ALGEBRAIC SOLUTIONS OF DIFFERENTIAL EQUATIONS OVER $\mathbb{P}^1 - \{0, 1, \infty\}$

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Abstract. The Grothendieck–Katz p-curvature conjecture predicts that an arithmetic differential equation whose reduction modulo p has vanishing p-curvatures for *almost all* p, has finite monodromy. It is known that it suffices to prove the conjecture for differential equations on $\mathbb{P}^1 - \{0, 1, \infty\}$. We prove a variant of this conjecture for $\mathbb{P}^1 - \{0, 1, \infty\}$, which asserts that if the equation satisfies a certain convergence condition for *all* p, then its monodromy is trivial. For those p for which the p-curvature makes sense, its vanishing implies our condition. We deduce from this a description of the differential Galois group of the equation in terms of p-curvatures and certain local monodromy groups. We also prove similar variants of the p-curvature conjecture for an elliptic curve with j-invariant 1728 minus its identity and for $\mathbb{P}^1 - \{\pm 1, \pm i, \infty\}$.

1. Introduction

The Grothendieck–Katz p-curvature conjecture was originally raised as a question on linear homogeneous systems of first-order differential equations (see Conjecture (I) in [Kat72, Introduction] for more details)

$$\frac{d\boldsymbol{y}}{dx} = A(x)\boldsymbol{y}.$$

Here $A(x)$ is a square matrix of rational functions of x with coefficients in some number field K and $\boldsymbol{y}$ is a vector-valued function. For all but finitely many primes $\mathfrak{p}$ of K, it makes sense to reduce this system modulo $\mathfrak{p}$ and to define an invariant, the p-curvature, in terms of the resulting system. According to the conjecture, if *almost all* (that is, all but finitely many) p-curvatures vanish, then the original system admits a full set of solutions in algebraic functions.

The conjecture generalizes to a smooth variety X equipped with a vector bundle with an integrable connection (M, ∇) defined over some number field K. It is known that the general version of the conjecture reduces to the case when $X = \mathbb{P}^1_K - \{0, 1, \infty\}$. (See [Bos01, 2.4.1], [Kat82, Thm. 10.5], and [And04, 7.1.4]).

André [And04, Sec. 6] and Bost [Bos01, 2.4.2] proved the conjecture in the case when the neutral connected component G°_{alg} of the algebraic monodromy group of (M, ∇) is *a priori* solvable. The key inputs are their generalizations, based on the work of D. V. and G. V. Chudnovsky, of the classical Borel–Dwork criterion for the rationality of formal power series.

In this paper, we apply Borel–Dwork type algebraicity results without assuming *a priori* that G°_{alg} is solvable to prove a variant of the conjecture for $\mathbb{P}^1_K - \{0, 1, \infty\}$ and for a certain elliptic curve minus a point. In our results, in addition to the

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assumption of vanishing p-curvature at almost all primes, we impose a condition $(*)_\mathfrak{p}$ on the $\mathfrak{p}$-adic radii of convergence of the horizontal sections of (M,∇) (see Assumptions 2.2.1, 6.1.3 for the precise statement) at every prime $\mathfrak{p}$ where p-curvature is either not defined or non-vanishing. When (M,∇) has an integral model at a prime $\mathfrak{p}$ so that one can make sense of its reduction mod p, the condition $(*)_\mathfrak{p}$ is *implied by* the vanishing of the p-curvature.

Theorem (Theorem 2.2.2)**.** *Let (M,∇) a vector bundle with a connection over $X=\mathbb{P}^1_K-\{0,1,\infty\}$. If the p-curvature of (M,∇) vanishes for almost all $\mathfrak{p}$ and the condition $(*)_\mathfrak{p}$ is satisfied for all other finite primes, then (M,∇) admits a full set of rational solutions, that is, $M^{\nabla=0}$ generates M as an $\mathcal{O}_X$-module.*

One can also make sense of the condition $(*)_\mathfrak{p}$ for vector bundles with connections over smooth algebraic curves equipped with a flat model over $\mathcal{O}_K$ such that there is a smooth $\mathcal{O}_K$-point or with a semistable model over $\mathcal{O}_K$. However, the condition $(*)_\mathfrak{p}$ at a fixed prime $\mathfrak{p}$ is not preserved under push-forward along finite maps from the curve in question to $\mathbb{P}^1-\{0,1,\infty\}$. Therefore, one cannot deduce from the above theorem the same result for arbitrary algebraic curves. Nevertheless, when X is an elliptic curve with j-invariant 1728 minus its identity point, we prove:

Theorem (Theorem 6.1.5)**.** *Let $X\subset\mathbb{A}^2_\mathbb{Z}$ be the affine curve defined by $y^2=x(x-1)(x+1)$ and let (M,∇) be a vector bundle with a connection over X. If the p-curvature of (M,∇) vanishes for almost all $\mathfrak{p}$ and the condition $(*)_\mathfrak{p}$ is satisfied for all other finite primes, then (M,∇) is étale locally trivial. Namely, there exists a finite étale map $f:Y\to X$ such that $f^*(M,\nabla)$ is isomorphic to $(\mathcal{O}_Y^{\mathrm{rk}M},d)$, where d is the differential operator on regular functions.*

Unlike the previous case, passing to a finite étale cover is necessary. We give an example of an (M,∇) with G_{gal} equal to $\mathbb{Z}/2\mathbb{Z}$.

Katz has shown in [Kat82, Thm. 10.2] that if the p-curvature conjecture holds, then for any vector bundle with an integrable connection (M,∇) on a smooth variety X over K as above, the Lie algebra $\mathfrak{g}_{\mathrm{gal}}$ of the differential Galois group G_{gal} of (M,∇) is in some sense generated by the p-curvatures. Namely, let $K(X)$ be the function field of X. The p-curvature conjecture implies that $\mathfrak{g}_{\mathrm{gal}}$ is the smallest algebraic Lie subalgebra of $\mathfrak{gl}_n(K(X))$ such that for almost all p the reduction of $\mathfrak{g}_{\mathrm{gal}}$ mod p contains the p-curvature.

We use Theorem 2.2.2 to prove a result analogous to Katz's theorem when $X=\mathbb{P}^1_K-\{0,1,\infty\}$. Of course, this result (Theorem 2.2.6) involves a condition at every prime $\mathfrak{p}$, but as a compensation we describe G_{gal} and not only its Lie algebra. In the geometric case, namely when (M,∇) is the relative de Rham cohomology with the Gauss–Manin connection, this extra local condition is often vacuous. We discuss the example of the Legendre family (Remark 3.3.2) and show that a variant of our result implies that $\mathfrak{g}_{\mathrm{gal}}$ is generated by the p-curvatures, which recovers a result of Katz.

The paper is organized as follows. In section 2 to 5, we will focus on the case when $X=\mathbb{P}^1_K-\{0,1,\infty\}$. In section 6, we discuss the case when X is the affine elliptic curve defined above.

In section 2, we formulate our main result for $X=\mathbb{P}^1_K-\{0,1,\infty\}$ (Theorem 2.2.2), and in particular the condition $(*)_\mathfrak{p}$, which substitutes for the vanishing of the p-curvature when it does not make sense to reduce (M,∇) mod $\mathfrak{p}$. We then use Theorem 2.2.2 to deduce a description of the differential Galois group following Katz.

In section 3, we use a theorem of André ([And04, Thm. 5.4.3]) to prove that a vector bundle with a connection (M, ∇), as in Theorem 2.2.2, is étale locally trivial on X. To do this, we apply André's criterion to the formal horizontal sections of (M, ∇) centered at a specific point x_0. We obtain a lower bound for André's analogue of their radii of convergence at archimedean places, using the uniformization of $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ by the unit disc, which arises from its interpretation as the moduli space of elliptic curves with level 2 structure. The chosen point x_0 corresponds to the elliptic curve with smallest stable Faltings height and we use the Chowla-Selberg formula to deduce the lower bound. We also discuss in this section some variants of our main theorem and an example of the Legendre family mentioned above.

In section 4, we apply the rationality criterion of Bost and Chambert-Loir ([BCL09, Thm. 7.8]) to prove Theorem 2.2.2. We give a lower bound for the local capacity of Ω, the image in $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ of a standard fundamental domain for $\Gamma(2)$ under the uniformization mentioned above. Together with the algebraicity of our formal solution proved in section 3, this allows us to apply the criterion in [BCL09], and deduce that the solutions of (M, ∇) are rational.

Section 5 is devoted to an interpretation of our computations in section 3 in terms of the stable Faltings height, obtained by relating our estimate for archimedean places to the Arakelov degree of the restriction of the tangent bundle to some point.

In section 6, we prove our theorem when X is the affine elliptic curve in Theorem 6.1.5 using André's criterion and ideas in section 3. As in section 2, we specify the local convergence condition $(*)_{\mathfrak{p}}$ at bad primes. Using the property of theta functions and Weierstrass-$\wp$ functions, we deduce from a result of Eremenko [Ere] a lower bound of the archimedean radii.

In section 7, we first give an example of an (M, ∇) over the affine elliptic curve in section 6 such that its p-curvatures vanish for all $\mathfrak{p}$ (with respect to a specific chosen model of (M, ∇) with good reduction everywhere) but its G_{gal} is $\mathbb{Z}/2\mathbb{Z}$. More precisely, (M, ∇) is the push-forward of $(\mathcal{O}, d)$ via the degree two self-isogeny of the elliptic curve. In the second half, we discuss a variant of our main theorems when X is $\mathbb{A}^1 - \{\pm 1, \pm i\}$ with the conclusion that (M, ∇) has finite monodromy. The proof relies on the result of Eremenko used in last section. We also give an example to show that even when (M, ∇) has good reduction everywhere and all its p-curvatures vanish, it can still have local monodromies of order two around the singular points $\pm 1, \pm i, \infty$.

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2. Statement of the main results

Let K be a number field and $\mathcal{O}_K$ its ring of integers. Without further indication, we use X to denote $\mathbb{P}^1_{\mathcal{O}_K} - \{0, 1, \infty\}$ and use x to denote the natural coordinate of X. Let M be a vector bundle with a connection $\nabla\colon M \to \Omega^1_{X_K} \otimes M$ over X_K. For a finite place v of K lying over a prime p, let K_v be the completion of K with respect to v and denote by $\mathcal{O}_v$ and k_v the ring of integers and residue field of K_v. For Σ a finite set of finite rational primes, we set $\mathcal{O}_{K,\Sigma} = \mathcal{O}_K[1/p]_{p\in\Sigma} \subset K$.

2.1. **The p-curvature and p-adic differential Galois groups.**

2.1.1. Let X be a curve (flat scheme of relative dimension 1 with smooth geometrically irreducible generic fiber) over $\mathcal{O}_K$ such that for any finite place v of K, the smooth locus of X_{k_v} is non-empty. For Σ, as above, sufficiently large, (M, ∇) extends to a vector bundle with connection (again denoted (M, ∇)) over $X_{\mathcal{O}_{K,\Sigma}}$. In particular, if $p \notin \Sigma$ we can consider the pull back of (M, ∇) to $X \otimes \mathbb{Z}/p\mathbb{Z}$. If D is a derivation on $X \otimes \mathbb{Z}/p\mathbb{Z}$, so is D^p. Let $\nabla(D)$ be the map $(D \otimes \mathrm{id}) \circ \nabla$. Then on $X \otimes \mathbb{Z}/p\mathbb{Z}$, the *p-curvature* is given by (see [Kat82, Sec. VII] for details) [1]

$$\psi_p(D) := \nabla(D^p) - \nabla(D)^p \in \mathrm{End}_{\mathcal{O}_{X\otimes\mathbb{Z}/p\mathbb{Z}}}(M \otimes \mathbb{Z}/p\mathbb{Z}).$$

In particular, $\psi_p\left(\frac{d}{dx}\right) = -\left(\nabla\left(\frac{d}{dx}\right)\right)^p$, where x is a (Zariski) local coordinate at some smooth point of X_{k_v}. Since $\psi_p(D)$ is p-linear in D, for $X = \mathbb{P}^1_{\mathcal{O}_K} - \{0, 1, \infty\}$, the equation $\psi_p \equiv 0$ is equivalent to $-\left(\nabla\left(\frac{d}{dx}\right)\right)^p \equiv 0$.

In general, the ψ_p depends on the choice of extension of (M, ∇) over $X_{\mathcal{O}_{K,\Sigma}}$. However, any two such extensions are isomorphic over $X_{\mathcal{O}_{K,\Sigma'}}$ for some sufficiently large Σ'.

