ON THE RENORMALIZED VOLUMES FOR CONFORMALLY COMPACT EINSTEIN MANIFOLDS

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Abstract. We study the renormalized volume of a conformally compact Einstein manifold. In even dimenions, we derive the analogue of the Chern-Gauss-Bonnet formula incorporating the renormalized volume. When the dimension is odd, we relate the renormalized volume to the conformal primitive of the Q-curvature.

0. Introduction

Recently, there is a series of work ([GZ],[FG-2] and [FH]) exploring the connection between scattering theory on asymptotically hyperbolic manifolds, the Q-curvature in conformal geometry and the "renormalized volume" of conformally compact Einstein manifolds. In particular, in [FG-2], a notion of Q-curvature was introduced for an odd-dimensional manifold as the boundary of a conformally compact Einstein manifold of even-dimension. In this note, in section 2 below, we will clarify the relation between the work of [FG-2] and the notion of Q-curvature in our earlier work [CQ] for the special case when the manifold is of dimension three. We then explore this relation to give a different proof of a result of Anderson [A] writing the Chern-Gauss-Bonnet formula for conformally compact Einstein 4-manifold with the renormalized volume. Our proof makes use of the special exhaustion function introduced in [FG-2] that yields remarkable simplification in computing the Q curvature. In section 3, using some recent result of Alexakis on Q-curvature, we generalize the Chern-Gauss-Bonnet formula involving the renormalized volume to all even dimensional conformally compact Einstein manifolds. The formula includes as special

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case the formula of Epstein (appendix A in [E]) for conformally compact hyperbolic manifolds. The formula also shows that the renormalized volume is a conformal invariant of the conformally compact structure when the dimension is odd. Finally in section 4, we obtain a formula similar to that in [FG-2] expressing the renormalized volume of a odd dimensional conformally compact Einstein manifold as the conformal primitive of the Q curvature, and in terms of the data of the scattering matrix.

1. Conformally compact Einstein manifolds and renormalized volumes

In this section, we will first recall some basic definitions and facts of conformally compact Einstein manifolds. We then state the main result in [FG-2].

Given a smooth manifold X^{n+1} of dimension $n+1$ with smooth boundary $\partial X = M^n$. Let x be a defining function for M^n in X^{n+1} as follows:

$$\begin{aligned} x > 0 &\;\text{ in } X; \\ x = 0 &\;\text{ on } M; \\ dx \neq 0 &\;\text{ on } M. \end{aligned}$$

A Riemannian metric g on X is conformally compact if $(X, x^2 g)$ is a compact Riemannian manifold with boundary. Conformally compact manifold (X^{n+1}, g) carries a well-defined conformal structure $[\hat{g}]$ on the boundary M^n; where each $\hat{g}$ is the restriction of $x^2 g$ for a defining function x. We call $(M^n, [\hat{g}])$ the conformal infinity of the conformally compact manifold (X^{n+1}, g). If, in addition, g satisfies $Ric_g = -ng$, where Ric_g denotes the Ricci tensor of the metric g, then we call (X^{n+1}, g) a conformally compact Einstein manifold.

A conformally compact metric is said to be asymptotically hyperbolic if its sectional curvature approach -1 at $\partial X = M$. It was shown ([FG-1], [GL]) that if g is an asymptotically hyperbolic metric on X, then a choice of metric $\hat{g}$ in $[\hat{g}]$ on M uniquely determines a defining function x near the boundary M and an identification of a neighborhood of M in X with $M \times (0, \epsilon)$ such that g has the normal form

$$g = x^{-2}(dx^2 + g_x) \tag{1.1}$$

where g_x is a 1-parameter family of metrics on M.

As a conformally compact Einstein metric g is clearly asymptotically hyperbolic, we have, as computed in [G-1] by Graham,

$$g_x = \hat{g} + g^{(2)}x^2 + (\text{even powers of } x) + g^{(n-1)}x^{n-1} + g^{(n)}x^n + \cdots, \tag{1.2}$$

when n is odd, and

$$g_x = \hat{g} + g^{(2)}x^2 + (\text{even powers of } x) + g^{(n)}x^n + hx^n \log x + \cdots , \tag{1.3}$$

when n is even. Where $\hat{g} = x^2 g|_{x=0}$, $g^{(2i)}$ are determined by $\hat{g}$ for $2i < n$. The trace part of $g^{(n)}$ is zero when n is odd; the trace part of $g^{(n)}$ is determined by $\hat{g}$ and h is traceless and determined by $\hat{g}$ too when n is even.

As a realization of the holography principle proposed in physics, one considers the asymptotic of the volume of a conformally compact Einstein manifold (X^{n+1}, g). Namely, if denote by x the defining function associated with a choice of a metric $\hat{g} \in [\hat{g}]$, we have

$$\mathrm{Vol}_g(\{x > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-1}\epsilon^{-1} + V + o(1) \tag{1.4}$$

for n odd, and

$$\mathrm{Vol}_g(\{x > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-2}\epsilon^{-2} + L \log \frac{1}{\epsilon} + V + o(1) \tag{1.5}$$

for n even. We call the constant term V in all dimensions the renormalized volume for (X^{n+1}, g). We recall that V in odd dimension and L in even dimension are independent of the choice $\hat{g}$ in the class $[\hat{g}]$ (cf. [HS] [G-1]).

Based on the work of [GZ], Fefferman and Graham [FG-2] introduced the following formula to calculate the renormalized volume V for a conformally compact Einstein manifold. Here we will quote a special case of their result. For odd n, upon a choice of a special defining function x, one sets

$$v = -\frac{d}{ds}|_{s=n}\wp(s)1,$$

where $\wp(s)$ denotes the Possion operator (see [GZ] or section 4 below for the definition of the operator) on X^{n+1}. The v solves

$$-\Delta v = n \quad \text{in } X^{n+1}, \tag{1.6}$$

and has the asymptotic behavior

$$v = \log x + A + Bx^n \tag{1.7}$$

in a neighborhood of M^n, where A, B are functions even in x, and $A|_{x=0} = 0$. Then Fefferman and Graham in [FG-2] defined

$$(Q_n)_{(g,\hat{g})} = k_n B|_{x=0} \tag{1.8}$$

where $k_n = 2^n \frac{\Gamma(\frac{n}{2})}{\Gamma(-\frac{n}{2})}$.

