Euler Equations: derivation, basic invariants and formulae

Mat 529, Lesson 1.

1 Derivation

The incompressible Euler equations are

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \tag{1}$$

coupled with

$$\nabla \cdot u = 0. \tag{2}$$

The unknown variable is the velocity vector $u = (u_1, u_2, u_3) = u(x, t)$, a function of $x \in \mathbb{R}^3$ (or $x \in \mathbb{T}^3$) and $t \in \mathbb{R}$. The pressure $p(x, t)$ is also an unknown. The notation $u \cdot \nabla u$ stands for

$$(u \cdot \nabla u)_i = \sum_j u_j \partial_j u_i$$

and $u \cdot \nabla u$ is the same as $(\nabla u)u$, the product of the square matrix (∇u) and the column vector u. We use summation convention unless explicitly stating the contrary. We also use the mechanics notation

$$u_{i,j} = \partial_j u_i,$$

sometimes even without the comma. The material derivative is denoted

$$D_t = \partial_t + u \cdot \nabla$$

when it is clear which u we are referring to. We use the notation ∇ for the gradient, $\nabla f = (\partial_1 f, \partial_2 f, \partial_3 f)$, sometimes indicating the variable ∇_x or ∇_a. The maps

$$X(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}^3, \quad a \mapsto X(a, t)$$

with

$$X(a,0) = a$$

obeying the ODEs

$$\frac{d}{dt}X(a,t) = u(X(a,t),t) \tag{3}$$

are the Lagrangian flow maps, and the map $t \mapsto X(a,t)$, for a fixed, is a Lagrangian trajectory, or path. We refer to a as a "label". When we consider $X(\cdot,t)$ as a change of variables, we have the following elementary but fundamental transport lemma:

Lemma 1. *Let $u(x,t) \in L^\infty([0,T],C^1(\mathbb{R}^3))$. Let $f \in L^\infty([0,T],L^1(\mathbb{R}^3))$ be given and smooth enough, $f \in C^1(\mathbb{R}^3 \times \mathbb{R})$. Let $\Omega_0 \subset \mathbb{R}^3$, and let $\Omega_t = \{x \mid \exists a \in \Omega_0,\ x = X(a,t)\}$. Then*

$$\frac{d}{dt}\int_{\Omega_t} f(x,t)dx = \int_{\Omega_t} (\partial_t f + \nabla \cdot (uf))(x,t)dx$$

Proof. We start by recalling that $X(a,t)$ is well defined for each a, differentiable in a, and that

$$\frac{d}{dt}(\nabla_a X) = (\nabla_x u(X(a,t),t)))(\nabla_a X) \tag{4}$$

holds for any a. Consequently

$$\frac{d}{dt}\det(\nabla_a X) = (Tr(\nabla u))\det(\nabla_a X). \tag{5}$$

From now on we will often drop arguments; $Tr(\nabla u)$ is in fact $Tr(\nabla_x u) \circ X$, i.e. it is the trace of the matrix ∇u computed at $X(a,t)$. We also denote by

$$Tr(\nabla u) = div\, u$$

Because the initial data for $\nabla_a X$ is the identity matrix $\mathbb{I}$, and $1 = \det \mathbb{I}$, we have that

$$det(\nabla_a X)(a,t) = \exp\left(\int_0^t (div\, u)(X(a,s),s)ds\right), \tag{6}$$

and so the determinant is positive for all time. Wee start by changing variables in the integral

$$\int_{\Omega_t} f dx = \int_{\Omega_0} (f \circ X)\det(\nabla_a X)da$$

and then we differentiate in time:

$$\frac{d}{dt}\int_{\Omega_t} f dx = \int_{\Omega_0} \left[\partial_t f(X(a,t),t) + \partial_t X(a,t)\cdot \nabla_x f(X(a,t))\right] \det \nabla_a X(a,t) da$$
$$+ \int_{\Omega_0} f \frac{d}{dt} \det(\nabla_a X) da = \int_{\Omega_0} \left[(\partial_t f + u\cdot\nabla f + (\nabla\cdot u) f)\circ X\right] \det(\nabla_a X) da$$
$$= \int_{\Omega_t} \left[\partial_t f + \nabla\cdot(uf)\right] dx,$$

and this ends the proof of the lemma.
We note from (6) that if $div\, u = 0$, then

$$\det(\nabla_a X) = 1, \tag{7}$$

and the flow map does not change volumes: it is incompressible. We derive now the incompressible Euler equations formally from an action principle. We consider two fixed time instants t_1, t_2 and the action

$$\mathcal{A}[X] = \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \left|\partial_t X \circ X^{-1}\right|^2 dxdt$$

