Nernst-Planck-Navier-Stokes Systems Far From Equilibrium

Peter Constantin, Mihaela Ignatova, and Fizay-Noah Lee

ABSTRACT. We consider ionic electrodiffusion in fluids, described by the Nernst-Planck-Navier-Stokes system. We prove that the system has global smooth solutions for arbitrary smooth data in bounded domains with smooth boundary in three space dimensions, in the following situations. We consider arbitrary positive Dirichlet boundary conditions for the ionic concentrations, arbitrary Dirichlet boundary conditions for the potential, arbitrary positive initial concentrations, and arbitrary regular divergence-free initial velocities. Global regularity holds for any positive, possibly different diffusivities of the ions, in the case of two ionic species, coupled to Stokes equations for the fluid. The result also holds in the case of Navier-Stokes coupling, if the velocity is regular. The same global smoothness of solutions is proved to hold for arbitrarily many ionic species as well, but in that case we require all their diffusivities to be the same.

1. Introduction

The Nernst-Planck-Navier-Stokes system describes the evolution of ions in a Newtonian fluid [**10**]. Several species of ions, with different valences $z_i \in \mathbb{R}$ diffuse with diffusivities $D_i > 0$, and are carried by an incompressible fluid with constant density and with velocity u, and by an electrical field generated by the local charge ρ and by voltage applied at the boundaries. The system of equations is

$$\partial_t c_i + u \cdot \nabla c_i = D_i \mathrm{div}\,(\nabla c_i + z_i c_i \nabla \Phi), \tag{1}$$

$i = 1, 2, \dots m$, coupled to the Poisson equation

$$-\epsilon \Delta \Phi = \sum_{i=1}^{m} z_i c_i = \rho, \tag{2}$$

and to the Navier-Stokes equations

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = -K\rho\nabla\Phi, \quad \nabla \cdot u = 0, \tag{3}$$

or to the Stokes equation,

$$\partial_t u - \nu \Delta u + \nabla p = -K\rho\nabla\Phi, \quad \nabla \cdot u = 0. \tag{4}$$

The function $c_i(x,t)$ represents the local concentration of the i-th species, and Φ is an electrical potential created by the charge density ρ. The positive constant ϵ is proportional to the square of the Debye length. The kinematic viscosity of the fluid is $\nu > 0$ and $K > 0$ is a coupling constant with units of energy per unit mass. This constant is proportional to the product of the Boltzmann's constant K_B and the absolute temperature T_K. The potential Φ has been nondimensionalized so that $\frac{k_B T_K}{e}\Phi$ is the physical electrical potential, where e is elementary charge. The charge density ρ has been nondimensionalized so that $e\rho$ is the physical electrical charge density.

The boundary conditions for c_i are inhomogeneous Dirichlet,

$$c_i(x,t)_{|\,\partial\Omega} = \gamma_i(x) \tag{5}$$

The boundary conditions for Φ are inhomogeneous Dirichlet

$$\Phi(x,t)_{|\,\partial\Omega} = W(x) \tag{6}$$

and the boundary conditions for the Navier-Stokes or Stokes equations are homogeneous Dirichlet,

$$u_{|\,\partial\Omega} = 0. \tag{7}$$

Key words and phrases. electroconvection, ionic electrodiffusion, Poisson-Boltzmann, Nernst-Planck, Navier-Stokes.
MSC Classification: 35Q30, 35Q35, 35Q92.

The bounded connected domain Ω need not be simply connected. The functions γ_i defined on the boundary of the domain are given, smooth positive time independent functions. We denote by Γ_i positive, smooth, time independent extensions of these functions in the interior

$$\gamma_i = \Gamma_{i|\,\partial\Omega} \tag{8}$$

The function W is also a given and time independent smooth function.

The NPNS system is a well posed semilinear parabolic system. The question we are discussing is whether solutions with any smooth initial data and obeying any smooth boundary conditions exist for all time and remain smooth.

As it is well known, semilinear parabolic scalar equations can blow up in finite time. The simplest such example is a semilinear heat equation ([**8**]). Semilinear systems in which there is a single concentration carried by the gradient of a potential can also blow up. Well known examples are chemotaxis equations, such as the Keller-Segal equation ([**9**]). The system we are discussing involves the Navier-Stokes equations, where the question of global regularity is a major open problem. However, even in the absence of fluid or in the case of Stokes flow coupled to Nernst-Planck equations, the problem of global existence of smooth solutions discussed here remains open in its full generality.

Boundary conditions play an essential role in the behavior the solutions of the NPNS system. No flux (blocking) boundary conditions for the concentrations model situations in which the boundaries are impermeable to the ions. Dirichlet (selective) boundary conditions for the concentrations discussed in this paper model situations in which boundaries maintain a certain concentration of ions. Global existence and stability of solutions of the Nernst-Planck equations, uncoupled to fluids has been obtained in [**1**], [**3**], [**7**] for blocking boundary conditions in two dimensions, or in three dimensions for small data, or in a weak sense. The system coupled to fluid equations was studied in [**14**] where global existence of weak solutions is shown in two and three dimensions for homogeneous Neumann boundary conditions on the potential, a situation without boundary current. In [**13**], homogeneous Dirichlet boundary conditions on the potential are considered, and global existence of weak solutions is shown in two dimensions for large initial data and in three dimensions for small initial data (small perturbations and small initial charge). In [**2**] the problem of global regularity in two dimensions was considered for Robin boundary conditions for the potential. In [**5**] global existence of smooth solutions for blocking boundary conditions and for uniform selective (special, stable Dirichlet) boundary conditions were obtained in two space dimensions. In [**6**] blocking and uniform selective boundary conditions were used to prove nonlinear stability of Boltzmann states in three space dimensions.

While blocking and uniformly selective boundary conditions lead to stable configurations, instabilities may occur for general selective boundary conditions. These instabilities have been studied in simplified models mathematically and numerically ([**12**], [**15**]) and observed in physical experiments [**11**]. In this paper we consider the system with large data, with general selective boundary conditions, in situations in which instabilities may occur. We prove global regularity of solutions for two cases: if there are only two species (cations and anions, $m = 2$) or if there are many species, but they all have the same diffusivities ($D_1 = \cdots = D_m$). The difficulty in three dimensions, even when there is no fluid, is in bounding the nonlinear growth of the concentrations.

This paper is organized as follows. In Section 2 we prove a necessary and sufficient condition for global regularity of solutions. In the case of the Nernst-Planck system coupled to Stokes equations this condition (Theorem 2) states that, if (and only if)

$$\sup_{0\le\tau<T}\int_0^\tau\left(\int_\Omega|\rho(x,t)|^2dx\right)^2dt = B(T) < \infty \tag{9}$$

is finite, then the solution is smooth on $[0,T]$. In the case of coupling to the Navier Stokes equation, the condition (9) is supplemented by a well known condition for regularity for the Navier-Stokes equations.

In Section 3 we introduce functionals of (c_i, Φ) which are used to cancel the contribution of electrical forces in the Navier-Stokes or Stokes energy balance, at the price of certain quadratic error terms. The first functional is a sum of relative entropies and a potential norm. The second one, which is just the potential part of the first one, is weaker and has a weaker dissipation, but it introduces better error terms.

Section 4 is devoted to quadratic bounds, which imply global regularity by the criterion established in Theorem 2. It is only here that the restriction to $m = 2$ (Theorem 3) or to $D_1 = \cdots = D_m$ (Theorem 4) is used. In these two special circumstances we show that there is a cubic dissipation term proportional to $\|\rho\|_{L^3}^3$ in the evolution of the quadratic norms that we are employing to control the concentrations. This cubic term is essential, because although it has a small prefactor, it can be used to absorb all the quadratic errors ocurring in the second energy.

