MOP experiment

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(See Dropbox: `https://www.dropbox.com/sh/kzf5l6uyzgkk2tj/AAA_xuAMQHX1dlswHAuwi5e8a` (or if/once that link breaks, my website, `http://web.mit.edu/vywang/www/`) for the latest version. Email me at vywang (at) mit.edu for errors/comments/suggestions or to discuss anything, e.g. where to look for more of these kinds of problems or topics.)

(Note that most of the problems have URL links/sources, but some are missing. If you find/create threads/papers I'm missing, please let me know and I'll add the links.)

Recommendation: Work on the topic/problems you think would help you the most.[1] Of course, if you don't feel like doing geometry that's fine; there's an algebra class tomorrow anyways.

Also, while it's important to be able to find certain ideas on your own, I encourage you to occasionally work in small groups, both in and out of class; I think you can gain a surprising amount of intuition just by talking to others. This might be easier to coordinate out of class, especially for the harder problems.

1 Algebra/polynomials, mostly with number theory

1. (Finite fields, concrete/elementary perspective[2]) Let f be a *monic* irreducible degree $d \geq 1$ polynomial *modulo* p (i.e. in $\mathbb{F}_p[x]$), for some prime p.

   (a) Show that $f(x) \mid g(x)^{p^d} - g(x)$ in $\mathbb{F}_p[x]$ for any $g \in \mathbb{F}_p[x]$.

   (b) Show that $x^{p^r} - x \mid g(x)^{p^r} - g(x)$ for any $g \in \mathbb{F}_p[x]$ and positive integer r.

   (c) For positive integers r, show that $f(x) \mid x^{p^r} - x$ modulo p if and only if $d \mid r$.

   (d) $f(x) \mid g(x)^p - g(x)$ in $\mathbb{F}_p[x]$ for some $g \in \mathbb{F}_p[x]$ if and only if $g(x) \pmod{f(x)}$ is a constant.

   (e) Show that $f(t) - \prod_{k=0}^{d-1}(t - x^{p^k}) \in (\mathbb{F}_p[x])[t]$ (i.e. a polynomial in t with coefficients in $\mathbb{F}_p[x]$) is the zero polynomial (in t) modulo $f(x)$, in the sense that its coefficients in t are all divisible by $f(x)$ modulo p (in $\mathbb{F}_p[x]$).

   (f) What do these mean, abstractly? (Don't worry if you haven't seen this before; you can probably find this interpretation in any standard abstract algebra book.)

   (g) (ELMO Shortlist 2013, W.) We define the *Fibonacci sequence* $\{F_n\}_{n\geq 0}$ by $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$; we define the *Stirling number of the second kind* $S(n,k)$ as the number of ways to partition a set of $n \geq 1$ distinguishable elements into $k \geq 1$ indistinguishable nonempty subsets.

   For every positive integer n, let $t_n = \sum_{k=1}^{n} S(n,k)F_k$. Let $p \geq 7$ be a prime. Prove that $t_{n+p^{2p}-1} \equiv t_n \pmod{p}$ for all $n \geq 1$.

   (h) (Putnam 2011) Let p be an odd prime. Show that $\sum_{k=0}^{p-1} k!x^k \in \mathbb{F}_p[x]$ has at most $(p-1)/2$ roots (modulo p).

[1] I remember last year I felt like I wasn't getting anything out of around half of the classes. It's a pity if you're not learning at least one really new/interesting thing every day.

[2] This is only one of many approaches to finite fields. Another common route is to consider the splitting field of $x^{p^r} - x$ (basically [up to isomorphism] the smallest field where it fully factors, and we can use our intuition from complex polynomials and FTA), which behaves nicely by the Frobenius endomorphism: $(x+y)^{p^r} = x^{p^r} + y^{p^r}$. (In the approach above Frobenius doesn't have as central a role.)

(i) (Polya, PFTB) Suppose that $(a_n)_{n\ge 1}$ is a linear recurrence sequence of integers such that n divides a_n for all positive integers n. Prove that (a_n/n) is also a linear recurrence sequence.

2. Are these related?

   (a) (Putnam 1983 B6) Let ω be a complex (2^n+1)th root of unity. Prove that there always exist polynomials $p(x), q(x)$ with integer coefficients such that $p(\omega)^2 + q(\omega)^2 = -1$.

   (b) (TSTST 2014; polynomial analog of Thue's lemma on $a \equiv \pm cb \pmod{m}$) Let F be a field[3], $M \in F[x]$ a nonzero polynomial of degree $d \ge 0$, and $C \in F[x]$ a polynomial relatively prime to M. Prove that there exist $A, B \in F[x]$ of degree at most $\frac{d}{2}$ such that $\frac{A(x)-C(x)B(x)}{M(x)} \in F[x]$.

3. (Putnam 1956 B7?) The nonconstant polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of the polynomials $P(z)+1$ and $Q(z)+1$. Prove that $P(z) = Q(z)$.

4. (I wish I knew more about valuations)

   (a) (Classical?) Let n be a positive integer and a a complex number. If na^k is an algebraic integer for all integers $k \ge 0$, show that a is an algebraic integer itself.

   (b) (MIT Problem-Solving Seminar) Let $f(x) = a_0 + a_1 x + \cdots \in \mathbb{Z}[[x]]$ be a formal power series with $a_0 \ne 0$. Suppose that $f'(x)f(x)^{-1} \in \mathbb{Z}[[x]]$. Prove or disprove that $a_0 \mid a_n$ for all $n \ge 0$.

   (c) What's the relation between (a) and (b)?

5. (W.) Let $\omega = e^{2\pi i/5}$ and $p > 5$ be a prime. Show that

$$\frac{1+\omega^p}{(1+\omega)^p} + \frac{(1+\omega)^p}{1+\omega^p}$$

   is an integer congruent to $2 \pmod{p^2}$.

6. (More practical "valuations", but take this with a grain of salt since I don't fully know what I'm talking about)

   (a) Let $\zeta = e^{2\pi i/p}$ for some prime p. From $(1-\zeta)(1-\zeta^2)\cdots(1-\zeta^{p-1}) = p$, what can you say about $\frac{(1-\zeta)^{p-1}}{p}$ as an algebraic number? (Something similar works for prime powers, but not for other numbers.)

   (b) (2012-2013 Winter OMO, W.) ω is a primitive 2013th root of unity. Find the number of ordered pairs of integers (a, b) with $1 \le a, b \le 2013$ such that $\frac{(1+\omega+\cdots+\omega^a)(1+\omega+\cdots+\omega^b)}{3}$ is an algebraic integer.

   (c) (1996 ISL) Let n be an even positive integer. In terms of n, determine the set of positive integers k such that $k = f(x)(x+1)^n + g(x)(x^n+1)$ for some polynomials $f, g \in \mathbb{Z}[x]$.

   (d) (St. Petersburg 2003, PFTB). Let p be a prime and let $n \ge p$ and $a_1, a_2, \ldots, a_n$ be integers. Define $f_0 = 1$ and f_k the number of subsets $B \subseteq \{1, 2, \ldots, n\}$ having k elements and such that p divides $\sum_{i\in B} a_i$. Show that $f_0 - f_1 + f_2 - \cdots + (-1)^n f_n$ is a multiple of p.

