Martinet's question on Hilbert 2-class field towers

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Background: Class group and Hilbert class field

- ► A number field K has a (finite abelian) *ideal class group* Cl(K) measuring the failure of unique factorization.
- ► Class field theory distinguishes the *Hilbert class field* K¹ (Galois, abelian over K) with Gal(K¹/K) ≃ Cl(K) (theme: extrinsic vs. intrinsic).
- Example: $K = \mathbb{Q}(\sqrt{3}) = \mathbb{Q}(\sqrt{(-4)(-3)})$ is a UFD, so $K^1 = K$.

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- Example: $K = \mathbb{Q}(\sqrt{3}) = \mathbb{Q}(\sqrt{(-4)(-3)})$ is a UFD, so $K^1 = K$.
- ► One definition of K¹: the maximal *abelian* extension L of K unramified at not just the usual (nonzero) finite primes ℘ ∈ Spec O_K, but also...
- "unramified at the *infinite* primes", i.e. no *real embedding* $K \hookrightarrow \mathbb{R}$ extends to an embedding $L \hookrightarrow \mathbb{C}$ with nonreal image.

Background: Hilbert class field towers and p^{∞} version

• Can iterate $K^{i+1} := (K^i)^1$ to get Hilbert class field tower

$$K \subseteq K^1 \subseteq K^2 \subseteq \cdots \subseteq K^\infty$$

(infinite tower of extensions), with top $K^{\infty} := \bigcup_{n>0} K^n$.

► We call the tower *finite* iff [K[∞] : K] < ∞ (tower stabilizes); otherwise *infinite*. (Aside: the tower is finite iff K can be embedded in a UFD.)

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- ► Our focus: analogous *p*-tower built from Hilbert *p*-class fields.
- Fix a prime p. Let K¹_(p) ≤ K¹ be the Hilbert p-class field of K, i.e. the maximal p-power-degree Galois sub-extension of K¹/K.

Background: History of towers

 History: Artin et al. (1920s) thought a refinement of Minkowski discriminant bound might prove uniform finiteness of towers (c.f. familiar Q case), despite

Theorem (Scholz (1929))

For any prime p and every integer $n \ge 1$, there exists a C_p -extension K/\mathbb{Q} such that $K_{(p)}^{n+1} \neq K_{(p)}^n$.

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Theorem (Vinberg/Gaschütz refinement of Golod–Shafarevich (1960s))

Fix a number field K (with unit group \mathcal{O}_{K}^{\times}) and a prime p. Then K has infinite p-tower if

$$\operatorname{rank}_{p}\operatorname{Cl}(\mathcal{K}) \geq 2 + 2\sqrt{1 + \operatorname{rank}_{p}(\mathcal{O}_{\mathcal{K}}^{\times}/(\mathcal{O}_{\mathcal{K}}^{\times})^{p})}.$$

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- rank_p(O[×]_K/(O[×]_K)^p) is easily computed using Dirichlet's unit theorem.
- ▶ rank_p Cl(K) is easy in special cases (esp. using genus theory).

Martinet's question (specialize to imaginary quadratics)

- ► Recap: **Golod–Shafarevich criterion** in terms of class group and unit group of *K*.
- Let's specialize Golod–Shafarevich to K an imaginary quadratic number field.

Corollary

An imaginary quadratic K has infinite 2-tower if rank₂ Cl(K) \geq 5, i.e. if the discriminant Δ_K has 6 prime factors (genus theory).

Question (Martinet (1978))

What if rank₂ Cl(K) = 4, i.e. if Δ_K has 5 prime factors? Must K still always have infinite 2-tower?

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Remark

No known counterexamples. Martinet was inspired also by Odlyzko (1976) "root discriminant" bounds, as root discriminant $|\Delta_{\mathcal{K}}|^{1/[\mathcal{K}:\mathbb{Q}]}$ is constant in unramified towers.

Background: Prime discriminants and genus theory

• Extrapolating from $\mathbb{Q}(\sqrt{3}) = \mathbb{Q}(\sqrt{(-4)(-3)})$ earlier: for any quadratic number field K with t (finite) ramified primes, we have a unique discriminant factorization

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 \blacktriangleright ... defined so that $2^* \in \{+8,-8,-4\}$ and

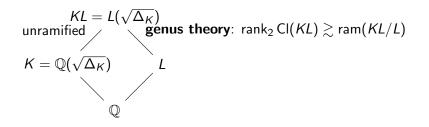
$$p^* = (-1)^{(p-1)/2} p \equiv 1 \pmod{4}$$

for odd primes p.

- ► Classical genus theory (dating back to Euler and Gauss) gives rank₂ Cl(K) ∈ {t − 1, t − 2}, and more.
- ► Relative genus theory relates 2-rank with ramification in general, even over base fields other than Q...

Main idea in literature for Martinet's question

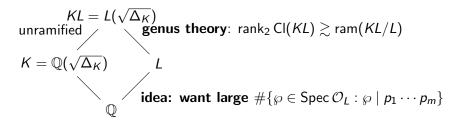
- Let K/Q be an imaginary quadratic with Δ_K = p₁^{*} · · · p₅^{*} < 0 (as in Martinet's question on infinitude of K₍₂₎[∞]/K).
- ► Take L/\mathbb{Q} a finite subfield of $K_{(2)}^{\infty}/\mathbb{Q}$ (the 2-tower of K).
- ▶ **Golod–Shafarevich** says that field $KL = L(\sqrt{\Delta_K})$, hence K, has an infinite 2-tower if rank₂ Cl(KL) $\geq 2 + 2\sqrt{1 + [L : \mathbb{Q}]}$.



