into two connected open sets. We denote the bounded one U_j. We orient Γ_j such that the vectors (n_j, τ_j) where n_j is the outer normal to U_j and τ_j is tangent to Γ_j define the same orientation in $\mathbb{R}^2$ as the standard basis (e_1, e_2). This is the same as saying that an observer traveling on Γ_j in the sense of the parameterization has U_j on his left side, or that $e^{i\frac{\pi}{2}} n_j = \tau_j$. (We identify $\mathbb{R}^2$ with $\mathbb{C}$.) We consider the vorticity

$$\omega = \sum_{k=1}^{N+1} \Omega_k \chi_{D_k} \tag{4}$$

with $\Omega_k \in \mathbb{R}$, $k = 1, \dots, N$, $\Omega_{N+1} = 0$ and χ_{D_k} the characteristic (indicator) function of D_k. As $\omega \in Y = L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, it is well-known ([**8**]) that the incompressible Euler equations with initial data like in (4) possess global unique weak solutions in Y. Moreover, the solution is given implicitly by

$$\omega(\cdot, t) = \sum_{k=1}^{N+1} \Omega_k \chi_{D_k(t)} \tag{5}$$

with $D_k(t)$ obtained from $D_k(0)$ by the Lagrangian transformation

$$D_k(t) = \Phi(D_k(0), t) \tag{6}$$

where

$$\frac{d}{dt}\Phi(\alpha, t) = u(\Phi(\alpha, t), t), \quad \Phi(\alpha, 0) = \alpha \tag{7}$$

and u us the velocity vector $u = \nabla^{\perp}\psi$, with $\nabla^{\perp} = e^{i\frac{\pi}{2}}\nabla$,

$$\nabla = \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix}$$

i.e.,

$$u = \nabla^{\perp}\psi = \begin{cases} -\partial_2\psi, \\ \quad \partial_1\psi \end{cases} \tag{8}$$

with

$$\Delta\psi = \omega(\cdot, t), \tag{9}$$

and boundary condition $\nabla\psi \to 0$, as $|x| \to \infty$, that is,

$$\psi = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\omega(y,t)dy + C. \tag{10}$$

As $e^{i\frac{\pi}{2}} n_k = \tau_k$, we obtain by the divergence theorem

$$u(x,t) = \frac{1}{2\pi}\sum_{k=1}^{N} \omega_k \oint_{\Gamma_k} \log|x-\zeta|\tau_k(\zeta)ds \tag{11}$$

where ω_k are the numbers

$$\omega_k = \lim_{\epsilon\downarrow 0}\left[\omega(x-\epsilon n_k, t) - \omega(x+\epsilon n_k, t)\right], \tag{12}$$

i.e., ω_k is the jump in $\omega(\cdot, t)$ as we cross from U_k to $\mathbb{R}^2 \setminus U_k$. Because each Γ_k intersects exactly two sets $\overline{D_j}$, there is no ambiguity in the definition. If Γ_k is parameterized by $z_k(s)$, with $s \in [0, 2\pi]$ and $z_k(0) = z_k(2\pi)$, $z_k \in \mathbb{C}$, then the integrals in (11) are

$$\frac{1}{2\pi}\int_0^{2\pi} \log|x - z_k(s)|\frac{dz_k(s)}{ds}ds = \frac{1}{2\pi}\oint_{\Gamma_k} \log|x-z|dz$$

The velocity field defined by (11) is Hölder continuous, and in particular (11) is well defined for $x \in \Gamma_j$. The vortex patch equations are the equations of evolution of the curves Γ_j. If

$$V(z, \Gamma_k) = \frac{1}{2\pi}\oint_{\Gamma_k} \log|z-\zeta|d\zeta \tag{13}$$

and

$$U(z) = U(z, \sum \omega_k\Gamma_k) = \sum_{k=1}^{N} \omega_k V(z, \Gamma_k) \tag{14}$$

then the vortex patch equations are

$$\frac{\partial z_j(\alpha,t)}{\partial t} = U(z_j, \sum \omega_k\Gamma_k) = \sum_{k=1}^{N} \omega_k V(z_j(\alpha,t), \Gamma_k) \tag{15}$$

i.e.

$$\frac{\partial z_j}{\partial t}(\alpha,t) = \sum_{k=1}^{N} \frac{\omega_k}{2\pi}\int_0^{2\pi} \log|z_j(\alpha,t) - z_k(\beta,t)|\frac{\partial z_k}{\partial\beta}(\beta,t)d\beta. \tag{16}$$

The center of the vorticity field is defined by

$$x(t) = \int_{\mathbb{R}^2} y\omega(y,t)dy.$$

We check that x is conserved during the motion:

$$x(t) = \int_{\mathbb{R}^2} y\omega(y,t)dy = \int_{\mathbb{R}^2} \Phi(\alpha,t)\omega_0(\alpha)d\alpha$$

by incompressibility ($\det \nabla_\alpha \Phi = 1$). Then

$$\begin{array}{l} \frac{dx}{dt} = \int_{\mathbb{R}^2} \partial_t \Phi(\alpha,t)\omega_0(\alpha)d\alpha = \int_{\mathbb{R}^2} u(\Phi(\alpha,t),t)\omega_0(\alpha)d\alpha \\ = \int_{\mathbb{R}^2} u(y,t)\omega(y,t)dy = \sum_{k=1}^N \Omega_k \int_{D_k} u(y,t)dy. \end{array}$$

Now

$$u(y,t) = \sum_{l=1}^N \frac{\Omega_l}{2\pi}\int_{D_l} \nabla^\perp \log|y-z|dz$$

and thus

$$\frac{dx}{dt} = \frac{1}{2\pi}\sum_{k=1,l=1}^N \Omega_k\Omega_l \int_{D_k\times D_l} \nabla_y^\perp \log|y-z|dzdy.$$

The expression

$$\int_{D_k\times D_l} \nabla_y^\perp \log|y-z|dzdy$$

is antisymmetric in k, l, and thus $\frac{dx}{dt} = 0$. We note that, if $\psi \in C^2$ then

$$u\omega = \left(\begin{array}{c} -\frac{1}{2}\partial_2(u_1^2 - u_2^2) + \partial_1(u_1u_2) \\ -\frac{1}{2}\partial_1(u_1^2-u_2^2) - \partial_2(u_1u_2) \end{array}\right) \tag{17}$$

and the integral $\int_{\mathbb{R}^2} u\omega dy = 0$ because u decays like $|y|^{-1}$. This argument requires though the compactly supported ω to be smoother than a vortex patch (C^α suffices). If the configuration of the Γ_k is a collection of concentric ellipses (in the geometric sense) then $(0,0)$, the center of the vorticity field coincides with the geometric center.