2.1.2. From now on, X is $\mathbb{P}^1_{\mathcal{O}_K} - \{0, 1, \infty\}$. Let L be a finite extension of K and w a place of L over the finite place v. We view L as a subfield of $\mathbb{C}_p$ via w. Fix an $x_0 \in X(\mathcal{O}_{L_w})$. Given a positive real number $r \leq 1$, we denote by $D(x_0, r)$ the open rigid analytic disc of radius r, with center x_0. The set of $\mathbb{C}_p$-points of the disc is

$$\{x \in X(\mathbb{C}_p) \mid |x - x_0|_p < r\},$$

where the norm $|\cdot|_p$ on $L_w \to \mathbb{C}_p$ is normalized so that $|p|_p = p^{-1}$. More precisely, this disc is the complement in $\mathbb{P}^1_{L_w}$ of the affinoid subdomain defined by the affinoid algebra[2]

$$\left\{\sum_{k=0}^{\infty} a_k(x - x_0)^{-k} \in L_w[[(x - x_0)^{-1}]] \mid \lim_{k\to\infty} |a_k|_p r^{-k} = 0\right\}.$$

Let $M^\vee$ be the dual vector bundle of M. It is naturally endowed with the connection such that for any local sections m, l of M and $M^\vee$ respectively,

$$d\langle l, m\rangle = \langle \nabla_{M^\vee}(l), m\rangle + \langle l, \nabla_M(m)\rangle.$$

[1] We could have defined the p-curvatures by considering derivations on X_{k_v} for v a place of K. For primes which are unramified in K, the two definitions are essentially equivalent, and the present definition will allow us to formulate the inequalities which arise below in a more uniform manner.

[2] This makes sense if one works with Berkovich spaces. If one works with classical rigid analytic spaces, one may view $D(x_0, r)$, being $\cup_{r'\leq r, r'\in p^{\mathbb{Q}}} D(x_0, r')$, as an admissible open subset. Nevertheless, for the discussion in this paper, it suffices to work with $r \in p^{\mathbb{Q}}$.

Definition 2.1.3. If (V, ∇) is a vector bundle with connection over some scheme or rigid space, we denote by $\langle V, \nabla\rangle^{\otimes}$, or simply $\langle V\rangle^{\otimes}$, if there is no risk of confusion regarding the connection ∇, the category of ∇-stable sub quotients of all the tensor products $V^{\otimes m} \otimes (V^{\vee})^{\otimes n}$ for $m, n \geq 0$. If the scheme (resp. rigid space) over which V is a vector bundle is connected (resp. has its associated Berkovich space to be connected), then this is a Tannakian category.

Definition 2.1.4. Let F_w be the field of fractions of the ring of all rigid analytic functions on $D(x_0, r)$

$$\left\{\sum_{k=0}^{\infty} a_k (x - x_0)^k \in L_w[[(x - x_0)]] \mid \forall r' < r, \lim_{k\to\infty} |a_k|_p (r')^k = 0\right\}$$

and $\eta_w \colon \mathrm{Spec}(F_w) \to X$ the natural map. Consider the fiber functor

$$\eta_w \colon \langle M|_{D(x_0,r)}\rangle^{\otimes} \to \mathrm{Vec}_{F_w}; \quad V \mapsto V_{\eta_w}.$$

The *p-adic differential Galois group* $G_w(x_0, r)$ is defined to be the automorphism group $\underline{\mathrm{Aut}}^{\otimes}\, \eta_w$ of η_w.

In other words, $G_w(x_0, r)$ is the subgroup of $\mathrm{GL}(M_{\eta_w})$ which stabilizes all objects in $\langle M|_{D(x_0,r)}\rangle^{\otimes}$.

For $v|p$ a finite place of K, we will say that (M, ∇) has *good reduction* at v if (M, ∇) extends to a vector bundle with connection on $X_{\mathcal{O}_v}$. The following lemma gives the basic relation between the p-curvature and the p-adic differential Galois group.

Lemma 2.1.5. *Let $x_0 \in X(\mathcal{O}_{L_w})$ and suppose that (M, ∇) has good reduction at v. If the p-curvature (defined in 2.1.1) vanishes, then the p-adic differential Galois group $G_w(x_0, p^{-\frac{1}{p(p-1)}})$ is trivial.*

Proof. To show that $G_w(x_0, p^{-\frac{1}{p(p-1)}})$ is trivial, we have to show that the restriction of M to $D(x_0, p^{-\frac{1}{p(p-1)}})$ admits a full set of solutions. It is well known that this is the case when $\psi_p \equiv 0$, but for the convenience of the reader we sketch the argument. See [Bos01, section 3.4.2, prop. 3.9] and [CL02, Lem. 7.6] for related arguments.

Assume there is an extension of (M, ∇) to a vector bundle with connection $(\mathcal{M}, \nabla)$ over $X_{\mathcal{O}_v}$. If m_0 is any section of $\mathcal{M}$, then a formal section in the kernel of ∇ is given by

$$m = \sum_{i=0}^{\infty} \nabla\left(\frac{d}{dx}\right)^i (m_0) \frac{(x - x_0)^i}{i!} (-1)^i.$$

Since $\psi_p \equiv 0$ (recall that this means the p-curvature vanishes on $X_{\mathcal{O}_v} \otimes \mathbb{Z}/p\mathbb{Z}$), we have $\nabla(\frac{d}{dx})^p(\mathcal{M}) \subset p\mathcal{M}$. Hence $\nabla(\frac{d}{dx})^i(m_0) \subset p^{[\frac{i}{p}]}\mathcal{M}$, where $[\frac{i}{p}]$ is the largest integer no greater than $\frac{i}{p}$, and one sees easily that the series defining m converges on $D(x_0, p^{-\frac{1}{p(p-1)}})$. Since this map is identity when restricted to M_{x_0}, this map defines a section of

$$M\Big|_{D(x_0, p^{-\frac{1}{p(p-1)}})} \to \left(M\Big|_{D(x_0, p^{-\frac{1}{p(p-1)}})}\right)^{\nabla = 0}.$$

$\square$

Remark 2.1.6. Let X_0 be a smooth geometrically irreducible curve over K. Let v be a finite place of K. Assume that $X_{0,\mathcal{O}_v}$ is a spread out of X_0 over $\mathcal{O}_v$ such that $X_{0,\mathcal{O}_v}$ is smooth at some point $x_0 \in X_{0,\mathcal{O}_v}(\mathcal{O}_v)$. Then the v-adic neighborhood of x_0 in X_0 is a disc of radius 1.

(1) Unlike the notion of p-curvature, the definition of $G_w(x_0, r)$ does not require (M, ∇) to have good reduction. It depends only on the choice of $\mathcal{O}_v$-model $X_{0,\mathcal{O}_v}$ of X_0 (which we of course always take to be $\mathbb{P}^1_{\mathcal{O}_v} - \{0, 1, \infty\}$ when $X_0 = \mathbb{P}^1_K - \{0, 1, \infty\}$), which is used to define $D(x_0, r)$, but not on how (M, ∇) is extended.
(2) If (M, ∇) admits a Frobenius structure with respect to some Frobenius lifting on $X_{0,\mathcal{O}_v}$, then $G_w(x_0, 1)$ is trivial whenever $x_0 \in X_{0,\mathcal{O}_v}(\mathcal{O}_v)$. See for example [Ked10, 17.2.2, 17.2.3].

2.2. **The main theorem and a Tannakian consequence.** From now on we set $x_0 = \frac{1+\sqrt{3}i}{2}$, which corresponds to the elliptic curve with smallest stable Faltings height. In section 5, we will give a theoretical explanation of why this choice gives the best possible estimates. We set $G_w = G_w\big(\frac{1+\sqrt{3}i}{2}, p^{-\frac{1}{p(p-1)}}\big)$, and we take L to be a number field containing $K(\sqrt{3}i)$.

By Lemma 2.1.5, the p-adic differential Galois group G_w is trivial when the vector bundle with connection (M, ∇) has good reduction over v, and $\psi_p \equiv 0$. This motivates the following assumption:

Assumption 2.2.1. The vector bundle with connection (M, ∇) satisfies that

(1) $\psi_p \equiv 0$ for all but finitely many finite primes $\mathfrak{p}$, and
(2) for every finite prime $\mathfrak{p}$, we have $(*)_\mathfrak{p}$: the group $G_w = \{1\}$ for all finite places w of L above $\mathfrak{p}$.

By definition, $(*)_\mathfrak{p}$ means that for every $w|\mathfrak{p}$, all horizontal sections of (M, ∇) centered at x_0 have convergence radii to be at least $p^{-\frac{1}{p(p-1)}}$. By what we have just seen, for all but finitely many $\mathfrak{p}$, the condition (1) makes sense, and implies (2). Thus (2) is only an extra condition at finitely many primes. In particular, if for all finite places, the p-curvatures are defined and vanish, then both conditions are satisfied. As above, the definition does not depend on the extension of (M, ∇) to $X_{\mathcal{O}_{K,\Sigma}}$ or the choice of primes Σ.

Theorem 2.2.2. *Let (M, ∇) be a vector bundle with a connection over $X_K = \mathbb{P}^1_K - \{0, 1, \infty\}$. If Assumption 2.2.1 holds for (M, ∇), then (M, ∇) admits a full set of rational solutions.*

The proof of this theorem is the subject of sections 3, 4.

Remark 2.2.3. André has pointed out that, if one replaces (2) in Definition 2.2.1 by the condition that the so called *generic radii* [3] of all formal horizontal sections of (M, ∇) are at least $p^{-\frac{1}{p(p-1)}}$, then the analogue of Theorem 2.2.2 admits an easier proof. Indeed if $w|p$, and the w-adic *generic radius* is at least $p^{-\frac{1}{p(p-1)}}$, then by [BS82, Sec. IV], p cannot divide the (finite by (1) and Katz's theorem [Kat70, Thm. 13.0]) order of the local monodromies of the complex local system corresponding to $(M_\mathbb{C}, \nabla)$ around $0, 1$ and ∞. If this condition holds for all w, then the local

[3] [Ked10, Def. 9.4.4] and see [Ked10, Prop. 9.7.5] for the geometric interpretation as the convergence radius at a generic point.

monodromies around $0, 1, \infty$ are all trivial and hence the global monodromy is trivial.

Once one uses (1) to show that the local monodromies are finite, this argument is 'prime by prime'. We do not know if Theorem 2.2.2 admits a similar proof, which avoids global arguments, although this seems to us unlikely. In any case, our method allows us to deal with some cases when X is an affine elliptic curve or the projective line minus more than three points. See Theorem 6.1.5 and Proposition 7.2.1. The conclusion of both results is that (M, ∇) has finite monodromy and we will give examples in section 7 with nontrivial monodromy. It seems unlikely that these results can be proved with a 'prime by prime' argument.

Applying Lemma 2.1.5, we have the following corollary:

Corollary 2.2.4. *If (M, ∇) is defined over $X_{\mathbb{Z}}$ and the p-curvature vanishes for all primes, then (M, ∇) admits a full set of rational solutions.*

2.2.5. As in [Kat82], we can use our main theorem to give a description of the differential Galois group of any vector bundle with a connection (M, ∇) over X_K.

Let $K(X)$ be the function field of X_K. Let ω be the fibre functor on $\langle M\rangle^{\otimes}$ given by restriction to the generic point of X_K. Write $G_{\mathrm{gal}} = \underline{\mathrm{Aut}}^{\otimes}\,\omega \subset \mathrm{GL}(M_{K(X)})$ for the corresponding differential Galois group (see [Kat82, Ch. IV] and [And04, 1.3, 1.4]).

Let G be the smallest closed subgroup of $\mathrm{GL}(M_{K(X)})$ such that:

(1) For almost all p, the reduction of $\mathrm{Lie}\, G$ mod p contains ψ_p.
(2) $G \otimes F_w$ contains G_w for all w, where, as above, F_w is the field of fractions of the ring of rigid analytic functions on $D\big(x_0, p^{-\frac{1}{p(p-1)}}\big)$.

Let $\mathfrak{g}$ be the smallest Lie subalgebra of $\mathrm{GL}(M_{K(X)})$ such that for almost all p, the reduction of $\mathfrak{g}$ mod p contains ψ_p. As proved in [Kat82, Prop. 9.3], $\mathfrak{g}$ is contained in $\mathrm{Lie}\, G_{\mathrm{gal}}$. Moreover, G_w is contained in $G_{\mathrm{gal}} \otimes F_w$ by definition. Hence G is a subgroup of G_{gal}. We will see from the proof of the following theorem that (in the presence of the condition (1)), to define G we only need to impose the condition (2) at finitely many primes.

