Remarks on type I blow up for the 3D Euler equations and the 2D Boussinesq equations

Dongho Chae* and Peter Constantin†

∗Department of Mathematics
Chung-Ang University
Seoul 06974, Republic of Korea
e-mail: dchae@cau.ac.kr
and
†Department of Mathematics
Princeton University
Princeton, NJ 08544, USA
e-mail: const@math.princeton.edu

Abstract

In this paper we derive kinematic relations for quantities involving the rate of strain tensor and the Hessian of the pressure for solutions of the 3D Euler equations and the 2D Boussinesq equations. Using these kinematic relations, we prove new blow up criteria and obtain conditions for the absence of type I singularity for these equations. We obtain both global and localized versions of the results. Some of the new blow up criteria and type I conditions improve previous results of [3].

AMS Subject Classification Number: 35Q31, 76B03
keywords: Euler equations, Boussinesq equations, kinematic relations, blow up criterion, type I singularity

1 The 3D Euler equations

1.1 Introduction and main results

We consider the homogeneous incompressible Euler equations on $\mathbb{R}^3 \times [0,T)$.

$$(E)\begin{cases} u_t + u\cdot\nabla u = -\nabla p, \\ \nabla\cdot u = 0, \quad u(x,0) = u_0(x) \end{cases}$$

where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the fluid velocity and $p = p(x,t)$ is the scalar pressure. We denote the initial velocity by $u_0(x) = u(x,0)$ where $x \in \mathbb{R}^3$. For the Cauchy problem of the system (E) with $u_0 \in W^{2,q}(\mathbb{R}^3)$, $q > 3$, $\nabla \cdot u_0 = 0$, there exists a local in time well-posedness result [11], but the question of the finite time blow up is a wide open problem. See e.g. [14, 7, 8, 9] and the references therein for detailed discussions of the problem. For important partial results we refer [1, 10]. See also [12, 13] and references therein for related numerical works.

We associate to a solution (u, p) of the Euler system (E) the $\mathbb{R}^{3\times3}$-valued functions $S = (S_{ij})$ and $P = (P_{ij})$, where

$$S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \quad P_{ij} = \partial_i \partial_j p.$$

For the vorticity $\omega = \nabla \times u$ we define the direction vectors

$$\xi = \omega/|\omega|, \qquad \zeta = S\xi/|S\xi|,$$

and the scalar functions

$$\alpha = S_{ij}\xi_i\xi_j, \quad \rho = P_{ij}\xi_i\xi_j,$$

where we used the convention of summing over repeated indices. In the case $\omega(x,t) = 0$ we set $\alpha(x,t) = \rho(x,t) = 0$. These quantities have been introduced previously [9, 14, 2]. Note that ξ is the *vorticity direction* vector, while ζ is the *vorticity stretching direction* vector. Below we also use the notations $[f]_+ = \max\{f, 0\}$ and $[f]_- = \max\{-f, 0\}$.

Theorem 1.1 *Let $(u, p) \in C^1(\mathbb{R}^3 \times (0,T))$ be a solution of the Euler equation (E) with $u \in C([0,T); W^{2,q}(\mathbb{R}^3))$, for some $q > 3$. Suppose the following holds. Either*

(i)

$$\int_0^T \exp\left(\int_0^t \int_0^s \|[\zeta \cdot P\xi]_-(\tau)\|_{L^\infty} d\tau ds\right) dt < +\infty, \tag{1.1}$$

or

$$\int_0^T \exp\left(\int_0^t \int_0^s \|[|S\xi|^2 - 2\alpha^2 - \rho]_+(\tau)\|_{L^\infty} d\tau ds\right) dt < +\infty,$$

then $\limsup_{t\to T} \|u(t)\|_{W^{2,q}} < +\infty$.

(ii) If either

$$\limsup_{t\to T} (T-t)^2 \|[\zeta \cdot P\xi]_-(t)\|_{L^\infty} < 1, \tag{1.2}$$

or

$$\limsup_{t\to T} (T-t)^2 \|[|S\xi|^2 - 2\alpha^2 - \rho]_+(t)\|_{L^\infty} < 1, \tag{1.3}$$

then $\limsup_{t\to T} \|u(t)\|_{W^{2,q}} < +\infty$.

Remark 1.1 In [3] we obtained the above theorem with $[\zeta \cdot P\xi]_-$ replaced by $|P|$, which is the matrix norm of the Hessian of the pressure. Since $|[\zeta \cdot P\xi]_-| \le |P|$ the above(and the localized version below) improve the results of Theorem 1.1 of [3]. Furthermore, the above theorem implies that the dynamical changes of the signs of the scalar quantities $\zeta \cdot P\xi$ and $|S\xi|^2 - 2\alpha^2 - \rho$ are important in the phenomena of blow up/regularity.

The following is a localized version of the above theorem.