Until now we have not concerned ourselves with X^{-1}; it is time we do, because it will play a major role in these notes. Because $a \mapsto X(a,t)$ is a map from Lagrangian coordinates called labels to Eulerian coordinates x, it is convenient to denote, when no confusion can arise,

$$X^{-1}(x,t) = A(x,t),$$

and call it the "back-to-labels" map. (Although I gave this name a bit tongue-in-cheek, it caught on, just as the "active" scalars name did. I gave the active scalars name in order to contrast them with passive scalars, which are tracers. More about active scalars later in the course.)
Let us remark that the flow map $a \mapsto X(a,t)$ is globally invertible under general conditions.

Lemma 2. *If $u \in L^1([0,T], L^\infty(\mathbb{R}^3)\cap C^2(\mathbb{R}^3))$ then $X(a,t)$ is invertible for $t \in [0,T]$.*

Note that we do not need $div\, u = 0$.
Proof. First of all, clearly, from the implicit function theorem, because $\nabla_a X$ is invertible, it follows that the flow map is locally injective. Now consider $R > \int_0^T \|u\|_{L^\infty(\mathbb{R}^3)} dt$. Let K_b be the closed ball of radius R around b. Notice that if $a \notin K_b$, then the equation $X(a,t) = b$ does not have any

solution for $0 \leq t \leq T$, simply because $X(a,t)$ starts at a and travels at most $\int_0^T \|u\|_{L^\infty(\mathbb{R}^3)} dt$ far from a. It follows that the number

$$\#\{a \mid X(a,t) = b\} = \#\{a \in K_b \mid X(a,t) = b\} = n_b(t)$$

is finite, continuous in t and locally constant in t. Indeed, the finiteness follows at fixed t from the implicit function theorem, because the set of solutions is discrete. In order to show continuity, we consider any fixed time t_0 and a fixed solution $X(a_0, t_0) = b$. We define now a path $\alpha(t)$ by the ODE

$$\frac{d}{dt}\alpha(t) = -(\nabla_a X)^{-1}(\alpha(t), t)(\partial_t X)(\alpha(t), t)$$

with initial data $\alpha(t_0) = a_0$. By ODE theory this $\alpha(t)$ exists and is unique for $t - t_0$ small. It follows that

$$\frac{d}{dt}X(\alpha(t), t) = 0$$

and thus $X(\alpha(t), t) = b$. This shows that $n_b(t)$ is continuous and locally constant. It always takes therefore the value it takes at $t = 0$, i.e. $n_b(t) = 1$ for any b.

We consider now variations of X, i.e. we take a family X_ϵ of flow maps. It is possible and mathematically more satisfactory to take these to be incompressible, but we will use a general deformation, so we consider given invertible flow maps $a \mapsto X_\epsilon(a,t)$ for small ϵ with only X_0 volume-preserving. We use the notation A_ϵ for the inverse. We assume that the end points are fixed $X_\epsilon(a, t_i) = X_0(a, t_i)$, $i = 1, 2$. This so that we do not incur boundary terms when we integrate by parts in t. We denote

$$u_\epsilon = \partial_t X_\epsilon \circ X_\epsilon^{-1}, \quad Y_\epsilon = \frac{d}{d\epsilon}X_\epsilon, \quad \eta_\epsilon = Y_\epsilon \circ X_\epsilon^{-1}, \quad D_t^\epsilon = \partial_t + u_\epsilon \cdot \nabla$$

Denote

$$\mathcal{A}_\epsilon = \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^3} |u_\epsilon|^2 dx dt$$