2. Preliminaries

We denote by C absolute constants. We denote C_Γ any constant that depends only on the parameters and boundary data of the problem, i.e. on $\nu, K, \epsilon, z_i, D_i$,, on the domain Ω itself, on norms of W and on norms of Γ_i. These constants may change form line to line, and they are explicitly computable. They do not depend on solutions, or initial data. We do not keep track of them to ease notation and focus on the ideas of the proofs.

We consider a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. We denote space $L^p(\Omega) = L^p$ and norms simply by $\|\cdot\|_{L^p}$. We denote by $H = L^2(\Omega)^3 \cap \{u \,|\, \operatorname{div} u = 0,\, \gamma_n u = 0\}$ the space of square integrable, divergence-free velocities, with vanishing normal component on the boundary, with norm $\|\cdot\|_H$. Here γ_n is the continuous linear map from $E(\Omega) = \{u \in L^2(\Omega)^3 \,|\, \operatorname{div} u \in L^2(\Omega)\}$ to $H^{-\frac{1}{2}}(\partial\Omega)$ that coincides with $u \mapsto u\cdot n$ for $u \in C^\infty(\overline{\Omega})^3$, see [**4**]. We denote by $V = H_0^1(\Omega)^3 \cap \{u \,|\, \operatorname{div} u = 0\}$ the space of divergence free vectors fields with components in $H_0^1(\Omega)$, with norm $\|\cdot\|_V$. We denote by $\mathbb{P}$ the Leray projector $\mathbb{P} : L^2(\Omega)^3 \to H$, and by A the Stokes operator

$$A = -\mathbb{P}\Delta, \qquad A : \mathcal{D}(A) \to H \tag{10}$$

where

$$\mathcal{D}(A) = H^2(\Omega)^3 \cap V. \tag{11}$$

DEFINITION 1. *We say that (c_i, Φ, u) is a strong solution of the system (1), (2), (3) or (1), (2), (4) with boundary conditions (5), (6), (7) on the time interval $[0,T]$ if $u \in L^\infty(0,T;V) \cap L^2(0,T;\mathcal{D}(A))$ and $c_i \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ solve the equations in distribution sense and the boundary conditions in trace sense.*

It is well known that strong solutions of Navier-Stokes equations are as smooth as the data permit ([**4**]). The same is true for the Nernst-Planck-Navier-Stokes equations.

THEOREM 1. *Let $c_i(0) - \Gamma_i \in H_0^1(\Omega)$, and $u(0) \in V$. There exists T_0 depending on $\|u_0\|_V$ and $\|c_i(0)\|_{H^1(\Omega)}$, the boundary conditions γ_i, W, and the parameters of the problem $(\nu, K, D_i, \epsilon, z_i)$, so that the system (1), (2), (3) (or the system (1), (2), (4)) with boundary conditions (5), (6), (7) has a unique strong solution (c_i, Φ, u) on the interval $[0, T_0]$.*

PROOF. We write

$$c_i = q_i + \Gamma_i. \tag{12}$$

and note that the equations (1) can be written as

$$\partial_t q_i + u\cdot\nabla q_i = D_i \operatorname{div}(\nabla q_i + z_i q_i \nabla\Phi) + F_i \tag{13}$$

where

$$F_i = -u\cdot\nabla\Gamma_i + D_i \operatorname{div}(\nabla\Gamma_i + z_i\Gamma_i\nabla\Phi). \tag{14}$$

The boundary conditions for q_i are homogeneous Dirichlet,

$$q_{i|\partial\Omega} = 0. \tag{15}$$

We sketch only the a priori bounds for the proof. The actual construction of solutions can be done via Galerkin approximations. Taking the scalar product of (13) with $-\Delta q_i$, we estimate the terms

$$\begin{array}{l} \left|z_iD_i\int_\Omega(\nabla q_i\cdot\nabla\Phi+q_i\Delta\Phi)\Delta q_idx\right| \le C_\Gamma\left(\|\nabla\Phi\|_{L^6}\|\nabla q_i\|_{L^3}+\|q_i\|_{L^4}\|\rho\|_{L^4}\right)\|\Delta q_i\|_{L^2} \\ \le C_\Gamma\left((\|\rho\|_{L^2}+1)\|\nabla q_i\|_{L^2}^{\frac12}\|\Delta q_i\|_{L^2}^{\frac32}+\|q_i\|_{L^2}^{\frac14}\|\rho\|_{L^2}^{\frac14}\|\rho\|_{L^6}^{\frac34}\|\nabla q_i\|_{L^2}^{\frac34}\|\Delta q_i\|_{L^2}\right) \end{array} \tag{16}$$

where we used

$$\|\nabla\Phi\|_{L^6}\le C_\Gamma(\|\rho\|_{L^2}+1) \tag{17}$$

(the inhomogeneous boundary conditions are accounted for in the added term C_Γ), embedding $H^1\subset L^6$ and interpolation. The advective term is estimated

$$\left|\int_\Omega(u\cdot\nabla q_i)\Delta q_idx\right|\le C_\Gamma\|u\|_V\|\nabla q_i\|_{L^2}^{\frac12}\|\Delta q_i\|_{L^2}^{\frac32} \tag{18}$$

The forcing term is estimated

$$\left|\int_\Omega F_i\Delta q_idx\right|\le C_\Gamma(\|u\|_H+\|\rho\|_{L^2}+1)\|\Delta q_i\|_{L^2} \tag{19}$$

where we used

$$\|\nabla\Phi\|_{L^p}\le C_\Gamma(\|\rho\|_{L^2}+1) \tag{20}$$

valid for $p\le 6$. Now we have

$$\|\rho\|_{L^p}\le C_\Gamma\sum_{i=1}^m(\|q_i\|_{L^p}+1) \tag{21}$$

and therefore, using also the Poincaré inequality we obtain

$$\frac{d}{dt}\sum_{i=1}^m\|\nabla q_i\|_{L^2}^2+\sum_{i=1}^m\|\Delta q_i\|_{L^2}^2\le C_\Gamma\left[\left(\sum_{i=1}^m\|\nabla q_i\|_{L^2}^2+1\right)^3+\|u\|_V^6\right]. \tag{22}$$

We take the scalar product of (3) with Au we obtain, using well known estimates for the NSE [**4**], a Hölder inequality, (17), (21) and the embedding $H^1\subset L^3$,

$$\begin{aligned} \frac{d}{dt}\|u\|_V^2+\nu\|Au\|_H^2 \le& C_\Gamma(\|u\|_V^6+\|\rho\nabla\Phi\|_{L^2}^2) \\ \le& C_\Gamma(\|u\|_V^6+\|\rho\|_{L^3}^2\|\nabla\Phi\|_{L^6}^2) \\ \le& C_\Gamma\left[\|u\|_V^6+\|\rho\|_{L^3}^2\left(1+\|\rho\|_{L^2}\right)^2\right] \\ \le& C_\Gamma\left[\|u\|_V^6+\left(1+\sum_{i=1}^m\|\nabla q_i\|_{L^2}^2\right)^2\right]. \end{aligned} \tag{23}$$

Adding to (22) we obtain short time control of the norms required by the definition of strong solutions. ☐

LEMMA 1. *Let (c_i,Φ,u) be a strong solution of the system (1), (2), (3) (or (1), (2), (4)) with boundary conditions (5), (6), (7) on the interval $[0,T]$. Let $p\in[2,6]$ be an even integer, take $c_i(0)\in L^p(\Omega)$ and consider the quantity*

$$\int_0^T\|\rho(t)\|_{L^2}^4dt=B(T)<\infty, \tag{24}$$

which is finite because (c_i,Φ,u) is a strong solution. Then $c_i\in L^\infty(0,T;L^p)$ and

$$\sum_{i=1}^m\|c_i(t)\|_{L^p}\le C_\Gamma\left(\sum_{i=1}^m\|c_i(0)\|_{L^p}+\int_0^T\|u\|_{L^p}dt+1\right)e^{C_\Gamma(T+B(T))}. \tag{25}$$

holds.