Let us observe that, for any vorticity in the Yudovitch class $Y = L^\infty(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$, we have that

$$\|u\|_{L^\infty} \le \sqrt{\frac{2}{\pi}\|\omega\|_{L^\infty(\mathbb{R}^2)}\|\omega\|_{L^1(\mathbb{R}^2)}}. \tag{18}$$

Indeed, this is easily verified by writing first

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)dy,$$

then splitting the integral in two pieces, one for $|x-y|\le R$ and one for $|x-y|\ge R$, and then optimizing in R. Note that if ω solves the Euler equations, then the right hand side of (18) is time-independent. On the other hand, it is easy to see that a velocity given by (13) is bounded by

$$|V(z,\Gamma)| \le \frac{1}{2\pi}|\Gamma| \tag{19}$$

where $|\Gamma|$ is the length of the curve Γ. Indeed, parameterizing

$$\zeta(\alpha) = z + r(s)e^{i\theta(s)}$$

with $s\in[0,|\Gamma|]$ and $r(0) = r(|\Gamma|)$, $\theta(0) = \theta(|\Gamma|)$ we have, denoting $\frac{d}{ds}$ by $'$ and integrating by parts twice

$$\begin{array}{l} V(z,\Gamma) = \frac{1}{2\pi}\int_0^{|\Gamma|} e^{i\theta}\left[r'\log r + i\theta' r\log r\right]ds \\ = \frac{1}{2\pi}\int_0^{|\Gamma|} e^{i\theta}\left[(r\log r - r)' + i\theta' r\log r\right]ds = \frac{1}{2\pi}\int_0^{|\Gamma|} e^{i\theta} i\theta' r ds \\ = -\frac{1}{2\pi}\int_0^{|\Gamma|} e^{i\theta} r' ds. \end{array}$$

The inequality (19) follows because $|r'|\le|\zeta'|$.

3. Elliptical vortex patches

An ellipse centered at the origin of cartesian coordinates in the plane can be represented as

$$z(\alpha) = z_1 e^{i\alpha} + z_2 e^{-i\alpha} \tag{20}$$

with $z_1, z_2 \in \mathbb{C}$, $\alpha \in [0, 2\pi]$. If we write $z_j = r_j e^{i\theta_j}$, then the ellipse is

$$z(\alpha) = e^{i\frac{\theta_1+\theta_2}{2}} \left(a\cos(\alpha+\phi) + ib\sin(\alpha+\phi)\right) \tag{21}$$

with

$$\begin{array}{l} a = r_1 + r_2 = |z_1| + |z_2|, \\ b = r_1 - r_2 = |z_1| - |z_2|, \\ \phi = \frac{\theta_1-\theta_2}{2}. \end{array} \tag{22}$$

Thus, $\frac{\theta_1-\theta_2}{2}$ is a phase shift, which of course is a redundant parameter, $\frac{\theta_1+\theta_2}{2}$ represents the angle the ellipse makes with the coordinate system, and a and b are major and minor semiaxes. The convention $|z_1| \geq |z_2|$ corresponds to a choice of positive trigonometric orientation (counter-clockwise). A Kirkhhoff ellipse is a solution of the 2D incompressible Euler equations whose vorticity is a nonzero constant Ω in a region bounded by an ellipse, and zero outside that region. The parametric representation of a Kirchhoff ellipse is ([**4**])

$$z(\alpha, t) = z_1(t)e^{i\alpha} + z_2(0)e^{-i\alpha} \tag{23}$$

with

$$z_1(t) = z_1(0)e^{-\frac{i}{2\pi}\frac{t\Omega A}{|z_1(0)|^2}} \tag{24}$$

where A is the area of the ellipse

$$A = \pi ab = \pi\left[|z_1|^2 - |z_2|^2\right]. \tag{25}$$

The Kirkhhoff ellipse has time independent $|z_1|$ and $|z_2|$, and therefore constant length of its semiaxes, and constant area. It rotates rigidly with angular velocity

$$\frac{d}{dt}\frac{\theta_1+\theta_2}{2} = -\frac{\Omega}{4}\frac{ab}{(a+b)^2} = -\frac{1}{4\pi}\frac{\Omega A}{|z_1(0)|^2} \tag{26}$$

4. Farfield perturbations of vortex patches

Let us consider a base vorticity

$$\omega = \Omega_1 \chi_{D_1} \tag{27}$$

and a perturbed vorticity

$$\eta = \Omega_1 \chi_{D_1} + \Omega_2 \chi_{D_2} \tag{28}$$

Because, by definition $D_2 \cap D_1 = \emptyset$, we have

$$\|\eta - \omega\|_Y = |\Omega_2|(1 + |D_2|)$$

where $|D_2|$ is the area of D_2. The boundaries Γ_1 and Γ_2 are described by functions $z_1(\alpha, t)$ and $z_2(\alpha, t)$ satisfying the vortex patch equations. We assume that z_2 is situated far

$$\sup_\alpha |z_1| \leq \epsilon \inf_\alpha |z_2| \tag{29}$$

with $\epsilon > 0$ very small. The fact that Γ_2 is far from Γ_1 does not stop η from being a small perturbation in Y of ω. The vortex patch system is

$$\frac{\partial z_1}{\partial t}(\alpha, t) = \frac{\omega_1}{2\pi}\int_0^{2\pi} \log|z_1(\alpha,t) - z_1(\beta,t)|\frac{\partial z_1}{\partial \beta}(\beta,t)d\beta + \frac{\omega_2}{2\pi}\int_0^{2\pi} \log|z_1(\alpha,t) - z_2(\beta,t)|\frac{\partial z_2}{\partial \beta}(\beta,t)d\beta, \tag{30}$$