Theorem 1.2 *Let $(u,p) \in C^1(B(x_0,r) \times (T-r,T))$ be a solution to (E) with $u \in C([T-r,T); W^{2,q}(B(x_0,r))) \cap L^\infty(T-r,T; L^2(B(x_0,r)))$ for some $q \in (3,\infty)$. We suppose*

$$\int_{T-r}^T \|u(t)\|_{L^\infty(B(x_0,r))} dt < +\infty,$$

and the following holds. Either

(i)

$$\int_{T-r}^T \exp\left(\int_0^t \int_0^s \|[\zeta \cdot P\xi]_-(\tau)\|_{L^\infty(B(x_0,r))} d\tau ds\right) dt < +\infty,$$

or

$$\int_{T-r}^T \exp\left(\int_{T-r}^t \int_{T-r}^s \|[|S\xi|^2 - 2\alpha^2 - \rho]_+(\tau)\|_{L^\infty(B(x_0,r))} d\tau ds\right) dt < +\infty.$$

Then for all $\varepsilon \in (0,r)$ $\limsup_{t \nearrow T} \|u(t)\|_{W^{2,q}(B(x_0,\varepsilon))} < +\infty$.

(ii) If (1.2) holds, and if either

$$\limsup_{t \to T} (T-t)^2 \|[\zeta \cdot P\xi]_-(t)\|_{L^\infty(B(x_0,r))} < 1, \tag{1.4}$$

or

$$\limsup_{t \to T} (T-t)^2 \|[|S\xi|^2 - 2\alpha^2 - \rho]_+(t)\|_{L^\infty(B(x_0,r))} < 1, \tag{1.5}$$

then for all $\varepsilon \in (0,r)$ $\limsup_{t \nearrow T} \|u(t)\|_{W^{2,q}(B(x_0,\varepsilon))} < +\infty$.

1.2 Kinematic relations

We use the particle trajectory mapping $a \mapsto X(a,t)$ from $\mathbb{R}^3$ into $\mathbb{R}^3$ generated by $u = u(x,t)$, which means the solution of the ordinary differential equation,

$$\begin{cases} \dfrac{\partial X(a,t)}{\partial t} = u(X(a,t),t) \quad \text{on} \quad (0,T), \\ X(a,0) = a \in \mathbb{R}^3. \end{cases}$$

The material derivative of $f = f(x,t)$ is defined by

$$D_t f := \partial_t f + u \cdot \nabla f.$$

We note that $(D_t f)(X(a,t),t) = \frac{\partial}{\partial t}\{f(X(a,t),t)\}$.

Proposition 1.1 *Let (u,p) be a solution of (E), which belongs to $C^1(\mathbb{R}^3\times(0,T))$. We use the above notations. Then, the followings hold true on $\mathbb{R}^3\times(0,T)$.*

$$D_t|S\omega| = -\zeta\cdot P\omega, \tag{1.6}$$
$$D_t^2\log|\omega| = |S\xi|^2 - 2\alpha^2 - \rho, \tag{1.7}$$
$$(D_t|\omega|)^2 + (|D_t\xi||\omega|)^2 = |S\omega|^2, \tag{1.8}$$
$$(D_t|S\omega|)^2 + (|D_t\zeta||S\omega|)^2 = |P\omega|^2, \tag{1.9}$$
$$\tag{1.10}$$

Remark 1.2 Applying the inequality, $a_1+\cdots+a_n\le\sqrt{n(a_1^2+\cdots+a_n^2)}$ to equations (1.8), (1.9) and (1.14) respectively, we obtain the following differential inequalities with the coefficients consisting of derivatives of the direction fields ξ and ζ,

$$D_t|\omega| + |D_t\xi||\omega| \le \sqrt{2}|S\omega|, \tag{1.11}$$
$$D_t|S\omega| + |D_t\zeta||S\omega| \le \sqrt{2}|P\omega|, \tag{1.12}$$
$$D_t|S\omega| + |D_t\zeta|D_t|\omega| + |D_t\zeta||D_t\xi||\omega| \le \sqrt{3}|P\omega|. \tag{1.13}$$

For an implication of (1.11) combined with (1.12), in particular, see Remark 1.3 below.

Proof of Proposition 1.1 Taking the gradient of (E), we find

$$D_t\nabla u = -(\nabla u)^2 - P. \tag{1.14}$$

We observe the decomposition of the matrix,

$$\nabla u = S + \Omega, \quad \text{where} \quad \Omega_{ij} = \frac{1}{2}(\partial_i u_j - \partial_j u_i) = \frac{1}{2}\epsilon_{ijk}\omega_k.$$

Here, ϵ_{ijk} is the totally skew-symmetric tensor with normalization $\epsilon_{123}=1$. Taking the skew symmetric part of (1.14), we obtain the vorticity equations

$$D_t\omega = \omega\cdot\nabla u = S\omega, \tag{1.15}$$

where we used the fact $\omega_j\partial_j u_i = \omega_j S_{ji} + \frac{1}{2}\epsilon_{jik}\omega_j\omega_k = \omega_j S_{ji}$. Taking the symmetric part of (1.14), on the other hand, we find

$$D_t S = -S^2 + \frac{1}{4}(|\omega|^2 I - \omega\otimes\omega) - P. \tag{1.16}$$

Contracting (1.15) with ω, and dividing the both sides by $|\omega|^2$, we have

$$D_t|\omega| = \alpha|\omega|. \tag{1.17}$$

From (1.15) and (1.17) we derive

$$D_t\xi = \frac{D_t\omega}{|\omega|} - \omega\frac{D_t|\omega|}{|\omega|^2} = S\xi - \alpha\xi. \tag{1.18}$$

