Compressible fluids and active potentials

Peter Constantin, Theodore D. Drivas, Huy Q. Nguyen, and Federico Pasqualotto

ABSTRACT. We consider a class of one dimensional compressible systems with degenerate diffusion coefficients. We establish the fact that the solutions remain smooth as long as the diffusion coefficients do not vanish, and give local and global existence results. The models include compressible Navier-Stokes equations, shallow water systems and lubrication approximation of slender jets. In all these models the momentum equation is forced by the gradient of a solution-dependent potential: the active potential. The method of proof uses the Bresch-Desjardins entropy and the analysis of the evolution of the active potential. March 9, 2018

1. Introduction

We consider a class of compressible fluid models in one space dimension with periodic boundary conditions:

$$\partial_t\rho + \partial_x(u\rho) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) = -\partial_x p(\rho) + \partial_x(\mu(\rho)\partial_x u) + \rho f, \tag{1.2}$$

$$(\rho, u)|_{t=0} = (\rho_0, u_0) \tag{1.3}$$

with constitutive laws given by

$$p(\rho) = c_p\rho^\gamma, \qquad \mu(\rho) = c_\mu\rho^\alpha, \qquad c_p \neq 0,\ c_\mu > 0. \tag{1.4}$$

Among these models are the one-dimensional barotropic compressible Navier-Stokes equations. In this description, ρ is the mass density, u is the fluid velocity, and $p(\rho)$, $\mu(\rho)$ are the fluid pressure and dynamic viscosity respectively. These are given by physical equations of state (1.4). For such systems, the specific heat at constant pressure is positive $c_p > 0$ so that $p(\rho)$ is non-negative. The viscosity is also assumed non-negative $c_\mu > 0$ but may be degenerate in the sense that it vanishes for $\rho = 0$.

Although the eqns. (1.1)–(1.3) describe cases of compressible Navier-Stokes equations, they serve also as models for a number of other physical systems if the basic variables and constitutive laws are appropriately defined. For example, a model for viscous incompressible motion of shallow water waves [**1, 2**] reads

$$\partial_t h + \partial_x(uh) = 0, \tag{1.5}$$

$$\partial_t(hu) + \partial_x(hu^2) + \frac{g}{2}\partial_x h^2 = 4\nu\partial_x(h\partial_x u) + hf \tag{1.6}$$

where

- h and u represent respectively the surface height and fluid velocity,
- g is gravity,
- $\nu > 0$ is the kinematic viscosity,
- f is the external force.

These equations are a special case of equations (1.1)-(1.2) with

$$p(\rho) = \frac{g}{2}\rho^2 \quad \text{and} \quad \mu(\rho) = 4\nu\rho.$$

Key words and phrases. compressible flow, shallow water, slender jet, global existence.
.

Equations (1.1)–(1.3) also appear in the theory of drop formation as the slender jet equations [**3, 4**]:

$$\partial_t h + u\partial_x h = -\frac{1}{2}\partial_x u h, \tag{1.7}$$

$$\partial_t u + u\partial_x u + \gamma\partial_x(\frac{1}{h}) = 3\nu\frac{\partial_x(h^2\partial_x u)}{h^2} - g, \tag{1.8}$$

where

- h and u represent respectively the neck radius and velocity of the jet,
- $\gamma > 0$ is the surface tension coefficient,
- $\nu > 0$ is the kinematic viscosity,
- $g > 0$ is gravity.

These equations arise as a reduction of the axisymmetric incompressible Navier-Stokes equations in two spatial dimensions governing a thin liquid threads with a moving boundary. Via the change of variables $\rho = h^2$, equations (1.7)-(1.8) become equations (1.1)-(1.2) with

$$p(\rho) = -\gamma\sqrt{\rho} \quad \text{and} \quad \mu(\rho) = 3\nu\rho.$$

Note that here the "pressure" that appears is non-positive in contrast with the Navier-Stokes descriptions.

In all the settings above, the one-dimensional equations (1.1)–(1.3) are approximate models of the underlying physical processes, whose quality may vary depending on the situation. In fact, although multi-dimensional analogues of (1.1)–(1.3) have been extensively employed in astrophysics [**5, 6**], they are not known to arise as an effective description by a controlled hydrodynamic limit and they have the defect that total energy is not conserved. This makes their interpretation as descriptions of dissipative molecular fluids evolving as nearly isolated systems dubious and must be considered as inherently approximate. Of course, they could be valid descriptions of fluid systems in other situations than these, as is the case of the shallow water and slender jet. Moreover, J. Eggers has argued that the slender jet equations described above become an exact description asymptotically close to drop pinch–off, justifying the use of the model (1.7), (1.8) in that context.

Four theorems are proved. The first result, Theorem 1.1, provides a blowup criterion for equations (1.1)–(1.3) with a wide range of constitutive pressure and viscosity laws (1.4). In what follows, we denote by $\mathbb{T}$ the interval $(0, 1]$ with periodic boundary conditions.

THEOREM 1.1. *Assume either*

(i) $c_p > 0$ *and* $\alpha > \frac{1}{2}$, $\gamma \neq 1$, $\gamma \geq \alpha - \frac{1}{2}$ *or*
(ii) $c_p < 0$ *and* $\frac{1}{2} < \alpha \leq \frac{3}{2}$, $\gamma < 1$, $0 < \gamma \leq \alpha$.

Let $k \geq 3$ *and assume further that*

$$f \in L^2(0, T; H^{k-1}(\mathbb{T})) \quad \textit{for all} \quad T > 0.$$

If (ρ, u) *is a solution of* (1.1)-(1.3) *on* $[0, T^*)$ *such that*

$$\rho \in C(0, T; H^k(\mathbb{T})), \quad u \in C(0, T; H^k(\mathbb{T})) \cap L^2(0, T; H^{k+1}(\mathbb{T})), \quad \forall T \in (0, T^*) \tag{1.9}$$

and

$$\inf_{t\in[0,T^*)} \min_{x\in\mathbb{T}} \rho(x, t) > 0,$$

then (ρ, u) *satisfies*

$$\sup_{T\in[0,T^*)} \|\rho\|_{L^\infty(0,T;H^k)} + \sup_{T\in[0,T^*)} \|u\|_{L^\infty(0,T;H^k)} + \sup_{T\in[0,T^*)} \|u\|_{L^2(0,T;H^{k+1})} < \infty \tag{1.10}$$

and can be continued in the class (1.9) *past* T^*.

Theorem 1.1 says that the only possible way for a singularity to form starting from smooth data is if the density becomes zero somewhere in the domain. This applies in particular to the viscous shallow water wave equations (1.5)-(1.6). In the slender jet equations (1.7)-(1.8) which model incompressible fluid drop formation, this says that singularities can only form at the onset of drop break-off. This answers a conjecture of P. Constantin recorded in [**3**].

REMARK 1.2. [**7**] proved that weak solutions of 1D compressible Navier-Stokes equations with constant viscosity do not exhibit vacuum states in finite time provided no vacuum states are present initially.

REMARK 1.3. Local well-posedness of (1.1)–(1.3) in the class (1.9) is established in Proposition B.1 of the Appendix B for arbitrary smooth $p(\rho)$ and smooth non-negative $\mu(\rho)$. This covers the special case of power law equations of state (1.4) in the entire parameters range in Theorem 1.1. Local existence of strong solution for 2D shallow water equations can be found in [**8, 9**]. We also refer to [**10, 11**] for classical results regarding equations of compressible viscous and heat-conductive fluids with constant viscosity.

Our next two theorems concern the long-time existence and persistence of regularity. Theorem 1.4 establishes global existence for arbitrarily large data, within a range of pressure and viscosity of the form (1.4).

THEOREM 1.4. *Assume*

$$c_p > 0, \quad \alpha \in (\frac{1}{2}, 1], \text{ and } \quad \gamma \geq 2\alpha.$$

Let $k \geq 3$ be an integer and let ρ_0 and u_0 belong to $H^k(\mathbb{T})$ such that $\rho_0(x) > 0$ for all $x \in \mathbb{T}$. Assume further that

$$f \in L^2(0, T; H^{k-1}(\mathbb{T})) \quad \text{for all} \quad T > 0.$$

Then there exists a unique global solution (ρ, u) to (1.1)-(1.3) such that

$$\rho \in C(0, T; H^k(\mathbb{T})), \quad u \in C(0, T; H^k(\mathbb{T})) \cap L^2(0, T; H^{k+1}(\mathbb{T}))$$

for all $T > 0$, and $\rho(x, t) > 0$ for all $(x, t) \in \mathbb{T} \times \mathbb{R}^+$.

This result applies to the viscous shallow water equations (1.5)-(1.6), giving an alternative proof to that of [**12**] in which only H^1 regularity is propagated. Moreover, Theorem 1.4 allows for more singular density dependence of the viscosity than in [**13**], which considers the case of $\alpha < \frac{1}{2}$ and $\gamma > 1$. In two dimensions, global stability of constant solutions to shallow water equations was proved in [**14, 15, 16**].

For more degenerate viscosity ρ^α allowing $\alpha > 1$, we prove global existence for a class of large initial data.

THEOREM 1.5. *Assume that*

$$c_p > 0, \quad \alpha > \frac{1}{2}, \quad \gamma \in [\alpha, \alpha + 1], \quad \gamma \neq 1 \tag{1.11}$$

and that

$$f(x, t) = f(t) \in L^2((0, T)) \quad \forall T > 0.$$

Let $k \geq 4$ be an integer and let u_0 and ρ_0 belong to $H^k(\mathbb{T})$ such that $\rho_0(x) > 0$ for all $x \in \mathbb{T}$ and

$$\partial_x u_0(x) \leq \frac{c_p}{c_\mu} \rho_0(x)^{\gamma - \alpha} \quad \forall x \in \mathbb{T}. \tag{1.12}$$

Then there exists a unique global solution (ρ, u) to (1.1)-(1.3) such that

$$\rho \in C(0, T; H^k(\mathbb{T})), \quad u \in C(0, T; H^k(\mathbb{T})) \cap L^2(0, T; H^{k+1}(\mathbb{T}))$$

for all $T > 0$, and $\rho(x, t) > 0$ for all $(x, t) \in \mathbb{T} \times \mathbb{R}^+$.

REMARK 1.6. The unique global solution in Theorem 1.4 satisfies

$$\partial_x u(x,t) \le \frac{c_p}{c_\mu}\rho(x,t)^{\gamma-\alpha}$$

for all $(x,t) \in \mathbb{T}\times\mathbb{R}^+$. Moreover, the proof provides a lower bound for the minimum of density ρ, see (6.11) and (6.14),

$$\min_{x\in\mathbb{T}} \rho(x,t) \ge \begin{cases} \left(\rho_m(0)^{\alpha-\gamma} + t\frac{c_p}{c_\mu}(\gamma-\alpha)\right)^{\frac{-1}{\gamma-\alpha}} & \text{when } \gamma > \alpha, \\ \rho_m(0)\exp\left(-t\frac{c_p}{c_\mu}\right) & \text{when } \gamma = \alpha. \end{cases}$$

Our last theorem establishes a bound on the time-averaged maximum density for a certain range of parameters assuming mean zero forcing.

THEOREM 1.7. *Assume that (ρ, u) is a sufficiently smooth solution to the system (1.1)–(1.3) on $[0, T^*)$. Assume that*

$$f = \partial_x g \tag{1.13}$$

for some periodic function g satisfying

$$g \in L^\infty(0,T^*;L^\infty(\mathbb{T})), \quad \text{and} \quad \partial_x g, \partial_t g \in L^\infty(0,T^*;L^\infty(\mathbb{T})).$$

Let us also assume that

$$\alpha \ge 1/2, \quad \gamma \in [\max\{2-\alpha,\alpha\}, \alpha+1], \quad \text{and} \quad c_p, c_\mu > 0.$$

Then, we have the following bound

$$\frac{1}{T}\int_0^T \|\rho(\cdot,t)\|_{L^\infty(\mathbb{T})}dt \le C_1 + \frac{1}{T}C_2, \tag{1.14}$$

where C_1 and C_2 are defined in equation (7.6). In particular, C_1 depends only on c_μ, c_p, α, γ, $\|\rho_0\|_{L^1}$, $\|\partial_x g\|_{L^\infty(0,T;L^\infty)}$, and $\|\partial_t g\|_{L^\infty(0,T;L^\infty)}$, whereas C_2 depends only on c_μ, c_p, γ, α, $\|\rho_0\|_{L^\infty}$, $\|\rho_0^{-1}\|_{L^\infty}$, $\|u_0\|_{L^2}$, $\|\partial_x\rho_0\|_{L^2}$, and $\|g\|_{L^\infty(0,T;L^\infty)}$. Consequently, if $T^* = \infty$ then

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|\rho(\cdot,t)\|_{L^\infty(\mathbb{T})}dt \le C_3 \tag{1.15}$$

where C_3 depends only on c_μ, c_p, α, γ, $\|\rho_0\|_{L^1}$, $\|\partial_x g\|_{L^\infty(0,\infty;L^\infty)}$, and $\|\partial_t g\|_{L^\infty(0,\infty;L^\infty)}$.

Theorem 1.7 applies for the viscous shallow water wave system (1.5),(1.6) for which global existence is established by Theorem 1.4. The interpretation of the bound (1.15) with $h \equiv \rho$ is that long-time average of the maximum surface height remains bounded, showing that, on average, no extreme events can develop.

The proofs are based on use of the Bresch-Desjardins entropy and analysis of the evolution of the active potential w. This object is the potential in the momentum equation (1.2): its gradient is the force

$$\rho D_t u = \partial_x w. \tag{1.16}$$

The potential

$$w = -p(\rho) + \mu(\rho)\partial_x u.$$

is unknown and combines the viscous stress with the pressure. As w depends on the unknowns and in turn determines their evolution, we refer to it as an *active potential*. Remarkably, w satisfies a *forced quadratic heat equation with linear drift and less degenerate diffusion* with the new dissipation term $\frac{\mu(\rho)}{\rho}\partial_x^2 w$. The active potential w contains one derivative of u and no derivative of ρ. On one hand, energy estimates for the coupled system of ρ and w allow us to control all the high Sobolev regularity of ρ and u as long as ρ is positive, leading to the proof of Theorem 1.1. On the other hand, the heat equation for w satisfies a maximum principle which enables us to obtain global regular solutions for a class of large data when the viscosity is strongly degenerate as in Theorem 1.5.

The fact that the active potential solves a nondegenerate evolution with a maximum principle was observed in [**17**] in the context of a 1D Hele Shaw model, where it served a similar role. The effective flux used in [**18**] is an active potential: there it was used by inverting the elliptic (nondegenerate) equation it solves at each fixed time.

2. A priori estimates: mass, energy and Bresch-Desjardins's entropy

Assume that (ρ, v) is a solution of (1.1)-(1.3) on the time interval $[0, T^*)$ such that

$$\rho \in C(0, T; H^3), \quad v \in C(0, T; H^3) \cap L^2(0, T; H^4)$$

for any $T < T^*$ and

$$\underline{\rho} := \inf_{t \in [0, T^*)} \min_{x \in \mathbb{T}} \rho(x, t) > 0. \tag{2.1}$$

In what follows we denote by $M(\cdot, \cdots, \cdot)$ a positive function that is increasing in each argument.

First, from the continuity equation (1.1), total mass is conserved:

$$\|\rho(\cdot, t)\|_{L^1(\mathbb{T})} = \|\rho_0\|_{L^1(\mathbb{T})}. \tag{2.2}$$

We have the following standard energy balance:

LEMMA 2.1 (Energy Balance). *Let $\bar{\rho} \geq 0$, and*

$$e := \frac{1}{2}\rho|u|^2 + \pi(\rho), \qquad \pi(\rho) = \rho \int_{\bar{\rho}}^{\rho} \frac{p(s)}{s^2} ds. \tag{2.3}$$

Then, the balance

$$\frac{d}{dt}\int_{\mathbb{T}} e(x, t) dx = -\int_{\mathbb{T}} \mu(\rho)|\partial_x u|^2 dx + \int_{\mathbb{T}} f\rho u dx \tag{2.4}$$

holds for any $t \in [0, T^*)$.

Using the equation of state for the density (1.4) and recalling that $\bar{\rho} \geq 0$ is an arbitrary constant that we are free to fix, we have an explicit formula for $\pi(\rho)$ from (2.3)

$$\pi(\rho) = c_p \rho \int_{\bar{\rho}}^{\rho} s^{\gamma - 2} ds = \begin{cases} \frac{c_p}{\gamma - 1}\rho^\gamma & \gamma > 1, \ \bar{\rho} = 0 \quad \text{or} \quad \gamma \in (0, 1), \ \bar{\rho} = \infty, \\ c_p \rho \log(\rho) & \gamma = 1, \ \bar{\rho} = 1. \end{cases} \tag{2.5}$$

Note that the function π satisfies

$$\pi''(\rho) = \frac{p'(\rho)}{\rho}.$$

LEMMA 2.2. *1. If $\gamma \in (1, \infty)$ and $c_p > 0$, then $\pi(\rho) \geq 0$ and*

$$\|e\|_{L^\infty(0,T;L^1)} + \|\mu(\rho)|\partial_x \rho|^2\|_{L^1(0,T;L^1)} \leq \Big(\|e(\cdot, 0)\|_{L^1} + \|f\|^2_{L^2(0,T;L^\infty)}\|\rho_0\|_{L^1(\mathbb{T})}\Big)\exp(2T). \tag{2.6}$$

2. If $\gamma \in (0, 1)$ and $c_p \neq 0$, then

$$\int_{\mathbb{T}} |\pi(\rho)| dx \leq \left|\frac{c_p}{\gamma - 1}\right| \int (\rho_0 + 1) dx \tag{2.7}$$

and there exists a positive constant $C = C(\gamma, \alpha, c_p, c_\mu)$ such that

$$\begin{aligned} &\|\rho u^2\|_{L^\infty(0,T;L^1)} + \|\mu(\rho)|\partial_x \rho|^2\|_{L^1(0,T;L^1)} \\ &\leq \Big(\|\rho_0 u_0^2\|_{L^1(\mathbb{T})} + C\big(1 + \|f\|^2_{L^2(0,T;L^\infty)}\big)\big(1 + \|\rho_0\|_{L^1(\mathbb{T})}\big)\Big)\exp(T). \end{aligned} \tag{2.8}$$

PROOF. First, using the mass conservation (2.2) we bound

$$\begin{aligned}\int_{\mathbb{T}} f\rho u \mathrm{d}x &\leq \frac{1}{2}\int_{\mathbb{T}} f^2\rho + \int_{\mathbb{T}}\frac{1}{2}\rho u^2 \\ &\leq \|f\|^2_{L^\infty(\mathbb{T})}\int_{\mathbb{T}}\rho + \int_{\mathbb{T}}\frac{1}{2}\rho u^2 \\ &\leq \|f\|^2_{L^\infty(\mathbb{T})}\|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}}\frac{1}{2}\rho u^2.\end{aligned} \tag{2.9}$$

1. If $\gamma \in (1,\infty)$ and $c_p > 0$, then we have $\pi(\rho) \geq 0$. It then follows from (2.9) that

$$\int_{\mathbb{T}} f\rho u \mathrm{d}x \leq \|f\|^2_{L^\infty(\mathbb{T})}\|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} e(x,t)\mathrm{d}x. \tag{2.10}$$

Ignoring the first term on the right hand side of (2.4), then using (2.10) and Grönwall's lemma we obtain

$$\|e\|_{L^\infty(0,T;L^1)} \leq \Big(\|e(\cdot,0)\|_{L^1} + \|f\|^2_{L^2(0,T;L^\infty)}\|\rho_0\|_{L^1(\mathbb{T})}\Big)\exp(T). \tag{2.11}$$