PROOF. We multiply the equation (13) by q_i^{p-1} and integrate. We estimate the term

$$\begin{aligned}\left|z_i D_i \int_\Omega \operatorname{div}(q_i \nabla\Phi) q_i^{p-1} dx\right| &= C_\Gamma \left|\int_\Omega q_i^{\frac{p}{2}} \nabla\Phi \cdot \nabla q_i^{\frac{p}{2}} dx\right| \\ &\le C_\Gamma \|\nabla\Phi\|_{L^6} \|q_i^{\frac{p}{2}}\|_{L^3} \|\nabla q_i^{\frac{p}{2}}\|_{L^2} \\ &\le C_\Gamma(\|\rho\|_{L^2}+1)\|q_i^{\frac{p}{2}}\|_{L^2}^{\frac{1}{2}} \|\nabla q_i^{\frac{p}{2}}\|_{L^2}^{\frac{3}{2}}\end{aligned} \tag{26}$$

where we used one integration by parts, allowed by the vanishing of q_i at the boundary and interpolation. We estimate the forcing term

$$\left|\int_\Omega F_i q_i^{p-1} dx\right| \le C_\Gamma(\|u\|_{L^p} + \|\nabla\Phi\|_{L^p} + \|\rho\|_{L^p} + 1)\|q_i\|_{L^p}^{p-1}. \tag{27}$$

Using the dissipative term

$$D_i \int_\Omega \Delta q_i q_i^{p-1} dx = -D_i \frac{4(p-1)}{p^2} \int_\Omega |\nabla q_i^{\frac{p}{2}}|^2 dx, \tag{28}$$

to absorb the corresponding term from the estimate in (26) and then discarding it, we obtain

$$\frac{d}{dt}\|q_i\|_{L^p} \le C_\Gamma(\|\rho\|_{L^2}^4 + 1)\|q_i\|_{L^p} + C_\Gamma(\|u\|_{L^p} + \|\nabla\Phi\|_{L^p} + \|\rho\|_{L^p} + 1), \tag{29}$$

and summing in i we obtain

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^m \|q_i\|_{L^p} &\le C_\Gamma(\|\rho\|_{L^2}^4 + 1)\sum_{i=1}^m \|q_i\|_{L^p} + C_\Gamma m(\|u\|_{L^p} + \|\nabla\Phi\|_{L^p} + \|\rho\|_{L^p} + 1) \\ &\le C_\Gamma(\|\rho\|_{L^2}^4 + 1)\sum_{i=1}^m \|q_i\|_{L^p} + C_\Gamma m(\|u\|_{L^p} + 1)\end{aligned} \tag{30}$$

where we used the estimate

$$\|\nabla\Phi\|_{L^p} \le C_\Gamma(\|\rho\|_{L^p} + 1) \le C_\Gamma(\sum_{i=1}^m \|q_i\|_{L^p} + 1), \tag{31}$$

from (21). Thus (25) follows from (30), concluding the proof. □

PROPOSITION 1. *Let (c_i, Φ, u) be a strong solution of the system (1), (2), (3) (or (1), (2), (4)) with boundary conditions (5), (6), (7) on the interval $[0,T]$. Consider the quantity $B(T)$ of (24). Then*

$$\sup_{t\in[0,T]} \sum_{i=1}^m \|c_i(t)\|_{H^1}^2 + \int_0^T \sum_{i=1}^m \|c_i\|_{H^2}^2\, dt \le C_\Gamma \left(\sum_{i=1}^m \|c_i(0)\|_{H^1}^2 + \int_0^T \|u\|_H^2 dt + 1\right) e^{C_\Gamma(T+R(T)+U(T))} \tag{32}$$

holds with

$$R(T) = \int_0^T \|\rho\|_{L^4}^2 dt \le C\left(\sum_{i=1}^m \|c_i(0)\|_{L^4} + \int_0^T \|u\|_{L^4} dt + 1\right)^2 e^{C_\Gamma(T+B(T))}, \tag{33}$$

and

$$U(T) = \int_0^T \|u\|_V^4 dt \tag{34}$$

PROOF. In view of (25) with $p = 4$ and the fact that $W^{1,p} \subset L^\infty$ for $p > 3$, we have that

$$\|\nabla\Phi\|_{L^\infty} \le C_\Gamma(\|\rho\|_{L^4} + 1) \le C_\Gamma\left(\sum_{i=1}^m \|c_i(0)\|_{L^4} + \int_0^T \|u\|_{L^4} dt + 1\right) e^{C_\Gamma(T+B(T))} \tag{35}$$

This is a quantitative bound in terms of the initial data and the constant $B(T)$. Using it together with $\|\rho(t)\|_{L^4}$ in the estimate of the evolution of $\|\nabla q_i\|_{L^2}$ we obtain

$$\begin{aligned}\left|z_iD_i\int_{\Omega}(\nabla q_i\cdot\nabla\Phi+q_i\Delta\Phi)\Delta q_idx\right| &\le C_{\Gamma}\left(\|\nabla\Phi\|_{L^\infty}\|\nabla q_i\|_{L^2}+\|q_i\|_{L^4}\|\rho\|_{L^4}\right)\|\Delta q_i\|_{L^2}\\ &\le C_{\Gamma}(\|\rho\|_{L^4}+1)\|\nabla q_i\|_{L^2}\|\Delta q_i\|_{L^2}\end{aligned} \tag{36}$$

instead of (16), and consequently together with (18), we obtain, after absorbing the $\|\Delta q_i\|_{L^2}$ terms from (18) and (36),

$$\frac{d}{dt}\sum_{i=1}^m\|\nabla q_i\|_{L^2}^2+\sum_{i=1}^m\|\Delta q_i\|_{L^2}^2\le C_{\Gamma}(\|\rho\|_{L^4}^2+\|u\|_V^4+1)\left(\sum_{i=1}^m\|\nabla q_i\|_{L^2}^2\right)+C_{\Gamma}(\|u\|_H^2+1) \tag{37}$$

instead of (22). Using (37) we obtain (32). □

THEOREM 2. *Let $T_1>0$ and let (c_i,Φ,u) be a strong solution of the Nernst-Planck-Stokes system (1), (2), (4) with boundary conditions (5), (6), (7) on all intervals $[0,T]$ with $T<T_1$. Assume that*

$$\sup_{T<T_1}\int_0^T\|\rho(t)\|_{L^2}^4dt<\infty. \tag{38}$$

Then there exists $T_2>T_1$ such that (c_i,Φ,u) can be uniquely continued as a strong solution on $[0,T_2]$.

Let $T_1>0$ and let (c_i,Φ,u) be a strong solution of the Nernst-Planck-Navier-Stokes system (1), (2), (3) with boundary conditions (5), (6), (7) on all intervals $[0,T]$ with $T<T_1$. Assume that

$$\sup_{T<T_1}\left[\int_0^T\|\rho(t)\|_{L^2}^4dt+\int_0^T\|u\|_V^4dt\right]<\infty. \tag{39}$$

Then there exists $T_2>T_1$ such that (c_i,Φ,u) can be uniquely continued as a strong solution on $[0,T_2]$.

The converse is obviously also true. Thus (38) (respectively, (39)) is a necessary and sufficient condition for regularity of the Nernst-Planck-Stokes system (respectively, of the Nernst-Planck-Navier-Stokes system).

