Diophantine Equations
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Loosely speaking, Diophantine equations (named after Diophantus of Alexandria) are equations over the integers (or some subset of the integers), typically representing a rich mixture of algebraic and number theoretic methods. Diophantine analysis can get very complicated[1], so we will restrict ourselves to the most common elementary techniques in this handout.

1 Prime Factorization

Observing the exponents in the prime factorization of a number can yield much useful information. The following examples illustrate this:

Example 1.1. *Find the number of triples of integers (x, y, z) such that $xyz = 720$.*

Solution 1.1.1. The prime factorization of 720 is $2^4 \cdot 3^2 \cdot 5$. If we look only at the prime factors of x, y, z, we see that the exponents of 2 in the factorizations of x, y, z must sum to 4, the exponents of 3 must sum to 2, and the exponents of 5 must sum to 0. The number of possible triples (a, b, c) that correspond to the exponents of 2 in the prime factorizations of x, y, z is simply the number of number of nonnegative solutions to $a + b + c = 4$, which is 15. Similarly, we find that the number of triples corresponding to the exponent of 3 is 6, and the number of triples corresponding to the exponent of 5 is 3. Because the exponents of 2, 3, and 5 are all independent of one another, the number of pairs (x, y, z) is simply $15 \cdot 6 \cdot 3 = 270$.

Example 1.2. *(Four numbers theorem) If $ab = cd$ for positive integers a, b, c, d, show that there exist positive integers x, y, z, w such that $a = xy$, $b = zw$, $c = xz$, and $d = yw$, where $\gcd(y, z) = 1$.*

Solution 1.2.1. As before, it's enough to just consider the exponents of primes individually. For a fixed prime p, we have $v_p(a) + v_p(b) = v_p(c) + v_p(d)$, and we need to find x, y, z, w such that $v_p(x) + v_p(y) = v_p(a)$, $v_p(z) + v_p(w) = v_p(b)$, $v_p(x) + v_p(z) = v_p(c)$, and $v_p(y) + v_p(w) = v_p(d)$, where $\min(v_p(y), v_p(z)) = 0$. Now it's just a simple system of equations; see if you can finish the argument.
There's also another perspective to this problem. Rewrite the equation as $a/c = d/b = y/z$, where $\gcd(y, z) = 1$ gives us the fraction $a/c = d/b$ in reduced form. Then because its in reduced form, $(a, c) = (xy, xz)$ for some x (when we reduce a fraction, we take out the common divisors of the numerator and denominator). Similarly, $(d, b) = (wy, wz)$ for some integer w.

Example 1.3. *If two numbers a and b are relatively prime, and ab is a perfect kth power, then a and b are both perfect kth powes.*

Solution 1.3.1. A number is a kth power if and only if each exponent in its prime factorization is divisible by k. If the prime factors of a are entirely distinct from the prime factors of b, and the exponents of each of the prime factors of ab are multiples of k, then the exponents of each of the prime factors of both a and b must be multiples of k as well; thus, a and b are both perfect squares.

[1] One well-known example is Fermat's Last Theorem, which states that $a^n + b^n = c^n$ has no solutions in positive integers a, b, c for all integers $n > 2$. Although Pierre de Fermat claimed to have "discovered a truly marvelous proof of this, which, however, the margin is not large enough to contain," proof eluded the best of mathematicians for over three centuries until 1994, when the British mathematician Andrew Wiles finally prevailed using the advanced machinery of elliptic curves and modular forms.

The following is an example of how the above can be applied:

Example 1.4. *Solve $y^2 = x^3 - x$ over the integers.*

Solution 1.4.1. Factor $x^3 - x$ as $x(x^2 - 1)$. Because x and $x^2 - 1$ are relatively prime integers that multiply to a square, they must both be squares. Thus, $x^2 - 1$ must be a square, say, $x^2 - 1 = z^2$. Then $(x - z)(x + z) = 1$, which can only happen when $x = \pm 1$ and $y = 0$.