and

$$\frac{\partial z_2}{\partial t}(\alpha, t) = \frac{\omega_1}{2\pi}\int_0^{2\pi} \log|z_2(\alpha,t) - z_1(\beta,t)|\frac{\partial z_1}{\partial \beta}(\beta,t)d\beta + \frac{\omega_2}{2\pi}\int_0^{2\pi} \log|z_2(\alpha,t) - z_2(\beta,t)|\frac{\partial z_2}{\partial \beta}(\beta,t)d\beta, \tag{31}$$

with ω_1 of order one and ω_2 very small. Let us write, in (30)

$$\log|z_1(\alpha) - z_2(\beta)| = \log|z_2(\beta)| + \log\left|1 - \frac{z_1(\alpha)}{z_2(\beta)}\right|$$

and in (31)

$$\log|z_1(\beta) - z_2(\alpha)| = \log|z_2(\alpha)| + \log\left|1 - \frac{z_1(\beta)}{z_2(\alpha)}\right|.$$

The system (30), (31) is thus

$$\frac{\partial z_1}{\partial t}(\alpha,t) = \frac{\omega_1}{2\pi}\int_0^{2\pi}\log|z_1(\alpha,t)-z_1(\beta,t)|\frac{\partial z_1}{\partial\beta}(\beta,t)d\beta + \frac{\omega_2}{2\pi}\int_0^{2\pi}\log\left|1 - \frac{z_1(\alpha,t)}{z_2(\beta,t)}\right|\frac{\partial z_2}{\partial\beta}(\beta,t)d\beta + U_1(t) \tag{32}$$

where

$$U_1(t) = \frac{\omega_2}{2\pi}\int_0^{2\pi}\log|z_2(\beta,t)|\frac{\partial z_2}{\partial\beta}(\beta,t)d\beta \tag{33}$$

and

$$\frac{\partial z_2}{\partial t}(\alpha,t) = \frac{\omega_1}{2\pi}\int_0^{2\pi}\log\left|1 - \frac{z_1(\beta,t)}{z_2(\alpha,t)}\right|\frac{\partial z_1}{\partial\beta}(\beta,t)d\beta + \frac{\omega_2}{2\pi}\int_0^{2\pi}\log|z_2(\alpha,t)-z_2(\beta,t)|\frac{\partial z_2}{\partial\beta}(\beta,t)d\beta, \tag{34}$$

Now we use the assumption that any $|z_2|$ is much larger than any $|z_1|$ and approximate the system by

$$\left\{\begin{array}{l} \frac{\partial z_1}{\partial t}(\alpha,t) = \omega_1 I_{11}(\alpha,t) - \frac{\omega_2}{2\pi}\int_0^{2\pi} Re\left(\frac{z_1(\alpha,t)}{z_2(\beta,t)}\right)\frac{\partial z_2(\beta,t)}{\partial\beta}d\beta + U_1(t),\\ \frac{\partial z_2}{\partial t}(\alpha,t) = -\frac{\omega_1}{2\pi}\int_0^{2\pi} Re\left(\frac{z_1(\beta,t)}{z_2(\alpha,t)}\right)\frac{\partial z_1(\beta,t)}{\partial\beta}d\beta + \omega_2 I_{22}(\alpha,t)\end{array}\right. \tag{35}$$

where

$$I_{jj}(\alpha,t) = \frac{1}{2\pi}\int_0^{2\pi}\log|z_j(\alpha,t) - z_j(\beta,t)|\frac{\partial z_j(\beta,t)}{\partial\beta}d\beta \tag{36}$$

and U_1 is given in (33). Let us make a few observations regarding quantities in (35). First,

$$\begin{array}{l} -\frac{\omega_2}{2\pi}\int_0^{2\pi} Re\left(\frac{z_1(\alpha,t)}{z_2(\beta,t)}\right)\frac{\partial z_2(\beta,t)}{\partial\beta}d\beta = -\frac{i}{2}\omega_2\left(\frac{1}{2\pi i}\oint_{\Gamma_2}\frac{d\zeta}{\zeta}\right)z_1(\alpha,t)\\ -\frac{\omega_2}{4\pi}\left(\int_0^{2\pi}(\overline{z_2}(\beta))^{-1}\frac{\partial z_2(\beta,t)}{\partial\beta}(\beta,t)d\beta\right)\overline{z_1}(\alpha,t)\\ = -\frac{i}{2}\omega_2 ind(0,\Gamma_2)z_1(\alpha,t) - \frac{\omega_2}{4\pi}\overline{z_1}(\alpha,t)\oint_{\Gamma_2}\overline{\zeta}^{-1}d\zeta\end{array}$$

where $ind(0,\Gamma_2)$ is the index (winding number) of Γ_2 at zero. Second,

$$\begin{array}{l} -\frac{\omega_1}{2\pi}\int_0^{2\pi} Re\left(\frac{z_1(\beta,t)}{z_2(\alpha,t)}\right)\frac{\partial z_1(\beta,t)}{\partial\beta}d\beta = -\frac{\omega_1}{4\pi}\left(\int_0^{2\pi}\overline{z_1}(\beta,t)\frac{\partial z_1(\beta,t)}{\partial\beta}d\beta\right)\overline{z_2}(\alpha,t)^{-1}\\ = -\frac{\omega_1}{4\pi}\overline{z_2}(\alpha,t)^{-1}\left[\int_0^{2\pi} Re(\overline{z_1}(\beta,t)\frac{\partial z_1(\beta,t)}{\partial\beta})d\beta + i\int_0^{2\pi} Im(\overline{z_1}(\beta,t)\frac{\partial z_1(\beta,t)}{\partial\beta})d\beta\right]\\ = -\frac{i\omega_1}{2}\overline{z_2}(\alpha,t)^{-1}A_1(t)\end{array} \tag{37}$$

where $A_j(t)$ is the normalized area of the region U_j bounded by the curve Γ_j:

$$A_j(t) = \frac{1}{2\pi}Im\int_0^{2\pi}\overline{z_j}(\beta,t)\frac{\partial z_j(\beta,t)}{\partial\beta}d\beta. \tag{38}$$

