AN A PRIORI ESTIMATE FOR A FULLY NONLINEAR EQUATION ON FOUR–MANIFOLDS

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In Memory of Thomas Wolff, our friend

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0. Introduction

In this paper we establish an *a priori* estimate for a fully nonlinear equation arising in conformal geometry (see [V-1], [CGY]). Although the equation is defined on any Riemannian manifold of dimension $n \geqslant 3$, four dimensions is of particular interest due to its connection with the Chern–Gauss–Bonnet integrand, and in estimating the eigenvalues of the Ricci curvature (see the introduction of [CGY]). For this reason we will restrict our attention to case $n = 4$, while introducing a variant of the equation.

Let us begin by introducing our notation. Let M^4 be a smooth compact four–manifold without boundary. Given a Riemannian metric g, we let W, Ric, and R denote respectively the Weyl curvature tensor, Ricci curvature, and scalar curvature of g. We also define the tensor

$$A = A_g = Ric - \frac{1}{6}Rg \ . \tag{0.1}$$

Note that A arises in a natural way when decomposing the Riemannian curvature tensor $Riem$:

$$Riem = W + \frac{1}{2}A \owedge g \ , \tag{0.2}$$

where $\owedge$ denotes the Kulkarni–Nomizu product of symmetric two–tensors; see [Be, 1G] for definitions.

To describe our equation we first note how the tensor A transforms under a conformal change of metric. Let $g = e^{2w}g_0$, and from now on let us designate quantities which depend on g_0 by attaching a sub– or superscript 0. Then

$$A_g = A_0 - 2\nabla_0^2 w + 2dw \otimes dw - |dw|^2 g_0 \ , \tag{0.3}$$

where ∇_0^2 denotes the Hessian. In [V-1], the following equations were introduced:

$$\sigma_k(A_g) = \sigma_k(A_0 - 2\nabla_0^2 w + 2dw \otimes dw - |dw|^2 g_0) = f, \tag{0.4}$$

where σ_k denotes the k^{th} elementary symmetric polynomial, applied to the eigenvalues of A_g. Note that when $k = 1$, $\sigma_1(A_g) = tr_g A_g = \frac{1}{3}R$, so (0.4) is the prescribed scalar curvature equation.

Our particular interest here is with the case $k = 2$. In terms of the Ricci and scalar curvature, $\sigma_2(A) = -\frac{1}{2}|Ric|^2 + \frac{1}{6}R^2$. Moreover, the decomposition (0.2) implies a splitting of the Euler form, so that the Chern–Gauss–Bonnet formula can be written

$$8\pi^2\chi(M^4) = \int \frac{1}{4}|W|^2 dvol + \int \sigma_2(A) dvol \ ,$$

where $|W|^2$ denotes the squared norm of the Weyl tensor as a (0,4) tensor. Thus, the Euler characteristic is expressed as the sum of two conformally invariant integrals. The second integral can be either positive or negative, but its positivity obviously has topological consequences. Furthermore, in [CGY], we were able to show that any metric g_0 satisfying

$$\left.\begin{aligned} &\int \sigma_2(A_{g_0}) dvol_{g_0} > 0\,, \\ &Y(g_0) > 0\,, \end{aligned}\right\} \tag{0.5}$$

where $Y(g_0)$ is the Yamabe invariant of g_0, is conformal to a metric g with $\sigma_2(A_g) > 0$. The positivity of $\sigma_2(A)$ is a kind of pinching condition on the Ricci curvature; in particular it implies

$$0 < Ric < \frac{1}{2} Rg$$

(see [CGY, Lemma 1.2]).

The present paper is, in a sense, the continuation of [CGY]. Once a conformal class admits a metric with $\sigma_2(A) > 0$, our goal is to show that it admits a metric normalizing $\sigma_2(A)$. To achieve this we need the following basic estimate:

Theorem. *Let (M^4, g_0) be a compact Riemannian four–manifold, which is not conformally equivalent to the round sphere. Suppose the conformal metric $g = e^{2w} g_0$ satisfies*

$$\frac{\sigma_2(A_g)}{R_g^\alpha} = f > 0\,, \tag{0.6}$$

where $\alpha = 0$ or 1 ; and $R_g > 0$. Then there is a constant $C = C(\|f\|_{C^2}, g_0)$ such that

$$\max_{M^4} \{e^w + |\nabla_0 w|\} \leqslant C\,. \tag{0.7}$$

Two corollaries follow from inequality (0.7). The first is technical, while the second is an existence result.

Corollary A. *Under the same assumptions, there is a constant $C = C(\|f\|_{C^2}, (\min f)^{-1}, g_0)$ such that*

$$\|w\|_\infty \leqslant C\,. \tag{0.8}$$

Corollary B. *Assume that (M^4, g_0) satisfies (0.5). If $\alpha = 0$, then given any positive (smooth) function $f > 0$, there exists a solution $g = e^{2w} g_0$ of (0.6). In particular, this is true if $f \equiv 1$.*

Remarks.

1. The assumption that (M^4, g_0) is not conformally equivalent to the round sphere is of course crucial, since (0.6) is invariant under the conformal group.

2. Taking $f \equiv 1$ in (0.6), the parallel with the Yamabe problem is apparent. We wish to emphasize, however, that the theorem above does *not* rely on the positive mass estimate.
3. Once the bound (0.8) is established for solutions of (0.6), an existence result like Corollary B follows from applying a standard procedure: first, (0.7) and (0.8) imply uniform C^1–bounds for solutions. In section 1 we show that one obtains $C^{1,1}$–bounds on solutions to (0.6) once C^1–bounds are known. Finally, invoking the concavity of equation (0.6) ([Ev], [Kr]) one can derive regularity estimates of any order. Applying the degree theoretic arguments, existence follows. This is explained in more detail in section 5.
4. The positivity of f implies that (0.6) is elliptic. This and other important properties of (0.4) are described in [CGY]. In [V-2] estimates like (0.7) are established, assuming however that f has a very special structure.
5. It is interesting to note that the equation

$$\sigma_2(A) \equiv 1$$

is variational *except in four dimensions* (due to conformal invariance; see [V-1]). When $n \neq 4$ it is the Euler–Lagrange equation of the functional

$$g \mapsto \int \sigma_2(A_g) dvol \, .$$

6. When $\alpha = 1$, the existence of solutions to (0.6) remains an open question. When $\alpha = 0$, one can rewrite (0.6) in a kind of divergence form (see section 5, (5.5)); this ultimately allows us to compute the degree of the equation as defined in [Li]. But the structure of (0.6) is more complicated when $\alpha = 1$, and a different argument is apparently required.

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1. The blow–up

In this section we begin the proof of the main theorem by doing a blow–up analysis. Namely, given a sequence of solutions to (0.6) for which the estimate (0.7) fails, we dilate them to construct a new sequence which converges to a smooth solution of (0.6) on $(\mathbb{R}^4, ds^2)$ with $f \equiv$ constant. In sections 2 and 3 we provide a classification of all such solutions; from this we eventually conclude that the manifold (M^4, g_0) is conformally equivalent to the round sphere (see section 4).

The aforementioned process is fairly standard, and is important whenever the PDE under consideration is invariant under some (non–compact) group of transformations. The main technical difficulty in our case is the absence of a Harnack inequality for solutions of (0.6). Consequently, if we simply dilate our solutions in order to obtain a sequence which is uniformly bounded above, we are unable to conclude that the sequence has a uniform lower bound (even locally). This makes it difficult to construct a non–trivial limiting solution to (0.6) on $\mathbb{R}^4$ from our original sequence.

To overcome this difficulty we dilate in a rather unusual way, which we now describe. Let $g_k = e^{2w_k} g_0$ be a sequence of solutions to

$$\frac{\sigma_2(A_{g_k})}{R^{\alpha}_{g_k}} = f_k \ , \tag{1.1}$$

where $\alpha = 0$ or 1, and we assume that $\{f_k\}$ satisfies

$$\left.\begin{aligned} 0 < c_0 \leqslant f_k \leqslant c_0^{-1}, \\ \|f_k\|_{C^2(M)} \leqslant c_1 \ . \end{aligned}\right\} \tag{1.2}$$

If inequality (0.7) fails, then

$$\max_M [|\nabla_0 w_k| + e^{w_k}] \to \infty \quad \text{as} \quad k \to \infty \ . \tag{1.3}$$

Let us assume that $P_k \in M$ is a point at which $(|\nabla_0 w_k| + e^{w_k})$ attains its maximum. By choosing normal coordinates $\{\Phi_k\}$ centered at P_k, we may identify the coordinate neighborhood of P_k in M with the unit ball $B(1) \subset \mathbb{R}^4$ such that $\Phi_k(P_k) = 0$. Given $\varepsilon > 0$, we define the dilations $T_\varepsilon : \mathbb{R}^4 \to \mathbb{R}^4$ by $x \mapsto \varepsilon x$, and consider the sequence $w_{k,\varepsilon} = T^\star_\varepsilon w_k + \log \varepsilon$. Note that

$$|\nabla_0 w_{k,\varepsilon}| + e^{w_{k,\varepsilon}} = \varepsilon(|\nabla_0 w_k| + e^{w_k}) \circ T_\varepsilon \ .$$

Thus, for each k we can choose ε_k so that

$$|\nabla_0 (w_{k,\varepsilon_k})| + e^{w_{k,\varepsilon_k}}\Big|_{x=0} = 1 \ . \tag{1.4}$$

Note that w_{k,ε_k} is defined in $B_{\frac{1}{\varepsilon_k}}(0)$, and

$$|\nabla_0 (w_{k,\varepsilon_k})| + e^{w_{k,\varepsilon_k}} \leqslant 1 \quad \text{on} \quad B_{\frac{1}{\varepsilon_k}}(0) \ . \tag{1.5}$$

To simplify notation, let us denote w_{k,ε_k} by w_k. Since from now on we view $\{w_k\}$ as a sequence defined on dilated balls in $\mathbb{R}^4$, there will be no danger of confusing the renormalized sequence with the original sequence. Note that $g^\star_k \equiv e^{2w_k} T^\star_{\varepsilon_k} g_0 \equiv e^{2w_k} g^k_0$ satisfies

$$\frac{\sigma_2(A_{g^\star_k})}{R^{\alpha}_{g^\star_k}} = f_k \circ T_{\varepsilon_k} \ , \tag{1.6}$$

where $\alpha = 0$ or 1. Furthermore, $g_0^k = T_{\varepsilon_k}^{\star} g_0 \to ds^2$ in $C^{2,\beta}$ on compact sets.

There are now two possibilities to consider, depending on the behavior of the exponential term in (1.4). First, suppose that

$$\lim_k e^{w_k(0)} = 0 \ .$$

After choosing a subsequence (also denoted $\{w_k\}$) with $e^{w_k(0)} \to 0$, we let $\overline{w}_k(x) = w_k(x) - w_k(0)$. Then $\bar{g}_k = e^{2\overline{w}_k} g_0^k$ satisfies

$$\left.\begin{aligned} \overline{w}_k(0) = 0 \,, \\ |d\overline{w}_k| \leqslant 1 \,, \\ \lim_k |d\overline{w}_k(0)| = 1 \ . \end{aligned}\right\} \tag{1.7}$$

Also

$$\frac{\sigma_2(A_{\bar{g}_k})}{R^{\alpha}_{\bar{g}_k}} = e^{2(2-\alpha)w_k(0)} f_k \circ T_{\varepsilon_k} \ . \tag{1.8}$$

Note that the bounds in (1.7) imply that given a fixed ball $B(\rho) = \{x \in \mathbb{R}^4 \big| |x| < \rho\} \subset \mathbb{R}^n$, then

$$\max_{B_\rho(0)} |\overline{w}_k| \leqslant \rho \tag{1.9}$$

so that $\{\overline{w}_k\}$ is bounded in the C^1–topology on compact sets in $\mathbb{R}^4$.

Alternatively, suppose

$$\limsup_k e^{w_k(0)} = \delta_0 > 0 \ .$$

Then there is a subsequence (also denoted $\{w_k\}$) satisfying

$$\left.\begin{aligned} -c_2 \leqslant w_k(0) \leqslant 0 \,, \\ |dw_k| \leqslant 1 \,, \end{aligned}\right\} \tag{1.10}$$

for some constant $c_2 > 0$. Note that (1.10) implies that for any $\rho > 0$, there is a constant C_ρ such that

$$\max_{B(\rho)} |w_k| \leqslant C_\rho \ . \tag{1.11}$$

We now want to show that both $\{\overline{w}_k\}$ and $\{w_k\}$ converge; but the type of convergence differs: for the sequence $\{\overline{w}_k\}$ the equation (1.8) will not be uniformly elliptic, because the RHS tends to zero as $k \to \infty$ (see [CGY, Prop. 1.5]). Consequently, the best estimate we can obtain is

$$\sup_{B(\rho)} |\nabla^2 \overline{w}_k| \leqslant C_\rho \ ; \tag{1.12}$$

see Proposition 1.1 below.