   (e) (China 2011) Show that for all positive integers r, $v_2\left(\sum_{k=-n}^{n} \binom{2n}{n+k} k^{2r}\right) \ge v_2((2n)!)$.

   (f) (W., adapted from Gabriel Dospinescu, PFTB, 2010 MR U160) Let p be a prime and let n, s be positive integers. Prove that $v_p\left(\sum_{p\mid k, 0\le k\le n} (-1)^k k^s \binom{n}{k}\right) \ge v_p(n!)$.

   (g) (Gabriel Dospinescu, PFTB) Let $p > 2$ be a prime number and let m and n be multiples of p, with n odd. For any function $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ satisfying $\sum_{k=1}^{m} f(k) \equiv 0 \pmod{p}$, consider the product $\prod_{k=1}^{m} f(k)$. Prove that the sum of these products is divisible by $\left(\frac{n}{p}\right)^m$. (Yes, we can strengthen it easily.)

[3] We don't lose any substance by specifying $F = \mathbb{R}$.

7. (PFTB-style polynomial algebra/NT; I wish I knew more commutative algebra/algebraic geometry)

   (a) (ELMO 2013, Andre Arslan) For what polynomials $P(n)$ with integer coefficients can a positive integer be assigned to every lattice point in $\mathbb{R}^3$ so that for every integer $n \geq 1$, the sum of the n^3 integers assigned to any $n \times n \times n$ grid of lattice points is divisible by $P(n)$?

   (b) (ELMO Shortlist 2012, David Yang) Prove that if m, n are relatively prime positive integers, $x^m - y^n$ is irreducible in the complex numbers. (A polynomial $P(x, y)$ is irreducible if there do not exist nonconstant polynomials $f(x, y)$ and $g(x, y)$ such that $P(x, y) = f(x, y)g(x, y)$ for all x, y.)

   (c) (ELMO 2012; related to Newton polygons; two-variable case of Eherenfeucht criterion) Let f, g be polynomials with complex coefficients such that $\gcd(\deg f, \deg g) = 1$. Suppose that there exist polynomials $P(x, y)$ and $Q(x, y)$ with complex coefficients such that $f(x) + g(y) = P(x, y)Q(x, y)$. Show that one of P and Q must be constant.

   (d) Prove that $\mathbb{C}[x, y]$ has unique factorization into irreducible (two-variable complex) polynomials (up to constant). (It suffices to show every *irreducible* polynomial is *prime*, i.e. if some nonconstant p is irreducible and $p \mid ab$, then $p \mid a$ or $p \mid b$.)

   (e) (Rookie Team Contest/ELMO Shortlist 2014, Yang Liu; AM-GM polynomial irreducibility) Find all positive integers $n \geq 2$ for which the n-variable polynomial $x_1^n + x_2^n + \cdots + x_n^n - nx_1x_2\cdots x_n$ is reducible over the complex numbers (formally, in $\mathbb{C}[x_1, x_2, \ldots, x_n]$).

   (f) (Ineffective Bézout's theorem) Prove that two relatively prime polynomials $f, g \in \mathbb{C}[x, y]$ share finitely many common zeros in the complex plane.

   (g) i. (Artin's *Algebra*) Let $x(t), y(t)$ be complex polynomials, not both constant. Show that there exists a polynomial $m \in \mathbb{C}[x, y]$ such that for $f \in \mathbb{C}[x, y]$, we have $m \mid f$ if and only if $f(x(t), y(t)) = 0$ for all $t \in \mathbb{C}$.

   ii. The following can be done in more than one order:
   (i) Prove that m is irreducible in $\mathbb{C}[x, y]$.
   (ii) Prove that m is unique up to a constant factor.
   (iii) For a point $(a, b) \in \mathbb{C}^2$, prove that $m(a, b) = 0$ if and only if there exists $t \in \mathbb{C}$ such that $(x(t), y(t)) = (a, b)$.

8. (a) (Putnam 2012 B6) Let p be an odd prime number such that $p \equiv 2 \pmod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \pmod p$. Show that π is an even permutation if and only if $p \equiv 3 \pmod 4$.

   (b) (2013-2013 Winter OMO, W.) Find the remainder when $\prod_{i=0}^{100}(1 - i^2 + i^4)$ is divided by 101.

   (c) (Noga Alon, Jean Bourgain; TST 2014) For a prime p, a subset S of residues modulo p is called a *sum-free multiplicative subgroup* of $\mathbb{F}_p$ if
   (i) there is a nonzero residue α modulo p such that $S = \{1, \alpha^1, \alpha^2, \ldots\}$ (all considered mod p), and
   (ii) there are no $a, b, c \in S$ (not necessarily distinct) such that $a + b \equiv c \pmod p$.
   Prove that for every integer N, there is a prime p and a sum-free multiplicative subgroup S of $\mathbb{F}_p$ such that $|S| \geq N$.

   (d) The TST problem, but with (ii) replaced by (ii') $0 \notin a_1S + a_2S + \cdots + a_kS$, for fixed integers a_i *with nonzero sum*. (In (ii) these integers are $+1, +1, -1$.)

   (e) (PFTB, AMM 10748) Let p, q be prime numbers and let r be a positive integer such that $q \mid p-1$, $q \nmid r$, and $p > r^{q-1}$. Let $a_1, \ldots, a_r$ be integers such that $a_1^{(p-1)/q} + \cdots + a_r^{(p-1)/q}$ is a multiple of p. Prove that at least one of the a_i's is a multiple of p.

9. (a) (ELMO 2012, Bobby Shen) A diabolical combination lock has n dials (each with c possible states), where $n, c > 1$. The dials are initially set to states $d_1, d_2, \ldots, d_n$, where $0 \leq d_i \leq c - 1$ for each $1 \leq i \leq n$. Unfortunately, the actual states of the dials (the d_i's) are concealed, and the initial settings of the dials are also unknown. On a given turn, one may advance each dial by an integer

amount c_i ($0 \le c_i \le c-1$), so that every dial is now in a state $d_i' \equiv d_i + c_i \pmod{c}$ with $0 \le d_i' \le c-1$. After each turn, the lock opens if and only if all of the dials are set to the zero state; otherwise, the lock selects a random integer k and cyclically shifts the d_i's by k (so that for every i, d_i is replaced by d_{i-k}, where indices are taken modulo n).

Show that the lock can always be opened, regardless of the choices of the initial configuration and the choices of k (which may vary from turn to turn), if and only if n and c are powers of the same prime.

(b) ($\mathbb{Q}[x]$ representations of n-sequences $\pmod{m}$, W.) Let $n, m > 1$ be positive integers. We say a sequence $(a_1, a_2, \ldots, a_n) \in (\mathbb{Z}/m\mathbb{Z})^n$ is *satisfied* by the integer-valued polynomial $f \in \mathbb{Q}[x]$ if $f(x) \equiv a_i \pmod{m}$ whenever $x \equiv i \pmod{n}$.