Collecting these observations, the system (35) becomes

$$\left\{\begin{array}{l} \frac{\partial z_1(\alpha,t)}{\partial t} = \omega_1 I_{11}(\alpha,t) - \frac{i}{2}\omega_2 ind(0,\Gamma_2)z_1(\alpha,t) - \frac{\omega_2}{4\pi}\overline{z_1}(\alpha,t)\oint_{\Gamma_2}(\overline{\zeta})^{-1}d\zeta + U_1(t)\\ \frac{\partial z_2(\alpha,t)}{\partial t} = -i\frac{\omega_1}{2}A_1(t)(\overline{z_2}(\alpha,t))^{-1} + \omega_2 I_{22}(\alpha,t)\end{array}\right. \tag{39}$$

Now we claim that solutions of (39) have constant normalized areas A_j. Indeed,

$$\frac{d}{dt}A_j(t) = -\frac{1}{\pi}Im\int_0^{2\pi}\partial_t z_j(\alpha,t)\frac{\partial\overline{z_j}(\alpha,t)}{\partial\alpha}d\alpha$$

The terms $I_{jj}(\alpha,t)$ in the integrals cancel because they lead to integrals

$$Im\int_0^{2\pi}\int_0^{2\pi}\log|z_j(\alpha,t)-z_j(\beta,t)|\frac{\partial\overline{z_j}(\alpha,t)}{\partial\alpha}\frac{\partial z_j(\beta,t)}{\partial\beta}d\alpha d\beta$$

which are zero because of the anti-symmetry of the integrand in (α,β). The rest of the terms cancel because they are integrals of derivatives of periodic functions:

$$Im\left(\frac{i}{2}\omega_2 ind(0,\Gamma_2)z_1(\alpha,t))\frac{\partial\overline{z_1}(\alpha,t)}{\partial\alpha}\right)=\frac{1}{4}\omega_2 ind(0,\Gamma_2)\partial_\alpha|z_1(\alpha,t)|^2,$$

$$\left(-\frac{\omega_2}{4\pi}\overline{z_1}(\alpha,t)\oint_{\Gamma_2}(\overline{\zeta})^{-1}d\zeta+U_1(t)\right)\frac{\partial\overline{z_1}(\alpha,t)}{\partial\alpha}=\partial_\alpha\left(-\frac{\omega_2}{2\pi}\overline{z_1}(\alpha,t)^2\oint_{\Gamma_2}(\overline{\zeta})^{-1}d\zeta+U_1(t)\overline{z_1}(\alpha,t)\right),$$

and

$$Im\left(-i\frac{\omega_1}{2}A_1(t)(\overline{z_2}(\alpha,t))^{-1}\frac{\partial\overline{z_2}(\alpha,t)}{\partial\alpha}\right)=-\partial_\alpha\left(\frac{\omega_1}{2}A_1(t)\log|z_2(\alpha,t)|\right)$$

Note the effect of the separation of Γ_2 from Γ_1: The equation for Γ_2 decouples

$$\partial_t z_2=-\frac{i\omega_1 A_1}{2}\left(\overline{z_2}\right)^{-1}+\omega_2 I_{22} \tag{40}$$

where A_1 is a constant, determined once for all from the area enclosed by the initial curve Γ_1. On the other hand z_2 is influences the evolution of z_1 only through constant (in α) terms, $U_1(t)$, given in (33), the winding number around zero of Γ_2, $ind(0,\Gamma_2)$ and

$$\zeta(t)=\frac{i}{2\pi}\int_0^{2\pi}\left(\overline{z_2}(\beta,t)\right)^{-1}\frac{\partial z_2(\beta,t)}{\partial\beta}d\beta. \tag{41}$$

$$\partial_t z_1=\omega_1 I_{11}-\frac{i}{2}\omega_2 ind(0,\Gamma_2)z_1(\alpha,t)+\frac{i\omega_2}{2}\zeta(t)\overline{z_1}(\alpha,t)+U_1(t). \tag{42}$$

The same decoupling occurs if we have a system of N widely separated curves where the vorticity jumps from one constant value to another. Now we are going to restrict our attention to the case in which the curve z_2 has antipodal reflection symmetry

$$z_2(\alpha+\pi,t)=-z_2(\alpha,t). \tag{43}$$

It is easy to see that if the initial curve $z_2(\cdot,0)$ has antipodal reflection symmetry, then the solution of (40) has antipodal reflection symmetry for all time. This follows because the derivative has also antipodal reflection symmetry. If a curve z has antipodal reflection symmetry then

$$\int_0^{2\pi}\log|z(\alpha)|\frac{\partial z(\alpha,t)}{\partial\alpha}d\alpha=0.$$

In particular, if z_2 has antipodal reflection symmetry then

$$U_1(t)=0. \tag{44}$$

If the winding number of the outer curve around the origin is 1, then the equation for z_1 becomes

$$\partial_t z_1=\omega_1 I_{11}-\frac{i\omega_2}{2}z_1(\alpha,t)+\frac{i\omega_2}{2}\zeta(t)\overline{z_1}(\alpha,t) \tag{45}$$

and this equation respects antipodal symmetry: if initial present, the symmetry persists as long as the solution is smooth. We note that if the winding number of the outer curve is nonzero, then, under our assumption of separation of contours, the outer curve surrounds the inner curve. Let us denote

$$U(0,t)=U(0,\omega_1\Gamma_1+\omega_2\Gamma_2)=\sum_{j=1}^{2}\frac{\omega_j}{2\pi}\int_0^{2\pi}\log|z_j(\beta)|\frac{\partial z_j}{\partial\beta}d\beta. \tag{46}$$

the velocity of the origin,

$$U(0,t)=\partial_t\Phi(0,t). \tag{47}$$

If the initial curves have antipodal reflection symmetry then $U(0,t) = 0$, because both integrals vanish. This means that 0 is a stagnation point, i.e., a fixed point of the Lagrangian path, for all time. An example of such a configuration is formed with two concentric ellipses (not necessarily aligned). The center of vorticity coincides with the origin in these cases. The system in which z_2 has antipodal reflection symmetry is therefore

$$\left\{ \begin{array}{l} \partial_t z_1 = \omega_1 I_{11} - \frac{i\omega_2}{2} z_1 + \frac{i\omega_2}{2}\zeta\,\overline{z_1}, \\ \partial_t z_2 = -\frac{i\omega_1 A_1}{2}\left(\overline{z_2}\right)^{-1} + \omega_2 I_{22} \end{array} \right. \tag{48}$$

with ζ given by (41).