Applying D_t to (1.15), using (1.16), we find

$$\begin{aligned} D_t^2\omega &= (D_tS)\omega + SD_t\omega = -S^2\omega - P\omega + S^2\omega \\ &= -P\omega, \end{aligned} \tag{1.19}$$

which was the key kinematic relation used in [3]. Multiplying (1.19) by $D_t\omega = S\omega$ from the left, we obtain

$$|D_t\omega|D_t|D_t\omega| = \frac{1}{2}D_t|D_t\omega|^2 = D_t\omega \cdot D_t^2\omega = -S\omega \cdot P\omega,$$

Dividing the both sides by $|D_t\omega| = |S\omega|$, we find

$$D_t|D_t\omega| = D_t|S\omega| = -\zeta \cdot P\omega, \tag{1.20}$$

and (1.6) is proved. Now we prove (1.7). Observing $\xi \cdot D_t\xi = 0$, we compute

$$\begin{aligned} D_t^2|\omega| &= D_t\{\xi \cdot D_t(|\omega|\xi)\} = D_t\xi \cdot D_t\omega + \xi \cdot D_t^2\omega \\ &= (S - \alpha I)\xi \cdot S\omega - \xi \cdot P\omega = \left(|S\xi|^2 - \alpha^2 - \rho\right)|\omega|. \end{aligned} \tag{1.21}$$

We divide (1.21) by $|\omega|$, then using (1.17), we deduce

$$|S\xi|^2 - \alpha^2 - \rho = \frac{D_t^2|\omega|}{|\omega|} = D_t\left(\frac{D_t|\omega|}{|\omega|}\right) + \frac{(D_t|\omega|)^2}{|\omega|^2} = D_t^2 \log|\omega| + \alpha^2.$$

The formula (1.7) is proved. Taking the square of (1.18), and multiplying it by $|\omega|^2$, we have

$$|S\omega|^2 = \alpha^2|\omega|^2 + |D_t\xi|^2|\omega|^2 = (D_t|\omega|)^2 + |D_t\xi|^2|\omega|^2, \tag{1.22}$$

and, (1.8) is proved. To show (1.9) we compute, using (1.19) and (1.20),

$$\begin{aligned} D_t\zeta &= \frac{D_t^2\omega}{|D_t\omega|} - \frac{D_t\omega D_t\left(|D_t\omega|\right)}{|D_t\omega|^2} = -\frac{P\omega}{|S\omega|} + \frac{S\omega(\zeta \cdot P\xi)|\omega|}{|S\omega|^2} \\ &= \frac{-P\xi + (\zeta \cdot P\xi)\zeta}{|S\xi|}. \end{aligned}$$

Because $D_t\zeta$ is perepependicular to ζ, in view of the fact that ζ has unit length, this yields an orthogonal decomposition of $P\xi$,

$$P\xi = (\zeta \cdot P\xi)\zeta - |S\xi|D_t\zeta = (\zeta \cdot P\xi)\zeta + \frac{(D_t\zeta \cdot P\xi)}{|D_t\zeta|^2}D_t\zeta, \tag{1.23}$$

which implies

$$\frac{D_t\zeta}{|D_t\zeta|} \cdot P\xi = -|S\xi||D_t\zeta|. \tag{1.24}$$

The decomposition (1.23), combined with (1.20) and (1.24), implies by the Pythagoras theorem

$$\begin{aligned} |P\omega|^2 &= (\zeta \cdot P\omega)^2 + \left(\frac{D_t\zeta}{|D_t\zeta|} \cdot P\omega\right)^2 \\ &= (D_t|S\omega|)^2 + |D_t\zeta|^2|S\omega|^2. \end{aligned} \tag{1.25}$$

The inequality (1.9) follows from this immediately. Substituting (1.22) into (1.25), we have (1.10). $\square$

1.3 Proofs of the main theorems

In order to prove Theorem 1.1 we shall use the following lemma.

Lemma 1.1 *Let $\alpha = \alpha(t)$ be a non-decreasing function, and $\beta = \beta(t) \geq 0$ on $[a, b]$.*

(i) Suppose $y = y(t)$ satisfies

$$y(t) \leq \alpha(t) + \int_a^t \beta(\tau) y(\tau) d\tau \quad \forall t \in [a, b].$$

Then, for all $t \in (a, b]$ we have

$$y(t) \leq \alpha(t) \exp\left(\int_a^t \beta(\tau) d\tau \right).$$

(ii) We assume furthermore $y(t) \geq 0$ on $[a, b]$. Suppose

$$y(t) \leq \alpha(t) + \int_a^t \int_a^s \beta(\tau) y(\tau) d\tau ds \quad \forall t \in [a, b].$$

Then, for all $t \in (a, b]$ we have

$$y(t) \leq \alpha(t) \exp\left(\int_a^t \int_a^s \beta(\tau) d\tau ds \right).$$

Proof In the case (i) from the well-known Gronwall inequality and the assumption of non-decreasing property of α we have

$$\begin{aligned} y(t) &\leq \alpha(t) + \int_a^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(\tau) d\tau \right) ds \leq \alpha(t) + \alpha(t) \int_a^t \beta(s) \exp\left(\int_s^t \beta(\tau) d\tau \right) ds \\ &= \alpha(t) - \alpha(t) \int_a^t \frac{d}{ds} \left\{ \exp\left(\int_s^t \beta(\tau) d\tau \right) \right\} ds = \alpha(t) \exp\left(\int_a^t \beta(\tau) d\tau \right). \end{aligned}$$

For the case (ii) we observe

$$y(t) \leq \alpha(t) + \int_a^t \int_0^s \beta(\tau) y(\tau) d\tau ds \leq \alpha(t) + \int_a^t \sup_{a<\tau<s} y(\tau) \int_a^s \beta(\tau) d\tau ds.$$