PROOF. The proof follows directly from Theorem 1, Lemma 1 and Proposition 1. We remark that in the case of (1), (2), (4), $U(T)$ is controlled by $B(T)$. □

PROPOSITION 2. *Let (c_i,Φ,u) be a strong solution of the system (1), (2), (3) (or (1), (2), (4)) with boundary conditions (5), (6), (7) on the interval $[0,T]$. If $c_i(x,0)\ge 0$, $i=1,\dots,m$ then $c_i(x,t)\ge 0$ a.e for $t\in[0,T]$.*

PROOF. In order to show this we take a convex function $F:\mathbb{R}\to\mathbb{R}$ that is nonnegative, twice continuously differentiable, identically zero on the positive semiaxis, and strictly positive on the negative axis. We also assume

$$F''(y)y^2\le CF(y) \tag{40}$$

with $C>0$ a fixed constant. Examples of such functions are

$$F(y)=\begin{cases} y^{2m} & \text{for}\quad y<0,\\ 0 & \text{for}\quad y\ge 0\end{cases} \tag{41}$$

with $m>1$. (In fact $m=1$ works as well, although we have only $F\in W^{2,\infty}(\mathbb{R})$ in that case.) We multiply the equation (1) by $F'(c_i)$ and integrate by parts using the fact that $F'(\gamma_i)=0$. We obtain

$$\frac{d}{dt}\int_{\Omega}F(c_i)dx=-D_i\int_{\Omega}F''(c_i)\left[|\nabla c_i|^2+z_ic_i\nabla\Phi\cdot\nabla c_i\right]dx. \tag{42}$$

Using a Schwarz inequality and the convexity of F, $F''\ge 0$, we have

$$\frac{d}{dt}\int_{\Omega}F(c_i(x,t))dx\le\frac{CD_i}{2}z_i^2\|\nabla\Phi\|_{L^\infty(\Omega)}^2\int_{\Omega}F(c_i(x,t))dx. \tag{43}$$

If $c_i(x,0) \geq 0$ then $F(c_i(x,0)) = 0$ and (43) above shows that $F(c_i(x,t))$ has vanishing integral. As F is nonnegative, it follows that $F(c_i(x,t)) = 0$ almost everywhere in x and because F does not vanish for negative values it follows that $c_i(x,t)$ is almost everywhere nonnegative. $\square$

From now on we consider only solutions with $c_i \geq 0$.

3. Energies

The Navier-Stokes and Stokes energy balance is

$$\frac{1}{2K}\frac{d}{dt}\int_\Omega |u|^2 dx + \frac{\nu}{K}\int_\Omega |\nabla u|^2 dx = -\int_\Omega \rho(u\cdot\nabla\Phi)dx. \tag{44}$$

We consider functionals of (c_i, Φ) which can be used to cancel the right hand side of the Navier-Stokes energy balance.

We denote $(-\Delta_D)^{-1}$ the inverse of the Laplacian with homogeneous Dirichlet boundary condition. We decompose

$$\Phi = \Phi_0 + \Phi_W \tag{45}$$

where Φ_W is harmonic and obeys the inhomogeneous boundary conditions,

$$\Delta\Phi_W = 0, \quad \Phi_{W|\partial\Omega} = W, \tag{46}$$

and

$$-\epsilon\Delta\Phi_0 = \rho, \quad \Phi_{0|\partial\Omega} = 0. \tag{47}$$

so that

$$\Phi_0 = \frac{1}{\epsilon}(-\Delta_D)^{-1}\rho. \tag{48}$$

We introduce

$$D = \min\{D_1, D_2, \dots D_m\}. \tag{49}$$

Let

$$\mathcal{E}_1 = \int_\Omega \left\{\sum_{i=1}^m \Gamma_i\left(\frac{c_i}{\Gamma_i}\log\left(\frac{c_i}{\Gamma_i}\right) - \left(\frac{c_i}{\Gamma_i}\right) + 1\right) + \frac{1}{2\epsilon}\rho(-\Delta_D)^{-1}\rho\right\} dx. \tag{50}$$

PROPOSITION 3. *Let (c_i, Φ, u) be a strong solution of the system (1), (2), (3) (or (1), (2), (4)) with boundary conditions (5), (6), (7) on the interval $[0,T]$. Then*

$$\frac{d}{dt}\mathcal{E}_1 + \mathcal{D}_1 \leq \int_\Omega \rho(u\cdot\nabla\Phi)dx + C_\Gamma\left(\sum_{i=1}^m \|c_i - \Gamma_i\|_{L^2} + 1\right)(\|u\|_H + 1) \tag{51}$$

holds on $[0,T]$, with

$$\mathcal{D}_1 = \frac{D}{2}\int_\Omega \left[\sum_{i=1}^m \left(c_i^{-1}|\nabla c_i|^2 + z_i^2 c_i|\nabla\Phi|^2\right) + \frac{1}{\epsilon}\rho^2\right] dx. \tag{52}$$

The term $\int_\Omega \rho(u\cdot\nabla\Phi)dx$ in the right hand side of (51) can be used to cancel the contribution of the electrical forces in the Navier-Stokes energy balance.

PROOF. We note that

$$\frac{1}{2\epsilon}\int_\Omega \rho(-\Delta_D)^{-1}\rho dx = \frac{1}{2}\int_\Omega \rho\Phi_0 dx. \tag{53}$$

In order to compute the time evolution of $\mathcal{E}_1$ we multiply the equations (1) by the factors $\log\left(\frac{c_i}{\Gamma_i}\right) + z_i\Phi_0$ and we integrate by parts, noting that these factors vanish at the boundary. Strictly speaking, we multiply by $\log\left(\frac{c_i+\delta}{\Gamma_i+\delta}\right) + z_i\Phi_0$ for small positive δ and after one integration by parts we let $\delta \to 0$:

$$
\begin{aligned}
&\int_\Omega ((\partial_t + u\cdot\nabla)c_i)\left(\log\left(\frac{c_i}{\Gamma_i}\right) + z_i\Phi_0\right) dx\\
=& - D_i\int_\Omega c_i\nabla\left(\log c_i + z_i\Phi\right)\cdot\nabla\left(\log\left(\frac{c_i}{\Gamma_i}\right) + z_i\Phi_0\right) dx\\
=& - D_i\int_\Omega c_i\left|\nabla\left(\log c_i + z_i\Phi\right)\right|^2 dx + D_i\int_\Omega c_i\nabla\left(\log c_i + z_i\Phi\right)\cdot\nabla\left(\log\Gamma_i + z_i\Phi_W\right) dx
\end{aligned}
\tag{54}
$$

We have thus

$$
\begin{aligned}
&\int_\Omega ((\partial_t + u\cdot\nabla)c_i)\left(\log\left(\frac{c_i}{\Gamma_i}\right) + z_i\Phi_0\right) dx\\
\le& - \frac{1}{2}D_i\int_\Omega c_i\left|\nabla\left(\log c_i + z_i\Phi\right)\right|^2 dx + \frac{1}{2}D_i\int_\Omega c_i|\nabla(\log\Gamma_i + z_i\Phi_W)|^2 dx
\end{aligned}
\tag{55}
$$

In view of the fact that

$$
((\partial_t + u\cdot\nabla)c_i)\log\left(\frac{c_i}{\Gamma_i}\right) = (\partial_t + u\cdot\nabla)\left(c_i\log\left(\frac{c_i}{\Gamma_i}\right) - c_i\right) + c_i u\cdot\nabla\log\Gamma_i,
\tag{56}
$$

summing in i, on the left hand side we have

$$
\begin{aligned}
&\sum_{i=1}^m\int_\Omega ((\partial_t + u\cdot\nabla)c_i)\left(\log\left(\frac{c_i}{\Gamma_i}\right) + z_i\Phi_0\right) dx\\
=&\frac{d}{dt}\int_\Omega\sum_{i=1}^m c_i(\log\left(\frac{c_i}{\Gamma_i}\right) - 1)dx + \int_\Omega(\partial_t\sum_{i=1}^m(z_ic_i))\Phi_0 dx\\
&+\int_\Omega(u\cdot\nabla(\sum_{i=1}^m z_ic_i))\Phi_0dx + \int_\Omega\sum_{i=1}^m c_iu\cdot\nabla\log\Gamma_i dx\\
=&\frac{d}{dt}\mathcal{E}_1 + \int_\Omega(u\cdot\nabla\rho)\Phi_0 dx + \sum_{i=1}^m\int_\Omega c_iu\cdot\nabla\log\Gamma_i dx.
\end{aligned}
\tag{57}
$$

