Euler Equations: local existence

Mat 529, Lesson 2.

1 Active scalars formulation

We start with a lemma.

Lemma 1. *Assume that w is a magnetization variable, i.e.*

$$\partial_t w + u \cdot \nabla w + (\nabla u)^* w = 0.$$

If $u = \mathbb{P}w$ then u solves the incompressible Euler equations.

Proof. Indeed, if $\bar{u} = w + \nabla q$ then

$$\begin{aligned} D_t\bar{u} &= D_t w + D_t \nabla q = D_t w + \nabla D_t q - (\nabla u)^* \nabla q \\ &= -(\nabla u)^*(w + \nabla q) + \nabla D_t q = -(\nabla u)^* \bar{u} + \nabla D_t q, \end{aligned}$$

and if $\bar{u} = u$ then $(\nabla u)^* \bar{u}$ is a gradient. This proves the lemma.

This implies

Theorem 1. *If $u \in C^{1,\mu}$ is given by*

$$u = (\mathbb{P})((\nabla A)^* v)$$

with $A \in C^{1,\mu}$, $v \in C^{1,\mu}$ solving

$$D_t A = 0 \tag{1}$$

and

$$D_t v = 0,$$

then u solves the incompressible Euler equations.

Proof. Clearly

$$w = (\nabla A)^* v$$

is a magnetization variable because

$$D_t(\nabla A)^* + (\nabla u)^*(\nabla A)^* = 0.$$

Because of the Weber formula, we have

Lemma 2. *A function $u(x,t)$ solves the incompressible Euler equations if and only if it can be represented in the form $u = u_A$ with*

$$u_A^i(x,t) = \phi^m\left(A(x,t)\right)\frac{\partial A^m(x,t)}{\partial x_i} - \frac{\partial n_A(x,t)}{\partial x_i} \tag{2}$$

and

$$\nabla \cdot u_A = 0 \tag{3}$$

where $A(x,t)$ solves the active scalars equation

$$\left(\partial_t + u_A \cdot \nabla\right) A = 0, \tag{4}$$

with initial data

$$A(x,0) = x.$$

The function ϕ represents the initial velocity and the function $n_A(x,t)$ is determined up to additive constants by the requirement of incompressibility, $\nabla \cdot u_A = 0$:

$$\Delta n_A(x,t) = \frac{\partial}{\partial x_i}\left\{\phi^m(A(x,t))\frac{\partial A^m(x,t)}{\partial x_i}\right\}.$$

The periodic boundary conditions are

$$A(x + Le_j, t) = A(x,t) + Le_j; \quad n_A(x + Le_j, t) = n_A(x,t) \tag{5}$$

with e_j the standard basis in $\mathbb{R}^3$. In this case

$$\delta_A(x,t) = x - A(x,t) \tag{6}$$

$n_A(x,t)$, and $u_A(x,t)$ are periodic functions in each spatial direction. One may consider also the case of decay at infinity, requiring that δ_A, u_A and n_A vanish sufficiently fast at infinity. The equation of state (2, 3) can be written as

$$u_A = \mathbb{P}\left\{\phi^m\left(A(\cdot,t)\right)\nabla A^m(\cdot,t)\right\} = \mathbb{P}\left\{(\nabla A)^* \phi(A)\right\}, \tag{7}$$

which is the Weber formula. The Eulerian pressure is determined, up to additive constants by

$$p(x,t) = \frac{\partial n_A(x,t)}{\partial t} + u_A(x,t)\cdot\nabla n_A(x,t) + \frac{1}{2}|u_A(x,t)|^2.$$

The Jacobian obeys

$$det\,(\nabla A(x,t)) = 1.$$

The vorticity

$$\omega_A(x,t) = \nabla\times u_A$$

satisfies the Helmholtz equation

$$D_t^A\omega_A = \omega_A\cdot\nabla u_A \tag{8}$$

and is given by the Cauchy formula

$$\omega_A(x,t) = [\nabla A(x,t)]^{-1}\,\zeta(A(x,t)) \tag{9}$$

where $\zeta = \nabla\times\phi$ is the initial vorticity.

2 Local existence

The proof of local existence of solutions to the Euler equations in the active scalars formulation is relatively simple and the result can be stated economically.