Theorem 2.2.6. *Let (M, ∇) be a vector bundle with a connection defined over $X_K = \mathbb{P}^1_K - \{0,\,1,\,\infty\}$. Then $G = G_{\mathrm{gal}}$.*

Proof. We follow the idea of the proof of Theorem 10.2 in [Kat82]. See also [And04, Prop. 3.2.2].

By a theorem of Chevalley, there exists W in $\langle M\rangle^{\otimes}$ and a line $L' \subset W_{K(X)}$ such that G is the intersection of G_{gal} with the stabilizer of L'. Let W' be the smallest ∇-stable submodule of $W_{K(X)}$ containing L'. Then W' has a $K(X)$-basis of the form $\{l, \nabla l, \cdots, \nabla^{r-1} l\}$ where $l \in L'$, $r = \mathrm{rk} W'$, and we have written $\nabla^i l$ for $\nabla(\frac{d}{dx})^i(l)$. Replacing W by $W' \cap W$, we may assume that $W_{K(X)} = W'$. Then $L = L' \cap W$ is a line bundle in W.

As above, let $\mathfrak{g}$ be the smallest algebraic Lie subalgebra of $\mathrm{GL}(M_{K(X)})$ such that for almost all p the reduction of $\mathfrak{g}$ mod p contains ψ_p. Let Σ be a finite set of primes of $\mathbb{Q}$ such that (M, ∇) extends to a vector bundle $\mathcal{M}$ with connection $\nabla\colon \mathcal{M} \to \mathcal{M} \otimes \Omega_{X_{\mathcal{O}_{K,\Sigma}}}$ over $X_{\mathcal{O}_{K,\Sigma}}$, and $\mathfrak{g}$ mod p contains ψ_p for $p \notin \Sigma$. We also assume that Σ contains all primes $p \le r$.

Let $U \subset X_{\mathcal{O}_{K,\Sigma}}$ be a non-empty open subset such that $l \in L|_U$, L and W extend to vector bundles with connection $\mathcal{L}$ and $\mathcal{W}$ respectively, in $\langle \mathcal{M}|_U\rangle^{\otimes}$, and

$\{l, \nabla l, \cdots, \nabla^{r-1}l\}$ forms a basis of $\mathcal{W}$. Let $\mathcal{N} := \mathrm{Sym}^r \mathcal{W} \otimes (\det \mathcal{W}^\vee)$ with the induced connection. The argument in [Kat82] implies that for $p \notin \Sigma$, the p-curvature of $(\mathcal{N}, \nabla)$ vanishes. Let $N := \mathcal{N}_{X_K \cap U}$. We will use the condition (2) in the definition of G to show that G_w acts trivially on N_{η_w}. We already know this for $p \notin \Sigma$, by Lemma 2.1.5. Thus we will only need to use (2) for $p \in \Sigma$. Assuming this for a moment, we can apply Theorem 2.2.2 to (N, ∇) and conclude that it has trivial global monodromy. Hence G_{gal} acts as a scalar on W. In particular, G_{gal} stabilizes L so, by the definition of L, $G_{\mathrm{gal}} = G$.

Let $D := D(x_0, p^{-\frac{1}{p(p-1)}})$. Recall that the category $\langle M|_{D(x_0,r)}\rangle^{\otimes} \otimes F_w$ is obtained from $\langle M|_{D(x_0,r)}\rangle^{\otimes}$ by taking the same collection of objects and tensoring the morphisms by F_w. By the definition of L, the group G_w acts as a character χ on L_{η_w}. The morphism $L_{\eta_w} \to W_{\eta_w}$ is a map between G_w-representations. By the equivalence of categories between $\langle M|_{D(x_0,r)}\rangle^{\otimes} \otimes F_w$ and the category of linear representations of G_w over F_w, this morphism is a finite F_w-linear combination of maps $L|_D \to W_D$ in $\langle M|_{D(x_0,r)}\rangle^{\otimes}$. In other words, there are a finite number of ∇-stable line bundles $W_i \subset W_D$, with G_w acting on W_{i,η_w} as χ such that $L|_D \subset \sum W_i$. In particular, $l|_D = \sum a_i \cdot w_i$, where $a_i \in F_w$ and $w_i \in W_i$. Since $\sum W_i$ is ∇-stable, $\nabla^n l \in \sum W_i$ and G_w acts as χ on $\nabla^n l|_D$. As W_{η_w} is generated by $\{l, \nabla l, \cdots, \nabla^{r-1}l\}|_D$, the group G_w acts as χ on W_{η_w}. Hence G_w acts trivially on N_{η_w}. □

Using the same idea as in the last paragraph of the proof above, we have the following lemma which is of independent interest.

Lemma 2.2.7. *Let $H_w \subset G_{\mathrm{gal}}$ be the smallest closed subgroup such that $G_w \subset H_w \otimes_{K(X)} F_w$. Then H_w is normal in G_{gal}.*

Proof. We need the following fact (see [And92, Lem. 1]): Assume that G is a algebraic group over some field E. Let $H \subset G$ be a closed subgroup and V an E-linear faithful algebraic representation of G. Then H is a normal subgroup of G if for every tensor space $V^{m,n} := V^{\otimes m} \otimes (V^\vee)^{\otimes n}$, and for every character χ of H over E, G stabilizes $(V^{m,n})^\chi$, the subspace of $V^{m,n}$ where H acts as χ. If G is connected, then these two conditions are equivalent.

We apply this result to $H_w \subset G_{\mathrm{gal}}$ and $V = M_{K(X)}$. Let $L \subset V^{m,n}$ be a line, and $W \subset V^{m,n}$ the smallest ∇-stable subspace containing L. It suffices to show that, if H_w acts via χ on L, then H_w acts via χ on W. This shows that $(V^{m,n})^\chi$ is ∇-stable, and hence that G_{gal} stabilizes $(V^{m,n})^\chi$.

As in the proof of the theorem above, G_w acts on W via χ. Hence H_w is contained in the subgroup of G_{gal} which acts on W via χ. □

3. Algebraicity: an application of André's theorem

The main goal of this section is to prove a weaker version of Theorem 2.2.2. Namely, that if (M, ∇) is a vector bundle with a connection over $X_K = \mathbb{P}^1_K - \{0, 1, \infty\}$ satisfying Assumption 2.2.1, then (M, ∇) admits a full set of algebraic solutions.

3.1. **André's algebraicity criterion.**

3.1.1. As the coordinate ring of X_K a principal ideal domain, M is free. Hence we may view ∇ as a system of first-order homogeneous differential equations. Thus

$M \cong \mathcal{O}_{X_K}^m$ and $\nabla(\frac{d}{dx})\boldsymbol{y} = \frac{d\boldsymbol{y}}{dx} - A(x)\boldsymbol{y}$, where $\boldsymbol{y}$ is a section of M, x is the coordinate of X, and $A(x)$ is an $m \times m$ matrix with entries in $\mathcal{O}_{X_K} = K[x^{\pm}, (x-1)^{\pm}]$.

As above, we set $x_0 = \frac{1}{2}(1+\sqrt{3}i)$. If $\boldsymbol{y}_0 \in L^m$, there exists $\boldsymbol{y} \in L[[x-x_0]]^m$ such that $\boldsymbol{y}(x_0) = \boldsymbol{y}_0$ and $\nabla(\boldsymbol{y}) = 0$. Our goal is to show that if (M, ∇) satisfies Assumption 2.2.1, then $\boldsymbol{y}$ is algebraic.

3.1.2. Now let $y \in K[[x]]$, and let v be a place of K. If v is finite, we denote by p the characteristic of the residue field. Let $|\cdot|_v$ be the v-adic norm normalized so that $|p|_v = p^{-\frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}}$ if v is finite, and $|x|_v = |x|_\infty^{-\frac{[K_v:\mathbb{R}]}{[K:\mathbb{Q}]}}$ for $x \in K$, if v is archimedean, where $|x|_\infty$ denotes the Euclidean norm on K_v. When there is no confusion, we will also write $|\cdot|$ for $|\cdot|_\infty$. For a positive real number R, we denote by $D_v(0,R)$ the rigid analytic z-disc of v-adic radius R. That is $D_v(0,R)$ is defined by the inequality $|z|_v < R$.

We first state the definition of v-adic uniformization and the associated radius R_v defined in André's paper ([And04, Def. 5.4.1]).

Definition 3.1.3.

(1) For $R \in \mathbb{R}^+$, a *v-adic uniformization* of y by $D_v(0,R)$ is a pair of meromorphic v-adic functions $g(z)$, $h(z)$ on $D_v(0,R)$ such that $h(0) = 0$, $h'(0) = 1$ and $y(h(z))$ is the germ at 0 of the meromorphic function $g(z)$.

(2) Let R_v be the supremum of the set of positive real R for which a v-adic uniformization of y by $D_v(0,R)$ exists. We call R_v the v-adic *radius (of uniformizability)*.

3.1.4. In order to state the algebraicity criterion, we need to introduce two constants $\tau(y)$, $\rho(y)$, which play similar roles as the global-boundedness condition in the Borel–Dwork rationality criterion. Let $y = \sum_{n=0}^\infty a_n x^n$. We define

$$\tau(y) = \inf_l \limsup_n \sum_{v,\, p \geq l} \frac{1}{n} \sup_{j \leq n} \log^+ |a_j|_v,$$

$$\rho(y) = \sum_v \limsup_n \frac{1}{n} \sup_{j \leq n} \log^+ |a_j|_v,$$

where $\log^+$ is the positive part of log, that is $\log^+(a) = \log(a)$ if $a > 1$ and is zero otherwise. The following is a slight reformulation of André's criterion.

Theorem 3.1.5 ([And04, Thm. 5.4.3, Cor. 5.4.5])**.** *Let $y \in K[[x]]$ such that $\tau(y) = 0$ and $\rho(y) < \infty$. For instance, y can be taken to be a (component of a) formal solution of (M, ∇) with vanishing p-curvatures for all but finitely many primes. Let R_v be the v-adic radius of y. If $\prod_{\textit{all places } v} R_v > 1$, then y is algebraic over $K(x)$.*

In our applications of this theorem, R_v will always be finite and non-zero and the infinite product (independent of the choice of the order of multiplication) converges to a finite number. In general, the v-adic radius R_v may be infinity and the product of all but finitely many places may converge to zero. We refer the reader to André's paper for a precise definition of the infinite product in such situations. We remark that we could have also used Thm. 6.1 and Prop. 5.15 in [BCL09] in place of André's Theorem.

3.2. **Estimate of the radii at archimedean places.** In this subsection, w is always an archimedean place. For any real number R, we use $D(0,R)$ to denote the complex analytic disc centered at 0 of radius R (with respect to Euclidean norm). We begin with the following simple lemma.

Lemma 3.2.1. *Suppose that $\phi\colon D(0,1) \to \mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}$ is a holomorphic map such that $\phi(0) = x_0$ and $\phi'(0) \neq 0$. Then for any the number field L where the connection and the initial conditions x_0, $\boldsymbol{y}_0$ are defined, the w-adic radius R_w of each component of the formal horizontal section $\boldsymbol{y}$ of (M,∇) such that $\boldsymbol{y}(x_0) = \boldsymbol{y}_0$ is no less than $|\phi'(0)|_w$.*

Proof. Let z be the complex coordinate on $D(0,1)$. Consider the formal power series $\phi^*\boldsymbol{y}$. The vector valued power series $\boldsymbol{g} = \phi^*\boldsymbol{y}$ is a formal solution of the differential equations $\frac{d\boldsymbol{g}}{dz} = (\phi'(z))A(\phi(z))\boldsymbol{g}$, which is associated to the vector bundle with connection $(\phi^*M, \phi^*\nabla)$. Since $D(0,1)$ is simply connected, $\boldsymbol{g}$ arises from a vector valued holomorphic function on $D(0,1)$ which we again denote by $\boldsymbol{g}$.

Consider the holomorphic map

$$\tilde{\phi} : D(0,R) \to \mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}, \quad x \mapsto \phi(x/\phi'(0)).$$

We have that $\tilde{\phi}'(0) = 1$ and the previous discussion shows that $\tilde{\phi}^*\boldsymbol{y}$ converges on $D(0,R)$. Therefore, by Definition 3.1.3, we have that $R_w \geq |\phi'(0)|_w$. □

3.2.2. Given x_0, the upper bound (in terms of x_0) of $|\phi'(0)|$ for all such ϕ in the above lemma has been studied by Landau and other people. Based on the work of Landau and Schottky, Hempel gave an explicit upper bound (see [Hem79, Thm. 4]) that can be reached when $x_0 = \frac{-1+\sqrt{3}i}{2}$. For the completeness of our paper, we give some details on the computation of $|\phi'(0)|$.