2 Modular arithmetic

Sometimes, it's difficult to analyze an equation just by algebraic manipulation, and we may need an extra boost. In these cases, it's often helpful to take it mod something, reducing our analysis down to a finite number of possibilities, so sometimes, when only a few cases are possible, we can get a lot of information (e.g. if we take an both sides of an equation modulo m, we may be able to place important restrictions on the variables modulo m).

Example 2.1. *Solve $x^2 + y^2 = 1003$ over the positive integers.*

Solution 2.1.1. Taking this equation mod 4, we have $x^2 + y^2 \equiv 3 \pmod 4$. Since x^2 and y^2 must be congruent to either 0 or 1 mod 4, their sum must be congruent to either 0, 1, or 2 mod 4. In particular, it cannot be congruent to 3 mod 4, so the equation has no solutions.

Here, we had squares $\pmod 4$ (i.e. quadratic residues). In general, if you have squares, you should consider modulo powers of 2 or primes, because there are relatively few quadratic residues and thus there are fewer possibilities and more restrictions (e.g. only 2 possibilities modulo 4). For other d^{th} powers in general, it turns out that primes p congruent to $1 \pmod d$ work well. For instance, the only cubic residues modulo 7 are $0, 1, -1$. Keep in mind that sometimes (especially parity) mods will be only a small part of the problem, but they can still make a big difference.

Example 2.2. *Do there exist integers $x_1, x_2, \ldots, x_{2011}$ such that*

$$x_1^7 + x_2^7 + \cdots + x_{2011}^7 = 123$$
$$x_1 + x_2 + \cdots + x_{2011} = 321$$

Solution 2.2.1. There do not exist such integers. Suppose for the sake of contradiction that a solution exists. By Fermat's little theorem, $x_i^7 \equiv x_i \pmod 7$. Thus, we have

$$321 = x_1 + x_2 + \cdots + x_{2011} \equiv x_1^7 + x_2^7 + \cdots + x_{2011}^7 = 123 \pmod 7,$$

which is false.

3 Algebraic Manipulation

Many times, Diophantine equations can be solved or simplified significantly through the use of algebraic manipulation.

Example 3.1. *Suppose that $a \equiv 1 \pmod p$. Show that $a^p \equiv 1 \pmod{p^2}$.*

Solution 3.1.1. Let $a = pk + 1$. Then by the binomial theorem,

$$a^p - 1 = (1 + pk)^p - 1 = \binom{p}{1}pk + \binom{p}{2}p^2k^2 + \cdots \equiv \binom{p}{1}pk = p^2k \equiv 0 \pmod{p^2},$$

as desired.

Example 3.2. *Find all pairs of integers (x, y) with $0 < x < y$ and $x^y = y^x$.*

Solution 3.2.1. There exists a solution to this using just number theory. We will not give it. Instead, we will use calculus.

Rearrange this to $x^{\frac{1}{x}} = y^{\frac{1}{y}}$. Let $f(x) = x^{\frac{1}{x}}$. We have $f'(x) = -x^{\frac{1}{x}-2}(\ln x - 1)$. Thus, $f(x)$ is decreasing for $x > e$ and increasing for $x < e$. Thus, if $f(x) = f(y)$ and $x \neq y$, we must have $x < e < y$. This means that $x = 1$ or $x = 2$.

Consider the equation as $x^y = y^x$ again. If $x = 1$, we must have $y = 1$ as well, which contradicts $x < 1$. Thus, we need $x = 2$, so $2^y = y^2$. We can show by induction that $2^y > y^2$ for $y > 4$, so either $y = 3$ or $y = 4$. The solution $y = 3$ fails, so we must have $y = 4$. Thus, our only solution is $(2, 4)$.