(i) Show that n and m are powers of the same prime if and only if every sequence $(a_1, a_2, \ldots, a_n) \in (\mathbb{Z}/m\mathbb{Z})^n$ is satisfied by some polynomial $f(x)$.

(ii) If $n = p^i$ and $m = p^j$ for a prime p and two positive integers i, j, find (in terms of p, i, j) the smallest positive integer M such that every sequence $(a_1, a_2, \ldots, a_n) \in (\mathbb{Z}/m\mathbb{Z})^n$ is satisfied by a polynomial $f(x)$ of degree at most M.

2 Analytic-flavored stuff

1. Let's pretend this is analytic number theory.

(a) (2013-2014 Spring OMO, W.) Warm-up. Classify pairs (m, n) of integers such that $x^3 + y^3 = m + 3nxy$ has infinitely many integer solutions (x, y).

(b) (Sierpinski, PFTB) Prove that for all N there exists a k such that more than N prime numbers can be written in the form $f(T) + k$ for some integer T, where $f \in \mathbb{Z}[x]$ is a nonconstant monic polynomial.

(c) (ISL 1991) Let a_n be the last nonzero digit of $n!$. Does a_n eventually become periodic?

(d) (China 2012) Given an integer $n \ge 4$. $S = \{1, 2, \ldots, n\}$. A, B are two subsets of S such that $ab + 1$ is a perfect square for all $a \in A$, $b \in B$. Prove that $\min\{|A|, |B|\} \le \log_2 n$.

(e) (ELMO 2014, Matthew Babbitt) Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3, 4, 5, 6\}$ and n is a positive integer. Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

(f) (heard from Lawrence Sun, who heard from Victor Reis?) Suppose we have a polynomial f with integer coefficients such that $f(p)$ is a square for all primes p. Does it follow that f is the square of some polynomial?

(g) (TST Proposal, Gabriel Carroll)

(a, Gabriel Carroll) Find all functions $f : \mathbb{N} \to \mathbb{Z}$ such that $f(m) - f(n) \mid m^2 - n^2$ for all positive integers m, n.

(b, W. and Mark Sellke) Let N be a positive integer. Find all functions $f : \mathbb{Z}_{\ge N} \to \mathbb{Z}$ such that $f(m) - f(n) \mid m^2 - n^2$ for all positive integers $m, n \ge N$.

(h) (Vaguely USAMO 2012/PFTB generalization) Find all $f : \mathbb{N} \to \mathbb{N}$ such that $\gcd(m, n) = 1 \implies \gcd(f(m), f(n)) = 1$ and $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

(i) (China 2009) Determine whether there exists an arithmetic progression consisting of 40 terms and each of whose terms can be written in the form $2^m + 3^n$ for some nonnegative integers m, n.

(j) (Generalization of classic harmonic numbers version, extension of ILL 1979) If a, b, n are positive integers, prove that $\sum_{k=0}^{n} \frac{1}{a+kb}$ cannot be an integer.

(k) (Russia 2002) Show that the numerator of the reduced fraction form of $H_n = 1/1 + 1/2 + \cdots + 1/n$ is infinitely often not a prime power.

(l) (a, ELMO Shortlist 2012, W.) Fix two positive integers $a, k \geq 2$, and let $f \in \mathbb{Z}[x]$ be a nonconstant polynomial. Suppose that for all sufficiently large positive integers n, there exists a rational number x satisfying $f(x) = f(a^n)^k$. Prove that there exists a polynomial $g \in \mathbb{Q}[x]$ such that $f(g(x)) = f(x)^k$ for all real x.

(b, I'm pretty sure this works/is a direct generalization but haven't checked too carefully, W.) Fix two positive integers $a, k \geq 2$, and let $f, p \in \mathbb{Z}[x]$ be two nonconstant polynomials. Suppose that for all sufficiently large positive integers n, there exists a rational number x satisfying $f(x) = p(a^n)^k$. Prove that there exists a polynomial $g \in \mathbb{Q}[x]$ such that $f(g(x)) = p(x)^k$ for all real x.

(m) (Polynomials with the same number of divisors) Let f, g be nonconstant integer-coefficient polynomials. If $f(n), g(n)$ have the same number of *positive integer divisors* for all sufficiently large positive integers n, prove that $f = \pm g$.

(n) (Pomerance, via harazi on AoPS) Find all positive integers n such that $n!$ divides the sum of the divisors of $n!$.

(o) (MathOverflow, triple with large LCM, solved by fedja) Does there exist $c > 0$ such that among any n positive integers one may find 3 with least common multiple at least cn^3?

2. First and second differences.

   (a) (constant Schwarz derivative implies quadratic) Let f be a continuous function such that at every point $x \in \mathbb{R}$, the Schwarz derivative $\lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$ exists and equals some constant L (independent of x). Prove that f is a quadratic function.

   (b) (from AoPS) A *bounded* sequence $\{x_n\}$ of real numbers satisfies $\lim_{n\to\infty}(x_n - 2x_{n+1} + x_{n+2}) = 0$. Prove that $\lim_{n\to\infty}(x_n - x_{n+1}) = 0$.

3. Polynomials with restricted coefficients.

   (a) (Putnam 1972 B6) The polynomial $p(x)$ has all coefficients 0 or 1, and $p(0) = 1$. Show that if the complex number z is a root, then $|z| \geq (\sqrt{5} - 1)/2$.

   (b) (TST 2011) A polynomial $P(x)$ is called *nice* if $P(0) = 1$ and the nonzero coefficients of $P(x)$ alternate between 1 and -1 when written in order. Suppose that $P(x)$ is nice, and let m and n be two relatively prime positive integers. Show that $Q(x) = P(x^n) \cdot \frac{(x^{mn}-1)(x-1)}{(x^m-1)(x^n-1)}$ is nice as well.

4. (TST 2011) Determine whether or not there exist two different sets A, B, each consisting of at most 2011^2 positive integers, such that every x with $0 < x < 1$ satisfies the following inequality: $|\sum_{a\in A} x^a - \sum_{b\in B} x^b| < (1-x)^{2011}$.

5. (a) (complex logarithm, bare hands) Prove that $\sum_{k\geq 1} \frac{z^k}{k}$ converges for all complex numbers $z \neq 1$ on the *closed* unit disk (except $z = 1$).

   (b) (example of Gibbs phenomenon in Fourier analysis) Prove that for any real x and positive integer n, we have $\sum_{k=1}^{n} \frac{\sin kx}{k} \leq 2\sqrt{\pi}$.

6. (China MO 2007; heard from IdeaMOP 2013) Given a bounded sequence of real numbers such that $a_n < \frac{1}{2n+2007} + \sum_{k=n}^{2n+2006} \frac{a_k}{k+1}$ for all $n \geq 1$, prove that $a_n < \frac{1}{n}$ for all positive integers n.

7. (heard from IdeaMOP 2013) Determine the minimum value of constant c such that for every n and reals $a_i, b_i \in [1, 2]$ $(1 \leq i \leq n)$ with $\sum a_i^2 = \sum b_i^2$, we have $\sum a_i^3/b_i \leq c \sum a_i^2$.