3.2.3. We recall the definition of θ-functions and their classical relation with the uniformization of $\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}$. Following the notation of [Igu62] and [Igu64], let

$$\theta_{00}(t) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 t),\ \theta_{01}(t) = \sum_{n\in\mathbb{Z}} \exp(\pi i(n^2t+n)), \theta_{10}(t) = \sum_{n\in\mathbb{Z}} \exp(\pi i(n+\frac{1}{2})^2 t)$$

These series converge pointwise to holomorphic functions on the upper half plane $\mathcal{H}$, which we denote by the same symbols.

Lemma 3.2.4. *(*[Igu64, p. 243]*) These holomorphic functions $\theta_{00}^4, \theta_{01}^4, \theta_{10}^4$ are modular forms of weight 2 and level $\Gamma(2)$. Moreover, there is an isomorphism from the ring of modular forms of level $\Gamma(2)$ to $\mathbb{C}[X,Y,Z]/(X-Y-Z)$ given by sending $\theta_{00}^4, \theta_{01}^4$ and θ_{10} to X, Y and Z respectively.*

3.2.5. Let $\lambda = \frac{\theta_{00}^4(t)}{\theta_{01}^4(t)} \colon \mathcal{H} \to \mathbb{P}^1(\mathbb{C})$ and $t_0 = \frac{1}{2}(-1+\sqrt{3}i)$. Then $\lambda : \mathcal{H} \to \mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}$ is a covering map with $\Gamma(2)$ as the deck transformation group ([Cha85], VII, §7). In particular, the projective curve defined by $v^2 = u(u-1)(u-\lambda(t))$ is an elliptic curve. Moreover, it is isomorphic to the elliptic curve $\mathbb{C}/(\mathbb{Z}+t\mathbb{Z})$ (see *loc. cit.*).

We need the following basic facts mentioned in [Igu62, p. 180] and [Igu64, p. 244] in this section and section 5:

Lemma 3.2.6.

(1) *Let η be the Dedekind eta function defined by $\eta = q^{1/24}\prod(1-q^n)$, where $q = e^{2\pi it}$. We have $2^8\eta^{24} = (\theta_{00}\theta_{01}\theta_{10})^8$. In particular, the holomorphic functions $\theta_{00}, \theta_{01}, \theta_{10}$ are everywhere nonzero on the upper half plane.*
(2) *The derivative $\lambda'(t_0)$ equals to $\pi i(\frac{\theta_{00}(t_0)\theta_{10}(t_0)}{\theta_{01}(t_0)})^4$.*
(3) *The holomorphic function $\frac{1}{2}(\theta_{00}^8 + \theta_{01}^8 + \theta_{10}^8)$ is the weight 4 Eisenstein form of level $\mathrm{SL}_2(\mathbb{Z})$ with constant term 1 in its Fourier expansion; the holomorphic function $\frac{1}{2}(\theta_{00}^4 + \theta_{01}^4)(\theta_{00}^4 + \theta_{10}^4)(\theta_{01}^4 - \theta_{10}^4)$ is the weight 6 Eisenstein form of level $\mathrm{SL}_2(\mathbb{Z})$ with constant term 1 in its Fourier expansion.*

Lemma 3.2.7. *The map λ sends t_0 to x_0.*

Proof. Since the automorphism group of the lattice $\mathbb{Z}+t_0\mathbb{Z}$, hence that of the elliptic curve $\mathbb{C}/(\mathbb{Z}+t_0\mathbb{Z})$ is of order 6, the automorphism group of the elliptic curve $v^2 = u(u-1)(u-\lambda(t_0))$ must also be of order 6. In particular, λ must send t_0 to either $\frac{1}{2}(1+\sqrt{3}i)$ or $\frac{1}{2}(1-\sqrt{3}i)$ (the roots of $0 = j(t_0) = 2^8\frac{(\lambda(t_0)^2-\lambda(t_0)+1)^3}{\lambda(t_0)^2(\lambda(t_0)-1)^2}$). Moreover, from the definition of θ, we can easily see that $\lambda(t_0)$ has positive imaginary part. □

Proposition 3.2.8. *Let y be a component of a formal horizontal section of (M,∇). Then $R_w^{\frac{[L:\mathbb{Q}]}{[L_w:\mathbb{R}]}} \geq \frac{3\Gamma(1/3)^6}{2^{8/3}\pi^3} = 5.632\cdots$.*

Proof. Consider the map $\lambda\circ\alpha\colon D(0,1)\to X_{\mathbb{C}}$, where $\alpha\colon D(0,1)\to\mathcal{H}$ is a holomorphic isomorphism such that $\alpha(0) = t_0$, that is, $\alpha\colon z\mapsto -\frac{1}{2}+\frac{\sqrt{3}i}{2}\frac{z+1}{1-z}$. We would like to apply Lemma 3.2.1 to the map $\lambda\circ\alpha$, which maps $0\in D(0,1)$ to x_0 since $\lambda(t_0) = \lambda(\frac{1}{2}(-1+\sqrt{3}i)) = x_0$ by Lemma 3.2.7.

Note that $|x_0| = |1-x_0| = 1$, so we have $|\theta_{00}(t_0)| = |\theta_{01}(t_0)| = |\theta_{10}(t_0)|$. By Lemma 3.2.6, we have

$$|\lambda'(t_0)| = |\pi i(\frac{\theta_{00}(t_0)\theta_{10}(t_0)}{\theta_{01}(t_0)})^4| = \pi|\theta_{00}(t_0)|^4 = \pi|2^8\eta^{24}(t_0)|^{1/6}.$$

We now apply the Chowla–Selberg formula (see [SC67]) to $\mathbb{Q}(\sqrt{3}i)$:

$$|\eta(t_0)|^4\Im(t_0) = \frac{1}{4\pi\sqrt{3}}\left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)^3.$$

Then we have

$$|\lambda'(t_0)| = \pi|2^8\eta^{24}(t_0)|^{1/6} = \frac{\pi 2^{4/3}}{4\pi\sqrt{3}\Im(t_0)}\left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)^3.$$

We get

$$|(\lambda\circ\alpha)'(0)| = |\lambda'(t_0)|\cdot|\alpha'(0)| = \frac{\pi 2^{4/3}}{4\pi\sqrt{3}\Im(t_0)}\left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)^3\cdot 2\Im(t_0) = \frac{3\Gamma(1/3)^6}{2^{8/3}\pi^3}$$

by the fact $\Gamma(1/3)\Gamma(2/3) = \frac{2\pi}{\sqrt{3}}$. □

3.3. **Algebraicity of formal solutions.**

Proposition 3.3.1. *Let (M,∇) be a vector bundle with a connection over $\mathbb{P}_K^1 - \{0,1,\infty\}$. If Assumption 2.2.1 holds for (M,∇), then (M,∇) is locally trivial with respect to the étale topology of $\mathbb{P}_K^1 - \{0,1,\infty\}$.*

Proof. Consider $\boldsymbol{y} \in L[[(x-x_0)]]$. By Proposition 3.2.8, we have

$$\prod_{w|\infty} R_w \geq 5.632\cdots.$$

If $w|p$ is a finite place of L, then since G_w is trivial, (M,∇) has a full set of solutions over $D(x_0, |p|^{\frac{1}{p(p-1)}})$. In particular, $\boldsymbol{y}$ is analytic on $D(x_0, |p|^{\frac{1}{p(p-1)}})$. Hence

$$\prod_{w|p} R_w \geq \prod_{w|p} |p|_w^{-\frac{1}{p(p-1)}} = p^{-\frac{1}{p(p-1)}}.$$

and

$$\log(\prod_w R_w) \geq \log 5.6325\cdots - \sum_p \frac{\log p}{p(p-1)} > 0.967\cdots.$$

Applying Theorem 3.1.5, we have that $\boldsymbol{y}$ is algebraic. Hence (M,∇) is étale locally trivial. □

Remark 3.3.2. It is possible to define G_w using different radii such that the proof of the above proposition continues to hold. Here are two examples:

(1) Set $G'_w := G_w(x_0, \frac{1}{4})$ for all primes $w|2$ and $G'_w = G_w(x_0, 1)$ for other w. In this situation, we have the following estimate of the w-adic radii of a formal solution.

$$\log(\prod_w R_w) \geq \log 5.6325\cdots - \log 4 > 0.342\cdots.$$

Then Theorem 3.1.5 implies that, if (M,∇) is a vector bundle with connection on X_K such that $\psi_p \equiv 0$ for almost all p, and $G'_w = \{1\}$ for all w, then (M,∇) has finite monodromy. This result cannot be proved 'prime by prime' because the condition at $w|2$ is too weak to imply that 2 does not divide the order of the local monodromies.

We define G' in the same way as G in section 2.2.5 but replacing G_w by G'_w. Applying the same argument as in Theorem 2.2.6, we have $\operatorname{Lie} G' = \operatorname{Lie} G_{\text{gal}}$.

The equality $\operatorname{Lie} G' = \operatorname{Lie} G_{\text{gal}}$ fails in general, if one drops condition (1) in section 2.2.5, and defines G' using just the analogue of condition (2) (that is with G_w replaced by G'_w). (The condition (1) is used to guarantee the assumption that $\tau(y) = 0, \rho(y) < \infty$ in Theorem 3.1.5.)

Here is an example. We consider the Gauss–Manin connection on H^1_{dR} of the Legendre family of elliptic curves. Since the Legendre family has good reduction at primes $w \nmid 2$, H^1_{dR} admits a Frobenius structure at such primes, so that $G_w = \{1\}$ (see Remark 2.1.6). For $w|2$ we have $G_w(x_0, \frac{1}{4}) = \{1\}$ by a direct computation: as in section 5.2 below, we see that the matrix giving the connection lies in $\frac{1}{2}\operatorname{End}(M_{\mathcal{O}_K}) \otimes \Omega^1_{X_{\mathcal{O}_K}}$ and a formal horizontal section of a general differential equation of this form will have convergence radius $\frac{1}{4}$. Hence, the smallest group containing all p-adic differential Galois groups is trivial while $\operatorname{Lie} G_{\text{gal}} = \mathfrak{sl}_2$. In particular, G' (defined with the condition (1)) is the smallest group containing almost all ψ_p and we recover a special case of [Kat82, thm. 11.2].

(2) We now consider a variant of our result when X equals to $\mathbb{P}^1$ minus more than three points. Let D be the union of $\{0\}$ and all 8-th roots of unity and let $X = \mathbb{A}^1 - D$. Let u_0 be one of the preimages of x_0 of the covering map $f : X \to \mathbb{P}^1 - \{0, 1, \infty\}, u \mapsto x = -\frac{1}{4}(u^4 + u^{-4} - 2)$. We may assume that the number field L contains u_0.

We consider the following weaker version of p-curvature conjecture:

Proposition 3.3.3. *Let (N, ∇) be a vector bundle with connection over X. Assume that the p-curvatures vanish for almost all $\mathfrak{p}$ and that for any finite place w, all the formal horizontal sections of (N, ∇) converges over the largest disc around u_0 in X_{L_w}. Then (N, ∇) must be étale locally trivial.*

By direct calculation, the w-adic distance from u_0 to D is $|2|_w^{\frac{1}{4}}$ when w is finite. Then our assumption means that all the formal horizontal sections of (N, ∇) centered at u_0 converge over $D(u_0, |2|_w^{\frac{1}{4}})$.

Proof of the proposition. By applying Theorem 3.1.5 to the formal horizontal sections around u_0, one only need to show that $\prod_{w|\infty} R_w \geq 2^{1/4}$. Fix an archimedean place w. Since the uniformization $\lambda \circ \alpha : D(0,1) \to \mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}$ factors through $f : \mathbb{A}^1(\mathbb{C}) - D \to \mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}$, then for the formal horizontal sections of (N, ∇), we have $R_w \geq |5.632\cdots|_w / |f'(u_0)|_w$ by the chain rule and Lemma 3.2.1. A direct computation shows that $\prod_{w|\infty} |f'(u_0)|_w = 4$ and then $\prod_{w|\infty} R_w \geq 2^{1/4}$ by the fact $5.6325... > 4 \cdot 2^{1/4}$. $\square$

We now formulate another possible proof of this proposition. The idea is to reduce the problem for (N, ∇) over X to $f_*(N, \nabla)$ over $\mathbb{P}^1 - \{0, 1, \infty\}$. Over $\mathbb{P}^1 - \{0, 1, \infty\}$, the assumption on (N, ∇) shows that for $f_*(N, \nabla)$, the p-adic differential group $G'_w := G_w(x_0, 1) = 1$ for $w \nmid 2$. Although for $w|2$, the 2-adic differential group $G'_w := G_w(x_0, 2^{-9/4})$ is not trivial, we still have $R_w \geq |2|_w^{9/4}$ by considering the uniformization $h(z) = -\frac{1}{4}((\frac{z}{4} + u_0)^4 + (\frac{z}{4} + u_0)^{-4} - 2)$. More precisely, by the assumption on (N, ∇), we can take $R = |4|_w \cdot |2|_w^{1/4}$ in Definition 3.1.3 and check that $|h'(0)|_w = 1$ and $h(0) = x_0$. Then we apply André's theorem and conclude that $f_*(N, \nabla)$ and hence (N, ∇) admit a full set of algebraic solutions.