In the last equality we used

$$
\frac{d}{dt}\frac{1}{2\epsilon}\int_\Omega\rho(-\Delta_D)^{-1}\rho dx = \int_\Omega(\partial_t\rho)\Phi_0 dx
\tag{58}
$$

because $(-\Delta_D)^{-1}$ is selfadjoint. Combining (55) and (57) we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_1 \le& - \frac{1}{2}\sum_{i=1}^m D_i\int_\Omega c_i\left|\nabla\left(\log c_i + z_i\Phi\right)\right|^2 dx + \frac{1}{2}\sum_{i=1}^m D_i\int_\Omega c_i|\nabla(\log\Gamma_i + z_i\Phi_W)|^2 dx\\
&- \sum_{i=1}^m\int_\Omega c_iu\cdot\nabla\log\Gamma_i dx - \int_\Omega(u\cdot\nabla\rho)\Phi_0 dx\\
=& - \frac{1}{2}\sum_{i=1}^m D_i\int_\Omega c_i\left|\nabla\left(\log c_i + z_i\Phi\right)\right|^2 dx + \frac{1}{2}\sum_{i=1}^m D_i\int_\Omega c_i|\nabla(\log\Gamma_i + z_i\Phi_W)|^2 dx\\
&- \sum_{i=1}^m\int_\Omega c_iu\cdot\nabla\log\Gamma_i dx + \int_\Omega\rho u\cdot\nabla(\Phi - \Phi_W) dx
\end{aligned}
\tag{59}
$$

We note that

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{m} D_i \int_{\Omega} c_i \left|\nabla\left(\log c_i + z_i\Phi\right)\right|^2 dx\\
\geq&\frac{D}{2}\sum_{i=1}^{m}\int_{\Omega} c_i \left|\nabla\left(\log c_i + z_i\Phi\right)\right|^2 dx\\
=&\frac{D}{2}\sum_{i=1}^{m}\int_{\Omega}(c_i^{-1}|\nabla c_i|^2 + z_i^2 c_i|\nabla\Phi|^2)dx + D\int_{\Omega}\sum_{i=1}^{m} z_i\nabla c_i\cdot\nabla\Phi dx\\
=&\frac{D}{2}\sum_{i=1}^{m}\int_{\Omega}(c_i^{-1}|\nabla c_i|^2 + z_i^2 c_i|\nabla\Phi|^2)dx + D\int_{\Omega}\sum_{i=1}^{m} z_i\nabla(c_i-\Gamma_i)\cdot\nabla\Phi dx + D\int_{\Omega}\sum_{i=1}^{m} z_i\nabla\Gamma_i\cdot\nabla\Phi dx\\
=&\frac{D}{2}\sum_{i=1}^{m}\int_{\Omega}(c_i^{-1}|\nabla c_i|^2 + z_i^2 c_i|\nabla\Phi|^2)dx + D\int_{\Omega}\sum_{i=1}^{m} z_i(c_i-\Gamma_i)(-\Delta\Phi) dx + D\int_{\Omega}\sum_{i=1}^{m} z_i\nabla\Gamma_i\cdot\nabla\Phi dx\\
=&\frac{D}{2}\sum_{i=1}^{m}\int_{\Omega}(c_i^{-1}|\nabla c_i|^2 + z_i^2 c_i|\nabla\Phi|^2)dx + D\frac{1}{\epsilon}\int_{\Omega}\sum_{i=1}^{m} z_i(c_i-\Gamma_i)\rho dx + D\int_{\Omega}\sum_{i=1}^{m} z_i\nabla\Gamma_i\cdot\nabla\Phi dx\\
=&\frac{D}{2}\sum_{i=1}^{m}\int_{\Omega}(c_i^{-1}|\nabla c_i|^2 + z_i^2 c_i|\nabla\Phi|^2)dx + D\frac{1}{\epsilon}\int_{\Omega}(\rho-\sum_{i=1}^{m} z_i\Gamma_i)\rho dx + D\int_{\Omega}\sum_{i=1}^{m} z_i\nabla\Gamma_i\cdot\nabla\Phi dx\\
\geq&\frac{D}{2}\sum_{i=1}^{m}\int_{\Omega}(c_i^{-1}|\nabla c_i|^2 + z_i^2 c_i|\nabla\Phi|^2)dx + \frac{D}{2\epsilon}\int_{\Omega}\rho^2 dx - \frac{D}{2\epsilon}\int_{\Omega}\left|\sum_{i=1}^{m} z_i\Gamma_i\right|^2 dx\\
&+D\int_{\Omega}\sum_{i=1}^{m} z_i\nabla\Gamma_i\cdot\nabla\Phi dx.
\end{aligned}
\tag{60}
$$

From (59) and (60) we obtain

$$
\frac{d}{dt}\mathcal{E}_1 + \mathcal{D}_1 \leq \mathcal{Q}_1 + \int_{\Omega}\rho u\cdot\nabla\Phi dx \tag{61}
$$

with $\mathcal{D}_1$ given in (52) and

$$
\begin{aligned}
\mathcal{Q}_1 =&\frac{1}{2}\sum_{i=1}^{m} D_i\int_{\Omega} c_i|\nabla(\log\Gamma_i + z_i\Phi_W)|^2 dx - \sum_{i=1}^{m}\int_{\Omega} c_i u\cdot\nabla\log\Gamma_i dx - \int_{\Omega}\rho u\cdot\nabla\Phi_W dx\\
&+\frac{D}{2\epsilon}\int_{\Omega}\left|\sum_{i=1}^{m} z_i\Gamma_i\right|^2 dx - D\int_{\Omega}\sum_{i=1}^{m} z_i\nabla\Gamma_i\cdot\nabla\Phi dx.
\end{aligned}
\tag{62}
$$

Note that $\mathcal{Q}_1$ is at most quadratic in terms of the unknowns u, c_i, in view of the fact that both ρ and Φ are affine in c_i. The inequality (51) follows by bounding Q_1. □

A useful energy is the potential part in $\mathcal{E}_1$:

$$
\mathcal{P} = \frac{1}{2\epsilon}\int_{\Omega}\rho(-\Delta_D)^{-1}\rho dx. \tag{63}
$$

PROPOSITION 4. *Let (c_i, Φ, u) be a strong solution of the system (1), (2), (3) (or (1), (2), (4)) with boundary conditions (5), (6), (7) on the interval $[0, T]$. Then*

$$
\frac{d}{dt}\mathcal{P} + \mathcal{D}_2 \leq \int_{\Omega}\rho(u\cdot\nabla\Phi)dx + C_\Gamma\left(\sum_{i=1}^{m}\|c_i-\Gamma_i\|_{L^2}+1\right)(\|\rho\|_{L^2}+1) + C_\Gamma\|\rho\|_{L^2}\|u\|_H \tag{64}
$$

holds on $[0,T]$*, with*

$$\mathcal{D}_2 = \frac{1}{2}\sum_{i=1}^m z_i^2 D_i \int_\Omega c_i \left|\nabla\Phi\right|^2 dx. \tag{65}$$

We note that the term $\int_\Omega \rho(u\cdot\nabla\Phi)dx$ in the right hand side of (64) can be used to cancel the contribution of electrical forces in the Navier-Stokes energy balance.