Theorem 2. *Let ϕ be a divergence free $C^{1,\mu}$ periodic vector valued function of three variables. There exists a time interval $[0,T]$ and a unique $C([0,T];C^{1,\mu})$ spatially periodic vector valued function $\delta(x,t)$ such that*

$$A(x,t) = x + \delta(x,t)$$

solves the active scalars formulation of the Euler equations, :

$$\frac{\partial A}{\partial t} + u\cdot\nabla A = 0,$$

$$u = \mathbb{P}\left\{(\nabla A(x,t))^*\phi(A(x,t))\right\}$$

with initial datum $A(x,0) = x$.

The same result holds if one replaces periodic with decaying at infinity. Differentiating the active scalars equation (4) we obtain the equation obeyed by the gradients

$$D_t^A\left(\frac{\partial A^m}{\partial x_i}\right)+\frac{\partial u_A^j}{\partial x_i}\frac{\partial A^m}{\partial x_j}=0. \tag{10}$$

It is useful to denote

$$\mathbb{P}_{jl}=\delta_{jl}-\partial_j\Delta^{-1}\partial_l \tag{11}$$

Differentiating in the representation (7) and using the fundamental property

$$\mathbb{P}_{jl}\frac{\partial f}{\partial x_l}=0$$

we obtain

$$\frac{\partial u_A^j}{\partial x_i}=\mathbb{P}_{jl}\left(Det\left[\zeta(A);\frac{\partial A}{\partial x_i};\frac{\partial A}{\partial x_l}\right]\right). \tag{12}$$

Recall that the function ζ is te curl of ϕ. This relation shows that the gradient of velocity can be expressed without use of second order derivatives of A and is the key to local existence: the equation (10) can be seen as a cubic quasi-local equation on characteristics. Let us make these ideas more precise. We will consider the periodic case first. We write $C^{j,\mu}$, $j=0,1$ to denote the Hölder spaces of real valued functions that are defined for all $x\in\mathbb{R}^3$ and are periodic with period L in each direction. We denote by $\|f\|_{0,\mu}$ the $C^{0,\mu}$ norm:

$$\|f\|_{0,\mu}=\sup_x|f(x)|+\sup_{x\neq y}\left\{|f(x)-f(y)|\left(\frac{L}{|x-y|}\right)^{\mu}\right\} \tag{13}$$

and by $\|f\|_{1,\mu}$ the $C^{1,\mu}$ norm:

$$\|f\|_{1,\mu}=\|f\|_{0,\mu}+L\|\nabla f\|_{0,\mu} \tag{14}$$

where the notation $|\cdots|$ refers to modulus, Euclidean norm, and Euclidean norm for matrices, as appropriate.

We break the solution of the problem in two parts, the map $\delta\to u$ and the map $u\to\delta$. We denote the first one W.

$$W[\delta,\phi](x,t)=\mathbb{P}\left\{(\mathbb{I}+\nabla\delta(x,t))^*\phi(x+\delta(x,t))\right\} \tag{15}$$

This map is linear in ϕ but nonlinear in δ.

Proposition 1. *The map $W[\delta,\phi]$ maps*

$$W : (C^{1,\mu})^3 \times (C^{1,\mu})^3 \to (C^{1,\mu})^3$$

continuously. There exist constants C depending on μ alone so that

$$\|W[\delta,\phi]\|_{0,\mu} \le C\|\phi\|_{0,\mu}\left\{1+\|\nabla\delta\|_{0,\mu}\right\}^2$$

and

$$\|\nabla W[\delta,\phi]\|_{0,\mu} \le C\|\nabla\times\phi\|_{0,\mu}\left\{1+\|\nabla\delta\|_{0,\mu}\right\}^3.$$

hold for any $\delta \in (C^{1,\mu})^3$, $\phi \in (C^{1,\mu})^3$.