Next, we integrate (2.4) in time and use (2.10), (2.11) together with the fact that $e(x,t) \geq 0$ to get

$$\begin{aligned}\|\mu(\rho)|\partial_x\rho|^2\|_{L^1(0,T;L^1)} &\leq \|e(\cdot,0)\|_{L^1} + \|f\|^2_{L^2(0,T;L^\infty)}\|\rho_0\|_{L^1(\mathbb{T})} + T\|e\|_{L^\infty(0,T;L^1)} \\ &\leq \Big(\|e(\cdot,0)\|_{L^1} + \|f\|^2_{L^2(0,T;L^\infty)}\|\rho_0\|_{L^1(\mathbb{T})}\Big)(1+T)\exp(T) \\ &\leq \Big(\|e(\cdot,0)\|_{L^1} + \|f\|^2_{L^2(0,T;L^\infty)}\|\rho_0\|_{L^1(\mathbb{T})}\Big)\exp(2T).\end{aligned}$$

2. If $\gamma \in (0,1)$ then

$$\int_{\mathbb{T}}|\pi(\rho)|dx \leq \left|\frac{c_p}{\gamma-1}\right|\int(\rho(t)+1)dx \leq \left|\frac{c_p}{\gamma-1}\right|\int(\rho_0+1)dx \tag{2.12}$$

where we used the fact that $\rho^\gamma \leq \max\{1,\rho\}$ together with the mass conservation (1.1). Ignoring the first term on the right hand side of (2.4) and using (2.12), (2.9) we find

$$\begin{aligned}\int_{\mathbb{T}}\frac{1}{2}\rho u^2(x,t)\mathrm{d}x &\leq \int_{\mathbb{T}}\frac{1}{2}\rho_0 u_0^2\mathrm{d}x + \int_{\mathbb{T}}\pi(\rho_0(x))\mathrm{d}x - \int_{\mathbb{T}}\pi(\rho(x,t))\mathrm{d}x + \int_0^t\int_{\mathbb{T}} f\rho u(x,s)\mathrm{d}x\mathrm{d}s \\ &\leq \int_{\mathbb{T}}\frac{1}{2}\rho_0u_0^2\mathrm{d}x + C(\|\rho_0\|_{L^1(\mathbb{T})}+1) + \|f(t)\|^2_{L^\infty(\mathbb{T})}\|\rho_0\|_{L^1(\mathbb{T})} + \int_0^t\int_{\mathbb{T}}\frac{1}{2}\rho u^2(x,s)\mathrm{d}x\mathrm{d}s\end{aligned}$$

for some positive constant $C = C(\gamma,\alpha,c_p,c_\mu)$. Grönwall's lemma then yields

$$\|\rho u^2\|_{L^\infty(0,T;L^1)} \leq \Big(\|\rho_0u_0^2\|_{L^1(\mathbb{T})} + C\big(1+\|f\|^2_{L^2(0,T;L^\infty)}\big)\big(1+\|\rho_0\|_{L^1(\mathbb{T})}\big)\Big)\exp(T). \tag{2.13}$$

Again, we integrate (2.4) in time and use (2.9), (2.13), (2.12) to arrive at

$$\|\mu(\rho)|\partial_x\rho|^2\|_{L^1(0,T;L^1)} \leq \Big(\|\rho_0u_0^2\|_{L^1(\mathbb{T})} + C\big(1+\|f\|^2_{L^2(0,T;L^\infty)}\big)\big(1+\|\rho_0\|_{L^1(\mathbb{T})}\big)\Big)\exp(2T).$$

□

If either $\gamma \in (1,\infty)$ and $c_p > 0$ or $\gamma \in (0,1)$ and $c_p \neq 0$, it follows from (2.5)-(2.8) that

$$\|\sqrt{\rho}u\|_{L^\infty(0,T;L^2)} \leq M(E_0,\|f\|_{L^2(0,T;L^\infty)},T), \tag{2.14}$$

$$\|\rho^{\frac{\alpha}{2}}\partial_x u\|_{L^2(0,T;L^2)} \leq M(E_0,\|f\|_{L^2(0,T;L^\infty)},T), \tag{2.15}$$

$$\|\rho\|_{L^\infty(0,T;L^{\max\{1,\gamma\}})} \leq M(E_0,\|f\|_{L^2(0,T;L^\infty)},T) \tag{2.16}$$

where

$$E_0 := \|\rho_0u_0^2\|_{L^1(\mathbb{T})} + \|\rho_0^\gamma\|_{L^1(\mathbb{T})} + \|\rho_0\|_{L^1(\mathbb{T})}. \tag{2.17}$$

LEMMA 2.3 (Bresch-Desjardins's Entropy [**19**]). *Let*

$$s := \frac{\rho}{2}\left|u + \frac{\partial_x \rho}{\rho^2}\mu(\rho)\right|^2 + \pi(\rho). \tag{2.18}$$

Then, the balance

$$\frac{d}{dt}\int_{\mathbb{T}} s(x,t)dx = -\int_{\mathbb{T}} |\partial_x \rho|^2 \mu(\rho)\frac{p'(\rho)}{\rho^2}dx + \int_{\mathbb{T}} f\rho\big(u + \frac{\partial_x \rho}{\rho^2}\mu(\rho)\big)dx \tag{2.19}$$

holds for any $t \in [0, T^*)$.

A proof of Lemma 2.3 can be found in [**19, 20, 21**] and is given for completeness in the appendix. The first term on the right hand side of (2.19) is negative whenever $c_p > 0$ and positive whenever $c_p < 0$.

LEMMA 2.4. *Define*

$$E_1 := E_0 + \|\partial_x(\rho_0^{\alpha-\frac{1}{2}})\|_{L^2(\mathbb{T})}. \tag{2.20}$$

1. If $c_p > 0$ *and* $\gamma \neq 1,\ \gamma \geq \alpha - \frac{1}{2},\ \alpha > \frac{1}{2}$*, then*

$$\|\rho\|_{L^\infty(0,T;L^\infty)} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T). \tag{2.21}$$

2. If $c_p < 0$ *and* $0 < \gamma \leq \alpha$*,* $\gamma < 1$*,* $\alpha \in (\frac{1}{2}, \frac{3}{2}]$*, then*

$$\|\rho\|_{L^\infty(0,T;L^\infty)} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T). \tag{2.22}$$

3. Under the conditions of 1. or 2., we have

$$\|\partial_x \rho\|_{L^\infty(0,T;L^2)} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T). \tag{2.23}$$

REMARK 2.5. The bound for (2.21) is independent of $\underline{\rho}$. This fact will be important in the proof of Theorem 1.4.

PROOF. 1. Since $c_p > 0$, the first term on the right hand side of (2.19) is negative, and thus

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} s(x,t)\mathrm{d}x &\leq \int_{\mathbb{T}} f\rho\big(u + \frac{\partial_x \rho}{\rho^2}\mu(\rho)\big)\mathrm{d}x \\
&\leq \frac{1}{2}\int_{\mathbb{T}} f^2\rho\mathrm{d}x + \int_{\mathbb{T}} \frac{1}{2}\rho\big(u + \frac{\partial_x \rho}{\rho^2}\mu(\rho)\big)^2\mathrm{d}x \\
&\leq \frac{1}{2}\|f(t)\|^2_{L^\infty(\mathbb{T})}\|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} \frac{1}{2}\rho\big(u + \frac{\partial_x \rho}{\rho^2}\mu(\rho)\big)^2\mathrm{d}x.
\end{aligned} \tag{2.24}$$

When $\gamma > 1$ we have $\pi(\rho) \geq 0$, hence $s > 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} s(x,t)\mathrm{d}x \leq \frac{1}{2}\|f(t)\|^2_{L^\infty(\mathbb{T})}\|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} s(x,t)\mathrm{d}x.$$

Grönwall's lemma then yields

$$\|s\|_{L^\infty(0,T;L^1)} \leq \Big(\|s(0,\cdot)\|_{L^1(\mathbb{T})} + \|f\|^2_{L^2(0,T;L^\infty)}\|\rho_0\|_{L^1(\mathbb{T})}\Big)\exp(T). \tag{2.25}$$

We combine (2.25) with (2.14) and the fact that

$$\|s(0,\cdot)\|_{L^1(\mathbb{T})} \leq \|\rho_0 u_0^2\|_{L^1(\mathbb{T})} + \|\partial_x(\rho_0^{\alpha-\frac{1}{2}})\|^2_{L^2(\mathbb{T})}. \tag{2.26}$$

In view of (2.15), this implies

$$\|\partial_x(\rho^{\alpha-\frac{1}{2}})\|_{L^\infty(0,T;L^2(\mathbb{T}))} \leq M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T) \tag{2.27}$$

with

$$E_1 = E_0 + \|\partial_x(\rho_0^{\alpha-\frac{1}{2}})\|_{L^2(\mathbb{T})}.$$

On the other hand, when $\gamma \in (0,1)$ we write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \frac{1}{2}\rho\big(u+\frac{\partial_x\rho}{\rho^2}\mu(\rho)\big)^2\mathrm{d}x \le \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \pi(\rho(x,t))\mathrm{d}x + \frac{1}{2}\|f(t)\|^2_{L^\infty(\mathbb{T})}\|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} \frac{1}{2}\rho\big(u+\frac{\partial_x\rho}{\rho^2}\mu(\rho)\big)^2\mathrm{d}x$$

where we recall from (2.7)

$$\int_{\mathbb{T}} |\pi(\rho)|dx \le \left|\frac{c_p}{\gamma-1}\right| \int (\rho_0+1)dx. \tag{2.28}$$

It follows from Grönwall's lemma that

$$\begin{aligned}
\sup_{t\in[0,T]} &\int_{\mathbb{T}} \frac{1}{2}\rho\big(u+\frac{\partial_x\rho}{\rho^2}\mu(\rho)\big)^2(x,t)\mathrm{d}x \\
&\le \Big(\int_{\mathbb{T}} \frac{1}{2}\rho\big(u+\frac{\partial_x\rho}{\rho^2}\mu(\rho)\big)^2(x,0)\mathrm{d}x + C(1+\|f\|^2_{L^2(0,T;L^\infty)})\big(1+\|\rho_0\|_{L^1(\mathbb{T})}\big)\Big)\exp(T) \\
&\le M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T).
\end{aligned}$$

Combined with (2.14), this implies the bound (2.27) when $\gamma \in (0,1)$.
Next, we recall from (2.16) the bound for $\|\rho^\gamma\|_{L^1(\mathbb{T})}$. By the assumption that $\gamma \ge \alpha - \frac{1}{2}$, we obtain

$$\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0,T;L^1)} \le C(1+\|\rho^\gamma\|_{L^\infty(0,T;L^1)} + \|\rho\|_{L^\infty(0,T;L^1)}) \le M(E_0, \|f\|_{L^2(0,T;L^\infty)}, T).$$

This combined with (2.27) and Nash's inequality

$$\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0,T;L^2)} \le C\|\rho^{\alpha-\frac{1}{2}}\|^{2/3}_{L^\infty(0,T;L^1)}\|\partial_x(\rho^{\alpha-\frac{1}{2}})\|^{1/3}_{L^\infty(0,T;L^2)} + C\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0,T;L^1)}$$

leads to

$$\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0,T;H^1)} \le M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T).$$

The stated bound (2.21) then follows by Sobolev embedding $H^1 \subseteq L^\infty$.

2. In this case, $c_p < 0$ and thus the first term on the right hand side of (2.19) is positive and is equal to

$$\begin{aligned}
-\gamma c_p c_\mu \int_{\mathbb{T}} |\rho^{(\gamma+\alpha-3)/2}\partial_x\rho|^2\mathrm{d}x &\le -2\gamma\frac{c_p}{c_\mu}\int_{\mathbb{T}} \rho^{\gamma-\alpha+1}\left(|u+c_\mu\rho^{\alpha-2}\partial_x\rho|^2 + |u|^2\right)\mathrm{d}x \\
&= -2\gamma\frac{c_p}{c_\mu}\int_{\mathbb{T}} \rho^{\gamma-\alpha}\left(s(x,t) - \pi(\rho) + \rho|u|^2\right)\mathrm{d}x.
\end{aligned}$$

Note that (2.24) provides the bound

$$\int_{\mathbb{T}} f\rho\big(u+\frac{\partial_x\rho}{\rho^2}\mu(\rho)\big)\mathrm{d}x \le \frac{1}{2}\|f(t)\|^2_{L^\infty(\mathbb{T})}\|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} \frac{1}{2}\rho\big(u+\frac{\partial_x\rho}{\rho^2}\mu(\rho)\big)^2\mathrm{d}x.$$

In addition, since $\gamma \in (0,1)$, part 2 of Lemma 2.2 provides a bound for $\pi(\rho)$ and ρu^2. Moreover, note that when $c_p < 0$ and $\gamma \in (0,1)$ we have $\pi(\rho), s \ge 0$. Using these together with the assumption that $\gamma \le \alpha$ we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} s(x,t)\mathrm{d}x &\leq -2\gamma\frac{c_p}{c_\mu}\int_{\mathbb{T}}\rho^{\gamma-\alpha}\left(s(x,t)-\pi(\rho)+\rho|u|^2\right)dx + \|f(t)\|^2_{L^\infty(\mathbb{T})}\|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} s(x,t)\mathrm{d}x. \\
&\leq -2\gamma\frac{c_p}{c_\mu}(\frac{1}{\underline{\rho}})^{\gamma-\alpha}\int_{\mathbb{T}}\left(s(x,t)-\pi(\rho)+\rho|u|^2\right)\mathrm{d}x + \|f(t)\|^2_{L^\infty(\mathbb{T})}\|\rho_0\|_{L^1(\mathbb{T})} + \int_{\mathbb{T}} s(x,t)\mathrm{d}x. \\
&\leq \Big(-2\gamma\frac{c_p}{c_\mu}(\frac{1}{\underline{\rho}})^{\gamma-\alpha}+1\Big)\int_{\mathbb{T}} s(x,t)\mathrm{d}x - 2\gamma\frac{c_p}{c_\mu}(\frac{1}{\underline{\rho}})^{\gamma-\alpha}\int_{\mathbb{T}}\left(-\pi(\rho)+\rho|u|^2\right)\mathrm{d}x \\
&\quad + \|f(t)\|^2_{L^\infty(\mathbb{T})}\|\rho_0\|_{L^1(\mathbb{T})} \\
&\leq \Big(-2\gamma\frac{c_p}{c_\mu}(\frac{1}{\underline{\rho}})^{\gamma-\alpha}+1\Big)\int_{\mathbb{T}} s(x,t)\mathrm{d}x + M(E_0,\|f\|_{L^2(0,T;L^\infty)},\frac{1}{\underline{\rho}},T) \\
&\quad + \|f(t)\|^2_{L^\infty(\mathbb{T})}\|\rho_0\|_{L^1(\mathbb{T})}
\end{aligned}
$$

for $t \leq T$. By Grönwall's lemma and (2.26), we deduce that

$$
\begin{aligned}
\|s\|_{L^\infty(0,T;L^1)} &\leq M(E_0+\|s(\cdot,0)\|_{L^1(\mathbb{T})},\|f\|_{L^2(0,T;L^\infty)},\frac{1}{\underline{\rho}},T) \\
&\leq M(E_1,\|f\|_{L^2(0,T;L^\infty)},\frac{1}{\underline{\rho}},T).
\end{aligned}
$$

Combining this with (2.14) gives

$$
\|\partial_x(\rho^{\alpha-\frac{1}{2}})\|_{L^\infty(0,T;L^2)} \leq M(E_1,\|f\|_{L^2(0,T;L^\infty)},\frac{1}{\underline{\rho}},T). \tag{2.29}
$$

Since $\alpha-\frac{1}{2}\in(0,1]$, the mass conservation (2.16) implies

$$
\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0,T;L^1)} \leq C(1+\|\rho_0\|_{L^1(\mathbb{T})}). \tag{2.30}
$$

Combined with (2.29), this yields

$$
\|\rho^{\alpha-\frac{1}{2}}\|_{L^\infty(0,T;H^1)} \leq M(E_1,\|f\|_{L^2(0,T;L^\infty)},\frac{1}{\underline{\rho}},T)
$$

from which (2.22) follows.

3. The bound (2.23) follows from (2.21) & (2.27) and (2.22) & (2.29) respectively. □

3. The active potential

We introduce in this section the 'active potential $w := -p(\rho)+\mu(\rho)\partial_x u$. This is a good unknown upon which much of the anlaysis is based. We first show that w satisfies a *forced quadratic heat equation with linear drift*.

PROPOSITION 3.1 (w–equation). *Let*

$$
w := -p(\rho)+\mu(\rho)\partial_x u. \tag{3.1}
$$

Then w satisfies

$$
\begin{aligned}
\partial_t w = \rho^{-1}\mu(\rho)\partial_x^2 w - (u+\mu(\rho)\frac{\partial_x\rho}{\rho^2})\partial_x w + \left(\rho\frac{p'(\rho)}{\mu(\rho)} - 2\frac{(\rho\mu'(\rho)+\mu(\rho))}{\mu(\rho)^2}p(\rho)\right)w \\
- \frac{(\rho\mu'(\rho)+\mu(\rho))}{\mu(\rho)^2}w^2 + \left(\rho\frac{p'(\rho)}{\mu(\rho)} - \frac{(\rho\mu'(\rho)+\mu(\rho))}{\mu(\rho)^2}p(\rho)\right)p(\rho) + \mu(\rho)\partial_x f.
\end{aligned} \tag{3.2}
$$

Moreover, the following balance holds

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}}\frac{1}{2}|w|^2(x,t)dx = &-\int_{\mathbb{T}}\rho^{-1}\mu(\rho)|\partial_x w|^2dx - \int_{\mathbb{T}}\left(u+\frac{\mu'(\rho)}{\rho}\partial_x\rho\right)w\partial_x w dx\\ &+\int_{\mathbb{T}}\left(\rho\frac{p'(\rho)}{\mu(\rho)} - 2\frac{(\rho\mu'(\rho)+\mu(\rho))}{\mu(\rho)^2}p(\rho)\right)|w|^2dx - \int_{\mathbb{T}}\frac{(\rho\mu'(\rho)+\mu(\rho))}{\mu(\rho)^2}w^3dx\\ &+\int_{\mathbb{T}}\left(\rho\frac{p'(\rho)}{\mu(\rho)} - \frac{(\rho\mu'(\rho)+\mu(\rho))}{\mu(\rho)^2}p(\rho)\right)p(\rho)wdx + \int_{\mathbb{T}}\mu(\rho)\partial_x f w dx.\end{aligned} \tag{3.3}$$

PROOF. From the definition of $w := -p(\rho) + \mu(\rho)\partial_x u$ given by (3.1), we compute

$$\partial_x w = (\partial_x\rho)(-p'(\rho)+\mu'(\rho)\partial_x u) + \mu(\rho)\partial_x^2 u. \tag{3.4}$$

Thus, we have

$$\begin{aligned}\partial_t w &= (\partial_t\rho)(-p'(\rho)+\mu'(\rho)\partial_x u) + \mu(\rho)\partial_t\partial_x u\\ &= -\partial_x(u\rho)(-p'(\rho)+\mu'(\rho)\partial_x u) + \mu(\rho)\partial_t\partial_x u\\ &= -\rho\partial_x u(-p'(\rho)+\mu'(\rho)\partial_x u) - u(\partial_x w - \mu(\rho)\partial_x^2 u) + \mu(\rho)\partial_t\partial_x u.\end{aligned} \tag{3.5}$$

The momentum equation (1.2) gives

$$\begin{aligned}\partial_t u &= -u\partial_x u + \rho^{-1}\partial_x w + f,\\ \partial_t\partial_x u &= -\partial_x u\partial_x u - u\partial_x^2 u - \frac{\partial_x\rho}{\rho^2}\partial_x w + \rho^{-1}\partial_x^2 w + \partial_x f.\end{aligned}$$

Combining the above results, we find

$$\begin{aligned}\partial_t w = &-\rho\partial_x u(-p'(\rho)+\mu'(\rho)\partial_x u) - u\partial_x w + u\mu(\rho)\partial_x^2 u\\ &-\mu(\rho)(|\partial_x u|^2 + u\partial_x^2 u) - \mu(\rho)\frac{\partial_x\rho}{\rho^2}\partial_x w + \rho^{-1}\mu(\rho)\partial_x^2 w + \mu(\rho)\partial_x f\\ = &\ \rho^{-1}\mu(\rho)\partial_x^2 w + \rho(\partial_x u)p'(\rho) - (\rho\mu'(\rho)+\mu(\rho))|\partial_x u|^2 - (u+\mu(\rho)\frac{\partial_x\rho}{\rho^2})\partial_x w + \mu(\rho)\partial_x f\\ = &\ \rho^{-1}\mu(\rho)\partial_x^2 w + \rho(w+p(\rho))\frac{p'(\rho)}{\mu(\rho)} - \frac{(\rho\mu'(\rho)+\mu(\rho))}{\mu(\rho)^2}(w+p(\rho))^2 - (u+\mu(\rho)\frac{\partial_x\rho}{\rho^2})\partial_x w + \mu(\rho)\partial_x f\end{aligned}$$

which, after rearrangement, establishes Eq. (3.2). For the energy, multiplying the equation (3.2) by w yields

$$\begin{aligned}\partial_t\left(\frac{1}{2}|w|^2\right) = &\ \partial_x(\frac{\mu(\rho)}{\rho}w\partial_x w) - \frac{\mu(\rho)}{\rho}|\partial_x w|^2 - \partial_x(\frac{\mu(\rho)}{\rho})w\partial_x w - \left(u+\frac{\mu(\rho)}{\rho^2}\partial_x\rho\right)w\partial_x w\\ &+\left(\rho\frac{p'(\rho)}{\mu(\rho)} - 2\frac{(\rho\mu'(\rho)+\mu(\rho))}{\mu(\rho)^2}p(\rho)\right)|w|^2 - \frac{(\rho\mu'(\rho)+\mu(\rho))}{\mu(\rho)^2}w^3\\ &+\left(\rho\frac{p'(\rho)}{\mu(\rho)} - \frac{(\rho\mu'(\rho)+\mu(\rho))}{\mu(\rho)^2}p(\rho)\right)p(\rho)w + \mu(\rho)\partial_x f w.\end{aligned}$$

Integrating in space yields the balance. □

Let us remark that in (3.2) the new viscosity coefficient is $\frac{\mu(\rho)}{\rho}$ which is less degenerate than the original viscosity $\mu(\rho)$ for the momentum equation. In particular, when $\mu(\rho) = c_\mu\rho^\alpha$ with $\alpha \leq 1$, $\frac{\mu(\rho)}{\rho}$ is not degenerate when ρ goes to 0. Energy estimates for the coupled system of ρ and w will allow us to control all the high Sobolev regularity of ρ and w as long as ρ is positive. This leads to the proof of our continuation criterion in Theorem 1.1: no singularity occurs before vacuum formation.