5. Inner ellipse

The system (48) has the remarkable property that if the initial curve Γ_1 is an ellipse,

$$z_1(\alpha, 0) = w_1 e^{i\alpha} + w_2 e^{-i\alpha}$$

then it remains an ellipse

$$z_1(\alpha, t) = w_1(t) e^{i\alpha} + w_2(t) e^{-i\alpha} \tag{49}$$

where $w_j(t)$ solve the ODE system

$$\left\{ \begin{array}{l} \frac{dw_1}{dt} = \frac{i\omega_1}{2}\left(\frac{|w_2|^2}{|w_1|^2} - 1\right) w_1 - \frac{i\omega_2}{2}\left(w_1 - \zeta\,\overline{w_2}\right) \\ \frac{dw_2}{dt} = -\frac{i\omega_2}{2}\left(w_2 - \zeta\,\overline{w_1}\right) \end{array} \right. \tag{50}$$

The proof of this fact is based on the following lemma:

LEMMA 1. *Let* $z(\alpha) = \zeta_1 e^{i\alpha} + \zeta_{-1} e^{-i\alpha}$ *with* $|\zeta_1| > |\zeta_{-1}|$. *Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|z(\alpha) - z(\beta)| \frac{\partial z(\beta)}{\partial \beta} d\beta = \frac{i}{2}\left(\frac{|\zeta_{-1}|^2}{|\zeta_1|^2} - 1\right)\zeta_1 e^{i\alpha}. \tag{51}$$

The proof of the lemma is based on a calculation done already in ([**4**]), but for the sake of completeness we present it below. The first observation is that

$$\frac{1}{2\pi}\int \log|z(\alpha) - z(\beta)| \frac{\partial z(\beta)}{\partial \beta} d\beta = \frac{1}{2}(Hz)(\alpha) + \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{z(\alpha) - z(\beta)}{e^{i\alpha} - e^{i\beta}}\right| \frac{\partial z(\beta)}{\partial \beta} d\beta$$

where

$$(Hf)(\alpha) = \frac{1}{2\pi} P.V. \int_0^{2\pi} \cot\left(\frac{\alpha - \beta}{2}\right) f(\beta) d\beta$$

is the circular Hilbert transform. This follows from the properties of the logarithm and

$$\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\alpha} - e^{i\beta}| \frac{\partial z(\beta)}{\partial \beta} d\beta = \frac{1}{2}(Hz)(\alpha)$$

which is obtained by integration by parts. Now we write

$$\frac{z(\alpha) - z(\beta)}{e^{i\alpha} - e^{i\beta}} = 1 + \delta(\alpha, \beta)$$

with

$$\delta(\alpha, \beta) = (\zeta_1 - 1) - \zeta_{-1} e^{-i\alpha} e^{-i\beta}$$

and expand

$$\log|1 + \delta| = Re\left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \delta^k\right).$$

Raising $\delta(\alpha, \beta)$ to a power k we obtain

$$\delta(\alpha, \beta)^k = (\zeta_1 - 1)^k - k(\zeta_1 - 1)^{k-1}\zeta_{-1} e^{-i\alpha} e^{-i\beta} + \dots$$

and the only nonzero contribution to the integral

$$\frac{1}{2\pi}\int_0^{2\pi} Re\left((-1)^{k+1}\frac{1}{k}(\delta(\alpha,\beta))^k\right)\frac{\partial z(\beta)}{\partial\beta}d\beta$$

comes from the second term, so

$$\frac{1}{2\pi}\int_0^{2\pi}\log|1+\delta(\alpha,\beta)|\frac{\partial z(\beta)}{\partial\beta}d\beta = \\ \sum_{k=1}^{\infty}\frac{i}{2}\zeta_1(-1)^k(\zeta_1-1)^{k-1}\zeta_{-1}e^{-i\alpha}+\sum_{k=1}^{\infty}\frac{i}{2}|\zeta_{-1}|^2\overline{(\zeta_1-1)}^{k-1}(-1)^{k-1}e^{i\alpha}$$

Therefore

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{z(\alpha)-z(\beta)}{e^{i\alpha}-e^{i\beta}}\right|\frac{\partial z(\beta)}{\partial\beta}d\beta=\frac{i}{2}\frac{|\zeta_{-1}|^2}{|\zeta_1|^2}\zeta_1 e^{i\alpha}-\frac{i}{2}\zeta_{-1}e^{-i\alpha}$$

and (51) follows from

$$Hz=-i\zeta_1 e^{i\alpha}+i\zeta_{-1}e^{-i\alpha}.$$

Now, this proof seems to work only if $|\zeta_1-1|+|\zeta_{-2}|<1$, but the linear scaling property

$$\frac{1}{2\pi}\int_0^{2\pi}\log|c(z(\alpha)-z(\beta)|\frac{c\partial z(\beta)}{\partial\beta}d\beta=c\frac{1}{2\pi}\int_0^{2\pi}\log|(z(\alpha)-z(\beta)|\frac{\partial z(\beta)}{\partial\beta}d\beta$$

valid for any $c\in\mathbb{C}$ reduces the problem to this case. Indeed, if $\zeta_1=re^{i\phi}$ and we choose $c=(r+\epsilon)^{-1}e^{-i\phi}$ then

$$|c\zeta_1-1|+|c\zeta_{-1}|\le\frac{\epsilon}{r+\epsilon}+\frac{|\zeta_{-1}|}{|\zeta_1|}\frac{r}{r+\epsilon}<1.$$

This concludes the proof of the lemma.