Since the function $t \mapsto \alpha(t) + \int_a^t \sup_{a<\tau<s} y(\tau) \int_a^s \beta(\tau) d\tau ds$ is non-decreasing on $[a, b]$, setting $h(t) = \int_a^t \beta(s) ds$ and $Y(t) = \sup_{a<\tau<t} y(\tau)$, we have

$$Y(t) \leq \alpha(t) + \int_a^t Y(s) h(s) ds.$$

Applying (i), we obatin

$$y(t) \leq Y(t) \leq \alpha(t) \exp\left(\int_a^t h(s) ds \right) = \alpha(t) \exp\left(\int_a^t \int_a^s \beta(\tau) d\tau ds \right)$$

for all $t \in [a,b]$. $\square$

Proof of Theorem 1.1 and Theorem 1.2 We integrate (1.6) along the trajectory for $t \in [0,s]$ to find

$$\frac{\partial}{\partial s}|\omega(X(a,s),s)| \leq \left|\frac{\partial}{\partial s}\omega(X(a,s),s)\right| = |(D_s\omega)(X(a,s),s)| = |S\omega(X(a,s),s)|$$
$$= |S_0(a)\omega_0(a)| - \int_0^s (\zeta \cdot P\xi)(X(a,\tau),\tau)|\omega(X(a,\tau),\tau|d\tau,$$

from which, after integrating with respect to s over $[0,t]$, we have

$$|\omega(X(a,t),t)| \leq |\omega_0(a)| + |S_0(a)\omega_0(a)|t + \int_0^t \int_0^s [\zeta \cdot P\xi]_-(X(a,\tau),\tau)|\omega(X(a,\tau),\tau)|d\tau ds.$$

Applying Lemma 2.1(ii) to solve this differential inequality, we find

$$|\omega(X(a,t),t)| \leq (|\omega_0(a)| + |S_0(a)\omega_0(a)|t)\times$$
$$\times \exp\left(\int_0^t \int_0^s [\zeta \cdot P\xi]_-(X(a,\tau),\tau)d\tau ds\right). \tag{1.26}$$

Taking the supremum over $a \in \mathbb{R}^3$, and integrating it with respect to t over $[0,T]$, we find

$$\int_0^T \|\omega(t)\|_{L^\infty}dt \leq (\|\omega_0\|_{L^\infty} + \|S_0\omega_0\|_{L^\infty}T)\times$$
$$\times \int_0^T \exp\left(\int_0^t \int_0^s \|[\zeta \cdot P\xi]_-(\tau)\|_{L^\infty}d\tau ds\right) dt. \tag{1.27}$$

Integrating (1.7) twice with respect to the time variable over $[0,s]$, we hvae

$$|\omega(X(a,t),t)| = |\omega_0(a)| \exp\left(\int_0^t \int_0^s [|S\xi|^2 - 2\alpha^2 - \rho]_+(X(a,\tau),\tau)d\tau ds\right), \tag{1.28}$$

and therefore

$$\int_0^T \|\omega(t)\|_{L^\infty}dt \leq \|\omega_0\|_{L^\infty} \int_0^T \exp\left(\int_0^t \int_0^s \|[|S\xi|^2 - 2\alpha^2 - \rho]_+(\tau)\|_{L^\infty}d\tau ds\right) dt.$$

Applying the well-known Beale-Kato-Majda criterion [1] to (1.27) and (1.28), we obtain the desired conclusion of Theorem 1.1(i). The argument of proof of Theorem 1.1(ii), using the result of (i) is the similar to [3], and we ommit it here.

The proof of Theorem 1.2, using the key pointwise estimates of the vorticity along the trajectories, (1.26) and (1.27) is similar to the corresponding ones in [3], and we do not repeat it here. $\square$

Remark 1.3 The linear differential inequalities (1.11) and (1.12) along the trajectory can be solved as

$$|\omega(X(a,t),t)| \le |\omega_0(a)| e^{-\int_0^t |D_t\xi(X(a,s),s)|ds} + \sqrt{2}\int_0^t |S\omega(X(a,s),s)| e^{-\int_s^t |D_\tau \xi(X(a,\tau),\tau)|d\tau} ds, \tag{1.29}$$

and

$$|S\omega(X(a,t),t)| \le |S_0\omega_0(a)| e^{-\int_0^t |D_t\zeta(X(a,s),s)|ds} + \sqrt{2}\int_0^t |P\xi(X(a,s),s)||\omega(X(a,s),s)| e^{-\int_s^t |D_\tau \zeta(X(a,\tau),\tau)|d\tau} ds \tag{1.30}$$

respectively. Parenthetically one can also use (1.10) to deduce

$$D_t|S\omega| + |D_t\zeta||D_t\xi||\omega| \le \sqrt{2}|P\omega|,$$

and then

$$|S\omega(X(a,t),t)| \le |S_0\omega_0(a)| e^{-\int_0^t |D_t\zeta(X(a,s),s)||D_t\xi(X(a,s),s)|ds} + \sqrt{2}\int_0^t |P\xi(X(a,s),s)||\omega(X(a,s),s)| e^{-\int_s^t |D_\tau\zeta(X(a,\tau),\tau)||D_\tau\xi(X(a,\tau),\tau)|d\tau} ds$$

instead of (1.30). Inserting (1.30) into (1.29), we find

$$|\omega(X(a,t),t)| \le |\omega_0(a)| + \sqrt{2}|S_0(a)\omega_0(a)| \int_0^t e^{-\int_0^s |D_\tau\zeta| d\tau} e^{-\int_s^t |D_\tau\xi|d\tau} ds + 2\int_0^t\int_0^s |P\xi(X(a,\sigma),\sigma)||\omega(X(a,\sigma),\sigma)| e^{-\int_\sigma^s |D_\tau\zeta|d\tau} e^{-\int_s^t |D_\tau\xi|d\tau} d\sigma ds \tag{1.31}$$