Furthermore, (3.2) can be regarded as a nonlinear heat equation with variable coefficients. Note that the zero-order term in (3.2) has the form $\lambda\rho^{2\gamma-\alpha}$ where λ depends only on c_μ and c_p. It can be readily seen that when the zero-order term and the forcing term in (3.2) are nonpositive, w remains nonpositive if it is nonpositive initially. This fact will be exploited as the key ingredient in proving the existence of global solutions in Theorem 1.5 when the viscosity is strongly degenerate.

4. Proof of Theorem 1.1

Throughout this section, we suppose that

$$0<\underline{\rho}\le\rho(x,t)\qquad t\in[0,T^*),\quad x\in\mathbb{T}. \tag{4.1}$$

and assume either

(i) $c_p>0$ and $\alpha>\frac{1}{2}, \gamma\ge\alpha-\frac{1}{2}, \gamma\neq 1$

(ii) $c_p<0$ and $\alpha\in(\frac{1}{2},\frac{3}{2}], 0<\gamma\le\alpha, \gamma<1$.

Under these assumptions, by Lemma 2.4, we have

$$\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{T}))}\le M(E_1,\|f\|_{L^2(0,T;L^\infty)},\frac{1}{\underline{\rho}},T), \tag{4.2}$$

and

$$\|\partial_x\rho\|_{L^\infty(0,T;L^2(\mathbb{T}))}\le M(E_1,\|f\|_{L^2(0,T;L^\infty)},\frac{1}{\underline{\rho}},T). \tag{4.3}$$

LEMMA 4.1.

$$\begin{aligned}\|w\|_{L^\infty(0,T;L^2)}+\|\partial_x w\|_{L^2(0,T;L^2)}+\|\partial_x u\|_{L^\infty(0,T;L^2)}+\|\partial_x^2 u\|_{L^2(0,T;L^2)}&\\ \le M(E_2,\|f\|_{L^2(0,T;H^1)},\frac{1}{\underline{\rho}},T),&\end{aligned} \tag{4.4}$$

where $E_2=E_1+\|\partial_x u_0\|_{L^2}$.

PROOF. As a consequence of (4.1), (4.2), and (3.3), there exist $c:=c(E_1,\|f\|_{L^2(0,T;L^\infty)},\frac{1}{\underline{\rho}},T)>0$ and $C:=C(E_1,\|f\|_{L^2(0,T;L^\infty)},\frac{1}{\underline{\rho}},T)>0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}\frac{1}{2}|w|^2(x,t)\mathrm{d}x\le&-\frac{1}{c}\int_{\mathbb{T}}|\partial_x w|^2dx+\int_{\mathbb{T}}(|u|+C|\partial_x\rho|)\,|w\partial_x w|dx\\ &+C\left(\int_{\mathbb{T}}|w|^2dx+\int_{\mathbb{T}}|w|^3dx+\int_{\mathbb{T}}|\partial_x f|^2dx+1\right).\end{aligned} \tag{4.5}$$

We bound

$$\int_{\mathbb{T}}|\partial_x w w u|\,\mathrm{d}x\le\|\partial_x w\|_{L^2}\|w\|_{L^2}\|u\|_{L^\infty}\le C_1\|\partial_x w\|_{L^2}\|w\|_{L^2}\|u\|_{H^1}\le\frac{1}{4c}\|\partial_x w\|_{L^2}^2+C\|w\|_{L^2}^2\|u\|_{H^1}^2$$

where C_1 denotes absolute constants throughout this proof. Next, applying Gagliardo-Nirenberg's inequality and Young's inequality implies

$$\int_{\mathbb{T}}|w|^3\,\mathrm{d}x\le\|w\|_{L^3}^3\le C_1(\|\partial_x w\|_{L^2}^{\frac{1}{2}}\|w\|_{L^2}^{\frac{5}{2}}+\|w\|_{L^2}^3)\le\frac{1}{4c}\|\partial_x w\|_{L^2}^2+C\|w\|_{L^2}^{\frac{10}{3}}+C\|w\|_{L^2}^3$$

and

$$\begin{aligned}\int_{\mathbb{T}} |\partial_x w w \partial_x \rho| \,\mathrm{d}x &\le \|\partial_x w\|_{L^2}\|w\|_{L^\infty}\|\partial_x\rho\|_{L^2}\\ &\le C_1\|\partial_x w\|_{L^2}(\|\partial_x w\|_{L^2}^{\frac12}\|w\|_{L^2}^{\frac12}+\|w\|_{L^2})\|\partial_x\rho\|_{L^2}\\ &\le C_1\|\partial_x w\|_{L^2}^{\frac32}\|w\|_{L^2}^{\frac12}\|\partial_x\rho\|_{L^2}+C_1\|\partial_x w\|_{L^2}\|w\|_{L^2}\|\partial_x\rho\|_{L^2}\\ &\le \frac{1}{4c}\|\partial_x w\|_{L^2}^2+C\|w\|_{L^2}^2\|\partial_x\rho\|_{L^2}^4+C\|w\|_{L^2}^2\|\partial_x\rho\|_{L^2}^2.\end{aligned}$$

Putting together the above bounds, and interpolating, yields the following inequality

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2+\frac{1}{4c}\|\partial_x w\|_{L^2}^2\le C\|w\|_{L^2}^2(\|w\|_{L^2}^2+\|\partial_x\rho\|_{L^2}^4+1)+C\|\partial_x f\|_{L^2}^2+C. \tag{4.6}$$

In view of (4.3), we have

$$\int_0^T \|\partial_x\rho(\cdot,t)\|_{L^2}^4\mathrm{d}t\le M(E_1,\|f\|_{L^2(0,T;L^\infty)},\frac{1}{\underline{\rho}},T).$$

Furthermore, using the definition of w together with bounds (4.2) & (2.15), we have

$$\|w\|_{L^2(0,T;L^2)}\le M(E_1,\|f\|_{L^2(0,T;L^\infty)},\frac{1}{\underline{\rho}},T).$$

The last two displays, together with Grönwall's lemma applied to (4.6), yields the bound

$$\begin{aligned}&\|w\|_{L^\infty(0,T;L^2(\mathbb{T}))}+\|\partial_x w\|_{L^2(0,T;L^2(\mathbb{T}))}\\ &\qquad\le M(\|w_0\|_{L^2},c,C,E_1,\|f\|_{L^1(0,T;H^1)},\frac{1}{\underline{\rho}},T)\le M(E_1,\|f\|_{L^1(0,T;H^1)},\frac{1}{\underline{\rho}},T).\end{aligned}$$

Here, we used the fact that

$$\|w_0\|_{L^2}^2\le 2c_p^2\|\rho_0\|_{L^\infty}^{2\gamma}+2c_\mu^2\|\rho_0\|_{L^\infty}^{2\alpha}\|\partial_x u_0\|_{L^2}^2.$$

The above bound can be used to obtain similar estimates for $\|\partial_x u\|_{L^\infty(0,T;L^2)}$ and $\|\partial_x^2 u\|_{L^2(0,T;L^2)}$ directly from the definition of w (3.1). □

LEMMA 4.2.

$$\begin{aligned}&\|\partial_x^2\rho\|_{L^\infty(0,T;L^2)}+\|\partial_x w\|_{L^\infty(0,T;L^2)}+\|\partial_x^2 w\|_{L^2(0,T;L^2)}\\ &\qquad+\|\partial_x^2 u\|_{L^\infty(0,T;L^2)}+\|\partial_x^3 u\|_{L^2(0,T;L^2)}\le M(E_3,\|f\|_{L^1(0,T;H^1)},\frac{1}{\underline{\rho}},T)\end{aligned} \tag{4.7}$$

where

$$E_3=E_2+\|\partial_x^2\rho_0\|_{L^2}+\|\partial_x^2 u_0\|_{L^2}.$$

PROOF. To prove this lemma, we obtain energy estimates for the mass equation (1.1) and the w–equation (3.2) simultaneously. The proof proceeds in 4 steps.

Step 1. Let $m\ge 2$ be an arbitrary integer. Differentiating equation (1.1) m times, then multiplying the resulting equation by $\partial_x^m\rho$ and integrating in space we get

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}|\partial_x^m\rho|^2&=-\int_T\partial_x^m(u\partial_x\rho)\partial_x^m\rho-\int_{\mathbb{T}}\partial_x^m(\rho\partial_x u)\partial_x^m\rho\\ &=-\int_{\mathbb{T}}u\partial_x\partial_x^m\rho\partial_x^m\rho-\int_{\mathbb{T}}\big([\partial_x^m,u]\partial_x\rho\big)\partial_x^m\rho-\int_{\mathbb{T}}\big([\partial_x^m,\rho]\partial_x u\big)\partial_x^m\rho-\int_{\mathbb{T}}\rho\partial_x^{m+1}u\partial_x^m\rho.\end{aligned}$$

Using the Kato-Ponce commutator estimate [**23**] and the inequality

$$\|\partial_x g\|_{L^\infty(\mathbb{T})}\le C\|\partial_x^2 g\|_{L^2(\mathbb{T})}\le C_n\|\partial_x^n g\|_{L^2(\mathbb{T})}\quad\forall n\ge 3,$$

we have

$$\|[\partial_x^m, u]\partial_x\rho\|_{L^2} \le C\|\partial_x u\|_{L^\infty}\|\partial_x^{m-1}\partial_x\rho\|_{L^2} + C\|\partial_x^m u\|_{L^2}\|\partial_x\rho\|_{L^\infty} \le C\|\partial_x^m u\|_{L^2}\|\partial_x^m\rho\|_{L^2}$$

and

$$\|[\partial_x^m, \rho]\partial_x u\|_{L^2} \le C\|\partial_x\rho\|_{L^\infty}\|\partial_x^{m-1}\partial_x u\|_{L^2} + C\|\partial_x^m\rho\|_{L^2}\|\partial_x u\|_{L^\infty} \le C\|\partial_x^m u\|_{L^2}\|\partial_x^m\rho\|_{L^2}.$$

In addition,

$$\left|\int_{\mathbb{T}} u\partial_x\partial_x^m\rho\partial_x^m\rho\right| = \frac{1}{2}\left|\int_{\mathbb{T}}\partial_x u|\partial_x^m\rho|^2\right| \le \frac{1}{2}\|\partial_x u\|_{L^\infty}\|\partial_x^m\rho\|_{L^2}^2 \le C\|\partial_x^m u\|_{L^2}\|\partial_x^m\rho\|_{L^2}^2.$$

We thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^m\rho\|_{L^2}^2 \le C\|\partial_x^m u\|_{L^2}\|\partial_x^m\rho\|_{L^2}^2 + \|\rho\|_{L^\infty}\|\partial_x^{m+1}u\|_{L^2}\|\partial_x^m\rho\|_{L^2}. \tag{4.8}$$

Step 2. Recall equation (3.2) with power-law pressure and viscosity

$$\begin{aligned}\partial_t w = c_\mu\rho^{\alpha-1}\partial_x^2 w - (u + c_\mu\rho^{\alpha-2}\partial_x\rho)\partial_x w + \frac{c_p}{c_\mu}(\gamma - 2(\alpha+1))\rho^{\gamma-\alpha}w \\ - \frac{1}{c_\mu}(\alpha+1)\rho^{-\alpha}w^2 + \frac{c_p^2}{c_\mu}(\gamma-(\alpha+1))\rho^{2\gamma-\alpha} + c_\mu\rho^\alpha\partial_x f.\end{aligned} \tag{4.9}$$

Differentiating in space, multiplying the resulting equation by $\partial_x w$ and integrating by parts in x leads to

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}|\partial_x w|^2 &= -c_\mu\int_{\mathbb{T}}\rho^{\alpha-1}|\partial_x^2 w|^2 + \int_{\mathbb{T}}(u + c_\mu\rho^{\alpha-2}\partial_x\rho)\partial_x w\partial_x^2 w + \frac{c_p}{c_\mu}(\gamma-2(\alpha+1))\int_{\mathbb{T}}|\partial_x w|^2\rho^{\gamma-\alpha} \\ &+ \frac{c_p}{c_\mu}(\gamma-\alpha)(\gamma-2(\alpha+1))\int_{\mathbb{T}} w\rho^{\gamma-\alpha-1}\partial_x w\partial_x\rho \\ &- \frac{2}{c_\mu}(\alpha+1)\int_{\mathbb{T}}\rho^{-\alpha}w|\partial_x w|^2 + \frac{\alpha}{c_\mu}(\alpha+1)\int_{\mathbb{T}} w^2\partial_x w\partial_x\rho\rho^{-\alpha-1} \\ &+ \frac{c_p^2}{c_\mu}(2\gamma-\alpha)(\gamma-(\alpha+1))\int_{\mathbb{T}}\rho^{2\gamma-\alpha-1}\partial_x w\partial_x\rho - c_\mu\int_{\mathbb{T}}\rho^\alpha\partial_x^2 w\partial_x f \\ &=: -c_\mu\int_{\mathbb{T}}\rho^{\alpha-1}|\partial_x^2 w|^2 + \sum_{j=1}^7 H_j.\end{aligned}$$

after integrating by parts. By virtue of (4.1) and (4.2), there exists $c := c(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T) > 0$ such that

$$c_\mu\int_{\mathbb{T}}\rho^{\alpha-1}|\partial_x^2 w|^2 \ge \frac{1}{c}\int_{\mathbb{T}}|\partial_x^2 w|^2.$$

Note, under our assumptions ρ and $1/\rho$ are bounded (see (4.1) and (4.2)). Therefore all coefficients involving L^∞ norms of ρ to some power can be bounded by some constant $C = M(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T, \gamma, \alpha)$. The constant may change line by line.

- Estimate for H_1:

$$\begin{aligned}\left|\int_{\mathbb{T}}(u + c_\mu\rho^{\alpha-2}\partial_x\rho)\partial_x w\partial_x^2 w\right| &\le \|\partial_x^2 w\|_{L^2}\|\partial_x w\|_{L^2}\|u\|_{L^\infty} + C\|\partial_x^2 w\|_{L^2}\|\partial_x w\|_{L^2}\|\partial_x\rho\|_{L^\infty} \\ &\le \frac{1}{10c}\|\partial_x^2 w\|_{L^2}^2 + C\|\partial_x w\|_{L^2}^2\|u\|_{H^1}^2 + C\|\partial_x w\|_{L^2}^2\|\partial_x^2\rho\|_{L^2}^2.\end{aligned}$$

- Estimate for H_2:

$$\left|\int_{\mathbb{T}}|\partial_x w|^2\rho^{\gamma-\alpha}\right| \le C\|\partial_x w\|_{L^2}^2.$$

- Estimate for H_3:

$$\left|\int_{\mathbb{T}} w\partial_x w\partial_x\rho\rho^{\gamma-\alpha-1}\right| \leq \|\rho^{\gamma-\alpha-1}\|_\infty\|w\|_{L^\infty}\|\partial_x w\|_{L^2}\|\partial_x\rho\|_{L^2}$$
$$\leq C\|w\|_{L^2}\|\partial_x w\|_{L^2}\|\partial_x\rho\|_{L^2} + C\|\partial_x w\|_{L^2}^2\|\partial_x\rho\|_{L^2}.$$

- Estimate for H_4:

$$\left|\int_{\mathbb{T}} \rho^{-\alpha}w|\partial_x w|^2\right| \leq \frac{1}{\underline{\rho}^\alpha}\|w\|_{L^\infty}\|\partial_x w\|_{L^2}^2 \leq \frac{1}{4\underline{\rho}^{2\alpha}}\|w\|_{L^\infty}^2 + C\|\partial_x w\|_{L^2}^4$$
$$\leq C\|w\|_{H^1}^2 + C\|\partial_x w\|_{L^2}^4.$$

- Estimate for H_5:

$$\left|\int_{\mathbb{T}} w^2\partial_x w\partial_x\rho\rho^{-\alpha-1}\right| \leq \frac{1}{\underline{\rho}^{1+\alpha}}\|\partial_x w\|_{L^2}\|w\|_{L^\infty}^2\|\partial_x\rho\|_{L^2}$$
$$\leq C\|\partial_x w\|_{L^2}\|w\|_{H^1}^2\|\partial_x\rho\|_{L^2}$$
$$\leq C\|\partial_x w\|_{L^2}\|w\|_{L^2}^2\|\partial_x\rho\|_{L^2} + C\|\partial_x w\|_{L^2}^3\|\partial_x\rho\|_{L^2}.$$

- Estimate for H_6:

$$\left|\int_{\mathbb{T}} \rho^{\gamma-\alpha-1}\partial_x w\partial_x\rho\right| \leq C\|\partial_x w\|_{L^2}\|\partial_x\rho\|_{L^2}.$$

- Estimate for H_7:

$$\left|\int_{\mathbb{T}} \rho^\alpha\partial_x^2 w\partial_x f\right| \leq \frac{1}{10c}\|\partial_x^2 w\|_{L^2}^2 + C\|\partial_x f\|_{L^2}^2.$$