On the other hand, the sequence $\{w_k\}$ does satisfy a (uniformly) elliptic equation; in addition to the bounds on the second derivatives as in (1.12), by the concavity of the equation ${\sigma_2}^{\frac{1}{2}}$ it follows from the work of Evans [Ev] and Krylov [Kr] that $\{w_k\}$ is bounded (on compact sets) in $C^{2,\beta}$. Then the Schauder estimates, along with standard elliptic estimates, give convergence in any C^k.

The preceding analysis is based on the following *a priori* estimate:

Proposition 1.1. *Let g_0 be a Riemannian metric on $B(\rho) \subset \mathbb{R}^4$. Suppose $g = e^{2w} g_0$ satisfies*

$$\frac{\sigma_2(A_g)}{R^\alpha} = \varphi \geqslant 0 \,, \tag{1.13}$$

$$R > 0 \,, \tag{1.14}$$

on $B(\rho)$, where $\alpha = 0$ or 1. Then there is a constant

$$C = C(\rho, \|g_0\|_{C^2(B(\rho))}, \ \|\varphi\|_{C^2(B(\rho))} \,, \ \|w\|_{L^\infty(B(\rho))} \,, \ \|\nabla w\|_{L^\infty(B(\rho))})$$

such that

$$\|\nabla^2 w\|_{L^\infty(B(\rho/2))} \leqslant C \,. \tag{1.15}$$

Remarks.

1. A careful examination of the proof for the Proposition indicates that one can modify the constant C in (1.15) to depend only on ρ, $\|g_0\|_{C^2(B(\rho))}$, and a lower bound of $\Delta_0 \varphi$ on $B(\rho)$ and an upper bound of $\|\nabla_0 \varphi\|$ on $B(\rho)$.
2. The C^k norm in the statement of Proposition 1.1 are with respect to the Euclidean metric.

Proof. The proof amounts to a localization of the estimates in [CGY]. Roughly speaking, the idea is to work intrinsically; that is, to derive bounds on the curvature of g on $B(\rho/2)$, then reinterpret these bounds in terms of the conformal factor w. For this reason, all covariant derivatives, curvature tensors, etc., which appear in the following are understood to be with respect to the metric g. If we need to refer to a quantity which depends on the background metric g_0, then we will denote this with a sub– or superscript 0. By convention, components of tensors are in normal coordinates. In particular, $W_{ijk\ell}$, R_{ij}, and B_{ij} denote the components of the the Weyl curvature, Ricci curvature, and Bach tensor (see [CGY, (1.18)]). Also, we let $E_{ij} = R_{ij} - \frac{1}{4} R g_{ij}$, $S_{ij} = -R_{ij} + \frac{1}{2} R g_{ij}$, and $A_{ij} = R_{ij} - \frac{1}{6} R g_{ij}$. We will need two identities from [CGY]:

$$\begin{aligned} S_{ij} \nabla_i \nabla_j R = 3\Delta\sigma_2(A_g) + 3(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2) \\ + 6 tr E^3 + R|E|^2 - 6 W_{ijk\ell} E_{ik} E_{j\ell} - 6 E_{ij} B_{ij} \end{aligned} \tag{1.16}$$

where $trE^3 = E_{ij}E_{ik}E_{jk}$ (see [CGY, Lemma 5.4]), and

$$\begin{aligned} S_{ij}\nabla_i\nabla_j V = & -\frac{1}{4}trE^3 + \frac{1}{48}R|E|^2 + \frac{1}{576}R^3 \\ & - \frac{1}{2}\langle\nabla w, \nabla\sigma_2(A)\rangle - \frac{1}{4}R|\nabla w|^4 - S_{ij}\nabla|\nabla w|^2\nabla w \\ & + W_{ijk\ell}S_{ik}\nabla_j w\nabla_\ell w - \frac{1}{2}S_{ij}A_{ik}A^\circ_{jk} + \frac{1}{4}S_{ij}A^\circ_{ik}A^\circ_{jk} \\ & - S_{ij}A^\circ_{ik}\nabla_i w\nabla_k w + \frac{1}{2}S_{ij}A^\circ_{ij}|\nabla w|^2 + \frac{1}{2}\nabla_k w\nabla_k A^\circ_{ij}S_{ij} \\ & + \frac{1}{2}RA^\circ_{ij}\nabla_i w\nabla_j w \ , \end{aligned} \tag{1.17}$$

where $V = \frac{1}{2}|\nabla w|^2$ (see [CGY, Cor. 5.15]). To accommodate the case $\alpha = 1$ in (1.13), we need to write these identities in a slightly different form. To this end, let $T_{ij} = S_{ij} - 3\frac{\sigma_2(A)}{R}g_{ij}$; by [CGY , Lemma 1.2], T_{ij} is positive semi–definite when $\sigma_2(A) \geqslant 0$ and $R > 0$.

Lemma 1.2. *Suppose $g = e^{2w}g_0$ satisfies (1.2) and (1.14) with $\alpha = 1$. Then*

$$\begin{aligned} T_{ij}\nabla_i\nabla_j R \geqslant & \ 3R\Delta\varphi + 6trE^3 + R|E|^2 \\ & - 6W_{ijk\ell}E_{ik}E_{j\ell} - 6B_{ij}E_{ij} \ , \end{aligned} \tag{1.18}$$

$$\begin{aligned} T_{ij}\nabla_i\nabla_j V = & -3\varphi\left\{\frac{1}{4}|A|^2 + \frac{1}{4}|A^\circ|^2 - \frac{1}{2}A_{ij}A^\circ_{ij} - \frac{1}{6}R|\nabla w|^2\right. \\ & \left. + \frac{1}{6}R_0|\nabla w|^2 + \frac{1}{6}\langle\nabla w, \nabla(R_0 e^{-2w})\rangle\right\} - \frac{1}{2}\langle\nabla w, \nabla\varphi\rangle R \\ & - \frac{1}{4}trE^3 + \frac{1}{48}R|E|^2 + \frac{1}{576}R^3 \\ & - \frac{1}{2}\langle\nabla w, \nabla\sigma_2(A)\rangle - \frac{1}{4}R|\nabla w|^4 - S_{ij}\nabla_i|\nabla w|^2\nabla_j w \\ & + W_{ijk\ell}S_{ik}\nabla_j w\nabla_\ell w - \frac{1}{2}S_{ij}A_{ik}A^\circ_{jk} + \frac{1}{4}S_{ij}A^\circ_{ik}A^\circ_{jk} \\ & - S_{ij}A^\circ_{ik}\nabla_i w\nabla_k w + \frac{1}{2}S_{ij}A^\circ_{ij}|\nabla w|^2 + \frac{1}{2}\nabla_k w\nabla_k A^\circ_{ij}S_{ij} \\ & + \frac{1}{2}RA^\circ_{ij}\nabla_i w\nabla_j w \ . \end{aligned} \tag{1.19}$$

Proof. Note that

$$T_{ij}\nabla_i\nabla_j R = S_{ij}\nabla_i\nabla_j R - 3\frac{\sigma_2(A)}{R}\Delta R \ .$$

By [CGY, Lemma 7.10],

$$3(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2) \geqslant -6\nabla\sigma_2(A)\frac{\nabla R}{R} + 6\sigma_2(A)\frac{|\nabla R|^2}{R^2} \ .$$

Substituting these into (1.16), we have

$$\begin{aligned} T_{ij}\nabla_i\nabla_j R &\geqslant 3\Delta\sigma_2(A) - 3\frac{\sigma_2(A)}{R}\Delta R - 6\nabla\sigma_2(A)\frac{\nabla R}{R} \\ &\quad + 6\sigma_2(A)\frac{|\nabla R|^2}{R^2} + 6trE^3 + R|E|^2 \\ &\quad - 6W_{ijk\ell}E_{ik}E_{j\ell} - 6B_{ij}E_{ij} \\ &= 3R\Delta\Big(\frac{\sigma_2(A)}{R}\Big) + 6trE^3 + R|E|^2 \\ &\quad - 6W_{ijk\ell}E_{ik}E_{j\ell} - 6B_{ij}E_{ij} \ . \end{aligned}$$

To establish (1.19), we first note

$$T_{ij}\nabla_i\nabla_j V = S_{ij}\nabla_i\nabla_j V - 3\frac{\sigma_2(A)}{R}\Delta V \ . \tag{1.20}$$

By the Bochner formula,

$$\Delta V = \frac{1}{2}\Delta|\nabla w|^2 = |\nabla^2 w|^2 + R_{ij}\nabla_i w\nabla_j w + \langle\nabla w, \nabla(\Delta w)\rangle \ . \tag{1.21}$$

From [CGY, (5.34)], we have

$$\nabla_i\nabla_j w = -\frac{1}{2}A_{ij} + \frac{1}{2}A^\circ_{ij} - \nabla_i w\nabla_j w + \frac{1}{2}|\nabla w|^2 g_{ij} \ , \tag{1.22}$$

so that

$$\begin{aligned} |\nabla^2 w|^2 &= \frac{1}{4}|A|^2 + \frac{1}{4}|A^\circ|^2 - \frac{1}{2}A_{ij}A^\circ_{ij} + |\nabla w|^4 \\ &\quad + A_{ij}\nabla_i w\nabla_j w - \frac{1}{6}R|\nabla w|^2 - A^\circ_{ij}\nabla_i w\nabla_j w \\ &\quad + \frac{1}{6}R_0|\nabla w|^2 \ . \end{aligned} \tag{1.23}$$

Also, tracing (1.22),

$$\Delta w = -\frac{1}{6}R + |\nabla w|^2 + \frac{1}{6}R_0 e^{-2w} \ ,$$

hence

$$\begin{aligned} \langle\nabla w, \nabla(\Delta w)\rangle &= -\frac{1}{6}\langle\nabla w, \nabla R\rangle + \langle\nabla w, \nabla|\nabla w|^2\rangle \\ &\quad + \frac{1}{6}\langle\nabla w, \nabla(R_0 e^{-2w})\rangle \\ &= -\frac{1}{6}\langle\nabla w, \nabla R\rangle + 2\nabla_i\nabla_j w\nabla_i w\nabla_j w \\ &\quad + \frac{1}{6}\langle\nabla w, \nabla(R_0 e^{-2w})\rangle \\ &= -\frac{1}{6}\langle\nabla w, \nabla R\rangle - A_{ij}\nabla_i w\nabla_j w + A^\circ_{ij}\nabla_i w\nabla_j w \\ &\quad - |\nabla w|^4 + \frac{1}{6}\langle\nabla w, \nabla(R_0 e^{-2w})\rangle \ . \end{aligned} \tag{1.24}$$

Therefore,

$$
\begin{aligned}
-3\frac{\sigma_2(A)}{R}\Delta V &= 3\varphi\Delta V \\
&= \frac{1}{2}\langle\nabla w, \nabla R\rangle\varphi - 3\varphi\left[\frac{1}{4}|A|^2 + \frac{1}{4}|A^\circ|^2\right. \\
&\quad - \frac{1}{2}A_{ij}A^\circ_{ij} - \frac{1}{6}R|\nabla w|^2 + \frac{1}{6}R_0|\nabla w|^2 \\
&\quad \left. + \frac{1}{6}\langle\nabla w, \nabla(R_0 e^{2w})\rangle\right] . \qquad (1.25)
\end{aligned}
$$

Notice the term $-\frac{1}{2}\langle\nabla w, \nabla\sigma_2(A)\rangle$ appearing in (1.17). since $\alpha = 1$ in (1.13),

$$
\begin{aligned}
-\frac{1}{2}\langle\nabla w, \nabla\sigma_2(A)\rangle &= -\frac{1}{2}\langle\nabla w, \nabla(R\varphi)\rangle \\
&= -\frac{1}{2}\langle\nabla w, \nabla R\rangle\varphi - \frac{1}{2}\langle\nabla w, \nabla\varphi\rangle R . \qquad (1.26)
\end{aligned}
$$

If we substitute (1.26) into (1.17), then add (1.25), we arrive at (1.19). □

Now let

$$F = R + 24V \ .$$

In the calculations which follow, C is a constant which depends at most on the quantities ρ, $\|\varphi\|_{C^2(B(\rho))}$, $\|g_0\|_{C^2(B(\rho))}$, $\|w\|_{C^1(B(\rho))}$. Since $\sigma_2(A) \geqslant 0$ and $R > 0$, note that we can bound the Ricci curvature by the scalar curvature; hence $|Ric| \lesssim R$, $|A| \lesssim R$, $|E| \lesssim R$. Also, observe that by (1.22), $|\nabla^2 w| \leqslant C|A| + C \leqslant CR + C$. With these facts in mind, from (1.13), (1.18), and (1.19), we see that

$$
\begin{aligned}
T_{ij}\nabla_i\nabla_j F &= T_{ij}\nabla_i\nabla_j R + 24T_{ij}\nabla_i\nabla_j V \\
&\geqslant 3R\Delta\varphi - \frac{1}{2}\langle\nabla w, \nabla\varphi\rangle R + \frac{1}{24}R^3 \\
&\quad - CR^2 - CR - C \ .
\end{aligned}
$$