If one replaces the assumption in Proposition 3.3.3 by that the generic radii of all formal horizontal sections of (N, ∇) are at least $|2|_w^{\frac{1}{4}}$ for all w finite, the results in [BS82] does not apply directly due to the fact that the points in D are too close to each other in L_w when $w|2$. However, one may modify the argument there, especially a modified version of eqn. (3) in *loc. cit.*, to see that the condition on generic radii would imply trivial monodromy of (N, ∇).

4. Rationality: an application of a theorem of Bost and Chambert-Loir

In this section, we will first review the rationality criterion due to Bost and Chambert-Loir for an algebraic formal function using capacity norms. Then we will use the moduli interpretation of X to compute the capacity norm and verify that in our situation this theorem is applicable.

4.1. **Review of the rationality criterion.** We will review the definition of adélic tube adapted to a given point, the definition of capacity norms for the special case we need, and the rationality criterion in [BCL09].

Definition 4.1.1. ([BCL09, Definition 5.16]) Let Y be a smooth projective curve over $\mathcal{O}_K[1/N]$, and let (x_0) be the divisor corresponding to a given point $x_0 \in Y(L)$ for some number field $L \supset K$. For each finite place w of L, let Ω_w be a rigid analytic open subset of Y_{L_w} containing x_0. For each archimedean place w, we choose one embedding $\sigma : L \to \mathbb{C}$ corresponding to w and we let Ω_w be an analytic open set of $Y_\sigma(\mathbb{C})$ containing x_0. The collection (Ω_w) is *an adélic tube* adapted to (x_0) if the following conditions are satisfied:

(1) If w is an archimedean place, the complement of Ω_w is non-polar (e.g. a finite collection of closed domains and line segments); if w is real, we further assume that Ω_w is stable under complex conjugation.
(2) If w is a finite place, the complement of Ω_w is a nonempty affinoid subset;[4]
(3) for almost all finite places w, Ω_w is the tube of the specialization of x_0 in the special fiber of Y. That is, Ω_w, is the open unit disc with center at x_0.

We call (Ω_w) a *weak* adélic tube if we drop the condition that Ω_w is stable under complex conjugation when w is real.

4.1.2. Now let $Y = \mathbb{P}^1_{\mathcal{O}_K}$. The weak adélic tube that we will use can be described as follows:

(1) For an archimedean place w, Ω_w will be an open simply connected domain inside $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$.
(2) For a finite place w, Ω_w will be chosen to be an open disc of form $D(x_0, \rho_w)$.
(3) For almost all finite places w, $\rho_w = 1$.

4.1.3. For Ω_w as above, Bost and Chambert-Loir have defined the local capacity norms $||\cdot||_w^{\mathrm{cap}}$ (see [BCL09, Chapter 5]). These are norms on the line bundle $T_{x_0}X$ over $\mathrm{Spec}(\mathcal{O}_L)$. The Arakelov degree[5] of $T_{x_0}X$ with respect to these norms plays the same role as $\log(\prod R_w)$ in section 3. This degree can be computed as a local sum after choosing a section of this bundle. We will use the section $\frac{d}{dx}$, in which case one has the following simple description of local capacity norms:

(1) For an archimedean place w, let $\phi\colon D(0,R) \to \Omega_w$ be a holomorphic isomorphism that maps 0 to x_0, then $||\frac{d}{dx}||_w^{\mathrm{cap}} = |R\phi'(0)|_w^{-1}$ (see [Bos99, Example 3.4]).
(2) For a finite place w, $||\frac{d}{dx}||_w^{\mathrm{cap}} = \rho_w^{-1}$ (see [BCL09, Example 5.12].

Now, we can state the rationality criterion:

Theorem 4.1.4. *(*[BCL09, Theorem 7.8]*) Let (Ω_w)be an adélic tube adapted to (x_0). Suppose y is a formal power series over X centered at x_0 satisfying the following conditions:*

(1) For all w, y extends to an analytic meromorphic function on Ω_w;
(2) The formal power series y is algebraic over the function field $K(X)$.
(3) The Arakelov degree of $T_{x_0}X$ defined as $\displaystyle\sum_w -\log(||\frac{d}{dx}||_w^{\mathrm{cap}})$ is positive.

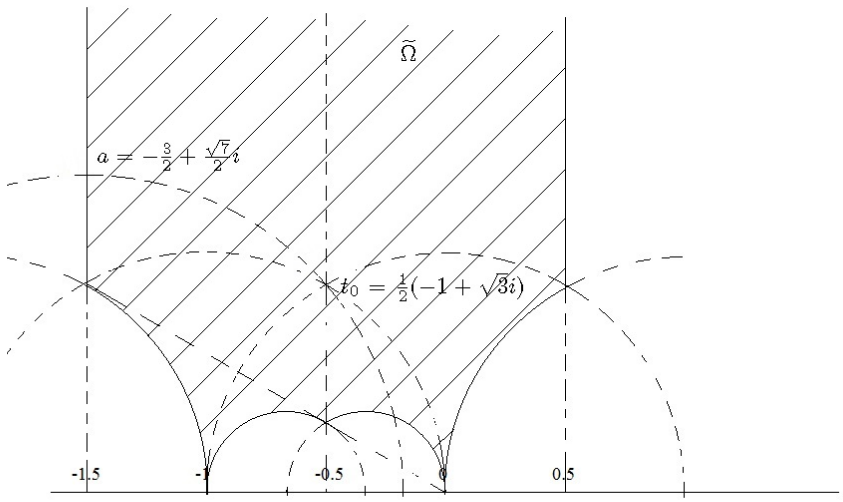
Then y is rational.

Corollary 4.1.5. *The theorem still holds if we only assume that (Ω_w) is a weak adelic tube.*

Proof. The idea is implicitly contained in the discussion in [Bos99, section 4.4]. We only need to prove that y is rational over $X_{L'}$, where L'/L is a finite extension

[4]In this paper, we only need to work with affinoids in rigid analytic spaces. We remark that the theorems that we cite remain true if one works with Berkovich spaces thanks to the work of Thuillier[Thu05].

[5]See section 5.1 for the definition of the Arakelov degree of an Hermitian line bundle. In [BCL09], they construct an integral model of X such that the local capacity norms coincide with the norms defined by the integral structure.



which we may assume does not have any real places. Let w be a place of L and w' a place of L' over w.

If w is archimedean, choose an embedding $\sigma' \colon L' \to \mathbb{C}$ corresponding to w' which extends a chosen embedding $\sigma : L \to L_w$ corresponding to w. We have a natural identification $Y_{\sigma'}(\mathbb{C}) = Y_\sigma(\mathbb{C})$, and we take $\Omega_{w'} := \Omega_w$. If w is a finite place, we set $\Omega_{w'} = \Omega_w \otimes_{L_w} L_{w'}$.

Since L' does not have any real places, the weak adélic tube $(\Omega_{w'})$ is an adélic tube. The first two conditions in Theorem 4.1.4 still hold and the Arakelov degree of $T_{x_0}X$ with respect to (Ω'_w) is the same as that of $T_{x_0}X$ with respect to (Ω_w). We can apply Theorem 4.1.4 to y over $X_{L'}$ and conclude that y is rational. □

4.2. **Proof of the main theorem.** Let y be the algebraic formal function which is one component of the formal horizontal section $\boldsymbol{y}$ of (M, ∇) over X_K.

Lemma 4.2.1. *Let y be as above. Then this formal power series centered at x_0 has convergence radius equal to 1 for almost all finite places.*

Proof. Since the covering induced by y is finite étale over X_L, by Proposition 3.3.1, it is étale over $X_{\mathcal{O}_w}$ at x_0 for almost all places. For such places, we have $\rho_w = 1$ by lifting criterion for étale maps. □

4.2.2. We now define an adélic tube (Ω_w) adapted to x_0. For an archimedean place w, we choose the embedding $\sigma : L \to \mathbb{C}$ corresponding to w such that $\sigma(x_0) = (1 + \sqrt{3}i)/2$. Let $\widetilde{\Omega}$ be the open region in the upper half plane cut out by the following six edges (see the attached figure): $\Re t = -\frac{3}{2}$, $|t + 2| = 1$, $|t + \frac{2}{3}| = \frac{1}{3}$, $|t+\frac{1}{3}| = \frac{1}{3}$, $|t-1| = 1$, and $\Re t = \frac{1}{2}$. This is a fundamental domain of the arithmetic group $\Gamma(2) \subset \mathrm{SL}_2(\mathbb{Z})$.

We define Ω_w to be $\lambda(\widetilde{\Omega})$. Here $\lambda : \mathcal{H} \to \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ is the map defined in 3.2.5.

For w finite, we choose Ω_w to be $D(x_0, 1)$ if y is étale over $X_{\mathcal{O}_w}$ at x_0; otherwise, we choose Ω_w to be $D(x_0, p^{-\frac{1}{p(p-1)}})$.

The collection (Ω_w) is a weak adélic tube and y extends to an analytic (in particular meromorphic) function on each Ω_w by Lemma 4.2.1, Lemma 3.2.1, and Lemma 2.1.5.

Lemma 4.2.3. *The Arakelov degree of $T_{x_0}X$ with respect to the adélic tube (Ω_w) defined above is positive.*

Proof. We want to give a lower bound of $(||\frac{d}{dx}||_w^{\mathrm{cap}})^{-1}$, the capacity of Ω_w. Let $a = -\frac{3}{2}+\frac{\sqrt{7}}{2}i$. On the line $\Re(t) = -\frac{3}{2}$, the point a is the closest point to $t_0 = \frac{1}{2}(-1+\sqrt{3}i)$ with respect to Poincaré metric because the geodesic passing through a and t_0 is perpendicular to the vertical line $\Im t = -3/2$. The stabilizer of t_0 in $\mathrm{SL}_2(\mathbb{Z})$ has order 3, and permutes the geodesics $\Re t = -\frac{3}{2}$, $|t+\frac{2}{3}| = \frac{1}{3}$, $|t-1| = 1$, and this action preserves the Poincaré metric. Using this, together with the fact that the distance to t_0 is invariant under $z \mapsto -1-\bar{z}$, one sees that the distance from any point on the boundary of $\widetilde{\Omega}$ to t_0 is at least that from a to t_0. Since $\alpha\colon D(0,1) \to \mathcal{H}$ (defined in the proof of Prop. 3.2.8) preserves the Poincaré metrics, $\alpha^{-1}(\widetilde{\Omega})$ contains a disc with respect to the Poincaré radius equal to the distance from t_0 to a.

In $D(0,1)$, a disc with respect to Poincaré metric is also a disc in the Euclidean sense. Hence $\alpha^{-1}(\widetilde{\Omega})$ contains a disc of Euclidean radius

$$|\alpha^{-1}(a)| = |(a-t_0)/(a-\bar{t}_0)| = 0.45685\cdots.$$

Since λ maps the fundamental domain $\widetilde{\Omega}$ isomorphically onto Ω_w, by 4.1.3, the local capacity $(||\frac{d}{dx}||_w^{\mathrm{cap}})^{-1}$ is at least $|(a-t_0)/(a-\bar{t}_0)| \cdot |\lambda'(\frac{1}{2}(-1+\sqrt{3}i))|$.

By 4.1.3, we have $-\log(||\frac{d}{dx}||_w^{\mathrm{cap}}) \geq -\frac{\log p}{p(p-1)}$ when $w|p$. Recall in Proposition 3.2.8 we have $|\lambda'(\frac{1}{2}(-1+\sqrt{3}i))| = 5.632\cdots$, hence the Arakelov degree of $T_{x_0}X$ is

$$\sum_w -\log(||\frac{d}{dx}||_w^{\mathrm{cap}}) > \log(5.6325\cdots \times 0.45685\cdots) - \sum_p \frac{\log p}{p(p-1)} > 0.184\cdots.$$

□

Now we are ready to prove Theorem 2.2.2:

Proof. Applying Proposition 3.3.1, we have a full set of algebraic solutions $\boldsymbol{y}$. Choosing the weak adélic tube as in 4.2.2 and applying Corollary 4.1.5 (the assumptions are verified by 4.2.2 and Lemma 4.2.3), we have that these algebraic solutions are actually rational.