We start by computing $\frac{d}{d\epsilon}X_\epsilon^{-1}$. Differentiating $a = X_\epsilon^{-1}(X_\epsilon(a))$ we obtain

$$(\frac{d}{d\epsilon}X_\epsilon^{-1})(X_\epsilon(a)) + (\nabla_x X_\epsilon^{-1})(X_\epsilon(a))Y_\epsilon(a) = 0,$$

and reading at $a = X_\epsilon^{-1}(x)$ we have

$$\frac{d}{d\epsilon}X_\epsilon^{-1} = -\eta_\epsilon \cdot \nabla X_\epsilon^{-1}.$$

Armed with this we compute $\frac{d}{d\epsilon}u_\epsilon$:

$$\begin{array}{l} \frac{d}{d\epsilon}u_\epsilon = \frac{d}{d\epsilon}\left(\partial_t X_\epsilon(X_\epsilon^{-1})\right) \\ = \partial_t Y_\epsilon(X_\epsilon^{-1}) + (\nabla_a(\partial_t X_\epsilon(X_\epsilon^{-1})))\frac{d}{d\epsilon}X_\epsilon^{-1} \\ = \partial_t Y_\epsilon(X_\epsilon^{-1}) - (\nabla_a(\partial_t X_\epsilon(X_\epsilon^{-1})))\eta_\epsilon \cdot \nabla X_\epsilon^{-1} \\ = \partial_t Y_\epsilon(X_\epsilon^{-1}) - \eta_\epsilon \cdot \nabla u_\epsilon \end{array}$$

Now we note that

$$\partial_t Y_\epsilon(X_\epsilon^{-1}) = (\partial_t + u_\epsilon \cdot \nabla)\eta_\epsilon$$

Indeed, this follows because

$$(\partial_t + u_\epsilon \cdot \nabla)Y_\epsilon(X_\epsilon^{-1}) = \partial_t Y_\epsilon(X_\epsilon^{-1}) + \nabla_a Y_\epsilon(X_\epsilon^{-1})\left(\partial_t + u_\epsilon \cdot \nabla\right)\left(X_\epsilon^{-1}\right)$$

and the fact that

$$(\partial_t + u_\epsilon \cdot \nabla)X_\epsilon^{-1} = 0$$

which is just the time derivative of $a = X_\epsilon^{-1}(X_\epsilon(a))$ read at $a = X_\epsilon^{-1}(x)$. In our notation we have thus

$$\frac{d}{d\epsilon}A_\epsilon = -\eta_\epsilon \cdot \nabla A_\epsilon$$

and

$$\frac{d}{d\epsilon}u_\epsilon = D_t^\epsilon \eta_\epsilon - \eta_\epsilon \cdot \nabla u_\epsilon$$

The action principle states that the Euler equations are obtained by seeking *least action among all volume preserving diffeomorphisms*. We treat "volume preserving" as a side constraint in a variational principle. Note that $div\, u = 0$ is also necessary not only sufficient for $det(\nabla_a X) = 1$, in view of (5). We use a Lagrange multiplier q that is independent of ϵ and compute

$$\frac{d}{d\epsilon}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\frac{1}{2}|u_\epsilon|^2 + q(div\, u_\epsilon)dxdt$$

and require it to vanish at $\epsilon = 0$. Stationarity with respect to q simply means $divu_{\epsilon|\,\epsilon=0} = 0$. The calculation gives thus

$$\begin{array}{l} \frac{d}{d\epsilon}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\frac{1}{2}|u_\epsilon|^2 + q(div\, u_\epsilon)dxdt \\ = \int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left(u_\epsilon \cdot \frac{d}{d\epsilon}u_\epsilon - \nabla q \cdot \frac{d}{d\epsilon}u_\epsilon\right)dxdt \\ = \int_{t_1}^{t_2}\int_{\mathbb{R}^3}(u_\epsilon - \nabla q)(D_t^\epsilon \eta_\epsilon - \eta_\epsilon \cdot \nabla u_\epsilon)dxdt \end{array}$$