Note that

$$
\begin{aligned}
|\Delta\varphi| &= |e^{-2w}\Delta_0\varphi + 2e^{-2w}\langle\nabla_0 w, \nabla_0\varphi\rangle| \\
&\leqslant C \ , \\
|\nabla\varphi| &= e^{-w}|\nabla_0\varphi| \leqslant C \ ,
\end{aligned}
$$

thus

$$T_{ij}\nabla_i\nabla_j F \geqslant \frac{1}{24}F^3 - CF^2 - CF - C \, .$$

Using the inequalities $-CF^2 \geqslant -\frac{1}{96}F^3 - C$, $-CF \geqslant -\frac{1}{96}F^3 - C$, we conclude

$$T_{ij}\nabla_i\nabla_j F \geqslant \frac{1}{48}F^3 - C\,. \tag{1.27}$$

When $\alpha = 0$ in (1.13) we still define $F = R + 24V$; using (1.16) and (1.17) directly and estimating as before, we find

$$S_{ij}\nabla_i\nabla_j F \geqslant \frac{1}{48}F^3 - C\,. \tag{1.28}$$

The conclusion of Proposition 1.1 now follows from applying the maximum principle to (1.27) and (1.28), after introducing a cut–off function η supported in $B(\rho)$, and estimating the maximum of ηF. Since the argument is quite standard, the details will be omitted. The result is an estimate for the scalar curvature R in $B(\rho/2)$; this implies a bound on $|A|$, and consequently bounds on $|\nabla^2 w|$, $|\nabla_0^2 w|$, and $|\partial_i\partial_j w|$. □

The same argument can be applied to give the following result:

Corollary 1.3. *Let g_0 be a Riemannian metric on $B(\rho)$ in R^4. Suppose $g = e^{2w}g_0$ satisfies*

$$\sigma_2(A_g) = \varphi_1 e^{-4w} + \varphi_2,$$
$$R > 0,$$

on $B(\rho)$, where $\varphi_1 \geq 0$ and $\varphi_2 \geq 0$. Then there is a a constant

$$C = C(\rho, \|g_0\|_{C^2(B(\rho))}, \|\varphi_1\|_{C^2(B(\rho))}\,, \|\varphi_2\|_{C^2(B(\rho))}\,, \|w\|_{L^\infty(B(\rho))}\,, \|\nabla w\|_{L^\infty(B(\rho))})$$

such that

$$\|\nabla^2 w\|_{L^\infty(B(\rho/2))} \leqslant C\,.$$

Applying the estimate of Proposition 1.1 to the sequences $g_k^\star = e^{2w_k}g_0^k$ and $\bar{g}_k = e^{2\overline{w}_k}g_0^k$ described above, we have

Corollary 1.4.

(i) *For the dilated and rescaled sequence $\bar{g}_k = e^{2\overline{w}_k}g_0^k$, there exists a $C^{1,1}$ conformal metric $\bar{g} = e^{2\overline{w}}ds^2$ with (a subsequence of) $w_k \to w$ in $C^{1,\beta}$ on compact subsets of $\mathbb{R}^4$, where $0 < \beta < 1$. Furthermore, $\bar{g}$ satisfies*

$$\sigma_2(A_{\bar{g}}) \equiv 0\,,$$

$$R_{\bar{g}} \geqslant 0\,, \tag{1.29}$$

$$|\nabla\overline{w}|(0) = 1\,. \tag{1.30}$$

(ii) *For the dilated sequence $g_k^\star = e^{2w_k}g_0^k$, there exists a C^∞ conformal metric $g = e^{2w}ds^2$ such that (a subsequence of) $w_k \to w$ in $C^{2,\beta}$ on compact subsets of $\mathbb{R}^4$. Furthermore, after possibly rescaling, g satisfies*

$$\frac{\sigma_2(A_g)}{R^\alpha} = \lambda_\alpha\,, \tag{1.31}$$
$$R > 0,$$

where $\lambda_\alpha = \frac{1}{2}(12)^{1-\alpha}$ and $\alpha = 0$ or 1.

2. Solutions on $\mathbb{R}^4$, Part 1

In this section we prove the following uniqueness result.

Theorem 2.1. *Suppose $g = e^{2w}ds^2$ is a conformal metric on $\mathbb{R}^4$ with $w \in C^{1,1}$, satisfying*

$$\sigma_2(A_g) = 0 \ , \tag{2.1}$$

$$R_g \geqslant 0 \ . \tag{2.2}$$

Then $w \equiv$ constant.

Remark. As a consequence of the above theorem, we see that the limiting metric $\bar{g} = e^{2\overline{w}}ds^2$, defined as the $C^{1,\beta}$–limit of the blow–up sequence described in Cor. 1.4(i), cannot occur.

Proof. Regularity considerations will complicate our arguments somewhat. We begin by deriving an estimate which holds for arbitrary smooth conformal metrics on $\mathbb{R}^4$; a limiting argument will imply that the same estimate holds for $C^{1,1}$ metrics. To this end, fix $\rho > 0$ and let η denote a cut–off function supported in $B(2\rho)$ satisfying $\eta \equiv 1$ on $B(\rho)$, $|\nabla \eta| \lesssim \rho^{-1}$, $|\nabla^2 \eta| \lesssim \rho^{-2}$. Let

$$\overline{w} = \frac{\int\limits_{B(2\rho)} w}{\int\limits_{B(2\rho)} 1}$$

denote the mean value of w on $B(2\rho)$. Since

$$A_g = -2\nabla^2 w + 2dw \otimes dw - |\nabla w|^2 \delta$$

where δ is the identity, we have

$$\begin{aligned} \sigma_2(A_g) &= \sigma_2(-2\nabla^2 w + 2dw \otimes dw - |\nabla w|^2 \delta) \\ &= e^{-4w}\{-2|\nabla^2 w|^2 + 2(\Delta w)^2 + 4\nabla^2 w(\nabla w, \nabla w) + 2\Delta w |\nabla w|^2\} \ . \end{aligned} \tag{2.3}$$

Using the Bochner identity

$$\tfrac{1}{2}\Delta|\nabla w|^2 = |\nabla^2 w|^2 + \langle \nabla w, \nabla(\Delta w)\rangle \ ,$$

we can rewrite (2.3) as

$$\begin{aligned} \sigma_2(A_g)e^{4w} = &-\Delta|\nabla w|^2 + 2\langle \nabla w, \nabla(\Delta w)\rangle + 2(\Delta w)^2 \\ &+ 4\nabla^2 w(\nabla w, \nabla w) + 2\Delta w|\nabla w|^2 \ . \end{aligned} \tag{2.4}$$

Now multiply both sides of (2.4) by $\eta^4(w-\overline{w})$ to get

$$\begin{aligned}\int (w-\overline{w})\sigma_2(A_g)e^{4w}\eta^4 = \int & -\eta^4(w-\overline{w})\Delta|\nabla w|^2 \\ & + 2\eta^4(w-\overline{w})\langle\nabla w, \nabla(\Delta w)\rangle + 2\eta^4(w-\overline{w})(\Delta w)^2 \\ & + 4\eta^4(w-\overline{w})\nabla^2 w(\nabla w, \nabla w) + 2\eta^4(w-\overline{w})\Delta w|\nabla w|^2 \; .\end{aligned} \tag{2.5}$$

Integrating by parts gives

$$\begin{aligned}\int -\eta^4(w-\overline{w})\Delta|\nabla w|^2 &= \int -|\nabla w|^2\Delta[\eta^4(w-\overline{w})] \\ &= \int -|\nabla w|^2[4\eta^3\Delta\eta(w-\overline{w}) + 12\eta^2(w-\overline{w})|\nabla\eta|^2 \\ &\quad + 8\eta^3\langle\nabla\eta, \nabla w\rangle + \eta^4\Delta w] \\ &= \int -4\eta^3\Delta\eta(w-\overline{w})|\nabla w|^2 - 12\eta^2|\nabla\eta|^2(w-\overline{w})|\nabla w|^2 \\ &\quad - 8\eta^3\langle\nabla\eta, \nabla w\rangle|\nabla w|^2 - \eta^4|\nabla w|^2\Delta w \; ,\end{aligned} \tag{2.6}$$

$$\begin{aligned}\int 2\eta^4(w-\overline{w})\langle\nabla w, \nabla(\Delta w)\rangle = \int & -2\eta^4(w-\overline{w})(\Delta w)^2 - 2\eta^4\Delta w|\nabla w|^2 \\ & - 8\eta^3(w-\overline{w})\langle\nabla\eta, \nabla w\rangle\Delta w \; ,\end{aligned} \tag{2.7}$$

$$\begin{aligned}\int 4\eta^4(w-\overline{w})\nabla^2 w(\nabla w, \nabla w) &= \int 2\eta^4(w-\overline{w})\langle\nabla w, \nabla|\nabla w|^2\rangle \\ &= \int -2\eta^4(w-\overline{w})\Delta w|\nabla w|^2 - 2\eta^4|\nabla w|^4 \\ &\quad - 8\eta^3(w-\overline{w})\langle\nabla\eta, \nabla w\rangle|\nabla w|^2 \; .\end{aligned} \tag{2.8}$$

Substituting (2.6)–(2.8) into (2.5) we have

$$\begin{aligned}\int (w-\overline{w})\sigma_2(A_g)e^{4w}\eta^4 = \int & -4\eta^3\Delta\eta(w-\overline{w})|\nabla w|^2 \\ & - 12\eta^2(w-\overline{w})|\nabla\eta|^2|\nabla w|^2 - 8\eta^3\langle\nabla\eta, \nabla w\rangle|\nabla w|^2 \\ & - 8\eta^3(w-\overline{w})\langle\nabla\eta, \nabla w\rangle|\nabla w|^2 - 8\eta^3(w-\overline{w})\langle\nabla\eta, \nabla w\rangle\Delta w \\ & - 3\eta^4|\nabla w|^2\Delta w - 2\eta^4|\nabla w|^4\end{aligned} \tag{2.9}$$

The last three terms in (2.9) require special attention. First, integrating by parts we can write

$$\begin{aligned}
&\int -8\eta^3(w-\overline{w})\langle\nabla\eta,\nabla w\rangle\Delta w \\
&\quad = \int -8\eta^3(w-\overline{w})\partial_i\eta\partial_i w\partial_j\partial_j w \\
&\quad = \int 8\eta^3(w-\overline{w})\partial_i\eta\partial_j\partial_i w\partial_j w + 8\eta^3(w-\overline{w})\partial_j\partial_i\eta\partial_i w\partial_j w \\
&\qquad + 8\eta^3\partial_j w\partial_i\eta\partial_i w\partial_j w + 24\eta^2(w-\overline{w})\partial_j\eta\partial_i\eta\partial_i w\partial_j w \\
&\quad = \int 4\eta^3(w-\overline{w})\langle\nabla\eta,\nabla|\nabla w|^2\rangle + 8\eta^3(w-\overline{w})\nabla^2\eta(\nabla w,\nabla w) \\
&\qquad + 8\eta^3|\nabla w|^2\langle\nabla\eta,\nabla w\rangle + 24\eta^2(w-\overline{w})|\langle\nabla\eta,\nabla w\rangle|^2 \\
&\quad = \int -4\eta^3(w-\overline{w})\Delta\eta|\nabla w|^2 - 4\eta^3|\nabla w|^2\langle\nabla\eta,\nabla w\rangle \\
&\qquad - 12\eta^2(w-\overline{w})|\nabla w|^2|\nabla\eta|^2 + 8\eta^3(w-\overline{w})\nabla^2\eta(\nabla w,\nabla w) \\
&\qquad + 8\eta^3|\nabla w|^2\langle\nabla\eta,\nabla w) + 24\eta^2(w-\overline{w})|\langle\nabla\eta,\nabla w\rangle|^2 \\
&\quad = \int -4\eta^3(w-\overline{w})\Delta\eta|\nabla w|^2 + 4\eta^3|\nabla w|^2\langle\nabla\eta,\nabla w\rangle \\
&\qquad - 12\eta^2(w-\overline{w})|\nabla w|^2|\nabla\eta|^2 + 8\eta^3(w-\overline{w})\nabla^2\eta(\nabla w,\nabla w) \\
&\qquad + 24\eta^2(w-\overline{w})|\langle\nabla\eta,\nabla w\rangle|^2 . \qquad (2.10)
\end{aligned}$$

For the last two terms in (2.9) we use the scalar curvature equation

$$\Delta w + |\nabla w|^2 + \tfrac{1}{6}Re^{2w} = 0 \qquad (2.11)$$

to conclude

$$\begin{aligned}
&\int -3\eta^4|\nabla w|^2\Delta w - 2\eta^4|\nabla w|^4 \\
&\quad = \int -3\eta^4|\nabla w|^2(-|\nabla w|^2 - \tfrac{1}{6}Re^{2w}) - 2\eta^4|\Delta w|^4 \\
&\quad = \int \tfrac{1}{2}Re^{2w}|\nabla w|^2\eta^4 + \eta^4|\nabla w|^4 . \qquad (2.12)
\end{aligned}$$

Substituting (2.10) and (2.12) into (2.9) we arrive at

$$\begin{aligned}
\int (w-\overline{w})\sigma_2(A_g)e^{4w}\eta^4 = \int & -8\eta^3\Delta\eta(w-\overline{w})|\nabla w|^2 \\
& - 24\eta^2(w-\overline{w})|\nabla\eta|^2|\nabla w|^2 + 24\eta^2(w-\overline{w})|\langle\nabla\eta,\nabla w\rangle|^2 \\
& + 8\eta^3(w-\overline{w})\nabla^2\eta(\nabla w,\nabla w) - 4\eta^3\langle\nabla\eta,\nabla w\rangle|\nabla w|^2 \\
& - 8\eta^3(w-\overline{w})\langle\nabla\eta,\nabla w\rangle|\nabla w|^2 \\
& + \tfrac{1}{2}Re^{2w}|\nabla w|^2\eta^4 + \eta^4|\nabla w|^4 . \qquad (2.13)
\end{aligned}$$

Using the properties of η we can estimate several of the terms on the RHS (2.13) in order to conclude that

$$\int \tfrac{1}{2}Re^{2w}|\nabla w|^2\eta^4 + \eta^4|\nabla w|^4 \leqslant \int (w-\overline{w})\sigma_2(A)e^{4w}\eta^4 + C\rho^{-2}\int_{A(\rho)} |w-\overline{w}||\nabla w|^2\eta^2 + c\rho^{-1}\int_{A(\rho)} |w-\overline{w}||\nabla w|^3\eta^3 \ , \tag{2.14}$$

where $A(\rho) = B(2\rho)\setminus B(\rho)$. In deriving (2.14) we have made no special assumptions about the metric $g = e^{2w}ds^2$, other than smoothness (in order to justify the integration by parts). We now begin to specialize to the case of interest.