8. (Putnam 1996; also in geometry) Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive reals x, y such that $\sum (a_i, b_i) x^{a_i} y^{b_i} = (0, 0)$.

9. (MOP 2007?) $\{a_n\}_{n=1}^{\infty}$ is a sequence satisfying $0 < a_n \leq a_{n+1} + a_{n^2}$ for all natural numbers n. Is $\sum_{n=1}^{\infty} a_n$ necessarily divergent?

10. (Continuous interpolation/extension: constraint on rationals, discussed at MOP 2014. The reason for having (a) is that there's at least one nice explicit construction that fails badly for (b).)

    (a) (Linus Hamilton) Prove that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) \notin \mathbb{Z}$ such that for any nonzero rational $q = a/b \neq 0$ in reduced form, $f(q)$ is a multiple of $1/b$.

    (b) For any nonzero rational $q = a/b \neq 0$ in reduced form, let S_q be a doubly-infinite set of reals with gaps at most $1/b$. (In other words, S_q is nonempty and if $u \in S_q$, then S_q has an element in $[u - 1/b, u)$ and an element in $(u, u + 1/b]$.) Prove that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for any nonzero rational $q \neq 0$, $f(q)$ lies in S_q.

    (c) (W., modest generalization) Let $q_1, q_2, \ldots$ be a (countable) sequence of distinct numbers in $[0, 1]$, and let $\epsilon_1, \epsilon_2, \ldots > 0$ be a sequence of positive numbers tending to 0. For each positive integer n, let S_n be a doubly-infinite set of reals with gaps at most ϵ_n. Prove that there exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that $f(q_n) \in S_n$ for any positive integer n.

11. (Putnam 1993 B4) $K(x, y)$, $f(x)$ and $g(x)$ are positive and continuous for $x, y \in [0, 1]$. $\int_0^1 f(y)K(x, y)dy = g(x)$ and $\int_0^1 g(y)K(x, y)dy = f(x)$ for all $x \in [0, 1]$. Show that $f = g$ on $[0, 1]$. (What's the discrete analog here?)

12. Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{C}[x]$ has roots $z_1, \ldots, z_n \in \mathbb{C}$ (not necessarily distinct).

    (a) (Mahler measure bound) Prove that $\prod_{k=1}^n \max(1, |z_k|) \leq \sqrt{1 + |a_{n-1}|^2 + \cdots + |a_0|^2}$.

    (b) (China TST 2003) Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{C}[x]$ has roots $z_1, \ldots, z_n \in \mathbb{C}$. If $\sum_{k=0}^{n-1} |a_k|^2 \leq 1$, show that $\sum_{k=1}^n |z_k|^2 \leq n$.

    (c) (MOP 2011) Prove that $\frac{1}{n} \sum_{k=1}^n |z_k|^2 < 1 + \max_{1 \leq k \leq n} |a_{n-k}|^2$.

13. (Romania TST 2004; also in geometry) Let D be a closed disk in the complex plane. Prove that for all positive integers n, and for all complex numbers $z_1, z_2, \ldots, z_n \in D$ there exists a $z \in D$ such that $z^n = z_1 z_2 \cdots z_n$.

14. (ELMO 2013, David Yang; also in combinatorics) Consider a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for every integer $n \geq 0$, there are at most $0.001n^2$ pairs of integers (x, y) for which $f(x + y) \neq f(x) + f(y)$ and $\max\{|x|, |y|\} \leq n$. Is it possible that for some integer $n \geq 0$, there are more than n integers a such that $f(a) \neq a \cdot f(1)$ and $|a| \leq n$?

15. (RMM 2012, Ben Elliott; also in combinatorics) Each positive integer is coloured red or blue. A function f from the set of positive integers to itself has the following two properties:

    (a) if $x \leq y$, then $f(x) \leq f(y)$; and

    (b) if x, y and z are (not necessarily distinct) positive integers of the same colour and $x + y = z$, then $f(x) + f(y) = f(z)$.

    Prove that there exists a positive number a such that $f(x) \leq ax$ for all positive integers x.

16. (komal.hu, based on RMM 2012; also in combinatorics) Is the previous problem still true as long as the number of colors is finite? (I don't know the solution to this one.)

17. (Iterated stuffs)

    (a) (Math Prize 2012) Define $L(x) = x - \frac{x^2}{2}$ for every real number x. Prove that $\lim_{n\to\infty} nL^n(17/n)$ exists, and compute its value explicitly.

    (b) (Putnam 2012) Suppose that $a_0 = 1$ and that $a_{n+1} = a_n + e^{-a_n}$ for $n = 0, 1, 2, \ldots$. Does $a_n - \ln n$ have a finite limit as $n \to \infty$?

    (c) (Putnam 2006) Let $k > 1$ be an integer. Suppose $a_0 > 0$ and define $a_{n+1} = a_n + a_n^{-1/k}$ for $n \geq 0$. Evaluate $\lim_{n\to\infty} a_n^{k+1}/n^k$.

18. (China 2012; also in geometry) Find the smallest possible value of a real number c such that for any 2012-degree monic polynomial $P(x) = x^{2012} + a_{2011}x^{2011} + \cdots + a_1x + a_0$ *with real coefficients*, we can obtain a new polynomial $Q(x)$ by multiplying some of its coefficients by -1 such that every root z of $Q(x)$ satisfies the inequality $|\Im z| \le c|\Re z|$.

19. (ELMO Shortlist 2012, David Yang; also in geometry) We have a compact set in $\mathbb{R}^n$. (For our purposes, just take compact to mean "closed and bounded"; closed basically means "contains the boundary.") We consider the set of directions (formally, unit vectors in the plane). Such a direction is called good if there is exactly one point in the set that is furthest along that direction. (For example, a northernmost point.)

    (a) Prove that there exists a good direction.

    (b) Prove that if $n = 2$, all but countably many directions are good.

20. ("tan9p", Math StackExchange; also in geometry) ABC is an equilateral triangle, and $AD = BE = CF$ for some distinct points D, E, F inside, with A, D, E collinear, B, E, F collinear, and C, F, D collinear. Prove that DEF is an equilateral triangle.

21. (ELMO Shortlist 2012, Calvin Deng) Let a, b, c be distinct positive real numbers, and let k be a positive integer greater than 3. Show that

$$\left|\frac{a^{k+1}(b-c)+b^{k+1}(c-a)+c^{k+1}(a-b)}{a^k(b-c)+b^k(c-a)+c^k(a-b)}\right| \ge \frac{k+1}{3(k-1)}(a+b+c)$$

and

$$\left|\frac{a^{k+2}(b-c)+b^{k+2}(c-a)+c^{k+2}(a-b)}{a^k(b-c)+b^k(c-a)+c^k(a-b)}\right| \ge \frac{(k+1)(k+2)}{3k(k-1)}(a^2+b^2+c^2).$$

22. (Real roots and interlacing, HroK's blog[4]) The intermediate value theorem and Rolle's theorem are ubiquitous principles in the analysis of real roots of continuous and differentiable functions, respectively.