Example 3.3. *Solve $y^2 = x^2 + y^3x - 1$ over the positive integers.*

Solution 3.3.1. This is a simple instance where the quadratic formula is the way to go, because it forces powerful restrictions upon us. Fix y. Then we have a quadratic equation $x^2 + y^3z - (1 + y^2) = 0$. We need the discriminant to be a perfect square, or else by the quadratic formula, x is irrational. So

$$\Delta_x = (y^3)^2 - 4(1)(-1 - y^2) = y^6 + 4y^2 + 4 = t^2$$

for some positive integer t. Once again, we use the discreteness of the integers: it's obvious that $t > y^3$. But we're working with integers, so this means that $t \geq y^3 + 1$, so

$$y^6 + 4y^2 + 4 = t^2 \geq (y^3 + 1)^2 = y^6 + 2y^3 + 1.$$

Clearly, this inequality cannot hold for large y, because once we subtract y^6 from both sides, the RHS is cubic in y and the LHS is quadratic in y. In fact, it holds only for $y = 1$ and $y = 2$ (prove this yourself!). If $y = 1$, then we get $x^2 + x - 2 = 0$, and $y = 2$, then we get $x^2 + 8x - 5 = 0$. Only $x^2 + x - 2$ has integer roots, so our only solution is $(1, 1)$.

4 Pythagorean Triples

A *Pythagoran triple* is a triple of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. For example, (3,4,5), (6,8,10), (5,12,13), and (8,15,17) are Pythagorean triples. The Pythagorean triples can be classified as follows:

Theorem 4.1. *Suppose (a, b, c) is a Pythagorean triple. Then $(a, b, c) = (k \cdot 2mn, k(m^2 - n^2), k(m^2 + n^2))$ or $(k(m^2 - n^2), k \cdot 2mn, k(m^2 + n^2))$ for some positive integer k and some relatively prime integers m, n.*

Solution 4.0.2. If a, b, c share a common factor $d > 1$, then if (a, b, c) is a Pythagorean triple, $(a/d, b/d, c/d)$ is also a Pythagorean triple. Thus, we can assume that $\gcd(a, b, c) = 1$. Taking the equation mod 4, we now see that c has to be odd, and exactly one of a, b has to be even. Without loss of generality, let a be even. We will show that $(a, b, c) = (2mn, m^2 - n^2, m^2 + n^2)$ for some relatively prime integers (m, n), which will complete the proof of this problem.

Let $a = 2a_0$. Rearrange $a^2 + b^2 = c^2$ as $4a_0^2 = c^2 - b^2 = (c - b)(c + b)$. If a prime p divides both $c - b$ and $c + b$, it must divide $(c - b) + (c + b) = 2c$ and $(c + b) - (c - b) = 2b$. If p divides b and c, p divides $a^2 = c^2 - b^2$ as well, meaning $\gcd(a, b, c) > 1$, a contradiction. Thus, we would need $p = 2$. Since c and b are both odd, $c - a$ and $c + b$ are both even. Thus, $(c - b)/2$ and $(c + b)/2$ are coprime. Since $a_0^2 - ((c - b)/2)((c + b)/2)$, $(c - b)/2$ and $(c + b)/2$ must both themselves be squares. Thus, $c + b = 2m^2$ and $c - b = 2n^2$ for some integers m, n. Solving, we have $c = m^2 + n^2$, $b = m^2 - n^2$, and $a = 2a_0 = 2mn$, as desired.

5 Infinite Descent

The method of infinite descent is essentially an extremal argument: given a solution (sometimes we take it to be minimal), we try to reduce it to either restrict the possible minimal solutions or show that there are no (nontrivial) solutions in the first place.

Example 5.1. *Show that $\sqrt{2}$ is irrational.*

Solution 5.1.1. Suppose for the sake of contradiction that $a/b = \sqrt{2}$ for some positive integers a, b with $a + b$ as small as possible. Then $a^2 = 2b^2$, so by Euclid's lemma $2 \mid a$. But then letting $c = a/2$, we have $2c^2 = b^2$, so $b/c = \sqrt{2}$, contradicting the minimality of $a + b$ (since $c + b < a + b$).

A special type of infinite descent is known as Vieta jumping, or root flipping, where we use in particular the fact that for a quadratic with integer coefficients, either both roots are integers or both roots are not integers (i.e. if one root is an integer, the other must be as well).