Putting together the above estimates gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x w\|_{L^2}^2 + \frac{1}{2c}\|\partial_x^2 w\|_{L^2}^2\\&\leq C\left(\|\partial_x w\|_{L^2}^2\|u\|_{H^1}^2 + \|\partial_x w\|_{L^2}^2\|\partial_x^2\rho\|_{L^2}^2 + \|\partial_x w\|_{L^2}^4 + \|\partial_x w\|_{L^2}^3\|\partial_x\rho\|_{L^2}\right) + G\end{aligned} \tag{4.10}$$

with

$$G = C\left(\|\rho\|_{L^\infty}\|\partial_x w\|_{L^2}^2 + \|w\|_{L^2}\|\partial_x w\|_{L^2}\|\partial_x\rho\|_{L^2} + \|\partial_x w\|_{L^2}^2\|\partial_x\rho\|_{L^2}\right.$$
$$\left.+\|w\|_{H^1}^2 + \|\partial_x w\|_{L^2}\|w\|_{L^2}^2\|\partial_x\rho\|_{L^2} + \|\partial_x w\|_{L^2}\|\partial_x\rho\|_{L^2} + \|\partial_x f\|_{L^2}^2\right).$$

By virtue of the estimates (4.2), (4.3) and (4.4) we deduce that

$$\|G\|_{L^1((0,T))} \leq M(E_2, \|f\|_{L^2(0,T;H^1)}, \frac{1}{\underline{\rho}}, T).$$

Step 3. Letting $m = 2$ in (4.8) and using the embedding $H^1(\mathbb{T}) \subset L^\infty(\mathbb{T})$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^2\rho\|_{L^2}^2 \leq C\|\partial_x^2 u\|_{L^2}\|\partial_x^2\rho\|_{L^2}^2 + C\|\rho\|_{H^1}\|\partial_x^3 u\|_{L^2}\|\partial_x^2\rho\|_{L^2}.$$

Recalling the definition (3.1) $w = -c_p\rho^\gamma + c_\mu\rho^\alpha\partial_x u$ we have

$$\begin{aligned}\partial_x^3 u &= \partial_x^2\left(\frac{w}{c_\mu\rho^\alpha} + \frac{c_p}{c_\mu}\rho^{\gamma-\alpha}\right)\\&= \frac{\partial_x^2 w}{c_\mu\rho^\alpha} - 2\alpha\frac{\partial_x w\partial_x\rho}{c_\mu\rho^{\alpha+1}} - \alpha\frac{w\partial_x^2\rho}{c_\mu\rho^{\alpha+1}} + \alpha(\alpha+1)\frac{w|\partial_x\rho|^2}{c_\mu\rho^{\alpha+2}}\\&\quad+\frac{c_p}{c_\mu}(\gamma-\alpha)\partial_x^2\rho\rho^{\gamma-\alpha-1} + \frac{c_p}{c_\mu}(\gamma-\alpha)(\gamma-\alpha-1)|\partial_x\rho|^2\rho^{\gamma-\alpha-2}.\end{aligned} \tag{4.11}$$

Consequently

$$\|\partial_x^3 u\|_{L^2} \le C \left(\|\partial_x^2 w\|_{L^2} + \|\partial_x w\|_{L^2}\|\partial_x\rho\|_{L^\infty} + \|w\|_{H^1}\|\partial_x^2\rho\|_{L^2} \right.$$
$$\left. +\|w\|_{L^\infty}\|\partial_x\rho\|_{L^2}\|\partial_x\rho\|_{L^\infty} + \|\rho^{\gamma-\alpha-1}\|_\infty\|\partial_x^2\rho\|_{L^2} + \|\rho^{\gamma-\alpha-2}\|_\infty\|\partial_x\rho\|_{L^2}\|\partial_x\rho\|_{L^\infty}\right).$$

Therefore, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^2\rho\|_{L^2}^2 \\
&\le C\left(\|\partial_x^2 u\|_{L^2}\|\partial_x^2\rho\|_{L^2}^2 + \|\rho\|_{H^1}\|\partial_x^2 w\|_{L^2}\|\partial_x^2\rho\|_{L^2} + \|\rho\|_{H^1}\|\partial_x w\|_{L^2}\|\partial_x^2\rho\|_{L^2}\|\partial_x\rho\|_{L^\infty}\right. \\
&\qquad + \|w\|_{H^1}\|\rho\|_{H^1}\|\partial_x^2\rho\|_{L^2}^2 + \|w\|_{L^\infty}\|\rho\|_{H^1}^2\|\partial_x\rho\|_{L^\infty}\|\partial_x^2\rho\|_{L^2} \\
&\qquad \left.+\|\rho\|_{H^1}\|\partial_x^2\rho\|_{L^2}^2 + \|\rho\|_{H^1}^2\|\partial_x^2\rho\|_{L^2}^2\right) \\
&\le \frac{1}{10c}\|\partial_x^2 w\|_{L^2}^2 + C\left(\|\partial_x^2 u\|_{L^2}\|\partial_x^2\rho\|_{L^2}^2 + \|\rho\|_{H^1}^2\|\partial_x^2\rho\|_{L^2}^2 + \|\rho\|_{H^1}\|\partial_x w\|_{L^2}\|\partial_x^2\rho\|_{L^2}^2\right. \\
&\qquad \left.+\|w\|_{H^1}\|\rho\|_{H^1}\|\partial_x^2\rho\|_{L^2}^2 + \|w\|_{H^1}\|\rho\|_{H^1}^2\|\partial_x^2\rho\|_{L^2}^2 + \|\rho\|_{H^1}\|\partial_x^2\rho\|_{L^2}^2 + \|\rho\|_{H^1}^2\|\partial_x^2\rho\|_{L^2}^2\right) \\
&\le \frac{1}{10c}\|\partial_x^2 w\|_{L^2}^2 + F\|\partial_x^2\rho\|_{L^2}^2,
\end{aligned} \tag{4.12}$$

with

$$\begin{aligned} F = \quad & C\left(\|\partial_x^2 u\|_{L^2} + \|\rho\|_{H^1}^2 + \|\rho\|_{H^1}\|\partial_x w\|_{L^2}\right. \\ & \left.+\|w\|_{H^1}\|\rho\|_{H^1} + \|w\|_{H^1}\|\rho\|_{H^1}^2 + \|\rho\|_{H^1} + \|\rho\|_{H^1}^2\right). \end{aligned}$$

Combining the estimates (4.2), (4.3) and (4.4) yields

$$\|F\|_{L^1((0,T))} \le M(E_2, \|f\|_{L^2(0,T;H^1(\mathbb{T}))}, \frac{1}{\underline{\rho}}, T).$$

Step 4. Adding (4.12) to (4.10) leads to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_x^2\rho\|_{L^2}^2 + \|\partial_x w\|_{L^2}^2) + \frac{1}{4c}\|\partial_x^2 w\|_{L^2}^2 &\le \|\partial_x w\|_{L^2}^2 H + \|\partial_x^2\rho\|_{L^2}^2(F + C\|\partial_x w\|_{L^2}^2) + G \\
&\le (\|\partial_x w\|_{L^2}^2 + \|\partial_x^2\rho\|_{L^2})(H + F + C\|\partial_x w\|_{L^2}^2) + G
\end{aligned} \tag{4.13}$$

with

$$H = C\left(\|u\|_{H^1}^2 + \|\partial_x w\|_{L^2}^2 + \|\partial_x w\|_{L^2}\|\partial_x\rho\|_{L^2}\right)$$

satisfying, in virtue of (4.2), (4.3) and (4.4),

$$\|H\|_{L^1((0,T))} \le M(E_2, \|f\|_{L^2(0,T;H^1)}, \frac{1}{\underline{\rho}}, T).$$

Finally, we integrate (4.13) in time, then apply Grönwall's lemma, the estimates for F, G and H, and the estimate (4.4) on $\|\partial_x w\|_{L^2(0,T;L^2)}$ to obtain

$$\begin{aligned}
&\|\partial_x^2\rho\|_{L^\infty(0,T;L^2)} + \|\partial_x w\|_{L^\infty(0,T;L^2)} + \frac{1}{c}\|\partial_x^2 w\|_{L^2(0,T;L^2)} \\
&\qquad \le M(E_2, \|f\|_{L^2(0,T;H^1)}, \frac{1}{\underline{\rho}}, T, \|\partial_x^2\rho_0\|_{L^2}, \|\partial_x w_0\|_{L^2}) \\
&\qquad \le M(E_3, \|f\|_{L^2(0,T;H^1)}, \frac{1}{\underline{\rho}}, T),
\end{aligned}$$

where

$$E_3 = E_2 + \|\partial_x^2\rho_0\|_{L^2} + \|\partial_x^2 u_0\|_{L^2}.$$

It then follows easily that

$$\|\partial_x^2 u\|_{L^\infty(0,T;L^2)} + \|\partial_x^3 u\|_{L^2(0,T;L^2)} \le M(E_3, \|f\|_{L^2(0,T;H^1)}, \frac{1}{\underline{\rho}}, T).$$

□

LEMMA 4.3. *For any $k \geq 2$ there exists M_k depending only on k such that*

$$\|\partial_x^k \rho\|_{L^\infty(0,T;L^2)} + \|\partial_x^{k-1} w\|_{L^\infty(0,T;L^2)} + \|\partial_x^k w\|_{L^2(0,T;L^2)} + \|\partial_x^k u\|_{L^\infty(0,T;L^2)} + \|\partial_x^{k+1} u\|_{L^2(0,T;L^2)} \leq M_k\big(E_{k+1}, \|f\|_{L^2(0,T;H^{k-1})}, \frac{1}{\underline{\rho}}, T\big) \tag{4.14}$$

where

$$E_{k+1} = E_k + \|\partial_x^k \rho_0\|_{L^2} + \|\partial_x^k u_0\|_{L^2}.$$

PROOF. The proof proceeds by induction in k. According to Lemma 4.2, (4.14) holds for $k = 2$. Assuming that (4.14) holds for $k-1$ with $k \geq 3$, to obtain it for k we perform H^k energy estimate for ρ and H^{k-1} energy estimate for w. This follows along the same lines as that of Lemma 4.2. We first apply (4.8) with $m = k$ to have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^k \rho\|_{L^2}^2 &\leq C\|\partial_x^k u\|_{L^2}\|\partial_x^k \rho\|_{L^2}^2 + \|\rho\|_{L^\infty}\|\partial_x^{k+1} u\|_{L^2}\|\partial_x^k \rho\|_{L^2} \\ &\leq M\big(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T\big)\Big(\|\partial_x^k u\|_{L^2}\|\partial_x^k \rho\|_{L^2}^2 + \|\partial_x^{k+1} u\|_{L^2}\|\partial_x^k \rho\|_{L^2}\Big).\end{aligned} \tag{4.15}$$

By differentiating k times the formula

$$\partial_x u = \frac{1}{c_\mu} w \rho^{-\alpha} + c_p \rho^{\gamma-\alpha}$$

and using the induction hypothesis together with the fact that $k \geq 3$ we obtain

$$\begin{aligned}\|\partial_x^{k+1} u\|_{L^2} &\leq C\|[\partial_x^k, \rho^{-\alpha}]w\|_{L^2} + C\|\rho^{-\alpha}\partial_x^k w\|_{L^2} + \|\partial_x^k \rho^{\gamma-\alpha}\|_{L^2} \\ &\leq C\|\partial_x \rho^{-\alpha}\|_{L^\infty}\|w\|_{H^{k-1}} + C\|\rho^{-\alpha}\|_{H^k}\|w\|_{L^\infty} + C\|\rho^{-\alpha}\|_{L^\infty}\|\partial_x^k w\|_{L^2} + \|\partial_x^k \rho^{\gamma-\alpha}\|_{L^2} \\ &\leq C\|\rho^{-\alpha}\|_{H^2}\|w\|_{H^{k-1}} + C\|\rho^{-\alpha}\|_{H^k}\|w\|_{H^1} + +C\|\rho^{-\alpha}\|_{H^1}\|\partial_x^k w\|_{L^2} + \|\rho^{\gamma-\alpha}\|_{H^k} \\ &\leq M\big(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T\big)\big(\|\partial_x^k w\|_{L^2} + \|\partial_x^k \rho\|_{L^2} + 1\big).\end{aligned}$$

It then follows from (4.15) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^k \rho\|_{L^2}^2 &\leq M\big(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T\big)\Big[\|\partial_x^k \rho\|_{L^2}^2\big(\|\partial_x^k u\|_{L^2} + 1\big) + \|\partial_x^k w\|_{L^2}\|\partial_x^k \rho\|_{L^2} + 1\Big] \\ &\leq \frac{1}{10c}\|\partial_x^k w\|_{L^2}^2 + M\big(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T\big)\Big[\|\partial_x^k \rho\|_{L^2}^2\big(\|\partial_x^k u\|_{L^2} + 1\big) + 1\Big]\end{aligned} \tag{4.16}$$

where $c = c(E_1, \|f\|_{L^2(0,T;L^\infty)}, \frac{1}{\underline{\rho}}, T) > 0$ be a positive number such that

$$\rho^{\alpha-1} \geq \frac{1}{c} \quad \forall (x,t) \in \mathbb{T} \times [0, T^*).$$

Next, we differentiate equation (4.9) $k-1$ times in x, multiply the resulting equation by $\partial_x^{k-1} w$ and integrate over $\mathbb{T}$. We estimate successively each resulting term on the right hand side of (4.9).

1. The dissipation term:

$$\begin{aligned}\int_{\mathbb{T}} \partial_x^{k-1}\big(\rho^{\alpha-1}\partial_x^2 w\big)\partial_x^{k-1} w &= -\int_{\mathbb{T}} \partial_x^{k-2}\big(\rho^{\alpha-1}\partial_x^2 w\big)\partial_x^k w \\ &= -\int_{\mathbb{T}} \rho^{\alpha-1}|\partial_x^k w|^2 - \int_{\mathbb{T}} \partial_x^k w \sum_{\ell=1}^{k-2} C_\ell \partial_x^\ell \rho^{\alpha-1}\partial_x^{k-\ell} w \\ &\le -\frac{1}{c}\|\partial_x^k w\|_{L^2}^2 + C\|\partial_x^k w\|_{L^2}\sum_{\ell=1}^{k-2} C_\ell \|\partial_x^\ell \rho^{\alpha-1}\|_{L^\infty}\|\partial_x^{k-\ell} w\|_{L^2} \\ &\le -\frac{1}{c}\|\partial_x^k w\|_{L^2}^2 + C\|\partial_x^k w\|_{L^2}\|\rho\|_{H^{k-1}}\big(\|\partial_x^{k-1} w\|_{L^2} + \|w\|_{L^2}\big) \\ &\le -\frac{1}{2c}\|\partial_x^k w\|_{L^2}^2 + C'\|\rho\|_{H^{k-1}}^2\big(\|\partial_x^{k-1} w\|_{L^2}^2 + \|w\|_{L^2}^2\big) \\ &\le -\frac{1}{2c}\|\partial_x^k w\|_{L^2}^2 + M\big(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T\big)\big(\|\partial_x^{k-1} w\|_{L^2}^2 + 1\big).\end{aligned}$$

2. The drift term. We have

$$\int_{\mathbb{T}} \partial_x^{k-1}\big(u\partial_x w + c_\mu \rho^{\alpha-2}\partial_x\rho\partial_x w\big)\partial_x^{k-1} w \quad = -\int_{\mathbb{T}} \partial_x^{k-2}\big(u\partial_x w\big)\partial_x^k w - c_\mu \int_{\mathbb{T}} \partial_x^{k-2}\Big(\partial_x \frac{\rho^{\alpha-1}}{\alpha-1}\partial_x w\Big)\partial_x^k w$$

where we adopted the convention $\frac{\rho^{\alpha-1}}{\alpha-1} = \ln\rho$ when $\alpha = 1$. Noting that $H^{k-2}(\mathbb{T})$ is an algebra for $k \ge 3$, we then bound

$$\begin{aligned}&\left|\int_{\mathbb{T}} \partial_x^{k-1}\big(u\partial_x w + c_\mu \rho^{\alpha-2}\partial_x\rho\partial_x w\big)\partial_x^{k-1} w\right| \\ &\quad\le C\|\partial_x^k w\|_{L^2}\|u\|_{H^{k-2}}\|w\|_{H^{k-1}} + C\|\partial_x^k w\|_{L^2}\Big\|\frac{\rho^{\alpha-1}}{\alpha-1}\Big\|_{H^{k-1}}\|w\|_{H^{k-1}} \\ &\quad\le \frac{1}{20c}\|\partial_x^k w\|_{L^2}^2 + C'\|u\|_{H^{k-2}}^2\|w\|_{H^{k-1}}^2 + C'\Big\|\frac{\rho^{\alpha-1}}{\alpha-1}\Big\|_{H^{k-1}}^2\|w\|_{H^{k-1}}^2 \\ &\quad\le \frac{1}{20c}\|\partial_x^k w\|_{L^2}^2 + M\big(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T\big)\big(\|\partial_x^{k-1} w\|_{L^2}^2 + 1\big)\end{aligned}$$

3. The nonlinearity term:

$$\begin{aligned}\left|\int_{\mathbb{T}} \partial_x^{k-1}\big(\rho^{-\alpha} w^2\big)\partial_x^{k-1} w\right| &= \left|\int_{\mathbb{T}} \partial_x^{k-2}\big(\rho^{-\alpha} w^2\big)\partial_x^k w\right| \\ &\le C\|\rho^{-\alpha}\|_{H^{k-2}}\|w\|_{H^{k-2}}^2\|\partial_x^k w\|_{L^2} \\ &\le \frac{1}{20c}\|\partial_x^k w\|_{L^2} + C'\|\rho^{-\alpha}\|_{H^{k-2}}^2\|w\|_{H^{k-2}}^4 \\ &\le \frac{1}{20c}\|\partial_x^k w\|_{L^2} + M\big(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T\big).\end{aligned}$$

4. The zero order term:

$$\begin{aligned}\left|\int_{\mathbb{T}} \partial_x^{k-1}(\rho^{2\gamma-\alpha})\partial_x^{k-1} w\right| &\le C\|\rho^{2\gamma-\alpha}\|_{H^{k-1}}\|\partial_x^{k-1} w\|_{L^2} \\ &\le M\big(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T\big)\|\partial_x^{k-1} w\|_{L^2}.\end{aligned}$$

5. The forcing term:

$$\left|\int_{\mathbb{T}} \partial_x^{k-1}(\rho^\alpha \partial_x f)\partial_x^{k-1} w\right| = \left|\int_{\mathbb{T}} \partial_x^{k-2}(\rho^\alpha \partial_x f)\partial_x^{k} w\right|$$
$$\leq C\|\rho^\alpha\|_{H^{k-2}}\|\partial_x f\|_{H^{k-2}}\|\partial_x^k w\|_{L^2}$$
$$\leq \frac{1}{20c}\|\partial_x^k w\|_{L^2}^2 + M\big(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T\big)\|f\|_{H^{k-1}}^2.$$

Putting the estimates 1. through 5. together, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\partial_x^{k-1} w\|_{L^2}^2 \leq \frac{-2}{5c}\|\partial_x^k w\|_{L^2}^2 + M\big(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T\big)\|\partial_x^{k-1} w\|_{L^2}^2$$
$$+ M\big(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T\big)\big(\|f\|_{H^{k-1}}^2 + 1\big).$$

Combining this with (4.16) and Grönwall's lemma leads to

$$\|\partial_x^k \rho\|_{L^\infty(0,T;L^2)}^2 + \|\partial_x^{k-1} w\|_{L^\infty(0,T;L^2)}^2 + \|\partial_x^k w\|_{L^2(0,T;L^2)}^2$$
$$\leq M\Big(\|\partial_x^k \rho_0\|_{L^2}^2 + \|\partial_x^{k-1} w_0\|_{L^2}^2 + \|f\|_{L^2(0,T;H^{k-1})}^2 + T\Big)\exp\Big(M\big(\|\partial_x^k u\|_{L^1(0,T;L^2)} + T\big)\Big)$$

where we denoted

$$M \equiv M\big(E_k, \|f\|_{L^2(0,T;H^{k-2})}, \frac{1}{\underline{\rho}}, T\big)$$

and used the fact that the $L^2(0,T;H^k)$ norm of u is controlled by M.