Theorem 1.1. *([FG-2]) When n is odd,*

$$V(X^{n+1}, g) = \frac{1}{k_n}\int_M (Q_n)_{(g,\hat{g})} dv_{\hat{g}}. \tag{1.9}$$

In section 2 and 3 below, we will apply Theorem 1.1 to identify the renormalized volume as part of the integral in the Gauss-Bonnet formula for (X^{n+1}, M^n, g) when n is odd. In section 4, we recall the work in [GZ] and [FG-2] relating the Q-curvature to data of scattering matrix on the asymptotically hyperbolic manifold X^{n+1} and derive a similar formula for the renormalized volume when n is even.

2. Chern-Gauss-Bonnet formula for $n = 3$

To motivate our discussions in this section we first recall some works in [CQ]. First we recall that the Paneitz operator defined on 4-manifold as:

$$P_{4/2} = \Delta^2 + \delta\{\frac{2}{3}Rg - 2\mathrm{Ric}\}d, \tag{2.1}$$

where R is the scalar curvature, Ric is the Ricci curvature. There are two important properties of the Paneitz operator:

$$(P_{4/2})_{g_w} = e^{-4w}(P_{4/2})_g.$$

$$(P_{4/2})_g w + (Q_4)_g = (Q_4)_{g_w} e^{4w}, \tag{2.2}$$

for any smooth function w defined on the 4-manifold, and where $g_w = e^{2w}g$ and where Q is the curvature function

$$Q_4 = \frac{1}{6}(-\Delta R + |R|^2 - 3|Ric|^2).$$

In [CQ], on a compact Riemannian 4-manifold (X^4, g) with boundary, a third order boundary operator P_b and a third order boundary curvature T were introduced as follows:

$$(P_b)_g = -\frac{1}{2}\frac{\partial}{\partial n}\Delta_g + \tilde{\Delta}\frac{\partial}{\partial n} + \frac{2}{3}H\tilde{\Delta} + L_{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta + \frac{1}{3}\tilde{\nabla}_\alpha H\cdot\tilde{\nabla}_\alpha - (F - \frac{1}{3}R)\frac{\partial}{\partial n} \tag{2.3}$$

and

$$T_g = \frac{1}{12}\frac{\partial R}{\partial n} + \frac{1}{6}RH - R_{\alpha N\beta N}L_{\alpha\beta} + \frac{1}{9}H^3 - \frac{1}{3}\mathrm{Tr}L^3 - \frac{1}{3}\tilde{\Delta}H, \tag{2.4}$$

where $\frac{\partial}{\partial n}$ is the outer normal derivative, $\tilde{\Delta}$ is the trace of the Hessian of the metric on the boundary, $\tilde{\nabla}$ is the derivative in the boundary, L is the second fundamental form of boundary, $H = \mathrm{Tr} L$, $F = R_{\alpha N \alpha N}$, and R is the scalar curvature all with respect to the metric g. P_b and T transform under conformal change of metric on the boundary of X^4 similar to that of $P_{4/2}$ and Q_4 on X^4 as follows:

$$\begin{aligned} (P_b)_{g_w} &= e^{-3w} (P_b)_g \\ (P_b)_g w + T_g &= T_{g_w} e^{3w} \end{aligned} \tag{2.5}$$

We remark that $(P_b)_g$ and T_g thus defined depend on the metric g on (X^4, g), and are not intrinsic quantities on the boundary of X^4.

In [CQ], we have also re-organized the terms in the integrand of Gauss-Bonnet formula for 4-manifolds with boundary into the following form:

$$\frac{1}{8\pi^2} \int_{X^4} (|\mathcal{W}|^2 + Q) dv + \frac{1}{4\pi^2} \int_{\partial X} (\mathcal{L} + T) d\sigma = \chi(X), \tag{2.6}$$

where

$$\mathcal{L} = \frac{1}{3} RH - FH + R_{\alpha N \beta N} L_{\alpha\beta} - R_{\alpha\gamma\beta\gamma} L_{\alpha\beta} + \frac{2}{9} H^3 - H|L|^2 + \mathrm{Tr} L^3. \tag{2.7}$$

We remark that the Weyl curvature $\mathcal{W}$ is a point-wise conformal invariant term on the 4-manifold, while $\mathcal{L}$ is a point-wise conformal invariant term on the boundary of the manifold.

We also remark that when the boundary is totally geodesic, the expressions of $(P_b)_g$ and T_g in (2.3) and (2.4) above become very simple;

$$(P_b)_g = -\frac{1}{2} \frac{\partial}{\partial n} \Delta_g \ + \ \tilde{\Delta} \frac{\partial}{\partial n} - (F - \frac{1}{3} R) \frac{\partial}{\partial n}, \quad T_g = \frac{1}{12} \frac{\partial R}{\partial n},$$

and in this case $\mathcal{L}$ vanishes.

Given a conformally compact Einstein manifold (X^{n+1}, g) and a choice of metric $\hat{g}$ in the conformal infinity $(M, [\hat{g}])$, we consider the compactification $(X^{n+1}, e^{2v} g)$, where v is the function which satisfies (1.6) and (1.7). We observe that

Lemma 2.1. *When n is odd, $(Q_{n+1})_{e^{2v} g} = 0$.*

Proof. The proof follows an observation made by Graham ([G-2], see also [Br]) that the Paneitz operator $P_{\frac{n+1}{2}}$ on an Einstein manifold is a polynomial of the Laplacian $\mathcal{P}(\Delta)$ and the polynomial $\mathcal{P}$ on the Einstein manifold is the same as the one on the constant curvature space with the constant the same as the constant of the scalar curvature of the Einstein manifold. Meanwhile the Q-curvature Q_{n+1} of an Einstein manifold is the same as the one on the constant curvature space. Therefore $(P_{\frac{n+1}{2}})_g = \mathcal{P}(\Delta_g)$ if $(P_{\frac{n+1}{2}})_{g_H} = \mathcal{P}(\Delta_{g_H})$, and $(Q_{n+1})_g = (Q_{n+1})_{g_H}$, where (H^{n+1}, g_H) is the hyperbolic space.

$$(P_{\frac{n+1}{2}})_{g_{H^{n+1}}} = \prod_{l=1}^{\frac{n+1}{2}} (-\Delta_{H^{n+1}} - C_l) \tag{2.8}$$

where $C_l = (\frac{n+1}{2} + l - 1)(\frac{n+1}{2} - l)$. Therefore

$$(P_{\frac{n+1}{2}})_g = \sum_{l=2}^{\frac{n+1}{2}} (-1)^{\frac{n+1}{2}-l} B_l (\Delta_g)^l - (-1)^{\frac{n-1}{2}} (n-1)! \Delta_g. \tag{2.9}$$

Meanwhile $(Q_{n+1})_{H^{n+1}} = (-1)^{\frac{n+1}{2}} n!$. Thus

$$(Q_{n+1})_g = (-1)^{\frac{n+1}{2}} n!. \tag{2.10}$$

Thus if v satisfies the equation (1.6), we have

$$(P_{\frac{n+1}{2}})_g v + (Q_{n+1})_g = 0. \tag{2.11}$$

It thus follows from equation (2.2) that $(Q_{n+1})_{e^{2v}g} = 0$.