Example 5.2. *(IMO 1988.6) Let a, b be positive integers such that $ab+1 \mid a^2+b^2$. Prove that $(a^2+b^2)/(ab+1)$ is a perfect square.*

Solution 5.2.1. Suppose for the sake of contradiction that there exists $k > 0$ not a perfect square such that $(a^2 + b^2)/(ab + 1) = k$; WLOG take $a + b$ to be minimal among all such pairs (a, b). Suppose WLOG that $a \geq b > 0$. Then $a^2 - (kb)a + (b^2 - k) = 0$. By Vieta's formulas, the product of the roots of the quadratic $x^2-(kb)x+(b^2-k) = 0$ are a and a' with $a' = kb-a \in \mathbb{Z}$ and $a' = (b^2-k)/a < a$. If $a' > 0$, then $a'+b < a+b$ contradicts minimality, so $a' \leq 0$. But k is not a perfect square, so $aa' = b^2-k \neq 0 \implies a' < 0 \implies a'b < 0$, so

$$0 < a'^2 + b^2 = k(a'b + 1) \leq 0,$$

which is absurd.

6 Linear Diophantines

A linear Diophantine equation is of the form

$$a_1x_1 + \cdots + a_nx_n = b$$

for some integers $a_1, \ldots, a_n, b$, where WLOG the a_i's are all nonzero (or else we can just take the zeros out). Bzout's identity (and its generalizations) show that solutions $(x_1, \ldots, x_n)$ exist to such a system if and only if $\gcd(a_1, \ldots, a_n) \mid b$. (See if you can prove this using induction on $n \geq 1$.) A simple implication of this is the Frobenius coin problem: for all sufficiently large integers $b > N$ divisible by $\gcd(a_1, \ldots, a_n)$, there exist *nonnegative* integers $(x_1, \ldots, x_n)$ that solve the equation.

Example 6.1. *(Chicken McNugget theorem) Let a, b be relatively prime positive integers. Then for every integer m, exactly one of m and $ab - a - b - m$ is representable as a nonnegative linear combination of a and b. As a corollary, we have that $ab - a - b$ is the largest unrepresentable number (since 0 is obviously representable).*

Solution 6.1.1. Hint: show that they are not both representable, but they are also not both unrepresentable. Use the fact that $ax + by = a(x - tb) + b(x + ta)$.

7 Problems

These are roughly ordered by difficulty (very roughly at the end). The problems in the beginning are more computational (e.g. AMC/AIME style).

1. Solve the two equations $x + 2y = 0$ and $x + 2y = 1$ in integers x, y.

2. Solve the equation $6x + 10y - 15z = 1$ in integers x, y, z.

3. Find the number of ordered pairs (x, y) of nonnegative integers such that $x + 2y = n$, where n is a fixed nonnegative integer.

4. Given an integer n and two relatively prime positive integers a, b, show that exactly one of n and $ab - a - b - n$ is expressible in the form $ax + by$ for some nonnegative integers x, y.

5. (IMO?) Let a, b, c be pairwise relatively prime positive integers. Show that $2abc - ab - bc - ca$ is the largest integer that cannot be expressed in the form $xbc + yca + zab$, where x, y, z are nonnegative integers.

6. Prove that there are arbitrarily long sequences of consecutive integers, none of which can be written as the sum of two perfect squares.

7. (2011 AIME) Find the number of positive integers m for which there exist nonnegative integers $x_0, x_1, \ldots, x_{2011}$ such that $m^{x_0} = \sum_{k=1}^{2011} m^{x_k}$.

8. (2011 AIME) For some integer m, the polynomial $x^3 - 2011x + m$ has the three integer roots a, b, and c. Find $|a| + |b| + |c|$.

9. (2009 AIME) The terms of the sequence (a_i) defined by $a_{n+2} = \dfrac{a_n + 2009}{1 + a_{n+1}}$ for $n \geq 1$ are positive integers. Find the minimum possible value of $a_1 + a_2$.

10. (2008 AIME) Find the largest integer n such that n^2 is the difference of two consecutive cubes and $2n + 79$ is a perfect square.