Noting that

$$|w_1|^2-|w_2|^2=A_1 \tag{52}$$

we can write (50) as

$$\begin{cases}\frac{dw_1}{dt} = -\frac{i}{2}\left(\frac{\omega_1 A_1}{|w_1|^2}+\omega_2\right)w_1+\frac{i\omega_2}{2}\zeta\,\overline{w_2},\\ \frac{dw_2}{dt} = -\frac{i\omega_2}{2}w_2+\frac{i\omega_2}{2}\zeta\,\overline{w_1}\end{cases} \tag{53}$$

The conservation in time of A_1 (given in (52) can be checked independently on (50). The system (48) now is reduced to the ODE system (53) where ζ is obtained from (41), coupling the ODE to the equation (40). Kirkhhoff ellipses are obtained by turning off the coupling, i.e., setting $\omega_2=0$. The ODE system reduces further by considering the variable

$$w=\frac{\overline{w_2}}{w_1} \tag{54}$$

This is a geometric quantity (see (21), (22)):

$$w=\frac{a_1-b_1}{a_1+b_1}e^{-i(\theta_1+\theta_2)}.$$

The system (53) implies

$$\frac{dw}{dt}=-\frac{i\omega_2}{2}\left(\overline{\zeta}+\zeta w^2\right)+\frac{i}{2}w\left(\frac{\omega_1 A_1}{|w_1|^2}+2\omega_2\right).$$

In view of (52)

$$\frac{A_1}{|w_1|^2}=1-|w|^2, \tag{55}$$

and therefore the equation for w is self-contained:

$$\frac{dw}{dt}=-\frac{i\omega_2}{2}\left(\overline{\zeta}+\zeta w^2\right)+\frac{i}{2}w\left(\omega_1(1-|w|^2)+2\omega_2\right). \tag{56}$$

The variables w_1 and w_2 are easily obtained once w is known. In view of the geometric interpretation, we expect $|w| = 1$ to be an invariant circle for the ODE. Indeed,

$$\frac{d}{dt}(1-|w|^2) = -\frac{\omega_2}{2}\left(Im(\zeta w)\right)(1-|w|^2) \tag{57}$$

This shows that $|w| = 1$ is an invariant circle for the equation. Moreover, in view of (52), if this set attracts a trajectory from inside ($|w| < 1$) this means that the inner ellipse evolves in time and degenerates into a line $|w_1| = \infty$. Indeed,

$$|w| = \frac{a_1 - b_1}{a_1 + b_1}$$

and $|w| = 1$ implies $b_1 = 0$, $a_1 = \infty$. (Because $A_1 = a_1 b_1$ is finite). This can happen only if

$$\limsup_{T\to\infty} \omega_2 \int_0^T Im(\zeta(t)w(t))dt = \infty$$

Let us write now

$$\zeta(t) = \gamma(t)e^{i\theta(t)} \tag{58}$$

with $\gamma(t) \in \mathbb{R}_+$ and consider the evolution of w in a co-moving frame, i.e., we introduce the variable

$$u = we^{i\theta}. \tag{59}$$

The equation (56) becomes

$$\frac{du}{dt} = -\frac{i\omega_2\gamma}{2}(1+u^2) + \frac{iu}{2}\left(\omega_1(1-|u|^2 + 2\omega_2 + 2\frac{d\theta}{dt}\right) \tag{60}$$

and the equation (57) becomes

$$\frac{d}{dt}(1-|u|^2) = \omega_2\gamma(Imu)(1-|u|^2). \tag{61}$$

We rescale time in order to have nondimensional quantities. Setting $\tau = \frac{\omega_1}{2}t$ we have

$$\frac{du}{d\tau} = -i\frac{\omega_2}{\omega_1}\gamma(1+u^2) + iu\left((1-|u|^2) + \frac{2\omega_2}{\omega_1} + \frac{d\theta}{d\tau}\right) \tag{62}$$

Writing $u = x + iy$, we arrive at

$$\begin{cases} \frac{dx}{d\tau} = 2\delta xy - y\left(1+\Delta - x^2 - y^2\right), \\ \frac{dy}{d\tau} = -\delta(1+x^2-y^2) + x(1+\Delta - x^2 - y^2) \end{cases} \tag{63}$$

where we denoted

$$\delta = \gamma\frac{\omega_2}{\omega_1}, \quad \Delta = 2\frac{\omega_2}{\omega_1} + \frac{d\theta}{d\tau}. \tag{64}$$

Recall that δ and Δ are computed from ζ, which is computed from the outer curve z_2. Our choice of variable u is motivated in the next section where we compute an approximation of ζ explicitly, and obtain δ and Δ explicitly and in addition, constant in time.

6. Two ellipses

We saw that if the initial curve $z_1(\cdot, 0)$ in (48) is an ellipse, then it remains an ellipse for all time, all be it with changing length of semiaxis. This is no longer the case for the evolution of z_2, unless the initial curve is a circle, in which case it stays a circle. If the initial data is an ellipse with small eccentricity, the evolution away from the ellipse will take a long time, the farther the curve and the smaller the eccentricty, the longer the time. More precisely if we start with

$$z_2(\alpha, 0) = \zeta_1(0)e^{i\alpha} + \zeta_2(0)e^{-i\alpha}, \tag{65}$$

the right hand side of the equation (40) (which is the same as the second equation of (48) introduces higher harmonics. These are introduced not by the nonlinear term, but by the term $-\frac{i\omega_1 A_1}{2}\overline{z_2}^{-1}$, which is small.