We will now combine the above observation to Theorem 1.1 of [FG-2] to give an alternative proof of a result of Anderson [A] (Theorem 2.3 below) for conformal compact Einstein 4-manifold (X^4, g). We first relate our curvature T to that of Q_3 as defined in (1.8).

Lemma 2.2.

$$T_{e^{2v}g} = 3B|_{x=0} = (Q_3)_{(g,\hat{g})}. \tag{2.12}$$

Proof. By the scalar curvature equation we have

$$\frac{1}{12} R_{e^{2v}g} = \frac{1}{2}(-\Delta_g e^v + \frac{1}{6} R_g e^v) e^{-3v}.$$

Therefore for v satisfies equation (1.6), we have

$$\frac{1}{12} R_{e^{2v}g} = \frac{1}{2}((e^{-v})^2 - |\nabla e^{-v}|^2).$$

We now apply the asymptotic expansion of v in (1.7) and write

$$e^{-2v} = \frac{1}{x^2} - 2A_2 - 2B_0 x + O(x^2)$$
$$|\nabla e^{-v}|^2 = \frac{1}{x^2} + 2A_2 + 4B_0 x + O(x^2),$$

where A_2 is the coefficient of x^2 of A and $B_0 = B|_{x=0}$. We get

$$T_{e^{2v}g} = -\frac{1}{12}\frac{\partial}{\partial x} R_{e^{2v}g}|_{x=0} = 3B_0 = Q_{(g,\hat{g})}.$$

This finishes the proof of the lemma.

Theorem 2.3. *[A] Suppose that (X^4, g) is a conformally compact Einstein manifold. Then*

$$\frac{1}{8\pi^2}\int_{X^4} |\mathcal{W}|_g^2 dv_g + \frac{3}{4\pi^2} V(X^4, g) = \chi(X^4). \tag{2.13}$$

Proof. Apply Lemma 2.1 to (2.6), we have

$$\frac{1}{8\pi^2}\int_{X^4} |\mathcal{W}|_{e^{2v}g}\, dv_{e^{2v}g} + \frac{1}{4\pi^2}\int_M (\mathcal{L} + T)_{(e^{2v}g,\hat{g})}\, dv_{\hat{g}} = \chi(X^4).$$

We now observe that as the boundary of M of X^4 is umbilical, $\mathcal{L}_{(e^{2v}g,\hat{g})} = 0$. Apply Lemma 2.2 and Theorem 1.1. we obtain (2.13) for the metric $e^{2v}g$. We then observe that once the formula (2.13) holds for the metric $e^{2v}g$, it holds for any metric $\tilde{g} \in [g]$ with $(X^{n+1}, \tilde{g})$ a conformally compact manifold as the term of the renormalized volume V is conformally invariant.

3. Chern-Gauss-Bonnet formula in higher dimensions

In higher dimensions when $n = 2k+1 > 3$ we wish to determine the analogous formula for the Euler characteristic. We continue to consider the metric $(X^{n+1}, e^{2v}g)$ where v satisfies the equations (1.6) and (1.7). We will find that the parity conditions imposed in (1.7) makes it possible to determine the local boundary invariants of order n for the compact manifold $(X^{n+1}, e^{2v}g)$. According to (1.1) and (1.7) we have the expansion of the metric $e^{2v}g$.

$$\begin{aligned} e^{2v}g = {} & H^2 dx^2 + \hat{g} + c^{(2)}x^2 + \text{even powers in } x \\ & + c^{(n-1)}x^{n-1} + (2B_0\hat{g} + g^{(n)})x^n + \cdots \end{aligned} \tag{3.1}$$

where

$$H = e^{A+Bx^n} = 1 + e_2 x^2 + \text{even powers in } x + e_{n-1}x^{n-1} + B_0 x^n + \cdots$$

and $c^{(2i)}$ for $1 \le i \le (n-1)/2$ are local invariants of $\hat{g}$. We remark that it is easy to see that the boundary of $(X^{n+1}, e^{2v}g)$ is totally geodesic.

Lemma 3.1.

$$(\partial_x \Delta^{\frac{n-3}{2}} R)_{e^{2v}g}|_{x=0} = -2nn! B_0. \tag{3.2}$$

Proof. We have

$$\begin{aligned}\Delta_{e^{2v}g} &= \frac{1}{H\sqrt{\det g_x}} \partial_\alpha (H\sqrt{\det g_x} g_x^{\alpha\beta} \partial_\beta) \\ &= Q_2^{(2)} \partial_x^2 + Q_2^{(1)} \partial_x + Q_2^{(0)}\end{aligned}$$

where the coefficients $Q^{(i)}$ have the following properties: $Q_2^{(2)}$ is a zeroth order differential operator, having an asymptotic expansion in powers of x in which the first nonzero odd power term is x^n. $Q_2^{(1)}$ is a zeroth order differential operator, having an expansion in which the first nonzero even degree term is x^{n-1}. $Q_2^{(0)}$ is differential operator of order 2 of purely tangential differentiations with coefficients which have expansion in powers of x in which the first nonzero odd term is x^n. Inductively, we see that, for $k \leq \frac{n-3}{2}$,

$$\Delta^k = Q_{2k}^{(2k)} \partial_x^{2k} + Q_{2k}^{(2k-1)} \partial_x^{2k-1} + \cdots + Q_{2k}^{(1)} \partial_x + Q_{2k}^{(0)} \tag{3.3}$$

where $Q_{2k}^{(i)}$ $(i \neq 0)$ is a differential operator of order $2k - i$ of purely tangential differentiations with coefficients having expansions in powers of x in which the first nonzero even terms are $x^{n-(2k-i)}$ if i is odd, and the first nonzero odd terms are $x^{n-(2k-i)}$ if i is even, and $Q_{2k}^{(0)}$ is a differential operator of order $2k$ of purely tangential differentiations with coefficients whose expansions in x have the first nonzero odd terms x^{n-2k+2}. Thus

$$\partial_x \Delta^k = F^{(2k+1)} \partial_x^{2k+1} + F^{(2k)} \partial_x^{2k} + \cdots + F^{(1)} \partial_x + F^{(0)} \tag{3.4}$$

where $F^{(2k+1)} = Q^{(2k)}$, $F^{(i)}$ $(0 < i < 2k+1)$ is a differential operator of order $2k - i + 1$ of purely tangential differentiations with coefficients whose expansions in x have the first nonzero even terms are $x^{n-(2k-i)-1}$ if i is even, and the first nonzero odd terms are $x^{n-(2k-i)-1}$ if i is odd, and $F^{(0)}$ is a differential operator of order $2k$ of purely tangential differentiations with coefficients whose expansions in x have the first nonzero even terms x^{n-2k+1}.