Proof. We note that W is made up from a number of operations. The first operation is the composition $\phi(x) \mapsto \phi(x+\delta(x))$. For a fixed $\delta \in (C^{1,\mu})^3$ the map $x \mapsto x+\delta$ is Lipschitz. Composition with a Lipschitz change of variables maps $C^{0,\mu}$ into itself continuously (we say that it is a continuous endomorphism). The joint continuity of $[\phi,\delta] \mapsto \phi(x+\delta)$ in $C^{1,\mu}$ follows naturally. The second operation is a sum of products of functions (a matrix applied to a vector). This is a continuous operation because the Hölder spaces $C^{j,\mu}$, $j=0,1$ we chose are Banach algebras. The third and last operation is the linear operator $\mathbb{P}$, which is bounded in Hölder spaces. We need to consider also derivatives of W. We use the formula (12) and note that the expression for the gradient is made of similar operations as above and apply the same kind of reasoning. This finishes the proof.

Time does not play any role in this proposition because the equation of state $(\delta,\phi) \mapsto W[\delta,\phi]$ is time independent. The second half of the procedure does depend on time. Let us denote by Θ the map that associates to two continuous paths $t \mapsto \delta(\cdot,t)$ and $t \mapsto \phi(\cdot,t)$ the path $t \mapsto \theta = \Theta[\delta,\phi]$ obtained by solving the partial differential equation

$$\frac{\partial\theta}{\partial t} + u\cdot\nabla\theta + u = 0 \tag{16}$$

where

$$u = W[\delta(\cdot,t),\phi(\cdot,t)],$$

periodic boundary conditions are imposed on θ and zero initial data

$$\theta(x,0) = 0$$

are required. The Euler equation requires only the use of a time independent ϕ, but allowing time dependent ϕ is very useful: one can then treat more equations, in particular the Navier-Stokes equation. Let us consider the space

$$\mathcal{P}_T = C([0,T],(C^{1,\mu})^3)$$

of continuous $(C^{1,\mu})^3$ -valued paths defined on a time interval $[0,T]$, endowed with the natural norm

$$\|\theta\|_{1,\mathcal{P}} = \sup_t \|\theta(\cdot,t)\|_{1,\mu}.$$

We will consider also the weaker norm

$$\|\theta\|_{0,\mathcal{P}} = \sup_t \|\theta(\cdot,t)\|_{0,\mu}.$$

Θ is nonlinear in both arguments.

Proposition 2. *The map $\Theta[\delta,\phi]$ maps*

$$\Theta : \mathcal{P}_T \times \mathcal{P}_T \to \mathcal{P}_T$$

and is continuous when the topology of the source space $\mathcal{P}_T \times \mathcal{P}_T$ is the natural product $C^{1,\mu}$ topology and the topology of the target space $\mathcal{P}_T$ is the weaker $C^{0,\mu}$ topology. Moreover, there exists a constant C depending on μ alone so that

$$\|\nabla\theta(\cdot,t)\|_{0,\mu} \le \left(\int_0^t \|\nabla u(\cdot,s)\|_{0,\mu} ds\right)\left\{\exp\{C\int_0^t \|\nabla u(\cdot,s)\|_{0,\mu} ds\}\right\}$$

holds for each $t \le T$ with $u = W[\delta,\phi]$ and $\theta = \Theta[\delta,\phi]$.

Proposition 2 states that the map Θ is bounded but not that it is continuous in the strong $C^{1,\mu}$ topology. The **proof** follows naturally from the idea to use the classical method of characteristics and ODE Gronwall type arguments. Similar ideas are needed below in the the slightly more difficult proof of Proposition 3 and we will sketch them there.

Let us take now a fixed ϕ, take a small number $\epsilon > 0$ and associate to it the set

$$\mathcal{I} \subset \mathcal{P}_T$$

defined by

$$\mathcal{I} = \{\delta(x,t); \delta(x,0) = 0, \|\nabla\delta(\cdot,t)\|_{0,\mu} \le \epsilon, \forall t \le T\}.$$

Combining the bounds in the two previous propositions one can choose, for fixed ϕ, a T small enough so that

$$\delta \mapsto \Theta[\delta, \phi] = \mathcal{S}[\delta]$$

maps

$$\mathcal{S} : \mathcal{I} \to \mathcal{I}.$$

Inspecting the bounds it is clear that it is sufficient to require

$$T\|\nabla \times \phi\|_{0,\mu} \le c\epsilon$$

with an appropriate c depending on μ alone. Leaving ϕ, ϵ and T fixed as above, the map $\mathcal{S}$ is Lipschitz in the weaker norm $C^{0,\mu}$:

Proposition 3. *There exists a constant C, depending on μ alone, such that, for every $\delta_1, \delta_2 \in \mathcal{I}$, the Lipschitz bound*

$$\|\mathcal{S}[\delta_1] - \mathcal{S}[\delta_1]\|_{0,\mathcal{P}} \le C\|\delta_1 - \delta_2\|_{0,\mathcal{P}}$$

holds.