We therefore approximate the evolution of z_2 by projecting it on the elliptical modes. This is done in order to compute $\zeta(t)$ explicitly. Thus, if $z_2 = \zeta_1 e^{i\alpha} + \zeta_2 e^{-i\alpha}$ then we approximate

$$(\overline{z_2})^{-1} = e^{i\alpha}(\overline{\zeta_1})^{-1}\left(1 + \overline{\left(\frac{\zeta_2}{\zeta_1}\right)} e^{2i\alpha}\right)^{-1} \approx e^{i\alpha}(\overline{\zeta_1})^{-1}$$

(Recall from (22) that circles correspond to $\zeta_2 = 0$ with our orientation convention that $|\zeta_1| \geq |\zeta_2|$.) With this the equation (40) with initial data (65) has solutions approximated by

$$z_2(\alpha, t) = \zeta_1(t)e^{i\alpha} + \zeta_2(t)e^{-i\alpha} \tag{66}$$

with

$$\begin{cases} \zeta_1(t) = \zeta_1(0)e^{\frac{-i(\omega_1 A_1 + \omega_2 A_2)t}{2|\zeta_1(0)|^2}} \\ \zeta_2(t) = \zeta_2(0) \end{cases} \tag{67}$$

Indeed, the approximate system is

$$\begin{cases} \frac{d\zeta_1}{dt} = -\frac{i\omega_1 A_1}{2}(\overline{\zeta_1})^{-1} + \frac{i\omega_2}{2}\left(\frac{|\zeta_2|^2}{|\zeta_1|^2} - 1\right)\zeta_1, \\ \frac{d\zeta_2}{dt} = 0, \end{cases}$$

so, from the second equation $|\zeta_2|^2 = |\zeta_2(0)|^2$, and substituting in the first equation we arrive at (67). Now, it is elementary to check that, if $z_2 = \zeta_1 e^{-i\alpha} + \zeta_2 e^{-i\alpha}$, then ζ given by (41) is computed by

$$\begin{aligned} &\tfrac{i}{2\pi}\oint_{\Gamma_2}(\overline{z})^{-1}\,dz = \tfrac{i}{2\pi}\int_0^{2\pi} i(\zeta_1 e^{i\alpha} - \zeta_2 e^{-i\alpha})e^{i\alpha}(\overline{\zeta_1})^{-1}\left(1 + \overline{\left(\tfrac{\zeta_2}{\zeta_1}\right)}e^{2i\alpha}\right)^{-1} d\alpha \\ &= \tfrac{i}{2\pi}\int_0^{2\pi} i(\zeta_1 e^{i\alpha} - \zeta_2 e^{-i\alpha})e^{i\alpha}(\overline{\zeta_1})^{-1}\left(1 - \overline{\left(\tfrac{\zeta_2}{\zeta_1}\right)}e^{2i\alpha} + \cdots\right) d\alpha \\ &= \zeta_2(\overline{\zeta_1})^{-1} \end{aligned}$$

because the series converges if $|\zeta_2| < |\zeta_1|$. We have that

$$\zeta(t) = \gamma e^{-i\frac{(1-\gamma^2)(\omega_1 A_1 + \omega_2 A_2)}{2A_2}} \tag{68}$$

where

$$\gamma = \frac{\zeta_2(0)}{\overline{\zeta_1}(0)} \tag{69}$$

and, without loss of generality we assumed that γ is real. Indeed, in view of (21), (22),

$$\gamma = \frac{a_2 - b_2}{a_2 + b_2}e^{i(\theta_1 + \theta_2)}$$

where a_2, b_2 are the major and minor semiaxes of $\Gamma_2(0)$ and $\frac{1}{2}(\theta_1 + \theta_2)$ is the angle $\Gamma_2(0)$ makes with the coordinate system. So, assuming that γ is real amounts to choosing the axis so that Ox is on the direction of the major semiaxis of $\Gamma_2(0)$. If γ is real, then

$$1 - \gamma^2 = 1 - \frac{|\zeta_2(0)|^2}{|\zeta_1(0)|^2} = \frac{A_2}{|\zeta_1(0)|^2} \tag{70}$$

and (68) follows from (67). Thus $\gamma = \frac{a_2 - b_2}{a_2 + b_2}$ is time-independent. The variable u defined in (59) describes the parameters of the inner ellipse $\Gamma_1(t)$ in a frame which rotates with angular velocity $-\frac{(1-\gamma^2)(\omega_1 A_1 + \omega_2 A_2)}{2A_2}$. In particular

$$|u| = \frac{a_1 - b_1}{a_1 + b_1} \tag{71}$$

gives the ration $\frac{a_1}{b_1}$ of the major to minor semiaxes of $\Gamma_1(t)$. The quantities δ and Δ defined in (64) and giving the coefficients in the system (63) which describes the evolution of $u = x + iy$, are,

$$\delta = \frac{(a_2 - b_2)}{a_2 + b_2}\frac{\omega_2}{\omega_1} \tag{72}$$

and

$$\Delta = \frac{2\omega_2}{\omega_1} - \left(1 - \left(\frac{a_2 - b_2}{a_2 + b_2}\right)^2\right) \frac{\omega_1 A_1 + \omega_2 A_2}{\omega_1 A_2}$$

i.e.

$$\Delta = \frac{\omega_2}{\omega_1}\left(1 + \left(\frac{a_2 - b_2}{a_2 + b_2}\right)^2\right) - \frac{A_1}{A_2}\left(1 - \left(\frac{a_2 - b_2}{a_2 + b_2}\right)^2\right). \tag{73}$$

They are constant because the lengths of the semiaxis of $\Gamma_2(t)$, a_2 and b_2 are constant in time.

7. The ODE system

We investigate the system (63),

$$\begin{cases} \frac{dx}{dt} = 2\delta xy - y\left(1 + \Delta - x^2 - y^2\right) \\ \frac{dy}{dt} = -\delta\left(1 + x^2 - y^2\right) + x\left(1 + \Delta - x^2 - y^2\right). \end{cases} \tag{74}$$