On the other hand, we have

$$R_{e^{2v}g} = -2n^2(n-1)B_0 x^{n-2} + \text{even powers of } x \text{ terms} + o(x^{n-2}). \tag{3.5}$$

Keeping track of the parity, we obtain (3.2).

Next we deal with all other boundary terms, these are contractions of one or more factors consisting of curvatures, covariant derivatives of curvatures, except $\partial_x^{n-2} R$ which is accounted in the above term $\partial_x \Delta^{\frac{n-3}{2}} R$. Since n is odd, and ∂x is the normal direction, each such term must contain at least one x index. In fact, the total number of x indices appearing in each of such terms must be odd. Thus one finds that each of such terms always contains a factor which is a covariant derivatives of curvature and in which x index appears odd number of times. Such factors, if we insist on taking ∇_x first, must appear as one of the following three different types

$$\nabla_\spadesuit \cdots \nabla_\spadesuit \nabla_x^{2k+1} R_{\spadesuit\spadesuit\spadesuit\spadesuit} \tag{I}$$

where $\spadesuit$ stands for indices other than x, in other words, tangential.

$$\nabla_\spadesuit \cdots \nabla_\spadesuit \nabla_x^{2k} R_{x\spadesuit\spadesuit\spadesuit} \tag{II}$$

and

$$\nabla_\spadesuit \cdots \nabla_\spadesuit \nabla_x^{2k-1} R_{x\spadesuit x\spadesuit}. \tag{III}$$

Note that in all three types $1 \leq 2k+1 \leq n-2$. Since the boundary is totally geodesic, we only need

Lemma 3.2. *All three types of boundary terms*

$$\nabla_x^{2k+1} R_{\spadesuit\spadesuit\spadesuit\spadesuit}, \quad \nabla_x^{2k} R_{x\spadesuit\spadesuit\spadesuit}, \quad \nabla_x^{2k-1} R_{x\spadesuit x\spadesuit}$$

vanish at the boundary for $1 \leq 2k+1 \leq n-2$.

Proof. We consider a point at the boundary and choose a normal coordinate on the boundary M^n in the special coordinates for X^{n+1}. Recall

$$\begin{aligned} R_{\alpha\beta\gamma\delta} = & \frac{1}{2}(-\partial_\beta\partial_\delta g_{\alpha\gamma} - \partial_\alpha\partial_\gamma g_{\beta\delta} + \partial_\beta\partial_\gamma g_{\alpha\delta} + \partial_\alpha\partial_\delta g_{\beta\gamma}) \\ & - g^{\eta\lambda}([\alpha\gamma,\eta][\beta\delta,\lambda] - [\beta\gamma,\eta][\alpha\delta,\lambda]), \end{aligned}$$

and

$$\nabla_x T_{\alpha\beta\cdots\delta} = \partial_x T_{\alpha\beta\cdots\delta} - \Gamma_{\alpha\ x}^{\lambda} T_{\lambda\beta\cdots\delta} - \Gamma_{\beta\ x}^{\lambda} T_{\alpha\lambda\cdots\delta} - \cdots - \Gamma_{\delta\ x}^{\lambda} T_{\alpha\beta\cdots\lambda}$$

where

$$\Gamma_{\beta\gamma}^{\alpha} = g^{\alpha\delta}[\beta\gamma,\delta]$$

and

$$[\alpha\beta,\gamma] = \frac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}).$$

For the simplicity of the notation we will use g to stand for $e^{2v}g$ if no confusion can arise. Each of the three types is a sum of products of factors that are of the form:

$$\partial_\alpha\partial_\beta\cdots\partial_\gamma g_{\lambda\mu}$$

or

$$\partial_\alpha\partial_\beta\cdots\partial_\gamma g^{\lambda\mu}.$$

We claim that each summand must has a factor that is one of the following

$$\partial_\spadesuit\cdots\partial_\spadesuit\partial_x^{2k+1}g_{\spadesuit\spadesuit},$$

$$\partial_\spadesuit\cdots\partial_\spadesuit\partial_x^{2k-1}g_{xx},$$

$$\partial_\spadesuit\cdots\partial_\spadesuit\partial_x^{2k+1}g^{\spadesuit\spadesuit}.$$

and

$$\partial_\spadesuit\cdots\partial_\spadesuit\partial_x^{2k-1}g^{xx}.$$

where $1 \le 2k+1 \le n-2$. To verify the claim, one needs to observe that, in writing the three types in local coordinates, the number of times the index x appears in each summand increases only when one sees

$$\Gamma^x_{\spadesuit\ x}T_{\alpha\beta\cdots\ x\cdots\delta},$$

where the number of x increases by 2. Thus, in the end, the total number of index x in each summand is still odd. Therefore one of the factors must have odd number of x. Finally one observes that for any individual factor arising here the number of x can not exceed $n-1$. So the proof is complete.

We now apply above results to derive a formula analogous to that of Theorem 2.3 for the renormalized volume on (X^{n+1}, g) for $n = 5$. In this case, we recall the formula of Graham [G-1]:

$$Q_6 = 64\pi^3 e - \frac{1}{6}J + \frac{1}{10}\Delta^2 R + \mathrm{Div}(T) \tag{3.6}$$

where e is the Euler class density whose integral over a compact 6-manifold gives its Euler number,

$$J = -3I + 7W_{ij}{}^{ab}W_{ab}{}^{pq}W_{pq}{}^{ij} + 4W_{ijkl}W^{iakb}W^{j\ \ l}_{\ a\ b},$$

$$I = |V|^2 - 16W^i{}_{jkl}C_{jkl,i} + 16P^{im}W_{ijkl}W_m{}^{jkl} + 16|C|^2,$$

$$V_{ijklm} = W_{ijkl,m} + g_{im}C_{jkl} - g_{jm}C_{ikl} + g_{km}C_{lij} - g_{lm}C_{kij},$$

$$C_{ijk} = P_{ij,k} - P_{ik,j},$$

$$P_{ij} = \frac{1}{4}(R_{ij} - \frac{R}{10}g_{ij}),$$

W_{ijkl} is the Weyl curvature, R_{ij} is the Ricci curvature, R is the scalar curvature, and Div(T) are divergence terms other than $\Delta^2 R$. Q_6 in this form is organized so that it is a sum of three types of terms: Euler class density, local conformal invariants, and divergence terms. G

