MOP 2018: ANALYTIC NUMBER THEORY (06/22, BK)

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1. Assorted problems

Problem 1.1 (2013-2014 Spring OMO)**.** Characterize all pairs (m, n) of integers such that $x^3 + y^3 = m + 3nxy$ has infinitely many integer solutions (x, y).

Problem 1.2. Prove by infinite descent that $(\frac{-3}{p}) = -1$ for odd primes $p \equiv 2 \pmod 3$.

Problem 1.3. Several different positive integers lie strictly between two successive squares. Prove that their pairwise products are also different.

Problem 1.4 (HMIC 2016/4)**.** Let P be an odd-degree integer-coefficient polynomial. Suppose that $xP(x) = yP(y)$ for infinitely many pairs x, y of integers with $x \neq y$. Prove that the equation $P(x) = 0$ has an integer root.

Problem 1.5 (Dan Schwarz, RMM 2010/1)**.** For a finite nonempty set of primes P, let $m(P)$ denote the largest possible number of consecutive positive integers, each of which is divisible by at least one member of P.

(1) Show that $|P| \leq m(P)$, with equality if and only if $\min(P) > |P|$.
(2) Show that $m(P) < (|P| + 1)(2^{|P|} - 1)$.

Remark 1.6. See ELMO 2013/3 for discussion on related "sieve-like" problems.

Problem 1.7 (Pell equation; special case of Dirichlet's unit theorem)**.** Let d be a positive squarefree integer. Prove that Pell's equation, $x^2 - dy^2 = 1$, has a nontrivial integer solution, $(x, y) \neq (\pm 1, 0)$.

Problem 1.8 (TST 2014)**.** Let $a_1, a_2, a_3, \dots$ be a sequence of integers, with the property that every consecutive group of a_i's averages to a perfect square. More precisely, for every positive integers n and k, the quantity

$$\frac{a_n + a_{n+1} + \cdots + a_{n+k-1}}{k}$$

is always the square of an integer. Prove that the sequence must be constant (all a_i are equal to the same perfect square).

Problem 1.9 (Russia 2002)**.** Show that the numerator of the reduced fraction form of $H_n = 1/1 + 1/2 + \cdots + 1/n$ is infinitely often not a prime power.

Problem 1.10 (USAMO 2012/3)**.** Determine which integers $n > 1$ have the property that there exists an infinite sequence $a_1, a_2, a_3, \dots$ of nonzero integers such that the equality

$$a_k + 2a_{2k} + \ldots + na_{nk} = 0$$

holds for every positive integer k.

Problem 1.11 (Miklos 2000/4)**.** Let $a < b < c$ be positive integers. Prove that there exist integers x, y, z, not all zero, such that $ax + by + cz = 0$ and $\max(|x|, |y|, |z|) \leq 1 + \frac{2}{\sqrt{3}}\sqrt{c}$, and show that the constant $\frac{2}{\sqrt{3}}$ cannot be improved.

2. Equidistribution

Let α be an irrational number. Let $e(x) := e^{2\pi i x}$.

Problem 2.1. Show that $\frac{1}{N}\sum_{n=1}^{N} e(\alpha n) \to 0$ as $N \to \infty$.

Problem 2.2. Show that $\frac{1}{N}\sum_{n=1}^{N} e(\alpha n^2) \to 0$ as $N \to \infty$.

These results are part of the subject of "estimating exponential sums".

3. Transcendence theory

Theorem 3.1 (Gelfond–Schneider theorem)**.** *If a and b are algebraic numbers with $a \neq 0$, $a \neq 1$, and b irrational, then* any value *of a^b is a transcendental number.*

Theorem 3.2 (Lindemann–Weierstrass Theorem, Baker's reformulation)**.** *If $a_1, \ldots, a_n$ are nonzero algebraic numbers, and $\alpha_1, \ldots, \alpha_n$ are distinct algebraic numbers, then $a_1 e^{\alpha_1} + \cdots + a_n e^{\alpha_n} \neq 0$.*

Baker's theorem (on "linear forms in logarithms") generalizes both results above.

4. Ideas in Dirichlet's theorem

Let $\chi\colon (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$ be a *multiplicative character* modulo m. By abuse of notation, we extend it to a periodic multiplicative function $\chi\colon \mathbb{Z} \to \mathbb{C}^\times$ such that $\chi(a) = 0$ if and only if $\gcd(a, m) > 1$. The extension $\mathbb{Z} \to \mathbb{C}^\times$ is known as a *Dirichlet character of modulus m.*

Definition 4.1. Define the *Dirichlet L-function* $L(s, \chi) := \sum_{n\geq 1} \frac{\chi(n)}{n^s}$ when $\Re(s) > 1$.

Problem 4.2. Study $L(s, \chi)$ for the two Dirichlet characters χ of modulus $m = 4$, especially as $s \to 1^+$ in $\mathbb{R}$, to prove Dirichlet's theorem for primes congruent to 1 or 3 modulo 4.

A general modulus m requires more work. Below, assume χ is *nontrivial.*

Problem 4.3. Extend the definition of $L(s, \chi)$ from $\Re(s) > 1$ to $\Re(s) > 0$.

Now, here are the most difficult steps in Monsky's elementary proof of Dirichlet's theorem.

Problem 4.4. If χ is *real-valued*, then $f(x) := \sum_{n\geq 1} \chi(n)\frac{x^n}{1-x^n}$ is unbounded as $x \to 1^-$.

Problem 4.5. If $L(1, \chi) = 0$, i.e. $\sum_{n\geq 1} \frac{\chi(n)}{n} = 0$, then $f(x)$ is bounded as $x \to 1^-$.

Consequently, $L(1, \chi) \neq 0$ for any (nontrivial) real-valued character χ. A standard argument, based on the product $\prod_{\chi \bmod m} L(s, \chi)$, shows that $L(1, \chi) \neq 0$ for complex-valued characters χ too.

Following Dirichlet's classical idea of finite Fourier analysis on $(\mathbb{Z}/m)^\times$ (generalizing the intended "second roots of unity filter" method above for $m = 4$), the non-vanishing of all the values $L(1, \chi)$ guarantees that for every residue class $\overline{a} \in (\mathbb{Z}/m)^\times$, we have

$$\sum_{\overline{p}=\overline{a}} \frac{1}{p^s} = \frac{1}{\phi(m)} \log\left(\frac{1}{s-1}\right) + O(1)$$

as $s \to 1^+$.