It follows easily from this that $\|\partial_x^k u\|_{L^\infty(0,T;L^2)}$ and $\|\partial_x^{k+1} u\|_{L^2(0,T;L^2)}$ can be controlled by the same bound. This finishes the proof of (4.14). □

In view of Lemmas 4.1, 4.2 and 4.3 we have proved that

$$\sup_{T\in[0,T^*)}\|\rho\|_{L^\infty(0,T;H^k)} + \sup_{T\in[0,T^*)}\|u\|_{L^\infty(0,T;H^k)} + \sup_{T\in[0,T^*)}\|u\|_{L^2(0,T;H^{k+1})}$$
$$\leq M_k\Big(\|(\rho_0,u_0)\|_{H^k\times H^k}, \|f\|_{L^2(0,T^*;H^{\max\{k-1,1\}})}, \frac{1}{\underline{\rho}}, T^*\Big) < \infty \tag{4.17}$$

for $k \geq 1$. Appealing to local existence, established by Prop. B.1, the solution can be extended past T^*.

5. Proof of Theorem 1.4

We assume here that $c_p > 0$ and that $\alpha \in (\frac{1}{2}, 1]$, $\gamma \geq 2\alpha$. By Prop. B.1, there exists a positive time T_0 such that problem (1.1)-(1.3) has a unique solution (ρ, u) on $[0, T_0]$ such that

$$\rho \in C(0,T_0;H^k), \quad u \in C(0,T_0;H^k)\cap L^2(0,T_0;H^{k+1}), \quad k \geq 3, \tag{5.1}$$

and $\rho > 0$ on $[0, T_0]$. Let T^* be the maximal lifetime of the classical solution (ρ, u), so that, by Thm. 1.1,

$$\inf_{t\in(0,T^*)}\min_{x\in\mathbb{T}}\rho(x,t) = 0. \tag{5.2}$$

We claim that $T^* = \infty$. We will argue by contradiction. Let us note that the H^k regularity, $k \geq 3$, of (ρ, u) suffices to justify all the calculations below. Recall from the proof of Lemma 2.3 in Appendix A, that

$$X = u + c_\mu \rho^{\alpha-2}\partial_x \rho, \tag{5.3}$$

defined also in Eq. (A.4), satisfies

$$\partial_t X + u\partial_x X = -\gamma\frac{c_p}{c_\mu}\rho^{\gamma-\alpha}(X-u) + f = -\gamma\frac{c_p}{c_\mu}\rho^{\gamma-\alpha}X + \gamma\frac{c_p}{c_\mu}\rho^{\gamma-\alpha}u + f. \tag{5.4}$$

By Lemma 2.4 1., we have

$$\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{T}))} \le M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T). \tag{5.5}$$

Since $\gamma \ge 2\alpha \ge \alpha + \frac{1}{2}$ for $\alpha \in (\frac{1}{2}, 1]$, combining the above estimate with (2.14), we have

$$\|\rho^{\gamma-\alpha} u\|_{L^\infty(0,T;L^2(\mathbb{T}))} \le M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T). \tag{5.6}$$

Note also

$$\partial_x(\rho^{\gamma-\alpha}u) = (\sqrt{\rho}\partial_x u)\rho^{\gamma-\alpha-\frac{1}{2}} + (\gamma-\alpha)\rho^{\gamma-2\alpha}(\rho^{\alpha-\frac{3}{2}}\partial_x\rho)(\sqrt{\rho}u)$$

Now, estimate (2.27) implies

$$\|(\rho^{\alpha-\frac{3}{2}}\partial_x\rho)\|_{L^2(0,T;L^2(\mathbb{T}))} \le M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T).$$

Putting together this, (2.14), (2.15), (5.5), and the assumption that $\gamma \ge 2\alpha$ we deduce that

$$\|\partial_x(\rho^{\gamma-\alpha}u)\|_{L^2(0,T;L^1(\mathbb{T}))} \le M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T).$$

which combined with (5.6) yields

$$\|\rho^{\gamma-\alpha}u\|_{L^2(0,T;W^{1,1})} \le M(E_1, \|f\|_{L^2(0,T;L^\infty)}, T). \tag{5.7}$$

Since (5.4) is a transport equation we then have

$$\begin{aligned}\|X\|_{L^\infty(0,T;L^\infty)} &\le \big(\|X_0\|_{L^\infty} + \gamma\frac{c_p}{c_\mu}\|\rho^{\gamma-\alpha}u\|_{L^1(0,T;L^\infty)} + \|f\|_{L^1(0,T;L^\infty)}\big)\exp\big(\gamma\frac{c_p}{c_\mu}\|\rho^{\gamma-\alpha}\|_{L^1(0,T;L^\infty)}\big)\\ &\le M(E_1, \|X_0\|_{L^\infty}, \|f\|_{L^2(0,T;L^\infty)}, T).\end{aligned} \tag{5.8}$$

Recall that $X = u + \frac{\partial_x\rho}{\rho^2}\mu(\rho) = u + c_\mu\rho^{\alpha-2}\partial_x\rho$, hence $X\rho^{\gamma-\alpha} = u\rho^{\gamma-\alpha} + c_\mu\rho^{\gamma-2}\partial_x\rho$. It then follows from (5.5), (5.7) and (5.8) that

$$\|\rho^{\gamma-2}\partial_x\rho\|_{L^2(0,T;L^\infty)} \le M(E_1, \|X_0\|_{L^\infty}, \|f\|_{L^2(0,T;L^\infty)}, T). \tag{5.9}$$

Using (1.1) and (1.2) we obtain

$$\partial_t u + (u - \frac{\mu'(\rho)\partial_x\rho}{\rho})\partial_x u = \frac{\mu(\rho)}{\rho}\partial_x^2 u - \frac{p'(\rho)\partial_x\rho}{\rho} + f = c_\mu\rho^{\alpha-1}\partial_x^2 u - c_p\gamma\rho^{\gamma-2}\partial_x\rho + f. \tag{5.10}$$

Using the maximum principle (see the argument leading to (6.6) below and a similar argument for the minimum) and the bound (5.9) gives

$$\begin{aligned}\|u\|_{L^\infty(0,T;L^\infty)} &\le \|u_0\|_{L^\infty} + c_p\gamma\|\rho^{\gamma-2}\partial_x\rho\|_{L^1(0,T;L^\infty)} + \|f\|_{L^1(0,T;L^\infty)}\\ &\le M(E_1, \|(X_0,u_0)\|_{L^\infty}, \|f\|_{L^2(0,T;L^\infty)}, T).\end{aligned} \tag{5.11}$$

From the definition of X and (5.8), this yields

$$\|\partial_x\rho^{\alpha-1}\|_{L^\infty(0,T;L^\infty)} \le M(E_1, \|(X_0,u_0)\|_{L^\infty}, \|f\|_{L^2(0,T;L^\infty)}, T) \tag{5.12}$$

when $\alpha < 1$, and

$$\|\partial_x \ln\rho\|_{L^\infty(0,T;L^\infty)} \le M(E_1, \|(X_0,u_0)\|_{L^\infty}, \|f\|_{L^2(0,T;L^\infty)}, T) \tag{5.13}$$

when $\alpha = 1$.

When $\alpha < 1$, the continuity equation implies

$$\partial_t(\rho^{\alpha-1}) = -(\alpha-1)\partial_x(u\rho)\rho^{\alpha-2}. \tag{5.14}$$

Integrating this in space and time and using the definition of X leads to

$$
\begin{aligned}
\int_{\mathbb{T}} \rho^{\alpha-1}(x,T)\mathrm{d}x &= \int_{\mathbb{T}} \rho_0^{\alpha-1}\mathrm{d}x + (\alpha-1)(\alpha-2)\int_0^t\int_{\mathbb{T}} (u\rho\rho^{\alpha-3}\partial_x\rho)(x,z)\mathrm{d}x\mathrm{d}z \\
&= \int_{\mathbb{T}} \rho_0^{\alpha-1}\mathrm{d}x + \frac{1}{c_\mu}(\alpha-2)(\alpha-1)\int_0^t\int_{\mathbb{T}} (uc_\mu\rho^{\alpha-2}\partial_x\rho)(x,z)\mathrm{d}x\mathrm{d}z \\
&\le \int_{\mathbb{T}} \rho_0^{\alpha-1}\mathrm{d}x + C\int_0^t\int_{\mathbb{T}} X^2(x,z)\mathrm{d}x\mathrm{d}z,
\end{aligned}
\tag{5.15}
$$

valid for $0 \le t \le T$.

Similarly, when $\alpha = 1$ we have

$$
\left|\int_{\mathbb{T}} \ln\rho(x,t)\mathrm{d}x\right| \le \left|\int_{\mathbb{T}} \ln\rho_0\mathrm{d}x\right| + C\int_0^t\int_{\mathbb{T}} X^2(x,z)\mathrm{d}x\mathrm{d}z, \quad 0 \le t \le T. \tag{5.16}
$$

Then by virtue of (5.8), (5.11), (5.12), (5.15), Poincaré-Wirtinger's inequality and Sobolev embedding we deduce that

$$
\|\rho^{\alpha-1}\|_{L^\infty(0,T;L^\infty)} \le M(E_1, \|(X_0,u_0)\|_{L^\infty}, \|\rho_0^{\alpha-1}\|_{L^1}, \|f\|_{L^2(0,T;L^\infty)}, T)
$$

if $\alpha < 1$.

On the other hand, if $\alpha = 1$, (5.5) combined with with (5.16), Poincaré-Wirtinger's inequality and Sobolev embedding, yields

$$
\|\ln\rho\|_{L^\infty(0,T;L^\infty)} \le M(E_1, \|(X_0,u_0)\|_{L^\infty}, \|\ln\rho_0\|_{L^1}, \|f\|_{L^2(0,T;L^\infty)}, T).
$$

Consequently

$$
\inf_{(x,t)\in\mathbb{T}\times[0,T]} \rho(x,t) \ge \mathcal{F}\left(M(E_0, \|(X_0,u_0)\|_{L^\infty}, \|\rho_0^{\alpha-1}\|_{L^1} + \|\ln\rho_0\|_{L^1}, \|f\|_{L^2(0,T;L^\infty)}, T)\right)
$$

where

$$
\mathcal{F}(z) = \begin{cases} z^{\frac{1}{\alpha-1}} & \text{if } \alpha < 1, \\ e^{-z} & \text{if } \alpha = 1. \end{cases} \tag{5.17}
$$

Therefore,

$$
\inf_{(x,t)\in\mathbb{T}\times[0,T^*)} \rho(x,t) \ge \mathcal{F}\left(M(E_0, \|(X_0,u_0)\|_{L^\infty}, \|\rho_0^{\alpha-1}\|_{L^1}, \|\ln\rho_0\|_{L^1}, \|f\|_{L^2(0,T^*;L^\infty)}, T^*)\right) > 0
$$

which contradicts (5.2).

6. Proof of Theorem 1.5

Recall the assumption (1.11)

$$
c_p > 0 \quad \text{and} \quad \gamma \in [\alpha, \alpha+1] \setminus \{1\} \quad \text{and} \quad \alpha > \frac{1}{2}. \tag{6.1}
$$

By Prop. B.1, there exists a positive time T_0 such that problem (1.1)-(1.3) has a unique solution (ρ, u) on $[0, T_0]$ such that

$$
\rho \in C(0,T_0;H^k), \quad u \in C(0,T_0;H^k) \cap L^2(0,T_0;H^{k+1}), \quad k \ge 4, \tag{6.2}
$$

and $\rho > 0$ on $[0, T_0]$. Let T^* be the maximal existence time. We claim that $T^* = \infty$. Assume by contradiction that T^* is finite. By Theorem 1.1 we have

$$
\inf_{t\in[0,T^*)} \min_{x\in\mathbb{T}} \rho(x,t) = 0. \tag{6.3}
$$

From Lemma 3.1, the w equation (3.2) is

$$\partial_t w = c_\mu \rho^{\alpha-1}\partial_x^2 w - (u + c_\mu \rho^{\alpha-2}\partial_x \rho)\partial_x w + \frac{c_p}{c_\mu}\left(\gamma - 2(\alpha+1)\right)\rho^{\gamma-\alpha} w$$
$$- c_\mu(\alpha+1)\rho^{-\alpha} w^2 + \frac{c_p^2}{c_\mu}\left(\gamma - (\alpha+1)\right)\rho^{2\gamma-\alpha}. \tag{6.4}$$

Note that the assumption $f(x,t) = f(t)$ was used to have $\partial_x f = 0$. It follows from (6.2) and the equation (6.4) that

$$w \in C(0,T;H^3) \cap L^2(0,T;H^4), \qquad \partial_t w \in C(0,T;H^1) \subset C(\mathbb{T}\times[0,T])$$

Thus, $w \in C^1(\mathbb{T}\times[0,T])$ and thus the function

$$w_M(t) := \max_{x\in\mathbb{T}} w(x,t) \tag{6.5}$$

is Lipschitz continuous on $[0,T]$. According to the Rademacher theorem, w_M is differentiable almost everywhere on $[0,T]$. There exists for each $t \in [0,T^*)$ a point x_t such that

$$w_M(t) = w(x_t,t).$$

Let $t \in (0,T)$ be a point at which w_M is differentiable. We have

$$\begin{aligned} w_M(t) &= \lim_{h\to 0^+} \frac{w_M(t+h) - w_M(t)}{h} \\ &= \lim_{h\to 0^+} \frac{w(x_{t+h},t+h) - w(x_t,t)}{h} \\ &\geq \lim_{h\to 0^+} \frac{w(x_t,t+h) - w(x_t,t)}{h} = \partial_t w(x_t,t). \end{aligned}$$

On the other hand,

$$\begin{aligned} w_M(t) &= \lim_{h\to 0^+} \frac{w_M(t) - w_M(t-h)}{h} \\ &= \lim_{h\to 0^+} \frac{w(x_t,t) - w(x_{t-h},t-h)}{h} \\ &\leq \lim_{h\to 0^+} \frac{w(x_t,t) - w(x_t,t-h)}{h} = \partial_t w(x_t,t). \end{aligned}$$

Thus, $w'_M(t) = \partial_t w(x_t,t)$ if w_M is differentiable at t. We deduce from this and equation (6.4) that for almost every $t \in (0,T)$,

$$\partial_t w_M \leq A(t) w_M + B(t) w_M^2 + C(t) \tag{6.6}$$

with

$$\begin{aligned} A(t) &:= \frac{c_p}{c_\mu}\left(\gamma - 2(\alpha+1)\right)\rho(x_t)^{\gamma-\alpha} \\ B(t) &:= -\frac{1}{c_\mu}(\alpha+1)\rho(x_t)^{-\alpha} \\ C(t) &:= \frac{c_p^2}{c_\mu}\left(\gamma - (\alpha+1)\right)\rho(x_t)^{2\gamma-\alpha}. \end{aligned}$$

where we used the facts that $\partial_x^2 w(x_t,t) \geq 0$ and $\partial_x w(x_t,t) = 0$. Note that $B(t) \leq 0$. In addition, the function C is nonpositive under the conditions (1.11). The condition on the initial data (1.12) is equivalent to $w_M(0) \leq 0$. We deduce that

$$w(t) \leq 0, \qquad \forall t < T^*. \tag{6.7}$$

At the point y_t where the density attains its minimum value $\rho_m := \rho(y_t, t)$, ρ_m satisfies

$$\partial_t \rho_m = -\partial_x u(y_t)\rho_m = -\frac{w(y_t)}{c_\mu}\rho_m^{1-\alpha} - \frac{c_p}{c_\mu}\rho_m^{\gamma-\alpha+1} \geq -\frac{c_p}{c_\mu}\rho_m^{\gamma-\alpha+1} \tag{6.8}$$

where we used (6.7). Provided that $\gamma \neq \alpha$, this implies the differential inequality

$$\frac{1}{(\alpha-\gamma)}\partial_t(\rho_m^{\alpha-\gamma}) \geq -\frac{c_p}{c_\mu}. \tag{6.9}$$

Since $\alpha < \gamma$, we find

$$\partial_t(\rho_m^{\alpha-\gamma}) \leq \frac{c_p}{c_\mu}(\gamma-\alpha) \tag{6.10}$$

which implies

$$\rho_m(t) \geq \left(\rho_m(0)^{\alpha-\gamma} + t\frac{c_p}{c_\mu}(\gamma-\alpha)\right)^{\frac{1}{\alpha-\gamma}}, \quad \forall t < T^* \tag{6.11}$$

Since $c_p/c_\mu > 0$, this implies that

$$\inf_{t\in[0,T^*)}\min_{x\in\mathbb{T}}\rho(x,t) \geq \left(\rho_m(0)^{\alpha-\gamma} + T^*\frac{c_p}{c_\mu}(\gamma-\alpha)\right)^{\frac{1}{\alpha-\gamma}} > 0 \tag{6.12}$$

which contradicts the assumption (6.3). We conclude that the solution (ρ, u) is global in time.

On the other hand, when $\alpha = \gamma$ we have

$$\partial_t \ln \rho_m \geq -\frac{c_p}{c_\mu} \tag{6.13}$$

and thus

$$\rho_m(t) \geq \rho_m(0)\exp\left(-t\frac{c_p}{c_\mu}\right) > 0 \tag{6.14}$$

which again leads to a contradiction with (6.3).

REMARK 6.1. With a more refined maximum principle argument, one can relax the regularity requirement of $k \geq 4$ which we used to conclude that (6.5) is Lipschitz continuous on $[0, T]$.

7. Proof of Theorem 1.7

In this section, we give an upper bound for the long-time average maximum density, assuming that the forcing has zero mean in space. This follows by an application of the Bresch-Desjardins's entropy and the following elementary lemma.