Applying Lemma 1.1 to (1.31), we obtain

$$|\omega(X(a,t),t)| \le \left(|\omega_0(a)| + \sqrt{2}|S_0(a)\omega_0(a)|t\right) \times \times \exp\left(2\int_0^t\int_0^s |P\xi(X(a,\sigma),\sigma)| e^{-\int_\sigma^s |D_\tau\zeta(X(a,\tau),\tau)|d\tau} e^{-\int_s^t |D_\tau\xi(X(a,\tau),\tau)|d\tau} d\sigma ds\right). \tag{1.32}$$

Since the quantities expressing the magnitudes of the changes of the two direction vectors $D_t\xi$ and $D_t\zeta$ contribute to the integral in the right hand side of (1.32) through factors like $e^{-\int_s^t |D_\tau\xi(X(a,\tau),\tau)|d\tau}$, they appear to have a desingularizing effect for the vorticity. We do not know, however a way to exploit this effect in the blow up criterion and the absence of the type I blow up. If we ignore the factor $e^{-\int_s^t |D_\tau\xi(X(a,\tau),\tau)|d\tau}$ in (1.32), taking supremum over $a \in \mathbb{R}^3$, and integrating over $t \in [0,T]$ then we have an estimate

$$\int_0^T \|\omega(t)\|_{L^\infty} dt \le \left(\|\omega_0\|_{L^\infty} + \sqrt{2}\|S_0\omega_0\|_{L^\infty} T\right) \times \times \int_0^T \exp\left(2\int_0^t\int_0^s \|P\xi(\tau)\|_{L^\infty} d\tau ds\right) dt$$

which yields a blow up criterion weaker than (1.1).

2 The 2D Boussinesq equations

Here we are concerned with the homogeneous incompressible Boussinesq equation on $\mathbb{R}^2$.

$$(B) \begin{cases} u_t + u \cdot \nabla u = -\nabla p + \theta e_2, \\ \theta_t + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \end{cases}$$

where $u(x,t) = (u_1(x,t), u_2(x,t))$ is the fluid velocity and $p = p(x,t)$ is the pressure, and $\theta = \theta(x,t)$ is the temperature. Let $u_0(x) = u(x,0), \theta_0(x,0)$ be the initial data of the system (B). The local well-posedness for the Boussinesq system for $(u_0, \theta_0) \in W^{2,q}(\mathbb{R}^2)$, $q > 2$, is well-known(see e.g.[4]), but the question of finite time blow up is a wide open problem similarly to the case of the 3D Euler equations. It is also well-known that there exists a strong similarity between (B) and the axisymmetric solution of the 3D Euler equations(see e.g.[14]).

For a solution (u, p, θ) of the system (B) let us introduce the $\mathbb{R}^{2\times 2}$-valued functions $U = (\partial_i u_j)$ and $P = (\partial_i \partial_j p)$. For the vector filed $\nabla^\perp \theta = (-\partial_2 \theta, \partial_1 \theta)$ we define the direction vectors

$$\xi = \nabla^\perp \theta / |\nabla^\perp \theta|, \qquad \zeta = U\nabla^\perp \theta / |U\nabla^\perp \theta|,$$

and the scalar functions

$$\alpha = \xi \cdot U\xi, \quad \rho = \xi \cdot P\xi.$$

Theorem 2.1 *Let $(u, p) \in C^1(\mathbb{R}^2 \times (0,T))$ be a solution of the Boussinesq equation (B) with $u \in C([0,T); W^{2,q}(\mathbb{R}^2))$, for some $q > 2$. Suppose the following holds. Either*

(i)

$$\int_0^T (T-t) \exp\left(\int_0^t \int_0^s \|[\zeta \cdot P\xi]_-(\tau)\|_{L^\infty} d\tau ds \right) dt < +\infty, \tag{2.1}$$

or

$$\int_0^T (T-t) \exp\left(\int_0^t \int_0^s \|[|U\xi|^2 - 2\alpha^2 - \rho]_+(\tau)\|_{L^\infty} d\tau ds \right) dt < +\infty, \tag{2.2}$$

then $\limsup_{t\to T} \|u(t)\|_{W^{2,q}} < +\infty$.

(ii) If either

$$\limsup_{t\to T} (T-t)^2 \|[\zeta \cdot P\xi]_-(t)\|_{L^\infty} < 2, \tag{2.3}$$

or

$$\limsup_{t\to T} (T-t)^2 \|[|U\xi|^2 - 2\alpha^2 - \rho]_+(t)\|_{L^\infty} < 2, \tag{2.4}$$

then $\limsup_{t\to T} \|u(t)\|_{W^{2,q}} < +\infty$.