It is essential that $\delta_j \in \mathcal{I}$, so that they are smooth and their gradients are small, but nevertheless this is a nontrivial statement. An inequality of the type

$$\|\mathcal{S}[\delta_1] - \mathcal{S}[\delta_1]\|_{0,\mathcal{P}} \le C\|\delta_1 - \delta_2\|_{1,\mathcal{P}}$$

is easier to obtain, but loses one derivative. This kind of loss of one derivative is a well-known difficulty in general compressible hyperbolic conservation laws. The situation is complicated in addition by the fact that the constitutive law W depends on gradients. As we shall see, incompressibility saves one derivative. The heart of the matter is

Proposition 4. *Let $\phi \in (C^{1,\mu})^3$ be fixed. There exists a constant depending on μ alone so that*

$$\|W[\delta_1, \phi] - W[\delta_2, \phi]\|_{0\mu} \le C\|\delta_1 - \delta_2\|_{0,\mu}\|\phi\|_{1,\mu}$$

holds for any $\delta_j \in C^{1,\mu}$ with $\|\delta_j\|_{1,\mu} \le 1$.

One could use the condition $\delta_j \in C^{1,\mu}$ with $\|\delta_j\|_{1,\mu} \leq M$ but then C would depend on M also.

Proof of Proposition 4. Denoting

$$u = W[\delta_1, \phi] - W[\delta_2, \phi],$$

$$\delta = \delta_1 - \delta_2,$$

$$\psi(x) = \frac{1}{2}\left(\phi(x + \delta_1(x)) + \phi(x + \delta_2(x))\right),$$

$$v(x) = \phi(x + \delta_1(x)) - \phi(x + \delta_2(x)),$$

$$\gamma = \frac{1}{2}(\delta_1 + \delta_2)$$

we write

$$u = u_1 + u_2$$

with

$$u_1 = \mathbb{P}\left\{(\nabla\delta)^*\psi\right\}$$

and

$$u_2 = \mathbb{P}\left\{(\mathbf{I} + \nabla\gamma)^* v\right\}$$

Now the bound

$$\|u_2\|_{0,\mu} \leq C\|\delta\|_{0,\mu}\|\phi\|_{1,\mu}$$

is obtained in the same way as the bound in Proposition 1. (Actually ϕ Lipschitz is enough here.)

The dangerous term is u_1 because it contains $\nabla\delta$. But here we can "integrate by parts" and write

$$u_1 = -\mathbb{P}\left\{(\nabla\psi)^*\delta\right\}$$

because of incompressibility. The matrix $\nabla\psi$ is bounded in $C^{0,\mu}$ and the bound follows again easily, as the bounds in Proposition 1. This ends the proof of Proposition 4.

We draw the attention to the fact that the presence of the $*$ (transpose) operation is essential for the "integration by parts" to be allowed.

Proof of Proposition 3. We denote $\theta_j = \mathcal{S}\delta_j$, $u_j = W(\delta_j, \phi)$, $u = u_1 - u_2$, $\theta = \theta_1 - \theta_2$ and write

$$\frac{\partial\theta}{\partial t} + \frac{u_1 + u_2}{2}\cdot\nabla\theta + u\cdot\nabla\left(\frac{\theta_1 + \theta_2}{2}\right) + u = 0$$

We consider the characteristics $X(a,t)$ defined by

$$\frac{dX}{dt} = \frac{u_1+u_2}{2}(X,t), \ \ X((a,0) = a$$

and note that in view of Proposition 1 and the assumption $\delta_j \in \mathcal{I}$, the characteristics are well defined for $0 \le t \le T$, their inverse $A(x,t) = X^{-1}(x,t)$ (the "back-to-labels" map) is defined too. Moreover,