LEMMA 7.1. *Let $m \geq \frac{1}{2}$. If $h^m \in W^{1,1}(\mathbb{T})$ then we have*

$$\|h\|_{L^\infty(\mathbb{T})} \leq 2\|\partial_x(h^m)\|_{L^1(\mathbb{T})}^{\frac{1}{m}} + 4\|h\|_{L^1(\mathbb{T})}. \tag{7.1}$$

PROOF OF LEMMA 7.1. Since $h \in W^{1,1}(\mathbb{T}) \subset C^0(\mathbb{T})$, we have $h \in C^0(\mathbb{T})$. In particular, there exists a point $x_0 \in \mathbb{T}$ such that $|h(x_0)| \leq \sqrt{2}\|h\|_{L^1(\mathbb{T})}$. For all $x \in \mathbb{T}$ we have

$$h^m(x) = \int_{x_0}^{x}\partial_y(h^m(y))dy + h^m(x_0),$$

hence

$$|h(x)|^m \leq \|\partial_x h^m\|_{L^1(\mathbb{T})} + |h(x_0)|^m \leq \|\partial_x(h^m)\|_{L^1(\mathbb{T})} + \sqrt{2}\|h\|_{L^1(\mathbb{T})}^m.$$

In view of the elementary inequality

$$(a+b)^{\frac{1}{m}} \leq 2a^{\frac{1}{m}} + 2b^{\frac{1}{m}}, \quad a,\ b,\ m > 0,$$

we thus obtain (7.1). □

PROOF OF THEOREM 1.7. Recall our assumptions

$$\gamma \in [\max\{2-\alpha,\alpha\},\alpha+1], \quad \alpha \geq 1/2, \quad \text{and} \quad c_p, c_\mu > 0. \tag{7.2}$$

Next, by Lemma 2.3, the entropy

$$s = \frac{\rho}{2}\left|u + \frac{\partial_x\rho}{\rho^2}\mu(\rho)\right|^2 + \pi(\rho). \tag{7.3}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} s(x,t)\mathrm{d}x = -\int_{\mathbb{T}} |\partial_x\rho|^2\mu(\rho)\frac{p'(\rho)}{\rho^2}dx + \int_{\mathbb{T}} f\rho\big(u + \frac{\partial_x\rho}{\rho^2}\mu(\rho)\big)\mathrm{d}x. \tag{7.4}$$

Integrating this in time yields

$$\int_{\mathbb{T}} s(x,T)\mathrm{d}x - \int_{\mathbb{T}} s(x,0)\mathrm{d}x + c_p c_\mu\gamma\int_0^T\int_{\mathbb{T}}\rho^{\alpha+\gamma-3}|\partial_x\rho|^2\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T\int_{\mathbb{T}} f\rho u\mathrm{d}x\mathrm{d}t + c_\mu\int_0^T\int_{\mathbb{T}} f\rho^{\alpha-1}\partial_x\rho\,\mathrm{d}x\mathrm{d}t.$$

Using the assumption (1.13) we calculate

$$\int_0^T\int_{\mathbb{T}} f\rho u\,\mathrm{d}x\mathrm{d}t = -\int_0^T\int_{\mathbb{T}} g\partial_x(\rho u)\mathrm{d}x\mathrm{d}t = \int_0^T\int_{\mathbb{T}} g\partial_t\rho\,\mathrm{d}x\mathrm{d}t$$
$$= \int_{\mathbb{T}}(g\rho)(x,T)\,\mathrm{d}x - \int_{\mathbb{T}}(g\rho)(x,0)\,\mathrm{d}x - \int_0^T\int_{\mathbb{T}}\rho\partial_t g\,\mathrm{d}x\mathrm{d}t.$$

This implies

$$\left|\int_0^T\int_{\mathbb{T}} f\rho u\,\mathrm{d}x\mathrm{d}t\right| \leq 2\|g\|_{L^\infty(0,T;L^\infty)}\|\rho_0\|_1 + \|\partial_t g\|_{L^1(0,T;L^\infty)}\|\rho_0\|_1$$
$$\leq 2\|g\|_{L^\infty(0,T;L^\infty)}\|\rho_0\|_1 + T\|\partial_t g\|_{L^\infty(0,T;L^\infty)}\|\rho_0\|_1.$$

On the other hand, using Cauchy–Schwarz, we have

$$\left|c_\mu\int_0^T\int_{\mathbb{T}} f\rho^{\alpha-1}\partial_x\rho\,\mathrm{d}x\mathrm{d}t\right| \leq \frac{1}{2}c_p c_\mu\gamma\int_0^T\int_{\mathbb{T}}\rho^{\alpha+\gamma-3}|\partial_x\rho|^2\mathrm{d}x\mathrm{d}t + C\int_0^T\int_{\mathbb{T}}\rho^{\alpha-\gamma+1}f^2\mathrm{d}x\mathrm{d}t$$
$$\leq \frac{1}{2}c_p c_\mu\gamma\int_0^T\int_{\mathbb{T}}\rho^{\alpha+\gamma-3}|\partial_x\rho|^2\mathrm{d}x\mathrm{d}t + CT(1+\|\rho_0\|_1)\|f\|^2_{L^\infty(0,T;L^\infty)}.$$

Here, C is a constant which depends only on c_γ, c_p and γ. We have used the assumption (7.2) that γ belongs to the range $\gamma \in [\max\{2-\alpha,\alpha\},\alpha+1]$ with $\alpha \geq 1/2$ to have $0 \leq \alpha-\gamma+1 \leq 1$.

Note that the allowed range of γ and α requires that $\gamma \geq 3/2$ always. Since, in particular $\gamma > 1$ we have $\pi(\rho) \geq 0$ and $s \geq 0$. Thus, putting all together, we obtain the bound

$$\frac{1}{2}c_p c_\mu\gamma\int_0^T\int_{\mathbb{T}}\rho^{\alpha+\gamma-3}|\partial_x\rho|^2\mathrm{d}x\mathrm{d}t$$
$$\leq 2\|g\|_{L^\infty(0,T;L^\infty)}\|\rho_0\|_1 + T\|\partial_t g\|_{L^\infty(0,T;L^\infty)}\|\rho_0\|_1 + CT(1+\|\rho_0\|_1)\|\partial_x g\|^2_{L^\infty(0,T;L^\infty)} + \int_{\mathbb{T}} s(x,0)\mathrm{d}x.$$

We thus obtain

$$\frac{1}{2}c_p c_\mu\gamma\int_0^T\int_{\mathbb{T}}\rho^{\alpha+\gamma-3}|\partial_x\rho|^2\mathrm{d}x\mathrm{d}t \leq M_1 T + M_0,$$

where M_0 is a constant which depends only on c_μ, c_p, γ, α, $\|\rho_0\|_{L^\infty}$, $\|\rho_0^{-1}\|_{L^\infty}$, $\|u_0\|_{L^2}$, $\|\partial_x\rho_0\|_{L^2}$, $\|g\|_{L^\infty(0,T;L^\infty)}$, and M_1 a constant which depends only on c_μ, c_p, γ, $\|\rho_0\|_{L^1}$, $\|\partial_t g\|_{L^\infty(0,T;L^\infty)}$, $\|\partial_x g\|_{L^\infty(0,T;L^\infty)}$.

In particular,

$$\int_0^T \int_{\mathbb{T}} |\partial_x(\rho^{\frac{1}{2}(\alpha+\gamma-1)})|^2 \mathrm{d}x\mathrm{d}t \leq M_3 T + M_2,$$

where $M_{i+2} = \frac{(\alpha+\gamma-1)^2}{2c_p c_\mu \gamma} M_i$, for $i = 0, 1$. Here, we used the fact that $\alpha + \gamma - 1 > 0$.

By assumption (7.2) we have that $\alpha + \gamma \geq 2\max\{1, \alpha\} \geq 2$ which implies $\frac{1}{m} \leq 2$. We now apply Lemma 7.1 with $m := \frac{1}{2}(\alpha + \gamma - 1)$. Use the embedding $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$, we obtain

$$\int_0^T \|\rho(\cdot, t)\|_{L^\infty} \mathrm{d}t \leq 2 \int_0^T \|\partial_x(\rho^m)\|_{L^2}^{\frac{1}{m}} \mathrm{d}t + 4T\|\rho_0\|_{L^1}.$$

Consequently,

$$\int_0^T \|\rho(\cdot, t)\|_{L^\infty} \mathrm{d}t \leq 2 \int_0^T (\|\partial_x(\rho^m)\|_{L^2}^2 + 1) \mathrm{d}t + 4T\|\rho_0\|_{L^1} \leq 2(M_3 T + M_2) + 2T + 4T\|\rho_0\|_{L^1}.$$

Hence,

$$\frac{1}{T} \int_0^T \|\rho(\cdot, t)\|_{L^\infty} \mathrm{d}t \leq (2M_3 + 2 + 4\|\rho_0\|_{L^1}) + \frac{2}{T} M_2, \tag{7.5}$$

and the claim follows, with the definition

$$C_1 = 2M_2, \qquad C_2 := 2M_3 + 2 + 4\|\rho_0\|_{L^1}. \tag{7.6}$$

□

Appendix A. Bresch-Desjardins's entropy

For the sake of completeness we present the proof of Lemma 2.3 which essentially follows from [**19, 20, 21**]. From the continuity equation (1.1), any smooth $\xi(\rho)$ satisfies

$$\partial_t \xi(\rho) = \partial_t \rho \xi'(\rho) = -\partial_x(u\rho)\xi'(\rho) = -u\partial_x \xi(\rho) - \rho(\partial_x u)\xi'(\rho) \tag{A.1}$$

Using equation (A.1) applied to the function $\partial_x \xi(\rho)$, we find the evolution of $\rho \partial_x \xi(\rho)$):

$$\begin{aligned}
\partial_t(\rho \partial_x \xi(\rho)) &= -\partial_x(\rho u)\partial_x \xi(\rho) + \rho \partial_t \partial_x \xi(\rho) \\
&= -\partial_x(\rho u)\partial_x \xi(\rho) - \rho \partial_x(u \partial_x \xi(\rho) + \rho(\partial_x u)\xi'(\rho)) \\
&= -\partial_x(\rho u)\partial_x \xi(\rho) - \rho \partial_x u \partial_x \xi(\rho) - \rho u \partial_x^2 \xi(\rho) - \rho \partial_x(\rho(\partial_x u)\xi'(\rho)) \\
&= -\partial_x(\rho u \partial_x \xi(\rho)) - \rho \partial_x u \partial_x \xi(\rho) - \rho \partial_x(\rho(\partial_x u)\xi'(\rho)) \\
&= -\partial_x(\rho u \partial_x \xi(\rho)) - \partial_x(\rho^2(\partial_x u)\xi'(\rho)).
\end{aligned} \tag{A.2}$$

Then, letting $X := u + \partial_x \xi(\rho)$, combining Eq. (A.2) with the momentum equation (1.2) yields

$$\partial_t(\rho X) = -\partial_x(\rho u X) - \partial_x p(\rho) + \partial_x(\mu(\rho)\partial_x u) - \partial_x(\rho^2(\partial_x u)\xi'(\rho)) + \rho f. \tag{A.3}$$

We now choose $\rho^2 \xi'(\rho) = \mu(\rho)$, so that the final two terms in (A.3) cancel. Thus with this choice,

$$X = u + \frac{\partial_x \rho}{\rho^2} \mu(\rho) \tag{A.4}$$

and, by (A.3), ρX satisfies

$$\partial_t(\rho X) = -\partial_x(\rho u X) - \partial_x p(\rho) + \rho f. \tag{A.5}$$

Whence, we obtain

$$\partial_t(\rho X^2) = -\partial_x(\rho u X^2) - 2X\partial_x p(\rho) + 2\rho f X. \tag{A.6}$$

Integrating in space

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}(\rho X^2)(x,t)\mathrm{d}x &= -\int_{\mathbb{T}}\rho u\frac{\partial_x p(\rho)}{\rho}\mathrm{d}x - \int_{\mathbb{T}}|\partial_x\rho|^2\mu(\rho)\frac{p'(\rho)}{\rho^2}\mathrm{d}x + \int_{\mathbb{T}} f\rho\big(u+\frac{\partial_x\rho}{\rho^2}\mu(\rho)\big)\mathrm{d}x\\ &= -\int_{\mathbb{T}}\rho u\,\partial_x\pi'(\rho)\mathrm{d}x - \int_{\mathbb{T}}|\partial_x\rho|^2\mu(\rho)\frac{p'(\rho)}{\rho^2}\mathrm{d}x + \int_{\mathbb{T}} f\rho\big(u+\frac{\partial_x\rho}{\rho^2}\mu(\rho)\big)\mathrm{d}x\\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}\pi(\rho)\mathrm{d}x - \int_{\mathbb{T}}|\partial_x\rho|^2\mu(\rho)\frac{p'(\rho)}{\rho^2}\mathrm{d}x + \int_{\mathbb{T}} f\rho\big(u+\frac{\partial_x\rho}{\rho^2}\mu(\rho)\big)\mathrm{d}x.\end{aligned}$$

The global balance (2.19) for entropy $s := \frac{1}{2}\rho X^2 + \pi(\rho)$ follows.

Appendix B. Local well-posedness

PROPOSITION B.1. *Assume that $p : \mathbb{R}^+ \to \mathbb{R}$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ are C^∞ functions away from zero. Let ρ_0 and u_0 belong to $H^k(\mathbb{T})$ for an integer $k \geq 1$, such that $r_0 := \min_{x\in\mathbb{T}}\rho_0 > 0$. Suppose that for all $T > 0$*

$$f \in L^2(0,T;H^{k-1}(\mathbb{T})).$$

Then, there exists a $T_0 > 0$ depending only on $\|(\rho_0,u_0)\|_{H^k(\mathbb{T})\times H^k(\mathbb{T})}$, r_0 and f, and a unique strong solution (ρ,u) to (1.1)-(1.3) on $[0,T_0]$ with data (ρ_0,u_0) such that

$$\rho \in C(0,T_0;H^k(\mathbb{T})), \quad u \in C(0,T_0;H^k(\mathbb{T}))\cap L^2(0,T_0;H^{k+1}(\mathbb{T}))$$

and $\rho(x,t) > \frac{r_0}{2}$ for all $(x,t) \in \mathbb{T}\times[0,T_0]$.

PROOF. **Step 0.** (Iteration Scheme) We are going to set up an iteration argument and prove that the iterates converge to the desired solution. Let us first suppose that the initial data ρ_0, u_0 are smooth, and let us define $r_0 := \min_{x\in\mathbb{T}}\rho_0$.

Let us initialize our scheme as follows:

$$\begin{aligned}&(\rho_0(x,t),u_0(x,t)) := (\rho_0(x),u_0(x)),\\ &\rho_1(x,t) = \rho_0(x),\end{aligned}$$

and we define $u_1(x,t)$ so that

$$\begin{aligned}&\partial_t u_1 - \frac{\mu(\rho_1)}{\rho_1}\partial_x^2 u_1 = -u_0\partial_x u_0 - \frac{1}{\rho_0}\partial_x p(\rho_0) + \frac{\partial_x\mu(\rho_0)}{\rho_0}\partial_x u_0 + f,\\ &u_1|_{t=0} = u_0(x,0).\end{aligned} \tag{B.1}$$

Let now $n \geq 2$. Given ρ_{n-1}, u_{n-1}, we iteratively define ρ_n first, and subsequently u_n as follows

$$\partial_t\rho_n + u_{n-1}\partial_x\rho_n = -\rho_{n-1}\partial_x u_{n-1}, \tag{B.2}$$

$$\partial_t u_n - \frac{\mu(\rho_n)}{\rho_n}\partial_x^2 u_n = -u_{n-1}\partial_x u_{n-1} - \frac{1}{\rho_{n-1}}\partial_x p(\rho_{n-1}) + \frac{\partial_x\mu(\rho_{n-1})}{\rho_{n-1}}\partial_x u_{n-1} + f, \tag{B.3}$$

$$(\rho_n,u_n)|_{t=0} = (\rho_0,u_0). \tag{B.4}$$

Let $k \geq 1$ be an integer. We let, for ease of notation,

$$A := \|\rho_0\|_{H^k} + \|u_0\|_{H^k}.$$

We are going to prove, by induction on n, that there exists $T_0 > 0$ such that the following assertions hold.

Step 1: There exists $u_1 \in C^\infty(\mathbb{T}\times[0,T_0])$ satisfying (B.1) and

$$\|u_1\|_{L^\infty(0,T_0;H^k)} \leq 2A, \qquad \int_0^{T_0}\int_{\mathbb{T}}\frac{\mu(\rho_1)}{\rho_1}(\partial_x^{k+1}u_1)^2\mathrm{d}x\mathrm{d}t \leq 8A. \tag{B.5}$$

Step 2: For $n \geq 2$, there exists $\rho_n \in C^\infty(\mathbb{T} \times [0, T_0])$ satisfying (B.2), (B.4), and

$$\rho_n(x,t) \geq \frac{r_0}{2} \text{ on } \mathbb{T} \times [0, T_0].$$

Furthermore,

$$\|\rho_n\|_{L^\infty(0,T_0;H^k)} \leq 2A.$$

Step 3: There exists $u_n \in C^\infty(\mathbb{T} \times [0, T_0])$ satisfying (B.3), (B.4), and

$$\|u_n\|_{L^\infty(0,T_0;H^k)} \leq 2A, \qquad \int_0^{T_0} \int_{\mathbb{T}} \frac{\mu(\rho_n)}{\rho_n} (\partial_x^{k+1} u_n)^2 \mathrm{d}x\mathrm{d}t \leq 8A.$$

Step 4: The sequence (ρ_n, u_n) is Cauchy in the space $L^\infty(0, T_0; L^2) \times \big(L^\infty(0, T_0; L^2) \cap L^2(0, T_0; H^1)\big)$.

Step 5: There exist

$$u \in C(0, T_0; H^k) \cap L^2(0, T_0; H^{k+1})$$

and

$$\rho \in C(0, T_0; H^k)$$

such that (ρ, u) is a strong solution to the system (1.1)–(1.2) with initial data (ρ_0, u_0). In particular, if $k = 3$, said solution is a classical solution.

Step 6: The constructed strong solution is unique.

Let us now turn to the details.

Step 1. This is the base case of the induction. The existence of u_1 in the conditions follows from the general theory of linear parabolic equations, using the fact that ρ_0 is bounded from below by r_0, and that all functions involved are smooth. The bound (B.5) is obtained exactly as in **Step 3**, and we omit the details here.

Step 2. Let $n \geq 2$. Let us adopt the following nomenclature:

$$\rho := \rho_n, \quad \eta := \rho_{n-1}, \quad u := u_n, \quad v := u_{n-1}.$$

We recall the induction hypotheses:

$$\begin{aligned} &\|v\|_{L^\infty(0,T_0;H^k)} \leq 2A, && \|\eta\|_{L^\infty(0,T_0;H^k)} \leq 2A, \\ &\int_0^{T_0} \int_{\mathbb{T}} \frac{\mu(\eta)}{\eta} (\partial_x^{k+1} v)^2 \mathrm{d}x\mathrm{d}t \leq 8A, && \inf_{t \in [0,T_0]} \inf_{x \in \mathbb{T}} \eta(x,t) \geq \frac{r_0}{2}. \end{aligned} \tag{B.6}$$

Existence up to time T_0 and smoothness for ρ_n follow from the method of characteristics.

In what follows, $M(\cdot, \ldots, \cdot)$ will always denote a positive, continuous function increasing in all its arguments. We first notice that, due to the mass equation (B.2) and the maximum principle, for all $k \geq 1$ and $0 \leq t \leq T_0$,

$$\inf_{\mathbb{T}} \rho(\cdot, t) \geq \inf_{\mathbb{T}} \rho_0 - \int_0^t \|\eta(\cdot, s)\partial_x v(\cdot, s)\|_{L^\infty} \mathrm{d}s \geq \inf_{\mathbb{T}} \rho_0 - M(A)\sqrt{t}\|\partial_x^2 v\|_{L^2(0,t;L^2)}. \tag{B.7}$$

Hence, restricting T_0 to be small only as a function of A and r_0, we have

$$\inf_{t \in [0,T_0]} \inf_{x \in \mathbb{T}} \rho(x,t) \geq \frac{r_0}{2}.$$

We have therefore recovered the last induction hypothesis in (B.6).