Remark 2.1 Note the relaxed smallness condition of for the nonexistence of type I blow up in (2.3) and (2.4) compared to (1.2) and (1.3) respectively in the case of 3D Euler equations. This is due to the extra factor, $(T-t)$ in (2.1) and (2.2), which originate from the non blow up criterion, $\int_0^T (T-t)\|\nabla^\perp\theta(t)\|_{L^\infty}dt < +\infty$ in [5, Theorem 1.2 (ii)].

The following is a localized version of the above theorem.

Theorem 2.2 *Let $(u,p) \in C^1(B(x_0,r)\times(T-r,T))$ be a solution to (E) with $u \in C([T-r,T);W^{2,q}(B(x_0,r)))\cap L^\infty(T-r,T;L^2(B(x_0,r)))$ for some $q\in(2,\infty)$. Let us assume*

$$\int_{T-r}^T \|u(t)\|_{L^\infty(B(x_0,r))}dt < +\infty.$$

If either

(i)

$$\int_{T-r}^T (T-t)\exp\left(\int_0^t\int_0^s \|[\zeta\cdot P\xi]_-(\tau)\|_{L^\infty(B(x_0,r))}d\tau ds\right)dt < +\infty, \tag{2.5}$$

or

$$\int_{T-r}^T (T-t)\exp\left(\int_{T-r}^t\int_{T-r}^s \|[|U\xi|^2-2\alpha^2-\rho]_+(\tau)\|_{L^\infty(B(x_0,r))}d\tau ds\right)dt < +\infty, \tag{2.6}$$

then for all $\varepsilon\in(0,r)$ $\limsup_{t\nearrow T}\|u(t)\|_{W^{2,q}(B(x_0,\varepsilon))} < +\infty$.

(ii) If either

$$\limsup_{t\to T}(T-t)^2\|[\zeta\cdot P\xi]_-(t)\|_{L^\infty(B(x_0,r))} < 2, \tag{2.7}$$

or

$$\limsup_{t\to T}(T-t)^2\|[|U\xi|^2-2\alpha^2-\rho]_+(t)\|_{L^\infty(B(x_0,r))} < 2, \tag{2.8}$$

then for all $\varepsilon\in(0,r)$ $\limsup_{t\nearrow T}\|u(t)\|_{W^{2,q}(B(x_0,\varepsilon))} < +\infty$.

Remark 2.2 Similarly to Remark 2.1 we also note here relaxed smallness condition of for the nonexistence of type I blow up in (2.7) and (2.8) compared to (1.4) and (1.5) respectively. This is due to the extra factor, $(T-t)$ in (2.5) and (2.6), which are from the local version of the non blow up criterion, $\int_0^T (T-t)\|\nabla^\perp\theta(t)\|_{L^\infty(B(x_0,r))}dt < +\infty$ in [6, Theorem 2.1].

2.1 Kinematic relations

Proposition 2.1 *Let (u,p,θ) be a solution of (B), which belongs to $C^1(\mathbb{R}^2\times(0,T))$. We use the above notations. Then, the followings hold true on $\mathbb{R}^2\times(0,T)$.*

$$D_t|U\nabla^\perp\theta| = -\zeta\cdot P\nabla^\perp\theta, \tag{2.9}$$

$$D_t^2\log|\nabla^\perp\theta| = |U\xi|^2-2\alpha^2-\rho, \tag{2.10}$$

$$(D_t|\nabla^\perp\theta|)^2 + (|D_t\xi||\nabla^\perp\theta|)^2 = |U\nabla^\perp\theta|^2, \tag{2.11}$$

$$(D_t|U\nabla^\perp\theta|)^2 + (|D_t\zeta||U\nabla^\perp\theta|)^2 = |P\nabla^\perp\theta|^2, \tag{2.12}$$

$$(D_t|U\nabla^\perp\theta|)^2 + (|D_t\zeta|D_t|\nabla^\perp\theta|)^2 + (|D_t\zeta||D_t\xi||\nabla^\perp\theta|)^2 = |P\nabla^\perp\theta|^2. \tag{2.13}$$

Remark 2.1 Although the above results look similar to those in Proposition 1.1 we have essentially different features because we do not use the symmetric part of U and because there exists no relation between $\nabla^\perp\theta$ and the skew symmetric part of U.

Proof of Proposition 2.1 Taking ∇ on the first equation of (B), we find

$$D_tU + U^2 = -P + \nabla(\theta e_2), \tag{2.14}$$

Taking $\nabla^\perp$ on the second equation of (B),

$$D_t\nabla^\perp\theta = U\nabla^\perp\theta. \tag{2.15}$$

Let us compute

$$\begin{aligned} D_t^2\nabla^\perp\theta &= D_tU\nabla^\perp\theta + UD_t\nabla^\perp\theta \\ &= -U^2\nabla^\perp\theta - P\nabla^\perp\theta + U^2\nabla^\perp\theta + \nabla^\perp\theta\cdot\nabla(\theta e_2) \\ &= -P\nabla^\perp\theta, \end{aligned} \tag{2.16}$$

where we used the fact

$$\nabla^\perp\theta\cdot\nabla(\theta e_2) = 0. \tag{2.17}$$

We multiply (2.16) by $D_t\nabla^\perp\theta$ to have

$$\begin{aligned} |D_t\nabla^\perp\theta|D_t|D_t\nabla^\perp\theta| &= \frac{1}{2}D_t\left(|D_t\nabla^\perp\theta|^2\right) = D_t\nabla^\perp\theta\cdot D_t^2\nabla^\perp\theta \\ &= -U\nabla^\perp\theta\cdot P\nabla^\perp\theta. \end{aligned}$$