Now

$$\begin{aligned}
&-\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla q(D_t^\epsilon\eta_\epsilon - \eta_\epsilon\cdot\nabla u_\epsilon)dxdt\\
&=\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(\nabla q(\eta_\epsilon\cdot\nabla u_\epsilon)+\eta_\epsilon\cdot\nabla D_t^\epsilon q-\eta_\epsilon\cdot(\nabla u_\epsilon)^*\nabla q)dxdt\\
&+\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(div\, u_\epsilon)(\eta_\epsilon\cdot\nabla q)dxdt\\
&=\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left[\eta_\epsilon\cdot\nabla D_t^\epsilon q+(div\, u_\epsilon)(\eta_\epsilon\cdot\nabla q)\right]dxdt
\end{aligned}$$

and therefore

$$\begin{aligned}
&\tfrac{d}{d\epsilon}\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\tfrac{1}{2}|u_\epsilon|^2+q(div\, u_\epsilon)dxdt\\
&=\int_{t_1}^{t_2}\int_{\mathbb{R}^3}[u_\epsilon\cdot(D_t^\epsilon\eta_\epsilon-\eta_\epsilon\cdot\nabla u_\epsilon)+\eta_\epsilon\cdot\nabla D_t^\epsilon q+(div\, u_\epsilon)(\eta_\epsilon\cdot\nabla q)]dxdt\\
&=\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left[\eta_\epsilon\cdot(-D_t^\epsilon u_\epsilon-(\nabla u_\epsilon)^*u_\epsilon+\nabla D_t^\epsilon q)+(div\, u_\epsilon)(\eta_\epsilon\cdot(\nabla q-u_\epsilon))\right]dxdt\\
&=-\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\eta_\epsilon\cdot\left[(D_t^\epsilon u_\epsilon+\nabla p_\epsilon)+(div\, u_\epsilon)(\eta_\epsilon\cdot(u_\epsilon-\nabla q)))\right]dxdt
\end{aligned}$$

where

$$p_\epsilon=\frac{1}{2}|u_\epsilon|^2-D_t^\epsilon q$$

Now we can set $\epsilon=0$, use $div\, u_0=0$, let η_0 be arbitrary and deduce by setting the derivative to zero for all η_0 that u must obey the Euler equations (1).

Note that

$$\frac{d}{d\epsilon}(\nabla_a X_\epsilon)=(\nabla_x\eta_\epsilon\circ X_\epsilon)(\nabla_a X_\epsilon)$$

(which follows by differentiating in a the tautology $Y_\epsilon(a)=\eta_\epsilon(X_\epsilon(a)))$ implies that

$$\frac{d}{d\epsilon}\det(\nabla_a X_\epsilon)=(div\,\eta_\epsilon)\det(\nabla_a X_\epsilon)$$

and therefore X_ϵ is incompressible if and only if η_ϵ is divergence-free. Thus, if we would have used deformations X_ϵ which are themselves incompressible, we wouldn't have needed the Lagrange multiplier q, and $div\, u_\epsilon=0$ would have been true for nonzero ϵ, but we would have deduced only that $D_t u+\nabla\frac{|u|^2}{2}$ is perpendicular on all divergence-free vectors, and then that would imply it is a gradient.

2 Basic invariants and formulae

Classical solutions of the Euler equations conserve energy

$$\frac{d}{2dt}\int_{\mathbb{R}^3}|u(x,t)|^2dx=0.$$

The vorticity

$$\omega(x,t) = \nabla \times u(x,t)$$

is an important field associated with the velocity. The gradient of velocity can be decomposed in its symmetric and anti-symmetric parts,

$$\nabla u = S + J$$

where

$$S = \frac{1}{2}\left((\nabla u) + (\nabla u)^*\right)$$

is the rate of strain, and

$$J = \frac{1}{2}\left((\nabla u) - (\nabla u)^*\right)$$

is given by the vorticity, i.e., for any vector v, the matrix J applied to v yields

$$2Jv = \omega \times v.$$

Taking the gradient of (1) we obtain the equation for the gradient matrix

$$(\partial_t + u\cdot\nabla)(\nabla u) + (\nabla u)^2 + (\nabla\nabla p) = 0. \tag{8}$$