First, note that if w is just $C^{1,1}$, then (2.14) holds for any mollification of w. A limiting argument then implies that (2.14) is equally valid for any $C^{1,1}$–metric $g = e^{2w}ds^2$. Moreover, if we assume that g satisfies (2.1) and (2.2), then

$$\int \eta^4|\nabla w|^4 \leqslant C\rho^{-2}\int_{A(\rho)} |w-\overline{w}||\nabla w|^2\eta^2 + C\rho^{-1}\int_{A(\rho)} |w-\overline{w}||\nabla w|^3\eta^3 \ . \tag{2.15}$$

We now proceed to estimate the last two integrals in (2.15). First,

$$C\rho^{-2}\int_{A(\rho)} |w-\overline{w}||\nabla w|^2\eta^2 \leqslant C\rho^{-2}\Big(\int_{A(\rho)} |w-\overline{w}|^2\Big)^{1/2}\Big(\int_{A(\rho)} \eta^4|\nabla w|^4\Big)^{1/2} \ . \tag{2.16}$$

By the Poincaré inequality on $B(2\rho)$, we see that

$$\int_{A(\rho)} (w-\overline{w})^2 \leqslant \int_{B(2\rho)} (w-\overline{w})^2 \leqslant C\rho^2 \int_{B(2\rho)} |\nabla w|^2 \ .$$

Let ξ be a cut–off function supported in $B(3\rho)$ satisfying $\xi \equiv 1$ on $B(2\rho)$, $|\nabla\xi| \lesssim \rho^{-1}$. Referring back to (2.11), we have

$$\int \xi^2\Delta w + \xi^2|\nabla w|^2 + \tfrac{1}{6}Re^{2w}\xi^2 = 0 \ .$$

Since $R \geqslant 0$,

$$\begin{aligned}\int \xi^2|\nabla w|^2 &\leqslant \int -\xi^2\Delta w \\ &= \int 2\xi\nabla\xi\nabla w \\ &\leqslant \int \tfrac{1}{2}\,\xi^2|\nabla w|^2 + 2|\nabla\xi|^2\end{aligned}$$

$\Rightarrow$

$$\int \xi^2|\nabla w|^2 \leqslant C\rho^2 \ .$$

Therefore,

$$\int_{B(2\rho)} |\nabla w|^2 \leqslant C\rho^2 \ , \tag{2.17}$$

so that

$$\int_{A(\rho)} (w - \overline{w})^2 \leqslant \int_{B(2\rho)} (w - \overline{w})^2 \leqslant C\rho^4 \ . \tag{2.18}$$

Substituting this into (2.16), we conclude

$$C\rho^{-2} \int_{A(\rho)} |w - \overline{w}||\nabla w|^2\eta^2 \leqslant C\Big(\int_{A(\rho)} \eta^4 |\nabla w|^4 \Big)^{1/2} \ . \tag{2.19}$$

For the last integral in (2.15) we have

$$C\rho^{-1} \int_{A(\rho)} |w - \overline{w}||\nabla w|^3\eta^3 \leqslant C\rho^{-1}\Big(\int_{A(\rho)} (w - \overline{w})^2 |\nabla w|^2 \eta^2 \Big)^{1/2} \Big(\int_{A(\rho)} \eta^4 |\nabla w|^4 \Big)^{1/2}. \tag{2.20}$$

Referring to (2.11) once again, and arguing as before, we find

$$\int (\Delta w + |\nabla w|^2 + \tfrac{1}{6} R e^{2w}) \eta^2 (w - \overline{w})^2 = 0$$

$\Rightarrow$

$$\begin{aligned} \int |\nabla w|^2 \eta^2 (w - \overline{w})^2 &\leqslant \int -\Delta w \, \eta^2 (w - \overline{w})^2 \\ &= \int 2\eta (w - \overline{w})^2 \nabla\eta \nabla w \\ &\qquad + 2(w - \overline{w})\eta^2 |\nabla w|^2 \\ &\leqslant \int \tfrac{1}{4} |\nabla w|^2 \eta^2 (w - \overline{w})^2 + 4|\nabla \eta|^2 (w - \overline{w})^2 \\ &\qquad + \tfrac{1}{4} |\nabla w|^2 \eta^2 (w - \overline{w})^2 + |\nabla w|^2 \eta^2 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \int |\nabla w|^2 \eta^2 (w - \overline{w})^2 &\lesssim \rho^{-2} \int_{A(\rho)} |w - \overline{w}|^2 + \int_{B(2\rho)} |\nabla w|^2 \\ &\lesssim \rho^2 \ , \end{aligned}$$

the last line following from (2.17) and (2.18). If we substitute this into (2.20),

$$C\rho^{-1} \int_{A(\rho)} |w - \overline{w}||\nabla w|^3\eta^3 \leqslant C\Big(\int_{A(\rho)} \eta^4 |\nabla w|^4 \Big)^{1/2} \ . \tag{2.21}$$

Combining (2.15), (2.19), and (2.21), we conclude

$$\int \eta^4 |\nabla w|^4 \leqslant C \Big(\int_{A(\rho)} \eta^4 |\nabla w|^4 \Big)^{1/2} . \tag{2.22}$$

From (2.22) it easily follows that $w \equiv$ constant. First, notice that (2.22) implies that

$$\int_{\mathbb{R}^4} |\nabla w|^4 < \infty .$$

In particular,

$$\int_{A(\rho)} |\nabla w|^4 = \int_{B(2\rho)\setminus B(\rho)} |\nabla w|^4 \to 0 \quad \text{as} \quad \rho \to \infty ,$$

which by (2.22) implies $|\nabla w| \equiv 0$. □

3. Solutions on $\mathbb{R}^4$, Part 2

In this section we provide a classification of all solutions on $\mathbb{R}^4$ of

$$\begin{aligned} &\frac{\sigma_2(A_g)}{R^\alpha} = \lambda_\alpha, \\ &g = u^2 ds^2, \end{aligned} \tag{3.1}$$

where $\lambda_\alpha = \frac{1}{2}(12)^{1-\alpha}$, $\alpha = 0$ or 1, and if $\alpha = 1$ we assume, in addition, that $R > 0$. As we shall see, all such solutions are obtained by pulling back the round metric on the sphere (and its images under the conformal group) under stereographic projection.

Our method is inspired by the corresponding uniqueness result of Obata [Ob] for the scalar curvature. Indeed, to emphasize the parallel between our arguments, it will be helpful to partially reproduce his. To this end, suppose $g = u^2 g_0$ is a conformal metric on S^4, where g_0 is the round metric. Assume that g has constant scalar curvature.

To begin, we write the formula which expresses the trace–free Ricci tensor E of g in terms of u:

$$E = -2u^{-1}\nabla_g^2 u + \tfrac{1}{2}u^{-1}\Delta_g u\, g . \tag{3.2}$$

Note that the Hessian and Laplacian in (3.2) are with respect to g, not g_0. If we pair both sides of (3.2) with uE and integrate over S^4 we obtain

$$\int_{S^4} |E|^2 u \, dvol_g = -2 \int_{S^4} g(E, \nabla_g^2 u) dvol_g .$$

Note that the second terms in (3.2) vanishes when we pair with uE because E *is trace–free.* We now apply the divergence theorem to conclude

$$\int_{S^4} |E|^2 u\, dvol_g = 2 \int_{S^4} g(\delta E, du) dvol_g \ .$$

The contracted second Bianchi identity says that $\delta E = \frac{1}{4} dR$, where R is the scalar curvature. Since R is constant, E *is divergence–free.* Thus

$$\int_{S^4} |E|^2 u\, dvol_g = 0 \ . \tag{3.3}$$

The uniqueness result follows, since (3.3) implies that g has constant curvature.

In our setting, we need a tensor which plays the same role that the trace–free Ricci tensor does in Obata's proof. The first step in describing such a tensor is the following result.

Proposition 3.1. *(See* [Gu, Thm. B]*). Suppose* (M^4, g) *is locally conformally flat. Define the symmetric two–tensor* L *by*

$$L = \tfrac{1}{4}|E|^2 g + \tfrac{1}{6} RE - E^2 \ .$$

Then L *satisfies*

$$tr_g L = 0, \tag{3.4}$$

$$\delta L = \tfrac{1}{2} d\, \sigma_2(A) \ . \tag{3.5}$$

Proof. Just take $\gamma_2 = 1$ and $\gamma_3 = -\frac{1}{12}$ in [Gu, Thm. B]. □

Now define the tensor

$$Z^\alpha = L - 2\alpha \tfrac{\sigma_2(A)}{R} E \ . \tag{3.6}$$

Lemma 3.2. *If* $g = u^2 ds^2$ *satisfies (3.1), then* Z^α *satisfies*

$$tr_g Z^\alpha = 0, \tag{3.7}$$

$$\delta Z^\alpha = 0 \,. \tag{3.8}$$

Proof. (3.7) is obvious from (3.4). Using (3.5),

$$\begin{aligned} \delta Z^\alpha &= \delta \left[L - 2\alpha \frac{\sigma_2(A)}{R} E \right] \\ &= \delta L - 2\alpha \frac{\sigma_2(A)}{R} \delta E \\ &= \tfrac{1}{2} d\, \sigma_2(A) - \frac{\alpha}{2} \frac{\alpha_2(A)}{R} dR \\ &= 0, \end{aligned}$$

since $\alpha = 0$ or 1. □

For our proof we will need two additional sharp inequalities involving Z^α.

Proposition 3.3. *Assume $\sigma_2(A) > 0$ and $R > 0$. Then*
(i) $g(Z^\alpha, E) \geqslant 0$, *with equality if, and only if,* $E = 0$.
(ii) $|Z^\alpha|^2 \leqslant \frac{1}{3} R g(Z^\alpha, E)$.

Proof.
(i) By (3.6),

$$\begin{aligned} g(Z^\alpha, E) &= g(L, E) - 2\alpha \frac{\sigma_2(A)}{R} |E|^2 \\ &= \tfrac{1}{6} R|E|^2 - trE^3 - 2\alpha \frac{\sigma_2(A)}{R}|E|^2 \ . \end{aligned} \tag{3.9}$$

Using the sharp inequality $|trE^3| \leqslant \frac{1}{\sqrt{3}}|E|^3$, we have

$$\begin{aligned} g(Z^\alpha, E) &\geqslant \tfrac{1}{6} R|E|^2 - \tfrac{1}{\sqrt{3}}|E|^3 - 2\alpha\frac{\sigma_2(A)}{R}|E|^2 \\ &= \tfrac{1}{6} R|E|^2 - \tfrac{1}{\sqrt{3}}|E|^3 - 2\alpha\frac{\sigma_2(A)}{R}(-\tfrac{1}{2}|E|^2 + \tfrac{1}{24}R^2) \\ &= \tfrac{1}{12}(2-\alpha)R|E|^2 - \tfrac{1}{\sqrt{3}}|E|^3 + \alpha\frac{|E|^4}{R} \\ &= \tfrac{1}{6}(1-\alpha)R|E|^2 - \tfrac{(1-\alpha)}{\sqrt{3}}|E|^3 \\ &\quad + \frac{\alpha}{R}(|E|^4 - \tfrac{1}{\sqrt{3}}R|E|^3 + \tfrac{1}{12}R^2|E|^2) \\ &= \tfrac{(1-\alpha)}{\sqrt{3}}|E|^2(-|E| + \tfrac{1}{2\sqrt{3}}R)^2 \\ &\quad + \alpha\frac{|E|^2}{R}(|E - \tfrac{1}{2\sqrt{3}}R)^2 \ . \end{aligned} \tag{3.10}$$

Now,

$$0 < 2\sigma_2(A) = -|E|^2 + \tfrac{1}{12}R^2 = \big(-|E| + \tfrac{1}{2\sqrt{3}}R\big)\big(|E| + \tfrac{1}{2\sqrt{3}}R\big) \ ,$$

so $-|E| + \frac{1}{2\sqrt{3}}R > 0$. Since $\alpha = 0$ or $\alpha = 1$, both terms on the RHS of (3.10) are non–negative, and their sum vanishes if, and only if, $E = 0$.