11. (2008 AIME) There exist r unique nonnegative integers $n_1 > n_2 > \cdots > n_r$ and r unique integers a_k ($1 \leq k \leq r$) with each a_k either 1 or -1 such that

$$a_1 3^{n_1} + a_2 3^{n_2} + \cdots + a_r 3^{n_r} = 2008.$$

Find $n_1 + n_2 + \cdots + n_r$.

12. (2000 AIME) A point whose coordinates are both integers is called a lattice point. How many lattice points lie on the hyperbola $x^2 - y^2 = 2000^2$?

13. Solve $x^2 + 1 = 2y^4$ over the integers.

14. Prove that the equation $x^4 = y^2 + z^2 + 4$ has no integer solutions.

15. (St. Petersburg) Prove that the equation $3^k = m^2 + n^2 + 1$ has infinitely many solutions in positive integers.

16. Let $p > 2$ be a prime. Prove that $p \equiv 1 \pmod 4$ iff there exist integers x, y such that $x^2 - py^2 = -1$.

17. Solve the equation $x^3 + 117y^3 = 5$ over the integers.

18. Find infinitely many integral solutions of $(x^2 + x + 1)(y^2 + y + 1) = z^2 + z + 1$.

19. Does $m^3 + 6m^2 + 5m = 27n^3 + 9n^2 + 9n + 1$ have any integer solutions?

20. Solve $1! + 2! + \cdots + x! = y^2$ over the positive integers.

21. Solve $x^2 + y^2 = 2z^2$ over the positive integers.

22. Solve $x^2 + y^2 = 3z^2$ over the positive integers.

23. (UK) Find all triples of positive integers such that $2 = \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(1 + \frac{1}{z}\right)$.

24. Show that $x^3 + 3y^3 = 9z^3$ has no nontrivial integer solutions.

25. Find all nonnegative integer solutions to $x^3 + 8x^2 - 6x + 8 = y^3$.

26. Several different positive integers lie strictly between two successive squares. Prove that their pairwise products are also different.

27. Solve $19x^3 - 84y^2 = 1984$ over the integers.

28. (IMO 1997) Find all pairs (a, b) of positive integers such that $a^{b^2} = b^a$.

29. If $n = a^2 + b^2 + c^2$ for positive integers a, b, c, show that there exist positive integers x, y, z such that $n^2 = x^2 + y^2 + z^2$.

30. Show that $x^2 + y^2 + z^2 = x^3 + y^3 + z^3$ has infinitely many integer solutions.

31. If $d > 1$ is a squarefree integer, show that $x^2 - dy^2 = c$ gives some bounds in terms of a fundamental solution.

32. Find all n such that $x^2 - y^2 = n$ has integer solutions.

33. An odd positive integer is the product of n distinct primes. In how many ways can it be represented as the difference of two squares?

34. If $3n + 1$ and $4n + 1$ are perfect squares, show that $56|n$.

35. (ISL 1985) Are there integers m and n such that $5m^2 - 6mn + 7n^2 = 1985$?

36. Prove that every Pythagorean triple contains a multiple of 5.

37. Solve $x^2 = 2^n + 3^n + 6^n$ over the positive integers.

38. Solve $x^3 + 3y^3 + 9z^3 - 3xyz = 0$ over the positive integers.

39. (Dirichlet) Solve $x^3 + 2y^3 + 4z^3 - 6xyz = 1$ over the positive integers.

40. (ISL) Let M denote the number of integral solutions to the equation $x^2 - y^2 = z^3 - t^3$ with $0 \leq x, y, z, t \leq 10^6$, and let N be the number of such solutions with $x^2 - y^2 = z^3 - t^3 + 1$. Show that $M > N$.

41. Solve $x^2 + 5y^2 = z^2$ over the integers.

42. Prove that $x^5 - y^2 = 4$ has no integer solutions.

43. Prove that the equation $4xy - x - y = z^2$ has no solutions in positive integers.

44. Show that $x^3 + y^3 + z^3 + t^3 = 1999$ has infinitely many integer solutions.

45. Show that $(x + 1)^2 + (x + 2)^2 + \cdots + (x + 99)^2 = y^z$ is not solvable in integers x, y, z with $z > 1$.

46. Find all positive integer solutions to $x^2 - y! = 2001$.

47. (IMO) Find all positive integer solutions to $a^{b^2} = b^a$.