Theorem 3.3. *For $n = 5$, we have*

$$\chi(X^6) = \frac{1}{128\pi^3}\int_{X^6}(\mathcal{J})_g dv_g - \frac{15}{8\pi^3}V(X^6, g), \tag{3.7}$$

where

$$\mathcal{J} = -|\nabla W|^2 + 8|W|^2 + \frac{7}{3}W_{ij}{}^{ab}W_{ab}{}^{pq}W_{pq}{}^{ij} + \frac{4}{3}W_{ijkl}W^{iakb}W^j{}_a{}^l{}_b$$

Proof. As a consequence of above lemmas, we have

$$\chi(X^6) = \frac{1}{128\pi^3}\int_{X^6}(\frac{1}{3}J_{e^{2v}g})dv_{e^{2v}g} - \frac{15}{8\pi^3}V(X^6, g). \tag{3.8}$$

Since J is a local conformal invariant and g is an Einstein metric, we obtain (3.7) directly from (3.8).

We can find a general formula for all higher dimensions. We recall a recent result of S. Alexakis [Al].

Theorem 3.4. *[Alexakis] On any compact Einstein m-dimensional manifold with m even, we have*

$$Q_m = a_m e + \mathcal{J} + Div(T_m). \tag{3.9}$$

where e is the Euler class density, $\mathcal{J}$ is a conformal invariant, and $Div(T_m)$ is a divergence term and a_m is some dimensional constant.

We also recall a fact we learned from Tom Branson [Br]:

Proposition 3.5. *On any compact m-dimensional manifold for m even, suppose Q_m is the curvature in the construction of [GJMS] and [GZ], [FG-2], then*

$$Q_m = b_m \Delta^{\frac{m-2}{2}} R + \textit{lower order terms}, \tag{3.10}$$

where

$$b_m = (-1)^{\frac{m-2}{2}} \frac{2^{m-1}(\frac{m}{2})!\Gamma(\frac{m+1}{2})}{\sqrt{\pi}(m-1)m!}.$$

Theorem 3.6. *When n is odd, we have*

$$\int_{X^{n+1}} (\mathcal{W}_{n+1})_g dv_g + (-1)^{\frac{n+1}{2}} \frac{\Gamma(\frac{n+2}{2})}{\pi^{\frac{n+2}{2}}} V(X^{n+1}, g) = \chi(X^{n+1}) \tag{3.11}$$

for some curvature invariant $\mathcal{W}_{n+1}$, which is a sum of contractions of Weyl curvatures and/or its covariant derivatives in an Einstein metric.

Proof. We first establish that equation (3.9) remains valid on a conformally Einstein manifold (X^{n+1}, g). Let $g_w = e^{2w} g$ be such a metric, then it follows from the Paneitz equation that for $m = n+1$,

$$\begin{aligned} (Q_m)_{g_w} e^v &= (P_m)_g v \ + \ (Q_m)_g \\ &= a_m e_g + \mathcal{J}_g + Div(T') \\ &= a_m e_{g_w} + \mathcal{J}_{g_w} + Div(T'') \end{aligned} \tag{3.9'}$$

the second equation follows from the fact that the Paneitz operator P_m is a divergence and Theorem 3.4. The third equation follows from the fact that the Pfaffians of any two Riemannian metrics on the same manifold differs by a divergence term and $\mathcal{J}$ is a conforaml invariant.

In order to apply this formula, we need to observe that the leading order term $\Delta^{\frac{m-2}{2}} R$ in formula (3.10) cannot appear in the conformally invariant term $\mathcal{J}$. In order to see this, we first recall that the $\mathcal{J}$ is a linear combination of terms of the form $Tr(\nabla^{I_1}\mathcal{R} \bigotimes \nabla^{I_2}\mathcal{R} ... \bigotimes \nabla^{I_k}\mathcal{R})$ of weight m where Tr denotes a suitably chosen pairwise contraction over all the indices. Observe that the conformal variation $\delta_w(\Delta^{\frac{m-2}{2}})R$, where δ_w denotes the variation of the metric g to g_w is of the form $\Delta^{\frac{m}{2}} w +$ lower order terms. Thus if $\Delta^{\frac{m-2}{2}} R$ does appear as a term in $\mathcal{J}$, its conformal variation must be cancelled by the conformal variations of the other terms in the linear combination, but it is clear that the conformal variations of the other possibilities of the curvature $\mathcal{R}$ other than the scalar curvature R cannot have order m in the number of derivatives of w and of the form $\Delta^{\frac{m}{2}} w$.

We can now apply the formula (3.9') to the metric $g_v = e^{2v}g$ where v is as in Lemma 2.1., thus by Lemma 2.1 the left hand side of (3.9') is identically zero, and we find

$$a_m\chi(X^{n+1}) = \int_{X^{n+1}} (\mathcal{J}_{g_v} - Div(T''))dv_{g_v}.$$

Among the divergence terms in $Div(T'')$, only the leading order term $b_m\Delta^{\frac{m-2}{2}}R$ has a non-zero contribution according to Lemma 3.2. The computation in Lemma 3.1 determines the precise contribution of this term as a mutiple of the renormalized volume. We also note that as g is an Einstein metric, we may assume that the terms which appear in the conformal invariant $\mathcal{J}$ are contractions of the Weyl curvature together with its covariant derivatives. We have thus finished the proof of Theorem 3.6.

Corollary 3.5. *When (X^{n+1}, g) is conformally compact hyperbolic, we have*

$$V(X^{n+1}, g) = \frac{(-1)^{\frac{n+1}{2}}\pi^{\frac{n+2}{2}}}{\Gamma(\frac{n+2}{2})}\chi(X). \tag{3.12}$$

One may compare (3.12) to a formula for renormalized volume given by Epstein in [E], where he has

$$V(X^{n+1}, g) = \frac{(-1)^m 2^{2m} m!}{(2m)!}\chi(X) \tag{3.13}$$

for $n = 2m - 1$.