PROOF. In order to compute the time evolution of (63) we take the equations (1), multiply by the factors $z_i\Phi_0$ and integrate by parts in view of the fact that Φ_0 vanishes on the boundary. We obtain

$$\begin{aligned}
\int_\Omega((\partial_t + u\cdot\nabla)c_i)z_i\Phi_0 dx =& - D_i z_i \int_\Omega (\nabla c_i + z_i c_i \nabla\Phi)\cdot\nabla\Phi_0 dx \\
=& - D_i z_i \int_\Omega \nabla c_i\cdot\nabla\Phi_0 dx - z_i^2 D_i \int_\Omega c_i\nabla\Phi\cdot\nabla\Phi_0 dx \\
=& - z_i D_i \int_\Omega \nabla(c_i - \Gamma_i)\cdot\nabla\Phi_0 dx - D_i z_i \int_\Omega \nabla\Gamma_i\cdot\nabla\Phi_0 dx \\
& - z_i^2 D_i \int_\Omega c_i \left|\nabla\Phi\right|^2 dx + z_i^2 D_i \int_\Omega c_i \nabla\Phi\cdot\nabla\Phi_W dx \\
=& - D_i z_i \epsilon^{-1}\int_\Omega (c_i - \Gamma_i)\rho dx - z_i^2 D_i \int_\Omega c_i\left|\nabla\Phi\right|^2 dx \\
& + z_i^2 D_i \int_\Omega c_i\nabla\Phi\cdot\nabla\Phi_W dx - D_i z_i \int_\Omega \nabla\Gamma_i\cdot\nabla\Phi_0 dx
\end{aligned} \tag{66}$$

In the last equality we used the fact that $c_i - \Gamma_i$ vanishes on the boundary and the fact that $-\epsilon\Delta\Phi_0 = \rho$. Summing in i, on the left hand side we have

$$\begin{aligned}
\sum_{i=1}^m \int_\Omega((\partial_t + u\cdot\nabla)c_i)z_i\Phi_0 dx =& \int_\Omega(\partial_t\rho)\Phi_0 dx + \int_\Omega(u\cdot\nabla\rho)\Phi_0 dx \\
=& \frac{d}{dt}\mathcal{P} + \int_\Omega (u\cdot\nabla\rho)\Phi_0 dx.
\end{aligned} \tag{67}$$

Putting together (66) and (67)

$$\begin{aligned}
\frac{d}{dt}\mathcal{P} + \sum_{i=1}^m z_i^2 D_i\int_\Omega c_i\left|\nabla\Phi\right|^2 dx =& -\sum_{i=1}^m D_i z_i\epsilon^{-1}\int_\Omega(c_i-\Gamma_i)\rho dx + \sum_{i=1}^m z_i^2 D_i \int_\Omega c_i\nabla\Phi\cdot\nabla\Phi_W dx \\
& - \sum_{i=1}^m D_i z_i \int_\Omega \nabla\Gamma_i\cdot\nabla\Phi_0 dx - \int_\Omega\rho(u\cdot\nabla\Phi_W)dx + \int_\Omega \rho u\cdot\nabla\Phi dx.
\end{aligned} \tag{68}$$

After a Schwarz inequality we obtain

$$\frac{d}{dt}\mathcal{P} + \mathcal{D}_2 \le \mathcal{Q}_2 + \int_\Omega \rho u\cdot\nabla\Phi dx \tag{69}$$

where $\mathcal{D}_2$ is given in (65) and

$$\begin{aligned}
\mathcal{Q}_2 =& -\sum_{i=1}^m D_i z_i \epsilon^{-1}\int_\Omega (c_i - \Gamma_i)\rho dx + \frac{1}{2}\sum_{i=1}^m z_i^2 D_i \int_\Omega c_i\left|\nabla\Phi_W\right|^2 dx \\
& - \sum_{i=1}^m D_i z_i \int_\Omega \nabla\Gamma_i\cdot\nabla\Phi_0 dx - \int_\Omega \rho(u\cdot\nabla\Phi_W)dx.
\end{aligned} \tag{70}$$

Unlike the term $\mathcal{Q}_1$ of (62), $\mathcal{Q}_2$ has no (u,c) quadratic terms. Instead, the only quadratic terms in it are of the type (c,ρ) or (u,ρ) (the (u,Φ_0) term is of (u,ρ) type in this accounting). Estimating Q_2 we obtain (64).

□

REMARK 1. *We note that if W and γ_i are time dependent, then Proposition 4 holds as stated above, with the very same Q_2 and still having only (c,ρ) and (u,ρ) quadratic terms.*

4. Quadratic bounds

We estimate the sum of L^2 norms of c_i. We take the scalar product of the equations (13) with $\frac{1}{D_i}q_i$ and add. We obtain first

$$\frac{d}{dt}\sum_{i=1}^m \frac{1}{2D_i}\int_\Omega q_i^2 dx + \sum_{i=1}^m \int_\Omega |\nabla q_i|^2 dx = -\frac{1}{2}\sum_{i=1}^m z_i \int_\Omega \nabla\Phi\cdot\nabla(q_i^2)dx + \sum_{i=1}^m \frac{1}{D_i}\int_\Omega F_i q_i dx. \tag{71}$$

The integartion by parts is justified because of (15). We integrate by parts one more time using the same boundary conditions and (2)

$$\frac{d}{dt}\sum_{i=1}^m \frac{1}{2D_i}\int_\Omega q_i^2 dx + \sum_{i=1}^m \int_\Omega |\nabla q_i|^2 dx = -\frac{1}{2\epsilon}\int_\Omega \rho\sum_{i=1}^m z_i q_i^2 dx + \sum_{i=1}^m \frac{1}{D_i}\int_\Omega F_i q_i dx. \tag{72}$$

THEOREM 3. *Consider $m=2$. Let $T>0$ be arbitrary. Let $c_i(\cdot,0)>0$, $c_i(\cdot,0)\in H^1$, $c_{i|\partial\Omega}=\gamma_i$ and $u_0\in V$ be given. Then the system (1), (2), (4)) with boundary conditions (5), (6), (7) has global nonnegative strong solutions on $[0,T]$. The system (1), (2), (3) has global nonnegative strong solutions if*

$$\int_0^T \|u\|_V^4 dt < \infty. \tag{73}$$

Moreover

$$\sup_{0\le t\le T}\sum_{i=1}^2 \|c_i(t)\|_{H^1}^2 + \int_0^T \sum_{i=1}^2 \|c_i(t)\|_{H^2}^2 dt \le C_\Gamma\left[\sum_{i=1}^2 \|c_i(0)\|_{H^1}^2 + \int_0^T \|u\|_H^2 dt + 1\right] e^{C_\Gamma(T+R(T)+U(T))} \tag{74}$$

holds for all T, where $R(T)$ is given by (33), $U(T)$ is given by (34), and with C_Γ depending only on the boundary conditions γ_i and W, domain Ω, and parameters ν, D_i, ϵ, K.

PROOF. We present the proof for the case $z_1=1$, $z_2=-1$, for simplicity of exposition. For general z_1, z_2 of opposite signs (with the notation convention that $z_1>0$, $z_2<0$), we multiply (13) by $\frac{|z_i|}{D_i}q_i$ and add. The main observation is that the corresponding cubic term in (72) is proportional to an integral of $\rho\sum_{i=1}^2 z_i|z_i|q_i^2 = \rho(|z_1|q_1+|z_2|q_2)(z_1q_1+z_2q_2)$ and this leads to a cubic dissipation term in terms of $|\rho|$. Specifically, when $z_1=1$, $z_2=-1$, then recalling $c_i=q_i+\Gamma_i$, we have

$$\sum_{i=1}^2 z_i q_i^2 = (\rho-\Gamma_1+\Gamma_2)(c_1+c_2-\Gamma_1-\Gamma_2). \tag{75}$$