48. Prove that $x^2 + (x + 1)^2 = y^2$ has infinitely many solutions in positive integers x, y.

49. (Fermat) If x, y are positive integers, prove that $x^2 - y^2$ and $x^2 + y^2$ cannot both be perfect squares.

50. (Fermat) Given 4 squares in arithmetic progression, show that they must all be equal.

51. Prove that $x^4 + y^4 = z^2$ has no solutions in positive integers.

52. Prove that $x^4 - y^4 = z^2$ has no solutions in positive integers.

53. Find all integer solutions to $x^4 - x^2y^2 + y^4 = z^2$.

54. Find all integer solutions to $x^4 + x^2y^2 + y^4 = z^2$.

55. (IMO 1982) Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers x, y, then it has at least three such solutions. Show that the equation has no solutions in integers for $n = 2891$.

56. Solve $1/x + 1/y = 1/z$ over the positive integers.

57. Solve $1/x^2 + 1/y^2 = 1/z^2$ over the positive integers.

58. Can the product of five consecutive positive integers be a perfect square?

59. (USAMO 2005) If $x^6 + x^3 + x^3y + y = 147^{157}$ and $x^3 + x^3y + y^2 + y + z^9 = 157^{147}$, prove that x, y, z are not all integers.

60. (IMO 1996) The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

61. (ISL 2001) Find the greatest value of the real constant m such that $m \le x/y$ for any positive integers x, y, z, u satisfying $x \ge y$, $x + y = z + u$, and $2xy = zu$.

62. Show that the product of two positive integers of the form $a^2 + ab + b^2$ is of the same form (i.e. the set of integers of this form is closed under multiplication).

63. If $\gcd(a, c) = \gcd(b, c) = 1$, show that $x^a + y^b = z^c$ has infinitely many positive integer solutions.

64. (ISL 1997) Let a, b, c be positive integers such that a and b are relatively prime and c is relatively prime either to a or b. Prove that there exist infinitely many triples (x, y, z) of distinct positive integers such that $x^a + y^b = z^c$.

65. (ISL 1999) Prove that every positive rational number is expressible in the form $\dfrac{a^3 + b^3}{c^3 + d^3}$ for some positive integers a, b, c, d.

66. Prove that $y^2 = x^3 + 7$ has no integral solutions.

67. (Engel) Find all integer solutions to $x^3 + x^2y + xy^2 + y^3 = 8(x^2 + xy + y^2 + 1)$.

68. (Engel) Find all integers m, n such that $(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n$.

69. (Engel) Find all integral solutions of $y^2 + y = x^4 + x^3 + x^2 + x$.

70. (Ljunggren) Find all positive integers $x, n > 1$ such that $\dfrac{x^n - 1}{x - 1}$ is an even perfect square.

71. (ISL?) If $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = 0$ with $a_i \in \{1, -1\}$ for all i, show that $4|n$.

72. If $p = a^2 + nb^2 = c^2 + nd^2$ for a prime p and positive integers a, b, c, d, n, show that $a = c$.

73. (Engel) Find the smallest positive integer of the form $|12^m - 5^n|$ for two integers m, n.

74. Find all primes p such that $(p - 1)! + 1$ is a perfect power of p.

75. (Brazil 2010) Solve $3^a - 1 = 2b^2$ over the positive integers.

76. (China 2010) Find all positive integers $m, n \ge 2$ such that $m + 1 \equiv 3 \pmod 4$ is a prime number and for some prime number p and nonnegative integer a,

$$\frac{m^{2^n - 1} - 1}{m - 1} = m^n + p^a.$$

77. (Russia 2011) For an integer a, let $P(a)$ be a largest prime positive divisor of $a^2 + 1$. Prove that there exist infinitely many triples of distinct positive integers (a, b, c) such that $P(a) = P(b) = P(c)$.