The rate of strain obeys

$$(\partial_t + u\cdot\nabla) S + S^2 + J^2 + (\nabla\nabla p) = 0, \tag{9}$$

where we can find that

$$J^2 = -\frac{1}{4}|\omega|^2 P_{\widehat{\omega}}^{\perp} \tag{10}$$

with $\widehat{\omega} = \frac{\omega}{|\omega|}$ and where

$$P_{\xi}^{\perp} v = v - (v\cdot\xi)\xi$$

for $\xi \in S^2$ and $v \in \mathbb{R}^3$. The equation obeyed by J is

$$(\partial_t + u\cdot\nabla) J + SJ + JS = 0, \tag{11}$$

and, in terms of ω this is

$$(\partial_t + u\cdot\nabla)\omega - \omega\cdot\nabla u = 0. \tag{12}$$

This equation is equivalent with the commutator equation

$$[\partial_t + u \cdot \nabla, \omega \cdot \nabla] = 0. \tag{13}$$

The fact that the commutator vanishes is the essence of a basic hydrodynamic fact, Ertel's theorem. Ertel's theorem says that if k is a constant of motion, i.e.

$$D_t k = 0$$

then $\omega \cdot \nabla k$ is also a constant of motion,

$$D_t(\omega \cdot \nabla k) = 0,$$

a fact that follows immediately from the commutation. A consequence of this is the fact that

$$\omega_3 = u_{2,1} - u_{1,2}$$

is conserved under two-dimensional flow, i.e. flow for which

$$D_t x_3 = 0.$$

One of the most important conservation laws in fluid mechanis is the conservation of circulation. This says that

$$\oint_{\gamma_t} u \cdot dx \tag{14}$$

is constant in time, where $\gamma_t = X(\gamma_0, t)$ is a closed loop transported by the flow. The proof of this fact is easy in Lagrangian coordinates. If γ_0 is given by a parameterization $a = \alpha(s)$ with $s \in [0,1]$, with $\alpha(0) = \alpha(1)$, then γ_t is given by $x = X(\alpha(s), t)$. Then integral is

$$\oint_{\gamma_t} u \cdot dx = \int_0^1 \partial_t X_j(\alpha(s), t)(\nabla_a X_j(\alpha(s), t)) \frac{d\alpha}{ds} ds$$

The Euler equations in Lagrangian form are just

$$\frac{d^2}{dt^2} X(a,t) = -\nabla_x p(X(a,t), t) \tag{15}$$

where ∇_x is Eulerian gradient. Differentiating

$$\frac{d}{dt}\left(\partial_t X \cdot \partial_{a_k} X\right) = -\nabla_x p \cdot \partial_{a_k} X + \frac{1}{2} \partial_{a_k} \left|\partial_t X\right|^2 = \partial_{a_k} q$$

with
$$q = -p \circ X + \frac{1}{2} |\partial_t X|^2 .$$
Therefore
$$\frac{d}{dt} \oint_{\gamma_t} u \cdot dx = \int_0^1 \frac{d}{ds} q(X(\alpha(s)), t) ds = 0$$
because the end points coincide.
The Cauchy formula is
$$\omega(X(a,t),t) = (\nabla_a X(a,t))\omega_0(a). \tag{16}$$
This is easily verified by checking that
$$\widetilde{\omega}(a,t) = \omega(X(a,t),t)$$
and
$$\zeta(a,t) = \nabla_a X(a,t)\omega_0(a)$$
obey the same ODE (in view of (4) and (12)) and have the same initial data, so they must coincide. An immediate and important consequence of Cauchy's formula is the Helmholtz theorem that states that *vortex lines are material*. Vortex lines are integral curves of the vector field $\omega(\cdot,t)$ at fixed time, that is, they are curves in space, such that the tangent at each point on the curve is parallel to the direction $\widehat{\omega}$ at that point. The fact that they are material means that they are transported by the flow $X(\cdot,t)$, i.e., the image of a vortex line under X is again a vortex line. This is clear from Cauchy's formula because if γ_0 is curve $a = \alpha(s)$ parameterized by some parameter $s \in [0,1]$ and if it is a vortex line, then $\frac{d\alpha}{ds} = c(s)\omega_0(\alpha(s))$ with some constant $c(s)$. At later time γ_t is given by $x = X(\alpha(s),t)$ and therefore its tangent is given by $\tau(s) = (\nabla_a X)\frac{d\alpha}{ds}$. This is then $\tau(s) = c(s)(\nabla_a X)\omega_0(\alpha(s))$ and by the Cauchy formula $\tau(s) = c(s)\omega(X(\alpha(s),t),t)$. The Cauchy invariants are
$$\epsilon_{ijk}\dot{X}_k^l X_k^l = \omega_0^i, \quad i = 1,2,3. \tag{17}$$
where ϵ_{ijk} is the signature of the permutation $(1,2,3) \mapsto (i,j,k)$, taken to be zero if two indices are the same, and we denoted for graphical ease
$$\dot{X}_k^l = \partial_t \partial_{a_k} X_l(a,t), \quad X_k^l = \partial_{a_k} X_l(a,t), \quad \omega_0 = \omega_0(a,t).$$