    As you are likely familiar with the most standard uses of these theorems, we briefly discuss the phenomenon of *interlacing* functions f, g with alternating real roots. (For a much more unified coverage, see Steve Fisk's paper Polynomials, roots, and interlacing.) In particular, we look at polynomial recurrences, which provide some of the most natural examples of interlacing.

    (a) Let's first give an example: The n^{th} Hermite polynomial $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ has all real roots (by induction one easily verifies that H_n is a polynomial of degree n).

    Indeed, we induct to show that for every $n \ge 1$, $h_n(x) = \frac{d^n}{dx^n} e^{-x^2}$ has exactly n roots, where the base case is obvious. But if for some $n > 1$ we assume that $h_{n-1}(x)$ has $n-1$ real roots $a_1 < \cdots < a_{n-1}$, then noting that $\pm\infty$ are also roots and $h_n = \frac{d}{dx} h_{n-1}$, we're done by Rolle's theorem. Obseve that h_n, h_{n-1} interlace, i.e. the roots of h_{n-1} lie in between those of h_n.

    Of course, there are several variations on the same idea. For example, if we have a recurrence like $p_n(x) = xp_{n-1}(x) + p_{n-2}(x)$ and we know that p_{n-1}, p_{n-2} interlace, then under some mild conditions we can show via the IVT that p_n, p_{n-1} and p_n, p_{n-2} also interlace (of course, we need p_n, p_{n-1}, p_{n-2} to have degrees within one of each other). This is in essence the idea behind, for instance, Sturm's theorem.

    Now for some problems, (very) roughly arranged in difficulty order!

    (b) (Steve Fisk) Suppose $\{a_i\}, \{b_i\}, \{c_i\}$ are sequences of reals where all a_i, c_i are positive and the b_i are unrestricted. Define a sequence of polynomials recursively by $p_{-1} = 0$, $p_0 = 1$, and $p_i = (a_ix + b_i)p_{i-1} - c_ip_{i-2}$ for $i > 1$. Show that $p_n(x)$ has all real roots for every positive integer n.

    (c) Prove Sturm's theorem.

[4] If you're interested in contributing to this excellent blog, ask me for the password.

(d) (ELMO Shortlist 2012, David Yang) Prove that any polynomial of the form $1+a_nx^n+a_{n+1}x^{n+1}+\cdots+a_kx^k$ ($k \geq n$) has at least $n-2$ non-real roots (counting multiplicity), where the a_i ($n \leq i \leq k$) are real and $a_k \neq 0$.

(e) Prove Newton's inequalities.

(f) (Descartes' rule of signs) For a polynomial $p \in \mathbb{R}[x]$, let $z(p)$ denote the number of positive zeros and $v(p)$ the number of sign changes.

(i) Show that $2 \mid z(p) - v(p)$.

(ii) Prove that $z(p) \leq v(p)$ by writing $p = (x-r)q$ for some positive real root r of $p(x)$ and inducting on $\deg p$.

(iii) Prove that $z(p) \leq v(p)$ by considering the derivative $p'(x)$ (assuming WLOG that $p(0) \neq 0$) and inducting on $\deg p$.

(g) (Classical) Show that out of all monic polynomials of a fixed degree n, $T_n(x)/2^{n-1}$ attains the smallest maximum absolute value on the interval $[-1,1]$, where $T_n(x)$ denotes the n^{th} Chebyshev polynomial of the first kind.

(h) (MOP 1999) Given n points on the unit circle such that the product of the distances from any point on the circle to the given points does not exceed 2, prove that the points must be vertices of a regular n-gon.

(i) (MathOverflow, hard generalization of previous problem, by fedja) Let $p_1, \ldots, p_n \geq 0$ be n non-negative weights summing up to n (average 1); let $z_1, \ldots, z_n$ be n points on the unit circle. Prove that there exists z on the unit circle such that $\prod |z - z_i|^{p_i} \geq 2$.

(j) (MOP 2001) Let $P(x)$ be a real-valued polynomial with $P(n) = P(0)$. Show that there exist at least n distinct (unordered) pairs of distinct real numbers $\{x, y\}$ such that $x - y \in \mathbb{Z}$ and $P(x) = P(y)$. Does this necessarily hold if we allow P to be any continuous function?

(k) (USAMO 2002) Prove that any monic polynomial of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

(l) (MOP 2007) Let a be a real number. Prove that every nonreal root of $f(x) = x^{2n} + ax^{2n-1} + \cdots + ax + 1$ lies on the unit circle and f has at most 2 real roots.

(m) (ELMO Shortlist 2011, Evan O'Dorney) If $a + b + c = a^n + b^n + c^n = 0$ for some positive integer n and complex a, b, c, show that two of a, b, c have the same magnitude.

(n) ("Enzo", MathOverflow) For $n \geq 1$, let

$$P_n(x) = x^{n+1} \left[\frac{\partial^{2n+1}}{\partial z^{2n+1}} \frac{\sinh(z)}{\cosh(z) - 1 + x} \right]_{z=0}.$$

Prove that $P_n(x)$ is a polynomial of degree n with every root real and strictly greater than 2.

3 Combinatorial stuff

1. (A certain flavor of combinatorial number theory)

(a) Farey sequences and Stern-Brocot trees (see also reference/paper by Dhroova Aiylam).

(b) (ELMO Shortlist 2010, Brian Hamrick) The numbers $1, 2, \ldots, n$ are written on a blackboard. Each minute, a student goes up to the board, chooses two numbers x and y, erases them, and writes the number $2x + 2y$ on the board. This continues until only one number remains. Prove that this number is at least $\frac{4}{9}n^3$.

(c) (2012-2013 Fall OMO, W.) The numbers $1, 2, \ldots, 2012$ are written on a blackboard. Each minute, a student goes up to the board, chooses two numbers x and y, erases them, and writes the number $2x+2y$ on the board. This continues until only one number N remains. Find the remainder when the maximum possible value of N is divided by 1000.

(d) (HMIC 2015 Problem 2) Let m, n be positive integers with $m \geq n$. Let S be the set of pairs (a, b) of relatively prime positive integers such that $a, b \leq m$ and $a + b > m$.

For each pair $(a, b) \in S$, consider the nonnegative integer solution (u, v) to the equation $au - bv = n$ chosen with $v \geq 0$ minimal, and let $I(a, b)$ denote the (open) interval $(v/a, u/b)$.

Prove that $I(a, b) \subseteq (0, 1)$ for every $(a, b) \in S$, and that any fixed irrational number $\alpha \in (0, 1)$ lies in $I(a, b)$ for exactly n distinct pairs $(a, b) \in S$.

(e) (USA TSTST 2013, Palmer Mebane) A finite sequence of integers $a_1, a_2, \ldots, a_n$ is called *regular* if there exists a real number x satisfying $\lfloor kx \rfloor = a_k$ for $1 \leq k \leq n$. Given a regular sequence $a_1, a_2, \ldots, a_n$, for $1 \leq k \leq n$ we say that the term a_k is *forced* if the following condition is satisfied: the sequence $a_1, a_2, \ldots, a_{k-1}, b$ is regular if and only if $b = a_k$. Find the maximum possible number of forced terms in a regular sequence with 1000 terms.