This shows that (M,∇) has a full set of rational solutions over X_L. Since formation of $\ker(\nabla)$ commutes with the finite extension of scalars $\otimes_K L$, this implies that (M,∇) has a full set of rational solutions over X_K. □

5. Interpretation using the Faltings height

In this section, we view $X_{\mathbb{Z}[\frac{1}{2}]}$ as the moduli space of elliptic curves with level 2 structure. Let $\lambda_0 \in X(\bar{\mathbb{Q}})$ and E the corresponding elliptic curve. Using the Kodaira–Spencer map, we will relate the Faltings height of E with our lower bound for the product of radii of uniformizability (see section 3) at archimedean places of the formal solutions in $\widehat{\mathcal{O}}_{X_K,\lambda_0}$. We will focus mainly on the case when $\lambda_0 \in X(\bar{\mathbb{Z}})$ and sketch how to generalize to $\lambda_0 \in X(\bar{\mathbb{Q}})$ at the end of this section. In this section, unlike the previous sections, we will use λ as the coordinate of X.

5.1. **Hermitian line bundles and their Arakelov degrees.**

5.1.1. Let K be a number field, and $\mathcal{O}_K$ its ring of integers. Recall that an *Hermitian line bundle* $(L, ||\cdot||_\sigma)$ over $\mathrm{Spec}(\mathcal{O}_K)$ is a line bundle L over $\mathrm{Spec}(\mathcal{O}_K)$, together with an Hermitian metric $||\cdot||_\sigma$ on $L \otimes_\sigma \mathbb{C}$ for each archimedean place $\sigma\colon K \to \mathbb{C}$.

Given an Hermitian line bundle $(L, ||\cdot||_\sigma)$, its (normalized) *Arakelov degree* is defined as:

$$\widehat{\deg}(L) := \frac{1}{[K:\mathbb{Q}]}\left(\log(\#(L/s\mathcal{O}_K)) - \sum_{\sigma: K\to\mathbb{C}} \log ||s||_\sigma\right),$$

where s is any section.

For a finite place v over p, the integral structure of L defines a norm $||\cdot||_v$ on L_{K_v}. More precisely, if s_v is a generator of $L_{\mathcal{O}_{K_v}}$ and n is an integer, we define $||p^n s_v||_v = p^{-n[K_v:\mathbb{Q}_p]}$. We obtain a norm on $\mathcal{O}_v$ by viewing it as the trivial line bundle. We will use $||\cdot||_v$ for the norms on different line bundle as no confusion would arise. We may rewrite the Arakelov degree using the p-adic norms:

$$\widehat{\deg}(L) = \frac{1}{[K:\mathbb{Q}]}\left(-\sum_v \log ||s||_v\right),$$

where v runs over all places of K. It is an immediate corollary of the product formula that the right hand side does not depend on the choice of s.

5.1.2. Let E be an elliptic curve over a number field K, and denote by $e\colon \mathrm{Spec}\, K \to E$ and $f\colon E \to \mathrm{Spec}\, K$ the identity and structure map respectively. For each $\sigma\colon K \to \mathbb{C}$, we endow $e^*\Omega^1_{E/K} = f_*\Omega_{E/K}$ with the Hermitian norm given by $||\alpha||_\sigma = (\frac{1}{2\pi}\int_{\sigma E} |\alpha \wedge \bar{\alpha}|)^{\frac{\epsilon_\sigma}{2}}$, where ϵ_σ is 1 for real embeddings and 2 otherwise.

This can be used to define the Faltings height of E, which we recall precisely only in the case when E has good reduction over $\mathcal{O}_K$. Denote by $f : \mathcal{E} \to \mathrm{Spec}\, \mathcal{O}_K$ the elliptic curve over $\mathcal{O}_K$ with generic fibre E, and again write e for the identity section of $\mathcal{E}$. The norms $||\alpha||_\sigma$ make $e^*\Omega^1_{\mathcal{E}/\mathrm{Spec}(\mathcal{O}_K)} = f_*\Omega^1_{\mathcal{E}/\mathrm{Spec}(\mathcal{O}_K)}$ into a Hermitian line bundle, and we define the *(stable) Faltings height* by

$$h_F(E_\lambda) = \widehat{\deg}(f_*\Omega^1_{\mathcal{E}/\mathrm{Spec}(\mathcal{O}_K)}).$$

Notice that $h_F(E_\lambda)$ does not depend on the choice of K. Here we use Deligne's definition for convenience [Del85, 1.2]. This differs from the original definition by Faltings (see [Fal86]) by a constant $\log(\pi)$.

In general, the elliptic curve E would have semi-stable reduction everywhere after some field extension. We assume this is the case and E has a Neron model $f : \mathcal{E} \to \mathrm{Spec}\, \mathcal{O}_K$ which endows $f_*\Omega^1_{\mathcal{E}/\mathrm{Spec}(\mathcal{O}_K)}$ a canonical integral structure. With the same Hermitian norm defined as above, we have a similar definition of Faltings height in the general case. See [Fal86] for details. As in the good reduction case, this definition does not depend on the choice of K.

5.1.3. We will assume that λ_0 and $\lambda_0 - 1$ are both units at each finite place. Given such a λ_0, consider the elliptic curve E_{λ_0} over $\mathbb{Q}(\lambda_0)$ defined by the equation $y^2 = x(x-1)(x-\lambda_0)$. Then E_{λ_0} has good reduction at primes not dividing 2, and potentially good reduction everywhere, since its j-invariant is an algebraic integer. Let K be a number field such that $(E_{\lambda_0})_K$ has good reduction everywhere. We denote by $\mathcal{E}_{\lambda_0}$ the elliptic curve over $\mathcal{O}_K$ with generic fiber E_{λ_0}.

5.1.4. To express our computation of radii in terms of Arakelov degrees, we endow the $\mathcal{O}_K$-line bundle $T_{\lambda_0}(X_{\mathcal{O}_K})$, the tangent bundle of $X_{\mathcal{O}_K}$ at λ_0, with the structure of an Hermitian line bundle as follows. For each archimedean place $\sigma\colon K \to \mathbb{C}$, we have the universal covering $\lambda : \mathcal{H} \to \sigma X$, introduced in 3.2.5. The $\mathrm{SL}_2(\mathbb{R})$-invariant metric $\frac{dt}{2\Im(t)}$ on the tangent bundle of $\mathcal{H}$ induces the desired metric on the tangent bundle via push-forward. As in the proof of Proposition 3.2.8, our lower bound on the radius of the formal solution is $|2\Im(t_0)\lambda'(t_0)|^{\epsilon_\sigma} = ||\frac{d}{d\lambda}||_\sigma^{-1}$, where t_0 is a point on $\mathcal{H}$ mapping to λ_0. It is easy to see the left hand side does not depend on the choice of t_0. Under the assumptions in 5.1.3, the tangent vector $\frac{d}{d\lambda}$ is an $\mathcal{O}_K$-basis vector for the tangent bundle $T_{\lambda_0}(X_{\mathcal{O}_K})$, and we have

$$\widehat{\deg}(T_{\lambda_0}X) = \frac{1}{[K:\mathbb{Q}]}(-\sum_{\sigma:K\to\mathbb{C}} \log||\frac{d}{d\lambda}||_\sigma) \leq \frac{1}{[K:\mathbb{Q}]}\log(\prod_\sigma R_\sigma),$$

where the R_σ are the radius of uniformization discussed in section 3.2.

5.2. **The Kodaira–Spencer map.** Consider the Legendre family of elliptic curves $E \subset \mathbb{P}^2_{\mathbb{Z}[\frac{1}{2}]} \times X_{\mathbb{Z}[\frac{1}{2}]}$ over $X_{\mathbb{Z}[\frac{1}{2}]}$ given by $y^2 = x(x-1)(x-\lambda)$. We have the Kodaira–Spencer map ([FC90, Ch. III,9],[Kat72, 1.1]):

$$KS\colon (f_*\Omega^1_{E/X_{\mathbb{Z}[\frac{1}{2}]}})^{\otimes 2} \to \Omega^1_{X_{\mathbb{Z}[\frac{1}{2}]}},\ \alpha\otimes\beta \mapsto \langle\alpha, \nabla\beta\rangle, \tag{5.2.1}$$

where ∇ is the Gauss–Manin connection and $\langle\cdot,\cdot\rangle$ is the pairing induced by the natural polarization.

5.2.2. Following Kedlaya's notes ([Ked, Sec. 1,3]), we choose $\{\frac{dx}{2y}, \frac{xdx}{2y}\}$ to be an integral basis of $H^1_{dR}(E/X)|_{\lambda_0}$ and compute the Gauss–Manin connection:

$$\nabla\frac{dx}{2y} = \frac{1}{2(1-\lambda)}\frac{dx}{2y}\otimes d\lambda + \frac{1}{2\lambda(\lambda-1)}\frac{xdx}{2y}\otimes d\lambda.$$

The Kodaira–Spencer map then sends $(\frac{dx}{2y})^{\otimes 2}$ to $\frac{1}{2\lambda(\lambda-1)}d\lambda$.

This computation shows:

Lemma 5.2.3. *Given v a finite place not lying over 2, the Kodaira–Spencer map (5.2.1) preserves the $\mathcal{O}_v$-generators of $(f_*\Omega^1_{E/X_{\mathbb{Z}[\frac{1}{2}]}})^{\otimes 2}|_{\lambda_0}$ and $\Omega^1_{X_{\mathbb{Z}[\frac{1}{2}]}}|_{\lambda_0}$ when λ_0 and $\lambda_0 - 1$ are both v-units.*

5.2.4. For the archimedean places σ, we consider $f_*\Omega^1_{\sigma E/\operatorname{Spec}\mathbb{C}}$ with the metrics $||\alpha||_\sigma$ defined in section 5.1, and we endow $\Omega^1_{X_\mathbb{Z}}|_{\lambda_0}$ the Hermitian line bundle structure as the dual of the tangent bundle.

To see the Kodaira–Spencer map preserves the Hermitian norms on both sides, one may argue as follows. Notice that the metrics on $(f_*\Omega^1_{\sigma E/\operatorname{Spec}\mathbb{C}})^{\otimes 2}$ and $\Omega^1_{X_\mathbb{Z}}$ are $\mathrm{SL}_2(\mathbb{R})$-invariant (see for example [ZP09, Remark 3 in Sec. 2.3]). Hence they are the same up to a constant and we only need to compare them at the cusps. To do this, one studies both sides for the Tate curve. See for example [MB90, 2.2] for a related argument and Lemma 3.2.6 (2) for relation between θ-functions and Ω^1_X.