(ii) We have

$$|Z^\alpha|^2 = |L|^2 + 4\alpha^2 \frac{\sigma_2(A)^2}{R^2}|E|^2 - 4\alpha\frac{\sigma_2(A)}{R} g(L, E) \tag{3.11}$$

while

$$|L|^2 = |E^2|^2 - \tfrac{1}{4}|E|^4 + \tfrac{1}{36}R^2|E|^2 - \tfrac{1}{3}RtrE^3 \ . \tag{3.12}$$

We will need the following lemma:

Lemma 3.4.

$$|E^2|^2 \leqslant \tfrac{7}{12}|E|^4 \ . \tag{3.13}$$

Proof. This amounts to a Lagrange–multiplier problem. Fix a point and diagonalize E:

$$E = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix}.$$

Since E is trace–free,

$$\lambda_4 = -\sum_{i=1}^{3} \lambda_i \ .$$

It suffices to establish (3.13) assuming $|E|^2 = 1$. Therefore, we must show that

$$f(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \equiv \lambda_1^4 + \lambda_2^4 + \lambda_3^4 + \left(\sum_{i=1}^{3} \lambda_i \right)^4 \leqslant \tfrac{7}{12}$$

subject to the constraint

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \left(\sum_{i=1}^{3} \lambda_i \right)^2 = 1 \ .$$

The associated Euler–Lagrange equation is then

$$\lambda_j^3 + \left(\sum_{i=1}^{3} \lambda_i \right)^3 = \mu \left(\lambda_j + \sum_{i=1}^{3} \lambda_i \right) , \tag{3.14}$$

$j = 1, 2, 3$; where μ is the Lagrange multiplier.

First, suppose

$$\lambda_j + \sum_{i=1}^{3} \lambda_i \neq 0 \tag{3.15}$$

for each $j = 1, 2, 3$. Using the common value of μ from (3.14) we find

$$\mu = \frac{\lambda_j^3 + (\Sigma\lambda_i)^3}{\lambda_j + \Sigma\lambda_i} = \lambda_j^2 - \lambda_j(\Sigma\lambda_i) + (\Sigma\lambda_i)^2 \ ,$$

so that

$$\lambda_j^2 - \lambda_j(\Sigma\lambda_i) + (\Sigma\lambda_i)^2 = \lambda_k^2 - \lambda_k(\Sigma\lambda_i) + (\Sigma\lambda_i)^2$$

for all j, k. Thus,

$$\lambda_1(\lambda_2 + \lambda_3) = \lambda_2(\lambda_1 + \lambda_3) = \lambda_3(\lambda_1 + \lambda_2) \ . \tag{3.16}$$

If one of the eigenvalues is zero, say $\lambda_1 = 0$, then by (3.16) we see that another eigenvalue must vanish — say $\lambda_2 = 0$. Then $\lambda_3 = \pm\frac{1}{\sqrt{2}}$, $\lambda_4 = -\lambda_3$, and $f(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{1}{2}$. However, if $\lambda_j \neq 0$, then from (3.16) we conclude that $\lambda_1 = \lambda_2 = \lambda_3$, and $f(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{7}{12}$.

Returning to the assumption (3.15), if for some j, say $j = 3$, we have

$$\lambda_3 + \sum \lambda_i = 0 \ ,$$

then rewriting (3.14) for $j = 1, 2$ we find

$$\lambda_j^3 - \lambda_3^3 = \mu(\lambda_j - \lambda_3) \ .$$

If $\lambda_1 = \lambda_3$, it follows that $\lambda_2 = -3\lambda_3$. If $\lambda_2 = \lambda_3$, it follows that $\lambda_1 = -3\lambda_3$. In either case E is conjugate to the matrix

$$E = \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & -3\lambda \end{pmatrix} .$$

If $\lambda_1 \neq \lambda_3$ and $\lambda_2 \neq \lambda_3$, we find, using the common value of μ,

$$\lambda_2(\lambda_2 + \lambda_3) = \lambda_1(\lambda_1 + \lambda_3) \ .$$

Hence either $\lambda_1 = \lambda_2$, so that $\lambda_3 = -\lambda_1$ and $f(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{1}{4} < \frac{7}{12}$, or $\lambda_1 + \lambda_2 = -\lambda_3$, then $\lambda_3 = 0$, and $\lambda_1 = -\lambda_2$. Then $f(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = 2{\lambda_1}^4$ with $|E|^2 = 2{\lambda_1}^2 = 1$. Thus $f(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{1}{4} < \frac{7}{12}$. □

Substituting (3.13) into (3.12), we find

$$|L|^2 \leqslant \tfrac{1}{3}|E|^4 + \tfrac{1}{36}R^2|E|^2 - \tfrac{1}{3}R\, tr E^3 \ . \tag{3.17}$$

Thus, from (3.11),

$$\begin{aligned} |Z^\alpha|^2 &\leqslant \tfrac{1}{3}|E|^4 + \tfrac{1}{36}R^2|E|^2 - \tfrac{1}{3}R\, tr E^3 \\ &\quad + 4\alpha^2 \frac{\sigma_2(A)^2}{R^2}|E|^2 - 4\alpha\frac{\sigma_2(A)}{R} g(L, E) \\ &= \tfrac{1}{3}|E|^4 + \tfrac{1}{36}R^2|E|^2 - \tfrac{1}{3}R\, tr E^3 \\ &\quad + 4\alpha^2 \frac{\sigma_2(A)^2}{R^2}|E|^2 - 4\alpha\frac{\sigma_2(A)}{R} g\left(Z^\alpha + 2\alpha\frac{\sigma_2(A)}{R}E, E\right) \\ &= \tfrac{1}{3}|E|^4 + \tfrac{1}{36}R^2|E|^2 - \tfrac{1}{3}R\, tr E^3 \\ &\quad - 4\alpha^2 \frac{\sigma_2(A)^2}{R^2}|E|^2 - 4\alpha\frac{\sigma_2(A)}{R} g(Z^\alpha, E) \\ &\leqslant \tfrac{1}{3}|E|^4 + \tfrac{1}{36}R^2|E|^2 - \tfrac{1}{3}R\, tr E^3 \ . \end{aligned}$$

Therefore,

$$|Z^\alpha|^2 - \tfrac{1}{3}R\, g(Z^\alpha, E) \leqslant \tfrac{1}{3}|E|^4 - \tfrac{1}{36}R^2|E|^2 + \tfrac{2}{3}\sigma_2(A)|E|^2 = 0 \ . \qquad \square$$

We are now prepared to prove the classification described at the beginning of the section. First, recall that when $\alpha = 1$ in (3.1) we assume that $R > 0$. However, if $\alpha = 0$, it turns out that the scalar curvature must be positive as well. Indeed, in either case it must be *strictly* positive.

Lemma 3.5. *Suppose $g = u^2 ds^2$ satisfies (3.1) with $\lambda_\alpha = \frac{1}{2}(12)^{1-\alpha}$, $\alpha = 0$ or 1, and if $\alpha = 1$ we assume, in addition, that $R > 0$. Then the scalar curvature satisfies*

$$R \geqslant 12 \tag{3.18}$$

Proof. If $R > 0$, then (3.1) implies

$$\tfrac{1}{24}R^2 \geqslant \sigma_2(Ag) = \lambda_\alpha R^\alpha \tag{3.19}$$

and (3.18) follows. Therefore, our only task is to show that $R > 0$ when $\alpha = 0$.

Note that (3.19) implies that either $R \geqslant 12$ or $R \leqslant -12$ on $\mathbb{R}^4$. To rule out the latter case, we appeal to the scalar curvature equation

$$\Delta u + \tfrac{1}{6}Ru^3 = 0 \ .$$

If $R \leqslant -12$, then

$$2u^3 \leqslant \Delta u \ . \tag{3.20}$$

Let $\rho \gg 1$ and η denote a cut–off function supported in $B(2\rho)$ satisfying $\eta \equiv 1$ on $B(\rho)$, $|\nabla\eta| \lesssim \rho^{-1}$. Multiplying both sides of (3.20) by $u\eta^4$ and integrating we find

$$\begin{aligned} \int 2u^4\eta^4 &\leqslant \int u\eta^4\Delta u = \int -4\eta^3 u\nabla\eta\nabla u - \eta^4|\nabla u|^2 \\ &\leqslant \int (4\eta^2u^2|\nabla\eta|^2 + \eta^4|\nabla u|^2) - \eta^4|\nabla u|^2 \\ &\lesssim \rho^{-2}\int_{\rho<|x|<2\rho} \eta^2u^2 \lesssim \rho^{-2}\left(\int_{\rho<|x|<2\rho}\eta^4u^4\right)^{1/2}\left(\int_{\rho<|x<2\rho}\right)^{1/2} \lesssim \left(\int_{\rho<|x|<2\rho}\eta^4u^4\right)^{1/2} . \end{aligned} \tag{3.21}$$

Now, (3.21) obviously implies that $u \in L^4(\mathbb{R}^4)$. But this in turn implies that the RHS of (3.21) approaches zero as $\rho \to \infty$. Since $u > 0$, we conclude that R cannot be negative. $\square$

Theorem 3.6. *Let $g = u^2 ds^2$ be a solution to (3.1), where $\alpha = 0$ or 1. Then $u(x) = (a|x|^2 + b_i x^i + c)^{-1}$ for constants a, b_i, c. In particular, g is obtained by pulling back to $\mathbb{R}^4$ the round metric on S^4 (or its image under a conformal map).*

Proof. To begin, fix $\rho > 1$ and let η denote a cut–off function supported in $B(2\rho)$ satisfying $\eta \equiv 1$ on $B(\rho)$, $|\nabla \eta| \lesssim \rho^{-1}$. As outlined above, we write the formula for the trace–free Ricci tensor E of g in terms of u:

$$E = -2u^{-1}\nabla_g^2 u + \tfrac{1}{2}u^{-1}\Delta_g u g \ . \tag{3.22}$$

Notice that in (3.22), the Hessian and Laplacian are with respect to g, not the Euclidean metric. Next we pair both sides with $-u\eta^2 Z^\alpha$ to get

$$\int g(Z^\alpha, E)u\eta^2 dvol_g = \int -2g(Z^\alpha, \nabla_g^2 u)\eta^2 dvol_g \ . \tag{3.23}$$

Note that in (3.23) we have used the fact that Z^α is trace–free. Applying the divergence theorem,

$$\begin{aligned} \int g(Z^\alpha, E)u\eta^2 dvol_g &= \int 2g(\delta Z^\alpha, du)\eta^2 dvol_g \\ &\quad + \int 2Z^\alpha(\nabla_g u, \nabla_g(\eta^2))dvol_g \ . \end{aligned}$$

By Lemma 3.2, $\delta Z^\alpha = 0$. Thus

$$\begin{aligned} \int g(Z^\alpha, E)u\eta^2 dvol_g &= \int 2Z^\alpha(\nabla_g u, \nabla_g(\eta^2))dvol_g \\ &= \int 4Z^\alpha(\nabla_g u, \nabla_g \eta)\eta \, dvol_g \\ &\leq \int 4|Z^\alpha||\nabla_g u||\nabla_g \eta|\eta \, dvol_g \ . \end{aligned}$$

Using Proposition 3.3 (ii), we conclude

$$\int g(Z^\alpha, E)u\eta^2 \, dvol_g \leqslant \tfrac{4}{\sqrt{3}} \int R^{1/2}(g(Z^\alpha, E))^{1/2}|\nabla_g u||\nabla_g \eta|\eta \, dvol_g \ .$$

By the Schwartz inequality,

$$\begin{aligned} \int g(Z^\alpha, E)u\eta^2 dvol_g &\leqslant \tfrac{4}{\sqrt{3}}\Big(\int_{supp|\nabla\eta|} g(Z^\alpha, E)u\eta^2 dvol_g \Big)^{1/2} \\ &\quad \times \Big(\int R|\nabla_g u|^2 u^{-1}|\nabla_g \eta|^2 dvol_g \Big)^{1/2} \ , \end{aligned} \tag{3.24}$$

which, of course, implies

$$\int g(Z^\alpha, E)u\eta^2 dvol_g \lesssim \int R|\nabla_g u|^2 u^{-1}|\nabla_g \eta|^2 dvol_g \ . \tag{3.25}$$