The Cauchy invariants are a direct consequence of Cauchy's formula. Here is the very short proof. We start with the useful formula

$$X_k^p = \frac{1}{2}\epsilon_{pmn}\epsilon_{krs}A_m^r A_n^s \tag{18}$$

where we denote A_m^i the matrix elements of $(\nabla_x X^{-1})\circ X$ which is the inverse of the matrix $\nabla_a X$. The identity above uses the fact that the determinat equals one. From this we obtain the formula

$$\epsilon_{mnp}A_m^i A_n^j = \epsilon_{ijk}X_k^p. \tag{19}$$

Indeed this is true by index orgy:

$$\begin{array}{l}\epsilon_{ijk}X_k^p = \frac{1}{2}\epsilon_{pmn}\epsilon_{ijk}\epsilon_{krs}A_m^r A_n^s \\ = \frac{1}{2}\epsilon_{pmn}\left(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}\right)A_m^r A_n^s \\ = \epsilon_{pmn}A_m^i A_n^j.\end{array}$$

Now, after applying the inverse matrix A_m^i to both sides of the Cauchy formula we have

$$\begin{array}{l}\omega_0^i = A_m^i\omega^m = A_m^i\epsilon_{mnp}\partial_n(u^p) = A_m^i\epsilon_{mnp}\dot{X}_j^p A_n^j \\ = \epsilon_{ijk}\dot{X}_j^p X_k^p\end{array}$$

by (19), and that gives the Cauchy invariant.
The helicity is the integral

$$\int_{\mathbb{R}^3}(u\cdot\omega)dx.$$

This is constant in time. Indeed, using the transport lemma, (1) and (12) we have

$$\frac{d}{dt}\int_{\mathbb{R}^3}(u\cdot\omega)dx = \int_{\mathbb{R}^3} div\,\left(\omega\left(-p+\frac{|u|^2}{2}\right)\right)dx = 0.$$

There exist local versions of this (on vortex tubes, i.e. on regions whose boundaries are foliated by vortex lines). A magnetization variable is a vector w that obeys

$$\partial_t w + u\cdot\nabla w + (\nabla u)^* w = 0.$$

For any such variable, the scalar $k = w\cdot\omega$ is conserved

$$(\partial_t + u\cdot\nabla)(w\cdot\omega) = 0.$$

The Weber formula is

$$u = \mathbb{P}((\nabla A)^* u_0(A)) \tag{20}$$

Here the Leray-Hodge matrix of operators $\mathbb{P}$ is given by

$$\mathbb{P}_{jl} = \delta_{jl} - \partial_j \Delta^{-1} \partial_l = \delta_{jl} + R_j R_l, \tag{21}$$

with

$$R_j = \partial_j (-\Delta)^{-\frac{1}{2}}$$

the Riesz operators. Note that $\mathbb{P}$ satisfies the basic property that

$$\mathbb{P}_{jl} \partial_l f = 0.$$

The derivation of the Weber formula is as follows: we start with (15) and apply $(\nabla_a X)^*$:

$$\frac{\partial^2 X(a,t)}{\partial t^2} (\nabla_a X(a,t))^* = -(\nabla_a \tilde{p})(a,t) \tag{22}$$

or, on components

$$\frac{\partial^2 X^j(a,t)}{\partial t^2} \frac{\partial X^j(a,t)}{\partial a_i} = -\frac{\partial \tilde{p}(a,t)}{\partial a_i}. \tag{23}$$

where

$$\tilde{p}(a,t) = p(X(a,t),t). \tag{24}$$

Pulling out a time derivative in the left-hand side we obtain

$$\frac{\partial}{\partial t} \left[\frac{\partial X^j(a,t)}{\partial t} \frac{\partial X^j(a,t)}{\partial a_i} \right] = -\frac{\partial \tilde{q}(a,t)}{\partial a_i} \tag{25}$$

where

$$\tilde{q}(a,t) = \tilde{p}(a,t) - \frac{1}{2} \left| \frac{\partial X(a,t)}{\partial t} \right|^2. \tag{26}$$