78. (Russia 2005) Integers $x > 2$, $y > 1$, $z > 0$ satisfy $x^y + 1 = z^2$. If p denotes the number of different prime divisors of x and q denotes the number of prime divisors of y, prove that $p \ge q + 2$.

79. (Victor Wang) Solve $y^2 = 8x^4 - 8x^2 + 1$, $y^2 = 20x^4 - 4x^2 + 1$, and $y^2 = 2x^4 - 2x^2 + 1$ over the integers.

80. Rational approximation dude. If d is a positive integer and not a perfect square, show that $x^2 - dy^2 = 1$ has a solution in positive integers x, y.

81. (nnosipov, AoPS) Find the least positive constant c for which

$$\frac{m}{n} < \sqrt{34} < \frac{m}{n} + \frac{c}{mn}$$

has infinitely many solutions in positive integers m, n.

82. (David Yang) If $(x-1)(y-1), xy, (x+1)(y+1)$ are all perfect squares for some positive integers $x, y > 1$, show that $x = y$.

83. (nnosipov, AoPS) Are there rational numbers x, y, z such that $x^2 - y^2 = 2011 = z^2 - x^2$?

84. Find all positive integer solutions satisfying both $x^2 + y^2 = zt$ and $z^2 - t^2 = 2xy$.

85. If $z^2 = 4 + \dfrac{x^2+1}{y^2}$ for integers x, y, z, show that $z^2 = 9$.

86. (nnosipov, AoPS) For some positive integers x and y, the number x^2 is divisible by $2xy + y^2 - 1$. Prove that $2x$ divides $y^2 - 1$.

87. Find all integers p, q such that $(p^2 - q^2 + 4pq)^2 - 12p^2q^2$ is a perfect square.

88. (Bulgaria) Let a, b, c be positive integers such that ab divides $c(c^2 - c + 1)$ and $a + b$ is divisible by $c^2 + 1$. Prove that $\{a, b\} = \{c, c^2 - c + 1\}$.

89. (ISL 2000) Prove that there exist infinitely many positive integers n such that $p = nr$ where p and r are the semiperimeter and the inradius of a triangle with integer side lengths, respectively.

90. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$,

$$(2^m + 1)f(n)f(2^m n) = 2^m f(n)^2 + f(2^m n)^2 + (2^m - 1)^2 n.$$

91. (China) Let $x < y$ be positive integers and

$$P = \frac{x^3 - y}{1 + xy}.$$

Find all integer values that P can take.

92. (China) Find all positive integer solutions to

$$(a + b)^x = a^y + b^y.$$

93. (IMO) Let $a > b > c > d$ be positive integers satisfying $ac + bd = (b + d + a - c)(b + d + c - a)$. Prove that $ab + cd$ is composite.

94. (MOP 2010) Let $a > b > c > d$ be positive integers satisfying $ac + bd = (b + d + a - c)(b + d + c - a)$. Compute the smallest possible number of total prime factors of $(ab + cd)(ac + bd)(ad + bc)$ (i.e. $p^i \| (ab + cd)(ac + bd)(ad + bc)$ adds i to the count).

95. (ISL 2009) Let k be a positive integer. If $a_0, a_1, \ldots$ is a sequence of integers such that

$$a_n = \frac{a_{n-1} + n^k}{n} \forall n \geq 1,$$

then show that $k - 2$ is divisible by 3.

96. (ISL 2009) Find all positive integers n such that there exists a sequence of positive integers $a_1, a_2, \ldots, a_n$ satisfying

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every k with $2 \leq k \leq n - 1$.

97. (Euler) Show that there are infinitely many quadruples of positive integers (a_1, a_2, a_3, a_4) such that $a_i a_j + 1$ is a perfect square for all $1 \le i < j \le 4$.

98. (Kiran Kedlaya) Given positive integers x, y, z such that $(xy+1)(yz+1)(zx+1)$ is a perfect square, show that each of $xy+1, yz+1, zx+1$ is a perfect square itself.