Dividing the both sides by $|D_t\nabla^\perp\theta| = |U\nabla^\perp\theta|$, we find

$$D_t|D_t\nabla^\perp\theta| = D_t|U\nabla^\perp\theta| = -\zeta\cdot P\xi|\nabla^\perp\theta|,$$

and (2.9) is proved. Multiplying (2.15) by $\nabla^\perp\theta$, we deduce

$$D_t|\nabla^\perp\theta| = \alpha|\nabla^\perp\theta|. \tag{2.18}$$

Using (2.15) and (2.18), we compute

$$D_t\xi = \frac{D_t\nabla^\perp\theta}{|\nabla^\perp\theta|} - \frac{\nabla^\perp\theta D_t|\nabla^\perp\theta|}{|\nabla^\perp\theta|^2} = U\xi - \alpha\xi. \tag{2.19}$$

This can be viewed as an orthogonal decomposition of $U\xi$,

$$U\xi = \alpha\xi + D_t\xi = \alpha\xi + \frac{D_t\xi\cdot U\xi}{|D_t\xi|^2}D_t\xi,$$

which shows

$$D_t\xi\cdot U\xi = |D_t\xi|^2 = |U\xi|^2 - \alpha^2 = |U\xi|^2 - \frac{(D_t|\nabla^\perp\theta|)^2}{|\nabla^\perp\theta|^2}. \tag{2.20}$$

Multiplying the both sides of (2.20) by $|\nabla^\perp\theta|^2$, the formula (2.11) follows immediately. Using (2.14) and (2.19), we compute

$$\begin{aligned}
D_t^2 \log|\nabla^\perp\theta| &= D_t\alpha = D_t(\xi\cdot U\xi)\\
&= D_t\xi\cdot U\xi + \xi\cdot D_tU\xi + \xi\cdot UD_t\xi\\
&= (U\xi-\alpha\xi)\cdot U\xi + \xi\cdot(-U^2-P)\xi + \xi\cdot U(U\xi-\alpha\xi)\\
&= |U\xi|^2 - 2\alpha^2 - \rho,
\end{aligned}$$

where we used $\xi\cdot\nabla(\theta e_2)=0$, which follows from (2.17). The formula (2.10) is proved. Using (2.16) and (2.9), we compute

$$\begin{aligned}
D_t\zeta &= \frac{D_t^2\nabla^\perp\theta}{|D_t\nabla^\perp\theta|} - \frac{D_t\nabla^\perp\theta D_t\left(|D_t\nabla^\perp\theta|\right)}{|D_t\nabla^\perp\theta|^2}\\
&= -\frac{P\nabla^\perp\theta}{|U\nabla^\perp\theta|} + \frac{U\nabla^\perp\theta(\zeta\cdot P\xi)|\nabla^\perp\theta|}{|U\nabla^\perp\theta|^2}\\
&= \frac{-P\xi+(\zeta\cdot P\xi)\zeta}{|U\xi|}. \qquad (2.21)
\end{aligned}$$

The formula (2.21) yields an orthogonal decomposition of $P\xi$,

$$P\xi = (\zeta\cdot P\xi)\zeta - |U\xi|D_t\zeta = (\zeta\cdot P\xi)\zeta + \frac{(D_t\zeta\cdot P\xi)}{|D_t\zeta|^2}D_t\zeta, \qquad (2.22)$$

which implies

$$\frac{D_t\zeta}{|D_t\zeta|}\cdot P\xi = -|U\xi||D_t\zeta|. \qquad (2.23)$$

The decomposition (2.22) also implies by the Pythagoras theorem, and then using (2.9) and (2.23),

$$\begin{aligned}
|P\nabla^\perp\theta|^2 &= (\zeta\cdot P\nabla^\perp\theta)^2 + \left(\frac{D_t\zeta}{|D_t\zeta|}\cdot P\nabla^\perp\theta\right)^2\\
&= (D_t|U\nabla^\perp\theta|)^2 + |D_t\zeta|^2|U\nabla^\perp\theta|^2,
\end{aligned}$$

which verifies (2.12). Inserting the expression of $|U\nabla^\perp\theta|^2$ in (2.12) into (2.13), we obtain (2.14). □

2.2 Proof of the main results

Proof of Theorem 2.1 and Theorem 2.2 The proof is similar to the case of 3D Euler equations. The main difference is that here we start from the kinematic relations of the Boussinesq equations in Proposition 2.1. Integrating (2.9) along the trajectory for $t\in[0,s]$, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial s}|\nabla^\perp\theta(X(a,s),s)| \le \left|\frac{\partial}{\partial s}\nabla^\perp\theta(X(a,s),s)\right| = |(D_s\nabla^\perp\theta)(X(a,s),s)| = |U\nabla^\perp\theta(X(a,s),s)|\\
&\quad = |S_0(a)\omega_0(a)| - \int_0^s (\zeta\cdot P\xi)(X(a,\tau),\tau)|\omega(X(a,\tau),\tau|d\tau.
\end{aligned}$$