Let us now differentiate the mass equation (B.2) k-times, multiply it by $\partial_x^k \rho$ and integrate by parts

$$\frac{1}{2}\partial_t \int_{\mathbb{T}} (\partial_x^k \rho)^2 \mathrm{d}x + \int_{\mathbb{T}} \partial_x^k \rho \, \partial_x^k (v \partial_x \rho) \mathrm{d}x = -\int_{\mathbb{T}} \partial_x^k \rho \, \partial_x^k (\eta \partial_x v). \tag{B.8}$$

If $k = 1$, we obtain

$$\frac{1}{2}\partial_t \|\rho\|_{L^2}^2 \le C\|\partial_x^2 v\|_{L^2}\|\rho\|_{L^2}^2 + \|\rho\|_{L^2}\|\eta\|_{L^\infty}\|\partial_x v\|_{L^2}, \tag{B.9}$$

$$\frac{1}{2}\partial_t \|\partial_x\rho\|_{L^2}^2 \le C\|\partial_x^2 v\|_{L^2}\|\partial_x\rho\|_{L^2}^2 + 2\|\partial_x\rho\|_{L^2}\|\partial_x\eta\|_{L^2}\|\partial_x v\|_{L^\infty} + \|\partial_x\rho\|_{L^2}\|\eta\|_{L^\infty}\|\partial_x^2 v\|_{L^2}. \tag{B.10}$$

Combining (B.9) and (B.10), integrating and using the induction hypotheses, we obtain, for suitable T_0 (depending only on A and r_0)

$$\|\rho\|_{L^\infty(0,T_0;H^1)} \le 2A. \tag{B.11}$$

If $k \ge 2$, in addition to previous estimate (B.9), we also have, for the terms appearing in (B.8),

$$\begin{aligned} &\left|\int_{\mathbb{T}} \partial_x^k\rho\,\partial_x^k(v\partial_x\rho)\mathrm{d}x\right| = \left|\frac{1}{2}\int_{\mathbb{T}} v\partial_x(\partial_x^k\rho)^2\mathrm{d}x + \int_{\mathbb{T}}\partial_x^k\rho\,([\partial_x^k, v]\partial_x\rho)\mathrm{d}x\right| \\ &\le \frac{1}{2}\|\partial_x v\|_{L^\infty}\|\rho\|_{H^k}^2 + \|\rho\|_{H^k}\|[\partial_x^k, v]\partial_x\rho\|_{L^2} \le C\|v\|_{H^2}\|\rho\|_{H^k}^2 + C\|\rho\|_{H^k}^2\|v\|_{H^k}. \end{aligned} \tag{B.12}$$

Furthermore,

$$\begin{aligned} &\left|\int_{\mathbb{T}} \partial_x^k\rho\,\partial_x^k(\eta\partial_x v)\right| \le \|\rho\|_{H^k}\|\eta\partial_x^{k+1}v\|_{L^2} + \|\rho\|_{H^k}\|[\partial_x^k,\eta]\partial_x v\|_{L^2} \\ &\le C\|\rho\|_{H^k}\left(\left\|\frac{\eta^3}{\mu(\eta)}\right\|_{L^\infty}^{\frac{1}{2}}\left\|\left(\frac{\mu(\eta)}{\eta}\right)^{\frac{1}{2}}\partial_x^{k+1}v\right\|_{L^2} + \|\eta\|_{H^2}\|v\|_{H^k} + \|v\|_{H^2}\|\eta\|_{H^k}\right). \end{aligned} \tag{B.13}$$

Now, due to our assumptions on μ and the induction hypothesis, we have

$$\left\|\frac{\eta^3}{\mu(\eta)}\right\|_{L^\infty}^{\frac{1}{2}} \le M(A, r_0^{-1}),$$

where M depends on μ and is an increasing function of its arguments.

Upon summation of (B.9) and (B.8), using (B.9) and (B.13),

$$\frac{1}{2}\partial_t\|\rho\|_{H^k}^2 \le C\|v\|_{H^k}\|\rho\|_{H^k}^2 + C\|\rho\|_{H^k}\|\eta\|_{H^k}\|v\|_{H^k} + M(A, r_0^{-1})\|\rho\|_{H^k}\left\|\left(\frac{\mu(\eta)}{\eta}\right)^{\frac{1}{2}}\partial_x^{k+1}v\right\|_{L^2}.$$

We now use the induction hypothesis (B.6) to obtain, for $0 \le t \le T_0$,

$$\partial_t\left(\|\rho\|_{H^k}\exp\left(-2CAt\right)\right) \le 4CA^2 + M(A, r_0^{-1})\left\|\left(\frac{\mu(\eta)}{\eta}\right)^{\frac{1}{2}}\partial_x^{k+1}v\right\|_{L^2}.$$

Upon integration, we obtain the following inequality:

$$\|\rho\|_{H^k} \le \exp\left(2CAt\right)\left(\|\rho_0\|_{H^k} + 4CA^2t + 8A\sqrt{t}M(A, r_0^{-1})\right).$$

It is now straightforward to choose T_0, depending only on A and r_0, such that the induction hypothesis

$$\|\rho\|_{L^\infty(0,T_0;H^k)} \le 2A$$

is recovered for ρ, in case $k \ge 2$.

Step 3. We now turn to the estimates on the momentum equation (B.3). Multiplying such equation by u and integrating by parts yields

$$\frac{1}{2}\partial_t\int_{\mathbb{T}} u^2\mathrm{d}x - \int_{\mathbb{T}}\frac{\mu(\rho)}{\rho}u\partial_x^2 u\mathrm{d}x = \int_{\mathbb{T}} u\cdot G_0\,\mathrm{d}x, \tag{B.14}$$

where $G_0 := -v\partial_x v - \frac{1}{\eta}\partial_x p(\eta) + \frac{\partial_x \mu(\eta)}{\eta}\partial_x v + f$. If $k \geq 1$, this implies

$$\begin{aligned}\frac{1}{2}\partial_t \|u\|_{L^2}^2 + \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho}(\partial_x u)^2 \mathrm{d}x &\leq M(A, r_0^{-1})\|\rho\|_{H^1}\|\partial_x u\|_{L^2}\|u\|_{L^\infty} \\ &+ C\|u\|_{L^2}\|v\|_{H^1}^2 + M(A, r_0^{-1})(\|\eta\|_{H^1}\|u\|_{L^2} + \|\eta\|_{H^1}\|v\|_{H^1}\|u\|_{H^1} + \|f\|_{L^2}\|u\|_{L^2}).\end{aligned} \tag{B.15}$$

Here, we used integration by parts and the following Lemma

LEMMA B.2. *Let f be a smooth function away from 0, and k be a positive integer. Let $u \in H^k(\mathbb{T}) \cap L^\infty(\mathbb{T})$, and suppose that there exists $r_0 > 0$ such that $u \geq r_0$ on $\mathbb{T}$. Then, there exists an increasing positive and continuous function M, depending only on f, k such that the following inequality holds:*

$$\|f \circ u\|_{H^k(\mathbb{T})} \leq M\big(\|u\|_{L^\infty(\mathbb{T})}, r_0^{-1}\big)\|u\|_{H^k(\mathbb{T})}. \tag{B.16}$$

PROOF OF LEMMA B.2. The proof of the lemma follows from Theorem 2.87 in [**22**], §2.8.2, and a straightforward cutoff argument. □

REMARK B.3. In what follows, we will always suppress the dependence of M on k and f, since they are fixed at the beginning of the argument.

Differentiating k-times ($k \geq 1$) equation (B.3), multiplying by $\partial_x^k u$, and integrating by parts yields

$$\frac{1}{2}\partial_t \int_{\mathbb{T}} (\partial_x^k u)^2 \mathrm{d}x - \int_{\mathbb{T}} (\partial_x^k u)\partial_x^k \left(\frac{\mu(\rho)}{\rho}\partial_x^2 u\right) \mathrm{d}x = -\int_{\mathbb{T}} (\partial_x^{k+1} u) \cdot G_k \, \mathrm{d}x. \tag{B.17}$$

Here, we defined

$$G_k := \partial_x^{k-1}\left(-v\partial_x v - \frac{1}{\eta}\partial_x p(\eta) + \frac{\partial_x \mu(\eta)}{\eta}\partial_x v + f\right), \qquad \text{for } k \geq 1.$$

When $k = 1$, the previous display (B.17) implies, upon integration by parts, an application of the Cauchy–Schwarz inequality, the induction hypotheses, Lemma B.2 and the bounds obtained in **Step 2**, that

$$\begin{aligned}\frac{1}{2}\partial_t \|\partial_x u\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{T}} \frac{\mu(\rho)}{\rho}(\partial_x^2 u)^2 \mathrm{d}x &\leq \int_{\mathbb{T}} \frac{\rho}{\mu(\rho)} G_1^2 \mathrm{d}x \\ &\leq M(A, r_0^{-1})(\|v\|_{H^1}^4 + \|\eta\|_{H^1}^2 + \|\eta\|_{H^1}^2\|\partial_x v\|_{L^2}\|\partial_x^2 v\|_{L^2} + \|f\|_{L^2}^2).\end{aligned} \tag{B.18}$$

Integrating (B.18) and, subsequently, (B.15), upon restricting T_0 to be sufficiently small only as a function of A and r_0, we have, in case $k = 1$,

$$\|u\|_{L^\infty(0,T_0;H^1)} \leq 2A, \qquad \int_0^{T_0} \frac{\mu(\rho)}{\rho}(\partial_x^2 u)^2 \mathrm{d}x\mathrm{d}t \leq 8A.$$

Let's focus now on the case $k \geq 2$. We have

$$\begin{aligned}&- \int_{\mathbb{T}} (\partial_x^k u)\partial_x^k \left(\frac{\mu(\rho)}{\rho}\partial_x^2 u\right) \mathrm{d}x \\ &= -\int_{\mathbb{T}} (\partial_x^k u)\partial_x^{k+1}\left(\frac{\mu(\rho)}{\rho}\partial_x u\right) \mathrm{d}x + \int_{\mathbb{T}} (\partial_x^k u)\partial_x^k \left(\partial_x \left(\frac{\mu(\rho)}{\rho}\right)\partial_x u\right) \mathrm{d}x \\ &= \int_{\mathbb{T}} \frac{\mu(\rho)}{\rho}(\partial_x^{k+1} u)^2 \mathrm{d}x \underbrace{+ \int_{\mathbb{T}} \partial_x^{k+1} u \left[\partial_x^k, \frac{\mu(\rho)}{\rho}\right](\partial_x u)\mathrm{d}x}_{(a)} \underbrace{- \int_{\mathbb{T}} (\partial_x^{k+1} u)\partial_x^{k-1}\left(\partial_x \left(\frac{\mu(\rho)}{\rho}\right)\partial_x u\right) \mathrm{d}x}_{(b)}.\end{aligned}$$

We estimate the last two terms in the previous display:

$$
\begin{aligned}
|(a)| &\leq \frac{1}{10}\int_{\mathbb{T}} \frac{\mu(\rho)}{\rho}(\partial_x^{k+1}u)^2 \mathrm{d}x + C\int_{\mathbb{T}} \frac{\rho}{\mu(\rho)}\left(\left[\partial_x^k, \frac{\mu(\rho)}{\rho}\right](\partial_x u)\right)^2 \mathrm{d}x \\
&\leq \frac{1}{10}\int_{\mathbb{T}} \frac{\mu(\rho)}{\rho}(\partial_x^{k+1}u)^2 \mathrm{d}x + M(A, r_0^{-1})\left\|\left[\partial_x^k, \frac{\mu(\rho)}{\rho}\right](\partial_x u)\right\|_{L^2} \\
&\leq \frac{1}{10}\int_{\mathbb{T}} \frac{\mu(\rho)}{\rho}(\partial_x^{k+1}u)^2 \mathrm{d}x \\
&\quad + M(A, r_0^{-1})\left(\left\|\partial_x \frac{\mu(\rho)}{\rho}\right\|_{L^\infty}\|\partial_x^k u\|_{L^2} + \|\partial_x u\|_{L^\infty}\left\|\partial_x^k \frac{\mu(\rho)}{\rho}\right\|_{L^2}\right) \\
&\leq \frac{1}{10}\int_{\mathbb{T}} \frac{\mu(\rho)}{\rho}(\partial_x^{k+1}u)^2 \mathrm{d}x + M\left(A, r_0^{-1}\right)\|u\|_{H^k}.
\end{aligned} \tag{B.19}
$$

Here, M is a continuous and increasing function of its arguments. We used the bounds obtained in **Step 2**, the Kato–Ponce commutator estimate, the fact that $k \geq 2$ and Lemma B.2 quoted below, applied to the function $\frac{\mu(\rho)}{\rho}$.

Similarly, the following estimate holds true, for $k \geq 2$:

$$
|(b)| \leq \frac{1}{10}\int_{\mathbb{T}} \frac{\mu(\rho)}{\rho}(\partial_x^{k+1}u)^2 \mathrm{d}x + M\left(A, r_0^{-1}\right)\|u\|_{H^k}. \tag{B.20}
$$

Again, M is a positive, continuous and increasing function of its arguments.

We now proceed to estimate the terms contained in the RHS of equation (B.17) (the terms named "G"), in case $k \geq 2$:

$$
\left|\int_{\mathbb{T}} (\partial_x^{k+1}u)\cdot G_k \,\mathrm{d}x\right| \leq \frac{1}{10}\int_{\mathbb{T}} \frac{\mu(\rho)}{\rho}(\partial_x^{k+1}u)^2 \mathrm{d}x + 5\int_{\mathbb{T}} \frac{\rho}{\mu(\rho)} G_k^2 \mathrm{d}x.
$$

Due to the bounds on ρ, we have

$$
\int_{\mathbb{T}} \frac{\rho}{\mu(\rho)} G_k^2 \mathrm{d}x \leq M\left(A, r_0^{-1}\right)\|G_k\|_{L^2}^2.
$$

Let us now define two auxiliary functions h (the thermodynamic enthalpy) and ζ in such a way that

$$
h'(x) = \frac{p'(x)}{x}, \qquad \zeta'(x) = \frac{\mu'(x)}{x}, \text{ for } x > 0.
$$

We now estimate:

$$
\|\partial_x^{k-1}(v\partial_x v)\|_{L^2}^2 \leq C\|v\|_{H^2}^2\|v\|_{H^k}^2 \leq CA^4.
$$

Furthermore,

$$
\|\partial_x^{k-1}\left(\frac{\partial_x p(\eta)}{\eta}\right)\|_{L^2}^2 \leq \|h(\eta)\|_{H^k}^2 \leq M(A, r_0^{-1}),
$$

where we used Lemma B.2, applied to the function h.

Finally, we have, since $k \geq 2$,

$$
\begin{aligned}
\left\|\partial_x^{k-1}\left(\frac{\partial_x \mu(\eta)}{\eta}\partial_x v\right)\right\|_{L^2}^2 &= \|\partial_x\zeta(\eta)\partial_x v\|_{H^{k-1}}^2 \leq C\left(\|\zeta(\eta)\|_{H^k}\|\partial_x v\|_{L^\infty} + \|v\|_{H^k}\|\partial_x\zeta(\eta)\|_{L^\infty}\right) \\
&\leq M(A, r_0^{-1}).
\end{aligned}
$$

Hence, for the term G_k, we have

$$
\left|\int_{\mathbb{T}} (\partial_x^{k+1}u)\cdot G_k \,\mathrm{d}x\right| \leq \frac{1}{10}\int_{\mathbb{T}} \frac{\mu(\rho)}{\rho}(\partial_x^{k+1}u)^2 \mathrm{d}x + M(A, r_0^{-1})(1 + \|f\|_{H^{k-1}}^2). \tag{B.21}
$$

Putting together estimates (B.14), (B.17), (B.19), (B.20), (B.21), and ignoring the positive integral term in the LHS, we obtain the inequality

$$\frac{1}{2}\partial_t \|u\|^2_{H^k} \leq M\left(A, r_0^{-1}\right)\|u\|_{H^k} + M\left(A, r_0^{-1}\right)(1 + \|f\|^2_{H^{k-1}}).$$

Using Grönwall's inequality, upon restricting T_0 to be small depending only on A, r_0 and f, we deduce that

$$\|u\|_{L^\infty(0,T_0;H^k)} \leq 2A. \tag{B.22}$$

We now revisit the same estimates without discarding the positive integral term in the LHS. We obtain, upon restricting T_0 to be smaller, depending only on A and r_0 and f, that

$$\int_0^{T_0}\int_{\mathbb{T}} \frac{\mu(\rho)}{\rho}(\partial_x^{k+1}u)^2 \mathrm{d}x\mathrm{d}t \leq 8A. \tag{B.23}$$

We have therefore recovered the induction hypotheses B.6, and in particular the sequence (ρ_n, u_n) is uniformly bounded in $L^\infty(0,T_0;H^k(\mathbb{T})) \times (L^\infty(0,T_0;H^k(\mathbb{T})) \cap L^2(0,T_0;H^{k+1}(\mathbb{T}))$.

Step 4. We now show that, for some T_0, depending only on A, r_0, the sequence (ρ_n, u_n) is Cauchy in the space $L^\infty(0,T_0;L^2) \times (L^\infty(0,T_0;L^2) \cap L^2(0,T_0;L^2))$.

Let's first consider the equation satisfied by $\delta u_n := u_{n+1} - u_n$:

$$\begin{aligned} &\partial_t(\delta u_n) - \frac{\mu(\rho_{n+1})}{\rho_{n+1}}\partial_x^2 u_{n+1} + \frac{\mu(\rho_n)}{\rho_n}\partial_x^2 u_n \\ &= \frac{1}{2}\partial_x(u_n^2 - u_{n-1}^2) + \partial_x(h(\rho_n) - h(\rho_{n-1})) + \partial_x\zeta(\rho_n)\,\partial_x u_n - \partial_x\zeta(\rho_{n-1})\,\partial_x u_{n-1}. \end{aligned} \tag{B.24}$$

Recall that we defined h and ζ so that the following equalities hold true:

$$\partial_x h(\rho) = \frac{\partial_x p(\rho)}{\rho}, \qquad \zeta(\rho) = \frac{\partial_x \mu(\rho)}{\rho}.$$

We now multiply equation (B.24) by δu_n and integrate by parts. We have:

$$\begin{aligned} &\int_{\mathbb{T}} (\delta u_n)\left(-\frac{\mu(\rho_{n+1})}{\rho_{n+1}}\partial_x^2 u_{n+1} + \frac{\mu(\rho_n)}{\rho_n}\partial_x^2 u_n\right)\mathrm{d}x \\ &= \underbrace{-\int_{\mathbb{T}} (\delta u_n)\frac{\mu(\rho_{n+1})}{\rho_{n+1}}\partial_x^2(\delta u_n)\mathrm{d}x}_{(a)} + \underbrace{\int_{\mathbb{T}}\left(\frac{\mu(\rho_n)}{\rho_n} - \frac{\mu(\rho_{n+1})}{\rho_{n+1}}\right)\partial_x^2 u_n(\delta u_n)\mathrm{d}x}_{(b)}. \end{aligned}$$

Note that, due to **Step 3**, there exists $c = c(A, r_0)$ such that, up to time T_0, there holds $\frac{\mu(\rho_i)}{\rho_i} \geq c$ for all integers $i \geq 0$.

Hence, for the term in (a), upon integration by parts,

$$\begin{aligned} (a) &\geq c\|\partial_x(\delta u_n)\|^2_{L^2} - \frac{1}{c}\|\partial_x \frac{\mu(\rho_n)}{\rho_n}\|_{L^2}\|\delta u_n\|_{L^\infty}\|\partial_x(\delta u_n)\|_{L^2} \\ &\geq c\|\partial_x(\delta u_n)\|^2_{L^2} - M(A, r_0^{-1})\Big(\|\delta u_n\|^{\frac{1}{2}}_{L^2}\|\partial_x(\delta u_n)\|^{\frac{3}{2}}_{L^2} + \|\delta u_n\|_{L^2}\|\partial_x(\delta u_n)\|_{L^2}\Big) \\ &\geq \frac{c}{2}\|\partial_x(\delta u_n)\|^2_{L^2} - M(A, r_0^{-1})\|\delta u_n\|^2_{L^2}. \end{aligned}$$

Here, we used Lemma B.2, the Gagliardo–Nirenberg–Sobolev inequality and the Young inequality.

We now estimate

$$(b) \geq -M(A, r_0^{-1})\|\delta\rho_n\|_{L^2}\|\partial_x^2 u_n\|_{L^2}\|\delta u_n\|^{\frac{1}{2}}_{L^2}\|\delta u_n\|^{\frac{1}{2}}_{H^1}.$$

Let us now turn to the terms appearing in the RHS of (B.24). We define

$$\underbrace{\int_{\mathbb{T}} \frac{1}{2}\partial_x(u_n^2 - u_{n-1}^2)(\delta u_n)\mathrm{d}x}_{(c)} + \underbrace{\int_{\mathbb{T}} (\delta u_n)\partial_x(h(\rho_n) - h(\rho_{n-1}))\mathrm{d}x}_{(d)}$$
$$+ \underbrace{\int_{\mathbb{T}} (\delta u_n)\left(\partial_x\zeta(\rho_n)\,\partial_x u_n - \partial_x\zeta(\rho_{n-1})\,\partial_x u_{n-1}\right)\mathrm{d}x}_{(e)}.$$

Then, for (c), we have, after integration by parts,

$$|(c)| \leq M(A)\|\partial_x(\delta u_n)\|_{L^2}\|\delta u_{n-1}\|_{L^2} \leq \frac{1}{10c}\|\partial_x(\delta u_n)\|_{L^2}^2 + M(A)\|\delta u_{n-1}\|_{L^2}^2.$$

Concerning the term (d), instead,

$$\begin{aligned}|(d)| &= \left|\int_{\mathbb{T}} \partial_x(\delta u_n)\,(h(\rho_n) - h(\rho_{n-1}))\mathrm{d}x\right| \\ &\leq \frac{1}{10c}\|\partial_x(\delta u_n)\|_{L^2}^2 + M(A, r_0^{-1})\|\delta\rho_{n-1}\|_{L^2}^2.\end{aligned}$$

Again, we used the fact that, due to the uniform bounds on ρ_n, h is Lipschitz of constant depending only on A and r_0.