4. Scattering theory and the renormalized volume

We now recall the connection between the renormalized volume and scattering theory introduced in [GZ]. Suppose that (X^{n+1}, g) is a conformally compact Einstein manifold and $(M^n, [\hat{g}])$ is its associated conformal infinity. And suppose that x is a defining function associated with a choice of metric $\hat{g} \in [\hat{g}]$ on M as before. One considers the asymptotic Dirichlet problem at infinity for the Poisson equation

$$(-\Delta_g - s(n-s))u = 0. \tag{4.1}$$

Based on earlier works on the resolvents, Graham and Zworski in [GZ] proved that there is a meromorphic family of solutions $u(s) = \wp(s)f$ such that

$$\wp(s)f = Fx^{n-s} + Gx^s \quad \text{if } s \notin n/2 + N \tag{4.2}$$

where $F, G, \in C^\infty(X)$, $F|_M = f$, and $F, G \bmod O(x^n)$ are even in x. Also if $n/2 - s$ is not an integer, then $G|_M$ is globally determined by f and g. The scattering operator is defined as:

$$S(s)f = G|_M. \tag{4.3}$$

Thus the function v satisfying (1.6) and (1.7) studied in [FG-2] is defined as:

$$v = -\frac{d}{ds}|_{s=n} \wp(s) 1. \tag{4.4}$$

Therefore when n is odd, we may rewrite (1.8) in Theorem 1.1 as

$$V(X^{n+1}, g) = -\int_M (\frac{d}{ds}|_{s=n} S(s)1)\, dv_{\hat{g}}. \tag{4.5}$$

We will now point out that formula similar to that (4.5) holds also when n is even. To do so, we first establish some notations.

When n is even and (X^{n+1}, g) conformal Einstein. For each $s < n$ and close to n, we consider the solution of the Possion equation $\wp(s)1 = u(s)$ as in (4.1). Then

$$u(s) = x^{n-s} F(x, s) + x^s G(x, s) \tag{4.6}$$

for functions F, G which are even in x. We denote the asymptotic expansion of F near boundary as

$$F(x, s) = 1 + a_2(s) x^2 + a_4(s) x^4 + \cdot\ + a_n(s) x^n + \cdot. \tag{4.7}$$

Denote $a'_k = \frac{d}{ds}|_{s=n} a_k(s)$. We have the following formula.

Theorem 4.1. *Suppose that (X^{n+1}, g) is a conformally compact Einstein manifold with even n. For a choice of metric $\hat{g} \in [\hat{g}]$ of the conformal infinity $(M, [\hat{g}])$, the renormalized volume is*

$$\begin{aligned} V(X^{n+1}, g, \hat{g}) = &-\int_M (\frac{d}{ds}|_{s=n} S(s)1)\, dv_{\hat{g}} \\ &- \frac{1}{n}\int_M 2a'_2 v^{(n-2)} dv_{\hat{g}} - \cdots - \frac{1}{n}\int_M (n-2) a'_{n-2} v^{(2)} dv_{\hat{g}} - \int_M a'_n dv_{\hat{g}}. \end{aligned} \tag{4.8}$$

Proof. Using the notations as set in (4.6) and (4.7), we have

$$v = -\frac{d}{ds}|_{s=n} u(s) = F(x, n) \log x - F' - x^n G' + G(x, n) x^n \log x.$$

Since

$$F(x,n) = 1 - c_{\frac{n}{2}} Q_n x^n + O(x^{n+1}),$$

and

$$G(x,n)|M = c_{\frac{n}{2}} Q_n = \lim_{s\to n} S(s)1,$$

where $c_k = (-1)^k (2^{2k} k!(k-1)!)^{-1}$, and

$$F' = \frac{dF}{ds}|_{s=n}, \quad \text{and } G' = \frac{dG}{ds}|_{s=n}.$$

We get

$$v \ = \ \log x - F' - x^n G' - 2c_{\frac{n}{2}} Q_n x^n \log x + O(x^{n+1}\log x).$$

Recall the expansion of the volume element

$$\begin{aligned} dv_{g_\epsilon} &= \sqrt{\frac{\det g_\epsilon}{\det \hat{g}}} dv_{\hat{g}} = \sqrt{\det(\hat{g}^{-1} g_\epsilon)} dv_{\hat{g}} \\ &= (1 + v^{(2)}\epsilon^2 + v^{(4)}\epsilon^4 + \cdots) dv_{\hat{g}} \end{aligned}$$

where $v^{(2)}, \cdots, v^{(n)}$ are determined by $\hat{g}$. We have

$$\mathrm{vol}(\{x > \epsilon\}) = \frac{1}{n}\int_{x>\epsilon} -\Delta v dv_g = \frac{1}{n}\int_{x=\epsilon} -\frac{\partial v}{\partial n} d\sigma_{g_\epsilon} = \frac{1}{n}\epsilon^{-n+1}\int_M \frac{\partial v}{\partial x}|_{x=\epsilon} dv_{g_\epsilon}$$

where

$$\begin{aligned} \frac{\partial v}{\partial x} \ &= \frac{1}{x} - \cdots - n a_n' x^{n-1} - n(\frac{d}{ds}|_{s=n} S(s)1) x^{n-1} \\ &\quad - 2c_{\frac{n}{2}} Q_n x^{n-1} - 2nc_{\frac{n}{2}} Q_n x^{n-1}\log x + o(x^{n-1}). \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{vol}(\{x > \epsilon\}) = \cdots + \frac{1}{n}\int_M (&v^{(n)} - 2a_2' v^{(n-2)} - \cdots - (n-2) v^{(2)} a_{n-2}' \\ &- n a_n' - n\frac{d}{ds}|_{s=n} S(s)1 - 2c_{\frac{n}{2}} Q_n) dv_{\hat{g}} + \cdots \end{aligned}$$

and the renormalized volume for (X^{n+1}, g) is

$$\begin{aligned} V(X^{n+1}, g, \hat{g}) = &-\frac{1}{n}\int_M 2a_2' v^{(n-2)} dv_{\hat{g}} - \cdots - \frac{1}{n}\int_M (n-2) a_{n-2}' v^{(2)} dv_{\hat{g}} \\ &- \int_M a_n' dv_{\hat{g}} - \int_M (\frac{d}{ds}|_{s=n} S(s)1)\, dv_{\hat{g}}. \end{aligned}$$

We have thus established the formula (4.8).

We remark that the coefficients a_k' for $k \le n$ in formula (4.8) can be explicitly computed and are curvatures of the metric of $\hat{g}$ of the conformal infinity $(M, \hat{g})$. In the following, we write down the formula for the cases $n = 2$ and $n = 4$.