$$\sup_{t,a}\left|\frac{\partial X}{\partial a}\right| \le C$$

and

$$\sup_{t,x}\left|\frac{\partial A}{\partial x}\right| \le C$$

holds with a constant C depending on μ alone. Consider now the function

$$F(x,t) = u\cdot\nabla\left(\frac{\theta_1+\theta_2}{2}\right) + u.$$

Solving by the method of characteristics we obtain

$$\theta(x,t) = -\int_0^t F(X(A(x,t),s),s)ds.$$

Using Proposition 4 in conjunction with the bounds in Propositions 1 and 2 we see that $F(x,t)$ is bounded (uniformly in time) in $C^{0,\mu}$:

$$\sup_t \|F(\cdot,t)\|_{0,\mu} \le C\|\phi\|_{1,\mu}\|\delta\|_{0,\mathcal{P}}$$

Compositions with the uniformly Lipschitz X and A are harmless and we obtain the desired result

$$\|\theta\|_{0,\mathcal{P}} \le C\|\delta\|_{0,\mathcal{P}}.$$

This ends the proof of Proposition 3.

Proof of Theorem 2. The proof follows now using successive approximations. Starting with a first guess $\delta_1 \in \mathcal{I}$ we define inductively

$$\delta_{n+1} = \mathcal{S}\delta_n \in \mathcal{I}.$$

Proposition 3 mplies that the sequence δ_n converges rapidly in the $C^{0,\mu}$ topology to a limit δ. Because $\mathcal{I}$ is convex it contains this weaker limit point, $\delta \in \mathcal{I}$. This is done for instance using The Littlewood-Paley characterization of C^s spaces.

Lemma 3. *Let $0 < s < 1$, and f_n be a sequence of functions uniformly bounded in $C^{1,s}$,*

$$\|f_n\|_{1,s} \leq C$$

and converging to f in C^s. Then $f \in C^{1,s}$ and

$$\|f\|_{C^{1,s}} \leq C.$$

The proof of the lemma is easy if one knows the fact that

$$\|f\|_{1,s} = \sup_j 2^{(1+s)j} \|\Delta_j f\|_{L^\infty}$$

where $f \sim \sum \Delta_j f$ is a Littlewood-Paley decomposition. Returning to the proof, because $\mathcal{S}$ has the weak Lipschitz property of Proposition 3 it follows that $\mathcal{S}\delta = \delta$. This actually means that $A = x + \delta(x,t)$ solves the active scalars formulation of the Euler equations and that $u = W[\delta, \phi]$ solves the usual Eulerian formulation.

Now let us consider the case of decay at infinity. It actually is instructive to look at this case because it illuminates the difference between ϕ, u, W on the one hand and x, δ, Θ on the other hand; the function spaces need to be modified in a natural fashion to accommodate this difference. The issue of decay at infinity is both a physical one – the total kinetic energy must be defined, and a mathematical one – $\mathbb{P}$ must be defined. But apart from this, the decay at infinity requirement does not hinder the proof in any respect.

Theorem 3. *Let ϕ be a $C^{1,\mu}$ velocity that is square integrable*

$$\int |\phi(x)|^2 dx < \infty$$

and whose curl is integrable to some power $1 < q < \infty$,

$$\int |\nabla \times \phi(x)|^q dx < \infty.$$

Then for ϵ sufficiently small there exists a time interval $[0,T]$ and a $C^{1,\mu}$ function $\delta(x,t)$ such that

$$\sup_t \|\nabla \delta(\cdot, t)\|_{0,\mu} \leq \epsilon$$

and such that $x + \delta(x,t)$ solves the active scalars formulation of the Euler equation. The velocity corresponding to this solution belongs to $C^{1,\mu}$, is square integrable and the vorticity is integrable to power q.

The proof follows along the same lines as above. Because ϕ enters linearly in the expression for W and because we control $\nabla\delta$ uniformly, issues of decay at infinity of do not arise. In other words the function space for velocities does not need to be a Banach algebra, rather a module over the Banach algebra of the δ variables, which variables need not decay at infinity.

Remark. The fact that $u \in C^{1,\mu}$ implies that the lagrangian flow map $X(a,t)$ exists and it is $C^{1,\mu}$. The proof of the theorem assures that $A(\cdot,t) \in C^{1,\mu}$. The fact that $A = X^{-1}$ follows by the chain rule. Thus, we do not need $u \in C^2$, $X \in C^2$ for invertibility of the Lagrangian map.