(f) (ISL 2007 A5) Let $c > 2$, and let $a_1, a_2, \ldots$ be a sequence of nonnegative real numbers such that $a_{m+n} \leq 2a_m + 2a_n$ for all $m, n \geq 1$, and $a_{2^k} \leq \frac{1}{(k+1)^c}$ for all $k \geq 0$. Prove that the sequence a_n is bounded.

(g) (IMO 2012, Dusan Djukic) Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

(h) (HMMT 2014 Team Problem 10, W. based on ISL 2007 A5) Fix a positive real number $c > 1$ and positive integer n. Initially, a blackboard contains the numbers $1, c, \ldots, c^{n-1}$. Every minute, Bob chooses two numbers a, b on the board and replaces them with $ca + c^2 b$. Prove that after $n - 1$ minutes, the blackboard contains a single number no less than

$$\left(\frac{c^{n/L} - 1}{c^{1/L} - 1}\right)^L,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $L = 1 + \log_\phi(c)$.

(i) (ISL 2013 N7, Sam Vandervelde) Let α be an irrational positive number, and let m be a positive integer. A pair of (a, b) of positive integers is called good if $a\lceil b\alpha \rceil - b\lfloor a\alpha \rfloor = m$. A good pair (a, b) is called excellent if neither of the pair $(a - b, b)$ and $(a, b - a)$ is good. Prove that the number of excellent pairs is equal to the sum of the positive divisors of m.

2. (Algebraic and additive combinatorics, vaguely)

(a) (Boris Bukh) Suppose $A_1, \ldots, A_m$ are $n \times n$ matrices with $m > n$ and $A_1 + \cdots + A_m$ invertible. Show that for some proper subset $S \subset \{1, 2, \ldots, m\}$ of indices, $\sum_{i \in S} A_i$ is also invertible.

(b) (TSTST 2011) Let n be a positive integer. Suppose we are given $2^n + 1$ distinct sets, each containing finitely many objects. Place each set into one of two categories, the red sets and the blue sets, so that there is at least one set in each category. We define the *symmetric difference* of two sets as the set of objects belonging to exactly one of the two sets. Prove that there are at least 2^n different sets which can be obtained as the symmetric difference of a red set and a blue set.

(c) (China 2011) Let ℓ, m, n be positive integers, and $A_1, A_2, \ldots, A_m, B_1, \ldots, B_n$ be $m + n$ pairwise distinct subsets of the set $\{1, 2, \ldots, \ell\}$. Suppose that the symmetric differences $A_i \Delta B_j := (A \cup B) \setminus (A \cap B)$ (for $1 \leq i \leq m, 1 \leq j \leq n$) cover each nonempty subset of $\{1, 2, \ldots, \ell\}$ exactly once. Find all possible values of m, n, in terms of ℓ.

(d) (TST 2014, W., based off of Evan O'Dorney's solution to IMO 2010.3) Find all functions $f : \mathbb{N} \to \mathbb{Z}$ such that $(f(m) - f(n))(m - n)$ is always a square.

(e) (China?) Let p be a prime and $a_1, \ldots, a_k$ be $k \geq 1$ distinct *nonzero* residues modulo p. Prove that there are at most $(p-1)/k$ numbers $n \in \{1, 2, \ldots, p-1\}$ such that $[na_1]_p < [na_2]_p < \cdots < [na_k]_p$, where $[x]_p$ denotes the (nonnegative) remainder of x modulo p.

(f) Let p be a prime and $b_1, \dots, b_k$ be $k \ge 1$ *nonzero* residues modulo p. Prove that there are at most $(p-1)/k$ numbers $n \in \{1, 2, \dots, p-1\}$ such that $[nb_1]_p + \cdots + [nb_k]_p = [n(b_1 + \cdots + b_k)]_p$, where $[x]_p$ denotes the (nonnegative) remainder of x modulo p.

(g) (Putnam 2012) Let q and r be integers with $q > 0$, and let A and B be intervals on the real line. Let T be the set of all $b + mq$ where b and m are integers with b in B, and let S be the set of all integers a in A such that ra is in T. Show that if the product of the lengths of A and B is less than q, then S is the intersection of A with some (possibly finite) arithmetic progression.

(h) (ISL 1999 C6) Suppose that every integer has been given one of the colours red, blue, green or yellow. Let x and y be odd integers so that $|x| \ne |y|$. Show that there are two integers of the same colour whose difference has one of the following values: $x, y, x + y$ or $x - y$.

(i) (based on China 2009; see also WOOT) If $X \subseteq \mathbb{Z}$ and a_i are integers such that $X + a_1, \dots, X + a_n$ partition $\mathbb{Z}$, then show that X is periodic: $X = X + N$ for some positive integer N. Furthermore, if $n = p$ is prime, show that the $a_1, \dots, a_p$ form a permutation of $C + D\{1, 2, \dots, p\}$ for some integer constants C, D with $D \ne 0$. (The original problem has $p = 3$.)

(What if n is not prime; say $n = 4$? How is this related to the previous problem (ISL 1999 C6)?)

(j) (ISL 2001 N6; Sidon subsets of $\{1, 2, \dots, n\}$; see also here) Is it possible to find 100 positive integers not exceeding 25000, such that all *pairwise* sums of them are different? Is 25000 asymptotically the best bound in the variable '100'?

(k) (IMO 2013, Alexander S. Golovanov and Mikhail A. Ivanov) Let $n \ge 3$ be an integer, and consider a circle with $n + 1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \dots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c. Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \le n$ and $\gcd(x, y) = 1$. Prove that $M = N + 1$.

(l) (ELMO 2013, David Yang; also in analysis) Consider a function $f : \mathbb{Z} \to \mathbb{Z}$ such that for every integer $n \ge 0$, there are at most $0.001n^2$ pairs of integers (x, y) for which $f(x + y) \ne f(x) + f(y)$ and $\max\{|x|, |y|\} \le n$. Is it possible that for some integer $n \ge 0$, there are more than n integers a such that $f(a) \ne a \cdot f(1)$ and $|a| \le n$?

(m) (RMM 2012, Ben Elliott; also in analysis) Each positive integer is coloured red or blue. A function f from the set of positive integers to itself has the following two properties:

(a) if $x \le y$, then $f(x) \le f(y)$; and

(b) if x, y and z are (not necessarily distinct) positive integers of the same colour and $x + y = z$, then $f(x) + f(y) = f(z)$.

Prove that there exists a positive number a such that $f(x) \le ax$ for all positive integers x.

(n) (komal.hu, based on RMM 2012; also in analysis) Is the previous problem still true as long as the number of colors is finite? (I don't know the solution to this one.)

(o) (China 2012) Prove that there exists a positive real number C with the following property: for any integer $n \ge 2$ and any subset X of the set $\{1, 2, \dots, n\}$ such that $|X| \ge 2$, there exist $x, y, z, w \in X$(not necessarily distinct) such that $0 < |xy - zw| < C\alpha^{-3}$, where $\alpha = \frac{|X|}{n}$. (The original problem asks for the weaker $C\alpha^{-4}$.)