Proposition 4.2. *Suppose that (X^{n+1}, g) is a conformally compact Einstein manifold with even n. For a choice of metric $\hat{g} \in [\hat{g}]$ of the conformal infinity $(M, [\hat{g}])$, we have*

$$V(X^3, g, \hat{g}) = -\int_{M^2} (\frac{d}{ds}|_{s=2} S(s)1)\, dv_{\hat{g}}, \text{ for } n = 2 \tag{4.9}$$

and

$$V(X^5, g, \hat{g}) = -\frac{1}{32 \cdot 36}\int_{M^4} R^2_{\hat{g}} dv_{\hat{g}} - \int_{M^4} (\frac{d}{ds}|_{s=4} S(s)1)\, dv_{\hat{g}}, \text{ for } n = 4 \tag{4.10}$$

where $R_{\hat{g}}$ is the scalar curvature for $(M^4, \hat{g})$.

Proof. For $n = 2$, one may calculate and obtain

$$a_2(s) = 2K_{\hat{g}},$$

where $K_{\hat{g}}$ is the Gaussian curvature of $(M^2, \hat{g})$, thus $a_2' = 0$, and (4.9) follows directly from (4.8). For $n = 4$, one calculate and obtain

$$a_2(s) = -\frac{4-s}{4(3-s)}\mathrm{Tr} g^{(2)},$$

thus

$$a_2' = -\frac{1}{4}\mathrm{Tr} g^{(2)} \tag{4.11}$$

and

$$\begin{aligned} a_4 = &-\frac{1}{8(4-s)}((4-s)(2\mathrm{Tr} g^{(4)} - |g^{(2)}|^2) \\ &- \frac{(4-s)(6-s)}{4(3-s)}(\mathrm{Tr} g^{(2)})^2 - \frac{(4-s)}{8(3-s)}\hat{\Delta}\mathrm{Tr} g^{(2)}) \\ = &-\frac{1}{8}(2\mathrm{Tr} g^{(4)} - |g^{(2)}|^2 - \frac{(6-s)}{4(3-s)}(\mathrm{Tr} g^{(2)})^2 - \frac{1}{4(3-s)}\hat{\Delta}\mathrm{Tr} g^{(2)}) \end{aligned}$$

where

$$g^{(2)} = -\frac{1}{2}(Ric_{\hat{g}} - \frac{1}{6}R_{\hat{g}}\hat{g}), \tag{4.12}$$

$$\mathrm{Tr} g^{(4)} = \frac{1}{4}|g^{(2)}|^2_{\hat{g}}. \tag{4.13}$$

Thus

$$a_4' = \frac{1}{32}(3(\mathrm{Tr}g^{(2)})^2 + \hat{\Delta}\mathrm{Tr}g^{(2)}) \tag{4.14}$$

Insert (4.11) and (4.12) and (4.14) to (4.8) and recall that when $n = 4$, $v^{(2)} = \frac{1}{2}\mathrm{Tr}g^{(2)}$, we obtain (4.10).

We end the section by deriving a variational formula, which indicates that when n is even, the scattering term in the renormalized volume is a conformal anomaly and is a conformal primitive of the Q-curvature Q_n. Namely,

Theorem 4.3. *Suppose that (X^{n+1}, g) is a conformally compact Einstein manifold with conformal infinity $(M, [\hat{g}])$, and that n is even. Then*

$$\frac{d}{d\alpha}|_{\alpha=0} \int_M \mathcal{S}_{e^{2\alpha w}\hat{g}}\, dv_{e^{2\alpha w}\hat{g}} = -2c_{\frac{n}{2}} \int_M w\,(Q_n)_{\hat{g}}\, dv_{\hat{g}} \tag{4.15}$$

where $\mathcal{S}_{\hat{g}} = \frac{d}{ds}|_{s=n} S_{(g,\hat{g})}(s)1$; and $S_{(g,\hat{g})}$ is the scattering operator as defined in (4.3).

To prove the theorem, we first list some elementary properties of scattering matrix under conformal change of metrics, we assume the same setting as in Theorem 4.3.

Lemma 4.4. *Denote $S(s) = S^{\hat{g}} = S_{(g,\hat{g})}(s)$ the scattering matrix, and $\hat{g}_w = e^{2w}\hat{g}$ a metric conformal to $\hat{g}$. Then*
(a)

$$S_{\hat{g}_w}(s) = e^{-sw} S(s) e^{(n-s)w}. \tag{4.16}$$

(b) S(s) has a simple pole at $s = n$ and its residue is $-c_{\frac{n}{2}} P_{\frac{n}{2}}$, i.e.

$$S(s) = -\frac{c_{\frac{n}{2}} P_{\frac{n}{2}}}{s-n} + T(s) \tag{4.17}$$

where where $T(s)$ is the regular part of the scattering matrix near $s = n$ and $P_{\frac{n}{2}}$ is the Paneitz operator.
(c)

$$S(n)1 = \lim_{s\to n} S(s)1 = T(n)1 = c_{\frac{n}{2}} Q_n. \tag{4.18}$$

(d)

$$\begin{aligned} e^{nw} S_{\hat{g}_w}(n)1 &= e^{nw} \lim_{s\to n} e^{-sw} S(s) e^{(n-s)w} \\ &= \lim_{s\to n}\Big(-\frac{c_{\frac{n}{2}} P_{\frac{n}{2}} e^{(n-s)w}}{s-n} + T(s)e^{(n-s)w}\Big) \\ &= c_{\frac{n}{2}} P_{\frac{n}{2}} w + S(n)1. \end{aligned} \tag{4.19}$$

Proof.
(a) is a simple consequence of the definition of the scattering matrix. (b) and (c) follow from the work of [GZ]. (d) is a consequence of the equation

$$(P_{\frac{n}{2}})_{\hat{g}} w + (Q_n)_{\hat{g}} = (Q_n)_{\hat{g}_w} e^{nw}$$

relating the Paneitz operator to Q -curvature on even dimensional manifolds.

We now compute the variation of

$$\int_M \frac{d}{ds}|_{s=n} S_{(g,\hat{g})}(s) 1 dv_{\hat{g}}$$

under the conformal change of metrics in $[\hat{g}]$. Denote by $\mathcal{S}_{\hat{g}} = \frac{d}{ds}|_{s=n} S_{(g,\hat{g})}(s)1$ with respect to the metric $\hat{g}$ on M.