We integrate (25) in time, fixing the label a:

$$\frac{\partial X^j(a,t)}{\partial t} \frac{\partial X^j(a,t)}{\partial a_i} = u^i_{(0)}(a) - \frac{\partial \tilde{n}(a,t)}{\partial a_i}, \tag{27}$$

where

$$\tilde{n}(a,t) = \int_0^t \tilde{q}(a,s) ds \tag{28}$$

and

$$u_{(0)}(a) = \frac{\partial X(a,0)}{\partial t} \tag{29}$$

is the initial velocity. Note that $\tilde{n}$ has dimensions of circulation or of kinematic viscosity (length squared per time). The conservation of circulation

$$\oint_\gamma \frac{\partial X(\gamma,t)}{\partial t} \cdot d\gamma = \oint_\gamma \frac{\partial X(\gamma,0)}{\partial t} \cdot d\gamma$$

follows directly from the form (27). Applying $[(\nabla_a X)^*]^{-1}$ to (27), and reading at $a = A(x,t)$, we obtain the formula

$$u^i(x,t) = \left(u^j_{(0)}(A(x,t))\right) \frac{\partial A^j(x,t)}{\partial x_i} - \frac{\partial n(x,t)}{\partial x_i} \tag{30}$$

where

$$n(x,t) = \tilde{n}(A(x,t)). \tag{31}$$

Because u is divergence-free, the Weber formula follows.

The equation (30) shows that the general Eulerian velocity can be written in a form that generalizes the Clebsch variable representation:

$$u = (\nabla A)^* B - \nabla n \tag{32}$$

where $B = u_{(0)}(A(x,t))$ and, consequently

$$D_t B = 0,$$

because of the basic

$$(\partial_t + u \cdot \nabla) A = 0. \tag{33}$$

Conversely, and somewhat more generally, if one is given a pair of M-uples of active scalars $A = (A^1(x,t), \cdots, A^M(x,t))$ and $B = (B^1(x,t), \cdots, B^M(x,t))$ of arbitrary dimension M, such that the active scalar equations $D_t A_i = D_t B_i = 0$ hold, and if u is given by

$$u(x,t) = \sum_{k=1}^{M} B^k(x,t) \nabla_x A^k(x,t) - \nabla_x n \tag{34}$$

with some function n, then it follows that u solves the Euler equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla \pi = 0$$

where

$$\pi = D_t n + \frac{1}{2}|u|^2$$

Indeed, the only thing one needs is the kinematic commutation relation

$$D_t \nabla f = \nabla D_t f - (\nabla u)^* \nabla f \tag{35}$$

that holds for any scalar function f. The kinematic commutation relation (35) is a consequence of the chain rule, so it requires no assumption other than smoothness. Differentiating (34) and using the active scalar equations it follows that

$$D_t(u) = -\sum_{k=1}^{M}((\nabla_x u)^* \nabla_x A^k) B^k - \nabla_x(D_t n) + (\nabla_x u)^* \nabla n =$$

$$-\nabla_x(D_t n) - (\nabla_x u)^* \left[\sum_{k=1}^{M} (\nabla_x A^k) B^k - \nabla_x n\right] =$$

$$-\nabla_x(D_t n) - (\nabla_x u)^* u = -\nabla_x(\pi).$$

Clebsch variables are obtained for $M = 1$. Note that for Clebsch variables (B^1, A^1) the vorticity is given by

$$\omega = \nabla B^1 \times \nabla A^1$$

and thus the helicity vanishes. Not all flows have zero helicity, and thus Clebsch variables do not represent all flows.