(p) (China 2011) Let $n > 1$ be an integer, and let k be the number of distinct prime divisors of n. Prove that there exists an integer a, $1 < a < \frac{n}{k} + 1$, such that $n \mid a^2 - a$.

(q) (ELMO 2013, W. [5]) Let $m_1, m_2, \dots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \dots, A_{2013}$ be 2013 (possibly empty) sets with $A_i \subseteq \{1, 2, \dots, m_i - 1\}$ for $i = 1, 2, \dots, 2013$. Prove that there is a positive integer N such that $N \le (2|A_1| + 1) \cdots (2|A_{2013}| + 1)$ and for each $i = 1, 2, \dots, 2013$, there does *not* exist $a \in A_i$ such that m_i divides $N - a$.

[5] For extensions/comments see the solutions file here. Actually, this proOblem is related to sieve theory—do you see why? See also exciting recent developments on the related "covering systems".

3. Enumerative combinatorics

   (a) (TST 2011) Let c_n be a sequence which is defined recursively as follows: $c_0 = 1$, $c_{2n+1} = c_n$ for $n \geq 0$, and $c_{2n} = c_n + c_{n-2^e}$ for $n > 0$ where e is the maximal nonnegative integer such that 2^e divides n. Prove that $\sum_{i=0}^{2^n-1} c_i = \frac{1}{n+2}\binom{2n+2}{n+1}$.

   (b) (Putnam 2013 B5) Let $X = \{1, 2, \ldots, n\}$, and let $k \in X$. Show that there are exactly $k \cdot n^{n-1}$ functions $f : X \to X$ such that for every $x \in X$ there is a $j \geq 0$ such that $f^j(x) \leq k$.

4. (ELMO Shortlist 2012, Linus Hamilton) Form the infinite graph A by taking the set of primes p congruent to $1 \pmod 4$, and connecting p and q if they are quadratic residues modulo each other. Do the same for a graph B with the primes $1 \pmod 8$. Show A and B are isomorphic to each other.

5. (a) (RMM 2012, Ilya Bogdanov) Given a positive integer $n \geq 3$, colour each cell of an $n \times n$ square array with one of $\lfloor (n+2)^2/3 \rfloor$ colours, each colour being used at least once. Prove that there is some 1×3 or 3×1 rectangular subarray whose three cells are coloured with three different colours.

   (b) Construct an example showing that $n^2/3 + O(n)$ colors are needed.

6. (Math Prize Olympiad 2010) Let S be a set of n points in the coordinate plane. Say that a pair of points is *aligned* if the two points have the same x-coordinate or y-coordinate. Prove that S can be partitioned into disjoint subsets such that (a) each of these subsets is a collinear set of points, and (b) at most $n^{3/2}$ unordered pairs of distinct points in S are aligned but not in the same subset.

7. Neighbor differences in arrays/matrices/squares/grids of numbers.

   (a) (The Art of Mathematics: Coffee Time in Memphis, 21. Neighbours in a Matrix; also classical) Every $n \times n$ matrix whose entries are $1, 2, \ldots, n^2$ in some order has two neighboring entries (in a row or in a column) that differ by at least n.

   (b) (IdeaMOP 2013, I don't know how to solve this) What's the largest number N of board squares one can label (in a $n \times n$ grid) from 1 to n^2 (distinct but not necessarily consecutive) without creating a difference of n or higher?

   (c) (RMM 2011, Dan Schwarz) The cells of a square 2011×2011 array are labelled with the integers $1, 2, \ldots, 2011^2$, in such a way that every label is used exactly once. We then identify the left-hand and right-hand edges, and then the top and bottom, in the normal way to form a torus (the surface of a doughnut).

   Determine the largest positive integer M such that, no matter which labelling we choose, there exist two neighbouring cells with the difference of their labels at least M. (Cells with coordinates (x, y) and (x', y') are considered to be neighbours if $x = x'$ and $y - y' \equiv \pm 1 \pmod{2011}$, or if $y = y'$ and $x - x' \equiv \pm 1 \pmod{2011}$.)

8. Airport problems.

   (a) (Korea?) Let n be a positive integer. There are n websites numbered $1, 2, \ldots n$, such that website i can link to website j only if $i < j$. At first, the only links are that for $1 \leq i \leq n-1$, website i has a link to website $i+1$. Prove that there is a way to add at most $3(n-1)\log_2(\log_2 n)$ links to the websites so that for any i, j with $i < j$, website j is reachable from website i by clicking at most three links.

   (b) (ISL 2013) In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The distance between two cities is defined to be the least possible numbers of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

9. Simple processes/algorithms, games.

(a) (Variant of 2012-2013 Fall OMO Problem 21) A game is played with 16 cards laid out in a row. Each card has a black side and a red side, and initially the face-up sides of the cards alternate black and red with the leftmost card black-side-up. A move consists of taking a consecutive sequence of AT LEAST TWO CARDS with leftmost card black-side-up and the rest of the cards red-side-up, and flipping all of these cards over. The game ends when a move can no longer be made. What is the maximum possible number of moves that can be made before the game ends?

(b) (Conway's recursive sequence A004073; ISL 1997) Define a sequence of strings as follows: Let $R_0 = 1$, and let R_n be the string obtained by replacing each instance of i in R_{n-1} with $123\dots i$ (1 through i), and finally adding $n+1$ at the end. For example, $R_1 = 12$, $R_2 = 1123$, $R_3 = 11121234$. Show that if we write R_n down and R_n backwards under that, then each column will contain exactly one 1.

(c) (Russia 2003) Ana and Bora start with the letters A and B, respectively. Every minute, one of them either prepends or appends to his/her own word the other person's word (not necessarily operating one after another). Prove that Ana's word can always be partitioned into two palindromes.

(d) (Bulgarian solitaire) Suppose we have $N = 1 + 2 + \cdots + n$ cards total among some number of stacks. In each move, Bob takes one card from each stack and forms a new stack with them. Show that Bob eventually ends up with $1, 2, \dots, n$ in some order.

(e) (MOP 2005?) Suppose n coins have been placed in piles on the integers on the real line. (A pile may contain zero coins.) Let T denote the following sequence of operations.

(a) Move piles $0, 1, 2, \dots$ to $1, 2, 3 \dots$, respectively.

(b) Remove one coin from each nonempty pile from among piles $1, 2, 3, \dots$, then place the removed coins in pile 0.

(c) Swap piles i and $-i$ for $i = 1, 2, 3, \dots$.

Prove that successive applications of T from any starting position eventually lead to some sequence of positions being repeated, and describe all possible positions that can occur in such a sequence.

(f) Nim, Sprague-Grundy theorem, motivation/intuition.

(g) (WOOT 2012, Game Theory Problem 5) Alberto and Barbara are playing the following game. Initially, there are several piles of stones on the table. With Alberto playing first, a player in turn performs one of the following two moves:

(a) take a stone from an arbitrary pile;

(b) select a pile and divide it into two nonempty piles.

The player who takes the last stone wins the game. Determine which player has a winning strategy in dependence of the initial state.