We now rewrite the integral on the RHS of (3.25) in terms of the Euclidean metric, using the identities

$$\begin{aligned} |\nabla_g u|^2 &= u^{-2}|\nabla u|^2 \ , \\ |\nabla_g \eta|^2 &= u^{-2}|\nabla \eta|^2 \ , \\ dvol_g &= u^4 dx \ . \end{aligned}$$

Thus,

$$\int R|\nabla_g u|^2 u^{-1}|\nabla_g \eta|^2 dvol_g = \int R|\nabla u|^2 u^{-1}|\nabla \eta|^2 dx \ .$$

Since $|\nabla\eta|^2 \lesssim \rho^{-2}$ and $supp\, \eta \subseteq B(2\rho) \setminus B(\rho)$, this implies

$$\int R|\nabla_g u|^2 u^{-1}|\nabla_g \eta|^2 dvol_g \lesssim \rho^{-2} \int_{B(2\rho)\setminus B(\rho)} R|\nabla u|^2 u^{-1} dx \ . \tag{3.26}$$

We claim that the RHS of (3.26) is bounded independent of ρ. Assuming this for the moment, we conclude by (3.25) that

$$\int_{\mathbb{R}^4} g(Z^\alpha, E)u\, dvol_g < \infty \ .$$

In particular,

$$\int_{supp|\nabla\eta|} g(Z^\alpha, E)\, u\, dvol_g \to 0 \quad \text{as} \quad \rho \to \infty \ . \tag{3.27}$$

Combining (3.27) with (3.24) and the boundedness of the integrals in (3.26) we conclude

$$\int g(Z^\alpha, E)u\eta^2\, dvol_g \to 0 \quad \text{as} \quad \rho \to \infty \ .$$

Therefore, $g(Z^\alpha, E) \equiv 0$ on $\mathbb{R}^4$, so by Proposition 3.3(i), $E \equiv 0$. The conclusion of Theorem 3.6 now follows from standard arguments. □

We now return to the estimate in (3.26)

Proposition 3.7. *There is a constant c_1 such that for any $\rho > 0$,*

$$\int_{B(2\rho)\setminus B(\rho)} R|\nabla u|^2 u^{-1} dx \leqslant c_1 \rho^2 \ . \tag{3.28}$$

Proof. It will simplify our calculations somewhat if we let $u = e^w$ and establish the equivalent estimate

$$\int_{B(2\rho)\setminus B(\rho)} Re^w |\nabla w|^2 \leqslant c_1 \rho^2 \; .$$

If $g = e^{2w} ds^2$, then

$$A = -2\nabla^2 w + 2dw \otimes dw - |\nabla w|^2 \delta \; ,$$

where δ is the identity. Therefore,

$$\begin{aligned} \sigma_2(A) &= \sigma_2(-2\nabla^2 w + 2dw \otimes dw - |\nabla w|^2 \delta) \\ &= e^{-4w} \left\{ -2|\nabla^2 w|^2 + 2(\Delta w)^2 + 4\nabla^2 w(\nabla w, \nabla w) + 2\Delta w |\nabla w|^2 \right\} \; . \end{aligned} \tag{3.29}$$

Using the Bochner identity

$$\tfrac{1}{2}\Delta |\nabla w|^2 = |\nabla^2 w|^2 + \langle \nabla w, \nabla(\Delta w) \rangle \; ,$$

we rewrite (3.29) as

$$\begin{aligned} \sigma_2(A) e^{4w} &= -\Delta |\nabla w|^2 + 2\langle \nabla w, \nabla(\nabla w) \rangle + 2(\Delta w)^2 \\ &\quad + 4\nabla^2 w(\nabla w, \nabla w) + 2\Delta w |\nabla w|^2 \; . \end{aligned} \tag{3.30}$$

Now, fix $\rho > 0$ and denote $A(\rho) = \{x \in \mathbb{R}^4 | \rho < |x| < 2\rho\}$. Let θ be a cut–off function satisfying

$$\begin{aligned} &\theta \equiv 1 \quad \text{on} \quad A(\rho) \; , \\ &\theta \equiv 0 \quad \text{on} \quad B(\frac{\rho}{2}) \; , \\ &\theta \equiv 0 \quad \text{outside} \quad B(3\rho) \; , \\ &|\nabla \theta| \lesssim \rho^{-1} \; , \\ &|\nabla^2 \theta| \lesssim \rho^{-2} \; . \end{aligned}$$

Multiplying both sides of (3.30) by $\theta^4 e^{-w}$, we find

$$\begin{aligned} \int \theta^4 \sigma_2(A) e^{3w} &= \int -\theta^4 e^{-w} \Delta |\nabla w|^2 + 2\theta^4 e^{-w} \langle \nabla w, \nabla(\Delta w) \rangle \\ &\quad + 2\theta^4 e^{-w} (\Delta w)^2 + 4\theta^4 e^{-w} \nabla^2 w(\nabla w, \nabla w) \\ &\quad + 2\theta^4 e^{-w} \Delta w |\nabla w|^2 \\ &= I_1 + I_2 + I_3 + I_4 + I_5 \; . \end{aligned} \tag{3.31}$$

We now analyze several of the terms in (3.31) more carefully.

Integrating by parts gives

$$
\begin{aligned}
I_1 &= \int -\theta^4 e^{-w}\Delta|\nabla w|^2 \\
&= \int -\Delta(\theta^4 e^{-w})|\nabla w|^2 \\
&= \int -[\Delta(\theta^4)e^{-w} + \theta^4\Delta(e^{-w}) + 2\langle\nabla(\theta^4),\nabla(e^{-w})\rangle]|\nabla w|^2 \\
&= \int -\Delta(\theta^4)e^{-w}|\nabla w|^2 - \theta^4[-e^{-w}\Delta w + e^{-w}|\nabla w|^2]|\nabla w|^2 \\
&\quad - 2\langle\nabla(\theta^4),\nabla(e^{-w})\rangle|\nabla w|^2 \\
&= \int -\Delta(\theta^4)e^{-w}|\nabla w|^2 + \theta^4 e^{-w}\Delta w|\nabla w|^2 - \theta^4 e^{-w}|\nabla w|^4 \\
&\quad + 2e^{-w}\langle\nabla(\theta^4),\nabla w\rangle|\nabla w|^2 \ . \qquad (3.32)
\end{aligned}
$$

Next,

$$
\begin{aligned}
I_2 &= \int 2\theta^4 e^{-w}\langle\nabla w,\nabla(\Delta w)\rangle \\
&= \int -2\theta^4 e^{-w}(\Delta w)^2 - 2\theta^4\langle\nabla(e^{-w}),\nabla w\rangle\Delta w \\
&\quad - 2e^{-w}\langle\nabla(\theta^4),\nabla w\rangle\Delta w \\
&= \int -2\theta^4 e^{-w}(\Delta w)^2 + 2\theta^4 e^{-w}|\nabla w|^2\Delta^w \\
&\quad - 2e^{-w}\langle\nabla(\theta^4),\nabla w\rangle\Delta w \ .
\end{aligned}
$$

For the last term above we integrate by parts again:

$$
\begin{aligned}
&\int -2e^{-w}\langle\nabla(\theta^4),\nabla w\rangle\Delta w \\
&\quad = \int -2e^{-w}\partial_i(\theta^4)\partial_i w\partial_k\partial_k w \\
&\quad = \int 2e^{-w}\partial_i(\theta^4)\partial_k\partial_i w\partial_k w + 2e^{-w}\partial_k\partial_i(\theta^4)\partial_i w\partial_k w \\
&\qquad + 2\partial_k(e^{-w})\partial_i(\theta^4)\partial_i w\partial_k w \\
&\quad = \int e^{-w}\partial_i(\theta^4)\partial_i|\nabla w|^2 + 2e^{-w}\nabla^2(\theta^4)(\nabla w,\nabla w) \\
&\qquad - 2e^{-w}|\nabla w|^2\langle\nabla(\theta^4),\nabla w\rangle \\
&\quad = \int -e^{-w}\partial_i\partial_i(\theta^4)|\nabla w|^2 - \partial_i(e^{-w})\partial_i(\theta^4)|\nabla w|^2 \\
&\qquad + 2e^{-w}\nabla^2(\theta^4)(\nabla w,\nabla w) - 2e^{-w}|\nabla w|^2\langle\nabla(\theta^4),\nabla w\rangle \\
&\quad = \int -e^{-w}\Delta(\theta^4)|\nabla w|^2 + 2e^{-w}\nabla^2(\theta^4)(\nabla w,\nabla w) \\
&\qquad - e^{-w}|\nabla w|^2\langle\nabla(\theta^4),\nabla w\rangle \ .
\end{aligned}
$$

Thus,

$$\begin{aligned} I_2 = \int & 2\theta^4 e^{-w}(\Delta w)^2 + 2\theta^4 e^{-w}\Delta w|\nabla w|^2 \\ & - e^{-w}\Delta(\theta^4)|\nabla w|^2 + 2e^{-w}\nabla^2(\theta^4)(\nabla w, \nabla w) \\ & - e^{-w}|\nabla w|^2\langle\nabla(\theta^4), \nabla w\rangle \ . \end{aligned} \tag{3.33}$$

Finally,

$$\begin{aligned} I_4 &= \int 4\theta^4 e^{-w}\nabla^2 w(\nabla w, \nabla w) \\ &= \int 2\theta^4 e^{-w}\nabla w \nabla|\nabla w|^2 \\ &= \int -2\theta^4 e^{-w}\Delta w|\nabla w|^2 - 2\theta^4\langle\nabla(e^{-w}), \nabla w\rangle|\nabla w|^2 \\ &\quad - 2e^{-w}\langle\nabla(\theta^4), \nabla w\rangle|\nabla w|^2 \\ &= \int -2\theta^4 e^{-w}\Delta w|\nabla w|^2 + 2\theta^4 e^{-w}|\nabla w|^4 \\ &\quad - 2e^{-w}\langle\nabla(\theta^4), \nabla w\rangle|\nabla w|^2 \ . \end{aligned} \tag{3.34}$$

Combining (3.31)–(3.34) we find

$$\begin{aligned} \int \theta^4\sigma_2(A)e^{3w} = \int & 3\theta^4 e^{-w}\Delta w|\nabla w|^2 + \theta^4 e^{-w}|\nabla w|^4 \\ & - e^{-w}\langle\nabla(\theta^4), \nabla w\rangle|\nabla w|^2 + 2e^{-w}\nabla^2(\theta^4)(\nabla w, \nabla w) \\ & - 2\Delta(\theta^4)e^{-w}|\nabla w|^2 \ . \end{aligned} \tag{3.35}$$

Now, as the scalar curvature satisfies

$$\Delta w + |\nabla w|^2 + \tfrac{1}{6}Re^{2w} = 0 \ ,$$

we have

$$\Delta w = -|\nabla w|^2 - \tfrac{1}{6}Re^{2w} \ ,$$

and substituting this into the first term in the RHS of (3.35) we find

$$\begin{aligned} \int \theta^4\sigma_2(A)e^{3w} = \int & -\tfrac{1}{2}\theta^4 Re^{w}|\nabla w|^2 - 2\theta^4 e^{-w}|\nabla w|^4 \\ & - e^{-w}\langle\nabla(\theta^4), \nabla w\rangle|\nabla w|^2 + 2e^{-w}\nabla^2(\theta^4)(\nabla w, \nabla w) \\ & - 2\Delta(\theta^4)e^{-w}|\nabla w|^2 \ . \end{aligned}$$

Using the fact that $\sigma_2(A) > 0$, and the properties of θ, we conclude

$$\int \tfrac{1}{2}\theta^4 R e^{2w}|\nabla w|^2 + 2\theta^4 e^{-w}|\nabla w|^4 \lesssim \int \rho^{-1}\theta^3 e^{-w}|\nabla w|^3 + \rho^{-2}\theta^2 e^{-w}|\nabla w|^2 \ . \tag{3.36}$$

By Hölder's inequality,

$$\int \rho^{-1}\theta^3 e^{-w}|\nabla w|^3 \leqslant \rho^{-1}\Big(\int \theta^4 e^{-w}|\nabla w|^4\Big)^{3/4}\Big(\int_{supp\ \theta} e^{-w}\Big)^{1/4} .$$

Using the inequality $xy \leqslant \frac{3}{4}\varepsilon^{4/3}x^{4/3} + \frac{1}{4}\varepsilon^{-4}y^4$, we have

$$\int \rho^{-1}\theta^3 e^{-w}|\nabla w|^3 \leqslant \tfrac{3}{4}\varepsilon^{4/3}\int \theta^4 e^{-w}|\nabla w|^4 + \tfrac{1}{4}\varepsilon^{-4}\rho^{-4}\int_{supp\ \theta} e^{-w} \ . \tag{3.37}$$