Here we give another argument:

Lemma 5.2.5. *The Kodaira–Spencer map preserves the Hermitian metrics:*

$$||(\frac{dx}{2y})^{\otimes 2}||_\sigma = ||\frac{d\lambda}{2\lambda_0(\lambda_0-1)}||_\sigma.$$

Proof. Let dz be an invariant holomorphic differential of $\mathbb{C}/(\mathbb{Z}\oplus t_0\mathbb{Z})$, where $\lambda(t_0) = \lambda_0$. By the theory of the Weierstrass-$\wp$ function, we have a map from the complex torus to the elliptic curve

$$u^2 = 4v^3 - g_2(t_0)v - g_3(t_0)$$

such that dz maps to $\frac{dv}{u}$. Here g_2 is the weight 4 modular form of level $\mathrm{SL}_2(\mathbb{Z})$ with $\frac{4\pi^4}{3}$ as the constant term in its Fourier series and g_3 is the weight 6 modular form with $\frac{8\pi^6}{27}$ as the constant term. Using Lemma 3.2.6 (3), we see that the right hand side has three roots: $\frac{\pi^2}{3}(\theta_{00}^4(t_0)+\theta_{01}^4(t_0)), -\frac{\pi^2}{3}(\theta_{00}^4(t_0)+\theta_{10}^4(t_0)), \frac{\pi^2}{3}(\theta_{10}^4(t_0)-\theta_{01}^4(t_0))$. Hence this curve is isomorphic to $y^2 = x(x-1)(x-\lambda_0)$ via the map

$$x = \frac{v - \frac{1}{3}\pi^2(\theta_{00}^4(t_0)+\theta_{01}^4(t_0))}{-\pi^2\theta_{01}^4(t_0)},\ y = \frac{u}{2(-\pi^2\theta_{01}^4(t_0))^{3/2}}, \tag{5.2.6}$$

and we have

$$\frac{dx}{2y} = \pi i\theta_{01}^2(t_0)\frac{dv}{u} = \pi i\theta_{01}^2(t_0)dz.$$

Hence

$$||(\frac{dx}{2y})^{\otimes 2}||_\sigma = |\pi^2\theta_{01}^4(t_0)\cdot(\frac{1}{2\pi}\int_{E(\mathbb{C})}|dz\wedge d\bar{z}|)|^{\epsilon_\sigma} = |\pi\theta_{01}^4(t_0)\Im(t_0)|^{\epsilon_\sigma}.$$

On the other hand, using Lemma 3.2.6 (2), we have

$$||\frac{d\lambda}{2\lambda_0(\lambda_0-1)}||_\sigma^{1/\epsilon_\sigma} = \left|\frac{2\Im(t_0)|\lambda'(t_0)|}{2\lambda_0(\lambda_0-1)}\right| = \left|\frac{\Im(t_0)\pi\theta_{00}^4(t_0)\theta_{10}^4(t_0)}{\theta_{01}^4(t_0)\lambda_0(\lambda_0-1)}\right| = |\pi\theta_{01}^4(t_0)\Im(t_0)|.$$

□

Proposition 5.2.7. *If λ_0 and $\lambda_0 - 1$ are both units at every finite places, we have $\widehat{\deg}(T_{\lambda_0}X) = -2h_F(E_{\lambda_0}) + \frac{\log 2}{3}$.*

Proof. By lemma 5.2.3 and lemma 5.2.5, we have

$$\begin{aligned}
-\widehat{\deg}(T_{\lambda_0}X) =&\widehat{\deg}(\Omega^1_{X_{O_K}}|_{\lambda_0})\\
=&\frac{1}{[K:\mathbb{Q}]}(-\sum_v \log||\frac{d\lambda}{2\lambda(\lambda-1)}||_v)\\
=&\frac{1}{[K:\mathbb{Q}]}(-\sum_{v|\infty}\log||\frac{d\lambda}{2\lambda(\lambda-1)}||_v - \sum_{v\text{ finite}}\log||\frac{d\lambda}{2\lambda(\lambda-1)}||_v)\\
=&\frac{1}{[K:\mathbb{Q}]}(-\sum_{v|\infty}\log||(\frac{dx}{2y})^{\otimes 2}||_v - \sum_{v\nmid 2,\infty}\log||(\frac{dx}{2y})^{\otimes 2}||_v\\
&-\sum_{v|2}\log||1/2||_v)\\
=&2h_F(E_{\lambda_0}) + \frac{1}{[K:\mathbb{Q}]}\sum_{v|2}\log||(\frac{dx}{2y})^{\otimes 2}||_v - \log 2.
\end{aligned} \tag{5.2.8}$$

Now we study $||(\frac{dx}{2y})^{\otimes 2}||_v$ given $v|2$. The sum $\frac{1}{[K:\mathbb{Q}]}\sum_{v|2}\log||(\frac{dx}{2y})^{\otimes 2}||_v$ does not change after extending K, hence we may assume that $\mathcal{E}_{\lambda_0}$ over $\mathcal{O}_v$ has the Deuring normal form $u^2 + auw + u = w^3$ (see [Sil09] Appendix A Prop. 1.3 and the proof of Prop. 1.4 shows in the good reduction case, a is a v-integer). An invariant differential generating $f_*\Omega^1_{\mathcal{E}_{\lambda_0}/\operatorname{Spec}\mathcal{O}_K[\frac{1}{3}]}$ is $\frac{dw}{2u+aw+1}$.

Because both $\frac{dw}{2u+aw+1}$ and $\frac{dx}{2y}$ are invariant differentials, we have $||\frac{dx}{2y}||_v = ||\Delta_1/\Delta_2||_v^{\frac{1}{12}}||\frac{dw}{2u+aw+1}||$, where Δ_1 and Δ_2 are the discriminant of the Deuring normal form and that of the Legendre form respectively. Since E has good reduction, $||\Delta_1||_v = 1$ (see the proof of *loc. cit.*). Hence $||\frac{dx}{2y}||_v = ||\frac{dw}{2u+aw+b}||_v \cdot ||1/16||_v^{1/12} = ||2||_v^{-1/3}$.

Hence $\widehat{\deg}(T_{\lambda_0}X) = -2h_F(E_{\lambda_0}) - \frac{2}{3}\log 2 + \log 2 = -2h_F(E_{\lambda_0}) + \frac{\log 2}{3}$. □

Remark 5.2.9. One may also use the formula of Faltings heights of elliptic curves due to Silverman ([Sil86, Prop. 1.1]) and Lemma 3.2.6 to deduce the above formula.

5.2.10. As pointed out by Deligne ([Del85, 1.5]), the point $\frac{1+\sqrt{3}i}{2}$ corresponds to the elliptic curve with smallest height. Hence, our choice $\frac{1+\sqrt{3}i}{2}$ gives the largest $\widehat{\deg}(T_{\lambda_0}X)$ among those λ_0 such that λ_0 and $\lambda_0 - 1$ are units at every prime.

5.3. **The general case.** For the general case when $\lambda_0 \in X(\bar{\mathbb{Q}})$, using a similar argument as in section 5.2, we have

(5.3.1)
$$\frac{1}{[K:\mathbb{Q}]}\Big(-\sum_{\sigma:K\to\mathbb{C}}\log||\frac{d}{d\lambda}||_\sigma\Big) \leq -2h_F(E_{\lambda_0}) + \frac{\log 2}{3} + \frac{1}{[K:\mathbb{Q}]}\Big(\sum_{v\text{ finite}}\log^+||\lambda_0||_v + \log(|\operatorname{Nm}\lambda_0(\lambda_0-1)|)\Big)$$

and equality holds if and only if $\lambda_0 \in X(\bar{\mathbb{Z}}_2)$. We remark that the difference between the general case and the case in section 5.2 is the contribution of finite places. From Silverman's formula ([Sil86, Prop. 1.1]), one observes that this contribution comes from the discriminant of the elliptic curve and is closely related to the valuations of λ_0 and $\lambda_0 - 1$. As discussed in 5.1.4, the left hand side is the sum of the logarithms of our estimates of the radii of uniformizability at archimedean places.

We also need to modify the estimate of the radii at finite places in Lemma 2.1.5. A possible estimate for R_v is $p^{-\frac{1}{p(p-1)}} \cdot \min\{||\lambda_0||_v, ||\lambda_0 - 1||_v, 1\}$. One explanation of the factor $\min\{||\lambda_0||_v, ||\lambda_0 - 1||_v, 1\}$ is that we cannot rule out the possibility that one has local monodromy at $0, 1, \infty$ merely from the information of p-curvature at v.

Compared to the case when $\lambda_0 \in X(\bar{\mathbb{Z}})$, our estimate for the sum of the logarithms of the archimedean radii increases by at most $\frac{1}{[K:\mathbb{Q}]}(\sum_{v\text{ finite}}\log^+||\lambda_0||_v + \log(|\operatorname{Nm}\lambda_0(\lambda_0-1)|))$, while the estimate for the sum of logarithms of the radii at finite places becomes smaller by $\sum_v \max\{\log^+||\lambda_0^{-1}||_v, \log^+||(\lambda_0-1)^{-1}||_v\}$. An explicit computation shows that the later is larger than the former. Hence the estimate for the product of the radii does not become larger than the case when $\lambda_0 \in X(\bar{\mathbb{Z}})$.

6. The affine elliptic curve case

Let $X \subset \mathbb{A}^2_{\mathbb{Z}}$ be the affine curve over $\mathbb{Z}$ defined by the equation $y^2 = x(x-1)(x+1)$. The generic fiber $X_{\mathbb{Q}}$ is an elliptic curve (with j-invariant 1728) minus its identity point. Given a vector bundle with connection over X_K, we first formulate a suitable p-adic convergence condition of its formal horizontal sections and then state our result in section 6.1. The proof is given in section 6.2.

6.1. **Formal horizontal sections and p-curvatures.**

6.1.1. We fix $x_0 = (0,0) \in X(\mathbb{Z})$ and denote by $(x_0)_K$ and $(x_0)_{k_v}$ the images of x_0 in $X(K)$ and $X(k_v)$. Let $y : X \to \mathbb{A}^1_{\mathbb{Z}}$ be the projection to the y-coordinate. It is easy to check that this map is étale along x_0 and hence induces isomorphisms between the tangent spaces $T_{x_0}X \cong T_0\mathbb{A}^1_{\mathbb{Z}}$ and between the formal schemes $\widehat{X_{K/(x_0)_K}} \cong \widehat{\mathbb{A}^1_{K/0}}$. In particular, we have an analytic section s_v of the projection y from $D(0,1) \subset \mathbb{A}^1(K_v)$ to $X(K_v)$ such that $s_v(0) = x_0$ for any finite place v by the lifting criterion for étale maps. By definition, the image $s_v(D(0,1))$ is the open rigid analytic disc in $X(K_v)$ which is the preimage of $(x_0)_{k_v}$ under the reduction map $X(K_v) \to X(k_v)$.

By choosing a trivialization of M in some neighborhood of $(x_0)_K$, we can view a formal horizontal section m of (M, ∇) around $(x_0)_K$ as a formal function in $\widehat{\mathcal{O}_{X_K,(x_0)_K}}^r \cong \widehat{\mathcal{O}_{\mathbb{A}^1_K,0}}^r$, where r is the rank of M. We denote $f \in \widehat{\mathcal{O}_{\mathbb{A}^1_K,0}}^r$ to be the image and the goal of this section is to prove that the formal power series f is algebraic.

Let U be $X - \{(0,1),(0,-1)\}$. It is a smooth scheme over $\mathbb{Z}$. Our chosen point x_0 is a $\mathbb{Z}$-point of U and $s_v(D(0,1)) \subset U(K_v)$. For $v|p$ a finite place of K, we say that (M,∇) has *good reduction* at v if (M,∇) extends to a vector bundle with connection on $U_{\mathcal{O}_v}$. The following lemma is similar to Lemma 2.1.5.

Lemma 6.1.2. *Suppose that (M,∇) has good reduction at v. If the p-curvature ψ_p vanishes*[6]*, then the formal function f is the germ of some meromorphic function on the disc $D(0, p^{-\frac{1}{p(p-1)}}) \subset \mathbb{A}^1$.*

Proof. Let $(\mathcal{M},\nabla)$ be an extension of (M,∇) over $X_{\mathcal{O}_v}$. Since y is étale, the derivation $\frac{\partial}{\partial y}$ is regular over some Zariski open neighborhood $\bar{V}$ of $x_0 \in X \otimes \mathbb{Z}/p\mathbb{Z}$. Let $V \subset X(K_v)$ be the preimage of $\bar{V}$ under reduction map. Since the p-curvature vanishes, we have $\nabla(\frac{\partial}{\partial y})^p(\mathcal{M}|_V) \subset p\mathcal{M}|_V$. Notice that $s_v(D(0,1)) \subset V$. Then the proof of Lemma 2.1.5 shows the existence of horizontal sections of M on $s_v(D(0,p^{-\frac{1}{p(p-1)}}))$. Via a local trivialization of M and the isomorphism of formal neighborhoods of x_0 and 0, we see that f is meromorphic over $D(0,p^{-\frac{1}{p(p-1)}})$. □

The following assumption is similar to Assumption 2.2.1.

Assumption 6.1.3. The vector bundle with connection (M,∇) satisfies that
(1) the p-curvature ψ_p vanishes for all but finitely many finite primes $\mathfrak{p}$, and
(2) all formal horizontal sections around x_0, when viewed as formal functions in $\widehat{\mathcal{O}_{\mathbb{A}^1_K,0}}^r$, are the germs of some meromorphic functions on $D(0,p^{-\frac{1}{p(p-1)}})$ for all finite places v.

Remark 6.1.4. The second condition does not depend on the choice of local trivialization of M. Moreover, for each v, this condition remains the same if we replace the projection y by any map $g : W_{\mathcal{O}_v} \to \mathbb{A}^1_{\mathcal{O}_v}$ such that $W_{\mathcal{O}_v}$ is a Zariski open neighborhood of $(x_0)_{\mathcal{O}_v}$ in $X_{\mathcal{O}_v}$ and that g is étale.

Theorem 6.1.5. *Let (M,∇) be a vector bundle with connection over X_K. If Assumption 6.1.3 holds for (M,∇), then (M,∇) is étale locally trivial.*

[6]This means $\psi_p \equiv 0$ on $X_{\mathcal{O}_v} \otimes \mathbb{Z}/p\mathbb{Z}$ as in section 2.1.1.