Lemma 4.5.

$$\int_M (\mathcal{S}_{e^{2w}\hat{g}} e^{nw} - \mathcal{S}_{\hat{g}}) dv_{\hat{g}} = -c_{\frac{n}{2}} \int_M (w\,(P_{\frac{n}{2}})_{\hat{g}} w + 2w (Q_n)_{\hat{g}})\, dv_{\hat{g}}. \tag{4.20}$$

Proof. By definition we have

$$\mathcal{S}_{\hat{g}e^{2w}} e^{nw} = \lim_{s \to n} \left(\frac{S_{e^{2w}\hat{g}}(s)1 - S_{e^{2w}\hat{g}}(n)1}{s-n}\right) e^{nw}. \tag{4.21}$$

Apply (4.16) and (4.19), denote $P_{\frac{n}{2}} = (P_{\frac{n}{2}})_{\hat{g}}$, $Q = Q_n = (Q_n)_{\hat{g}}$ and $T = T_{(g,\hat{g})}$, we obtain

$$\begin{aligned} e^{nw} \frac{S_{e^{2w}\hat{g}}(s)1 - S_{e^{2w}\hat{g}}(n)1}{s-n} &= \frac{e^{(n-s)w} S(s) e^{(n-s)w} - c_{\frac{n}{2}} P_{\frac{n}{2}} w - c_{\frac{n}{2}} Q_{\hat{g}}}{s-n} \\ &= \frac{(e^{(n-s)w} - 1) S(s) e^{(n-s)w} + S(s) e^{(n-s)w} - c_{\frac{n}{2}} P_{\frac{n}{2}} w - c_{\frac{n}{2}} Q_{\hat{g}}}{s-n} \\ &= \frac{e^{(n-s)w} - 1}{s-n} S(s) e^{(n-s)w} + \frac{S(s) e^{(n-s)w} - c_{\frac{n}{2}} P_{\frac{n}{2}} w - c_{\frac{n}{2}} Q_{\hat{g}}}{s-n}. \end{aligned} \tag{4.22}$$

We now claim that after taking limit we have

$$\mathcal{S}_{e^{2w}\hat{g}} e^{nw} - \mathcal{S}_{\hat{g}} = -c_{\frac{n}{2}} (w P_{\frac{n}{2}} w + w Q_{\hat{g}} - \frac{1}{2} P_{\frac{n}{2}} w^2) - T(n) w. \tag{4.23}$$

To see the claim we observe that

$$\lim_{s \to n} \frac{e^{(n-s)w} - 1}{s-n} = -w, \tag{4.24}$$

$$\lim_{s\to n} S(s)e^{(n-s)w} = \lim_{s\to n}\left(-\frac{c_{\frac{n}{2}} P_{\frac{n}{2}} e^{(n-s)w}}{s-n} + T(s)e^{(n-s)w}\right) \tag{4.25}$$
$$= c_{\frac{n}{2}} P_{\frac{n}{2}} w + c_{\frac{n}{2}} Q_n,$$

$$\lim_{s\to n} \frac{S(s)e^{(n-s)w} - c_{\frac{n}{2}} P_{\frac{n}{2}} w - c_{\frac{n}{2}} Q_{\hat{g}}}{s-n}$$
$$= \lim_{s\to n}\left(\frac{S(s)(e^{(n-s)w}-1) - c_{\frac{n}{2}} P_{\frac{n}{2}} w}{s-n} + \frac{S(s)1 - S(n)1}{s-n}\right) \tag{4.26}$$
$$= \lim_{s\to n} \frac{S(s)(e^{(n-s)w}-1) - c_{\frac{n}{2}} P_{\frac{n}{2}} w}{s-n} + \mathcal{S}_{\hat{g}},$$

and

$$\lim_{s\to n} \frac{S(s)(e^{(n-s)w}-1) - c_{\frac{n}{2}} P_{\frac{n}{2}} w}{s-n}$$
$$= \lim_{s\to n} \frac{\frac{-c_{\frac{n}{2}} P_{\frac{n}{2}} (e^{(n-s)w}-1)}{s-n} + T(s)(e^{(n-s)w}-1) - c_{\frac{n}{2}} P_{\frac{n}{2}} w}{s-n} \tag{4.27}$$
$$= \lim_{s\to n} -c_{\frac{n}{2}} P_{\frac{n}{2}} \frac{e^{(n-s)w} - 1 - (n-s)w}{(s-n)^2} + \lim_{s\to n} T(s)\frac{e^{(n-s)w}-1}{s-n}$$
$$= -\frac{1}{2} c_{\frac{n}{2}} P_{\frac{n}{2}} w^2 - T(n)w.$$

Thus the claim (4.23) follows from the formulas (4.24) to (4.27).

Due to the fact that both operators $P_{\frac{n}{2}}$ and $T(n)$ are self-adjoint, we have

$$\int_M P_{\frac{n}{2}} w^2 dv_{\hat{g}} = \int_M w^2 P_{\frac{n}{2}} 1 dv_{\hat{g}} = 0$$

and

$$\int_M T(n)w dv_{\hat{g}} = \int_M wT(n)1 dv_{\hat{g}} = c_{\frac{n}{2}} \int_M wQ_n dv_{\hat{g}}.$$

Thus integrating (4.23), we get

$$\int_M \mathcal{S}_{e^{2w}\hat{g}} dv_{e^{2w}\hat{g}} - \int_M \mathcal{S}_{\hat{g}} dv_{\hat{g}} = -c_{\frac{n}{2}} \int_M (wP_{\frac{n}{2}} w + 2wQ_n) dv_{\hat{g}}.$$

This is the desired formula (4.15).

Theorem 4.3 follows from Lemma 4.5 by a simple integration.

We remark that (4.10) and (4.15) give another proof of the fact observed in [HS] and [G-1] that when $n = 2$, $V(X^3, g)$ is the conformal primitive of the Gaussian curvature $K_{\hat{g}}$ on $(M^2, \hat{g})$; while for $n = 4$, $V(X^5, g)$ is the conformal primitive of $\frac{1}{16}\sigma_2(\hat{g})$ on $(M^4, \hat{g})$, where

$$\sigma_2(\hat{g}) = \frac{1}{6}(R_{\hat{g}}^2 - 3|Ric|_{\hat{g}}^2)$$

and the relation of $\sigma_2(\hat{g})$ to $(Q_4)_{\hat{g}}$ is given by

$$(Q_4)_{\hat{g}} = \frac{1}{6}(-\Delta R + R^2 - 3|Ric|^2)_{\hat{g}} = \sigma_2(\hat{g}) - \frac{1}{6}\Delta_{\hat{g}} R_{\hat{g}}, \tag{4.28}$$

where $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$ for the Euclidean metric. We remark that the term σ_2 plays an important role in some recent work [CGY] in conformal geometry, where the sign of $\int_M (\sigma_2)_{\hat{g}} dv_{\hat{g}}$ is used to study existence of metrics with positive Ricci curvature on compact, closed manifolds of dimension 4; also the relation (4.28) between σ_2 and Q_4 plays a crucial role in the proof of the results in [CGY].

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