Say L/\mathbb{Q} is unramified at $m \ge 1$ primes p_1, \ldots, p_m dividing Δ_K . Then the main (i.e. non-archimedean) contribution to $\operatorname{ram}(KL/L)$ is **splitting** $\#\{\wp \in \operatorname{Spec} \mathcal{O}_L : \wp \mid p_1 \cdots p_m\}$ in L/\mathbb{Q} .

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- ► Recap: For Martinet's question, suffices to find suitable L with large splitting count #{℘ ∈ Spec O_L : ℘ | p₁ · · · p_m}.
- Mouhib (2010), improving on Sueyoshi (2004), proved infinite 2-towers when Δ_K = p₁^{*} · · · p₅^{*} has exactly 1 *negative* prime discriminant, say p₅^{*}, using L ≈ Q(√p₁^{*}, . . . , √p₄^{*}) (totally real, so hard to extend method).

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The other best results mimic Martinet (1978), by taking *L* inside the *(narrow)* genus field $\mathbb{Q}(\sqrt{p_1^*}, \ldots, \sqrt{p_5^*})$ of $K = \mathbb{Q}(\sqrt{\Delta_K})$.

- ► Hajir (1996, 2000), Benjamin (2001, 2002), and Sueyoshi (2004, 2009, 2010) systematically established infinite 2-towers in many *Rédei matrix* (1930s) cases, i.e. by casework on pairwise Kronecker symbols (^{*P*^{*}_i}/_{*P*_i}).
- However, many cases remain open, especially for small rank₄ Cl(K).

Schmithals' 2-class field idea for Martinet's question

- ▶ Goal (recap): find $L/\mathbb{Q} \subseteq K^{\infty}_{(2)}/\mathbb{Q}$ with lots of **splitting**.
- Schmithals' idea (1980): take $L = F_{(2)}^1$ (Hilbert 2-class field) for a (quadratic) field F.
- ► Motivation: decomposition law, e.g. if a rational prime p is inert in F/Q, then the prime ideal pO_F is principal, hence totally split in F¹/F, so also totally split in L/F.

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- ► (In fact, it is *harder* to guarantee lots of splitting in L/Q when p splits in F/Q...)

Theorem (W. (2015))

For distinct primes $\ell_1, \ell_2 \equiv 1 \pmod{4}$, let $F = \mathbb{Q}(\sqrt{\ell_1 \ell_2})$. If p is prime with $\left(\frac{\ell_1}{p}\right) = \left(\frac{\ell_2}{p}\right) = -1$, then (p) splits into exactly 2 primes in the extension L/\mathbb{Q} .

Remark

Dominguez, Miller, and Wong (2013) used a similar result to prove infinitude of imaginary quadratic fields with #CI(F) of any given 2-adic valuation.

Progress on Martinet's question from Schmithals' 2-class field idea

- Goal (recap): find suitable L/Q ⊆ K[∞]₍₂₎/Q with lots of splitting, i.e. #{℘ ∈ Spec O_L : ℘ | p₁ ··· p_m} should be large.
- Recap: Schmithals' idea (1980) of looking at L = F¹₍₂₎ for a choice of quadratic field F.

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- Recap: Schmithals' idea (1980) of looking at L = F¹₍₂₎ for a choice of quadratic field F.
- Choosing among F ⊆ K[∞]₍₂₎ with 4 prime discriminants, Benjamin (2015) partially addressed several open cases with rank₄ Cl(K) ∈ {1,2}.
- Choosing among F ⊆ K[∞]₍₂₎ with 3 or 2 prime discriminants, we (2015) do the same when rank₄ Cl(K) ∈ {0,2}, using tools of the following flavor.

Lemma (W. (2015))

Say $p_4^*, p_5^* > 0$, and let $F := \mathbb{Q}(\sqrt{p_4^* p_5^*})$. If $8 \mid \#Cl(F)$ and at least 1 of p_1, p_2, p_3 is inert in F/\mathbb{Q} , then K has infinite 2-tower.

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Take $K = \mathbb{Q}(\sqrt{(-7)(-3)(-8)(+29)(+5)})$, with $(\frac{+29}{7}) = +1$ and $(\frac{+5}{7}) = -1$, so 7 is inert in $F := \mathbb{Q}(\sqrt{(+29)(+5)})$. Here F has class number 4, so its Hilbert 2-class field $L := F_{(2)}^1$ coincides with its Hilbert class field F^1 , which can be computed in SageMath. (The Lemma fails here since #Cl(F) = 4.)

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► The genus theory input gives an a priori lower bound on rank₂ Cl(KL) from the splitting of 7, 3, 2 in L/Q:

rank₂ Cl(*KL*) ≥ #{
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≥ 4 + 2 + 2 - 1 = 7.

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- In fact, here the bound is tight: the class group Cl(KL) has cyclic direct sum decomposition (336, 336, 4, 4, 2, 2, 2) (assuming the generalized Riemann hypothesis for a reasonable SageMath run-time), so 2-rank exactly 7—just shy of the 2 + 2√8 + 1 = 8 needed for Golod–Shafarevich.
- But Golod-Shafarevich does not take into account the 4-rank of 4, or the 8- and 16- ranks of 2, so it would be nice to have a strengthening incorporating such data.

Thanks for listening!