Recall that $u = x + iy$, where u parameterizes the inner ellipse via (49), (54), (59). In view of (71), we are interested in $x^2 + y^2 \le 1$. The quantities δ and Δ are constants, fixed by the outer ellipse via (72) and (73). The fixed points of (74) are given by

$$y = 0, \quad \text{and} \quad x^3 + \delta x^2 - (1 + \Delta)x + \delta = 0, \tag{75}$$

and

$$x = \frac{\Delta}{2\delta}, \quad \text{and} \quad x^2 + y^2 = 1. \tag{76}$$

PROPOSITION 1. *The invariant set*

$$\{(x, y) \mid \; x^2 + y^2 = 1\}$$

for (74) attracts trajectories of (74) if and only if

$$\left|\frac{\Delta}{2\delta}\right| < 1. \tag{77}$$

By "attracts trajectories" we mean that there exist (x_0, y_0) with $x_0^2 + y_0^2 < 1$ such that the solution $(x(t), y(t))$ of (74) with initial data (x_0, y_0), satisfies

$$\limsup_{t\to\infty}\,(x(t)^2 + y(t)^2) = 1.$$

Proof. The quantity

$$K = \frac{\Delta - 2\delta x}{1 - x^2 - y^2} - \log(1 - x^2 - y^2) \tag{78}$$

is a conserved quantity for (74), as it is easily verified. If we assume

$$\left|\frac{\Delta}{2\delta}\right| > 1$$

then, from the the hypothesis $x_0^2 + y_0^2 < 1$, $\limsup_{t\to\infty}(x^2(t) + y^2(t)) = 1$ we derive a contradiction. Indeed, on the trajectory $(x(t), y(t))$, K takes the finite value K_0 computed at (x_0, y_0). Multiplying by $(1 - x^2 - y^2)$ we obtain

$$(1 - x^2 - y^2)K_0 = \Delta - 2\delta x - (1 - x^2 - y^2)\log(1 - x^2 - y^2)$$

and, on a sequence $t_j \to \infty$ on which $x(t_j)^2 + y(t_j)^2 \to 1$ we deduce that $\frac{\Delta}{2\delta} = \lim_{j\to\infty} x(t_j) \in [-1, 1]$ which is absurd. On the other hand, if (77) is satisfied, then the fixed points

$$x = \frac{\Delta}{2\delta}, \quad y = \pm\sqrt{1 - \left(\frac{\Delta}{2\delta}\right)^2} \tag{79}$$

lie on $x^2 + y^2 = 1$. Analyzing the linear stability we find that the linearized system is

$$\begin{cases} \frac{d\xi}{dt} = 2y(\delta + x)\xi + (2y^2)\eta, \\ \frac{d\eta}{dt} = -2x^2\xi + 2y(\delta - x)\eta \end{cases} \tag{80}$$

and the fixed point (79) is stable if $\delta y < 0$, and unstable if $\delta y > 0$. By ODE theory, the stable fixed point has a nonempty open basin of attraction which therefore intersects $x^2 + y^2 < 1$. This finishes the proof of the theorem except for the borderline case of $\left|\frac{\Delta}{2\delta}\right| = 1$. This case necessitaes a further study of the phase portrait of (74) which we perform for other reasons as well. Before we do so, let us note that (77) holds if and only if

$$\frac{b_2}{a_2} < \frac{\omega_1 A_1}{\omega_2 A_2} < \frac{a_2}{b_2} \tag{81}$$

Note that (81) does not involve the aspect ratio of the inner ellipse. We investigate further the system. We take $\delta > 0$: in view of (72) and (81) this is the only possible case for instability if ω_2 has the same sign as ω_1. We need to find out how many solutions of the cubic equation in (75) lie in $x^2 \le 1$. We look therefore for intersections of the curves $f(x) = x - x^3$ and $g(x) = \delta x^2 - \Delta x + \delta$ in $-1 \le x \le 1$. The minimum of g is attained at $x = \frac{\Delta}{2\delta}$ and is positive if $g_{min} = \delta - \frac{\Delta^2}{4\delta} > 0$. The maximum of f is obtained at $x = \frac{1}{\sqrt{3}}$ and equals $f_{max} = \frac{2}{3\sqrt{3}}$. All intersections will be in $0 \le x \le 1$. There will be two intersections if and only if the point $(\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}})$ is situated above the graph of the parabola $y = g(x)$. The reason for this is that $f(0) = 0 < g(0) = \delta$ and $f(1) = 0 < g(1) = 2\delta - \Delta$. In this case there will be two roots, $0 \le x_1 < \frac{1}{\sqrt{3}} < x_2 \le 1$. If the point $(\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}})$ is situated below the graph of the parabola there will be no intersections, and if it is on the parabola, there will be one intersection point. The conditions are thus

$$\begin{cases} \frac{2}{3\sqrt{3}} < \frac{4\delta}{3} - \frac{\Delta}{\sqrt{3}} \Leftrightarrow \text{two solutions } \; x_1 < x_2, \\ \frac{2}{3\sqrt{3}} = \frac{4\delta}{3} - \frac{\Delta}{\sqrt{3}} \Leftrightarrow \text{one solution } \; x_1 = x_2, \\ \frac{2}{3\sqrt{3}} > \frac{4\delta}{3} - \frac{\Delta}{\sqrt{3}} \Leftrightarrow \text{no solutions} \end{cases} \tag{82}$$

When there are two solutions, then the smaller one x_1 is stable, the larger one x_2 is unstable. Numerically it is easy then to see that there is a homoclinic orbit connecting x_2 to itself and surrounding x_1. The circle $x^2 + y^2 = 1$ is composed of two heteroclinic orbits going from the unstable fixed point on the circle to the stable one. There are also heteroclinic orbits connecting the unstble fixed point on the circle to x_2 and x_2 to the stable fixed point on the circle. If there is only one fixed solution then the previous picture simplifies, and, in addition to the two heteroclinic orbits on the unit circle there are only heteroclinic orbits connecting the unstable fixed point on the circle to $x_1 = x_2$, and connecting the latter to the stable fixed point on the circle. If there is no solution inside then all orbits connect the unstable fixed point on the circle to the stable one. If $\left|\frac{\Delta}{2\delta}\right| = 1$ then there are no orbits connecting the circle with the interior of the disk.

In view of the fact that $|u| = |w|$ (see 59), the upshot is that in all the cases obeying (77), when the unit circle attracts trajectories, it follows that $\limsup_{t\to\infty} |w| = 1$ where w is related to (49) by (54). Consequently, there is unbounded growth of the inner ellipse. Indeed from the conservation of A_1 and from (55) it follows that $\limsup_{t\to\infty} |w_1| = \infty$, and that means, in view of (22) that the sum of lengths of semiaxes of the inner ellipse diverges.

Acknowledgment. Research partially supported by NSF-DMS grants 1209394 and 1265132.

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