Finally, concerning (e),

$$\begin{aligned}|(e)| &\leq \left|\int_{\mathbb{T}}(\delta u_n)\partial_x\zeta(\rho_n)\partial_x(\delta u_{n-1})\mathrm{d}x\right| + \left|\int_{\mathbb{T}}(\delta u_n)\partial_x(\zeta(\rho_n) - \zeta(\rho_{n-1}))\partial_x u_{n-1}\mathrm{d}x\right| \\ &\leq \|\delta u_n\|_{L^\infty}\|\partial_x\zeta(\rho_n)\|_{L^2}\|\partial_x(\delta u_{n-1})\|_{L^2} + \left|\int_{\mathbb{T}}(\zeta(\rho_n) - \zeta(\rho_{n-1}))\partial_x((\delta u_n)\partial_x u_{n-1})\mathrm{d}x\right| \\ &\leq M(A, r_0^{-1})\Big(\|\delta u_n\|_{L^2}^{\frac{1}{2}}\|\partial_x(\delta u_n)\|_{L^2}^{\frac{1}{2}}\|\partial_x(\delta u_{n-1})\|_{L^2} + \|\partial_x(\delta u_{n-1})\|_{L^2}\|\delta u_n\|_{L^2}\Big) \\ &\quad + M(A, r_0^{-1})(\|\delta\rho_{n-1}\|_{L^2}\|\partial_x\delta u_n\|_{L^2}\|\partial_x^2 u_n\|_{L^2}^{\frac{1}{2}} + \|\delta\rho_{n-1}\|_{L^2}\|\delta u_n\|_{L^\infty}\|\partial_x^2 u_n\|_{L^2})\end{aligned}$$

where $\delta\rho_{n-1} := \rho_n - \rho_{n-1}$. Putting together the estimates on the momentum equation, we have

$$\begin{aligned}&\frac{1}{2}\partial_t\|\delta u_n\|_{L^2}^2 + \frac{1}{10c}\|\partial_x(\delta u_n)\|_{L^2}^2 \\ &\leq M(A, r_0^{-1})(\|\delta u_n\|_{L^2}^2 + \|\delta u_{n-1}\|_{L^2}^2 + \|\delta\rho_{n-1}\|_{L^2}^2) \\ &\quad + M(A, r_0^{-1})\|\delta\rho_n\|_{L^2}\|\partial_x^2 u_n\|_{L^2}\|\delta u_n\|_{L^2}^{\frac{1}{2}}\|\partial_x(\delta u_n)\|_{L^2}^{\frac{1}{2}} \\ &\quad + M(A, r_0^{-1})(\|\delta u_n\|_{L^2}^{\frac{1}{2}}\|\partial_x(\delta u_n)\|_{L^2}^{\frac{1}{2}}\|\partial_x(\delta u_{n-1})\|_{L^2} + \|\partial_x(\delta u_{n-1})\|_{L^2}\|\delta u_n\|_{L^2}) \\ &\quad + M(A, r_0^{-1})(\|\delta\rho_n\|_{L^2}\|\partial_x\delta u_n\|_{L^2}\|\partial_x^2 u_n\|_{L^2}^{\frac{1}{2}} + \|\delta\rho_n\|_{L^2}\|\delta u_n\|_{L^\infty}\|\partial_x^2 u_n\|_{L^2}).\end{aligned}$$

Upon integration between time $s = 0$ and $s = t$, using Hölder's inequality and the bounds obtained in **Step 1**,

$$\begin{aligned}
&\frac{1}{2}\|(\delta u_n)(\cdot,t)\|_{L^2}^2 + \frac{1}{10c}\|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)}^2 \\
&\quad \le M(A, r_0^{-1})(\|\delta u_n\|_{L^2(0,t;L^2)}^2 + \|\delta u_{n-1}\|_{L^2(0,t;L^2)}^2 + \|\delta \rho_{n-1}\|_{L^2(0,t;L^2)}^2) \\
&\quad + M(A, r_0^{-1})t^{\frac{1}{4}}\|\delta\rho_n\|_{L^\infty(0,t;L^2)}\|\delta u_n\|_{L^\infty(0,t;L^2)}^{\frac{1}{2}}\|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)}^{\frac{1}{2}} \\
&\quad + M(A, r_0^{-1})t^{\frac{1}{4}}\|\delta u_n\|_{L^\infty(0,t;L^2)}^{\frac{1}{2}}\|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)}^{\frac{1}{2}}\|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)} \\
&\quad + M(A, r_0^{-1})t^{\frac{1}{2}}\|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)}\|\delta u_n\|_{L^\infty(0,t;L^2)} \\
&\quad + M(A, r_0^{-1})t^{\frac{1}{4}}\|\delta\rho_{n-1}\|_{L^\infty(0,t;L^2)}\|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)} \\
&\quad + M(A, r_0^{-1})t^{\frac{1}{4}}\|\delta\rho_{n-1}\|_{L^\infty(0,t;L^2)}\|\partial_x \delta u_n\|_{L^2(0,t;L^2)} \\
&\quad + M(A, r_0^{-1})t^{\frac{1}{4}}\|\delta\rho_{n-1}\|_{L^\infty(0,t;L^2)}\|\delta u_n\|_{L^\infty(0,t;L^2)}^{\frac{1}{2}}\|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)}^{\frac{1}{2}} \\
&\quad \le \frac{1}{20c}\|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)}^2 + M(A, r_0^{-1})t^{\frac{1}{4}}(\|\delta u_n\|_{L^\infty(0,t;L^2)}^2 + \|\delta u_{n-1}\|_{L^\infty(0,t;L^2)}^2 + \\
&\qquad\qquad \|\delta\rho_{n-1}\|_{L^\infty(0,t;L^2)}^2 + \|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)}^2).
\end{aligned} \tag{B.25}$$

Let us now calculate the equation satisfied by differences of ρ_n:

$$\partial_t(\delta\rho_n) = -u_n\partial_x\rho_{n+1} + u_{n-1}\partial_x\rho_n - \rho_n\partial_x u_n + \rho_{n-1}\partial_x u_{n-1}. \tag{B.26}$$

Multiplying equation (B.26) by $\delta\rho_n$, we obtain

$$\begin{aligned}
\frac{1}{2}\partial_t\|\delta\rho_n\|_{L^2}^2 = &\underbrace{-\int_{\mathbb{T}}(\delta\rho_n)(u_n\partial_x\rho_{n+1} - u_{n-1}\partial_x\rho_n)\mathrm{d}x}_{(a)} \\
&\underbrace{-\int_{\mathbb{T}}(\delta\rho_n)(\rho_n\partial_x u_n - \rho_{n-1}\partial_x u_{n-1})\mathrm{d}x}_{(b)}.
\end{aligned}$$

Considering (a), we have, integrating by parts, using Gagliardo–Nirenberg–Sobolev and Hölder's inequality,

$$\begin{aligned}
|(a)| &\le \left|\int_{\mathbb{T}}(\delta\rho_n)(\delta u_{n-1})\partial_x\rho_{n+1}\mathrm{d}x\right| + \left|\int_{\mathbb{T}}\partial_x(\delta\rho_n)(\delta\rho_n)u_{n-1}\mathrm{d}x\right| \\
&\le M(A)(\|\delta\rho_n\|_{L^2}\|\delta u_{n-1}\|_{H^1}^{\frac{1}{2}}\|\delta u_{n-1}\|_{L^2}^{\frac{1}{2}} + \|\delta\rho_n\|_{L^2}^2\|\partial_x^2 u_n\|_{L^2}^{\frac{1}{2}}).
\end{aligned}$$

On the other hand, (b) yields

$$\begin{aligned}
|(b)| &\le \left|\int_{\mathbb{T}}(\delta\rho_n)(\delta\rho_{n-1})\partial_x u_n\mathrm{d}x\right| + \left|\int_{\mathbb{T}}(\delta\rho_n)\partial_x(\delta u_{n-1})\rho_{n-1}\mathrm{d}x\right| \\
&\le M(A)(\|\delta\rho_n\|_{L^2}^2 + \|\delta\rho_{n-1}\|_{L^2}^2)\|\partial_x^2 u_n\|_{L^2}^{\frac{1}{2}} + M(A)\|\partial_x(\delta u_{n-1})\|_{L^2}\|\delta\rho_n\|_{L^2}.
\end{aligned}$$

Putting together the estimates on the mass equation yields

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\delta\rho_n\|_{L^2}^2 \\
&\le M(A)\Big(\|\delta\rho_n\|_{L^2}\|\partial_x(\delta u_{n-1})\|_{L^2}^{\frac{1}{2}}\|\delta u_{n-1}\|_{L^2}^{\frac{1}{2}} + \|\delta\rho_n\|_{L^2}^2\|\partial_x^2 u_n\|_{L^2}^{\frac{1}{2}}\Big) + M(A)\|\delta\rho_n\|_{L^2}\|\delta u_{n-1}\|_{L^2} \\
&\quad + M(A)(\|\delta\rho_n\|_{L^2}^2 + \|\delta\rho_{n-1}\|_{L^2}^2)\|\partial_x^2 u_n\|_{L^2}^{\frac{1}{2}} + M(A)\|\partial_x(\delta u_{n-1})\|_{L^2}\|\delta\rho_n\|_{L^2}.
\end{aligned}$$

Upon integration, the previous display yields

$$\begin{aligned}
\frac{1}{2}\|\delta\rho_n(t,\cdot)\|_{L^2}^2 &\le M(A)t^{\frac{3}{4}}\|\delta\rho_n\|_{L^\infty(0,t;L^2)}\|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)}^{\frac{1}{2}}\|\delta u_{n-1}\|_{L^\infty(0,t;L^2)}^{\frac{1}{2}} \\
&\quad + M(A)t^{\frac{3}{4}}\|\delta\rho_n\|_{L^\infty(0,t;L^2)}^2 + M(A)t(\|\delta\rho_n\|_{L^\infty(0,t;L^2)}^2 + \|\delta u_{n-1}\|_{L^\infty(0,t;L^2)}^2) \\
&\quad + M(A)t^{\frac{3}{4}}(\|\delta\rho_n\|_{L^\infty(0,t;L^2)}^2 + \|\delta\rho_{n-1}\|_{L^\infty(0,t;L^2)}^2) \\
&\quad + M(A)t^{\frac{1}{2}}\|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)}\|\delta\rho_n\|_{L^\infty(0,t;L^2)} \\
&\le M(A)t^{\frac{1}{2}}\Big(\|\delta\rho_n\|_{L^\infty(0,t;L^2)}^2 + \|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)}^2 + \|\delta u_{n-1}\|_{L^\infty(0,t;L^2)}^2 \\
&\qquad + \|\delta\rho_{n-1}\|_{L^\infty(0,t;L^2)}^2\Big).
\end{aligned}$$

Combining now (B.25) and (B.27), we obtain, for suitably small t depending only on A and r_0,

$$\begin{aligned}
&\frac{1}{4}\|\delta\rho_n\|_{L^\infty(0,t;L^2)}^2 + \frac{1}{4}\|\delta u_n\|_{L^\infty(0,t;L^2)}^2 + \frac{1}{20c}\|\partial_x(\delta u_n)\|_{L^2(0,t;L^2)}^2 \\
&\le M(A,r_0^{-1})t^{\frac{1}{4}}(\|\partial_x(\delta u_{n-1})\|_{L^2(0,t;L^2)}^2 + \|\delta u_{n-1}\|_{L^\infty(0,t;L^2)}^2 + \|\delta\rho_{n-1}\|_{L^\infty(0,t;L^2)}^2).
\end{aligned}$$

Upon suitable choice of T_0, this implies that the sequence (ρ_n, u_n) is Cauchy in the space $L^\infty(0,T_0;L^2) \times (L^\infty(0,T_0;L^2) \cap L^2(0,T_0;H^1))$.

Step 5. Denote

$$X^m = L^\infty(0,T_0;H^m) \times \big(L^\infty(0,T_0;H^m) \cap L^2(0,T_0;H^{m+1})\big)$$

a Banach space with its canonical norm. We have proved in the previous steps that (ρ_n, u_n) is bounded in X^k and Cauchy in X^{k-1}. The latter implies that (ρ_n, u_n) converges to some (ρ, u) in X^{k-1}. The former implies that some subsequence (ρ_{n_j}, u_{n_j}) converges weak-* to some (ρ_*, u_*) in X^k. Since both weak-* convergence in X^k and strong convergence in X^{k-1} imply convergence in the sense of distributions we deduce that $(\rho, u) = (\rho_*, u_*) \in X^k$. It can be easily verified that (ρ, u) is a strong solution to the system (1.1)–(1.2). Moreover, since $\rho_n \to \rho$ strongly in $L^2(0,T_0;L^2)$ and (ρ_n) is bounded in $L^\infty(0,T_0;H^1)$ it follows by interpolation that $\rho_n \to \rho$ strongly in $L^\infty(0,T_0;H^{3/4})$, and hence in $L^\infty(0,T_0;L^\infty)$. This combined with the fact that $\rho_n(x,t) \ge \frac{r_0}{2}$ for all $(x,t) \in \mathbb{T} \times [0,T_0]$ (see **Step 2**) yields

$$\rho(x,t) \ge \frac{r_0}{2} \quad \forall (x,t) \in \mathbb{T} \times [0,T_0].$$

Step 6. We now establish uniqueness of strong solutions. Consider two solutions (ρ_1, u_1) and (ρ_2, u_2), such that

$$\rho_i \in C(0,T_0;H^k(\mathbb{T})), \quad u_i \in C(0,T_0;H^k(\mathbb{T})) \cap L^2(0,T_0;H^{k+1}(\mathbb{T})), \text{ for } i = 1,2.$$

and let $(\delta\rho, \delta u) = (\rho_1 - \rho_2, u_1 - u_2)$. We have

$$\partial_t \delta u + \delta u \partial_x u_1 + u_2 \partial_x \delta u = -\partial_x((\rho_1) - (\rho_2)) + \rho_1^{-1}\partial_x(\mu(\rho_1)\partial_x u_1) - \rho_2^{-1}\partial_x(\mu(\rho_2)\partial_x u_2), \tag{B.27}$$

$$\partial_t \delta\rho + \partial_x(u_1 \delta\rho + \rho_2 \delta u) = 0, \tag{B.28}$$

$$(\delta\rho, \delta u)|_{t=0} = (0,0) \tag{B.29}$$

We now notice that equation (B.27) is the same as equation (B.24), upon formally substituting $n = 1$ in the LHS, and $n = 2$ in the RHS. Similarly, recalling (B.26), we have

$$\underbrace{\partial_t(\delta\rho_n)}_{(a)} = \underbrace{-u_n}_{(b)}\underbrace{\partial_x\rho_{n+1}}_{(a)} + \underbrace{u_{n-1}}_{(b)}\underbrace{\partial_x\rho_n}_{(a)}\underbrace{-\rho_n\partial_x u_n + \rho_{n-1}\partial_x u_{n-1}}_{(b)}.$$

Formally substituting $n = 1$ in terms (a), and $n = 2$ in terms (b), we obtain (B.28). It is then straightforward to see that the same estimates as in **Step 4** yield uniqueness of strong solutions. □

Acknowledgment. We thank Toan Nguyen for interesting comments. The research of PC is partially supported by NSF grant DMS-1713985. The research of TD is partially supported by NSF grant DMS-1703997.

References

[1] J.-F. Gerbeau and B. Perthame. Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. *Discrete Contin. Dyn. Syst. Ser. B*, 1(1): 89–102, 2001.

[2] Fabien Marche. Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *Eur. J. Mech. B Fluids*, 26(1):49–63, 2007.

[3] J. Eggers and T. F. Dupont. Drop formation in a one-dimensional approximation of the Navier-Stokes equation. *J. Fluid Mech.* 262, 205–221, 1994.

[4] J. Eggers and M. A Fontelos. *Singularities: formation, structure, and propagation*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2015. xvi+453 pp.

[5] Kritsuk, A. G., Norman, M. L., Padoan, P., and Wagner, R. (2007). The statistics of supersonic isothermal turbulence. *The Astrophysical Journal*, 665(1), 416.

[6] Kritsuk, A. G., Wagner, R., and Norman, M. L. (2013). Energy cascade and scaling in supersonic isothermal turbulence. *J. Fluid Mech*, 729.

[7] David Hoff and Joel Smoller. Non-formation of vacuum states for compressible Navier-Stokes equations. *Comm. Math. Phys.*, 216(2):255–276, 2001.

[8] An Ton Bui. Existence and Uniqueness of a Classical Solution of an Initial-Boundary Value Problem of the Theory of Shallow Waters. *SIAM J. Math. Anal.*, 12(2), 229–241, 1981.

[9] Yachun Li, Ronghua Pan, and Shengguo Zhu. On classical solutions to 2D shallow water equations with degenerate viscosities. *J. Math. Fluid Mech.*, 19(1):151–190, 2017.

[10] Akitaka Matsumura and Takaaki Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, 20(1):67–104, 1980.

[11] Akitaka Matsumura and Takaaki and Nishida. Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. *Comm. Math. Phys.*, 89(4):445–464, 1983.

[12] Haspot, Boris. Existence of global strong solution for the compressible Navier-Stokes equations with degenerate viscosity coefficients in 1D. arXiv preprint arXiv:1411.5503, 2014.

[13] A. Mellet and A. Vasseur. Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations. *SIAM J. Math. Anal.*, 39(4):1344–1365, 2007/08.

[14] Linda Sundbye. Global existence for the Dirichlet problem for the viscous shallow water equations. *J. Math. Anal. Appl.*, 202(1):236–258, 1996.

[15] Linda Sundbye. Global existence for the Cauchy problem for the viscous shallow water equations. *Rocky Mountain J. Math.*, 28(3):1135–1152, 1998.

[16] P. E. Kloeden. Global existence of classical solutions in the dissipative shallow water equations. *SIAM J. Math. Anal.*, 16(2):301–315, 1985.

[17] Constantin, P., Elgindi, T., Nguyen, H., and Vicol, V. (2017). On singularity formation in a Hele-Shaw model. arXiv preprint arXiv:1708.08490.

[18] David Hoff. Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. *J. Differential Equations*, 120(1): 215–254, 1995.

[19] Didier Bresch and Benoît Desjardins. Existence of Global Weak Solutions for a 2D Viscous Shallow Water Equations and Convergence to the Quasi-Geostrophic Model. *Communications in Mathematical Physics*, 238(1), 211–223, 2003. 2003.

[20] Didier Bresch, Benoît Desjardins, and Chi-Kun Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations*, 28(3-4):843–868, 2003.

[21] Didier Bresch, Benoît Desjardins, and Guy Métivier. Recent mathematical results and open problems about shallow water equations. In *Analysis and simulation of fluid dynamics*, Adv. Math. Fluid Mech., pages 15–31. Birkhäuser, Basel, 2007.

[22] H. Bahouri, J. -Y. Chemin and R. Danchin. *Fourier analysis and nonlinear partial differential equations*. Grundlehren der Mathematischen Wissenschaften 343. Springer, New York, 2011.

[23] T. Kato and G. Ponce. “Commutator estimates and the Euler and Navier–Stokes equations.” Communications on Pure and Applied Mathematics 41.7 (1988): 891-907.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544

Email address: const@math.princeton.edu

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544
Email address: tdrivas@math.princeton.edu

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544
Email address: qn@math.princeton.edu

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544
Email address: fp2@princeton.edu