Similarly,

$$\begin{aligned}\int \rho^{-2}\theta^2 e^{-w}|\nabla w|^2 &\leqslant \rho^{-2}\Big(\int \theta^4 e^{-w}|\nabla w|^4\Big)^{1/2}\Big(\int_{supp\ \theta} e^{-w}\Big)^{1/2}\\ &\leqslant \frac{\varepsilon}{2}\int \theta^4 e^{-w}|\nabla w|^4 + \frac{1}{2\varepsilon}\rho^{-4}\int_{supp\ \theta} e^{-w} \ .\end{aligned} \tag{3.38}$$

Substituting (3.37) and (3.38) into (3.36), and choosing $\varepsilon > 0$ sufficiently small, we conclude

$$\begin{aligned}\int_{B(2\rho)\setminus B(\rho)} R|\nabla u|^2 u^{-1} &= \int_{B(2\rho)\setminus B(\rho)} Re^{w}|\nabla w|^2\\ &\leqslant \int \theta^4 Re^{w}|\nabla w|^2\\ &\lesssim \rho^{-4}\int_{supp\ \theta} e^{-w} \ .\end{aligned} \tag{3.39}$$

Then (3.28) follows from (3.39) and the following technical lemma:

Lemma 3.8. *If $g = u^{4/n-2}ds^2 = e^{2w}ds^2$ is a conformal metric on $\mathbb{R}^n$ with scalar curvature $R \geqslant c_0 > 0$, then there is a constant c_2 such that for all $|x|$ sufficiently large,*

$$u^{-1}(x) = e^{-\frac{(n-2)}{2}w(x)} \leqslant c_2|x|^{n-2} \ .$$

Proof. The scalar curvature equation

$$-\Delta u = \tfrac{(n-2)}{4(n-1)} R u^{\frac{n+2}{n-2}}$$

is invariant under the Kelvin transform $x \mapsto \frac{x}{|x|^2}$. Thus, if $\widehat{u}(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right)$, $\widehat{R}(x) = R\left(\frac{x}{|x|^2}\right)$, then $\widehat{u}$ satisfies

$$-\Delta \widehat{u} = \frac{(n-2)}{4(n-1)} \widehat{R} \widehat{u}^{\frac{n+2}{n-2}} \qquad \text{on} \qquad \mathbb{R}^n - \{0\} \; .$$

Since $\widehat{R}(x) \geqslant c_0 > 0$, it follows that

$$-\Delta \widehat{u} \geqslant c_n \widehat{u}^{\frac{n+2}{n-2}} \qquad \text{on} \qquad \mathbb{R}^n - \{0\} \; .$$

Then the arguments of [KMPS, Lemma 1] imply that

$$-\Delta \widehat{u} \geqslant 0$$

on $\mathbb{R}^n$ in the sense of distributions. From this we conclude that $\widehat{u}(x) \geqslant c_1 > 0$ on $B(1)\backslash\{0\}$. □

4. The Proof of the main theorem

The proof of the main theorem begins by assuming that for some sequence of conformal metrics $\{g_k = e^{2w_k} g_0\}$ the bound (0.7) fails. We then dilate as described in section 1 to obtain a sequence $\{g_k^\star = T_{\varepsilon_k}^\star (e^{2w_k} g_0) \equiv e^{2w_k} g_0^k\}$ on $\mathbb{R}^4$ satisfying

$$\max\{e^{w_k} + |\nabla_0 w_k|\} = e^{w_k(0)} + |\nabla_0 w_k(0)| = 1 \; . \tag{4.1}$$

Recall that if $\lim\limits_k e^{w_k(0)} = 0$, then by Cor. 1.4(i) (a subsequence of) the rescaled sequence $\overline{w}_k = w_k - w_k(0)$ converges to $w \in C^{1,1}(\mathbb{R}^4)$, with $\bar{g} = e^{2\overline{w}} ds^2$ satisfying

$$\begin{aligned} \sigma_2(A_{\bar{g}}) &\equiv 0 \,, \\ R_{\bar{g}} &\geqslant 0 \; , \\ |\nabla \overline{w}(0)| &= 1 \, . \end{aligned} \tag{4.2}$$

However, Theorem 2.1 implies that $\overline{w} \equiv$ constant, contradicting (4.2). Consequently, we may assume that $\limsup e^{w_k(0)} > 0$. In this case, by Cor. 1.4(ii) (a subsequence of) $w_k \to w \in C^\infty(\mathbb{R}^4)$. After rescaling if necessary, the limiting metric $g = e^{2w} ds^2$ satisfies

$$\begin{aligned} &\frac{\sigma_2(A_g)}{R^\alpha} = \lambda_\alpha \,, \\ &R_g > 0 \,, \end{aligned}$$

where $\lambda_\alpha = \frac{1}{2}(12)^{1-\alpha}$ and $\alpha = 0$ or 1. According to Theorem 3.1, g is obtained by pulling back the round metric on S^4 via stereographic projection. In particular, it follows that

$$\int_{\mathbb{R}^4} \sigma_2(A_g) dvol_g = 16\pi^2 \,, \tag{4.3}$$

$$\int_{\mathbb{R}^4} \frac{\sigma_2(A_g)}{R} dvol_g = \frac{4}{3}\pi^2 \,, \tag{4.4}$$

$$\int_{\mathbb{R}^4} dvol_g = \frac{8}{3}\pi^2 \,. \tag{4.5}$$

At this point, it will simplify the exposition if we consider the cases where $\alpha = 0$ and $\alpha = 1$ separately, beginning with $\alpha = 0$.

Now, given any fixed ball $B(\rho) \subset \mathbb{R}^4$,

$$\int_{B(\rho)} \sigma_2(A_{g_k^\star}) dvol_{g_k^\star} \to \int_{B(\rho)} \sigma_2(A_g) dvol_g \tag{4.6}$$

as $k \to \infty$. On the other hand, since $g_k^\star = T_{\varepsilon_k}^\star g_k$, we have

$$\begin{aligned} \int_{B(\rho)} \sigma_2(A_{g_k^\star}) dvol_{g_k^\star} &= \int_{B(\rho)} T_{\varepsilon_k}^\star (\sigma_2(A_{g_k}) dvol_{g_k}) \\ &= \int_{T_{\varepsilon_k}(B(\rho))} \sigma_2(A_{g_k}) dvol_{g_k} \\ &\leqslant \int_{M^4} \sigma_2(A_{g_k}) dvol_{g_k} \,. \end{aligned} \tag{4.7}$$

Since the RHS of (4.7) is conformally invariant, this implies

$$\int_{B(\rho)} \sigma_2(A_{g_k^\star}) dvol_{g_k^\star} \leqslant \int_{M^4} \sigma_2(A_{g_0}) dvol_{g_0} \,. \tag{4.8}$$

If we let $k \to \infty$ in (4.8), then let $\rho \to \infty$, it follows from (4.6) and (4.3) that

$$16\pi^2 \leqslant \int_{M^4} \sigma_2(A_{g_0}) dvol_{g_0} \,. \tag{4.9}$$

However, by [G, Theorem B], $\int \sigma_2(A_{g_0}) dvol_{g_0} \leqslant 16\pi^2$ with equality if, and only if, (M^4, g_0) is conformally equivalent to the round sphere. Thus, the bound (0.7) holds unless (M^4, g_0) is conformally the round sphere, as claimed.

When $\alpha = 1$ we argue similarly. First, for any fixed ball $B(\rho)$,

$$\begin{aligned}\Big(\int_{B(\rho)} \frac{\sigma_2(A_{g_k^\star})}{R_{g_k^\star}} dvol_{g_k^\star}\Big)\Big(\int_{B(\rho)} dvol_{g_k^\star}\Big)^{-1/2}\\ \to \Big(\int_{B(\rho)} \frac{\sigma_2(A_g)}{R_g} dvol_g\Big)\Big(\int_{B(\rho)} dvol_g\Big)^{-1/2}\end{aligned} \tag{4.10}$$

as $k \to \infty$. By the Schwartz inequality,

$$\begin{aligned}&\Big(\int_{B(\rho)} \frac{\sigma_2(A_{g_k^\star})}{R_{g_k^\star}} dvol_{g_k^\star}\Big)\Big(\int_{B(\rho)} dvol_{g_k^\star}\Big)^{-1/2}\\ &\leqslant \Big(\int_{B(\rho)} \sigma_2(A_{g_k^\star}) dvol_{g_k^\star}\Big)^{-1/2}\Big(\int_{B(\rho)} \frac{\sigma_2(A_{g_k^\star})}{R^2_{g_k^\star}} dvol_{g_k^\star}\Big)^{-1/2}\Big(\int_{B(\rho)} dvol_{g_k^\star}\Big)^{-1/2}.\end{aligned} \tag{4.11}$$

From the definition of σ_2 we notice

$$\begin{aligned}\frac{\sigma_2(A_{g_k^\star})}{R^2_{g_k^\star}} &= \frac{1}{R^2_{g_k^\star}}\left[-\frac{1}{2}|E_{g_k^\star}|^2 + \frac{1}{24}R^2_{g_k^\star}\right]\\ &\leqslant \frac{1}{24},\end{aligned}$$

which by (4.11) implies

$$\Big(\int_{B(\rho)} \frac{\sigma_2(A_{g_k^\star})}{R_{g_k^\star}} dvol_{g_k^\star}\Big)\Big(\int_{B(\rho)} dvol_{g_k^\star}\Big)^{-1/2} \leqslant \frac{1}{2\sqrt{6}}\Big(\int_{B(\rho)} \sigma_2(A_{g_k^\star}) dvol_{g_k^\star}\Big)^{-1/2}.$$

By (4.7) and (4.8) we conclude

$$\Big(\int_{B(\rho)} \frac{\sigma_2(A_{g_k^\star})}{R_{g_k^\star}} dvol_{g_k^\star}\Big)\Big(\int_{B(\rho)} dvol_{g_k^\star}\Big)^{-1/2} \leqslant \frac{1}{2\sqrt{6}}\Big(\int_{M^4} \sigma_2(A_{g_0}) dvol_{g_0}\Big)^{-1/2}. \tag{4.12}$$

If we let $k \to \infty$ in (4.12), then let $\rho \to \infty$, it follows from (4.10), (4.4), and (4.5) that

$$\Big(\frac{4}{3}\pi^2\Big)\Big(\frac{8}{3}\pi^2\Big)^{-1/2} \leqslant \frac{1}{2\sqrt{6}}\Big(\int_{M^4} \sigma_2(A_{g_0}) dvol_{g_0}\Big)^{1/2}.$$

In other words, (4.9) holds. Once again, we see that (0.7) holds unless (M^4, g_0) is conformally the round sphere. This completes the case $\alpha = 1$ and the proof of the main theorem.

To prove Corollary A when $\alpha = 0$, we first observe that by (0.7), w is bounded from above. We integrate (0.6) with respect to the volume form of $g = e^{2w} g_0$:

$$\int \sigma_2(A_g) dvol_g = \int f dvol_g = \int f e^{4w} dvol_{g_0} \tag{4.13}$$

by the conformal invariance of the integral on the left in (4.13), we see that

$$0 < \int \sigma_2(A_{g_0}) dvol_{g_0} = \int \sigma_2(A_g) dvol_g \leqslant \|f\|_\infty vol(g_0) e^{4 \max w} .$$

Therefore, $\max w \geqslant C(g_0, \|f\|_\infty)$. Since (0.7) implies $|\nabla_0 w| \leqslant C(g_0, \|f\|_{C^2})$, we also know that

$$|\max w - \min w| \leqslant C(g_0, \|f\|_{C^2}), \tag{4.14}$$

and (0.8) follows.

When $\alpha = 1$, we integrate (0.6) once again:

$$\int \sigma_2(A_g) dvol_g = \int R\, f\, dvol_g \leqslant \|f\|_\infty \int R\, dvol_g . \tag{4.15}$$

Using the scalar curvature equation, and integrating by parts, we find

$$\begin{aligned} \int R dvol_g &= \int R e^{4w} dvol_{g_0} \\ &= \int (-6\Delta_0 w - 6|\nabla_0 w|^2 + R_0) e^{4w} dvol_{g_0} \\ &= \int (12 e^{2w} |\nabla_0 w|^2 + R_0 e^{2w}) dvol_{g_0} \\ &\leqslant C(g_0, \|f\|_{C^2}) e^{2 \max w} . \end{aligned} \tag{4.16}$$

Note that in deriving (4.16) we used the bound (0.7). From (4.15) and (4.16) we conclude $\max w \geqslant C(g_0, \|f\|_{C^2})$. Appealing to (4.14) once again, it follows that $\|w\|_\infty \leqslant C(g_0, \|f\|_{C^2})$.

5. Existence of solutions

In this section, we prove Corollary B.

We will apply the degree theory developed for fully nonlinear equations in [L]. The key aspect of this theory is that the degree remains invariant under continuous deformations of the equation as long as there is a uniform a priori estimates for all solutions of the equation, and a uniform bound for the ellipticity.