Remark 6.1.6. This theorem cannot be deduced from applying Theorem 2.2.2 to the push-forward of (M,∇) via some finite étale map from an open subvariety of the affine elliptic curve to $\mathbb{P}^1_K-\{0,1,\infty\}$. Unlike the $\mathbb{P}^1_K-\{0,1,\infty\}$ case, the conclusion here allows the existence of (M,∇) with finite nontrivial monodromy. See section 7.1.

6.2. **Estimate at archimedean places and algebraicity.** Let $\sigma : K \to \mathbb{C}$ be an archimedean place. Let $\phi : D(0,1) \to X(\mathbb{C})$ be a uniformization map such that $\phi(0)=x_0$. We have the following lemma whose proof is the same as that of Lemma 3.2.1:

Lemma 6.2.1. *The σ-adic radius R_σ (see Definition 3.1.3) of the formal functions f in 6.1.1 would be at least $|(y\circ\phi)'(0)|_\sigma$.*

Let $t_0=\frac{1+i}{2}$. A direct manipulation of the definition shows $\lambda(t_0)=-1$, where λ is defined in 3.2.5. Let $F : D(0,1)\to\mathbb{C}-(\mathbb{Z}+t_0\mathbb{Z})$ be a uniformization map such that $F(0)=\frac{1}{2}$.

Lemma 6.2.2 (Eremenko)**.** *The derivative* $|F'(0)|=2^{-3/2}\pi^{-3/2}\Gamma(1/4)^2=0.8346...$

Proof. From [Ere, Sec. 2], we have $F'(0)=\frac{2^{5/2}}{B(1/4,1/4)}|(\lambda^{-1})'(i)|$[7], where B is Beta function. By Lemma 3.2.6, the Chowla-Selberg formula ([SC67])

$$|\eta(i)|=2^{-1}\pi^{-3/4}\Gamma(1/4),\tag{6.2.3}$$

and $\theta_{00}^4(i)=2\theta_{01}^4(i)=2\theta_{10}^4(i)$, we have

$$|(\lambda^{-1})'(i)|=|\pi i(\frac{\theta_{01}(i)\theta_{10}(i)}{\theta_{00}(i)})^4|=\pi|\eta(i)|^4=\frac{\Gamma(1/4)^4}{2^4\pi^2}.$$

We obtain the desired formula by noticing that $B(1/4,1/4)=\pi^{-1/2}\Gamma(1/4)^2$. □

Lemma 6.2.4. *Let α be the constant $2(-\pi^2\theta_{01}^4(t_0))^{3/2}$ and $\wp$ be the Weierstrass-$\wp$ function. We have $y\circ\phi=\alpha^{-1}\wp'\circ F$, up to some rotation on $D(0,1)$.*

Proof. The map $g:=(\wp,\wp')$ maps $\mathbb{C}-(\mathbb{Z}+t_0\mathbb{Z})$ to the affine curve $u^2=4v^3-g_2(t_0)v-g_3(t_0)$. Let s be the isomorphism from this affine curve to $X(\mathbb{C})$ given by (5.2.6). Since both $s\circ g(1/2)$ and x_0 are the unique point fixed by the four automorphisms of $X(\mathbb{C})$, we have $s\circ g(1/2)=x_0$. Hence $s\circ g\circ F(0)=x_0=\phi(0)$ and then the uniformizations $s\circ g\circ F$ and ϕ are the same up to some rotation. We have $y\circ\phi=y\circ s\circ g\circ F=\alpha^{-1}\wp'\circ F$ by (5.2.6). □

Proposition 6.2.5. *The σ-adic radius* $R_\sigma^{\frac{[K:\mathbb{Q}]}{[K_\sigma:\mathbb{R}]}}\geq 2^{-5/2}\pi^{-2}\Gamma(1/4)^4=3.0949\cdots$.

Proof. Differentiate both sides of $(\wp'(z))^2=4(\wp(z))^3-g_2(t_0)\wp(z)-g_3(t_0)$, we have

$$\wp''(1/2)=6\wp(1/2)^2-g_2(t_0)/2=-g_2(t_0)/2,$$

where the second equality follows from that

$$\wp(1/2)=\pi^2(\theta_{00}^4(t_0)+\theta_{01}^4(t_0))/3=\pi^2\theta_{01}^4(t_0)(\lambda(t_0)+1)/3=0.$$

By Lemma 3.2.6 and $\theta_{00}^4(t_0)=-\theta_{01}^4(t_0)=\theta_{10}^4(t_0)/2$, we have

$$|g_2(t_0)|=\frac{4\pi^4}{3}\cdot\frac{1}{2}|\theta_{00}^8(t_0)+\theta_{01}^8(t_0)+\theta_{10}^8(t_0)|=4\pi^4|\theta_{01}^8(t_0)|.$$

[7]The choice of λ there is different. We have $\lambda(i)=2$ here.

Then by Lemma 6.2.4 the absolute value of the derivative of $y \circ \phi$ at 0 would be

$$
\begin{aligned}
|\alpha^{-1}\wp''(1/2) \cdot F'(0)| &= 2^{-1}\pi^{-3}|\theta_{01}(t_0)|^{-6} \cdot 2\pi^4|\theta_{01}(t_0)|^8 \cdot |F'(0)| \\
&= \pi|\theta_{01}(t_0)|^2 \cdot 2^{-3/2}\pi^{-3/2}\Gamma(1/4)^2 \text{ (by Lemma 6.2.2)} \\
&= 2\pi \cdot 2^{-2}\pi^{-3/2}\Gamma(1/4)^2 \cdot 2^{-3/2}\pi^{-3/2}\Gamma(1/4)^2 \\
&= 2^{-5/2}\pi^{-2}\Gamma(1/4)^4 = 3.0949\cdots,
\end{aligned}
\tag{6.2.6}
$$

where the third equality follows from

$$|\theta_{01}(t_0)| = 2^{-1/12}|\theta_{00}(t_0)\theta_{01}(t_0)\theta_{10}(t_0)|^{1/24} = 2^{1/4}|\eta(t_0)| = 2^{1/2}|\eta(i)|,$$

and (6.2.3). □

Proof of Theorem 6.1.5. By Proposition 6.2.5, we have the following estimate of v-adic radii at archimedean places of f in paragraph 6.1.1: $\prod_{v|\infty} R_v \geq 3.0949\cdots$. By Definition 6.1.3, we have the following estimate of the v-adic radii at finite places of f: $\log(\prod_{v\nmid\infty} R_v) \geq -\sum_p \frac{\log p}{p(p-1)} = -0.761196\cdots$. Hence

$$\log(\prod_v R_v) \geq \log 3.0949\cdots - 0.761196\cdots = 0.3685\cdots > 0.$$

We conclude by applying Theorem 3.1.5. □

7. Examples

In this section, we first give an example of (M, ∇) with good reduction and vanishing p-curvature for all $\mathfrak{p}$ but with nontrivial global monodromy over the affine elliptic curve in section 6. Then we discuss a variant of our main theorems with X being the affine line minus all 4-th roots of unity.

7.1. **An example with vanishing p-curvature for all $\mathfrak{p}$ and nontrivial G_{gal}.** Let K be $\mathbb{Q}(\sqrt{-1})$, $X \subset \mathbb{A}^2_{\mathbb{Z}}$ be the affine curve defined by $y^2 = x(x-1)(x+1)$, E be the elliptic curve defined as the compactification of X_K, and $f : E \to E$ be a degree two self isogeny of E. We will also use f to denote the restriction of f to $X_K \backslash \{P\}$, where P is the non-identity element in the kernel of f.

Let (M, ∇) be $f_*(\mathcal{O}_{X_K \backslash \{P\}}, d)$. By definition, G_{gal} is $\mathbb{Z}/2\mathbb{Z}$.

Proposition 7.1.1. *The vector bundle with connection (M, ∇) satisfies Assumption 6.1.3.*

Proof. Notice that f extends to a degree two étale cover from E to E over $\mathbb{Z}[\frac{i}{2}]$. For $\mathfrak{p} \nmid 2$, we endow (M, ∇) with the natural integral structure of $(\mathcal{O}_X, d)$ via push-forward and then the p-curvature of (M, ∇) coincides with that of $f^*(M, \nabla)$ by the fact that p-curvatures remain the same under étale pull back[8]. Hence the p-curvature of (M, ∇) vanishes as $f^*(M, \nabla)$ is trivial.

For $\mathfrak{p}|2$, we write (M, ∇) out explicitly. Without loss of generality, we may assume that f from the curve $y^2 = x(x-1)(x+1)$ to the curve $s^2 = t(t-1)(t+1)$ is given by $t = -\frac{i}{2}(x - \frac{1}{x})$ and $s = \frac{1+i}{4}\frac{y}{x}(x + \frac{1}{x})$. Locally around $(t, s) = (0, 0)$,

[8]Because $p \neq 2$ is unramified in K and (M, ∇) has good reduction at $\mathfrak{p}$, the notion of p-curvature here is classical.

$1, x$ is an $\mathcal{O}_{X_K}$ basis of $f_*\mathcal{O}_{X_K}$ and this basis gives rise to a natural Zariski local extension of (M, ∇) over $X_{\mathcal{O}_\mathfrak{p}}$. Direct calculation shows that

$$\nabla(1) = 0, \nabla(x) = \frac{2s}{(t^2-1)(3t^2-1)}ds + \frac{2st(1+2i)}{(t^2-1)(3t^2-1)}xds.$$

Therefore, $\nabla(f_1 + f_2 x) \equiv df_1 + x df_2 \pmod 2$ and the p-curvature of (M, ∇) vanishes. □

Remark 7.1.2. In the above proof, we show that there is an extension of (M, ∇) over $U_{\mathcal{O}_K}$ such that its p-curvatures are all vanishing. However, given the argument for $\mathfrak{p} \nmid 2$, in order to apply Theorem 6.1.5, we do not need to construct an extension of (M, ∇) but only need to check that x, locally as a formal power series of s, converges on $D(0, 2^{-1/2})$ for $v|2$. This is not hard to see: x, as a power series of t, converges when $|t|_v < |2|_v$; and t, as a power series of s, converges when $|s|_v < |2|_v^{1/2}$ and the image of $|s|_v < |2|_v^{1/2}$ is contained in $|t|_v < |2|_v$.

7.2. **A variant of the main theorems.** In this section, we will prove a variant of the main theorems when $X = \mathbb{A}^1_\mathbb{Q} - \{\pm 1, \pm i\}$. Similar to Theorem 6.1.5, the conclusion is that (M, ∇) has finite monodromy and we give an example with nontrivial finite monodromy.

To define the p-adic convergence conditions for bad primes, we take $x_0 = 0$. We say (M, ∇) satisfies $(*)_\mathfrak{p}$ if all its horizontal sections centered at x_0 have $\mathfrak{p}$-adic convergence radii no less than $p^{-\frac{1}{p(p-1)}}$.

Proposition 7.2.1. *Let (M, ∇) be a vector bundle with connection over X_K with p-curvature vanishes for almost all primes and satisfying $(*)_\mathfrak{p}$ for all other finite primes $\mathfrak{p}$. We further assume that the formal horizontal sections around x_0 converge over $D(x_0, 1)$ for $v|15$. Then (M, ∇) is étale locally trivial.*

Proof. By Lemma 6.2.2, we have $R_\infty \geq 2 \cdot 0.8346\cdots$. By the assumptions on finite places, we have $\log(\prod_{v \nmid \infty} R_v) \geq -\sum_{p \neq 3,5} \frac{\log p}{p(p-1)} = -0.4976\cdots$. Then we conclude by applying Theorem 3.1.5. □

Example 7.2.2. Let s be $(1-x^4)^{1/2}$. It is the solution of the differential equation $\frac{ds}{dx} = \frac{-2x^3}{1-x^4}$. Consider the connection on $\mathcal{O}_X$ given by $\nabla(f) = df + \frac{2x^3}{1-x^4}dx$. It has p-curvature vanishing for all p: $\nabla(f) \equiv df \pmod 2$ and $\nabla(f) \equiv df + (p+1)\frac{2x^3}{1-x^4}dx \pmod p$ with solution $s \equiv (1-x^4)^{(p+1)/2} \pmod p$ when $p \neq 2$. . In conclusion, $(\mathcal{O}_X, \nabla)$ satisfies the assumptions in the above proposition while it has nontrivial monodromy of order two.

Remark 7.2.3. If we replace our assumption by similar conditions on generic radii, the above example shows that one could have order two local monodromy around $\pm 1, \pm i$. The reason is [BS82, III eqn. (3)] does not hold in this situation and a modification of their argument would show that an order two local monodromy is possible.

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