After integrating this again with respect to s over $[0,t]$, we find

$$|\nabla^\perp\theta(X(a,t),t)| \le |\nabla^\perp\theta_0(a)| + |\nabla^\perp\theta_0(a)\cdot\nabla u_0(a)|t$$
$$+\int_0^t\int_0^s [\zeta\cdot P\xi]_-(X(a,\tau),\tau)|\nabla^\perp\theta(X(a,\tau),\tau)|d\tau ds.$$

Thanks to Lemma 2.1(ii) we find

$$|\nabla^\perp\theta(X(a,t),t)| \le (|\nabla^\perp\theta_0(a)| + |\nabla^\perp\theta_0\cdot\nabla u_0(a)(a)|t)\times$$
$$\times\exp\left(\int_0^t\int_0^s [\zeta\cdot P\xi]_-(X(a,\tau),\tau)d\tau ds\right). \tag{2.24}$$

Taking the supremum over $a\in\mathbb{R}^2$, and integrating it with respect to t over $[0,T]$ after multiplying by $T-t$, we find

$$\int_0^T (T-t)\|\nabla^\perp\theta(t)\|_{L^\infty}dt \le (\|\nabla^\perp\theta_0(a\|_{L^\infty} + \|\nabla^\perp\theta_0\cdot\nabla u_0\|_{L^\infty}T)\times$$
$$\times\int_0^T (T-t)\exp\left(\int_0^t\int_0^s \|[\zeta\cdot P\xi(\tau)]_-\|_{L^\infty}d\tau ds\right)dt. \tag{2.25}$$

Integrating (2.10) twice with respect to the time variable over $[0,s]$, we hvae

$$|\nabla^\perp\theta(X(a,t),t)| \le |\nabla^\perp\theta_0(a)|\exp\left(\int_0^t\int_0^s [|U\xi|^2-2\alpha^2-\rho]_+(X(a,\tau),\tau)d\tau ds\right), \tag{2.26}$$

and from which we also deduce

$$\int_0^T (T-t)\|\nabla^\perp\theta(t)\|_{L^\infty}dt \le \|\nabla^\perp\theta_0\|_{L^\infty}\times$$
$$\times\int_0^T (T-t)\exp\left(\int_0^t\int_0^s \|[|U\xi|^2-2\alpha^2-\rho]_+(\tau)\|_{L^\infty}d\tau ds\right)dt. \tag{2.27}$$

Applying the blow up criterion of [5, Theorem 1.2 (ii)] to (2.25) and (2.27), we obtain the desired conclusion of Theorem 2.1(i). The proof of Theorem 2.1(ii), using the result of (i) is the similar to the one in [3].

The proof of Theorem 2.2, using the key estimates (2.24) and (2.26) is also similar to the corresponding ones in [3]. The essential point here is that we apply the local version of the blow up criterion [6, Theorem 2.1]. □

Acknowledgements

Chae was partially supported by NRF grant 2016R1A2B3011647, while the work of Constantin was partially supported by the Simons Center for Hidden Symmetries and Fusion Energy.

Conflicts of Interest Statement

The authors declare that there is no conflict of interest.

References

[1] J. T. Beale, T. Kato and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations,* Comm. Math. Phys., **94**, pp. 61-66, (1984).

[2] D. Chae, *On the Finite-Time Singularities of the 3D Incompressible Euler Equations,* Comm. Pure Appl. Math., vol. LX, pp. 0597-0617, (2007).

[3] D. Chae and P. Constantin, *On a Type I singularity condition in terms of the pressure for the Euler equations in* $\mathbb{R}^3$, Int. Math. Res. Notices, rnab014, (2021).

[4] D. Chae and H-S. Nam, *Local existence and blow-up criterion for the Boussinesq equations,* Proc. Roy. Soc. Edinburgh Sect. A **127**, no. 5, pp. 935-946, (1997).

[5] D. Chae and J. Wolf, *On the Regularity of Solutions to the 2D Boussinesq Equations Satisfying Type I Conditions*, J. Nonlinear Sciences, **29**, pp. 643-654, (2019).

[6] D. Chae and J. Wolf, *Removing Type II Singularities Off the Axis for the Three Dimensional Axisymmetric Euler Equations,* Arch. Rational Mech. Anal., **234**, pp. 1041-1089, (2019).

[7] P. Constantin, *Analysis of Hydrodynamic Models,* CBMS-NSF Regional Conference Series in Applied Mathematics, **90,** (2017).

[8] P. Constantin, *On the Euler equations of incompressible fluids,* Bull. Amer. Math. Soc., **44** no. 4, pp. 603-621,(2007).

[9] P. Constantin, *Geometric statistics in turbulence,* SIAM Rev. **36**, pp. 73-98, (1994).

[10] P. Constantin, C. Fefferman, A. Majda, *Geometric constraints on potential singularity formulation in the 3-D Euler equations,* Comm. PDE, **21**, no. 3-4, pp. 559-571,(1996).

[11] T. Kato and G. Ponce, *Commutator estimates and the Euler and NavierStokes equations,* Commun. Pure Appl. Math. **41**(7), pp. 891-907, (1988).

[12] R. M. Kerr, *Evidence for a singularity of the three-dimensional, incompressible Euler equations,* Phys. Fluids. A, 5(7), pp. 1725-1746, (1993).

[13] G. Luo and T. Y. Hou, *Formation of finite-time singularities in the 3D axisymmetric Euler equations: a numerics guided study,* SIAM Rev. **61**, no. 4, pp. 793-835. (2019).

[14] A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow,* Cambridge Univ. Press. (2002).