Thus

$$\begin{aligned}\rho\sum_{i=1}^2 z_i q_i^2 =& \rho^2(c_1+c_2) - (\Gamma_1-\Gamma_2)\rho(q_1+q_2) - \rho^2(\Gamma_1+\Gamma_2)\\ \ge& |\rho|^3 - (\Gamma_1-\Gamma_2)\rho(q_1+q_2) - \rho^2(\Gamma_1+\Gamma_2)\end{aligned} \tag{76}$$

because

$$c_1+c_2 \ge |\rho|. \tag{77}$$

Now we use Hölder and Young inequalities to bound in (72)

$$\begin{aligned}\frac{1}{2\epsilon}\int_\Omega \rho\sum_{i=1}^2 z_i q_i^2 dx \geq &\frac{1}{2\epsilon}\int_\Omega |\rho|^3 dx - \frac{1}{2\epsilon}(\|q_1\|_{L^2(\Omega)} + \|q_2\|_{L^2(\Omega)})\|\Gamma_2-\Gamma_1\|_{L^6(\Omega)}\|\rho\|_{L^3(\Omega)}\\ &-\frac{1}{2\epsilon}[\|\Gamma_1\|_{L^3(\Omega)} + \|\Gamma_2\|_{L^3(\Omega)}]\|\rho\|^2_{L^3(\Omega)}\\ \geq &\frac{1}{4\epsilon}\|\rho\|^3_{L^3} - \frac{1}{4L^2}\|q_1\|^2_{L^2} - \frac{1}{4L^2}\|q_2\|^2_{L^2} - C_\Gamma\end{aligned} \tag{78}$$

with L the constant in the Poincaré inequality

$$\|\nabla q\|^2_{L^2(\Omega)} \geq L^{-2}\|q\|^2_{L^2(\Omega)}. \tag{79}$$

Using the Poincaré inequality (79), we absorb the terms $\|q_i\|^2_{L^2}$ from (78) into the dissipation terms $\|\nabla q_i\|^2_{L^2}$ from (72). Thus from (72), (78) and (79) we obtain

$$\frac{d}{dt}\sum_{i=1}^2\frac{1}{2D_i}\int_\Omega q_i^2 dx + \frac{3}{4}\sum_{i=1}^2\int_\Omega |\nabla q_i|^2 dx + \frac{1}{4\epsilon}\|\rho\|^3_{L^3} \leq C_\Gamma + \sum_{i=1}^2\frac{1}{D_i}\int_\Omega F_i q_i dx. \tag{80}$$

We have

$$\left|\int_\Omega F_i q_i dx\right| \leq C_\Gamma(\|\rho\|_{L^2} + \|u\|_H + 1)\|q_i\|_{L^2} \tag{81}$$

and therefore we have that

$$\mathcal{E}_3 = \sum_{i=1}^2\frac{1}{D_i}\int_\Omega q_i^2 dx \tag{82}$$

obeys

$$\frac{d}{dt}\mathcal{E}_3 + \mathcal{D}_3 \leq C_\Gamma + \widetilde{C_\Gamma}\|u\|^2_H \tag{83}$$

with

$$\mathcal{D}_3 = \sum_{i=1}^2\frac{1}{2}\int_\Omega |\nabla q_i|^2 dx + \frac{1}{4\epsilon}\|\rho\|^3_{L^3}. \tag{84}$$

We singled out the coefficient $\widetilde{C_\Gamma}$ of $\|u\|^2_H$ because we use it next. We take a constant

$$\delta = \frac{\nu}{2KL^2\widetilde{C_\Gamma}} \tag{85}$$

such that the dissipation in the Navier-Stokes energy balance exceeds twice the contribution from $\|u\|^2_H$ in the right hand side of (83) when the latter is multiplied by δ,

$$\frac{\nu}{K}\int_\Omega |\nabla u|^2 dx \geq 2\delta\widetilde{C_\Gamma}\|u\|^2_H. \tag{86}$$

We consider

$$\mathcal{F} = \frac{1}{2K}\|u\|^2_H + \mathcal{P} + \delta\mathcal{E}_3 \tag{87}$$

and, using (44), (64) and (83) multiplied by δ we obtain

$$\begin{aligned}&\frac{d}{dt}\mathcal{F} + \frac{\nu}{2K}\int_\Omega |\nabla u|^2 dx + \frac{\delta}{2}\sum_{i=1}^2\|\nabla q_i\|^2_{L^2} + \frac{\delta}{4\epsilon}\|\rho\|^3_{L^3}\\ &\leq C_\Gamma[\|\rho\|_{L^2}(\sum_{i=1}^2\|q_i\|_{L^2} + 1) + \|\rho\|_{L^2}\|u\|_H + \sum_{i=1}^2\|q_i\|_{L^2} + 1].\end{aligned} \tag{88}$$

The positive cubic term in ρ on the left hand side together with the rest of positive quadratic dissipative terms on the left hand side can be used to absorb all the quadratic terms on the right hand side, because they all involve at least one ρ, and the linear terms are also absorbed using Poincaré inequalities for both q_i and for u. This results in

$$\frac{d}{dt}\mathcal{F} + c_\Gamma \mathcal{F} \le C_\Gamma \tag{89}$$

with $c_\Gamma > 0$. It follows that

$$\mathcal{F}(t) \le \mathcal{F}(0)e^{-c_\Gamma t} + C_\Gamma \tag{90}$$

This implies in particular that

$$\|\rho(t)\|_{L^2}^2 \le C_\Gamma \mathcal{F}(0)e^{-c_\Gamma t} + C_\Gamma \tag{91}$$

and, squaring and integrating in time, (24) holds

$$B(T) \le C_\Gamma(\mathcal{F}(0)^2 + T). \tag{92}$$

Moreover, the dissipation is time integrable,

$$\int_0^T \left\{ \frac{\nu}{2K}\int_\Omega |\nabla u|^2 dx + \frac{\delta}{2}\sum_{i=1}^2 \|\nabla q_i\|_{L^2}^2 + \frac{\delta}{4\epsilon}\|\rho\|_{L^3}^3 \right\} dt \le C_\Gamma(\mathcal{F}(0) + T). \tag{93}$$

It follows from (32) that (74) holds. □

THEOREM 4. *Let $m > 2$ and assume $D_1 = D_2 = \cdots = D_m = D > 0$. Let $T > 0$ be arbitrary. Let $c_i(\cdot, 0) > 0$, $c_i(\cdot, 0) \in H^1$, $c_{i|\partial\Omega} = \gamma_i$ and $u_0 \in V$ be given. Then the system (1), (2), (4)) with boundary conditions (5), (6), (7) has global nonnegative strong solutions on $[0, T]$. The system (1), (2), (3) has global nonnegative strong solutions if*

$$\int_0^T \|u\|_V^4 dt < \infty. \tag{94}$$

Moreover

$$\sup_{0\le t\le T}\sum_{i=1}^m \|c_i(t)\|_{H^1}^2 + \int_0^T \sum_{i=1}^m \|c_i(t)\|_{H^2}^2 dt \le C_\Gamma \left[\sum_{i=1}^m \|c_i(0)\|_{H^1}^2 + \int_0^T \|u\|_H^2 dt + 1\right] e^{C_\Gamma(T+R(T)+U(T))} \tag{95}$$

holds for all T, where $R(T)$ is given by (33), $U(T)$ is given by (34), and with C_Γ depending only on the boundary conditions γ_i and W, domain Ω, and parameters ν, D_i, ϵ, K.