(h) (Putnam 2013 B6) Let $n \geq 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of n spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

(a) places a stone in an empty space, or

(b) removes a stone from a nonempty space s, places a stone in the nearest empty space to the left of s (if such a space exists), and places a stone in the nearest empty space to the right of s (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?

10. Some graph theory.

(a) (TST 2014, Zoltán Füredi) Let n be an even positive integer, and let G be an n-vertex graph with exactly $\frac{n^2}{4}$ edges, where there are no loops or multiple edges (each unordered pair of distinct vertices is joined by either 0 or 1 edge). An unordered pair of distinct vertices $\{x, y\}$ is said to be *amicable* if they have a common neighbor (there is a vertex z such that xz and yz are both edges). Prove that G has at least $2\binom{n/2}{2}$ pairs of vertices which are amicable.

(b) (MOP 2010) Fix n points in space in such a way that no four of them are in the same plane, and choose any $\lfloor n^2/4 \rfloor + 1$ segments determined by the given points. Determine the least number of points that are the vertices of a triangle formed by the chosen segments.

11. Consider a directed graph G with n vertices, where 1-cycles and 2-cycles are permitted. For any set S of vertices, let $N^+(S)$ denote the out-neighborhood of S (i.e. set of successors of S), and define $(N^+)^k(S) = N^+((N^+)^{k-1}(S))$ for $k \geq 2$.

    For fixed n, let $f(n)$ denote the maximum possible number of distinct sets of vertices in $\{(N^+)^k(X)\}_{k=1}^{\infty}$, where X is some subset of $V(G)$.

    (a) (ELMO Shortlist 2012, Linus Hamilton[6]) Show that $f(n)$ is *sub-exponential*, in the sense that for any $c > 1$, we have $f(n) < c^n$ for all sufficiently large n (certainly depending on c).

    (b) (Linus) Let $g(n)$ denote Landau's function: the largest least common multiple of any partition of n. Prove that $g(n) \leq f(n) \leq (n + c)g(n) + n^d$ for some constants $c, d > 0$ independent of n.

    (c) (W.) Prove that $g(n) \leq f(n) \leq g(n) + n^d$ for some constant $d > 0$ independent of n.

4 A touch of geometry

1. (MOP 2007, for James Tao) In triangle ABC, point L lies on side BC. Extend segment AB through B to M such that $\angle ALC = 2\angle AMC$. Extend segment AC through C to N such that $\angle ALB = 2\angle ANB$. Let O be the circumcenter of triangle AMN. Prove that $OL \perp BC$.

2. (Yang Liu, David Yang, based on USAMO 2014) Construct P_0, P_1, P_{-1}, P_2, P_{-2}, P_3, P_{-3}, P_4, P_{-4}, P_5, P_{-5} in order such that

    (i) P_0, P_1, P_{-1} are not collinear;

    (ii) $P_2 \in P_0P_{-1}$ only (i.e. it doesn't lie on any other line so far except the ones specified, in this case P_0P_{-1});

    (iii) $P_{-2} \in P_1P_2$ only;

    (iv) $P_3 \in P_0P_{-2}$ only;

    (v) $P_{-3} \in P_1P_3$ only;

    (vi) $P_4 \in P_0P_{-3}, P_{-1}P_{-2}$ only;

    (vii) $P_{-4} \in P_1P_4, P_2P_3$ only;

    (viii) $P_5 \in P_0P_{-4}, P_{-1}P_{-3}$ only;

    (ix) $P_{-5} \in P_1P_5, P_2P_4$ only.

    Prove that lines P_0P_{-5}, $P_{-1}P_{-4}$, $P_{-2}P_{-3}$ have nonempty intersection.

3. (Putnam 1996; also in analysis) Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive reals x, y such that $\sum (a_i, b_i) x^{a_i} y^{b_i} = (0, 0)$.

4. (ELMO Shortlist 2012, David Yang; also in analysis) We have a compact set in $\mathbb{R}^n$. (For our purposes, just take compact to mean "closed and bounded"; closed basically means "contains the boundary.") We consider the set of directions (formally, unit vectors in the plane). Such a direction is called good if there is exactly one point in the set that is furthest along that direction. (For example, a northernmost point.)

    (a) Prove that there exists a good direction.

    (b) Prove that if $n = 2$, all but countably many directions are good.

[6] Linus said he thought of this problem with Mitchell Lee and David Yang by trying to optimize their answer to the "half(L)" problem on this page, which I haven't looked at but may be interesting.

5. (Putnam 1992 A6) Four points are chosen independently and at random on the surface of a sphere (using the uniform distribution). What is the probability that the center of the sphere lies inside the resulting tetrahedron?

6. (Putnam 2000 A6) Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

7. ("tan9p", Math StackExchange; also in analysis) ABC is an equilateral triangle, and $AD = BE = CF$ for some distinct points D, E, F inside, with A, D, E collinear, B, E, F collinear, and C, F, D collinear. Prove that DEF is an equilateral triangle.

8. (Motzkin-Rabin) Let $\mathcal{S}$ be a finite set of points in the plane (not all collinear), each colored red or blue. Show that there exists a monochromatic line passing through at least two points of $\mathcal{S}$.

9. (heard from Razvan class, MOP 2013) Prove that there are at most n area-halving lines through a fixed point O inside a convex n-gon with no two parallel sides.

10. (The boundaries of disk intersections and unions.)

    (a) The boundary of the *intersection* of $n \geq 2$ (not necessarily intersecting) disks consists of at most $2n - 2$ arcs.

    (b) The *outside* boundary of the *union* of $n \geq 2$ (not necessarily intersecting) disks consists of at most $2n - 2$ arcs.

    (c) (Ashwin Sah, WOOT 2014 Practice Olympiad 7) The boundary of the *union* of $n \geq 3$ (not necessarily intersecting) disks consists of at most $6n - 12$ arcs.

11. (Romania TST 2004; also in analysis) Let D be a closed disk in the complex plane. Prove that for all positive integers n, and for all complex numbers $z_1, z_2, \ldots, z_n \in D$ there exists a $z \in D$ such that $z^n = z_1 z_2 \cdots z_n$.

12. (China 2012; also in analysis) Find the smallest possible value of a real number c such that for any 2012-degree monic polynomial $P(x) = x^{2012} + a_{2011}x^{2011} + \cdots + a_1 x + a_0$ *with real coefficients*, we can obtain a new polynomial $Q(x)$ by multiplying some of its coefficients by -1 such that every root z of $Q(x)$ satisfies the inequality $|\Im z| \leq c|\Re z|$.

13. (V. Galperin and G. Galperin, also various olympiads) Darkness has descended on the plane of Mopok. All electricity has been cut-off by the Trydan Corporation. All they have to light the plane are k lighthouses that run on cooking oil where $k \geq 1$ is a positive integer. They can be kept going indefinitely, the Mopoks love their french fries, but each lighthouse can only illuminate a sector of $360/k$ degrees. The lamp of each lighthouse can be rotated but it must be fixed before the light is turned on. Can the lights be rotated so that the whole plane is covered?

    (The lighthouses positions are given, you don't get to choose where to put them.)

    (Assuming that k is even makes the problem a bit easier.)