Given (M^4, g_0), with the metric g_0 satisfying the conditions (0.5), the main result of [CGY] asserts the existence of a conformal metric $g = e^{2w} g_0$ for which the equation

$$\sigma_2(A_g) = f \tag{5.1}$$

holds for some positive function f. Assume in addition that (M^4, g_0) is not conformally equivalent to the standard 4-sphere, then the main theorem of this paper asserts the existence of an a priori bound for any solution w of the equation. In particular, given a (smooth) positive function h, there is a constant c independent of t so that all solutions $g = e^{2w} g_0$ of the equation

$$\sigma_2(A_g) = tf + (1-t)h \qquad (\Sigma_t)$$

with $R(g) > 0$ satisfies the bounds

$$\left.\begin{array}{l} \|w\|_{4,\alpha} \leqslant c, \\ S_{ij}(g)\xi_i\xi_j \geqslant \frac{1}{c}\,|\xi|^2\,. \end{array}\right\} \tag{5.2}$$

We denote by the set O_c

$$O_c = \{w \in C^{4,\alpha} | (5.2) \text{ holds }\} \cap \{w \in C^{4,\alpha} | \sigma_2(A_{g_w}) > 0;\ R_{g_w} > 0\}\,.$$

We denote the degree of the equation (Σ_t) by $\deg(\Sigma_t, O_c, 0)$. The degree theory of [L] implies that

$$deg(\Sigma_0, O_c, 0) = deg(\Sigma_1, O_c, 0). \tag{5.3}$$

We need to do a calculation verifying that for $t = 1$ the degree of the equation is non-zero. In order to do this, we deform the equation to one whose degree is easy to determine. First, it is useful to re-write the equation (5.1) in a suggestive form. Suppose $g = e^{2w} g_0$, denote

$$M_{ij}(w) = 2S^0_{ij} + 2\nabla^0_i\nabla^0_j w - 2\Delta_0 w g^0_{ij} - 2\nabla^0_i w\nabla^0_j w. \tag{5.4}$$

Then, after some computation, the equation (5.1) may be written in the form

$$-\nabla^0_i\{M_{ij}(w)\nabla^0_j w\} + \sigma_2(A_{g_0}) = \sigma_2(A_g)e^{4w} \;=\; fe^{4w}. \tag{5.5}$$

It is important to note the identity

$$M_{ij}(w) = S_{ij} + S^0_{ij} + |\nabla_0 w|^2 g^0_{ij}$$

so that it is clear that when both $\sigma_2(A_g) > 0, R_g > 0$ and $\sigma_2(A_{g_0}) > 0, R_{g_0} > 0$, then M_{ij} is positive definite.

It is also convenient to re-formulate the equation (5.1), on account of the conformal covariance property, using the solution metric g of the equation as the background metric:

$$-\nabla_i\{M_{ij}(v)\nabla_j v\} + f = fe^{4v} \tag{5.6}$$

so that $v = 0$ is a solution to this equation satisfying $R > 0$.

We now use the following deformation:

$$-\nabla_i\{M_{ij}(v)\nabla_j v\} + f = \sigma_2(A_{g_v})e^{4v} = (1-t)f\int e^{4v} + tfe^{4v}. \tag{5.7}$$

where $\int e^{4v} = \int e^{4v} dvol_g$. We label this equation by Γ_t. When $t = 1$ we recover the equation (5.6), and when $t = 0$ we have the "linear" equation

$$-\nabla_i\{M_{ij}(v)\nabla_j v\} + f = f\int e^{4v}. \tag{5.8}$$

To analyze the equation (5.8), we integrate it over the manifold to find that $\int e^{4v} dvol_g = 1$. Hence the equation reduces to

$$-\nabla_i\{M_{ij}(v)\nabla_j v\} = 0.$$

By the positive definiteness of M_{ij}, $v = 0$ is the unique solution. To calculate the degree of this equation, we assume for the moment that we have established the a priori estimates for all solutions of the equations Γ_t for all $0 \le t \le 1$; thus $deg(\Gamma_1, O_C, 0) = deg(\Gamma_0, O_C, 0)$.

We need to find the linearization of the equation (5.8) at the unique solution $v = 0$:

$$\begin{aligned}\mathcal{L}\dot{v} &= -2S_{ij}\nabla_i\nabla_j\,\dot{v} - f\int e^{4v}4\dot{v}dvol_g \\ &= -2S_{ij}\nabla_i\nabla_j\dot{v} - 4f\int \dot{v}dvol_g\,.\end{aligned} \tag{5.9}$$

To see that $\mathcal{L}$ has no kernel, we set $\mathcal{L}\varphi = 0$. Then upon integration we find $\int \varphi dvol_g = 0$, and the maximum principle shows $\varphi = 0$.

If μ is an eigenvalue of $\mathcal{L}$, say with eigenfunction φ; then upon integration

$$-4(\int f dvol_g)(\int \varphi dvol_g) = \mu\int \varphi dvol_g\,.$$

So either $\mu = -4\int f dvol_g$ or $\int \varphi dvol_g = 0$. In the latter case, we can multiply equation (5.9) by φ and integrate to conclude that $\mu > 0$. Of course, constant functions φ are eigenfunctions with eigenvalue $-4\int f dvol_g$. Hence $\deg(\Gamma_0, \Omega_c, 0) = -1$.

To complete this argument, in the remaining part of this section we will prove *a priori* estimates for solutions of the equation Γ_t which verify the condition $R > 0$.

A preliminary observation is the volume bound

$$c_1 \leqslant \int e^{4v} dvol_g \leqslant c_2 \tag{5.10}$$

for solutions of the equation Γ_t , where c_1, c_2 are constants with $c_i = c_i(maxf, inff)$.

An immediate consequence is the bound for the mean value

$$\overline{v} = \int v dvol_g \leqslant c\,. \tag{5.11}$$

We divide our consideration of Γ_t into two cases: (I) $0 < t_0 \leqslant t \leqslant 1$, and (II) $0 \leqslant t \leqslant t_0$, where t_0 is to be determined later.

Case I: $0 < t_0 \leq t \leq 1$. We first observe that from (5.7)

$$\sigma_2(A_{g_v}) = (1-t) f e^{-4v} \int e^{4v} \; + t f. \tag{5.12}$$

Thus $\sigma_2(A_{g_v}) \geq t_0(\inf f)$. We claim that all solutions of (5.12) are uniformly bounded in C^1. Suppose this is not the case, we will modify the blowup argument in section one of this paper to obtain a contradiction. Assume that there is a sequence $\{v_k\}$ with $\sup(|\nabla v_k| + e^{v_k}) \to \infty$. Denote $g_k = e^{2v_k} g$ and let

$$f_k = \sigma_2(A_{g_k}) = (1-t) f e^{-4v_k} \int e^{4v_k} \; + t f$$

Choose P_k to be the maximum point of $|\nabla v_k| + e^{v_k}$ on M, and $\epsilon_k \to 0$ as in section one. Denote $v_{k,\epsilon_k} = T^*_{\epsilon_k} v_k + log \epsilon_k$, so that

$$|\nabla v_{k,\epsilon_k}| + e^{v_{k,\epsilon_k}}|_{x=0} = 1.$$

Again, there are two possibilities depending on whether $\lim_{k\to\infty} e^{v_{k,\epsilon_k}(0)} = 0$. We first observe that if $\lim_{k\to\infty} e^{v_{k,\epsilon_k}(0)} = \delta_0 > 0$ then $v_k(P_k) \to \infty$, so that denoting for simplicity $g_k = e^{2v_{k,\epsilon_k}} g$, for the dilated metric $\sigma_2(A_{g_k})(0) \geq t f(0)$, and a subsequence converges to a solution of the equation $\sigma_2(A_{g_\infty}) = t f(0)$ on $\mathbb{R}^4$. By Theorem 3.6 and the argument in the proof of the main theorem in section four, this implies that M is conformally equivalent to S^4, a contradiction to our assumption.

The other possibility is that $\lim_{k\to\infty} e^{v_{k,\epsilon_k}(0)} = 0$. In this case, $|\nabla v_{k,\epsilon_k}(0)| \to 1$; as before we rescale the dilated sequence

$$\tilde{v}_{k,\epsilon_k} = v_{k,\epsilon_k} - v_{k,\epsilon_k}(0). \tag{5.13}$$

Then denoting by $\tilde{g}_k = g_{\tilde{v}_{k,\epsilon_k}}$ we have $\sigma_2(A_{\tilde{g}_k}) = \tilde{f}_k$, with

$$\tilde{f}_k = \sigma_2(A_{\tilde{g}_k}) = (T^*_{\epsilon_k} f)[(1-t)(\int e^{4v_k})\epsilon_k^4 e^{-4\tilde{v}_{k,\epsilon_k}} + t e^{4v_{k,\epsilon_k}(0)}]. \tag{5.14}$$

Because of the bound (5.10), we can apply Corollary 1.3 to obtain bounds (on any fixed ball) on the second derivatives of $\tilde{v}_{k,\epsilon_k}$. Therefore $\tilde{v}_{k,\epsilon_k} \to v$ in $C^{1,\alpha}$ on compact sets. Since the RHS of (5.14) approaches 0 as $k \to \infty$, the limiting metric $\overline{g} = e^{2v} ds^2$ satisfies $\sigma_2(A_{\overline{g}}) \equiv 0$. It follows from Theorem 2.1 that v is a constant; but this contradicts the assumption above that $|\nabla v_{k,\epsilon_k}(0)| \to 1$ as $k \to \infty$. Thus we have the a priori bound for solutions of Γ_t, as claimed.

Case II: $0 < t \leq t_0 < 1$

In this case we will prove directly that the metric g_v satisfying the equation (5.7) has $||v||_{C^{4,\alpha}}$ norm bounded for some $\alpha > 0$.

To do this, we first recall a special case of the Trudinger-Moser inequality ([M], [F]): On (M^4, g), there exists some constant c_0 so that for all $v \in W^{1,4}(M^4)$, we have

$$\int \exp\Big(c_4 \frac{|v - \overline{v}|}{D(v)}\Big)^{4/3} dvol_g \leqslant c_0, \tag{5.15}$$

where $c_4 = 4|S_3|^{4/3}$, $\overline{v} = \frac{\int v dvol_g}{\int dvol_g}$ and $D(v) = \left(\int |\nabla v|^4\right)^{1/4}$.

Multiply the equation (5.15) $v - \overline{v}$ and integrate; we get

$$\int \big(S_{ij}\nabla_i v \nabla_j v + \tilde{S}_{ij}\nabla_i v \nabla_j v + |\nabla v|^4\big) dvol_g = \int f(v-\overline{v})(1-t) \int e^{4v} + t \int f(v-\overline{v})e^{4v}\,. \tag{5.16}$$

The concavity of the logarithm implies:

$$\begin{aligned} \int f(v - \overline{v})e^{4v} &\leqslant (\sup f)\Big(\int e^{4v}\Big) \log \int e^{|v-\overline{v}|}e^{4v} \\ &\leqslant (\sup f)\Big(\int e^{4v}\Big)\Big(\log \int e^{5|v-\overline{v}|} + 4\overline{v}\Big)\,. \end{aligned} \tag{5.17}$$

Using the simple inequality

$$5t \leqslant \frac{c_4}{D^{4/3}}\, t^{4/3} + \alpha D^4$$

where

$$\alpha = \Big(\frac{15}{4c_4}\Big)^3 \Big(5 - \frac{15}{4c_4}\Big)\,,$$

we find

$$\begin{aligned} \log \int e^{5|v-\overline{v}|} &\leqslant \log \int \exp\bigg(c_4 \Big(\frac{|v - \overline{v}|}{D}\Big)^{4/3} + \alpha D^4\bigg) \\ &\leqslant c_0 + \alpha D^4\,. \end{aligned} \tag{5.18}$$

Combine (5.17), (5.18) and substituting into (5.16), recall (5.12) and (5.13), we get

$$D^4 \leqslant c + tc(c_0 + \alpha D^4)\,.$$

Thus for $t \leqslant t_0$ sufficiently small, we find $D \leqslant c$. As a consequence,

$$\int e^{4v} \cdot \int e^{-4v} = \int e^{4(v-\overline{v})} \cdot \int e^{-4(v-\overline{v})} \leqslant ce^{cD} \leq c.$$

We find from (5.12) that

$$e^{-4\overline{v}} \leq \int e^{4v} \leq c.$$

Hence $-c_1 \leqslant \overline{v} \leqslant c$. The positivity of R_{g_v} then yields a lower bound for v.

From the expression of $\sigma_2(A_{g_v})$ in (5.14), we can then apply the argument of section 5 in [CGY] to conclude that there is an *a priori* estimate for $||v||_{W^{2,p}}$ for some $p > 4$; hence $||v||_{C^{1,\alpha}} \leq c$ for some constant c. At this point, the regularity theory of Evans [Ev] and Krylov [Kr] applies to show that we have uniform bounds:

$$\|v\|_{2,\alpha} \leqslant c\,, and \;\; S_{ij} \geq \frac{3\sigma_2}{R} \geq c\,.$$

The uniform ellipticity of the equations then easily yields

$$\|v\|_{4,\alpha} \leqslant c.$$

This finishes the proof of the *a priori* estimate for Case II, hence for the equation Γ_t, and consequently the proof of Corollary B.

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