PROOF. We consider the auxiliary variables

$$S = \sum_{i=1}^m q_i \tag{96}$$

and

$$Z = \sum_{i=1}^m z_i q_i. \tag{97}$$

Summing in (13) we have

$$\begin{cases} (\partial_t + u\cdot\nabla) S = D\left(\Delta S + \operatorname{div}(Z\nabla\Phi)\right) + F_S \\ (\partial_t + u\cdot\nabla) Z = D((\Delta Z + \operatorname{div}(S\nabla\Phi)) + F_Z, \end{cases} \tag{98}$$

with

$$F_S = \sum_{i=1}^m F_i, \tag{99}$$

$$F_Z = \sum_{i=1}^m z_i F_i, \tag{100}$$

and F_i given in (14) and with D in (49). Multiplying by S and Z and integrating by parts (twice in the nonlinear term, once in linear terms) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}(S^2+Z^2)dx + D\int_{\Omega}(|\nabla S|^2+|\nabla Z|^2)dx = -\frac{D}{\epsilon}\int_{\Omega} SZ\rho dx + \int_{\Omega}(SF_S+ZF_Z)\,dx \tag{101}$$

Now we use

$$Z = \rho - \Gamma_Z \tag{102}$$

and

$$S = \sum_{i=1}^{m} c_i - \Gamma_S \tag{103}$$

with

$$\Gamma_S = \sum_{i=1}^{M}\Gamma_i, \tag{104}$$

and

$$\Gamma_Z = \sum_{i=1}^{m} z_i\Gamma_i, \tag{105}$$

together with

$$\sum_{i=1}^{m} c_i \ge |\rho|, \tag{106}$$

to deduce

$$SZ\rho \ge |\rho|^3 - \Gamma_S\rho^2 - S\rho\Gamma_Z. \tag{107}$$

Let us note the relationships

$$\begin{cases} F_S = -u\cdot\nabla\Gamma_S + D(\Delta\Gamma_S + \mathrm{div}\,(\Gamma_Z\nabla\Phi)),\\ F_Z = -u\cdot\nabla\Gamma_Z + D(\Delta\Gamma_Z + \mathrm{div}\,(\Gamma_S\nabla\Phi)). \end{cases} \tag{108}$$

We deduce that

$$\begin{aligned} \left|\int_{\Omega}(SF_S+ZF_Z)dx\right| \le & C_\Gamma(\|u\|_H+1)(\|S\|_{L^2}+\|Z\|_{L^2}) + C_\Gamma\|\nabla\Phi\|_{L^2}(\|\nabla S\|_{L^2}+\|\nabla Z\|_{L^2}) \\ \le & C_\Gamma + C_\Gamma\|u\|_H^2 + \frac{D}{2}(\|\nabla S\|_{L^2}^2+\|\nabla Z\|_{L^2}^2) + \frac{D}{2\epsilon}\|\rho\|_{L^3}^3 \end{aligned} \tag{109}$$

where in the second line, we use Poincaré inequalities to bound $\|S\|_{L^2}, \|Z\|_{L^2}$. We also use (31) to bound $\|\nabla\Phi\|_{L^2}^2 \le C_\Gamma\|\nabla\Phi\|_{L^3}^2 \le C_\Gamma(1+\|\rho\|_{L^3}^2)$ and then use a Young's inequality. Using (101) and (107) to absorb terms on the right hand side of (109), we obtain

$$\frac{d}{dt}\int_{\Omega}(S^2+Z^2)dx + D\int_{\Omega}(|\nabla S|^2+|\nabla Z|^2)dx + \frac{D}{\epsilon}\int_{\Omega}|\rho|^3dx \le C_\Gamma + \widetilde{C_\Gamma}\|u\|_H^2. \tag{110}$$

We take δ defined in (85) with the current $\widetilde{C_\Gamma}$ and consider the functional

$$\mathcal{G} = \frac{1}{2K}\|u\|_H^2 + \mathcal{P} + \delta\int_{\Omega}(S^2+Z^2)dx \tag{111}$$

and obtain from (44), (64) and (110)

$$\begin{aligned} &\frac{d}{dt}\mathcal{G} + \frac{\nu}{2K}\int_{\Omega}|\nabla u|^2dx + D\delta(\|\nabla S\|_{L^2}^2+\|\nabla Z\|_{L^2}^2) + \frac{D\delta}{\epsilon}\|\rho\|_{L^3}^3 \\ &\le C_\Gamma[\|\rho\|_{L^2}(\sum_{i=1}^{m}\|c_i-\Gamma_i\|_{L^2}+1) + \|\rho\|_{L^2}\|u\|_H + \sum_{i=1}^{m}\|c_i-\Gamma_i\|_{L^2}+1], \end{aligned} \tag{112}$$

where the contribution from $\delta\widetilde{C_\Gamma}\|u\|_H^2$ coming from (110) is abosrbed in the dissipation term coming from (44). Now we note that

$$0 \le c_i \le \sum_{i=1}^m c_i = S + \Gamma_S \tag{113}$$

implies that

$$\|c_i - \Gamma_i\|_{L^2} \le \|S\|_{L^2} + C_\Gamma \tag{114}$$

and the Poincaré inequality for S implies

$$\|\nabla S\|_{L^2}^2 \ge \frac{1}{L^2}\|c_i - \Gamma_i\|_{L^2}^2 - C_\Gamma. \tag{115}$$

Therefore we obtain using Poincaré, Hölder and Young inequalities to absorb all the time-dependent terms on the right hand side of (112) into the left,

$$\frac{d}{dt}\mathcal{G} + c_\Gamma\mathcal{G} \le C_\Gamma \tag{116}$$

with $c_\Gamma > 0$. The rest of the proof follows as in the proof of Theorem 3. □

REMARK 2. *Theorems 3 and 4 hold if γ_i and W are smooth bounded functions of time. There are natural modifications to the formula (14) for F_i and the constants in the estimates, but, in view of Remark 1, the scheme of proof remains unchanged.*

Acknowledgment. The work of PC was partially supported by NSF grant DMS- 171398. We thank the anonymous referees for useful questions and suggestions.

References

[1] P. Biler, J. Dolbeault. Long time behavior of solutions to Nernst-Planck and Debye-Hckel drift-diffusion systems. Ann. Henri Poincare **1**(2000) 461-472.

[2] D. Bothe, A. Fischer, J. Saal, Global well-posedness and stability of electrokinetic flows, SIAM J. Math. Anal, **46** 2, (2014), 1263-1316.

[3] Y.S. Choi, and R. Lui, Multi-Dimensional Electrochemistry Model, Arch. Rational Mech. Anal. **130** (1995), 315-342.

[4] P. Constantin , C. Foias, *Navier-Stokes equations*, Chicago University Press, Chicago (1988).

[5] P. Constantin, M. Ignatova. On the Nernst-Planck-Navier-Stokes system. Archive for Rational Mechanics and Analysis **232**, No. 3, (2018), 1379 -1428.

[6] P. Constantin, M. Ignatova, F-N Lee, Nernst-Planck-Navier-Stokes systems near equilibrium, preprint (2020).

[7] H. Gajewski, K. Groger, Reaction-diffusion processes of electrically charged species. Math. Nachr., **177** (1996), 109-130.

[8] Y. Giga, R. V. Kohn, Nondegeneracy of blowup for semilinear heat equations, Commun. Pure Applied Math, **42** (1989), 854-884.

[9] B. Perthame, *Transport equations in biology*, Frontiers in Mathematics, Birkhauser Verlag, Basel, (2007)

[10] I. Rubinstein, *Electro-Diffusion of Ions*, SIAM Studies in Applied Mathematics, SIAM, Philadelphia 1990.

[11] S. M. Rubinstein, G. Manukyan, A. Staicu, I. Rubinstein, B. Zaltzman, R.G.H. Lammertink, F. Mugele, M. Wessling, Direct observation of a nonequilibrium electro-osmotic instability. Phys. Rev. Lett. **101**, (2008) 236101-236105.

[12] I. Rubinstein, B. Zaltzman, Electro-osmotically induced convection at a permselective membrane, Phys. Rev. E **62** (2000) 2238-2251.

[13] R. Ryham, Existence, uniqueness, regularity and long-term behavior for dissipative systems modeling electrohydrodynamics. arXiv:0910.4973v1, (2009).

[14] M. Schmuck. Analysis of the Navier-Stokes-Nernst-Planck-Poisson system. Math.Models MethodsAppl., **19** (2009), 993-1014.

[15] B. Zaltzman, I. Rubinstein, Electro-osmotic slip and electroconvective instability. J. Fluid Mech. **579**, (2007) 173-226.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544
Email address: const@math.princeton.edu

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122
Email address: ignatova@temple.edu

PROGRAM IN APPLIED AND COMPUTATIONAL MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544
Email address: fizaynoah@princeton.edu